Theory of mixed motives

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1. From Grothendieck's theory of motives to the theory of mixed motives.

One of the main motivations for A. Grothedieck for laying the foundations of algebraic geometry and developing the theory of étale cohomology was the Weil conjecture on the zeta function. This conjecture concerns the congruence zeta function of an algebraic variety defined over a finite field. If X is a smooth projective variety over a finite field $\mathbb{F}_q$, its zeta function $Z(X;t) \in \mathbb{Q}[[t]]$ is by definition the formal power series $\exp(\sum_{n\geq 1} N_n t^n/n)$, encoding the number of rational points N_n of X over $\mathbb{F}_{q^n}$ for $n \geq 1$. The Weil conjecture asserts that (a) $Z(X;t)$ is a rational function in t; (b) a functional equation is satisfied for $Z(X;t)$; and (c) an analogue of the Riemann hypothesis should hold.

It was known to A. Weil that if there was a cohomology theory for varieties over $\mathbb{F}_q$ with suitable properties, then the Weil conjecture follows. After years of search for such a cohomology theory, the ℓ-adic étale cohomology was introduced by M. Artin and Grothendieck, and was shown to satisfy most of the required properties. To be more precise, for a prime integer ℓ and for a variety X over a field with characteristic $p \neq \ell$, one can define a vector space over the field $\mathbb{Q}_\ell$ of ℓ-adic numbers, $H^i_{ét}(X, \mathbb{Q}_\ell)$, called the ℓ-adic étale cohomology; one may view it as an analogue of the Betti (namely singular) cohomology, say with $\mathbb{Q}$-coefficients, for topological spaces. As for Betti cohomology, one has properties such as finite dimensionality, Poincaré duality, the Künneth formula (k is assumed separably closed for these properties), and an algebraic cycle has its associated class in the étale cohomology.

To explain the connection to the zeta function, let X be a smooth projective variety over a finite field $\mathbb{F}_q$, $\overline{X} = X \otimes \overline{\mathbb{F}}_q$, and let $F : \overline{X} \to \overline{X}$ be the Frobenius endomorphism (the map that sends a point with coordinates x to the point with coordinates x^q). There is then the induced action F^* on the ℓ-adic cohomology $H^i_{ét}(\overline{X}, \mathbb{Q}_\ell)$, and one can consider the characteristic polynomial

$$P_i(t) = \det(1 - tF^* | H^i_{ét}(\overline{X}, \mathbb{Q}_\ell) \in \mathbb{Q}_\ell[t]$$

for $i = 0, \dots, 2\dim X$. Using the trace formula (of Lefschetz type) it can be shown that the zeta function factors as an alternating product $Z(X;t) = \prod_{i=0,\cdots,2\dim X} P_i(t)^{(-1)^{i+1}}$ over $\mathbb{Q}_\ell$. With this it is not hard to verify part of the Weil conjecture, namely the rationality and the functional equation. The remaining part, the analogue of the Riemann hypothesis, can be deduced from the following statement regarding the eigenvalues of the Frobenius endomorphism acting on the ℓ-adic cohomology of the variety: The polynomial $P_i(t)$ has integer coefficients, independent of ℓ, whose reciprocal roots are of absolute value $q^{i/2}$. It was the conjecture in this form that was finally proven by P. Deligne (see [De-3]).

Going back to the 1950's, in his letter addressed to Weil, J.-P. Serre formulated and proved the following analogue of the Weil conjecture for complex varieties (see [Se]): Let X be a smooth projective variety over $\mathbb{C}$, and $F : X \to X$ a map. Assume there is an integer $q > 0$ and an ample divisor H such that F^*H is algebraically equivalent to qH. Then the eigenvalues of the endomorphism F^* on $H^r(X, \mathbb{C})$ induced by F has $q^{r/2}$ for absolute values. The proof uses that the cohomology has Hodge structure, along with the Lefschetz "primitive" decomposition (which follows from the hard Lefschetz theorem) and the polarization (which exists by the Hodge index theorem).

Inspired by this work, Grothendieck formulated the "standard conjectures" on the cohomology classes of algebraic cycles. These conjectures can be formulated for any *Weil cohomology* theory, including the "usual" cohomology, such as Betti, ℓ-adic, or the algebraic de Rham theory. The Betti cohomology is another name for the singular cohomology of an algebraic variety over $\mathbb{C}$. The algebraic de Rham cohomology $H^i_{DR}(X/k)$ of a smooth projective variety X over a field k (assumed of characteristic zero) is by definition the Zariski hypercohomology of the algebraic de Rham complex $\Omega^\bullet_X$ of the variety. By a Weil cohomology we mean any abstract cohomology theory for smooth projective varieties over a field k, namely a contravariant functor $X \mapsto H^*(X)$, that takes values in graded K-vector spaces, with K a field of characteristic zero, subject to a set of axioms such as finite dimensionality, Poincaré duality, the Künneth formula and the cycle class map. The standard conjectures consist of the conjectures of Lefschetz type, and those of Hodge type; the former asserts that certain cohomology classes be algebraic, and the latter is an analogue of the Hodge index theorem for "primitive" algebraic classes. Grothendieck observed that these conjectures for any Weil cohomology over a finite field imply the Weil conjecture via an argument similar to Serre's. But even further, the same conjectures for any Weil cohomology over a field k imply the existence of a "good" theory of *pure motives*. Briefly speaking, through a Weil cohomology one can construct an additive category $\mathcal{M}(k)$ of pure motives over k. The conjectures imply that $\mathcal{M}(k)$ is independent of the choice of a Weil cohomology, that it is a semi-simple abelian category, and that there is a contravariant functor $X \mapsto h(X) \in \mathcal{M}(k)$, called the motivic cohomology, from the category of smooth projective varieties over k, such that each Weil cohomology functor factors through h; in other words various cohomology theories (Betti, ℓ-adic, and de Rham) can be "unified" to motivic cohomology. (An abelian category is semi-simple if any sub-object of an object is a direct summand.) This is **Grothendieck's theory of motives**. See [Kl-1],[Kl-2], [Ma] for this; we will briefly mention it in §4.

Before Grothendieck's theory evolved into the theory of mixed motives, there were several developments in algebraic geometry; the principal ones being, as we review below, the theory of weights on cohomology, algebraic K-theory and the values of zeta functions, and the study of Chow groups.

Firstly, P. Deligne established the theory of weights of the cohomology of a variety over $\mathbb{C}$ or over a finite field. Recall that for a smooth projective variety X over $\mathbb{C}$, the cohomology $H^i(X(\mathbb{C}), \mathbb{Q})$ has so-called Hodge structure of weight i (namely there is the Hodge decomposition $H^i(X, \mathbb{Q}) \otimes \mathbb{C} = \bigoplus_{p+q=i} H^{p,q}$, $\overline{H^{p,q}} = H^{q,p}$). Generalizing this, Deligne [De-2] showed that, for any quasi-projective variety X, its cohomology has a *mixed Hodge structure*; informally, a mixed Hodge structure is an iterated extension of Hodge structures of different weights. Hodge structure of weight i is often called pure Hodge structure in order to distinguish it from mixed Hodge structure. For varieties over a finite field, generalizing the Weil conjecture, Deligne developed the theory of pure ℓ-adic sheaves, [De-4]. Subsequently the theory was extended to the theory of mixed sheaves in [BBD], which is an ℓ-adic analogue of mixed Hodge theory.

Secondly, algebraic K-theory for exact categories was developed in the 1970's by D. Quillen. For a noetherian scheme X there correspond the groups $K_n(X)$ for $n \geq 0$, the algebraic K-group of the exact category of vector bundles on X; for $n = 0$, $K_0(X)$ is the Grothendieck group of the commutative monoid of the isomorphism classes of vector bundles on X. In [Bo], A. Borel showed the algebraic K-group of a number field F and the zeta function of F are connected as follows. He defined the *regulator map*, $K_{2m-1}(F) \to V_m$ for integers $m \geq 2$, where V_m is a certain finite dimensional $\mathbb{R}$-vector space, and related this map to the value (more precisely the order and the leading coefficient) of $\zeta_F(s)$ at $s = m$. Subsequently S. Bloch ([Bl-3]) defined a regulator map for an elliptic curve E over $\mathbb{C}$, $K_2(E) \to \mathbb{C}$; when E has complex multiplication, defined over an imaginary quadratic field, then he showed that the value of the L-function $L(E, s)$ at $s = 2$ can be expressed using this regulator map. This should be contrasted to the Birch-Swinnerton-Dyer conjecture on the value of $L(E, s)$ at $s = 1$, which has to do with the Mordell-Weil group.

Thirdly, we look at some works on algebraic cycles. For a smooth projective variety X over $\mathbb{C}$, one has the Chow group $\mathrm{CH}^r(X)$ of algebraic cycles of codimension r modulo rational equivalence. If $r = 1$, it is an extension of an abelian variety $\mathrm{Pic}^0(X)$ by a finitely generated abelian group. In contrast, for $r \geq 2$, D. Mumford has shown that $\mathrm{CH}^r(X)$ is too large so that it does not have any finite dimensional algebro-geometric structure. But S. Bloch ([Bl-1]) postulated that the Chow group (tensored with $\mathbb{Q}$) have a certain filtration by subgroups, in such a way that the successive quotients are controlled by the usual cohomology of X. This conjecture, which has far-reaching consequences, has since served as a guiding principle for the study of Chow groups.

Pushing this circle of ideas further, A. Beilinson proposed the framework of the abelian category of **mixed motives** over a field, [Be]. In an ideal form, one may summarize the theory as follows.

(1) it contains the theory of Grothendieck's motives over a field;

(2) it unifies the theory of weights for not-necessarily smooth quasi-projective varieties;

(3) it naturally explains the existence of the filtration on the Chow group as conjectured by Bloch;

(4) it gives rise to the theory of motivic cohomology of varieties; and

(5) it clarifies the meaning of the regulator maps for varieties.

We now give further explanation for these conjectures, in particular (1)-(4), although the details are postponed to later sections. As for (5), we point out that the theory of mixed motives is loosely connected to the so-called Beilinson conjectures: he made conjectures on the values of zeta functions of varieties over number fields, generalizing the works of Borel, of Bloch, and the conjecture of Birch-Swinnerton-Dyer.

The hypothetical abelian category of mixed motives over a field k is an $\mathbb{Q}$-linear abelian category $M\mathcal{M}(k)$, with tensor product structure and Tate objects $\mathbb{Q}(r)$, $r \in \mathbb{Z}$, satisfying the following properties:

- There is a weight structure on $M\mathcal{M}(k)$. Namely each object M of $\mathcal{M}(k)$ comes equipped with a finite increasing filtration (the weight filtration) $W_\bullet M$, and each morphism in $M\mathcal{M}(k)$ is strictly compatible with the weight filtration. The category of Grothendieck motives $\mathcal{M}(k)$, defined through a Weil cohomology, is an exact full subcategory of $M\mathcal{M}(k)$, and it coincides with the full subcategory of semi-simple objects.
- Let $D^b(M\mathcal{M}(k))$ be the bounded derived category of $M\mathcal{M}(k)$. To each quasi-projective variety X over k there corresponds an object $h(X)$ in $D^b(M\mathcal{M}(k))$, and the association $X \mapsto h(X)$ forms a contravariant functor from the category of quasi-projective varieties over k to

$D^b(M\mathcal{M}(k)\,)$. Thus for integers i, r with $r \geq 0$, one can define

$$H^i_{\mathcal{M}}(X, \mathbb{Q}(r)) := \mathrm{Hom}_{D^b(M\mathcal{M}(k))}(\mathbb{Q}(0), h(X) \otimes \mathbb{Q}(r)[i])$$

which we call the **motivic cohomology group** of X.

Since $h(X)$ is represented by a complex in $M\mathcal{M}(k)$, one has the "canonical" increasing filtration $\tau_{\leq i} h(X)$, and one has cohomology

$$\underline{H}^i(X) := H^i(h(X)) = \tau_{\leq i} h(X) / \tau_{\leq i-1} h(X) \ .$$

- If X is smooth over k, $H^i_{\mathcal{M}}(X, \mathbb{Q}(r))$ coincides with the group $K_{2r-i}(X)^{(r)}_{\mathbb{Q}}$, the r-th eigenspace of $K_{2r-i}(X) \otimes \mathbb{Q}$ with respect to the action of the Adams operation.
- If X is smooth and $i = 2r$, the group $H^{2r}_{\mathcal{M}}(X, \mathbb{Q}(r))$ coincides with $\mathrm{CH}^r(X) \otimes \mathbb{Q}$. The filtration $\tau_{\leq i} h(X)$ on $h(X)$ induces a filtration $F^\bullet$ on the Chow group given by the formula

$$\begin{aligned} & F^\nu \,\mathrm{CH}^r(X)_{\mathbb{Q}} \\ = \ & \mathrm{Im}[\mathrm{Hom}_{D^b(M\mathcal{M}(k))}(\mathbb{Q}(0), \tau_{\leq 2r-\nu} h(X) \otimes \mathbb{Q}(r)[2r]) \to \mathrm{Hom}_{D^b(M\mathcal{M}(k))}(\mathbb{Q}(0), h(X) \otimes \mathbb{Q}(r)[2r])] \ . \end{aligned}$$

- If X is smooth and projective, there is a non-canonical direct sum decomposition

$$h(X) = \bigoplus_{i=0,\ldots,2\dim X} \underline{H}^i(X)[-i] \ .$$

Hence one has the following formula for the graded pieces of the Chow group

$$Gr^\nu_F \,\mathrm{CH}^r(X) = \mathrm{Hom}_{D^b(M\mathcal{M}(k))}(\mathbb{Q}(0), \underline{H}^{2r-\nu}(X) \otimes \mathbb{Q}(r)[\nu]) \ .$$

2. The triangulated category of mixed motives over a field k

Regarding this section we refer the reader to [Ha-1, 2, 3] for details.

Let k be a field and let (Smooth Proj $/k$) denote the category of smooth projective algebraic varieties over k. The triangulated category of mixed motives is closely related to this category.

We review some categorical notions. Recall that a *triangulated category* is an additive category $\mathcal{C}$, together with an additive functor $T : \mathcal{C} \to \mathcal{C}$ called the shifting functor, and a collection of diagrams of the form $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ called the distinguished triangles, that satisfy certain axioms (see [Har], for example). We will usually write $X[1]$ for TX. The basic examples are the following. For an abelian category $\mathcal{A}$, let $K(\mathcal{A})$ be the homotopy category of complexes with values in $\mathcal{A}$; it has the structure of a triangulated category. If we localize it with respect to the class of quasi-isomorphisms, we obtain the derived category of $\mathcal{A}$, denoted $D(\mathcal{A})$, which again has the structure of a triangulated category.

A *tensor product structure* on an additive category $\mathcal{C}$ is a biadditive functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $(X, Y) \mapsto X \otimes Y$, satisfying the conditions of associativity, commutativity, and the existence of the unit object (as are satisfied for the tensor product in the category of modules). A triangulated category equipped with a tensor product structure is called a *tensor triangulated category* if the two structures are compatible (see [Ha 2, §3]).

An additive category is *pseudo-abelian* if for any object X and an endomorphism $p : X \to X$ such that $p \circ p = p$, the kernel of p exists.

We can construct a theory of mixed motives based on higher Chow groups:

Theorem 2.1 ([Ha-2, §4, §5]) *There is a pseudo-abelian triangulated category $\mathcal{D}(k)$, called the* **triangulated category of mixed motives**, *satisfying the following properties.*

(1) There is a tensor product structure on $\mathcal{D}(k)$, that is compatible with the structure of triangulated category. There are objects $\mathbb{Z}(n)$ for $n \in \mathbb{Z}$, called the Tate objects, such that $\mathbb{Z}(0)$ is the unit object for the tensor product, $\mathbb{Z}(-1)$ the inverse of $\mathbb{Z}(1)$ with respect to the tensor product, and for $n > 0$, one has $\mathbb{Z}(n) = \mathbb{Z}(1)^{\otimes n}$ and $\mathbb{Z}(-n) = \mathbb{Z}(-1)^{\otimes n}$.

(2) There is a functor h : $(\text{Smooth Proj}\,/k)^{opp} \to \mathcal{D}(k)$ such that one has a functorial isomorphism, for X smooth projective,

$$\mathrm{Hom}_{\mathcal{D}(k)}(\mathbb{Z}(0), h(X) \otimes \mathbb{Z}(r)[2r-n]) = \mathrm{CH}^r(X, n)\ .$$

The right hand side is the higher Chow group, as defined by S. Bloch.

(3) If k is a subfield of $\mathbb{C}$, there is a functor, called the Betti cohomology functor, H_B^* : $\mathcal{D}(k) \to gr\,Vect_{\mathbb{Q}}$ such that the composition with h coincides with the Betti cohomology $X \mapsto H^*(X(\mathbb{C}),\ \mathbb{Q})$. Here $gr\,Vect_{\mathbb{Q}}$ denotes the abelian category of finite dimensional graded $\mathbb{Q}$-vector spaces. Similarly for any field and a prime $\ell \neq \mathrm{char}\,k$, there is the ℓ-adic étale cohomology functor H_ℓ^* : $\mathcal{D}(k) \to gr\,Vect_{\mathbb{Q}_\ell}$ such that the composition with h coincides with the ℓ-adic étale cohomology $H^*_{\acute{e}t}(X \otimes_k \bar{k},\ \mathbb{Q}_\ell)$.

The higher Chow group of a not necessarily smooth quasi-projective algebraic variety X is defined as follows (cf. [Bl-2]). There are two versions of the theory, one using simplicies and the other using cubes. We will use the latter, as we need the product defined at the chain level.

Let $\square^1 = \mathbb{P}^1_k - \{1\}$ and $\square^n = (\square^1)^n$ with coordinates $(x_1, \cdots, x_n)$. A codimension one face of $\square^n$ is a divisor of the form $\square^{n-1}_{i,a} = \{x_i = a\}$ where $a = 0$ or ∞; an intersection of codimension one faces is called a face. A face of dimension m is canonically isomorphic to $\square^m$.

Let X be an equi-dimensional variety (or a scheme). Let $\mathcal{Z}^r(X \times \square^n)$ be the free abelian group on the set of codimension r irreducible subvarieties of $X \times \square^n$ meeting each $X \times$ face properly, namely satisfying the condition $\mathrm{codim}_{X\times\square^n}(V \cap (X \times F)) \geq r$. The inclusions of codimension one faces $\delta_{i,a} : \square^{n-1}_{i,a} \hookrightarrow \square^n$ induce the map

$$\partial = \sum (-1)^i (\delta^*_{i,0} - \delta^*_{i,\infty}) : \mathcal{Z}^r(X \times \square^n) \to \mathcal{Z}^r(X \times \square^{n-1})\ .$$

One has $\partial \circ \partial = 0$. Let $\pi_i : X \times \square^n \to X \times \square^{n-1}$, $i = 1, \cdots, n$ be the projections, and $\pi_i^* : \mathcal{Z}^r(X\times\square^{n-1}) \to \mathcal{Z}^r(X\times\square^n)$ be the pull-backs. Let $\mathcal{Z}^r(X, n)$ be the quotient of $\mathcal{Z}^r(X\times\square^n)$ by the sum of the images of π_i^*. Thus an element of $\mathcal{Z}^r(X, n)$ is represented uniquely by a cycle whose irreducible components are non-degenerate (not a pull-back by π_i). The map ∂ induces a map $\partial : \mathcal{Z}^r(X, n) \to \mathcal{Z}^r(X, n-1)$, and $\partial \circ \partial = 0$. The complex $\mathcal{Z}^r(X, \bullet)$ thus defined is the *cycle complex* of X in codimension r. The higher Chow groups are the homology groups of this complex:

$$\mathrm{CH}^r(X, n) = H_n \mathcal{Z}^r(X, \bullet)\ .$$

Note if $n = 0$ we get the Chow group $\mathrm{CH}^r(X, 0) = \mathrm{CH}^r(X)$.

Bloch proved that there is a functorial isomorphism

$$\mathrm{CH}^r(X,\ n)_{\mathbb{Q}} \ \cong\ K'_n(X)^{(r)}_{\mathbb{Q}}$$

(version of Riemann-Roch theorem). Here, for an abelian group A, set $A_{\mathbb{Q}} = A \otimes_{\mathbb{Z}} \mathbb{Q}$. The group $K'_n(X)$ on the right is the algebraic K-group of the exact category of coherent sheaves on X, and it is known the Admas operators ψ^k acting on $K'_n(X)_{\mathbb{Q}}$ splits it to the direct sum of eigenspaces $K'_n(X)_{\mathbb{Q}}^{(r)}$.

We outline the construction of the triangulated category $\mathcal{D}(k)$. There is a triangulated subcategory $\mathcal{D}_{\text{finite}}(k)$ such that $\mathcal{D}(k)$ is its pseudo-abelian completion. Since the difference between the two categories is slight, we focus on the construction of $\mathcal{D}_{\text{finite}}(k)$.

A *finite symbol* K over k is a finite formal sum $\bigoplus_\alpha (X_\alpha, r_\alpha)$, where X_α is a smooth projective variety and r_α an integer. One has the direct sum and the tensor product for finite symbols: $(X, r) \otimes (X', r') = (X \times X', r + r')$. For finite symbols K, L, one has a complex of abelian groups $\mathrm{Hom}(K, L)^\bullet$. If $K = (X, r)$ and $L = (Y, s)$, then we set

$$\mathrm{Hom}((X, r), (Y, s)\,)^\bullet = \mathcal{Z}^{\dim X + s - r}(X \times Y, -\bullet) \ .$$

In general, $\mathrm{Hom}(K, L)^\bullet$ is defined by extending this by linearity. For finite symbols K, L and M, there is a partially defined, associative composition map

$$\mathrm{Hom}(L, M)^\bullet \otimes \mathrm{Hom}(K, L)^\bullet \dashrightarrow \mathrm{Hom}(K, M)^\bullet$$

that sends $v \otimes u$ to $v \circ u$. If $K = (X, r)$, $L = (Y, s)$ and $M = (Z, t)$, it is the map

$$\mathcal{Z}(Y \times Z, m) \otimes \mathcal{Z}(X \times Y, n) \to \mathcal{Z}(X \times Z, n + m)$$

given by

$$v \otimes u \mapsto v \circ u = p_{XZ\,*}(p_{YZ}^* v \cdot p_{XY}^* u) \ .$$

Here p_{XY}, for example, is the projection $X \times Y \times Z \to X \times Y$, p^*, p_* are the inverse image and direct image of cycles, and $p_{YZ}^* v \cdot p_{XY}^* u$ is the intersection of cycles. The intersection is not always defined, so the composition map is only partially defined. (This will not present an essential problem, since one can show that there is quasi-isomorphic subcomplex of $\mathrm{Hom}(L, M)^\bullet \otimes \mathrm{Hom}(K, L)^\bullet$ on which the composition is defined. This is an example of "moving" lemmas, that are systematically developed in [Ha-2].) For general symbols one can extend the above definition linearly and define the composition map.

An object of $\mathcal{D}_{finite}(k)$ is of the form $K = (K^m; f^{m,n})$. Here K_m is a sequence of symbols, one for each $m \in \mathbb{Z}$, such that $K^m = 0$ for all but a finite number of m's. For $m, n \in \mathbb{Z}$ with $m < n$, there corresponds $f^{m,n} \in \mathrm{Hom}(K^m, K^n)^{-(n-m-1)}$, and they are subject to the condition

$$(-1)^n \partial f^{m,n} + \sum_{m < \ell < n} f^{\ell,n} \circ f^{m,\ell} = 0 \ .$$

Here the composition $f^{\ell,n} \circ f^{m,\ell}$ is assumed to be defined. Note that, if $f^{m,n} = 0$ for $n - m \geq 2$, then the condition becomes $f^{m+1,m+2} \circ f^{m,m+1} = 0$, which is the same as the condition of differentials of a complex of abelian groups.

Let $(L^m; g^{m,n})$ be another object. A morphism $u : K \to L$ is represented by a collection of elements $u^{m,n} \in \mathrm{Hom}(K^m, L^n)^{-n+m}$ for $m \leq n$ such that

$$(-1)^n \partial u^{m,n} - \sum (-1)^{m+\ell} u^{\ell,n} \circ f^{m,\ell} + \sum (-1)^{\ell+n} g^{\ell,n} \circ u^{m,\ell} = 0 \ .$$

It defines the zero morphism if there exist $U^{m,n} \in \mathrm{Hom}(K^m, L^n)^{-n+m-1}$ for $m \leq n$ such that

$$u^{m,n} = (-1)^n \partial U^{m,n} + \sum (-1)^{m+\ell} U^{\ell,n} \circ f^{m,\ell} + \sum (-1)^{\ell+n} g^{\ell,n} \circ U^{m,\ell} \ .$$

One thus obtains an additive category. The reader should observe the similarity to the construction of the homotopy category $K(\mathcal{A})$ of an abelian category. Tensor product can be defined, and we can show that the category has the structure of a triangulated category, compatible with the tensor structure.

We next turn to the problem of how to extract an abelian category from $\mathcal{D}(k)$. For this we need to assume the following three conjectures.

Let X be a smooth projective variety over k, and let $H^i(X)$ stand for one of the following: the Betti cohomology $H^i(X(\mathbb{C}),\ \mathbb{Q})$, the ℓ-adic étale cohomology $H^i_{\acute{e}t}(X \otimes_k \bar{k},\ \mathbb{Q}_\ell)$, or the de Rham cohomology $H^i_{DR}(X/k)$. There is the cycle class map $cl : \mathrm{CH}^r(X) \to H^{2r}(X)$. Denote by $H^{2r}_{alg}(X)$ the linear subspace of $H^{2r}(X)$ generated over $\mathbb{Q}$ by the image of cl; an element in the subspace is called *algebraic*. **Grothendieck's standard conjectures** consists of two parts.

(1) The subspace $H^{2r}_{alg}(X)$ satisfies the following analogue of the hard Lefschetz theorem. If $L \in H^2(X)$ is the class of an hyperplane section, then for $r < n$ the map $L^{n-r} : H^{2r}_{alg}(X) \to H^{2n-2r}_{alg}(X)$ is an isomorphism.

(2) The primitive part of the subspace $H^{2r}_{alg}(X)$ satisfies an analogue of the Hodge index theorem.

The conjecture (1) asserts there should be "enough" algebraic cycles (thus similar to the Hodge conjecture in that regard). For example it implies that following statement: If $\Delta_X \subset X \times X$ is the diagonal and $cl(\Delta_X) \in H^{2\dim X}(X \times X)$ its cycle class, then its Künneth components are all algebraic. The conjecture (2) is known if $k = \mathbb{C}$ as the Hodge index theorem. Note that for both conjectures properties special to the cohomology theory in question (such as the Frobenius action or the Hodge structure) are irrelevant. The Weil conjecture can be deduced from the standard conjectures, and Grothendieck repeatedly stressed the importance of the latter. The theory of pure motives based on the standard conjectures was a model for later development of the theory of mixed motives.

To explain the **Bloch-Beilinson-Murre filtration conjecture**, let X be a smooth projective variety, and consider the group $\mathrm{CH}^r(X)_{\mathbb{Q}} = \mathrm{CH}^r(X) \otimes \mathbb{Q}$; we will abbreviate it to $\mathrm{CH}^r(X)$ in this paragraph. The conjecture states that there be a finite (exhaustive and separated) filtration

$$\mathrm{CH}^r(X) = F^0 \supset F^1 \supset \cdots \supset F^{r+1} = (0)$$

which is functorial for the action of algebraic correspondences, and whose successive quotients $F^\nu \mathrm{CH}^r(X)/F^{\nu+1}\mathrm{CH}^r(X)$ "depends only on" $H^{2r-\nu}(X)$ in a sense that can be made precise. The existence of such a filtration was first suggested by Bloch. Subsequently Beilinson gave an "explanation" on how the filtration be given in the framework of the hypothetical theory of the abelian category of mixed motives, as we already mentioned in §1. On the other hand J.P. Murre formulated a conjecture as to the existence of the decomposition of the diagonal class in the ring of self-correspondences $\mathrm{CH}^{\dim X}(X \times X)$, and related it to the filtration as Bloch conjectured to exist. For these, see [Bl-1], [Be], [Ja], and [Mu].

The **Beilinson-Soulé vanishing conjecture** asserts that for X smooth projective, one has

$$\mathrm{CH}^r(X,\ n)_{\mathbb{Q}} = 0$$

if $n > 0$ and $2r - n \leq 0$.

We need to recall another categorical notion. A *t-structure* of a triangulated category D is a pair of full additive subcategories $(D^{\leq 0},\ D^{\geq 0})$ satisfying certain axioms, [BBD]. Given a

t-structure, the full subcategory $D^{\leq 0} \cap D^{\geq 0}$ is indeed an abelian category, which we call the *heart* of the triangulated category. One also has the functor ("cohomology" with respect to the t-structure) $H^i : D \to D^{\leq 0} \cap D^{\geq 0}$ for $i \in \mathbb{Z}$, which takes an exact triangle to a long exact sequence in the heart. If D and D' are triangulated categories with t-structures, $F : D \to D'$ is an exact functor of triangulated categories, we say F respects the t-structures if F sends $D^{\leq 0}$ (resp. $D^{\geq 0}$) to $D'^{\leq 0}$ (resp. $D'^{\geq 0}$).

For example, the derived category $D(\mathcal{A})$ of an abelian category $\mathcal{A}$ has a natural structure of a triangulated category, where $D(\mathcal{A})^{\leq 0}$ (resp. $D(\mathcal{A})^{\geq 0}$) is the subcategory of objects K with $H^i(K) = 0$ for $i > 0$ (resp. $i < 0$). Its heart can be identified with $\mathcal{A}$. Likewise the abelian category of finite dimensional graded $\mathbb{Q}$-vector spaces $gr\,Vect_{\mathbb{Q}}$ has a t- structure by the grading.

The additive category $\mathcal{D}(k)_{\mathbb{Q}}$ by definition has the same objects as $\mathcal{D}(k)$, and has the homomorphism groups

$$\mathrm{Hom}(X, Y) = \mathrm{Hom}_{\mathcal{D}(k)}(X, Y) \otimes \mathbb{Q} \ .$$

This also is a triangulated category, and one has all the properties that $\mathcal{D}(k)$ has as stated in Theorem (2.1), except that $\mathrm{CH}^r(X, n)$ should be replaced with $\mathrm{CH}^r(X, n) \otimes \mathbb{Q}$. In this category the object $\mathbb{Z}(r)$ will be written $\mathbb{Q}(r)$. For the rest of this section only, we let $\mathcal{D}(k)$ stand for $\mathcal{D}(k)_{\mathbb{Q}}$.

Theorem 2.2([Ha-3, Theorem (3.4)]). *Assume the above three conjectures. Then there is a unique t-structure on $\mathcal{D}(k)$ such that the Betti and the étale cohomology functor $H^* : \mathcal{D}(k) \to gr\,Vect_K$ (where K is the coefficient field of the cohomology: $K = \mathbb{Q}, \mathbb{Q}_\ell$) respects the t-structures.*

In particular, under the same conjectures, the heart of the t-structure would be an abelian category, which we call the *abelian* category of mixed motives.

The triangulated subcategory generated by $\mathbb{Q}(r)$ is denoted $\mathcal{DT}(k)$, and called the triangulated category of *mixed Tate motives* over k. For this subcategory, the existence of the t-structure (so that the cohomology functors respect the t-structures) is equivalent to the vanishing conjecture for $\mathrm{Spec}\, k$; this can be shown by repeating part of the proof for the above theorem. Since the vanishing conjecture is known for a number field k, one has the abelian category of mixed Tate motives over k.

§3. Motivic cohomology

For this section we refer to [Ha-MP], [Ha-HC]. We start with an analogy. In topology, recall that there are four homology theories, say with $\mathbb{Z}$-coefficients, for good topological spaces. They are the singular homology H_*, singular cohomology H^*, Borel-Moore homology H_*^{BM} (the homology of the complex of infinite chains), and singular cohomology with compact support, H_c^*. Among these four theories, there are two types of duality. One is the "obvious" duality between the singular homology and cohomology as well as between the Borel-Moore homology and singular cohomology with compact support. If the space is a topological manifold, one has Poincaré duality between singular homology and Borel Moore homology (of complementary dimension) as well as between cohomology and compactly supported cohomology.

The higher Chow group for an algebraic variety has properties similar to the Borel-Moore homology; for example it is contravariant for proper maps and contravariant for open immersions. One may thus call the higher Chow group the **motivic Borel-Moore homology**, and write $H_i^{\mathcal{M},BM}(X,\mathbb{Z}(r))$ for $\mathrm{CH}^{\dim X-r}(X,i-2r)$. As shown in [Ha-HC], starting from the higher Chow groups one can construct the other three theories, which are the **motivic cohomology** $H^*_{\mathcal{M}}(X,\mathbb{Z}(r))$, **motivic homology** $H^{\mathcal{M}}_*(X,\mathbb{Z}(r))$, and **motivic homology with compact support**, $H^*_{\mathcal{M},c}(X,\mathbb{Z}(r))$. In this section we explain the construction of one of them, the motivic cohomology. Throughout this section we assume that the characteristic of the field is zero, so there is resolution of singularities. The next theorem contains the existence of the motivic cohomology. Denote by *(Q-Proj/k)* the category of not necessarily smooth quasi-projective varieties over k.

Theorem 3.1([Ha-HC, Theorems I, II]). *(1) The functor h in Theorem 2.1 can be extended to a functor*

$$h\ :(Q\text{-}Proj/k)^{opp}\to\mathcal{D}(k)$$

from (Q-Proj/k).

(2) For integers i,r with $r\geq 0$, there is a contravariant functor, called the motivic cohomology, taking values in abelian groups

$$X\mapsto H^i_{\mathcal{M}}(X,\ \mathbb{Z}(r));$$

it coincides with $\mathrm{CH}^r(X,2r-i)$ for X smooth.

(3) There is a functorial isomorphism of abelian groups

$$H^i_{\mathcal{M}}(X,\ \mathbb{Z}(r))=\mathrm{Hom}_{\mathcal{D}(k)}(\mathbb{Z}(0),\ h(X)\otimes\mathbb{Z}(r)[i])\ .$$

The proof of the theorem consists of showing (2) first, then constructing the functor h so that (3) holds. In this article we focus on the definition of the functor $H^i_{\mathcal{M}}(X,\ \mathbb{Z}(r))$.

For a quasi-projective variety X, consider its *cubical hyperresolution*

$$a:X_\bullet\to X$$

as in [GNPP]. This is a truncated strict simplicial scheme, consisting of smooth quasi-projective varieties, with an augmentation a to X, satisfying certain conditions.

A *truncated strict simplicial scheme* $X_\bullet$ is a collection of schemes X_n, one for each n with $0\leq n\leq p$ (for some fixed integer $p\geq 0$), along with a set of maps $d_i:X_n\to X_{n-1}$ for $0\leq i\leq n$, such that $d_id_j=d_{j-1}d_i$ for $i<j$. An augmentation to X is a map $a:X_0\to X$ such that $ad_0=ad_1$.

For a truncated strict simplicial scheme $X_\bullet$ with an augmentation to X be a cubical hyperresolution, it is required that each X_n be smooth and a certain condition be satisfied for X_n with respect to the maps d_i. Referring to [GNPP] for the precise condition, we just give the basic idea. Let $p:\tilde{X}\to X$ be a desingularization of X, and $Y\subset X$ be closed subset such that $X-Y$ is open dense and p is an isomorphism outside Y. Set $W=p^{-1}(Y)_{red}$. One has closed immersions $i:Y\to X$, $i':W\to\tilde{X}$, the map $p'=p|_W:W\to Y$, and the following commutative square

$$\begin{array}{ccc} W & \xrightarrow{i'} & \tilde{X} \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{i} & X\,. \end{array}$$

This diagram, which has X at the end, is called a *2-resolution* of X. Set $X_0 = \tilde{X} \coprod Y$, $X_1 = W$; let $d_0, d_1 : X_1 \to X_0$ be the maps given by i' and p', and let $a : X_0 \to X$ be the map (p, i). Then one obtains a truncated strict simplicial scheme (with $p = 1$) $X_\bullet$ with an augmentation to X. If Y and W are smooth, then $X_\bullet$ gives a cubical hyperresolution (note it arises forms a "square", which is a cube of dimension two, whence the name).

If Y or W is not smooth, one takes 2-resolutions $W_\bullet \to W$ of W, $Y_\bullet \to Y$ of Y so that there is a map $W_\bullet \to Y_\bullet$ over $p' : W \to Y$. When combined, the 2-resolutions form a "cube" with W and Y on one edge. By replacing these W and Y with $\tilde{X}$ and X, respectively, one forms another "cube" diagram with X at the end vertex. As a square does, the cube gives rise to a truncated strict simplicial scheme (with $p = 2$) $X_\bullet$ augmented to X. If all the varieties X_n are smooth, this gives a cubical hyperresolution of X. If not, we perform the same procedure on one "face" of the cube, obtaining a 4-dimensional "cube" with X at the end. After a finite number of such procedures, we arrive at an n-dimensional "cube" $X_\bullet \to X$ with all X_n smooth. The associated truncated strict simplicial scheme is by definition a cubical hyperresolution of X.

An important property of a cubical hyperresolution is that the cohomology of $X_\bullet$ and of X are isomorphic, namely $a^* : H^*(X) \xrightarrow{\sim} H^*(X_\bullet)$. Recall that a hypercovering appearing in the mixed Hodge theory of Deligne also satisfies the same property. So cubical hyperresolutions are analogous to hypercoverings. The main theme of [GNPP] is the application of cubical hyperresolutions to the mixed Hodge theory.

In [Ha-HC] we apply the technique of hyperresolutions to the cycle theory and define the motivic cohomology. Let X be a quasi-projective variety and $a : X_\bullet \to X$ be its hyperresolution. For each X_n take its cycle complex $\mathcal{Z}^r(X_n, \bullet)$, and form a double complex

$$0 \to \mathcal{Z}^r(X_0, \bullet) \to \mathcal{Z}^r(X_1, \bullet) \to \cdots$$

where $\mathcal{Z}^r(X_0, 0)$ is placed in degree 0 and the horizontal differentials are given by $\sum(-1)^i d_i^*$, the alternating sums of the pull-backs by d_i. Let $\mathcal{Z}^r(X_\bullet, \bullet)^*$ be its total complex and set

$$H^i_{\mathcal{M}}(X, \mathbb{Z}(r)) = H^{2r-i}\mathcal{Z}^r(X_\bullet, \bullet)^* .$$

In [Ha-HC] it is proven that the right hand side is independent, up to canonical isomorphism, of the choice of hyperresolution, and that the assignment $X \mapsto H^i_{\mathcal{M}}(X, \mathbb{Z}(r))$ forms a contravariant functor. The proof rests on the properties of the category of hyperresolutions as discussed in [GNPP].

Properties of the motivic cohomology

- The motivic cohomology satisfies the homotopy invariant property, namely the projection $X \times \mathbb{A}^1 \to X$ induces an isomorphism

$$H^i_{\mathcal{M}}(X, \mathbb{Z}(r)) \xrightarrow{\sim} H^i_{\mathcal{M}}(X \times \mathbb{A}^1, \mathbb{Z}(r)) .$$

Also one has the Mayer-Vietoris exact sequence for an open covering $X = U \cup V$:

$$\begin{aligned} &\to H^i_{\mathcal{M}}(X, \mathbb{Z}(r)) \to H^i_{\mathcal{M}}(U, \mathbb{Z}(r)) \bigoplus H^i_{\mathcal{M}}(V, \mathbb{Z}(r)) \to H^i_{\mathcal{M}}(U \cap V, \mathbb{Z}(r)) \\ &\to H^{i+1}_{\mathcal{M}}(X, \mathbb{Z}(r)) \to \cdots . \end{aligned}$$

• The functor h of Theorem 3.1 also satisfies the homotopy invariance $h(X) \xrightarrow{\sim} h(X \times \mathbb{A}^1)$. One has the Mayer-Vietoris property: for an open covering $X = U \cup V$, namely a distinguished triangle of the form

$$h(X) \longrightarrow h(U) \oplus h(V) \longrightarrow h(U \cap V) \xrightarrow{[1]} .$$

• In place of the functor h, one can also consider the functor h_c of *motives with compact support*. Let $(\ \textit{Q-Proj}/k;\ \textit{proper})$ be the category of quasi-projective varieties over k and proper maps between them. Then one can show:

(1) The functor h of Theorem 2.1 extends to a functor

$$h_c : (\ \textit{Q-Proj}/k;\ \textit{proper})^{opp} \to \mathcal{D}(k)\ .$$

(2) There is a contravariant functor, called the motivic cohomology with compact support, taking values in abelian groups

$$X \mapsto H^i_{\mathcal{M},c}(X,\ \mathbb{Z}(r))\ ,$$

which coincides with $\mathrm{CH}^r(X, 2r-i)$ for X smooth projective.

(3) There is a functorial isomorphism $H^i_{\mathcal{M},c}(X,\ \mathbb{Z}(r)) = \mathrm{Hom}_{\mathcal{D}(k)}(\mathbb{Z}(0),\ h_c(X) \otimes \mathbb{Z}(r)[i])$. There is also a functorial isomorphism relating it to the higher Chow group:

$$\mathrm{CH}^{\dim X - s}(X, n) = \mathrm{Hom}_{\mathcal{D}(k)}(h_c(X) \otimes \mathbb{Z}(s)[2s-n], \mathbb{Z}(0)\ )\ .$$

§4. The theory of relative Chow motives.

There is the theory of pure (or classical) motives over a field k, more specifically those of **Chow motives** and **Grothendieck motives** (see [Kl-1], [Kl-2], [Ma]). In [CH] this was extended to the theory of pure motives over an algebraic variety. As we will describe below, to a quasi-projective variety S over k, there corresponds a category of pure motives over S. The classical theory over k is the special case where we take $S = \operatorname{Spec} k$.

Consider a smooth quasi-projective variety X over k, equipped with a (not necessarily smooth) projective map $p : X \to S$; let $(\mathrm{Smooth}/k,\ \mathrm{Proj}/S)$ be the category of such varieties X/S. For a pair of objects $p : X \to S$ and $q : Y \to S$ in this category, consider the Chow group $\mathrm{CH}_*(X \times_S Y)$; an element in this of group is an *algebraic correspondence* from X to Y over S. Given three such objects one can define a map

$$\mathrm{CH}_a(X \times_S Y) \otimes \mathrm{CH}_b(Y \times_S Z) \to \mathrm{CH}_{a+b-\dim Y}(X \times_S Z),$$
$$u \otimes v \mapsto v \circ u,$$

called *composition*; it is associative, namely one has

$$w \circ (v \circ u) = (w \circ v) \circ u\ .$$

In particular $\mathrm{CH}_{\dim X}(X \times_S X)$ has the structure of a non-commutative ring with respect to composition, in which the class of the diagonal $[\Delta_X]$ is the identity.

If $S = \operatorname{Spec} k$, the operation of composition is easier to explain. We take smooth projective varieties X, Y over k and consider the Chow group $\mathrm{CH}_*(X \times Y)$. The composition $\mathrm{CH}_a(X \times Y) \otimes \mathrm{CH}_b(Y \times Z) \to \mathrm{CH}_{a+b-\dim Y}(X \times Z)$ is defined by

$$v \circ u = p_{XZ\,*}(p^*_{YZ} v \cdot p^*_{XY} u)$$

where for example p_{XY} is the projection $X \times Y \times Z \to X \times Y$. In other words, $v \circ u$ is obtained by pulling back u and v to $X \times Y \times Z$, taking intersection there, and then pushing forward by the projection $X \times Y \times Z \to X \times Z$. The associativity can be easily verified. For a general S, the definition of composition is more involved, using the refined Gysin map as in [Fu].

The pseudo-abelian category of **Chow motives** over S, denoted $\mathcal{CHM}(S)$, consists of objects of the form $(X/S, r, P)$ where X/S is an object of (Smooth/k, Proj/S), r is an integer, and if $X = \coprod_i X_i$ is the decomposition into irreducible components, P is an element of $\bigoplus_i \mathrm{CH}_{\dim X_i}(X \times_S X_i)$ satisfying $P \circ P = P$. If $(Y/S, s, Q)$ is another object and $Y = \coprod Y_j$ the irreducible decomposition, we define the homomorphism group by

$$\mathrm{Hom}\left((X, r, P), (Y, s, Q)\right) = Q \circ \left(\bigoplus_j \mathrm{CH}_{\dim Y_j - s + r}(X \times_S Y_j)\right) \circ P,$$

and the composition as the map induced by the composition of correspondences. We set $h(X/S) = (X/S, 0, \Delta_X)$, and call it the motive of X/S. An object of $\mathcal{CHM}(S)$ is called a **Chow motive** over S.

In the case $S = \mathrm{Spec}\, k$, we obtain the category $\mathcal{CHM}(k)$ of Chow motives over k. An object there is of the form (X, r, P), where X is a smooth projective variety over k, $r \in \mathbb{Z}$, and $P \in \bigoplus_i \mathrm{CH}_{\dim X_i}(X \times X_i)$ satisfies $P \circ P = P$. We have $h(X) = (X, 0, \Delta_X)$, the Chow motive of X.

In the above construction, one could use the group $\mathrm{CH}(X \times_S Y) \otimes \mathbb{Q}$ in place of $\mathrm{CH}(X \times_S Y)$. The resulting pseudo-abelian $\mathbb{Q}$-linear category is denoted $\mathcal{CHM}(S)_{\mathbb{Q}}$. In the rest of this section, the latter will be simply written $\mathcal{CHM}(S)$.

Assume now $k \subset \mathbb{C}$, and let $D^b_c(S) = D^b_c(S(\mathbb{C}), \mathbb{Q})$ be the derived category of constructible sheaves of $\mathbb{Q}$-vector spaces on the topological space $S(\mathbb{C})$. There is a functor "Betti cohomology realization" $\rho : \mathcal{CHM}(S) \to D^b_c(S)$ such that

$$\rho\left(\,(X/S, r, \Delta_X)\,\right) = Rp_* \mathbb{Q}_X[2r]$$

where $p : X \to S$ is the structure map.

Let us now briefly recall the concept of perverse sheaves ([BBD]). There is a t-structure on $D^b_c(S, \mathbb{Q})$, called the *perverse t-structure*. The heart of this t-structure is a $\mathbb{Q}$-linear abelian category denoted $Perv(S)$. An object of $Perv(S)$ is called a *perverse sheaf*. There is thus the *perverse cohomology* functor

$${}^p\mathcal{H}^i : D^b_c(S) \to Perv(S)\ .$$

Set ${}^p\mathcal{H}^* = \bigoplus_i {}^p\mathcal{H}^i : D^b_c(S) \to Perv(S)$, the total perverse cohomology. Composing with the realization functor we obtain a functor

$${}^p\mathcal{H}^* \rho : \mathcal{CHM}(S) \to Perv(S)\ .$$

If $S = \mathrm{Spec}\, k$, one has the realization functor $\rho : \mathcal{CHM}(k) \to D^b(Vect_{\mathbb{Q}})$ such that $\rho(\,(X, r, \Delta_X)\,) = R\Gamma(X(\mathbb{C}), \mathbb{Q})[2r]$. The latter is a complex computing $H^*(X) = H^*(X(\mathbb{C}), \mathbb{Q})$, the Betti cohomology of X.

We next proceed to define the pseudo-abelian category $\mathcal{M}(S)$ of **Grothendieck motives** over S. By a *symbol* over S we mean a pair $M = (X/S, r)$ consisting of an object X/S of (Smooth/k, Proj/S) and an integer r. Let $N = (Y/S, s)$ be another symbol. We may view M

and N as objects of $\mathit{CH}\mathcal{M}(S)$, so we have the map of homomorphism groups associated with the functor ${}^p\mathcal{H}^*\rho$,

$$\mathrm{Hom}_{\mathit{CH}\mathcal{M}(S)}(M,N) \to \mathrm{Hom}_{Perv(S)}({}^p\mathcal{H}^*\rho(M), {}^p\mathcal{H}^*\rho(N)) \ ;$$

we denote the image of this map by $\mathrm{Hom}_{Perv(S)}({}^p\mathcal{H}^*\rho(M), {}^p\mathcal{H}^*\rho(N))_{alg}$. For a third symbol $L = (Z/S, t)$ the composition

$$\begin{array}{rl} & \mathrm{Hom}_{Perv(S)}({}^p\mathcal{H}^*\rho(M), {}^p\mathcal{H}^*\rho(N))_{alg} \otimes \mathrm{Hom}_{Perv(S)}({}^p\mathcal{H}^*\rho(N), {}^p\mathcal{H}^*\rho(L))_{alg} \\ \to & \mathrm{Hom}_{Perv(S)}({}^p\mathcal{H}^*\rho(M), {}^p\mathcal{H}^*\rho(L))_{alg} \end{array}$$

sending $u \otimes v$ to $v \circ u$ is defined, and the associativity holds. In particular if $M = N$ the group $\mathrm{Hom}_{Perv(S)}({}^p\mathcal{H}^*\rho(M), {}^p\mathcal{H}^*\rho(M))_{alg}$ is a ring with respect to the composition.

If $S = \mathrm{Spec}\,\mathbb{C}$, $Perv(S)$ is identified with the category of finite dimensional $\mathbb{Q}$-vector spaces $Vect_{\mathbb{Q}}$, the functor ${}^p\mathcal{H}^*(X)$ with Betti cohomology $H^*(X)$. We take $M = (X, r)$, $N = (Y, s)$, and the above map can be identified with the cycle class map

$$\mathrm{CH}^{\dim X + s - r}(X \times Y) \to H^{2(\dim X + s - r)}(X \times Y) \ .$$

The image $\mathrm{Hom}_{Vect_{\mathbb{Q}}}(H^*(X), H^*(Y))_{alg} = H^{2(\dim X+s-r)}(X \times Y)_{alg}$ is the same as the group of algebraic cycles on $X \times Y$ modulo homological equivalence. The category $\mathcal{M}(k)$ has as objects triples (X, r, p) where $M = (X, r)$ is a symbol over k, and p is an element of the ring $\mathrm{Hom}(H^*(X), H^*(X))_{alg}$, satisfying $p \circ p = p$. The homomorphism group is given by

$$\begin{array}{rl} & \mathrm{Hom}_{\mathcal{M}(S)}((X,r,p),(Y,s,q)) \\ = & q \circ \mathrm{Hom}_{Vect_{\mathbb{Q}}}(H^*(X), H^*(Y)\,)_{alg} \circ p \end{array}$$

and the composition induced from the composition for $\mathrm{Hom}_{Vect_{\mathbb{Q}}}(H^*(X), H^*(Y)\,)_{alg}$.

As can be seen from the constructions, there are a natural *full* functor $cano : \mathit{CH}\mathcal{M}(S) \to \mathcal{M}(S)$ and a *faithful* functor (cohomology realization) $\rho : \mathcal{M}(S) \to Perv(S)$. Via the realization functors ρ, the functor *cano* and the perverse cohomology functor ${}^p\mathcal{H}^*$ are compatible, namely the following diagram commutes:

$$\begin{array}{ccc} \mathit{CH}\mathcal{M}(S) & \xrightarrow{cano} & \mathcal{M}(S) \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho} \\ D^b_c(S) & \xrightarrow{{}^p\mathcal{H}^*} & Perv(S) \ . \end{array}$$

Note that if $S = \mathrm{Spec}\,k$, the construction of $\mathcal{M}(k)$ is parallel to that of $\mathit{CH}\mathcal{M}(k)$, using homological equivalence of cycles instead of rational equivalence. Since Grothendieck focused mostly on $\mathcal{M}(k)$, it is called the category of Grothendieck motives over k.

In what follows we assume $k = \mathbb{C}$. When $S = \mathrm{Spec}\,\mathbb{C}$, if we assume Grothendieck's standard conjectures and the Bloch-Beilinson-Murre conjecture, we have: (1) For a smooth projective variety X over $\mathbb{C}$, there exists a direct sum decomposition $h(X) = \bigoplus h^i(X)$ in $\mathit{CH}\mathcal{M}(\mathbb{C})$ such that $\rho(h^i(X)) = H^i(X)$. (2) Such a decomposition is not unique, but the filtration of $h(X)$ given by $\bigoplus_{j \le i} h^j(X)$ is uniquely determined. Parts (1) and (2) of the following theorem generalize this.

Theorem 4.1([CH, Theorem 7.2]) *Assume Grothendieck's standard conjectures and the Bloch-Beilinson-Murre conjecture for smooth projective varieties over k. Let $p : X \to S$ be an object of*

($Smooth/\mathbb{C}$, $Proj/S$). There is then an algebraic Whitney stratification $\{S_\alpha\}$ of S with respect to which the following holds:

(1) There are a direct sum decomposition

$$h(X/S) = \bigoplus_{j,\alpha} h^j_\alpha(X/S)$$

in $CH\mathcal{M}(S)$, $\mathbb{Q}$-local systems $\mathcal{V}^j_\alpha$ on smooth strata $S_\alpha - S_{\alpha+1}$, and isomorphisms in $D^b_c(S)$

$$\rho\left(h^j_\alpha(X/S)\right) \cong IC_{S_\alpha}\left(\mathcal{V}^j_\alpha\right)[-j + \dim S_\alpha].$$

Here $IC(\mathcal{V})$ is the intersection complex of a local system $\mathcal{V}$.

(2) For each i, the subobject

$$\bigoplus_{j \leq i,\, \alpha} h^j_\alpha(X/S)$$

(the sum over $j \leq i$ and all α) is uniquely determined, independent of the choice of the decomposition of $h(X/S)$.

(3) The category $\mathcal{M}(S)$ is a semi-simple abelian category. The functor $\rho : \mathcal{M}(S) \to Perv(S)$ is exact and faithful.

By an algebraic Whitney stratification we mean a decreasing sequence of Zariski closed sets $S = S_0 \supset S_1 \supset \cdots \supset S_{\dim S}$ such that $S_\alpha - S_{\alpha+1}$ is smooth and of codimension α, satisfying a certain local condition called the *Whitney condition*. The notion of the intersection complex of a local system is due to M. Goresky and R. MacPherson, [GM]. Let S be an algebraic variety over $\mathbb{C}$, U a smooth dense open subset of S, and $\mathcal{L}$ a $\mathbb{Q}$-local system on U. Then the intersection complex $IC_S(\mathcal{L})$ is determined as an object in the derived category $D^b_c(S)$. It is characterized by the following conditions: (i) $IC_S(\mathcal{L})|_U = \mathcal{L}$, (ii) $IC_S(\mathcal{L})[\dim S]$ is a perverse sheaf on S, (iii) there are no subobjects or quotient objects of $IC_S(\mathcal{L})[\dim S]$ in $Perv(S)$, with support on $S - U$.

According to the decomposition theorem in the theory of perverse sheaves (see [BBD]), for a map $p : X \to S$ satisfying the condition of the theorem, there are an algebraic Whitney stratification $\{S_\alpha\}$, local systems $\mathcal{V}^j_\alpha$ on $S_\alpha - S_{\alpha+1}$, and a direct sum decomposition

$$Rp_*\mathbb{Q}_X \cong \bigoplus_{j,\alpha} IC_{S_\alpha}\left(\mathcal{V}^j_\alpha\right)[-j + \dim S_\alpha]\ .$$

So the decomposition in (1) of the above theorem is a "lifting" of this to the category of motives over S; we thus refer to it as a **relative motivic decomposition** of X/S. If $S = \operatorname{Spec}\mathbb{C}$, which we mentioned just before the theorem, it is called an **absolute motivic decomposition** of X.

There are interesting examples of fibrations $p : X \to S$ for which the statement (1) of the above theorem can be proven without assuming any conjectures. In [GHM-1], we showed the existence of the relative decomposition in case S is a Hilbert modular variety and X a self-product of the universal family of abelian varieties over S. In this case one can further show the existence of the absolute motivic decomposition of X, verifying in particular that the algebraicity of the Künneth components of the diagonal class of X (see [GHM-2]).

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