# Quasi DG categories and mixed motivic sheaves

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**Abstract**

We introduce the notion of a quasi DG category, generalizing that of a DG category. To a quasi DG category satisfying certain additional conditions, we associate another quasi DG category, the quasi DG category of $C$-diagrams. We then show the homotopy category of the quasi DG category of $C$-diagrams has the structure of a triangulated category. This procedure is then applied to produce a triangulated category of mixed motives over a base variety.

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In this paper we introduce the notion of a *quasi DG category*, and give a procedure to construct a triangulated category associated to it. Then we apply it to the construction of the triangulated category of mixed motivic sheaves over a base variety. If the base variety is the Spec of the ground field, this coincides with the triangulated category of motives as in [6].

The notion of a quasi DG category is a generalization of that of a DG category. Recall that a DG category is an additive category $\mathcal{C}$, such that for a pair of objects $X, Y$ the group of homomorphisms $F(X,Y)$ has the structure of a complex, and the composition $F(X,Y)\otimes F(Y,Z)\to F(X,Z)$ is a map of complexes. (See §0 for the sign convention for the tensor product of complexes.)

A quasi DG category (which is not really a category) also consisits of a class of objects, and for a pair of objects there corresponds a complex $F(X,Y)$. But there is no composition map $F(X,Y)\otimes F(Y,Z)\to F(X,Z)$. Instead, we have the following structure:

(1) There is a quasi-isomorphic subcomplex

$$\iota : F(X,Y)\hat{\otimes}F(Y,Z)\hookrightarrow F(X,Y)\otimes F(Y,Z)\ .$$

(2) There is another complex $F(X,Y,Z)$ and a surjective quasi-isomorphism

$$\sigma : F(X,Y,Z)\to F(X,Y)\hat{\otimes}F(Y,Z)\ .$$

(3) There is a map of complexes $\varphi : F(X,Y,Z)\to F(X,Z)$ .

In the derived category at least, one has an induced map $F(X,Y)\otimes F(Y,Z)\to F(X,Z)$ obtained by composing $\iota^{-1}$, $\sigma^{-1}$, and $\varphi$. In particular there is a map $\psi_Y : H^0F(X,Y)\otimes H^0F(Y,Z)\to H^0F(X,Z)$.

The above pattern persists for more than three objects as follows.

(1) For each sequence of objects, $X_1, \cdots X_n$ $(n \geq 2)$, there corresponds a complex $F(X_1, \cdots, X_n)$. For a subset of integers $S = \{i_1, \cdots, i_{a-1}\} \subset (1, n)$, let $i_0 = 1$, $i_a = n$ and

$$F(X_1, \cdots, X_n \,\overline{|}\, S) := F(X_{i_0}, \cdots, X_{i_1}) \otimes F(X_{i_1}, \cdots, X_{i_2}) \otimes \cdots \otimes F(X_{i_{a-1}}, \cdots, X_{i_a}) \ .$$

There is a complex $F(X_1, \cdots, X_n|S)$ and an injective quasi-isomorphism

$$\iota_S : F(X_1, \cdots, X_n|S) \hookrightarrow F(X_1, \cdots, X_n \,\overline{|}\, S) \ .$$

We assume $F(X_1, \cdots, X_n|\emptyset) = F(X_1, \cdots, X_n)$.

(2) For $S \subset S'$ there is a surjective quasi-isomorphism

$$\sigma_{S\,S'} : F(X_1, \cdots, X_n|S) \to F(X_1, \cdots, X_n|S') \ .$$

For $S \subset S' \subset S''$, $\sigma_{S\,S''} = \sigma_{S'\,S''}\sigma_{S\,S'}$. In particular we have $\sigma_S := \sigma_{\emptyset\,S} : F(X_1, \cdots, X_n) \to F(X_1, \cdots, X_n|S)$.

(3) For $K = \{k_1, \cdots, k_b\} \subset (1, n)$ disjoint from $S$, there is a map

$$\varphi_K : F(X_1, \cdots, X_n|S) \to F(X_1, \cdots, \widehat{X_{k_1}}, \cdots, \widehat{X_{k_b}}, \cdots, X_n|S) \ .$$

If $K$ is the disjoint union of $K'$ and $K''$, one has $\varphi_K = \varphi_{K'}\varphi_{K''}$.

If $K$ and $S'$ are disjoint $\sigma_{S\,S'}$ and $\varphi_K$ commute. The injection $\iota_S$ and the maps $\sigma$, $\varphi$ are compatible (see §1 for what this means).

These are the main data of a quasi DG category. A few other conditions are required, as we briefly mention below ( see (1.5) for details).

- There is direct sum for objects: $X \oplus Y$. There is the zero object $O$. The complexes $F(X_1, \cdots, X_n|S)$ are required to be additive in each variable, in an appropriate sense.
- For each object $X$ there is the identity element $1_X$ in $H^0F(X, X)$.

A DG category may be regarded as a quasi DG category. Indeed take $F(X_1, \cdots, X_n) = F(X_1, X_2) \otimes \cdots \otimes F(X_{n-1}, X_n)$, $F(X_1, \cdots, X_n|S) = F(X_1, \cdots, X_n)$ for any $S$, $\sigma_{S\,S'}$ to be identities, and $\varphi_K$ to be the composition at $X_k$, $k \in K$.

To a quasi DG category $\mathcal{C}$ one can associate an additive category $Ho(\mathcal{C})$, called the *homotopy category* of $\mathcal{C}$; it is the category in which the objects are the same as for $\mathcal{C}$, and the homomorphism group is $H^0F(X, Y)$, and the composition is the map $\psi_Y$ above.

In §§2 and 3, we take a quasi DG category $\mathcal{C}$ satisfying two additional conditions on the complexes $F(X_1, \cdots, X_n)$. These are the existence of diagonal elements and diagonal extension (1.5)(iv), and the existence of a generating set, notion of proper intersection, and distinguished subcomplexes, (1.5)(v). For such $\mathcal{C}$, we produce another quasi DG category $\mathcal{C}^\Delta$, where the objects are what we call *C-diagrams* in $\mathcal{C}$, see (1.8). We must also define the complexes $\mathbb{F}(K_1, \cdots, K_n)$ for a sequence of $C$-diagrams, together with maps $\sigma$ and $\varphi$. We do this in §2, and also verify other axioms of a quasi DG category.

We can thus consider the homotopy category of $\mathcal{C}^\Delta$; in §3 we prove:

**Main Theorem.** *Let $\mathcal{C}$ be a quasi DG category satisfying the conditions (iv),(v) of (1.5). Let $\mathcal{C}^\Delta$ be the quasi DG category of $C$-diagrams in $\mathcal{C}$, and $Ho(\mathcal{C}^\Delta)$ its homotopy category. Then*

*$Ho(\mathcal{C}^\Delta)$ has the structure of a triangulated category. For an object $X$ of $\mathcal{C}$ there corresponds an object (written by the same $X$) in $\mathcal{C}^\Delta$. For two such objects and $n \in \mathbb{Z}$ we have*

$$\mathrm{Hom}_{Ho(\mathcal{C}^\Delta)}(X, Y[n]) = H^n F(X, Y) \ .$$

So far there is no geometry involved. For us the main example of a quasi DG category is that of symbols over a quasi-projective variety $S$, denoted $Symb(S)$, see [8] for details. A typical object of $Symb(S)$ is of the form $(X, r)$, where $r$ is an integer and $X$ a smooth variety with a projective map to $S$. For two such objects $(X, r)$ and $(Y, s)$, the corresponding complex $F((X, r), (Y, s))$ is quasi-isomorphic to $\mathcal{Z}_{\dim Y - s + r}(X \times_S Y, \bullet)$, the cycle complex of the fiber product $X \times_S Y$ (See [1], [2], [3] for the cycle complex and higher Chow groups.) We refer to [8] for the construction of the complexes

$$F((X_1, r_1), \cdots, (X_n, r_n)|S)$$

and the maps $\iota_S$, $\sigma_{S S'}$, and $\varphi_K$. The additional conditions (1.5), (iv) and (v) are satisfied for $Symb(S)$.

In §4 we apply this construction of §§2 and 3 to $Symb(S)$. The resulting triangulated category $\mathcal{D}(S) := Ho(Symb(S)^\Delta)$ is by definition the triangulated category of mixed motives over $S$. For $S = \mathrm{Spec}\, k$ the construction of the triangulated category in [6] is similar but simpler since then $Symb(S)$ is (almost) a DG category, and the notion of $C$-diagrams is simpler. (Essentially the same idea appeared in [9], preceding [6].) It is useful to have a construction of $\mathcal{D}(k)$ via $C$-diagrams, because one can construct objects concretely using cycles; see for example [11].

In case $S = \mathrm{Spec}\, k$, the work [6], [7] deals with not only the construction of the triagulated category $\mathcal{D}(k)$, but also the cohomology realization functor and the functor of cohomological motives (which associates to each quasi-projective variety $X$ its motive $h(X)$ in $\mathcal{D}(k)$). We will discuss these problems in a separate paper.

We collected basic notions in §0; at the beginning of each section we indicated which is needed from §0. On the technical aspect, we point out two problems for the reader. The first is the delicate question of signs; wherever it is an issue we elaborated on it. The second is the question of general positions (choice of distinguished subcomplexes); one may find it cumbersome at first, but it can always be managed for our purposes. On this also we have given enough details.

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# 0 Basic notions

(0.1) *Multiple complexes.* By a complex of abelian groups we mean a graded abelian group $A^\bullet$ with a map $d$ of degree one satisfying $dd = 0$. If $u : A \to B$ and $v : B \to C$ are maps of complexes, we define $u \cdot v : A \to C$ by $(u \cdot v)(x) = v(u(x))$. So $u \cdot v$ is $v \circ u$ in the usual notation. As usual we also write $vu$ for $v \circ u$ (but not for $v \cdot u$).

A double complex $A = (A^{i,j}; d', d'')$ is a bi-graded abelian group with differentials $d'$ of degree $(1, 0)$, $d''$ of degree $(0, 1)$, satisfying $d'd'' + d''d' = 0$. Its total complex $\mathrm{Tot}(A)$ is the complex with $\mathrm{Tot}(A)^k = \bigoplus_{i+j=k} A^{i,j}$ and the differential $d = d' + d''$. In contrast a "double" complex $A = (A^{i,j}; d', d'')$ is a bi-graded abelian group with differentials $d'$ of degree $(1,0)$, $d''$ of degree $(0,1)$, satisfying $d'd'' = d''d'$. Its total complex $\mathrm{Tot}(A)$ is given by $\mathrm{Tot}(A)^k = \bigoplus_{i+j=k} A^{i,j}$ and the differential $d = d' + (-1)^i d''$ on $A^{i,j}$.

Let $(A, d_A)$ and $(B, d_B)$ be complexes. Then $(A^{i,j} = A^j \otimes B^i; 1 \otimes d_B, d_A \otimes 1)$ is a "double" complex; notice the first grading comes from the grading of $B$. Its total complex has differential $d$ given by

$$d(x \otimes y) = (-1)^{\deg y} dx \otimes y + x \otimes dy \ .$$

Note this differs from the usual convention.

More generally $n \geq 2$ one has the notion of $n$-tuple complex and "$n$-tuple" complex. An $n$-tuple (resp. "$n$-tuple") complex is a $\mathbb{Z}^n$-graded abelian group $A^{i_1,\cdots,i_n}$ with differentials $d_1, \cdots, d_n$, $d_k$ raising $i_k$ by 1, such that for $k \neq \ell$, $d_k d_\ell + d_\ell d_k = 0$ (resp. $d_k d_\ell = d_\ell d_k$). A single complex $\mathrm{Tot}(A)$, called the total complex, is defined in either case. As a variant one can define partial totalization: For a subset $S = [k, \ell] \subset [1, n]$ with cardinality $\geq 2$, one can "totalize" in degrees in $S$, so the result $\mathrm{Tot}^S(A)$ is an $m$-tuple (resp. "$m$-tuple") complex, where $m = n - |S| + 1$.

For $n$ complexes $A_1^\bullet, \cdots, A_n^\bullet$, the tensor product $A_1^\bullet \otimes \cdots \otimes A_n^\bullet$ is an "$n$-tuple" complex.

The difference between $n$-tuple and "$n$-tuple" complexes is slight, so we often do not make the distinction. There is an obvious notion of maps of $n$-tuple ("$n$-tuple") complexes.

If $A$ is an $n$-tuple complex and $B$ an $m$-tuple complex, and when $S = [k, \ell] \subset [1, n]$ with $m = n - |S| + 1$ is specified, one can talk of maps of $m$-tuple complexes $\mathrm{Tot}^S(A) \to B$. When the choice of $S$ is obvious from the context, we just say maps of multiple complexes $A \to B$. For example if $A$ is an $n$-tuple complex and $B$ an $(n-1)$-tuple complex, for each set $S = [k, k+1]$ in $[1, n]$ one can speak of maps of $(n-1)$-tuple complexes $\mathrm{Tot}^S(A) \to B$; if $n = 2$ there is no ambiguity.

(0.1.1) *Multiple subcomplexes of a tensor product complex.* Let $A$ and $B$ be complexes. A double subcomplex $C^{i,j} \subset A^i \otimes B^j$ is a submodule closed under the two differentials. If $\mathrm{Tot}(C) \hookrightarrow \mathrm{Tot}(A \otimes B)$ is a quasi-isomorphism, we way $C^{\bullet\bullet}$ is a quasi-isomorphic subcomplex. It is convenient to let $A^\bullet \hat{\otimes} B^\bullet$ denote such a subcomplex. (Note it does not mean the tensor product of subcomplexes of $A$ and $B$.) Likewise a quasi-isomorphic multiple subcomplex of $A_1^\bullet \otimes \cdots \otimes A_n^\bullet$ is denoted $A_1^\bullet \hat{\otimes} \cdots \hat{\otimes} A_n^\bullet$.

(0.2) *Finite ordered sets, partitions and segmentations.* Let $I$ be a non-empty finite totally ordered set (we will simply say a finite ordered set), so $I = \{i_1, \cdots, i_n\}, i_1 < \cdots < i_n$, where $n = |I|$. The *initial* (resp. *terminal*) element of $I$ is $i_1$ (resp. $i_n$); let $\mathrm{in}(I) = i_1$, $\mathrm{tm}(I) = i_n$. If $n \geq 2$, let $\mathring{I} = I - \{\mathrm{in}(I), \mathrm{tm}(I)\}$.

If $I = \{i_1, \cdots, i_n\}$, a subset $I'$ of the form $[i_a, i_b] = \{i_a, \cdots, i_b\}$ is called a *sub-interval.*

In the main body of the paper, for the sake of concreteness we often assume $I = [1, n] = \{1, \cdots, n\}$, a subset of $\mathbb{Z}$. More generally a finite subset of $\mathbb{Z}$ is an example of a finite ordered set.

A *partition* of $I$ is a disjoint decomposition into sub-intervals $I_1, \cdots, I_a$ such that there is a sequence of integers $i < i_1 < \cdots < i_{a-1} < j$ so that $I_k = [i_{k-1}, i_k - 1]$, with $i_0 = i$ and $i_a = j + 1$.

So far we have assumed $I$ and $I_i$ to be of cardinality $\geq 1$. In some contexts we allow only finite ordered sets with at least two elements. There instead of partition the following notion plays a role. Given a subset of $\mathring{I}$, $\Sigma = \{i_1, \cdots, i_{a-1}\}$, where $i_1 < i_2 < \cdots < i_{a-1}$, one has a decomposition of $I$ into the sub-intervals $I_1, \cdots, I_a$, where $I_k = [i_{k-1}, i_k]$, with $i_0 = i_1$, $i_a = i_n$. Thus the sub-intervals satisfy $I_k \cap I_{k+1} = \{i_k\}$ for $k = 1, \cdots, a-1$. The sequence of sub-intervals $I_1, \cdots, I_a$ is called the *segmentation* of $I$ corresponding to $\Sigma$. (The terminology is adopted to distinguish it from the partition).

Finite ordered sets of cardinality $\geq 1$ and partitions appear in connection with a sequence of fiberings. On the other hand, finite ordered sets of cardinality $\geq 2$ and segmentations appear when we consider a sequence of varieties (or an associated sequence of fiberings). See below.

(0.3) *Tensor product of "Double" complexes.* Let $A^{\bullet,\bullet} = (A^{a,p}; d'_A, d''_A)$ be a "double" complex (so $d'$ has degree $(1,0)$, $d''$ has degree $(0,1)$, and $d'd' = 0$, $d''d'' = 0$ and $d'd'' = d''d'$). The associated total complex $\mathrm{Tot}(A)$ has differential $d_A$ given by $d_A = d' + (-1)^a d''$ on $A^{a,p}$. The association $A \mapsto \mathrm{Tot}(A)$ forms a functor. Let $(B^{b,q}; d'_B, d''_B)$ be another "double" complex. Then the tensor product of $A$ and $B$ as "double" complexes, denoted $A^{\bullet,\bullet} \times B^{\bullet,\bullet}$, is by definition the "double" complex $(E^{c,r}; d'_E, d''_E)$, where

$$E^{c,r} = \bigoplus_{a+b=c\,,p+q=r} A^{a,p} \otimes B^{b,q}$$

and $d'_E = (-1)^b d'_A \otimes 1 + 1 \otimes d'_B$, $d''_E = (-1)^q d''_A \otimes 1 + 1 \otimes d''_B$.

The tensor product complex $\mathrm{Tot}(A) \otimes \mathrm{Tot}(B)$ and the total complex of $A^{\bullet,\bullet} \times B^{\bullet,\bullet}$ are related as follows. There is an isomorphism of complexes

$$u : \mathrm{Tot}(A) \otimes \mathrm{Tot}(B) \to \mathrm{Tot}(A^{\bullet,\bullet} \times B^{\bullet,\bullet})$$

given by $u = (-1)^{aq} \cdot id$ on the summand $A^{a,p} \otimes B^{b,q}$.

Let $A$, $B$, $C$ be "double" complexes. One has an obvious isomorphism of "double" complexes $(A \times B) \times C = A \times (B \times C)$; it is denoted $A \times B \times C$. The following diagram commutes:

$$\begin{array}{ccc} \mathrm{Tot}(A) \otimes \mathrm{Tot}(B) \otimes \mathrm{Tot}(C) & \xrightarrow{u \otimes 1} & \mathrm{Tot}(A \times B) \otimes \mathrm{Tot}(C) \\ {\scriptstyle 1\otimes u}\downarrow & & \downarrow{\scriptstyle u} \\ \mathrm{Tot}(A) \otimes \mathrm{Tot}(B \times C) & \xrightarrow{u} & \mathrm{Tot}(A \times B \times C)\ . \end{array}$$

The composition defines an isomorphism $u : \mathrm{Tot}(A) \otimes \mathrm{Tot}(B) \otimes \mathrm{Tot}(C) \xrightarrow{\sim} \mathrm{Tot}(A \times B \times C)$.

One can generalize this to the case of tensor product of more than two "double" complexes. If $A_1, \cdots, A_n$ are "double" complexes, there is an isomorphism of complexes

$$u_n : \mathrm{Tot}(A_1) \otimes \cdots \otimes \mathrm{Tot}(A_n) \to \mathrm{Tot}(A_1 \times \cdots \times A_n)$$

which coincides with the above $u$ if $n = 2$, and is in general a composition of $u$'s in any order. As in case $n = 3$, one has commutative diagrams involving $u$'s; we leave the details to the reader.

Let $A$, $B$, $C$ be "double" complexes and $\rho : A^{\bullet,\bullet} \times B^{\bullet,\bullet} \to C^{\bullet,\bullet}$ be a map of "double" complexes, namely it is bilinear and for $\alpha \in A^{a,p}$ and $\beta \in B^{b,q}$,

$$d'\rho(\alpha \otimes \beta) = \rho((-1)^b d'\alpha \otimes \beta + \alpha \otimes d'\beta)$$

and

$$d''\rho(\alpha \otimes \beta) = \rho((-1)^q d''\alpha \otimes \beta + \alpha \otimes d''\beta)\ .$$

Composing $\mathrm{Tot}(\rho) : \mathrm{Tot}(A \times B) \to \mathrm{Tot}(C)$ with $u : \mathrm{Tot}(A) \otimes \mathrm{Tot}(B) \xrightarrow{\sim} \mathrm{Tot}(A \times B)$, one obtains the map

$$\hat{\rho} : \mathrm{Tot}(A) \otimes \mathrm{Tot}(B) \to \mathrm{Tot}(C) \ ;$$

it is given given by $(-1)^{aq} \cdot \rho$ on the summand $A^{a,p} \otimes B^{b,q}$.

The same holds for a map of "double" complexes $\rho : A_1 \times \cdots \times A_n \to C$.

(0.4) *The subcomplex* $\Phi A$. Let $(A^{\bullet\bullet}; d_1, d_2)$ be a "double" complex satisfying

(i) $A^{a,p} = 0$ if $p < 0$,

(ii) The sequence of complexes

$$A^{\bullet\, 0} \xrightarrow{d_2} A^{\bullet\, 1} \xrightarrow{d_2} \cdots$$

is exact.

We then let $\Phi A$ be the kernel of $d_2 : A^{\bullet\, 0} \to A^{\bullet\, 1}$. Then one has an exact sequence of complexes

$$0 \to (\Phi A)^\bullet \to A^{\bullet,0} \xrightarrow{d_2} A^{\bullet,1} \xrightarrow{d_2} \cdots \ .$$

$\Phi A$ a complex with differential $d_1$, and the inclusion $\Phi A \hookrightarrow \mathrm{Tot}(A^{\bullet\bullet})$ is a quasi-isomorphism. (The differential of $\mathrm{Tot}(A)$ is now defined to be $u \mapsto (-1)^{\deg_2 u} d_1(u) + d_2(u)$.) The association $A \mapsto \Phi A$ forms an exact functor from the category of "double" complexes satisfying the conditions (i), (ii) to the category of complexes. $\Phi A$ is so to speak the peripheral complex of $A^{\bullet\bullet}$ in the second direction.

This can be generalized to the case of "$n$-tuple" complexes $(A^{\bullet \cdots \bullet}; d_1, \cdots, d_n)$ satisfying the conditions similar to (i), (ii) with respect to the last degree and differential. Then $\Phi(A) = \mathrm{Ker}(d_n)$ is an "$(n-1)$-tuple" complex.

If $A^{\bullet\bullet}$ and $B^{\bullet\bullet}$ are "double" complexes satisfying (i), (ii), then $A^{\bullet\bullet} \otimes B^{\bullet\bullet}$ is a "quadruple" complex. By the Künneth formula, $\mathrm{Tot}_{24}(A^{\bullet\bullet} \otimes B^{\bullet\bullet})$ is a "triple" complex satisfying (i), (ii) with respect to the third degree, and one has

$$\Phi(A^{\bullet\bullet}) \otimes \Phi(B^{\bullet\bullet}) = \Phi(\mathrm{Tot}_{24}(A^{\bullet\bullet} \otimes B^{\bullet\bullet})\,)$$

as a "double" complex.

This generalizes to "$n$-tuple" complexes. For example if $A^{\bullet\bullet\bullet}$ and $B^{\bullet\bullet\bullet}$ are "triple" complexes satisfying (i), (ii), then $A^{\bullet\bullet\bullet} \otimes B^{\bullet\bullet\bullet}$ is a "6-tuple" complex. $\mathrm{Tot}_{36}(A^{\bullet\bullet\bullet} \otimes B^{\bullet\bullet\bullet})$ is a "5-tuple" complex satisfying (i), (ii) with respect to the last degree, and one has

$$\Phi(A^{\bullet\bullet\bullet}) \otimes \Phi(B^{\bullet\bullet\bullet}) = \Phi(\mathrm{Tot}_{36}(A^{\bullet\bullet\bullet} \otimes B^{\bullet\bullet\bullet})\,)$$

as a "quadruple" complex.

We now explain a procedure to give a quasi-isomorphic subcomplex of the last complex (this will be used in §2). Assume we have a quasi-isomorphic "6-tuple" subcomplex $A^{\bullet\bullet\bullet} \hat{\otimes} B^{\bullet\bullet\bullet} \hookrightarrow A^{\bullet\bullet\bullet} \otimes B^{\bullet\bullet\bullet}$, see (0.1.1) for notation. Then $\Phi\, \mathrm{Tot}_{36}(A^{\bullet\bullet\bullet} \hat{\otimes} B^{\bullet\bullet\bullet})$ is a quasi-isomorphic subcomplex of $\Phi\, \mathrm{Tot}_{36}(A^{\bullet\bullet\bullet} \otimes B^{\bullet\bullet\bullet}) = \Phi(A^{\bullet\bullet\bullet}) \otimes \Phi(B^{\bullet\bullet\bullet})$. We set

$$\Phi(A^{\bullet\bullet\bullet}) \hat{\otimes} \Phi(B^{\bullet\bullet\bullet}) := \Phi\, \mathrm{Tot}_{36}(A^{\bullet\bullet\bullet} \hat{\otimes} B^{\bullet\bullet\bullet}) \ .$$

# 1 Quasi DG categories.

We refer to (0.1) for multiple complexes and tensor product of complexes. In this section we will consider sequences of objects indexed by $[1, n] = \{1, \cdots, n\}$, or more generally by a finite (totally) ordered set $I$. For notions related to finite ordered sets see (0.2).

(1.1) A *DG category* $\mathcal{C}$ is an additive category such that for objects $X, Y$ the group $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has the structure of a complex of abelian groups, written $F(X, Y)^\bullet$, and the composition of arrows

$$F(X, Y)^\bullet \otimes F(Y, Z)^\bullet \to F(X, Z)^\bullet$$

which sends $u \otimes v$ to $u \cdot v$, is a map of complexes. Here to $u : X \to Y$ and $v : Y \to Z$ there corresponds the product $u \cdot v : X \to Z$, which is the composition $v \circ u$ in the usual notation.

A complex of abelian groups $A^\bullet$ is said to be degree-wise $\mathbb{Z}$-free if for each $p$ there is a set $\mathcal{S}_A^p$ such that $A^p = \mathbb{Z}\mathcal{S}_A^p$.

(1.2) **Definition.** A *weak quasi DG category* $\mathcal{C}$ consists of the following data.

(i) The class of objects $Ob(\mathcal{C})$. There is a distinguished object $O$, called the zero object. There is direct sum of objects $X \oplus Y$, and one has $(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z)$.

(ii) For each sequence of irreducible objects $X_1, \cdots, X_n$ ($n \geq 2$), a complex of abelian groups $F(X_1, \cdots, X_n)$.

It is *additive* in each variable, by which we mean the following. If a variable $X_i = O$, then it is zero. If $X_1 = Y_1 \oplus Z_1$, then one has a direct sum decomposition of complexes

$$F(Y_1 \oplus Z_1, X_2, \cdots, X_n) = F(Y_1, \cdots, X_n) \oplus F(Z_1, \cdots, X_n) \ .$$

The same for $X_n$. If $1 < i < n$ and $X_i = Y_i \oplus Z_i$, then there is a direct sum decomposition of complexes

$$\begin{aligned} & F(X_1, \cdots, X_{i-1}, Y_i \oplus Z_i, X_{i+1}, \cdots, X_n) \\ = \ & F(X_1, \cdots, Y_i, \cdots, X_n) \oplus F(X_1, \cdots, Z_i, \cdots, X_n) \\ & \oplus F(X_1, \cdots, Y_i) \otimes F(Z_i, \cdots, X_n) \oplus F(X_1, \cdots, Z_i) \otimes F(Y_i, \cdots, X_n) \ . \end{aligned}$$

We often refer to the last two terms as the *cross terms*. Note if $F(X, Y)$ is a complex, additive in each variable, then the tensor product $F(X_1, X_2) \otimes F(X_2, X_3) \otimes \cdots \otimes F(X_{n-1}, X_n)$ is additive in each variable in the above sense. So additivity here means "quadratic additivity", so to speak.

(iii) Two types of maps as follows. For $1 < k < n$ a map of complexes

$$\tau_{X_k} : F(X_1, \cdots, X_n) \to F(X_1, \cdots, X_k) \otimes F(X_k, \cdots, X_n) \ ,$$

and for $1 < \ell < n$ a map of complexes

$$\varphi_{X_\ell} : F(X_1, \cdots, X_n) \to F(X_1, \cdots, \widehat{X_\ell}, \cdots, X_n) \ .$$

$\tau_{X_k}$ is assumed to be a quasi-isomorphism.

$\tau_k$ is additive in each variable in the following sense: If $X_k = Y_k \oplus Z_k$, then $\tau_k(X_1, \cdots, X_n)$ is the direct sum of

$$\tau_k(X_1, \cdots, Y_k, \cdots, X_n) : F(X_1, \cdots, Y_k, \cdots, X_n) \to F(X_1, \cdots, Y_k) \otimes F(Y_k, \cdots, X_n) \ ,$$

$$\tau_k(X_1,\cdots,Z_k,\cdots,X_n) : F(X_1,\cdots,Z_k,\cdots,X_n) \to F(X_1,\cdots,Z_k)\otimes F(Z_k,\cdots,X_n)\ ,$$

and the identity maps on the cross terms

$$F(X_1,\cdots,Y_k)\otimes F(Z_k,\cdots,X_n)\oplus F(X_1,\cdots,Z_k)\otimes F(Y_k,\cdots,X_n)\ .$$

If $i<k$ and $X_i = Y_i\oplus Z_i$, then $\tau_k(X_1,\cdots,X_n)$ is the direct sum of $\tau_{X_k}(X_1,\cdots,Y_i,\cdots,X_n)$, $\tau_{X_k}(X_1,\cdots,Z_i,\cdots,X_n)$, and in addition, if $i>0$, the following maps on the cross terms:

$$1\otimes\tau_{X_k} : F(X_1,\cdots,Y_i)\otimes F(Z_i,\cdots,X_n) \to F(X_1,\cdots,Y_i)\otimes F(Z_i,\cdots,X_k)\otimes F(X_k,\cdots,X_n)\ ,$$

$$1\otimes\tau_{X_k} : F(X_1,\cdots,Z_i)\otimes F(Y_i,\cdots,X_n) \to F(X_1,\cdots,Z_i)\otimes F(Y_i,\cdots,X_k)\otimes F(X_k,\cdots,X_n)\ .$$

Similarly if $i>k$.

The $\varphi_{X_\ell}$ is additive in each variable, as follows. Assume $X_i=Y_i\oplus Z_i$. Then $\varphi_{X_\ell}(X_1,\cdots,X_n)$ is the direct sum of $\varphi_{X_\ell}(X_1,\cdots,Y_i,\cdots,X_n)$, $\varphi_{X_\ell}(X_1,\cdots,Z_i,\cdots,X_n)$, and the following maps on the cross terms: If $i<\ell$, the maps

$$1\otimes\varphi_{X_\ell} : F(X_1,\cdots,Y_i)\otimes F(Z_i,\cdots,X_\ell,\cdots,X_n)\to F(X_1,\cdots,Y_i)\otimes F(Z_i,\cdots,\widehat{X_\ell},\cdots,X_n)$$

and $1\otimes\varphi_{X_\ell}$ on $F(X_1,\cdots,Z_i)\otimes F(Y_i,\cdots,\widehat{X_\ell},\cdots,X_n)$; if $i=\ell$, the zero maps; if $i>\ell$ similar to the case $i<\ell$.

These maps satisfy the conditions below.

(1) For two elements $k<\ell$ in $(1,n)$, $(1\otimes\tau_{X_\ell})\tau_{X_k} = (\tau_{X_k}\otimes 1)\tau_{X_\ell}$, namely the following square commutes:

$$\begin{array}{ccc}
F(X_1,\cdots,X_n) & \xrightarrow{\tau_{X_k}} & F(X_1,\cdots,X_k)\otimes F(X_k,\cdots,X_n)\\
\downarrow{\scriptstyle \tau_{X_\ell}} & & \downarrow{\scriptstyle 1\otimes\tau_{X_\ell}}\\
F(X_1,\cdots,X_\ell)\otimes F(X_\ell,\cdots,X_n) & \xrightarrow{\tau_{X_k}\otimes 1} & F(X_1,\cdots,X_k)\otimes F(X_k,\cdots,X_\ell)\otimes F(X_\ell,\cdots,X_n)\ .
\end{array}$$

(2) For two elements $k<\ell$ in $(1,n)$, $\varphi_{X_\ell}\varphi_{X_k}=\varphi_{X_k}\varphi_{X_\ell}$, namely the following commutes:

$$\begin{array}{ccc}
F(X_1,\cdots,X_n) & \xrightarrow{\varphi_{X_k}} & F(X_1,\cdots,\widehat{X_k},\cdots,X_n)\\
\downarrow{\scriptstyle \varphi_{X_\ell}} & & \downarrow{\scriptstyle \varphi_{X_\ell}}\\
F(X_1,\cdots,\widehat{X_\ell},\cdots,X_n) & \xrightarrow{\varphi_{X_k}} & F(X_1,\cdots,\widehat{X_k},\cdots,\widehat{X_\ell},\cdots,X_n)\ .
\end{array}$$

(3) For distinct elements $k$ and $\ell$ in $(1,n)$, $\tau_{X_\ell}\varphi_{X_k}=(\varphi_{X_k}\otimes 1)\tau_{X_\ell}$ if $k<\ell$, and $\tau_{X_\ell}\varphi_{X_k} = (1\otimes\varphi_{X_k})\tau_{X_\ell}$ if $k>\ell$. The following diagram is for $k<\ell$.

$$\begin{array}{ccc}
F(X_1,\cdots,X_n) & \xrightarrow{\varphi_{X_k}} & F(X_1,\cdots,\widehat{X_k},\cdots,X_n)\\
\downarrow{\scriptstyle \tau_{X_\ell}} & & \downarrow{\scriptstyle \tau_{X_\ell}}\\
F(X_1,\cdots,X_\ell)\otimes F(X_\ell,\cdots,X_n) & \xrightarrow{\varphi_{X_k}\otimes 1} & F(X_1,\cdots,\widehat{X_k},\cdots,X_\ell)\otimes F(X_\ell,\cdots,X_n)
\end{array}$$

We often simply write $\tau_k$, $\varphi_k$ for $\tau_{X_k}$, $\varphi_{X_k}$. Then the equality in (1) reads $(1\otimes\tau_\ell)\tau_k = (\tau_k\otimes 1)\tau_\ell$. Note $\tau_\ell$ is not necessarily the $\ell$-th $\tau$-map.

(4) Existence of the identity in the ring $H^0F(X,X)$. This condition will be stated in (1.4).

For a finite ordered set $I$ and a collection of objects $(X_i)_{i\in I}$ indexed by $I$, one can define the complex $F((X_i)_{i\in I})$, or $F(I)$ for short. Then the conditions (1)-(3) can be stated more naturally.

(1.3) It is convenient to reformulate the above conditions as follows. For a subset $I = \{\ell_1, \cdots, \ell_a\} \subset [1,n]$, write $F(I)$ in place of $F(X_{\ell_1}, \cdots, X_{\ell_a})$ for abbreviation. For a subset $S = \{i_1, \cdots, i_a\} \subset (1,n)$, let

$$F(X_1, \cdots, X_n \intercal S) := F(X_1, \cdots, X_{i_1}) \otimes F(X_{i_1}, \cdots, X_{i_2}) \otimes \cdots \otimes F(X_{i_a}, \cdots, X_n)$$

be the tensor product complex; in other words, if $I_1, \cdots, I_a$ is the segmentation (see (0.2)) corresponding to $S$, $F(X_1, \cdots, X_n \intercal S) = F(I_1) \otimes \cdots \otimes F(I_a)$. Note $F(X_1, \cdots, X_n \intercal \emptyset) = F(X_1, \cdots, X_n)$. We also write $F([1,n] \intercal S)$ for $F(X_1, \cdots, X_n \intercal S)$.

More generally for $I \subset [1,n]$ let $\overset{\circ}{I} = I - \{\mathrm{in}(I), \mathrm{tm}(I)\}$. For a subset $S$ of $\overset{\circ}{I}$ one defines the complex $F(I \intercal S)$ in a similar manner.

The complex $F(X_1, \cdots, X_n \intercal S)$ is additive in each variable. If a variable $X_i = O$, then it is zero. If $X_i = Y_i \oplus Z_i$, then

$$\begin{aligned} & F(X_1, \cdots, X_{i-1}, Y_i \oplus Z_i, X_{i+1}, \cdots, X_n \intercal S) \\ = \ & F(X_1, \cdots, Y_i, \cdots, X_n \intercal S) \oplus F(X_1, \cdots, Z_i, \cdots, X_n \intercal S) \\ & \oplus F(X_1, \cdots, Y_i \intercal S_1) \otimes F(Z_i, \cdots, X_n \intercal S_2) \oplus F(X_1, \cdots, Z_i \intercal S_1) \otimes F(Y_i, \cdots, X_n \intercal S_2) \ , \end{aligned}$$

where $S_1, S_2$ is the partition of $S$ by $i$, namely $S_1 = S \cap (1,i)$, $S_2 = S \cap (i,n)$. (The last two terms are not there if $i = 1$ or $n$.)

For subsets $S \subset S'$ of $\overset{\circ}{I}$ we will define a map

$$\tau_{S\,S'} : F(I \intercal S) \to F(I \intercal S') \ .$$

Let

$$\tau_S : F(I) \to F(I \intercal S)$$

be the composition of $\tau_{X_k}$'s for $k \in S$. Define for $S \subset S'$ the map $\tau_{S\,S'}$ as follows. If $S = \emptyset$, let $\tau_{\emptyset\,S'} = \tau_{S'}$. In general let $I_1, \cdots, I_a$ be the segmentation of $I$ corresponding to $S$, $S'_i = S' \cap \overset{\circ}{I}_i$, and $\tau_{S'_i} : F(I_i) \to F(I_i \intercal S'_i)$ be the map just defined. Then

$$\tau_{SS'} := \bigotimes_i \tau_{S'_i} : \bigotimes_i F(I_i) \to \bigotimes_i F(I_i \intercal S'_i) = F(I \intercal S) \ .$$

The $\tau_{SS'}(X_1, \cdots, X_n)$ is additive in each variable, namely if $X_i = Y_i \oplus Z_i$, then $\tau_{SS'}(X_1, \cdots, X_n)$ is the direct sum of the maps $\tau_{SS'}(X_1, \cdots, Y_i, \cdots, X_n)$, $\tau_{SS'}(X_1, \cdots, Z_i, \cdots, X_n)$, and the maps

$$\tau_{S_1 S'_1} \otimes \tau_{S_2 S'_2} : F(X_1, \cdots, Y_i \intercal S_1) \otimes F(Z_i, \cdots, X_n \intercal S_2) \to F(X_1, \cdots, Y_i \intercal S'_1) \otimes F(Z_i, \cdots, X_n \intercal S'_2) \ ,$$

$$\tau_{S_1 S'_1} \otimes \tau_{S_2 S'_2} : F(X_1, \cdots, Z_i \intercal S_1) \otimes F(Y_i, \cdots, X_n \intercal S_2) \to F(X_1, \cdots, Y_i \intercal S'_1) \otimes F(Z_i, \cdots, X_n \intercal S'_2)$$

on the cross terms.

For $K = \{k_1, \cdots, k_b\} \subset (1,n)$ disjoint from $S$, we define a map

$$\varphi_K : F(X_1, \cdots, X_n \intercal S) \to F(X_1, \cdots, \widehat{X_{k_1}}, \cdots, \widehat{X_{k_b}}, \cdots, X_n \intercal S) \ ;$$

More generally for $I \subset \mathring{I}$ and $K \subset \mathring{I}$ disjoint from $S$, we have $\varphi_K : F(I \intercal S) \to F(I-K \intercal S)$. If $S = \emptyset$, $\varphi_K$ is the composition of $\varphi_k$ for $k \in K$; in general, let $I_1, \cdots, I_a$ be the segmentation of $[1,n]$ corresponding to $S$, and

$$\varphi_K := \bigotimes_i \varphi_{K \cap I_i} : \bigotimes_i F(I_i) \to \bigotimes_i F(I_i - K) \ .$$

It is additive in each variable in the following sense: If $X_i = Y_i \oplus Z_i$, then the map $\varphi_K(X_1, \cdots, X_n)$ is the sum of $\varphi_K(X_1, \cdots, Y_i, \cdots, X_n)$, $\varphi_K(X_1, \cdots, Z_i, \cdots, X_n)$, and, if $i \notin K$, the maps

$$\varphi_{K_1} \otimes \varphi_{K_2} \text{ on } F(X_1, \cdots, Y_i \intercal S_1) \otimes F(Z_i, \cdots, X_n \intercal S_2) \ ,$$
$$\varphi_{K_1} \otimes \varphi_{K_2} \text{ on } F(X_1, \cdots, Z_i \intercal S_1) \otimes F(Y_i, \cdots, X_n \intercal S_2)$$

on the cross terms ($K_1, K_2$ is the partition of $K$ by $i$), and if $i \in K$, the zero maps on the cross terms.

These maps satisfy the following properties. Conversely given maps $\tau_{SS'}$ and $\varphi_K$ satisfying those properties, the maps $\tau_k = \tau_{\emptyset,k}$ and $\varphi_\ell$ satisfy the conditions (1)-(3) in (1.2), and the derived maps from them coincide with $\tau_{SS'}$ and $\tau_K$.

(1) $\tau_{SS'}$ is a quasi-isomorphism. For $S \subset S' \subset S''$, $\tau_{S'S''}\tau_{SS'} = \tau_{SS''}$. The map $\tau_{SS'}$ is compatible with tensor product as follows: For a subset $T \subset S$, $I_1, \cdots, I_c$ the segmentation corresponding to $T$, and $S_i = S \cap \mathring{I}_i$; note $F(I \intercal S) = F(I_1 \intercal S_1) \otimes \cdots \otimes F(I_c \intercal S_c)$. For $S \subset S'$, with $S'_i = S' \cap \mathring{I}_i$ the following commutes:

$$\begin{array}{ccc} F(I \intercal S) & = & F(I_1 \intercal S_1) \otimes \cdots \otimes F(I_c \intercal S_c) \\ {\scriptstyle \tau_{SS'}}\downarrow & & \downarrow{\scriptstyle \otimes \tau_{S_i S'_i}} \\ F(I \intercal S') & = & F(I_1 \intercal S'_1) \otimes \cdots \otimes F(I_c \intercal S'_c) \ . \end{array}$$

(2) If $K$ is the disjoint union of $K'$ and $K''$, $\varphi_K = \varphi_{K''}\varphi_{K'}$. Also $\varphi_K$ is compatible with tensor product, namely under the same assumption as in (1) and with $K_i = K \cap I_i$, the following diagram commutes:

$$\begin{array}{ccc} F(I \intercal S) & = & F(I_1 \intercal S_1) \otimes \cdots \otimes F(I_c \intercal S_c) \\ {\scriptstyle \varphi_K}\downarrow & & \downarrow{\scriptstyle \otimes \varphi_{K_i}} \\ F(I - K \intercal S) & = & F(I_1 - K_1 \intercal S_1) \otimes \cdots \otimes F(I_c - K_c \intercal S_c) \ . \end{array}$$

(3) If $K$ and $S'$ are disjoint the following commutes:

$$\begin{array}{ccc} F(I \intercal S) & \xrightarrow{\varphi_K} & F(I - K \intercal S) \\ {\scriptstyle \tau_{SS'}}\downarrow & & \downarrow{\scriptstyle \tau_{SS'}} \\ F(I \intercal S') & \xrightarrow{\varphi_K} & F(I - K \intercal S') \ . \end{array}$$

(1.4) A weak quasi DG category $\mathcal{C}$ is not a category in the usual sense, since the composition is not defined. Nevertheless, one has composition in a weak sense.

For three objects $X$, $Y$ and $Z$, let

$$\psi_Y : F(X,Y) \otimes F(Y,Z) \to F(X,Z)$$

be the map in the derived category defined as the composition $\varphi_Y \circ (\sigma_Y)^{-1}$ where the maps are as in

$$F(X,Y)\otimes F(Y,Z) \xleftarrow{\sigma_Y} F(X,Y,Z) \xrightarrow{\varphi_Y} F(X,Z) \ .$$

The map $\psi_Y$ is verified to be associative, namely the following commutes in the derived category:

$$\begin{array}{ccc} F(X,Y)\otimes F(Y,Z)\otimes F(Z,W) & \xrightarrow{\psi_Y\otimes id} & F(X,Z)\otimes F(Z,W) \\ {\scriptstyle id\otimes\psi_Z}\downarrow & & \downarrow{\scriptstyle \psi_Z} \\ F(X,Y)\otimes F(Y,W) & \xrightarrow{\psi_Y} & F(X,W) \ . \end{array}$$

Let $H^0F(X,Y)$ be the 0-th cohomology of $F(X,Y)$. $\psi_Y$ induces a map

$$\psi_Y : H^0F(X,Y)\otimes H^0F(Y,Z) \to H^0(\,F(X,Y)\otimes F(Y,Z)\,) \xrightarrow{H^0\psi_Y} H^0F(X,Z) \ ,$$

which is associative. In particular $H^0F(X,X)$ is a ring. We often write $u\cdot v$ for $\psi_Y(u\otimes v)$.

The last condition (4) for a weak quasi DG category is:

(4) For each $X$ there is an element $1_X \in H^0F(X,X)$ such that $1_X\cdot u = u$ for any $u \in H^0F(X,Y)$ and $u\cdot 1_X = u$ for $u\in H^0F(Y,X)$. (Call such $1_X$ the *identity*.)

Thus one can associate to $\mathcal{C}$ an additive category, the associated *homotopy category*, denoted by $Ho(\mathcal{C})$. Objects of $Ho(\mathcal{C})$ are the same as the objects of $\mathcal{C}$, and $\mathrm{Hom}(X,Y) := H^0F(X,Y)$. Composition of arrows is the map induced from $\psi_Y$. The object 0 is the zero object, and the direct sum $X\oplus Y$ is the direct sum in the categorical sense. $1_X$ gives the identity $X\to X$.

*Remark.* More generally we have maps $\psi_Y : H^mF(X,Y)\otimes H^nF(Y,Z) \to H^{m+n}F(X,Z)$ for $m,n\in\mathbb{Z}$, defined in a similar manner. The groups $H^mF(X,Y)$ and the composition maps for them play roles when we consider the quasi DG category $\mathcal{C}^\Delta$.

(1.5) A *quasi DG category* $\mathcal{C}$ consists of data (i)-(iii), satisfying the conditions (1)-(5). When necessary we will also impose additional structure (iv),-(v).

(i) *The class of objects* $Ob(\mathcal{C})$. There is a distinguished object $O$, called the zero object. There is direct sum of objects $X\oplus Y$, and one has $(X\oplus Y)\oplus Z = X\oplus(Y\oplus Z)$.

(ii) *Multiple complexes* $F(X_1,\cdots,X_n)$. For each sequence of irreducible objects $X_1,\cdots X_n$ $(n\geq 2)$, a complex of abelian groups $F(X_1,\cdots,X_n)$.

For a subset $S\subset(1,n)$, we have $F(X_1,\cdots,X_n \,\overline{|}\, S) := F(I_1)\otimes\cdots\otimes F(I_a)$ as in (1.3); this is an $a$-tuple complex. As before for a finite ordered set $I$ and a sequence of objects $(X_i)_{i\in I}$, one has $F(I) = F(I;X)$ and $F(I|S) = F(I|S;X)$.

(iii) *Multiple complexes* $F(X_1,\cdots,X_n|S)$ *and maps* $\iota_S$, $\sigma_{SS'}$ *and* $\varphi_K$.

(1) We require given a quasi-isomorphic multiple subcomplex of abelian groups

$$\iota_S : F(X_1,\cdots,X_n|S) \hookrightarrow F(X_1,\cdots,X_n \,\overline{|}\, S) \ .$$

We assume $F(X_1,\cdots,X_n|\emptyset) = F(X_1,\cdots,X_n)$. The complex $F(X_1,\cdots,X_n|S)$ is additive in each variable, by which we mean the same property as in (1.3) for $F(X_1,\cdots,X_n \,\overline{|}\, S)$. The inclusion $\iota_S$ is compatible with the additivity.

For a subset $T\subset S$, if $I_1,\cdots,I_c$ is the segmentation corresponding to $T$, and $S_i = S\cap \mathring{I}_i$, one requires there is an inclusion of multiple complexes

$$F(I|S) \subset F(I_1|S_1)\otimes\cdots\otimes F(I_c|S_c) \qquad (1.5.1)$$

where the latter group is viewed as a subcomplex of $F(I \,\overline{|}\, S)$ by the tensor product of the inclusions $\iota_{S_i} : F(I_i|S_i) \hookrightarrow F(I_i \,\overline{|}\, S_i)$.

(2) For $S \subset S'$ given a surjective quasi-isomorphism of multiple complexes

$$\sigma_{S\,S'} : F(X_1, \cdots, X_n|S) \to F(X_1, \cdots, X_n|S') \ .$$

For $S \subset S' \subset S''$, $\sigma_{S\,S''} = \sigma_{S'\,S''}\sigma_{S\,S'}$. The $\sigma_{SS'}$ is additive in each variable variable, namely it satisfies the same condition as in (1.3) for $\tau$ and $F(X_1, \cdots, X_n \,\overline{|}\, S)$.

$\sigma$ is assumed compatible with the inclusion in (1.5.1): If $S \subset S'$ and $S'_i = S' \cap \overset{\circ}{I_i}$ the following commutes:

$$\begin{array}{ccc} F(I|S) & \hookrightarrow & F(I_1|S_1) \otimes \cdots \otimes F(I_1|S_1) \\ {\scriptstyle \sigma_{SS'}}\downarrow & & \downarrow{\scriptstyle \otimes \sigma_{S_i\,S'_i}} \\ F(I|S') & \hookrightarrow & F(I_1|S'_1) \otimes \cdots \otimes F(I_1|S'_1) \ . \end{array}$$

We write $\sigma_S = \sigma_{\emptyset\,S} : F(I) \to F(I|S)$. The composition of $\sigma_S$ and $\iota_S$ is denoted $\tau_S : F(I) \to F(I \,\overline{|}\, S)$.

(3) For $K = \{k_1, \cdots, k_b\} \subset (1, n)$ disjoint from $S$, a map of multiple complexes

$$\varphi_K : F(X_1, \cdots, X_n|S) \to F(X_1, \cdots, \widehat{X_{k_1}}, \cdots, \widehat{X_{k_b}}, \cdots, X_n|S) \ .$$

If $K = K' \amalg K''$ then $\varphi_K = \varphi_{K''}\varphi_{K'} : F(I|S) \to F(I-K|S)$. The $\varphi_K$ is additive in each variable variable, namely it satisfies the same condition as in (1.3) for $\varphi_K$ and $F(X_1, \cdots, X_n \,\overline{|}\, S)$.

$\varphi_K$ is assumed to be compatible with the inclusion in (1.5.1): With the same notation as above and $K_i = K \cap I_i$, the following commutes:

$$\begin{array}{ccc} F(I|S) & \hookrightarrow & F(I_1|S_1) \otimes \cdots \otimes F(I_c|S_c) \\ {\scriptstyle \varphi_K}\downarrow & & \downarrow{\scriptstyle \otimes \varphi_{K_i}} \\ F(I-K|S) & \hookrightarrow & F(I_1 - K_1|S_1) \otimes \cdots \otimes F(I_c - K_c|S_c) \ . \end{array}$$

If $K$ and $S'$ are disjoint the following commutes:

$$\begin{array}{ccc} F(I|S) & \xrightarrow{\varphi_K} & F(I-K|S) \\ {\scriptstyle \sigma_{SS'}}\downarrow & & \downarrow{\scriptstyle \sigma_{SS'}} \\ F(I|S') & \xrightarrow{\varphi_K} & F(I-K|S') \ . \end{array}$$

(4) (acyclicity of $\sigma$) For disjoint subsets $R, J$ of $\overset{\circ}{I}$ with $|J| \neq \emptyset$, consider the following sequence of complexes, where the maps are alternating sums of $\sigma$, and $S$ varies over subsets of $J$:

$$F(I|R) \xrightarrow{\sigma} \bigoplus_{\substack{|S|=1 \\ S \subset J}} F(I|R \cup S) \xrightarrow{\sigma} \bigoplus_{\substack{|S|=2 \\ S \subset J}} F(I|R \cup S) \to \cdots \to F(I|R \cup J) \to 0. \tag{1.5.2}$$

Then the sequence is exact.

*Remarks to (4).* (i) If $|J| = 1$ and $|S'| = |S| + 1$, this says $\sigma_{S\,S'}$ is a surjective quasi-isomorphism, which was already assumed in (2).

(ii) Since each $\sigma_{S\,S'}$ is a quasi-isomorphism, the total complex of the double complex (1.5.2) is acyclic, see [8], (2.4). Thus the exactness implies that the induced map

$$\sigma : F(I|R) \to \mathrm{Ker}[\bigoplus_{\substack{|S|=1 \\ S \subset J}} F(I|R \cup S) \xrightarrow{\sigma} \bigoplus_{\substack{|S|=2 \\ S \subset J}} F(I|R \cup S)]$$

is a quasi-isomorphism.

(5) (existence of the identity in the ring $H^0F(X,X)$) There are composition maps for $H^0F(X,Y)$ as in (1.4). We assume there exists the identity element $1_X$ in $H^0F(X,X)$.

This concludes the definition of a quasi DG category. For the purpose of constructing a related category $\mathcal{C}^\Delta$, we need additional structure (iv) and (v) below.

(iv) *Diagonal elements and diagonal extension.*

(6) For each irreducible object $X$ and a constant sequence of objects $i \mapsto X_i = X$ on a finite ordered set $I$ with $|I| \geq 2$, there is a distinguished element, called the *diagonal element*

$$\mathbf{\Delta}_X(I) \in F(I) = F(X, \cdots, X)$$

of degree zero and coboundary zero. In particular for $|I| = 2$ we write $\Delta_X = \mathbf{\Delta}_X(I) \in F(X,X)$. One requires:

(6-1) If $S \subset \overset{\circ}{I}$, and $I_1, \cdots, I_c$ the corresponding segmentation, one has

$$\tau_S(\mathbf{\Delta}_X(I)\,) = \mathbf{\Delta}_X(I_1) \otimes \cdots \otimes \mathbf{\Delta}_X(I_c)$$

in $F(I \rceil S) = F(I_1) \otimes \cdots \otimes F(I_c)$.

(6-2) For $K \subset \overset{\circ}{I}$, $\varphi_K(\mathbf{\Delta}_X(I)\,) = \mathbf{\Delta}_X(I - K)$.

(7) Let $I$ be a finite ordered set, $k \in I$, $m \geq 2$, and $I^\sim$ be the finite ordered set obtained by replacing $k$ by a finite ordered set with $m$ elements $\{k_1, \cdots, k_m\}$. If $I = [1, n]$, $I^\sim$ is $\{1, \cdots, k-1, k_1, \cdots, k_m, k+1, \cdots, n\}$.

There is given a map of complexes, called the *diagonal extension*,

$$\mathrm{diag}(I, I^\sim) : F(I) \to F(I^\sim)$$

subject to the following conditions (for simplicity assume $I = [1, n]$):

(7-1) If $k' \neq k$, $\varphi_{k'}\,\mathrm{diag}(I, I^\sim) = \mathrm{diag}(I - \{k'\}, I^\sim - \{k'\})\varphi_{k'}$, namely the following square commutes:

$$\begin{tikzcd} F(I) \arrow[r, "\mathrm{diag}(I,I^\sim)"] \arrow[d, "\varphi_{k'}"'] & F(I^\sim) \arrow[d, "\varphi_{k'}"] \\ F(I - \{k\}) \arrow[r, "\mathrm{diag}(I-\{k'\},I^\sim-\{k'\})"] & F(I^\sim - \{k'\}) \,. \end{tikzcd}$$

If $\ell \in \{k_1, \cdots, k_m\}$, $\varphi_\ell\,\mathrm{diag}(I, I^\sim) = \mathrm{diag}(I, I^\sim - \{\ell\})$. If $m = 2$ the right side is the identity.

(7-2) If $k = n$, $\ell \in \{n_1, \cdots, n_m\}$, let $I'_1, I''$ be the segmentation of $I^\sim$ by $\ell$. Then the following diagram commutes:

$$\begin{tikzcd} F(I) \arrow[r, "\mathrm{diag}(I,I^\sim)"] \arrow[d, "\mathrm{diag}(I,I'_1)"'] & F(I^\sim) \arrow[d, "\tau_\ell"] \\ F(I'_1) \arrow[r] & F(I'_1) \otimes F(I'') \,. \end{tikzcd}$$

The lower horizontal map is $u \mapsto u \otimes \mathbf{\Delta}(I'')$. Note $I''$ parametrizes a constant sequence of objects, so one has $\mathbf{\Delta}(I'') \in F(I'')$. Similarly in case $k = 1$, $\ell \in \{1_1, \cdots, 1_m\}$.

If $1 < k < n$ and $\ell \in \{k_1, \cdots, k_m\}$, let $I_1, I_2$ be the segmentation of $I$ by $k$, and $I'_1, I'_2$ of $I^{\sim}$ by $\ell$. One then has a commutative diagram:

$$\begin{array}{ccc} F(I) & \xrightarrow{\mathrm{diag}(I, I^{\sim})} & F(I^{\sim}) \\ {\scriptstyle \tau_k}\downarrow & & \downarrow{\scriptstyle \tau_\ell} \\ F(I_1) \otimes F(I_2) & \longrightarrow & F(I'_1) \otimes F(I'_2) \end{array},$$

where the lower horizontal arrow is $\mathrm{diag}(I_1, I'_1) \otimes \mathrm{diag}(I_2, I'_2)$.

(v) *The set of generators, notion of proper intersection, and distinguished subcomplexes with respect to constraints.*

(8)(the generating set) For a sequence $X$ on $I$, the complex $F(I) = F(I; X)$ is degree-wise $\mathbb{Z}$-free on a given set of generators $\mathcal{S}_F(I) = \mathcal{S}_F(I; X)$. More precisely $\mathcal{S}_F(I) = \amalg_{p \in \mathbb{Z}} \mathcal{S}_F(I)^p$, where $\mathcal{S}_F(I)^p$ generates $F(I)^p$. This set is compatible with direct sum in each variable: Assume for an element $k \in I$ one has $X_k = Y_k \oplus Z_k$; let $X'_i$ (resp. $X''_i$ be the sequence such that $X'_i = X_i$ for $i \neq k$, and $X'_k = Y_k$ (resp. $X''_i = X_i$ for $i \neq k$, and $X''_k = Z_k$). Then $\mathcal{S}_F(I; X) = \mathcal{S}_F(I; X') \amalg \mathcal{S}_F(I; X'')$.

(9) (notion of proper intersection.) Let $I$ be a finite ordered set, $I_1, \cdots, I_r$ be almost disjoint sub-intervals of $I$, namely one has $\mathrm{tm}(I_i) \leq \mathrm{in}(I_{i+1})$ for each $i$. Assume given a sequence of objects $X_i$ on $I$. Let $\alpha_i \in \mathcal{S}_F(I_i)$ be a set of elements where $i$ varies over a subset $A$ of $\{1, \cdots, r\}$. We are given the condition whether the set $\{\alpha_i\}$ is *properly intersecting.* The following condition is to be satisfied.

- If $\{\alpha_i |\ i \in A\}$ is properly intersecting, for any subset $B$ of $A$, $\{\alpha_i |\ i \in B\}$ is properly intersecting.
- Let $A$ and $A'$ be subsets of $\{1, \cdots, r\}$ such that $\mathrm{tm}(A) < \mathrm{in}(A')$. If $\{\alpha_i |\ i \in A\}$ and $\{\alpha_i |\ i \in A'\}$ are both properly intersecting sets, the union $\{\alpha_i |\ i \in A \cup A'\}$ is also properly intersecting.
- If $\{\alpha_1, \cdots, \alpha_r\}$ is properly intersecting, then for any $i$, writing $\partial \alpha_i = \sum c_{i\nu} \beta_\nu$ with $\beta_\nu \in \mathcal{S}_F(I_i)$, each set
$$\{\alpha_1, \cdots, \alpha_{i-1}, \beta_\nu, \alpha_{i+1}, \cdots, \alpha_r\}$$
is properly intersecting.
- The condition of proper intersection is compatible with direct sum in each variable. To be precise, under the same assumption as in (8), for a set of elements $\alpha_i \in \mathcal{S}_F(I_i; X')$ for $i = 1, \cdots, r$, the set $\{\alpha_i \in \mathcal{S}_F(I_i; X')\}_i$ is properly intersecting if and only if the set $\{\alpha_i \in \mathcal{S}_F(I_i; X)\}_i$ is properly intersecting.

*Remark.* For $I_i$ almost disjoint and elements $\alpha_i \in F(I_i)$, one defines $\{\alpha_i \in F(I_i) | i \in A\}$ to be properly intersecting if the following holds. Write $\alpha_i = \sum c_{i\nu} \alpha_{i\,\nu}$ with $\alpha_{i\,\nu} \in \mathcal{S}_F(I_i)$, then for any choice of $\nu_i$ for $i \in A$, the set $\{\alpha_{i\,\nu_i} | i \in A\}$ is properly intersecting.

Further, if $S_i \subset \overset{\circ}{I_i}$, one can define the condition of proper intersection for $\{\alpha_i \in F(I_i|S_i) | i \in A\}$ by writing each $\alpha_i$ as a sum of tensors of elements in the generating set.

(10) (description of $F(I|S)$ ) When $I_1, \cdots, I_r$ is a segmentation of $I$, namely when $\mathrm{in}(I_1) = \mathrm{in}(I)$, $\mathrm{tm}(I_i) = \mathrm{in}(I_{i+1})$ and $\mathrm{tm}(I_r) = \mathrm{tm}(I)$, the subcomplex of $F(I_1) \otimes \cdot \otimes F(I_r)$ generated by

$\alpha_1 \otimes \cdots \otimes \alpha_r$ with $\{\alpha_i\}$ properly intersecting is denoted by $F(I_1)\hat{\otimes} \cdots \hat{\otimes} F(I_r)$. If $S \subset \overset{\circ}{I}$ is the subset corresponding to the segmentation, this subcomplex coincides with $F(I|S)$.

*Remark.* The property (1) is an obvious consequence of this description.

(11)(distinguished subcomplexes) Let $I$ be a finite ordered set, $L_1, \cdots, L_r$ be almost disjoint sub-intervals such that $\cup L_i = I$; equivalently, $\mathrm{in}(L_1) = \mathrm{in}(I)$, $\mathrm{tm}(L_i) = \mathrm{in}(L_{i+1})$ or $\mathrm{tm}(L_i)+1 = \mathrm{in}(L_{i+1})$, and $\mathrm{tm}(L_r) = \mathrm{tm}(I)$. Assume given a sequence of objects $X_i$ on $I$. Let *Dist* be the smallest class of subcomplexes of $F(L_1) \otimes \cdots \otimes F(L_r)$ satisfying the conditions below. It is then required that each subcomplex *Dist* is a quasi-isomorphic subcomplex.

(11-1) A subcomplex obtained as follows is in *Dist*. Let $I_1, \cdots, I_c$ be a set of almost disjoint sub-intervals of $I$ with union $I$, that is coarser than $L_1, \cdots, L_r$; let $S_i \subset \overset{\circ}{I_i}$ such that the segmentations of $I_i$ by $S_i$, when combined for all $i$, give precisely the $L_i$'s. Let $I \hookrightarrow \mathbb{I}$ be an inclusion into a finite ordered set $\mathbb{I}$ such that the image of each $I_a$ is a sub-interval. Assume given an extension of $X$ to $\mathbb{I}$. Let $J_1, \cdots, J_s \subset \mathbb{I}$ be sub-intervals of $\mathbb{I}$ such that the set $\{I_i, J_j\}_{i,j}$ is almost disjoint, and $f_j \in F(J_j|T_j)$, $j = 1, \cdots, s$ be a properly intersecting set. Then one defines the subcomplex

$$[F(I_1|S_1) \otimes \cdots \otimes F(I_c|S_c)]_{\mathbb{I};f} \ ,$$

as the one generated by $\alpha_1 \otimes \cdots \otimes \alpha_c$, $\alpha_i \in F(I_i|S_i)$, such that $\{\alpha_1, \cdots, \alpha_c, f_j\,(j = 1, \cdots, s)\,\}$ is properly intersecting. We require it is in *Dist*.

The data consisting of $I \hookrightarrow \mathbb{I}$, $X$ on $\mathbb{I}$, $J_i \subset \mathbb{I}$, and $f_j \in F(J_j|T_j)$ is called a *constraint*, and the corresponding subcomplex the distinguished subcomplex for the constraint.

(11-2) Tensor product of subcomplexes in *Dist* is again in *Dist*. For this to make sense, note complexes of the form $F(L_1) \otimes \cdots \otimes F(I_r)$ are closed under tensor products: If $I'$ is another finite ordered set and $L'_1, \cdots, L'_s$ are almost disjoint sub-intervals with union $I'$, then the tensor product

$$F(L_1) \otimes \cdots \otimes F(I_r) \otimes F(L'_1) \otimes \cdots \otimes F(I'_s)$$

is associated with the ordered set $I \amalg I'$ and almost disjoint sub-intervals $(L_1, \cdots, L_r, L'_1, \cdots, L'_s)$.

(11-3) A finite intersection of subcomplexes in *Dist* is again in *Dist*.

(1.6) Note that a quasi DG category is a weak quasi DG category. Indeed one shows there exist unique maps $\tau_{S\,S'} : F(I \,\overline{|}\, S) \to F(I \,\overline{|}\, S')$ satisfying the conditions (1.3) (1), and compatible with $\sigma_{SS'}$ via $\iota_S$ and $\iota_{S'}$, namely making the following diagram commute

$$\begin{tikzcd} F(I|S) \arrow[r, "\iota_S"] \arrow[d, "\sigma_{S\,S'}"'] & F(I \,\overline{|}\, S) \arrow[d, "\tau_{S\,S'}"] \\ F(I|S') \arrow[r, "\iota_{S'}"] & F(I \,\overline{|}\, S') \ . \end{tikzcd}$$

There are also maps $\varphi_K : F(I \,\overline{|}\, S) \to F(I - K \,\overline{|}\, S)$ satisfying (1.3)(2) and compatible with $\varphi_K : F(I|S) \to F(I - K|S)$ via the $\iota_S$'s, namely the following commutes:

$$\begin{tikzcd} F(I|S) \arrow[r, "\iota_S"] \arrow[d, "\varphi_K"'] & F(I \,\overline{|}\, S) \arrow[d, "\varphi_K"] \\ F(I - K|S) \arrow[r, "\iota_S"] & F(I - K \,\overline{|}\, S) \ . \end{tikzcd}$$

The property (1.3)(3) is satisfied as well.

When we assume the structure (iv), the condition (5) is redundant. From (6) and (7) it follows that $[\Delta_X] \in H^0F(X,X)$ is the identity in the sense of (1.4). Indeed for $u \in H^0F(X,Y)$ take its representative $\underline{u} \in F(X,Y)^0$, then take its diagonal extension $\mathrm{diag}(\underline{u}) \in F(X,X,Y)^0$. Then one has $\tau_2(\mathrm{diag}(\underline{u})) = \Delta_X \otimes \underline{u}$ and $\varphi_2(\mathrm{diag}(\underline{u})\,) = \underline{u}$. The same argument shows the following, which is stronger than (5):

(5)' For each $u \in H^nF(X,Y)$, $n \in \mathbb{Z}$, one has $1_X \cdot u = u$. Similarly for $u \in H^nF(Y,X)$, $u \cdot 1_X = u$.

(1.7) **Example.** Let $S$ be a quasi-projective variety. Let $(\mathrm{Smooth}/k\,,\mathrm{Proj}/S)$ be the category of smooth varieties $X$ equipped with projective maps to $S$. A *symbol* over $S$ is an object the form

$$\bigoplus_{\alpha\in A}(X_\alpha/S, r_\alpha)$$

where $X_\alpha$ is a collection of objects in $(\mathrm{Smooth}/k\,,\mathrm{Proj}/S)$ indexed by a finite set $A$, and $r_\alpha \in \mathbb{Z}$.

In [8]we defined

- the complexes $F(K_1,\cdots,K_n|S)$ for a sequence of symbols $K_i$ and $S \subset (1,n)$,
- the maps $\iota$, $\sigma$ and $\varphi$,
- the diagonal elements $\mathbf{\Delta}_K(I)$ and the diagonal extension $\mathrm{diag}(I, I')$.

and showed the properties (1)-(11), except the additivity.

The additivity of $F(X_1,\cdots,X_n|S)$ can be easily shown as follows. For the complex $\mathcal{F}(X_1,\cdots,X_n|\Sigma)$ as in [8], when $X_i = Y_i \amalg Z_i$ one has for $i \in \Sigma$,

$$\mathcal{F}(X_1,\cdots,X_n|\Sigma) = \mathcal{F}(X_1,\cdots,Y_i,\cdots,X_n|\Sigma) \oplus \mathcal{F}(X_1,\cdots,Z_i,\cdots,X_n|\Sigma)\ ,$$

and for $i \notin \Sigma$,

$$\begin{aligned} & \mathcal{F}(X_1,\cdots,X_n|\Sigma) \\ = \; & \mathcal{F}(X_1,\cdots,Y_i,\cdots,X_n|\Sigma) \oplus \mathcal{F}(X_1,\cdots,Z_i,\cdots,X_n|\Sigma) \\ & \oplus \mathcal{F}(X_1,\cdots,Y_i|\Sigma_1) \otimes \mathcal{F}(Z_i,\cdots,X_n|\Sigma_2) \oplus \mathcal{F}(X_1,\cdots,Z_i|\Sigma_1) \otimes \mathcal{F}(Y_i,\cdots,X_n|\Sigma_2) \end{aligned}$$

where $\Sigma_1, \Sigma_2$ is the partition of $\Sigma$ by $i$. It follows that $F(X_1,\cdots,X_n|S) = \bigoplus_{\Sigma\supset S}\mathcal{F}(X_1,\cdots,X_n|\Sigma)$ satisfies the required additivity as a module. Since the product map $\rho_i$ is zero on the cross terms $\mathcal{F}(X_1,\cdots,Y_i|\Sigma_1) \otimes \mathcal{F}(Z_i,\cdots,X_n|\Sigma_2)$ and $\mathcal{F}(X_1,\cdots,Z_i|\Sigma_1) \otimes \mathcal{F}(Y_i,\cdots,X_n|\Sigma_2)$, one checks the additivity holds as a complex as well.

We refer to this as the quasi DG category $Symb(S)$.

(1.8) Let $\mathcal{C}$ be a quasi DG category. We will construct another quasi DG category $\mathcal{C}^\Delta$ out of $\mathcal{C}$. An object of $\mathcal{C}^\Delta$ is of the form $K = (K^m; f(m_1,\cdots,m_\mu)\,)$, where $(K^m)$ is a sequence of objects of $\mathcal{C}$ indexed by $m \in \mathbb{Z}$, almost all of which are zero, and

$$f(m_1,\cdots,m_\mu) \in F(K^{m_1},\cdots,K^{m_\mu})^{-(m_\mu-m_1-\mu+1)}$$

is a collection of elements indexed by sequences $(m_1 < m_2 < \cdots < m_\mu)$ with $\mu \geq 2$. We require the following conditions:

(i) For each $j = 2, \cdots, \mu - 1$

$$\sigma_{K^{m_j}}(f(m_1, \cdots, m_\mu)) = f(m_1, \cdots, m_j) \otimes f(m_j, \cdots, m_\mu)$$

in $F(K^{m_1}, \cdots, K^{m_j}) \otimes F(K^{m_j}, \cdots, K^{m_\mu})$.

(ii) For each $(m_1, \cdots, m_\mu)$, one has

$$\partial f(m_1, \cdots, m_\mu) + \sum_{1 \le t < \mu} \sum_{m_t < k < m_{t+1}} (-1)^{m_\mu + \mu + k + t} \varphi_{K^{m_k}}(f(m_1, \cdots, m_t, k, m_{t+1}, \cdots, m_\mu)) = 0 \ .$$

We call an object of $\mathcal{C}^\Delta$ a *C-diagram* in $\mathcal{C}$. For $C$-diagrams $K_1, \cdots, K_n$, we will define complexes of abelian groups $\mathbb{F}(K_1, \cdots, K_n)$ together with maps $\sigma_{K_i}$ and $\varphi_{K_i}$. So we have a quasi $DG$ category. Its homotopy category $Ho(\mathcal{C}^\Delta)$ is a triangulated category.

## 2 Function complexes $\mathbb{F}(K_1, \cdots, K_n)$.

In (2.6) the operation $\Phi$ of (0.4) is used. In (2.9) we refer to (0.3) for tensor product of "double" complexes, and to [8], (3.6)-(3.9) for a particular construction of distinguished complexes.

Given a quasi-DG category, we have defined the notion of $C$-diagrams in the category. For a sequence off $C$-diagrams $K_1, \cdots, K_n$, we will define the complexes $\mathbb{F}(K_1, \cdots, K_n)$ and the maps $\varphi$ and $\sigma$ among them, and show they satisfy the axioms of a quasi DG category.

(2.1) In this section a *sequence* is a pair $(M|M')$ consisting of a finite increasing sequence of integers $M = (m_1, \cdots, m_\mu)$ where $m_1 < \cdots < m_\mu$ with $\mu \ge 2$, and a subset $M'$ of $M - \{m_1, m_\mu\}$. We allow $M'$ to be empty. For simplicity we also use the notation $\mathbb{M}$ for $(M|M')$. When there is no confusion denote $(M|\emptyset)$ by $M$.

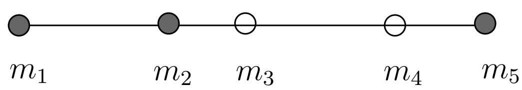
Let $\mathrm{in}(M) = m_1$, $\mathrm{tm}(M) = m_\mu$, $\overset{\circ}{M} = M - \{m_1, m_\mu\}$. Let $|M| = \mu$. For sequences $(M|M')$ and $(N|N')$ with $\mathrm{tm}(M) = \mathrm{in}(N)$, let

$$\mathbb{M} \circ \mathbb{N} = (M \cup N | M' \cup \{\mathrm{tm}(M)\} \cup N') \ .$$

A *double sequence* a quadruple $(M_1|M_1'; M_2|M_2')$. Here $M_1$, $M_2$ are finite sequences of integers, each of cardinality $\ge 1$, and $M_1'$ and $M_2'$ are subsets of $M_1 - \{\mathrm{in}(M_1)\}$, $M_2 - \{\mathrm{tm}(M_2)\}$, respectively. A double sequence may be viewed as a map defined on $[1, 2]$, which sends $i$ to $\mathbb{M}_i := (M_i|M_i')$. To be specific, we will say it is a double sequence on the set $[1, 2]$. (Note however that $\mathbb{M}_i$ is not a sequence in the sense just defined, since $M_i$ may have cardinality one, and even if $|M_1| \ge 2$, $M_1'$ may contain $\mathrm{tm}(M_1)$.) We also use a single letter $A$ to denote a double sequence,

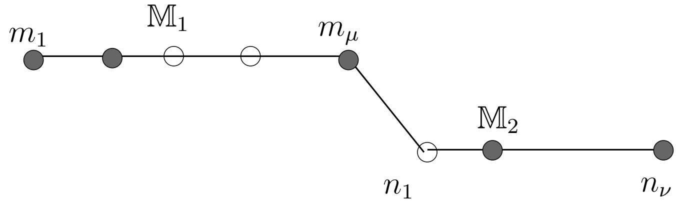
$$A = (\mathbb{M}_1; \mathbb{M}_2) = (M_1|M_1'; M_2|M_2') \ .$$

The following figure illustrates a sequence, where the line segment is $[m_1, m_\mu]$, the solid dots indicate the set $M$ and the hollow dots the subset $M'$.




$$\mathbb{M} = (M|M') = (\{m_1, \cdots, m_5\}|\{m_3, m_4\})$$

The next figure illustrates a double sequence. In the first line lies $\mathbb{M}_1$ which is a line segment with solid and hollow dots, and in the second lies $\mathbb{M}_2 = (M_2 = \{n_1, \cdots, n_\nu\}|M_2')$.




(2.2) *The complex $F(M|M')$.* Let $\mathcal{C}$ be a quasi DG category. Let $(K^m)$ be a sequence of objects indexed by integers $m$, all but a finite number of them being zero. To a sequence $(M|M')$, one can associate the complex

$$F(\mathbb{M}) = F(M|M') := F(K^{m_1}, \cdots, K^{m_\mu}|M') \ .$$

If $\mathbb{M} = (M|\emptyset)$, we simply write $F(M) = F(M|\emptyset)$; in general, if $M_1, \cdots, M_r$ is the segmentation of $M$ given by $M'$, $F(M|M') = F(M_1)\hat{\otimes}\cdots\hat{\otimes}F(M_r) \subset F(M_1)\otimes\cdots\otimes F(M_r)$. In this section the differential of $F(\mathbb{M})$ is denoted $\partial$. For $k \in \overset{\circ}{M} - M'$ there is the corresponding map of complexes $\varphi_k : F(M|M') \to F(M - \{k\}|M')$. There is also the map $\sigma_k : F(M|M') \to F(M|M' \cup \{k\})$. The maps $\varphi_k$ commute with each other, $\sigma_k$ commute with each other, and $\varphi_k$ and $\sigma_\ell$ commute with other.

(2.3) *The complex $\bigoplus F(M|M')$.* We will define the structure of a complex on $\bigoplus F(M|M')$, the direct sum over all sequences $(M|M')$.

For $M = (m_1, \cdots, m_\mu)$, let

$$\gamma(M) = m_\mu - m_1 - \mu + 1 \ .$$

If $M$ and $N$ are sequences with $\operatorname{tm} M = \operatorname{in} N$, then $\gamma(M \cup N) = \gamma(M) + \gamma(N)$.

If $k \in \overset{\circ}{M}$, let $M_{\leq k} := \{m_i \in M \mid m \leq k\}$. In this section, for an integer $d$, write $\{d\} := (-1)^d$; this is useful when $d$ is a complicated expression. For $u \in F(M|M')$, let $|u| = \deg u$ (the degree in $F(M|M')$)).

We will define a map $\boldsymbol{\partial} : F(M|M') \to F(M|M')$ of degree 1, and a map $\boldsymbol{\varphi} : \bigoplus F(M|M') \to \bigoplus F(M|M')$ of degree 0. They are obtained from $\partial$ and $\varphi$ by putting appropriate signs.

For $u \in F(M|\emptyset)$ define $\boldsymbol{\partial}(u) := \partial u$. For $k \in \overset{\circ}{M}$, define $\boldsymbol{\varphi}_k : F(M|\emptyset) \to F(M - \{k\}|\emptyset)$ by

$$\boldsymbol{\varphi}_k(u) := \{|u| + \gamma(M_{\leq k})\}\varphi_k(u) \ .$$

In general let $M_1, \cdots, M_r$ be the segmentation of $M$ corresponding to $M'$, so that $F(M|M') = F(M_1)\hat{\otimes}\cdots\hat{\otimes}F(M_r)$. For $u = u_1 \otimes \cdots \otimes u_r \in F(M|M')$, define

$$\boldsymbol{\partial}(u) = \sum_i \{\sum_{j>i}(|u_j| + \gamma(u_j)\,)\}u_1 \otimes \cdots \otimes (\boldsymbol{\partial} u_i) \otimes \cdots \otimes u_r \ .$$

Here $\gamma(u_i) := \gamma(M_i)$ if $u_i \in F(M_i)$. For $k \in \overset{\circ}{M} - M'$, let $i$ be such that $k \in M_i$, and define $\boldsymbol{\varphi}_k : F(M|M') \to F(M - \{k\}|M')$ by

$$\boldsymbol{\varphi}_k(u) = \sum_{j>i} \{(|u_j| + \gamma(u_j))\} u_1 \otimes \cdots \otimes \boldsymbol{\varphi}_k(u_i) \otimes \cdots \otimes u_r \ .$$

Now let

$$\boldsymbol{\varphi}(u) := \sum_k \boldsymbol{\varphi}_k(u) \ ,$$

the sum over $k \in \overset{\circ}{M} - M'$.

One verifies the following equalities:

$$\boldsymbol{\partial\partial}(u) = 0, \quad \boldsymbol{\varphi\varphi}(u) = 0, \quad \boldsymbol{\partial\varphi}(u) + \boldsymbol{\varphi\partial}(u) = 0 \ .$$

For $u \otimes v \in F(M|M')$, where $u \in F(M_1)\hat{\otimes} \cdots \hat{\otimes} F(M_s)$, $v \in F(M_{s+1})\hat{\otimes} \cdots \hat{\otimes} F(M_r)$,

$$\boldsymbol{\partial}(u \otimes v) = \{|v| + \gamma(v)\} \boldsymbol{\partial} u \otimes v + u \otimes \boldsymbol{\partial} v \ , \quad \boldsymbol{\varphi}(u \otimes v) = \{|v| + \gamma(v)\} \boldsymbol{\varphi}(u) \otimes v + u \otimes \boldsymbol{\varphi}(v) \ .$$

[The last two equalities are obvious. Using them, one may assume $u \in F(M|\emptyset)$ to prove the first three equalities. The verification is straightforward.]

Let $\bigoplus F(M|M')$ be the direct sum over all sequences $(M|M')$, and

$$\delta := \boldsymbol{\partial} + \boldsymbol{\varphi} : \bigoplus F(M|M') \to \bigoplus F(M|M') \ .$$

Define the *first degree* of $u \in F(M|M')$ by

$$\deg_1(u) = |u| + \gamma(M) \ .$$

Then $\delta$ increases the first degree by 1. We have the following proposition, so $\bigoplus F(M|M')$ is a complex with degree $\deg_1$ and differential $\delta$, and the differential is compatible with tensor product.

(2.3.1) **Proposition.** (1) $\delta\delta = 0$.

(2) For $u \otimes v \in F(M|M')$, where $u \in F(M_1)\hat{\otimes} \cdots \hat{\otimes} F(M_s)$, $v \in F(M_{s+1})\hat{\otimes} \cdots \hat{\otimes} F(M_r)$, one has $\delta(u \otimes v) = \{|v| + \gamma(v)\} \delta u \otimes v + u \otimes \delta v$ .

If one fixes $M'$ and takes the sum over $(M|M')$ where only $M$ varies, one still obtains a complex; taking then the sum over $M'$ gives the complex discussed above.

In particular, $\bigoplus F(M) = \bigoplus F(M|\emptyset)$ is a complex, which appears in the following subsection.

(2.4) In the complex $\bigoplus F(M)$, an element $f = (f(M)) \in \bigoplus F(M)$ is of first degree 0 if $\deg f(M) + \gamma(M) = 0$. It satisfies $\delta(f) = 0$ iff for each $M = (m_1, \cdots, m_\mu)$,

$$\partial f(M) + \sum_k \boldsymbol{\varphi}_k(f(M \cup \{k\})) = 0$$

where $k$ varies over the set $[\mathrm{in}\, M, \mathrm{tm}\, M] - M$. Concretely

$$\partial f(m_1, \cdots, m_\mu) + \sum_{t=1}^{\mu-1} \sum_{m_t < k < m_{t+1}} (-1)^{m_\mu + \mu + k + t} \varphi_k(f(m_1, \cdots, m_t, k, m_{t+1}, \cdots, m_\mu) = 0 \ .$$

We now restate the definition of a $C$-diagram.

(2.4.1) **Definition.** A *$C$-diagram* $K = (K^m; f(M))$ in the quasi DG category is a sequence of objects $K^m$ indexed by $m \in \mathbb{Z}$, all but a finite number of them being zero, and a set of elements $f(M) \in F(M)^{-\gamma(M)}$ indexed by $M = (m_1, \cdots, m_\mu)$, satisfying the following conditions:

(i) For each $k \in \overset{\circ}{M}$, $\sigma_k(f(M)) = f(M_{\leq k}) \otimes f(M_{\geq k})$ in $F(M_{\leq k}) \otimes F(M_{\geq k})$. (To be precise one should write $\tau_k$ for $\sigma_k$, but we may not make the distinction.)

(ii) $f = (f(M)) \in \bigoplus F(M)$ satisfies $\delta(f) = 0$.

For an object $X$ and $n \in \mathbb{Z}$, there is a $C$-diagram $K$ with $K^n = X$, $K^m = 0$ if $m \neq n$, and $f(M) = 0$ for all $M$. We write $X[-n]$ for this.

(2.5) *The differential $\boldsymbol{\sigma}$.* Under the same assumption, we define, for each $k \in \overset{\circ}{M} - M'$, the map $\boldsymbol{\sigma}_k : F(M|M') \to F(M|M' \cup \{k\})$ as follows. For $u \in F(M|\emptyset)$, if $\sigma_k(u) = \sum u' \otimes u''$ where $u' \in F(M_{\leq k})$ and $u'' \in F(M_{\geq k})$, let

$$\boldsymbol{\sigma}_k(u) = \sum \{\deg_1(u')) \cdot \gamma(u'')\} u' \otimes u'' \ .$$

In general, for $u = u_1 \otimes u_2 \otimes \cdots \otimes u_r \in F(M|M')$,

$$\boldsymbol{\sigma}(u) := \sum_k (-1)^{|M'_{>k}|} u_1 \otimes \cdots \otimes \boldsymbol{\sigma}_k(u_i) \otimes \cdots \otimes u_r$$

where $k$ varies over the set $\overset{\circ}{M} - M'$. (Here $M'_{>k}$ denotes the subset of $M'$ of elements $> k$.) For $u \in F(M|M')$, define $\tau(u) := |M'|$. Note $\boldsymbol{\sigma}$ increases $\tau(u)$ by one.

(2.5.1) **Proposition.** (1) $\boldsymbol{\sigma}\boldsymbol{\sigma}(u) = 0$.

(2) $\delta\boldsymbol{\sigma}(u) = \boldsymbol{\sigma}\delta(u)$.

(3) $\boldsymbol{\sigma}(u \otimes v) = \{\tau(v) + 1\}\boldsymbol{\sigma}(u) \otimes v + u \otimes \boldsymbol{\sigma}(v)$ .

*Proof.* (3) is obvious from the definitions. For (1) one may thus assume $u \in F(M|\emptyset)$, and verify it directly. (2) is also reduced to the case $u \in F(M|\emptyset)$, and one can show (after some calculation) the identities $\boldsymbol{\partial}\boldsymbol{\sigma}_k = \boldsymbol{\sigma}_k\boldsymbol{\partial}$ and $\boldsymbol{\sigma}_k\boldsymbol{\varphi}_\ell = \boldsymbol{\varphi}_\ell\boldsymbol{\sigma}_k$ for $k \neq \ell$.

(2.6) *The complexes $\mathbb{H}(K, L)$, $\mathbb{G}(K, L)$ and $\mathbb{F}(K, L)$.* Let $K = (K^m; f_K(M))$ and $L = (L^m; f_L(M))$ be $C$-diagrams. To a double sequence $A = (M|M'; N|N')$ one associates the complex

$$F(A) = F(M|M'; N|N') = F(K^{m_1}, \cdots, K^{m_\mu}; L^{n_1}, \cdots, L^{n_\nu}|M' \cup N') \ .$$

To be precise, consider the finite ordered set $M \amalg N$ (where $m < n$ if $m \in M$ and $n \in N$), and the sequence of objects on it.

If $M_1, \cdots, M_r$ is the segmentation of $M$ given by $M'$, and $N_1, \cdots, N_s$ that of $N$ given by $N'$, then

$$F(M|M'; N|N') = F(M_1)\hat{\otimes} \cdots \hat{\otimes} F(M_r \cup N_1)\hat{\otimes} F(N_2)\hat{\otimes} \cdots \hat{\otimes} F(N_s) \ .$$

We refer to $M_1, \cdots, M_{r-1}, M_r \cup N_1, N_2, \cdots, N_s$ as the segmentation of $M \cup N$ by $M' \cup N'$.

Let us say the double sequence is *free* when $M'$ and $N'$ are empty; then the corresponding complex is free of tensor products.

As for $F(M|M')$, one has maps $\partial, \varphi$ and $\sigma$ among the $F(A)$. Set $|A| = |M| + |N|$, $\gamma(A) = \gamma(M) + \gamma(N) + (n_1 - m_\mu)$, and $\tau(A) = |M'| + |N'|$. For $u \in F(A)$, let $\gamma(u) = \gamma(A)$ and $\tau(u) = \tau(A)$. One can then define the maps $\boldsymbol{\partial}$ and $\boldsymbol{\varphi}$ as before, as well as the sum

$$\delta = \boldsymbol{\partial} + \boldsymbol{\varphi} : \bigoplus F(A) \to \bigoplus F(A) \ .$$

Specifically if $u \in F(A)$ with $A = (M|\emptyset; N|\emptyset)$, then $\boldsymbol{\partial}(u) = \partial(u)$. If $u = u_1 \otimes \cdots \otimes u_r \otimes u_{r+1} \otimes \cdots \otimes u_{r+s-1} \in F(M_1)\hat{\otimes} \cdots \hat{\otimes} F(N_s)$ as above, then

$$\boldsymbol{\partial}(u) = \sum_i \left\{ \sum_{j>i} (|u_j| + \gamma(u_j)\,) \right\} u_1 \otimes \cdots (\boldsymbol{\partial} u_i) \otimes \cdots \otimes u_{r+s-1} \ .$$

Similarly for $\boldsymbol{\varphi}$.

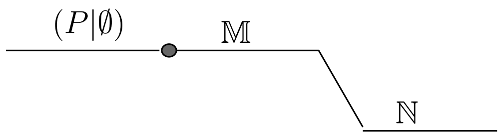
One also has the map $\boldsymbol{\sigma} : \bigoplus F(A) \to \bigoplus F(A)$. These maps satisfy the same identities as before. In addition, we will define maps $\mathbf{f}_K$ and $\mathbf{f}_L$.

For this purpose one needs to invoke (1.5), (v) and take appropriate quasi-isomorphic subcomplexes. There is a distinguished subcomplex $[F(M|M'; N|N')]_f \hookrightarrow F(M|M'; N|N')$ satisfying the following conditions:

- If $\mathrm{tm}(P) = \mathrm{in}(M)$, then the map

$$f_K(P) \otimes (-) : [F(\mathbb{M}; \mathbb{N})]_f \to [F(P \circ \mathbb{M}; \mathbb{N})]_f$$

is defined. Here $P \circ \mathbb{M} := (P \cup M | \{\mathrm{tm}(M)\} \cup M')$.




- Similarly if $\mathrm{tm}(N) = \mathrm{in}(Q)$, then

$$(-) \otimes f_L(Q) : [F(\mathbb{M}; \mathbb{N})]_f \to [F(\mathbb{M}; \mathbb{N} \circ Q)]_f$$

is defined.

For the existence of such a subcomplex, with reference to (1.5),(11-1), take $\{f_K(P)\}$ and $\{f_L(Q)\}$ as the set of constraints. In the rest of this paper we write $F(\mathbb{M}; \mathbb{N})$ for $[F(\mathbb{M}; \mathbb{N})]_f$.

For a sequence $P = (P|\emptyset)$ with $\mathrm{tm}(P) = \mathrm{in}(M)$ and $u \in F(\mathbb{M}; \mathbb{N})$, let

$$\mathbf{f}_K(P) \otimes u = \{|\tau(u)| + 1\} f_K(P) \otimes u \in F(P \circ \mathbb{M}; \mathbb{N}) \ .$$

Let $\mathbf{f}_K \otimes u := \sum \mathbf{f}_K(P) \otimes u$ where $P$ varies over sequences with $\mathrm{tm}(P) = \mathrm{in}(M)$. When there is no confusion, we also write $\mathbf{f}_K(u) = \mathbf{f}_K \otimes u$ like an operator.

If $\mathrm{in}(Q) = \mathrm{tm}(N)$ let

$$u \otimes \mathbf{f}_L(Q) = -u \otimes f_L(Q) \in F(\mathbb{M}; \mathbb{N} \circ Q) \ ,$$

and $u \otimes \mathbf{f}_L = \sum u \otimes \mathbf{f}_L(Q)$, the sum over $Q$ with $\mathrm{in}(Q) = \mathrm{tm}(N)$. We also write $\mathbf{f}_L(u) = u \otimes \mathbf{f}_L$. These operations are subject to the following identities.

(2.6.1) **Proposition.** *(1)* $\delta\mathbf{f}_K = \mathbf{f}_K\delta$. $\delta\mathbf{f}_L = \mathbf{f}_L\delta$.

*(2)* $\boldsymbol{\sigma}(\mathbf{f}_K \otimes u) + \mathbf{f}_K \otimes (\mathbf{f}_K \otimes u) + \mathbf{f}_K \otimes \boldsymbol{\sigma}(u) = 0$. $\boldsymbol{\sigma}(u \otimes \mathbf{f}_L) + (u \otimes \mathbf{f}_L) \otimes \mathbf{f}_L + \boldsymbol{\sigma}(u) \otimes \mathbf{f}_L = 0$. *(With the operator-like notation,* $\boldsymbol{\sigma}\mathbf{f}_K + \mathbf{f}_K\boldsymbol{\sigma} + \mathbf{f}_K\mathbf{f}_K = 0$*, etc. )*

*(3)* $\mathbf{f}_K \otimes (u \otimes v) = \{\tau(v) + 1\}(\mathbf{f}_K \otimes u) \otimes v$. *(Here* $u \in F(\mathbb{M}) = F(K^{\mathbb{M}})$, $v \in F(\mathbb{M}'; \mathbb{N})$ *with* $\mathrm{tm}(M) = \mathrm{in}(M')$*; or either,* $u \in F(\mathbb{M}; \mathbb{N})$, $v \in F(\mathbb{N}') = F(L^{\mathbb{N}'})$ *with* $\mathrm{tm}(N) = \mathrm{in}(N')$*. ) Similarly* $(u \otimes v) \otimes \mathbf{f}_L = u \otimes (v \otimes \mathbf{f}_L)$.

*(4)* $\mathbf{f}_K\mathbf{f}_L + \mathbf{f}_L\mathbf{f}_K = 0$.

*Proof.* Direct verification.

For $u \in F(A)$ define
$$d'(u) = \boldsymbol{\sigma}(u) + \mathbf{f}_K(u) + \mathbf{f}_L(u) \ .$$
It increases $\tau(u)$ by one, $d'd' = 0$ and $\delta d' = d'\delta$. Further, $d'(u\otimes v) = \{\tau(v)+1\}(d'u)\otimes v + u\otimes(d'v)$. Thus the direct sum $\bigoplus_A F(A)$ is a "double" complex with respect to the two gradings
$$\deg_1 u = |u| + \gamma(u), \quad \deg_2(u) = \tau(u) + 1$$
and the commuting differentials $\delta$, $d'$. Denote it by $H^{\bullet\bullet}(K, L)$. Let
$$\mathbb{H}(K, L) = \mathrm{Tot}(H^{\bullet\bullet}(K, L)\,)$$
be the associated total complex. The total degree is of $u \in F(A)$ is given by $\deg_{\mathbb{H}}(u) = |u| + \gamma(u) + 1$, and the total differential $d_{\mathbb{H}}$ is given by
$$d_{\mathbb{H}} = (-1)^{\deg_2 u}\delta + d'$$
on $u \in F(A)$. This coincides with the convention for the total complex (0.1) if we view $d'$ as the first differential and $\delta$ the second, so it would be more legitimate if we called $d'$ (resp. $\delta$) the first differential (resp. the second). But as we will see in the next section, the differential $\delta$ plays the primary role, hence the name the first differential.

One can show, for each $a$, the complex with respect to $d'$, $H^{a,1} \to H^{a,2} \to \cdots$ is acyclic. For the proof consider the filtration given by the sum of terms with $\mathrm{in}\, M \le a$ and $\mathrm{tm}\, N \ge b$, for varying $a, b$; in the subquotients the maps $\mathbf{f}_K$ and $\mathbf{f}_L$ are zero, so the differentials are just $\boldsymbol{\sigma}$, and the claim follows from (1.5) (4).

Applying the operation $\Phi$ in (0.4) (with a shift to the second degree), we obtain a complex $\mathbb{G}^{\bullet}(K, L) := \Phi H^{\bullet\bullet}(K, L)$. So the degree and differential are given by $\deg_{\mathbb{G}}(u) = \deg_1(u) + 1$ and $d_{\mathbb{G}} = -\delta$ on $u \in F(A)$. Set finally
$$\mathbb{F}(K, L) = \mathbb{G}(K, L)[1] \ .$$
The degree and the differential are given as follows:
$$\deg_{\mathbb{F}}(u) = \deg_1(u) \ , \quad d_{\mathbb{F}}(u) = \delta(u).$$
If $K = X[0]$ and $L = Y[n]$, we have $\mathbb{F}(K, L) = F(X, Y)[n]$.

(2.7) *The complexes* $\mathbb{H}(I)$ *and* $\mathbb{G}(I)$. Let $n \ge 2$. Assume given a sequence of $C$-diagrams $K_i = (K_i^m; f_{K_i}(M))$ for $i = 1, \cdots, n$. We will define complexes $\mathbb{H}(K_1, \cdots, K_n)$ and $\mathbb{G}(K_1, \cdots, K_n)$, generalizing $\mathbb{H}(K, L)$ and $\mathbb{G}(K, L)$ in case $n = 2$. As in the case $|I| = 2$, $\mathbb{G}$ is a quasi-isomorphic subcomplex of $\mathbb{H}$.

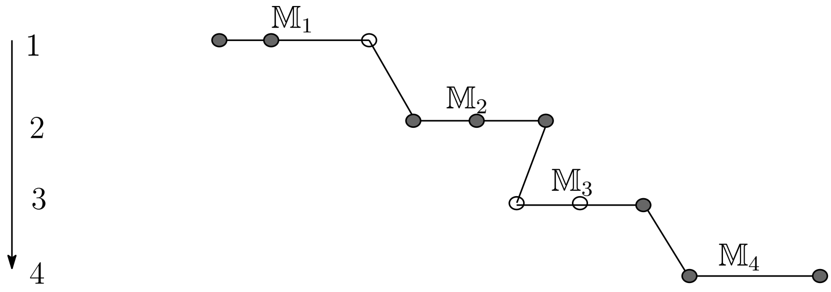
A *multi-sequence* on $[1, n]$ is a $2n$-tuple of finite sequences

$$A = (\mathbb{M}_1; \cdots ; \mathbb{M}_n) = (M_1|M_1'; M_2|M_2'; \cdots , M_{n-1}|M_{n-1}'; M_n|M_n')$$

satisfying the following conditions:

- Each $M_i$ is non-empty;
- $M_j' \subset M_j$ for $j = 1, \cdots , n$, and $\mathrm{in}(M_1) \notin M_1'$, and $\mathrm{tm}(M_n) \notin M_n'$.

The following illustrates a multi-sequence. The vertical direction is for $[1, n]$, and the horizontal direction for the $\mathbb{M}_i$.




Associated to $A$ is a finite ordered set $M_1 \cup \cdots \cup M_n$ (disjoint union) and $(K_i^m)$ defines a sequence of objects on it; so there corresponds the complex

$$F(A) = F(\mathbb{M}_1; \cdots ; \mathbb{M}_n) = F(M_1 \cup \cdots \cup M_n | M_1' \cup \cdots \cup M_n') \ .$$

Let $\partial$ be its differential. Set $|A| = \sum |M_i|$,

$$\gamma(A) = \sum \gamma(M_i) + \sum_{i=1}^{n-1} (\mathrm{in}\, M_{i+1} - \mathrm{tm}\, M_i) \ ,$$

and $\tau(A) = \sum |M_i'|$.

Now consider

$$\bigoplus_A F(A)$$

the direct sum over all multi-sequences $A$ on $[1, n]$. We make it into a "triple" complex, denoted $H^{\bullet\bullet\bullet} = H^{\bullet\bullet\bullet}(K_1, \cdots , K_n)$. It is analogous to the double complex $H^{\bullet\bullet}$ in the previous subsection. Set

$$M_{int}' = \cup_{1<i<n} M_i' \qquad M_{out}' = M_1' \cup M_n' \ .$$

- The first degree is $\deg_1(u) = |u| + \gamma(u)$, and the first differential is $d_1 = \delta$, to be defined as follows.

As in the case $n = 2$, one has the map $\boldsymbol{\partial}$. For each $k \in M_i - M_i'$ with $|M_i| \geq 2$ and $k \notin \{\mathrm{in}\, M_1, \mathrm{tm}\, M_n\}$, one has the map $\boldsymbol{\varphi}_k : F(\mathbb{M}_1; \cdots ; \mathbb{M}_n) \to F(\mathbb{M}_1; \cdots ; (M_i - \{k\}|M_i'); \cdots ; \mathbb{M}_n)$. (If $M_i$ consists of a single element, the target of $\varphi_k$ is $F(\mathbb{M}_1; \cdots , \widehat{\mathbb{M}_i}, \cdots ; \mathbb{M}_n)$, which is not associated with a multi-sequence on $[1, n]$. Thus we need to require $M_i$ to contain at least two elements.) So $\boldsymbol{\varphi} = \sum \boldsymbol{\varphi}_k$ and $\delta = \boldsymbol{\partial} + \boldsymbol{\varphi}$ are endomorphisms of $\bigoplus F(A)$.

• The second degree is $\deg_2(u) = |M'_{int}| + 1$, and the second differential is $d_2 = \boldsymbol{\sigma}_{int}$ defined by (with $\boldsymbol{\sigma}_k$ as before)

$$\boldsymbol{\sigma}_{int}(u) := \sum (-1)^{|(M'_{int})_{>k}|} u_1 \otimes \cdots \otimes \boldsymbol{\sigma}_k(u_j) \otimes \cdots \otimes u_r$$

the sum over $k \in \cup_{1<i<n}(M_i - M'_i)$.

• The third degree is $\deg_3(u) = |M'_{out}|$, and the third differential is $d_3 = \boldsymbol{\sigma}_{out} + \mathbf{f}_{K_1} + \mathbf{f}_{K_n}$, where the three operators are defined as follows.

We let

$$\boldsymbol{\sigma}_{out}(u) := \sum (-1)^{|(M'_{out})_{>k}|} u_1 \otimes \cdots \otimes \boldsymbol{\sigma}_k(u_j) \otimes \cdots \otimes u_r$$

the sum over $k \in M_1 \cup M_n - M'_1 \cup M'_n$.

Set $\mathbf{f}_{K_1}(P) \otimes u = \{|M'_{out}| + 1\} f_{K_1}(P) \otimes u$, and $\mathbf{f}_{K_1}(u) = \sum_P \mathbf{f}_{K_1}(P) \otimes u$. Similarly $u \otimes \mathbf{f}_{K_n}(Q) = -u \otimes f_{K_n}(Q)$, and $\mathbf{f}_{K_n}(u) = \sum_Q u \otimes \mathbf{f}_{K_n}(Q)$.

By the following result, which can be shown as in the previous section, we have a "triple" complex.

(2.7.1) **Proposition.** *(1) $\boldsymbol{\sigma}_{int}$ is a differential, and commutes with $\delta$. Similarly $\boldsymbol{\sigma}_{out}$ is a differential, and commutes with $\delta$. The differentials $\boldsymbol{\sigma}_{int}$ and $\boldsymbol{\sigma}_{out}$ commute.*

*(2) $d_3$ is a differential, and commutes with $\delta$ and $\boldsymbol{\sigma}_{int}$.*

Let $\mathbb{H}(K_1, \cdots, K_n)$ be the total complex of $H^{\bullet\bullet\bullet}$. The total differential is

$$d_{\mathbb{H}} = (-1)^{\deg_2 + \deg_3}\delta + (-1)^{\deg_3} d_2 + d_3 \ .$$

As in the case $|I| = 2$, we have

(2.7.2) **Proposition.** *The complex*

$$H^{\bullet\bullet 0} \xrightarrow{d_3} H^{\bullet\bullet 1} \xrightarrow{d_3} \cdots$$

*is exact.*

Let now $G^{\bullet\bullet}(K_1, \cdots, K_n) = \Phi(H^{\bullet\bullet\bullet}(K_1, \cdots, K_n))$; it is a "double" complex. We also define $\mathbb{G}(K_1, \cdots, K_n)$ to be its total complex. Note the differential of this complex is

$$d_{\mathbb{G}} = (-1)^{\deg_2 u}\delta + \boldsymbol{\sigma}_{int}$$

when acting on $u$. It is a quasi-isomorphic subcomplex of $\mathbb{H}(K_1, \cdots, K_n)$. When the sequence $K_1, \cdots, K_n$ is understood, write them as $G^{\bullet\bullet}([1, n])$ and $\mathbb{G}([1, n])$, respectively.

The same construction applies to any finite totally ordered set $I$ and a sequence of $C$-diagrams indexed by $I$. If $|I| = 2$ and $I = \{i_1, i_2\}$, one has $\mathbb{G}(I) = \mathbb{G}(K_{i_1}, K_{i_2})$, as defined in the previous subsection.

(2.8) **Proposition.** *$\mathbb{G}(I)$ is acyclic if $|I| \geq 3$.*

*Proof.* We show the acyclicity of $\mathbb{H}(I)$. For each $(a, b)$ with $a \leq b$, the sum

$$\bigoplus_{\text{in } M_1 \leq a,\, \text{tm } M_n \geq b} F(\mathbb{M}_1; \cdots; \mathbb{M}_n)$$

is a subcomplex of $\mathbb{H}(I)$, and gives its filtration (increasing in $a$, decreasing in $b$). In a successive quotient, which is the form

$$\bigoplus_{\mathrm{in}\, M_1 = a,\, \mathrm{tm}\, M_n = b} F(\mathbb{M}_1; \cdots ; \mathbb{M}_n)\ ,$$

the maps $\mathbf{f}_K$ are zero, hence $d' = \boldsymbol{\sigma}$. Consider its filtration

$$Fil^c = \bigoplus_{|M| \le c} F(\mathbb{M}_1; \cdots ; \mathbb{M}_n)\ .$$

In the successive quotients one has $\boldsymbol{\varphi} = 0$, so its is a sum of complexes of the following form

$$F(\mathcal{I}|\emptyset) \xrightarrow{\ \sigma\ } \bigoplus_{|S|=1} F(\mathcal{I}|S) \xrightarrow{\ \sigma\ } \bigoplus_{|S|=2} F(\mathcal{I}|S) \xrightarrow{\ \sigma\ } \cdots \to F(\mathcal{I}|\mathring{\mathcal{I}}) \to 0$$

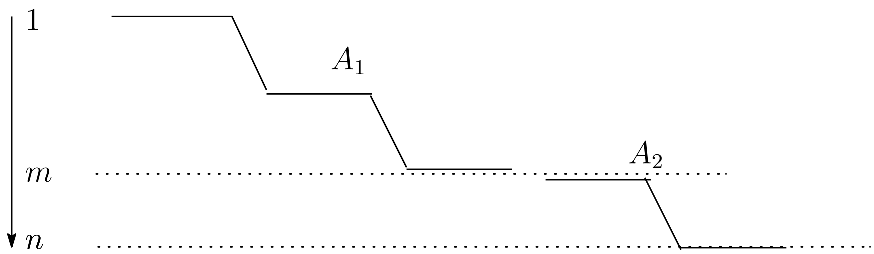
where $\mathcal{I} = \amalg M_i$ (so $|\mathcal{I}| \ge 3$) and $S$ varies over subsets of $\mathring{\mathcal{I}}$. Since $|\mathcal{I}| \ge 3$, $\mathring{\mathcal{I}}$ is non-empty, so these complexes are acyclic; we are done.

(2.9) *The complex* $\mathbb{G}(I|\Sigma)$ Let $I$ be a finite totally ordered set, and $I_1, \cdots, I_c$ be a segmentation of $I$. We will define a quasi-isomorphic multiple subcomplex denoted

$$\mathbb{G}(I_1)\tilde{\otimes} \cdots \tilde{\otimes}\mathbb{G}(I_c) \subset \mathbb{G}(I_1) \otimes \cdots \otimes \mathbb{G}(I_c)\ .$$

The usage of the symbol $\tilde{\otimes}$ is made in line with the convention (0.1.1) for a quasi-isomorphic multiple subcomplex, but to avoid confusion with $\hat{\otimes}$.

For simplicity let $I_1 = [1, m]$ and $I_2 = [m, n]$. Let $A_1 = (\mathbb{M}_1; \cdots ; \mathbb{M}_m)$ and $A_2 = (\mathbb{N}_m; \cdots ; \mathbb{N}_n)$ be multi-sequences on $[1, m]$ and $[m, n]$, respectively. There are cases $\mathrm{tm}\, A_1 < \mathrm{in}\, A_2$, $\mathrm{tm}\, A_1 = \mathrm{in}\, A_2$ and $\mathrm{tm}\, A_1 > \mathrm{in}\, A_2$.




Let

$$F(A_1)\tilde{\otimes}F(A_2)$$

be the distinguished subcomplex of $F(A_1) \otimes F(A_2)$ prescribed to have the following properties:

- If $\mathrm{tm}(A_1) = \mathrm{in}(A_2)$, one has

$$F(A_1)\tilde{\otimes}F(A_2) \subset F(A_1)\hat{\otimes}F(A_2)\ .$$

- If $\mathrm{tm}(P) = \mathrm{in}(A_1)$, the map

$$(f_{K_1}(P) \otimes (-)\,) \otimes id : F(A_1)\tilde{\otimes}F(A_2) \to F(P \circ A_1)\tilde{\otimes}F(A_2)\ ,$$

which sends $u \otimes v$ to $f_{K_1}(P) \otimes u \otimes v$ is defined.

- If $\mathrm{tm}(A_1) = \mathrm{in}(P)$,

$$((-) \otimes \ f_{K_m}(P)) \otimes id : F(A_1)\tilde{\otimes} F(A_2) \to F(A_1 \circ P)\tilde{\otimes} F(A_2)$$

is defined.

- The same conditions for the maps $id \otimes (f_{K_m}(P) \otimes (-)\,)$ and $id \otimes (\,(-) \otimes f_{K_n}(P)\,)$.
- If $\mathrm{tm}(A_1) > \mathrm{in}(A_2)$, one has

$$F(A_1)\tilde{\otimes} F(A_2) = [F(A_1)]_f \otimes [F(A_2)]_f \ .$$

Here $[F(A_1)]_f \subset F(A_1)$ is the distinguished subcomplex as in (2.6).

One can verify the subcomplex consisting of the elements satisfying these conditions is a distinguished subcomplex of $F(A_1) \otimes F(A_2)$. Indeed, with reference to (1.5), (11-1), each condition corresponds to a constraint.

Alternatively we can view this as an example of the special type of distinguished subcomplexes in [8], (3.6)-(3.9). To exactly refer to the formulation there, we must specify the constraints as follows. Let

$$I^1 = M_1 \amalg \cdots \amalg M_m \ ,$$

$$\mathbb{I}^1 = M_1^- \amalg M_2 \amalg \cdots \amalg M_{m-1} \amalg M_m^+ \ ,$$

where $M_1^- = \{k \in \mathbb{Z} | N \le k < \mathrm{in}(M_1)\} \amalg M_1$ with $N$ small enough such that $K_1^k = 0$ for $k < N$ (it is the set obtained by "extending $M_1$ to the left"), and similarly $M_m^+ = M_m \amalg \{k \in \mathbb{Z} |\, \mathrm{tm}(M_m) < k \le N'\}$ with $N'$ large enough. There is a sequence of objects indexed by $\mathbb{I}^1$, given by $K_1, K_2, \cdots$ on $M_1^-$, $M_2$, ..., respectively. Similarly let

$$I^2 = N_m \amalg \cdots \amalg N_n \ ,$$

$$\mathbb{I}^2 = N_m^- \amalg N_{m+1} \amalg \cdots \amalg N_n^+ \ ;$$

there is a sequence of objects indexed by $\mathbb{I}^2$. If $J \subset \mathbb{I}^1$ is to the left (resp. to the right) of $I^1$, one has $f_{K_1}(J) \in F(J; K_1)$ (resp. $f_{K_m}(J) \in F(J; K_m)$ ). Similarly if $J \subset \mathbb{I}^2$ is to the left (resp. right) of $I^2$, there is $f_{K_m}(J) \in F(J; K_m)$ (resp. $f_{K_n}(J) \in F(J; K_n)$ ). The distinguished subcomplex of $[F(A_1)\tilde{\otimes} F(A_2)]_f$ with respect to these constraints was defined in (3.6)-(3.9), which we now simply write $F(A_1)\tilde{\otimes} F(A_2)$, satisfies the required conditions.

Let

$$H^{\bullet\bullet\bullet}([1,m])\tilde{\otimes} H^{\bullet\bullet\bullet}([m,n]) \subset H^{\bullet\bullet\bullet}([1,m]) \otimes H^{\bullet\bullet\bullet}([m,n])$$

be the "6-ple" subcomplex defined as the sum $\bigoplus F(A_1)\tilde{\otimes} F(A_2)$. It is a quasi-isomorphic subcomplex. Let

$$G^{\bullet\bullet}([1,m])\tilde{\otimes} G^{\bullet\bullet}([m,n]) := \Phi\, \mathrm{Tot}_{36}(H^{\bullet\bullet\bullet}([1,m])\tilde{\otimes} H^{\bullet\bullet\bullet}([m,n])\,)\,,$$

a quasi-isomorphic "quadruple" subcomplex of $G^{\bullet\bullet}([1,m]) \otimes G^{\bullet\bullet}([m,n])$. From this, we obtain

$$\mathbb{G}([1,m])\tilde{\otimes}\mathbb{G}([m,n]) := \mathrm{Tot}_{12}\, \mathrm{Tot}_{34}(G^{\bullet\bullet}([1,m])\tilde{\otimes} G^{\bullet\bullet}([m,n])\,)$$

a quasi-isomorphic "double" subcomplex of $\mathbb{G}([1,m]) \otimes \mathbb{G}([m,n])$, and

$$G^{\bullet\bullet}([1,m])\tilde{\times} G^{\bullet\bullet}([m,n]) = \mathrm{Tot}_{13}\, \mathrm{Tot}_{24}(G^{\bullet\bullet}([1,m])\tilde{\otimes} G^{\bullet\bullet}([m,n])\,)$$

a quasi-isomorphic "double" subcomplex of $G^{\bullet\bullet}([1,m]) \times G^{\bullet\bullet}([m,n])$. (From (0.3) recall for "double" complexes $A^{\bullet\bullet}$ and $B^{\bullet\bullet}$, their tensor product as a "double" complex $A^{\bullet\bullet} \times B^{\bullet\bullet}$ is defined as $\mathrm{Tot}_{13}\,\mathrm{Tot}_{24}(A^{\bullet\bullet} \otimes B^{\bullet\bullet})$. ) There is an isomorphism of complexes (cf. (0.3) )

$$u : \mathbb{G}([1,m])\tilde{\otimes}\mathbb{G}([m,n]) \to \mathrm{Tot}(G^{\bullet\bullet}([1,m])\tilde{\times}G^{\bullet\bullet}([m,n])\,)\ .$$

If $c > 2$ the definition is similar. For multi-sequences $A_1, \cdots, A_c$ on $I_1, \cdots, I_c$ respectively, one can define a distinguished subcomplex

$$F(A_1)\tilde{\otimes}F(A_2)\tilde{\otimes}\cdots\tilde{\otimes}F(A_c)$$

subject to similar conditions. We have a "$3c$-ple" subcomplex

$$H^{\bullet\bullet\bullet}(I_1)\tilde{\otimes}\cdots\tilde{\otimes}H^{\bullet\bullet\bullet}(I_c)$$

(also denoted $H^{\bullet\bullet\bullet}(I|\Sigma)$ ) and from this a "2c-ple" subcomplex

$$G^{\bullet\bullet}(I_1)\tilde{\otimes}\cdots\tilde{\otimes}G^{\bullet\bullet}(I_c) = \Phi\,\mathrm{Tot}_{3,6,\cdots,3c}(\text{above})\ .$$

Then we get "$c$-ple" subcomplex

$$\mathbb{G}(I_1)\tilde{\otimes}\cdots\tilde{\otimes}\mathbb{G}(I_c) := \mathrm{Tot}_{12}\,\mathrm{Tot}_{34}\cdots\mathrm{Tot}_{2c-1,2c}(G^{\bullet\bullet}(I_1)\tilde{\otimes}\cdots\tilde{\otimes}G^{\bullet\bullet}(I_c))$$

of $\mathbb{G}(I_1) \otimes \cdots \otimes \mathbb{G}(I_c) = \mathbb{G}(I|\Sigma)$.

The last one will play a major role. If $\Sigma \subset \overset{\circ}{I}$ corresponds to $I_1, \cdots, I_c$, we write it $\mathbb{G}(I|\Sigma)$.

(2.10) *The maps $\rho$ and* $\Pi$. We define the product map (for $m \in \Sigma$)

$$\rho_m : \mathbb{G}(I|\Sigma) \to \mathbb{G}(I|\Sigma - \{m\})\ .$$

For simplicity of notation let us consider the case $\Sigma = \{m\}$, where the map is of the form $\rho_m : \mathbb{G}([1,m])\tilde{\otimes}\mathbb{G}([m,n]) \to \mathbb{G}([1,n])$.

First consider a map

$$\rho_m : H^{\bullet\bullet\bullet}([1,m])\tilde{\otimes}H^{\bullet\bullet\bullet}([m,n]) \to H^{\bullet\bullet\bullet}([1,n])$$

defined as follows. On each direct summand $F(A_1)\tilde{\otimes}F(A_2)$ with $\mathrm{tm}(A_1) = \mathrm{in}(A_2)$, $\rho_m$ is the inclusion map $F(A_1)\tilde{\otimes}F(A_2) \to F(A_1 \circ A_2)$. (The inclusion is obvious, since the latter has constraint with respect to $f_{K_1}(P) \otimes (-)$ and $(-) \otimes f_{K_n}(P)$, whereas the former has in addition with respect to $(-) \otimes f_{K_m}(P) \otimes id$ and $id \otimes f_{K_m}(P) \otimes (-)$.) Otherwise the map $\rho_m$ is set to be zero. This is not a map of "triple" complexes.

Taking $\Phi\,\mathrm{Tot}_{36}$ on the left and $\Phi$ on right, we get an induced map $\rho_m : G^{\bullet,\bullet}([1,m])\tilde{\times}G^{\bullet,\bullet}([m,n]) \to G^{\bullet,\bullet}([1,n])$.

(2.10.1) **Lemma.** *The map*

$$\rho_m : G^{\bullet\bullet}([1,m])\tilde{\times}G^{\bullet\bullet}([m,n]) \to G^{\bullet,\bullet}([1,n])$$

*is a map of "double" complexes.*

*Proof.* We must show

$$\delta\rho(u \otimes v) = (-1)^{\deg_1 v}\rho(\delta u \otimes v) + \rho(u \otimes \delta v)$$

and

$$d'\rho(u \otimes v) = (-1)^{\deg_2 v}\rho(d'u \otimes v) + \rho(u \otimes d'v)\ .$$

Let $u \otimes v \in F(A_1)\tilde{\otimes}F(A_2)$. If $\mathrm{tm}(A_1) \neq \mathrm{in}(A_2)$, these hold trivially (all terms are zero). If $\mathrm{tm}(A_1) = \mathrm{in}(A_2)$, then $\rho$ acts as identities; the above equalities can be verified taking note that the point $\mathrm{tm}\, A_1$ is in the set $M'_{int}$ for $u \otimes v$.

Taking the total complexes and using the isomorphism $u$ as in (0.3) one obtains the map of complexes $\rho_m : \mathbb{G}([1, m])\tilde{\otimes}\mathbb{G}([m, n]) \to \mathbb{G}([1, n])$; it sends the sum of terms $u \otimes v \in F(A_1)\tilde{\otimes}F(A_2)$ (with $\mathrm{tm}(A_1) = \mathrm{in}(A_2)$) to the sum of $\{(\deg_2 u) \cdot (\deg_1 v)\}u \otimes v$.

The case $\Sigma$ contains more than one element is similar, so one has $\rho_m : \mathbb{G}(I|\Sigma) \to \mathbb{G}(I|\Sigma - \{m\})$. Explicitly, it is induced from the map (if $I_i \cap I_{i+1} = \{m\}$)

$$\rho_m : H^{\bullet\bullet\bullet}(I_1)\tilde{\otimes}\cdots\tilde{\otimes}H^{\bullet\bullet\bullet}(I_c) \to H^{\bullet\bullet\bullet}(I_1)\tilde{\otimes}\cdots\tilde{\otimes}H^{\bullet\bullet\bullet}(I_i \cup I_{i+1})\tilde{\otimes}\cdots\tilde{\otimes}H^{\bullet\bullet\bullet}(I_c)$$

which sends $u_1 \otimes \cdots \otimes u_c \in F(A_1)\tilde{\otimes}\cdots\tilde{\otimes}F(A_c)$ to itself if $\mathrm{tm}(A_i) = \mathrm{in}(A_{i+1})$ and to zero otherwise. From this we get a map

$$\rho_m : G^{\bullet\bullet}(I_1)\tilde{\otimes}\cdots\tilde{\otimes}G^{\bullet\bullet}(I_c) \to G^{\bullet\bullet}(I_1)\tilde{\otimes}\cdots\tilde{\otimes}G^{\bullet\bullet}(I_i \cup I_{i+1})\tilde{\otimes}\cdots\tilde{\otimes}G^{\bullet\bullet}(I_c)$$

which is, as before, a map of "$2(c-1)$-ple" complexes. For $m$, $m'$ distinct, $\rho_m\rho_{m'} = \rho_{m'}\rho_m$.

For $k \in \mathring{I} - \Sigma$, define a map of complexes

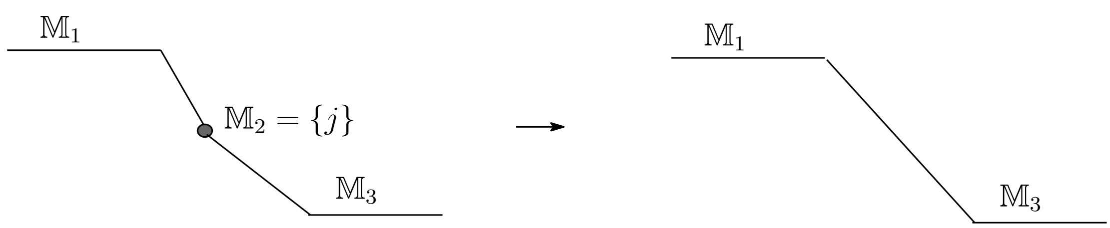
$$\Pi_k : \mathbb{G}(I|\Sigma) \to \mathbb{G}(I - \{k\}|\Sigma)$$

as follows. Let $I = [1, n]$ for simplicity.

First assume $\Sigma = \emptyset$. On the direct sum $F(A)$ where $A = (\mathbb{M}_1; \cdots; \mathbb{M}_n)$ is a multi-sequence on $[1, n]$, let $\Pi_k = 0$ unless $\mathbb{M}_k = (M_k|\emptyset)$ with $|M_k| = 1$. For such $\mathbb{M}_k$, letting $M_k = \{j\}$ and $A|_{[1,n]-\{k\}} = (\mathbb{M}_1; \cdots; \widehat{\mathbb{M}_k}; \cdots; \mathbb{M}_n)$, we define

$$\Pi_k : F(A) \to F(A|_{[1,n]-\{k\}})\ ,$$

by $\Pi_k(u) = (-1)^j\varphi_j(u)$ (*not* a typo for $(-1)^j\boldsymbol{\varphi}_j(u)$). The following figure is for the case $n = 3$ and $k = 2$.




Taking the sum over $A$'s, we obtain a map $\Pi_k : H^{\bullet\bullet\bullet}(I) \to H^{\bullet\bullet\bullet}(I - \{k\})$. One verifies

(2.10.2) **Lemma.** *(1) The map* $\Pi_k : H^{\bullet\bullet\bullet}(I) \to H^{\bullet\bullet\bullet}(I-\{k\})$ *is a map of "triple" complexes.*

*(2) If* $k \neq k'$, $\Pi_k\Pi_{k'} = \Pi_{k'}\Pi_k$.

*Proof.* (1) For $A = (\mathbb{M}_1; \cdots ; \mathbb{M}_n)$ with $\mathbb{M}_k = (M_k|\emptyset)$, $|M_k| = 1$ and $M_k = \{j\}$, if $u \in F(A)$, one has

$$\boldsymbol{\partial}\varphi_j(u) = \varphi_j\boldsymbol{\partial}(u) \ ,$$

$$\boldsymbol{\varphi}\varphi_j(u) = \varphi_j\boldsymbol{\varphi}(u) \ .$$

To show the first equality, let $A_1, \cdots , A_r$ be the segmentation of $M = M_1\cup\cdots\cup M_n$ by $M'$, and assume $u = u_1 \otimes \cdots \otimes u_r \in F(A)$ with $u_i \in F(A_i)$. The notion of segmentaton was discussed in (2.6) in case $c = 2$. Then

$$\boldsymbol{\partial}(u) = \sum_i \{\sum_{a>i} \deg_1(u_a)\} u_1 \otimes \cdots \otimes (\partial u_i) \otimes \cdots \otimes u_r \ .$$

One also has

$$\varphi_j(u_1 \otimes \cdots \otimes u_r) = u_1 \otimes \cdots \otimes \varphi_j(u_k) \otimes \cdots u_r$$

by the compatibility of $\varphi_j$ and the tensor product, (1.5)(3). Since $\deg_1(u_k) = \deg_1 \varphi_j(u_k)$, the first equality follows. The proof of the second equality is similar.

We also have

$$\boldsymbol{\sigma}_a\varphi_j(u) = \varphi_j\boldsymbol{\sigma}_a(u) \ .$$

Indeed if $a \in M - M'$, $a \neq j$, and if $\sigma_a(u) = \sum u' \otimes u''$, then $\boldsymbol{\sigma}_a(u) = \sum\{\deg_1(u')\cdot\gamma(u'')\} u'\otimes u''$. If $a > j$, $u'$ and $\varphi_j(u')$ have the same $\deg_1$; if $a < j$, $u''$ and $\varphi_j(u'')$ have the same $\deg_1$. Hence the equality follows. Thus

$$\boldsymbol{\sigma}_{int}\varphi_j(u) = \varphi_j\boldsymbol{\sigma}_{int}(u) \ ,$$

$$\boldsymbol{\sigma}_{out}\varphi_j(u) = \varphi_j\boldsymbol{\sigma}_{out}(u) \ .$$

As for $\mathbf{f}$, one easily verifies

$$\mathbf{f}_{K_1}\varphi_j(u) = \varphi_j\mathbf{f}_{K_1}(u) \ , \quad \mathbf{f}_{K_n}\varphi_j(u) = \varphi_j\mathbf{f}_{K_n}(u) \ .$$

Now we look at $(-1)^j\varphi_j$. Since $\varphi_j$ commutes with $\partial$, One has $\boldsymbol{\partial}((-1)^j\varphi_j) = ((-1)^j\varphi_j)\boldsymbol{\partial}$ and $\boldsymbol{\varphi}_i((-1)^j\varphi_j) = ((-1)^j\varphi_j)\boldsymbol{\varphi}_i$ for $i \neq j$. In addition, if $j < j'$ and $M_k = \{j, j'\}$, $M'_k = \emptyset$, then for $u \in F(A)$,

$$((-1)^{j'}\varphi_{j'})\boldsymbol{\varphi}_j + \boldsymbol{\varphi}_{j'}((-1)^j\varphi_j) = 0 \ .$$

To prove this assume $A_s$ contains $M_k$, so that

$$\boldsymbol{\varphi}_j(u) = \sum_i \{\sum_{a>i} \deg_1(u_a)\} u_1 \otimes \cdots \otimes \boldsymbol{\varphi}_j(u_i) \otimes \cdots \otimes u_r \ ,$$

and $\boldsymbol{\varphi}_j(u_i) = \{|u_i| + \gamma((A_s)_{\leq j})\}\varphi_j(u_i)$. The identity follows from $\gamma((A_s)_{\leq j'}) - \gamma((A_s)_{\leq j}) = j' - j - 1$.

It follows $\Pi_k$ commutes with $\delta$, $\boldsymbol{\sigma}_{int}$, and $d_3$.

(2) is obvious.

Taking $\Phi$ and then Tot we get an induced map $\Pi_k : \mathbb{G}(I) \to \mathbb{G}(I - \{k\})$.

We extend this to a map of complexes $\Pi_k : \mathbb{G}(I|\Sigma) \to \mathbb{G}(I - \{k\}|\Sigma)$ as follows. First consider the map

$$H^{\bullet\bullet\bullet}(I_1)\tilde{\otimes}\cdots\tilde{\otimes}H^{\bullet\bullet\bullet}(I_c) \to H^{\bullet\bullet\bullet}(I_1)\tilde{\otimes}\cdots\tilde{\otimes}H^{\bullet\bullet\bullet}(I_i - \{k\})\tilde{\otimes}\cdots\tilde{\otimes}H^{\bullet\bullet\bullet}(I_c)$$

which sends $u = u_1 \otimes \cdots \otimes u_c \in \mathbb{G}(I_1)\tilde{\otimes}\cdots\tilde{\otimes}\mathbb{G}(I_c)$ to

$$u_1 \otimes \cdots \otimes \Pi_k(u_i) \otimes \cdots \otimes u_c \ .$$

Applying Tot's and $\Phi$, we get a map of "$c$-ple" complexes $\Pi_k : \mathbb{G}(I|\Sigma) \to \mathbb{G}(I - \{k\}|\Sigma)$. Then $\Pi_k$ satisfies properties similar to the above, and in addition (3) below. Indeed (1), (2) hold at the level of "triple" complexes $H^{\bullet\bullet\bullet}(I|\Sigma)$, and (3) holds at the level of $G^{\bullet\bullet}(I_1)\tilde{\otimes}\cdots\tilde{\otimes}G^{\bullet\bullet}(I_c)$. The proof is straightforward.

(2.10.3) **Lemma.** *(1) The* $\Pi_k : \mathbb{G}(I|\Sigma) \to \mathbb{G}(I - \{k\}|\Sigma)$ *is a map of complexes.*
*(2) If* $k \neq k'$, $\Pi_k\Pi_{k'} = \Pi_{k'}\Pi_k$.
*(3) If* $m \neq k$, $\Pi_k\rho_m = \rho_m\Pi_k$.

For a subset $K \subset \overset{\circ}{I}$, define $\Pi_K : \mathbb{G}(I|\Sigma) \to \mathbb{G}(I - K|\Sigma)$ by composing $\Pi_k$ for $k \in K$. If $K$ is the disjoint union of $K'$ and $K''$, then $\Pi_K = \Pi_{K'}\Pi_{K''}$.

(2.11) *The complexes* $\mathbb{G}(I,T)$. For $I = [1,n]$, and a multi-sequence $A = (\mathbb{M}_1; \cdots ; \mathbb{M}_n)$, the *type* of $A$ is the subset of $\overset{\circ}{I}$ defined by

$$T = \{i \in \overset{\circ}{I} | M'_i \neq \emptyset\}.$$

For $T \subset \overset{\circ}{I}$, consider the complex

$$\bigoplus_{A \text{ of type } T} F(A) \ .$$

We make it into a "triple" complex denoted $H^{\bullet\bullet\bullet}(I,T)$ as follows.

The first degree and differential are the same as before. The second degree is $\deg_2 = |M'_{int}| + 1$ as before, and the differential is

$$d_2(u) = \boldsymbol{\sigma}_{int,T}(u) := \sum (-1)^{|(M'_{int})_{>k}|} u_1 \otimes \cdots \otimes \boldsymbol{\sigma}_k(u_j) \otimes \cdots \otimes u_r$$

the sum over $k \in \cup_{i\in T}(M_i - M'_i)$. (The restriction on $k$ is imposed so that the target is still of type $T$). The third degree and differential are the same as before: $\deg_3(u) = |M'_{out}|$, $d_3 = \boldsymbol{\sigma}_{out} + \mathbf{f}_{K_1} + \mathbf{f}_{K_n}$.

The complex $H^{\bullet\bullet\bullet}(I)$ has a filtration by subcomplexes indexed by types. For a given type $T$ the corresponding subcomplex $Fil^T$ is given as the sum $\bigoplus F(A)$, where $A$ varies over multi-sequences $A$ with type containing $T$. The subquotient (at $T$) in the filtration is the complex $H^{\bullet\bullet\bullet}(I,T)$ introduced above.

Let $\mathbb{G}(I,T)$ be the complex obtained from this by applying $\Phi$ and shifting the degree by $|T| + 1$:

$$\mathbb{G}(I,T) = \mathrm{Tot}(\Phi H^{\bullet\bullet\bullet}(I,T)\,)[\,|T| + 1] \ .$$

Namely

$$\deg_{\mathbb{G}(I,T)}(u) = \deg_{\mathbb{G}(I)}(u) - |T| - 1$$

and the differential $d_{\mathbb{G}(I,T)}$ is $(-1)^{|T|+1}$ times the original differential $(-1)^{\deg_2}\delta + \boldsymbol{\sigma}_{int,T}$.

For $T \subset T'$ with $|T'| = |T| + 1$, let

$$\boldsymbol{\sigma}_{T,T'} := (-1)^{|T_{>i}|} \sum (-1)^{|(M'_{out})_{>k}|} \boldsymbol{\sigma}_k : \mathbb{G}(I,T) \to \mathbb{G}(I,T') \ ,$$

the sum over $k \in M_i$, $T' = T \cup \{i\}$. Then $\boldsymbol{\sigma}_{T,T'}$ is a map of complexes.

(2.11.1) **Lemma.** *(1) If $i, j$ are distinct elements not in $T$, letting $T_1 = T \cup \{i\}$, $T_2 = T \cup \{j\}$, one has $\boldsymbol{\sigma}_{T_1 T''} \boldsymbol{\sigma}_{T T_1} = \boldsymbol{\sigma}_{T_2 T''} \boldsymbol{\sigma}_{T T_2}$.*

*(2) $\boldsymbol{\sigma}_{T,T'}$ is a quasi-isomorphism.*

*Proof.* (1) Immediate from the definition and the property $\boldsymbol{\sigma}\boldsymbol{\sigma} = 0$, (2.5.1).

(2) For each $A = (\mathbb{M}_1; \cdots; \mathbb{M}_n)$ of type $T$, $T' = T \cup \{i\}$, the total complex of the following double comlex is acyclic (by condition (4) of (1.5) ):

$$\begin{aligned} F(A) &\xrightarrow{\boldsymbol{\sigma}} \bigoplus_{|M'_i|=1} F(\mathbb{M}_1; \cdots; (M_i|M'_i); \cdots; \mathbb{M}_n) \\ &\xrightarrow{\boldsymbol{\sigma}} \bigoplus_{|M'_i|=2} F(\mathbb{M}_1; \cdots; (M_i|M'_i); \cdots; \mathbb{M}_n) \xrightarrow{\boldsymbol{\sigma}} \cdots \to F(\mathbb{M}_1; \cdots; (M_i|M_i); \cdots; \mathbb{M}_n) \ . \end{aligned}$$

Taking the sum over $A$ and applying $\Phi$, the first term $F(A)$ gives $\mathbb{G}(I,T)$, and the terms from the second to the last give $\mathbb{G}(I,T')$. Hence the assertion.

Consider the "double" complex

$$\mathbb{G}(I,\emptyset) \xrightarrow{\tilde{\boldsymbol{\sigma}}} \bigoplus_{|T|=1} \mathbb{G}(I,T) \xrightarrow{\tilde{\boldsymbol{\sigma}}} \cdots \xrightarrow{\tilde{\boldsymbol{\sigma}}} \mathbb{G}(I,\mathring{I}) \ , \tag{2.11.2}$$

where $\tilde{\boldsymbol{\sigma}}$ is the sum of $(-1)^{|T_{>i}|}\boldsymbol{\sigma}_{T,T'}$ for $T' = T \cup \{i\}$, and $\mathbb{G}(I,\emptyset)$ is placed in degree 1. To be precise, the first degree for $\mathbb{G}(I,T)$ is $|T|+1$, the first differential is $\tilde{\boldsymbol{\sigma}}$, and the second degree and differential are the ones for $\mathbb{G}(I,T)$.

Note

$$\tilde{\boldsymbol{\sigma}} = \sum_k (-1)^{|(M'_{out})_{>k}|} \boldsymbol{\sigma}_k$$

the sum over $k \in M_i$ with $i \notin T$. Using this one verifies $\mathbb{G}(I)$ is the total complex of the "double" complex (2.11.2). Indeed the total degree of $u \in \mathbb{G}(I,T)^p$ equals $p + |T| + 1$, which equals its total degree in $\mathbb{G}(I)$. The total differential acting on $u \in \mathbb{G}(I,T)^p$ is, according to (0.3),

$$\tilde{\boldsymbol{\sigma}} + (-1)^{|T|+1} \cdot (-1)^{|T|+1}((-1)^{\deg_2 u}\delta + \boldsymbol{\sigma}_{int,T}) \ ,$$

which equals the total differential in $\mathbb{G}(I)$.

These definitions make sense for any subset $I$ of $[1,n]$.

*Remark.* The passage from $\mathbb{G}(I,T)$ to $\mathbb{G}(I)$ is the same as the construction in [8], (2.3). We examined the signs carefully in order to demonstrate this. Otherwise the signs will not be too important for the rest of this paper.

(2.12) *The complexes* $\mathbb{G}(I,T|\Sigma)$. It is obvious how to refine (2.10) taking the type into account. For $\Sigma \subset \overset{\circ}{I}$ and $T \subset \overset{\circ}{I} - \Sigma$, letting $I_1, \cdots, I_c$ be the corresponding segmentation of $I$ and $T_i = T \cap \overset{\circ}{I}_i$, let

$$\begin{aligned} H^{\bullet\bullet\bullet}(I,T|\Sigma) &= H^{\bullet\bullet\bullet}(I_1,T_1)\tilde{\otimes}\cdots\tilde{\otimes}H^{\bullet\bullet\bullet}(I_c,T_c) \\ &= \bigoplus F(A_1)\tilde{\otimes}\cdots\tilde{\otimes}F(A_c)\ , \end{aligned}$$

the sum over $(A_1, \cdots, A_c)$ where $A_i$ is a multi-sequence on $I_i$ of type $T_i$. From this we get a "$c$-ple" complex as

$$\mathbb{G}(I,T|\Sigma) := \mathrm{Tot}_{12}\cdots\mathrm{Tot}_{2c-1,2c}\,\Phi\,\mathrm{Tot}_{3,\cdots,3c}(H^{\bullet\bullet\bullet}(I,T|\Sigma)\,)[|T_1|+1,\cdots,|T_c|+1]\ .$$

Here $\mathrm{Tot}_{12}\cdots\mathrm{Tot}_{2c-1,2c}\,\Phi\,\mathrm{Tot}_{3,\cdots,3c}$ is the same procedure as before to get a "$c$-ple" complex, and $[|T_1|+1,\cdots,|T_c|+1]$ is a shift of degrees applied to it.

Clearly there is an injective quasi-isomorphism $\iota_\Sigma : \mathbb{G}(I,T|\Sigma) \to \mathbb{G}(I,T\rceil\Sigma)$. The map $\rho_m$ defined in (2.10) is the direct sum of the maps $\rho_m : \mathbb{G}(I,T|\Sigma) \to \mathbb{G}(I, T\cup\{m\}|\Sigma - \{m\})$. Composing $\rho_m$'s one obtains maps

$$\rho_K : \mathbb{G}(I,T|\Sigma) \to \mathbb{G}(I,T\cup K|\Sigma - K)\ .$$

The map $\Pi_k : \mathbb{G}(I|\Sigma) \to \mathbb{G}(I-\{k\}|\Sigma)$ defined in (2.10) is the sum of $\Pi_k : \mathbb{G}(I,T|\Sigma) \to \mathbb{G}(I-\{k\},T|\Sigma)$. Composing $\Pi_k$'s one obtains

$$\Pi_K : \mathbb{G}(I,T|\Sigma) \to \mathbb{G}(I-K,T|\Sigma)\ .$$

One also has $\boldsymbol{\sigma}_{T,T'} : \mathbb{G}(I,T|\Sigma) \to \mathbb{G}(I,T'|\Sigma)$ the sum of which is $\boldsymbol{\sigma}_{T,T'}$ in the previous section.

Thus Assumption (A) in [8], (2.10) is satisfied for $\mathbb{G}(I,T|\Sigma)$, and the maps $\iota_\Sigma$, $\boldsymbol{\sigma}_{T,T'}$, $\rho_K$ and $\Pi_K$ (the last three playing the roles of $r$, $\rho$, $\pi$, respectively). We do not recall Assumption (A) here; it is enough to say that all the required conditions have been proven in (2.7)-(2.12). The comparison of data is as follows:

| this section | [8], §2 |
|---|---|
| $\mathbb{G}(I,T)$ | $A(I,\mathfrak{J})$ |
| $\mathbb{G}(I\|\Sigma)$ | $A(I\|\Sigma)$ |
| $\tilde{\otimes}$ | $\hat{\otimes}$ |
| $\boldsymbol{\sigma}_{T,T'}$ | $r_{\mathfrak{J},\mathfrak{J}'}$ |
| $\tilde{\boldsymbol{\sigma}}$ | $r$ |
| $\rho_m$ | $\rho_m$ |
| $\Pi_k$ | $\pi_k$ |
| $\mathbb{F}(I\|S)$ | $B(I\|S)$ . |

(2.13) *The complex* $\mathbb{F}(I|S)$. Now that we have Assumption (A) satisfied, we can apply the construction of [8], (2.10) to obtain the bar complex $\mathbb{F}(I|S)$ for a subset $S \subset \overset{\circ}{I}$. We recall the construction, for the convenience of the reader, and refer to [8] for the proofs of the properties.

Let $I$ be a finite ordered set and $S \subset \overset{\circ}{I}$; set

$$\mathbb{F}(I|S) := \bigoplus_{\Sigma \supset S} \mathbb{G}(I|\Sigma) \ ,$$

where $\Sigma$ varies over subsets containing $S$. The degree of $u = u_1 \otimes \cdots \otimes u_c \in \mathbb{G}(I|\Sigma) = \mathbb{G}(I_1)\tilde{\otimes} \cdots \tilde{\otimes}\mathbb{G}(I_c)$ is $\deg_{\mathbb{F}}(u) = \sum(\epsilon_j - 1)$, $\epsilon_j = \deg_{\mathbb{G}} u_j$. The differential $d_{\mathbb{F}}$ is the sum $\bar{d}_{\mathbb{G}} + \bar{\rho}$ of the maps given as follows. If $I_1, \cdots, I_c$ is the partition of $I$ corresponding to $\Sigma$, on an element $u = u_1 \otimes \cdots \otimes u_c \in \mathbb{G}(I|\Sigma)$ with $\epsilon_j = \deg(u_j) - 1$,

$$\bar{d}_{\mathbb{G}}(u_1 \otimes \cdots \otimes u_c) = -\sum (-1)^{\sum_{j>i}\epsilon_j} u_1 \otimes \cdots \otimes u_{i-1} \otimes d_{\mathbb{G}}(u_i) \otimes \cdots \otimes u_c \ ,$$

$$\bar{\rho}(u_1 \otimes \cdots \otimes u_c) = \sum_{1 \le i \le c-1} (-1)^{\sum_{j \ge i}\epsilon_j} \rho_{k_i}(u)$$

with $k_i = \mathrm{tm}(I_i)$. If $S = \emptyset$, let $\mathbb{F}(I) = \mathbb{F}(I|\emptyset)$. If $|I| = 2$, $\mathbb{F}(I)$ coincides with $\mathbb{F}(K, L)$ defined before.

One has the obvious surjection $\sigma_{S\,S'} : \mathbb{F}(I|S) \to \mathbb{F}(I|S')$ for $S \subset S'$; it is easy to see $\sigma_{S\,S'}$ is a quasi-isomorphism. There is an injective quasi-isomorphism $\iota_S : \mathbb{F}(I|S) \to \mathbb{F}(I \top S)$, defined as the sum of the injections $\mathbb{G}(I|\Sigma) \to \mathbb{G}(I \top \Sigma)$.

For $K \subset \overset{\circ}{I}$ disjoint from $S$, let $\varphi_K : \mathbb{F}(I|S) \to \mathbb{F}(I - K|\Sigma)$ be the sum of the maps

$$\Pi_K : \mathbb{G}(I|\Sigma) \to \mathbb{G}(I - K|\Sigma)$$

if $K$ is disjoint from $\Sigma$, and zero otherwise.

In the next proposition we state the properties of $\mathbb{F}(I|S)$. The proof is in [8], (2.11)-(2.12).

(2.14) **Proposition.** *(1) $\mathbb{F}(I)$ is a complex of free $\mathbb{Z}$-modules. For $S \subset \overset{\circ}{I}$ corresponding to a segmentation $I_1, \cdots, I_c$ of $I$, let $\mathbb{F}(I \top S) = \mathbb{F}(I_1) \otimes \cdots \otimes \mathbb{F}(I_c)$. $\mathbb{F}(I|S)$ is a complex of free $\mathbb{Z}$-modules together with an injective quasi-isomorphism $\iota_S : \mathbb{F}(I|S) \hookrightarrow \mathbb{F}(I \top S)$. If $S = \emptyset$, $\mathbb{F}(I|\emptyset) = \mathbb{F}(I)$. If $S = \overset{\circ}{I}$, $I = [1, n]$ and $I_i = [i, i+1]$,*

$$\begin{aligned} \mathbb{F}(I|\overset{\circ}{I}) &= \mathbb{G}(I_1)[1]\tilde{\otimes} \cdots \tilde{\otimes}\mathbb{G}(I_{n-1})[1] \\ &= \mathbb{G}(I_1, \emptyset)\tilde{\otimes} \cdots \tilde{\otimes}\mathbb{G}(I_{n-1}, \emptyset) = \mathbb{G}(I, \emptyset|\overset{\circ}{I}). \end{aligned}$$

*If $S \supset S'$, $S'$ gives the segmentation $I_1, \cdots, I_c$ and $S_i = \overset{\circ}{I_i} \cap S$, then one has inclusion*

$$\mathbb{F}(I|S) \subset \mathbb{F}(I_1|S_1) \otimes \cdots \otimes \mathbb{F}(I_c|S_c) \subset \mathbb{F}(I \top S) \ .$$

*(2) For subsets $S \subset S'$ there corresponds a surjective quasi-isomorphism $\sigma_{S\,S'} : \mathbb{F}(I|S) \to \mathbb{F}(I|S')$. One has $\sigma_{S\,S''} = \sigma_{S'\,S''}\sigma_{S\,S'}$. The $\sigma$ is compatible with the inclusion $\mathbb{F}(I|S) \subset \mathbb{F}(I_1|S_1) \otimes \cdots \otimes \mathbb{F}(I_c|S_c)$, namely if $S \subset S''$ and $S''_i = S'' \cap \overset{\circ}{I_i}$, the following commutes:*

$$\begin{tikzcd} \mathbb{F}(I|S) \arrow[r, hook] \arrow[d, "\sigma_{S\,S''}"'] & \mathbb{F}(I_1|S_1) \otimes \cdots \otimes \mathbb{F}(I_c|S_c) \arrow[d, "\otimes\sigma_{S_i\,S''_i}"] \\ \mathbb{F}(I|S'') \arrow[r, hook] & \mathbb{F}(I_1|S''_1) \otimes \cdots \otimes \mathbb{F}(I_c|S''_c) \ . \end{tikzcd}$$

*(3) There are maps* $\varphi_K : \mathbb{F}(I|S) \to \mathbb{F}(I-K|S)$ *which satisfy* $\varphi_K = \varphi_{K''}\varphi_{K'}$ *if* $K = K' \amalg K''$ *and are compatible with* $\sigma_{S\,S'}$*, namely the following diagram commutes:*

$$\begin{tikzcd} \mathbb{F}(I|S) \arrow[r, "\sigma_{S\,S'}"] \arrow[d, "\varphi_K"'] & \mathbb{F}(I|S') \arrow[d, "\varphi_K"] \\ \mathbb{F}(I-K|S) \arrow[r, "\sigma_{S\,S'}"] & \mathbb{F}(I-K|S')\ . \end{tikzcd}$$

*The following square commutes in the derived category.*

$$\begin{tikzcd} \mathbb{F}(I|S) \arrow[r, "\sigma"] \arrow[d, "\varphi_K"'] & \mathbb{G}(I,\emptyset|\mathring{I}) \arrow[r, "\iota"] & \mathbb{G}(I,\emptyset\top\mathring{I}) \arrow[d, "\varphi_K"'] \\ \mathbb{F}(I-K|S) \arrow[r, "\sigma"] & \mathbb{G}(I-K,\emptyset|\mathring{I}-K) \arrow[r, "\iota"] & \mathbb{G}(I-K,\emptyset\top\mathring{I}-K) \end{tikzcd}$$

*(Here the right vertical map is defined as follows. Consider the diagram*

$$\begin{tikzcd} & \mathbb{G}(I,\emptyset|\Sigma) \arrow[d, "\rho_K"] \\ \mathbb{G}(I,\emptyset|\Sigma-K) \arrow[r, "\boldsymbol{\sigma}_{\emptyset,K}"] \arrow[d, "\pi_K"'] & \mathbb{G}(I,K|\Sigma-K) \\ \mathbb{G}(I-K,\emptyset|\Sigma-K) & \end{tikzcd}$$

*Inverting the quasi-isomorphism* $\boldsymbol{\sigma}$*, we get a map in the derived category of abelian groups* $\varphi_K : \mathbb{G}(I,\emptyset\top\Sigma) \to \mathbb{G}(I-K,\emptyset\top\Sigma-K)$*.)*

*(4) Let* $R$*,* $J$ *be disjoint subsets of* $\mathring{I}$*, with* $J$ *non-empty. Then the following sequence of complexes is exact (the maps are alternating sums of the quotient maps* $\sigma$*)*

$$\mathbb{F}(I|R) \xrightarrow{\sigma} \bigoplus_{S\subset J\ ,|S|=1} \mathbb{F}(I|R\cup S) \xrightarrow{\sigma} \bigoplus_{S\subset J\ ,|S|=2} \mathbb{F}(I|R\cup S) \xrightarrow{\sigma} \cdots \to \mathbb{F}(I|R\cup J) \to 0\ .$$

*Moreover the total complex of the sequence is acyclic. (Equivalently, the induced map* $\sigma$ *:* $\mathbb{F}(I|R) \to \operatorname{Ker}\Big(\sigma : \bigoplus_{S\subset J\ ,|S|=1} \mathbb{F}(I|R\cup S) \to \bigoplus_{S\subset J\ ,|S|=2} \mathbb{F}(I|R\cup S)\Big)$ *is a surjective quasi-isomorphism.)*

# 3 The quasi DG category $\mathcal{C}^\Delta$.

(3.1) Let $\mathcal{C}$ be a quasi DG category. We will define another quasi DG category $\mathcal{C}^\Delta$, where the objects are $C$-diagrams in $\mathcal{C}$ as defined in §2. For a sequence of $C$-diagrams $K_1, \cdots, K_n$, we have defined complexes $\mathbb{F}(K_1, \cdots, K_n|S)$ and maps $\sigma_{S\,S'}$, $\varphi_K$ satisfying the conditions of (1.5), (1)-(4), except for the existence of direct sum. We will verify the remaining properties, in particular (5).

In (3.10)-(3.12) we show that the homotopy category of $\mathcal{C}^\Delta$ has the structure of a triangulated category, concluding the proof of Main Theorem.

(3.2) *The subcomplex* $\mathbb{F}(K, L)$. We first examine the composition in the homotopy category. Let us recall $\mathbb{F}(K, L)$ is given by

$$\mathbb{F}(K,L) = \Phi H^{\bullet\bullet} \subset H^{\bullet 1}$$

where

$$H^{\bullet 1} = \bigoplus F(M|\emptyset; N|\emptyset)$$

the sum over double sequences $(M|\emptyset; N|\emptyset)$, which we will abbreviate to $(M; N)$. So an element of $H^{\bullet 1}$ is of the form $u = (u(M; N)) \in \bigoplus F(M; N)$. The differential $d_{\mathbb{F}}$ acting on $u$ is nothing but $\delta = \boldsymbol{\partial} + \boldsymbol{\varphi}$ as defined in §2. $u$ has degree ($= \deg_{\mathbb{F}}$) zero if $u(M; N) \in F(M; N)^{-\gamma(M;N)}$.

(i) An element $u \in H^{\bullet 1}$ of degree zero is in $\mathbb{F}(K, L)^0$ if the following condition ($\sigma$-*consistency*) is satisfied. For $k \in M$, $k \neq \mathrm{in}(M)$,

$$\sigma_k(u(M; N)) = f_K(M_{\leq k}) \otimes u(M_{\geq k}; N) \ .$$

For $k \in N$, $k \neq \mathrm{tm}(N)$, $\sigma_k(u(M; N)) = u(M; N_{\leq k}) \otimes f_L(N_{\geq k})$.

(ii) It is $\delta$-closed if

$$\partial(u(M; N)) + \sum \boldsymbol{\varphi}_k(u(M \cup \{k\}; N)) + \sum \boldsymbol{\varphi}_k(u(M; N \cup \{k\})) = 0 \ .$$

Here $k$ in the first sum varies over the set $[\mathrm{in}\, M, \infty) - M$, and $k$ in the second sum over $(-\infty, \mathrm{tm}\, N] - N$.

Recall from §1 we have the homotopy category $Ho(\mathcal{C}^{\Delta})$. A morphism $u : K \to L$ in the homotopy category is represented by $u \in \mathbb{F}(K, L)^0$ which is $\delta$-closed.

(3.3) *The complex* $\mathbb{F}(K, L, M)$. For a finite sequence of $C$-diagrams $K_1, \cdots, K_n$, we defined in (2.13) the complex $\mathbb{F}(K_1, \cdots, K_n)$. We now examine the case $n = 3$. Let $K$, $L$, $M$ be three $C$-diagrams. The complex $\mathbb{F}(K, L, M)$ is of the form

$$\begin{tikzcd} & \mathbb{G}([1,3], \emptyset|\{2\}) \arrow[d, "\bar{\rho}"] \\ \mathbb{G}([1,3], \emptyset) \arrow[r, "-\tilde{\boldsymbol{\sigma}}"] & \mathbb{G}([1,3], \{2\}) \end{tikzcd}$$

which we also write

$$\begin{tikzcd} & \mathbb{G}(K, L)\tilde{\otimes}\mathbb{G}(L, M) \arrow[d, "\bar{\rho}"] \\ \mathbb{G}(K, L, M, \emptyset) \arrow[r, "-\tilde{\boldsymbol{\sigma}}"] & \mathbb{G}(K, L, M, \{2\}) \ . \end{tikzcd}$$

- The differential $d_{\mathbb{F}}$ is equal to $\bar{d}_{\mathbb{G}} + \bar{\rho}$, to be specified below.
- Recall in general the complex $\mathbb{G}(I)$ is the direct sum of $\mathbb{G}(I, T)$, each $\mathbb{G}(I, T)$ is a complex with differential $d_{\mathbb{G}(I,T)}$, and on $\mathbb{G}(I, T)$

$$d_{\mathbb{G}(I)} = \sum (-1)^{|T|+1} d_{\mathbb{G}(I,T)} + \tilde{\boldsymbol{\sigma}}$$

where $\tilde{\boldsymbol{\sigma}} = \sum \boldsymbol{\sigma}_k$, the sum over $k \in M_i$ with $i \notin T$. In particular $\mathbb{G}([1, 3])$ is of the form

$$\mathbb{G}([1,3], \emptyset) \xrightarrow{\tilde{\boldsymbol{\sigma}}} \mathbb{G}([1,3], \{2\}) \ .$$

- For $u \otimes v \in \mathbb{G}(K, L)\tilde{\otimes}\mathbb{G}(L, M)$, $\deg_{\mathbb{F}}(u \otimes v) = \deg_{\mathbb{G}}(u) + \deg_{\mathbb{G}}(v) - 2$, and $d_{\mathbb{F}}(u \otimes v) = \bar{d}_{\mathbb{G}}(u \otimes v) + \bar{\rho}(u \otimes v)$, where

$$\bar{d}_{\mathbb{G}}(u \otimes v) = -(-1)^{\deg_{\mathbb{G}}(v)-1} d_{\mathbb{G}}(u) \otimes v - u \otimes d_{\mathbb{G}}(v) \ ,$$

$$\bar{\rho}(u \otimes v) = (-1)^{\deg_{\mathbb{G}}(v)-1}\rho(u \otimes v) \ .$$

Further, $\rho(u \otimes v) = (-1)^{\deg_2(u)\cdot\deg_1(v)}u \otimes v$ if $u \in F(A)$, $v \in F(B)$ with $\mathrm{tm}(A) = \mathrm{in}(B)$, and zero otherwise.

Note if $u \otimes v \in \mathbb{G}(K, L)^1\tilde{\otimes}\mathbb{G}(L, M)^1$ one has $(-1)^{\deg_{\mathbb{G}}(v)-1} = 1$ and, since $\deg_1(v) = 0$, $(-1)^{\deg_2(u)\cdot\deg_1(v)} = 1$.

- For $W \in \mathbb{G}(\emptyset) := \mathbb{G}(K, L, M, \emptyset)$, $\deg_{\mathbb{F}}(W) = \deg_{\mathbb{G}}(W) - 1 = \deg_{\mathbb{G}(\emptyset)}(W)$, and

$$d_{\mathbb{F}}(W) = -d_{\mathbb{G}}(W) = d_{\mathbb{G}(\emptyset)}(W) - \tilde{\boldsymbol{\sigma}}(W) \ .$$

- For $z \in \mathbb{G}(\{2\}) = \mathbb{G}(K, L, M, \{2\})$, $\deg_{\mathbb{F}}(W) = \deg_{\mathbb{G}}(W) - 1 = \deg_{\mathbb{G}(\{2\})}(W) + 1$, and

$$d_{\mathbb{F}}(z) = -d_{\mathbb{G}}(z) = -d_{\mathbb{G}(\{2\})}(z) \ .$$

In particular, for a degree 0 element of $\mathbb{F}(K, L, M)$ is of the form $(u \otimes v, W, z)$ with

$$u \otimes v \in \mathbb{G}(K, L)^1\tilde{\otimes}\mathbb{G}(L, M)^1 \ , W \in \mathbb{G}(K, L, M, \emptyset)^0 \ , z \in \mathbb{G}(K, L, M, \{2\})^{-1} \ ,$$

then one has

$$\begin{aligned} d_{\mathbb{F}}(u \otimes v, W, z) \quad = \quad & -d_{\mathbb{G}}(u) \otimes v - u \otimes d_{\mathbb{G}}(v) + \rho(u \otimes v) \\ & +d_{\mathbb{G}(\emptyset)}(W) - \tilde{\boldsymbol{\sigma}}(W) \\ & -d_{\mathbb{G}}(z) \ . \end{aligned}$$

We also recall there is a map of complexes $\Pi_2 = \Pi_L : \mathbb{G}(K, L, M, \emptyset) \to \mathbb{G}(K, M)[1]$.

(3.4) **Proposition.** *Let $u : K \to L$ and $v : L \to M$ be morphisms. Assume $u$ is represented by $u \in \mathbb{G}(K, L)^1$ and $v$ represented by $v \in \mathbb{G}(L, M)^1$, and $u \otimes v \in \mathbb{G}(K, L)\tilde{\otimes}\mathbb{G}(L, M)$. Then*

*(1) There are an element $W \in \mathbb{G}(K, L, M, \emptyset)^0$, $d$-closed in $\mathbb{G}(K, L, M, \emptyset)$, and an element $z \in \mathbb{G}(K, L, M, \{2\})^{-1}$ such that*

$$\rho(u \otimes v) - \tilde{\boldsymbol{\sigma}}(W) - d_{\mathbb{G}}(z) = 0 \ .$$

*In this case $(u \otimes v, W, z) \in \mathbb{F}(K, L, M)^0$ is $d_{\mathbb{F}}$-closed. (Note the degree in $\mathbb{G}(K, L, M)$ is shifted, so $\deg_{\mathbb{G}}(W) = 1$, and $\deg_{\mathbb{G}}(z) = 1$.)*

*(2) If (1) is satisfied, the element $\Pi_L(W) \in \mathbb{G}(K, M)^1$ is $d$-closed and represents the morphism $u \cdot v : K \to M$.*

*Proof.* (1) Since $u \otimes v$ is $d$-closed in $\mathbb{G}(K, L)\tilde{\otimes}\mathbb{G}(L, M)$, $\rho(u \otimes v) \in \mathbb{G}(K, L, M, \{2\})^0$ is $d$-closed. Since $\tilde{\boldsymbol{\sigma}}$ is a quasi-isomorphism, the claim follows.

(2) By definition the composition is induced by the projection $\mathbb{F}(K, L, M) \to \mathbb{G}(K, L)\tilde{\otimes}\mathbb{G}(L, M)$ and the composition of the maps as

$$\begin{aligned} \mathbb{F}(K, L, M) \quad & \xrightarrow{\mathrm{proj}} \mathbb{G}(K, L, M, \emptyset) \\ & \xrightarrow{\Pi_L} \mathbb{G}(K, M)[1] = \mathbb{F}(K, M) \ . \end{aligned}$$

So the assertion is obvious.

(3.5) *The identity map.* Let $K = (K^m; f(M))$ be an object. For a non-decreasing sequence $M = (m_1, \cdots, m_\mu)$, $m_1 \leq m_2 \leq \cdots \leq m_\mu$, let $M'$ be the increasing sequence obtained by eliminating repetitions. Set $\gamma(M) = \gamma(M')$. We have the diagonal extension map diag : $F(M') \to F(M)$. Let $f(M) \in F(M)$ be the image of $f(M') \in F(M')$. (We say that $f(M)$ is obtained from $f(M')$ by means of diagonal extension.)

These elements

$$f(M) = f(m_1, \cdots, m_\mu) \in F(K^{m_1}, \cdots, K^{m_\mu})^{-\gamma(M)}$$

satisfy the following properties:

(i) For each $M$,

$$\partial f(M) + \sum_a \boldsymbol{\varphi}_a(f(M \cup \{a\})\,) = 0$$

where $a$ varies over $[\mathrm{in}(M), \mathrm{tm}(M)] - M = [\mathrm{in}(M'), \mathrm{tm}(M')] - M'$.

(ii) For $k = m_i$, $k \neq m_1, m_\mu$ one has

$$\sigma_{m_i}(f(M)\,) = f(m_1, \cdots, m_i) \otimes f(m_i, \cdots, m_\mu)$$

or $\sigma_k(f(M)) = f(M_{\le k}) \otimes f(M_{\ge k})$ for short.

(iii) For $k = m_i$ which is repeated in $M$,

$$\varphi_{m_i}(f(m_1, \cdots, m_\mu)\,) = f(m_1, \cdots, \widehat{m_i}, \cdots, m_\mu)\ .$$

(iv) If $m_1 = \cdots = m_\mu = m$, then $f(m, \cdots, m) = \boldsymbol{\Delta}(m, \cdots, m) \in F(K^{m_1}, \cdots, K^{m_\mu})$.

For a sequence $(M; N) = (m_1, \cdots, m_\mu; n_1, \cdots, n_\nu)$ with $m_1 < \cdots < m_\mu \le n_1 < \cdots < n_\nu$, let

$$\begin{aligned} \tilde{f}(M; N) &= \tilde{f}(m_1, \cdots, m_\mu; n_1, \cdots, n_\nu) \\ &= \begin{cases} (-1)^{n_1} f(m_1, \cdots, m_\mu, n_1, \cdots, n_\nu) \in F(M; N)^{-\gamma(M;N)} & \text{if } m_\nu = n_1 \\ 0 & \text{if } m_\nu < n_1\ . \end{cases} \end{aligned}$$

One may simply write $\tilde{f}(m_1, \cdots, m_\mu; n_1, \cdots, n_\nu) = (-1)^{n_1} \delta_{m_\mu n_1} f(m_1, \cdots, m_\mu, n_1, \cdots, n_\nu)$. Note the repetition of indices can occur only in the first case for $m_\mu$ and $n_1$, and $\tilde{f}(M; N) \in F(M; N)^{-\gamma(M;N)}$. The collection $(\tilde{f}(M; N)\,)$, as $(M; N)$ varies, is an element in $H^{\bullet\,1}(K, K)$.

**Proposition.** *The element $(\tilde{f}(M; N)\,)$ is contained in $\mathbb{F}(K, K)^0$ and $\delta$-closed.*

*Proof.* We verify the two conditions in (3.2). The first condition is obvious. To show the identity

$$\partial(\tilde{f}(M; N)\,) + \sum \boldsymbol{\varphi}_k(\tilde{f}(M \cup \{k\}; N)\,) + \sum \boldsymbol{\varphi}_k(\tilde{f}(M; N \cup \{k\})\,) = 0\ ,$$

there are cases $m_\mu = n_1$ and $m_\mu < n_1$. If $m_\mu = n_1$ it holds by property (i) for $f(M)$. If $m_\mu < n_1$, the first term is zero and the last two terms are

$$\boldsymbol{\varphi}_{n_1}(\{n_1\} f(m_1, \cdots, m_\mu, n_1; n_1, \cdots, n_\nu)\,) + \boldsymbol{\varphi}_{m_\mu}(\{n_1\} f(m_1, \cdots, m_\mu; m_\mu, n_1, \cdots, n_\nu)\,)$$

which is zero by property (iii) above.

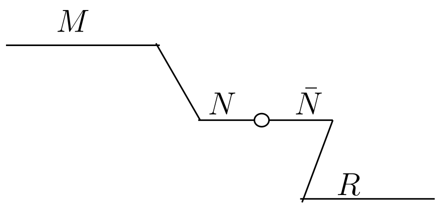
Let $\iota_K : K \to K$ be the morphism represented by $(\tilde{f}(M;N)\,)$. The following shows it is the identity map.

(3.6) **Proposition.** *For any morphism $u : K \to L$, one has $\iota_K \cdot u = u$. Similarly for any $v : K \to L$, $v \cdot \iota_L = v$.*

*Proof.* Let $u$ be represented by $\underline{u} = (\underline{u}(M;N)) \in \mathbb{F}(K,L)^0$. Then $\rho(\tilde{f} \otimes \underline{u})$ equals

$$\sum_{\operatorname{tm} N = \operatorname{in} \bar{N}} \tilde{f}(M;N) \otimes \underline{u}(\bar{N};R) \in \oplus F(K^M;K^N) \tilde{\otimes} F(K^{\bar{N}};L^R) \subset \mathbb{G}(K,K,L,\{2\})\ .$$

Here $(M;N)$ and $(\bar{N};R)$ are free double sequences. The restriction $\operatorname{tm} N = \operatorname{in} \bar{N}$ occurs since $\rho$ acts as identity in that case, and as zero otherwise.




By means of diagonal extension, for any $(M;N) = (m_1, \cdots, m_\mu; n_1, \cdots, n_\nu)$ with $m_1 \le \cdots \le m_\mu$, $n_1 \le \cdots \le n_\nu$, one obtains $\underline{u}(M;N) \in F(M;N)$ satisfying properties similar to (i)-(iv) for $f(M)$ above. Let

$$\underline{W} = (\underline{W}(M;N;R)\,) \in \bigoplus F(K^M;K^N;L^R) \subset \mathbb{G}(K,K,L,\emptyset)$$

where for each free triple sequence $(M;N;R)$,

$$\underline{W}(M;N;R) = \underline{u}(M_\vartriangle N;R) := \begin{cases} (-1)^\ell \underline{u}(M,N;R) & \text{if } \operatorname{tm} M = \ell = \operatorname{in} N\ , \\ 0 & \text{if } \operatorname{tm} M \neq \operatorname{in} N\ . \end{cases}$$

The sign $_\vartriangle$ indicates applying diagonal extension if $\operatorname{tm} M = \ell = \operatorname{in} N$ and putting sign $(-1)^\ell$. Then $\underline{W}$ is $\sigma$-consistent, namely

$$\sigma_k(\underline{W}(M;N;R)\,) \quad = \quad \begin{cases} f(M_{\le k}) \otimes \underline{W}(M_{\ge k};N;R) & \text{if } k \in M - \{\operatorname{in} M\} \\ \tilde{f}(M;N_{\le k}) \otimes \underline{u}(N_{\ge k};R) & \text{if } k \in N \\ \underline{W}(M;N;R_{\le k}) \otimes f_L(R_{\ge k}) & \text{if } k \in R - \{\operatorname{tm} R\}\ . \end{cases}$$

Also, $\underline{W}$ is $\delta$-closed, as can be shown by the same argument as in the proof of the above proposition. Further one has

$$\boldsymbol{\sigma}_2(\underline{W}) = \sigma_2(\underline{W}) = \rho(\tilde{f} \otimes \underline{u})\ .$$

The second equality is clear, and the first one holds because $\deg_1 \tilde{f}(M;N) = 0$. By Proposition (3.4), $\iota \cdot u$ is represented by $\Pi_2(\underline{W})$, which equals $\underline{u}$:

$$\begin{aligned} \Pi_2(\underline{W})(M;R) &= (-1)^\ell \varphi_\ell(\underline{W}(M;\{\ell\};R)\,) \\ &= \underline{u}(M;R)\ . \end{aligned}$$

The argument for the second statement is similar.

(3.7) *$\sigma$-consistent prolongation.* For $a = 1, 2$, assume given a finite sequence of objects $i \mapsto X_i^a$. Let $X_i = X_i^1 \oplus X_i^2$. For a finite set of integers $M$ and a map $\alpha : M \to \{1,2\}$, let $X^{\alpha,M}$ denote the sequence of objects $X_i^{\alpha(i)}$ indexed by $i \in M$, and $F(X^{\alpha,M}) := F(M, X_i^{\alpha(i)})$ the corresponding complex. Similarly one has $F(X^M) = F(M, X_i)$. Then by additivity we have

$$F(X^M) = \bigoplus F(X^{\alpha_1,M_1}) \otimes \cdots \otimes F(X^{\alpha_c,M_c})$$

where the sum is over all segmentations $M_1, \cdots, M_c$ of $M$ and functions $\alpha_i : M_i \to \{1,2\}$ such that at each $k = \operatorname{tm} M_i = \operatorname{in} M_{i+1}$, $\alpha_i(k) \neq \alpha_{i+1}(k)$. Note $\bigoplus_\alpha F(X^{\alpha,M})$, the sum over functions $\alpha$ on $M$, is a direct summand.

Now assume for each subset $M$ and a function $\alpha : M \to \{1,2\}$, given an element $f(\alpha, M) \in F(X^{\alpha,M})$ such that for $k \in \overset{\circ}{M}$,

$$\sigma_k(f(\alpha, M)) = f(\alpha_{\le k}, M_{\le k}) \otimes f(\alpha_{\ge k}, M_{\ge k}) \ ,$$

where $\alpha_{\le k}$ (resp. $\alpha_{\ge k}$) is the restriction of $\alpha$ to $M_{\le k}$ (resp. $M_{\ge k}$)). This is a kind of $\sigma$-consistency. We then define an element $f(M) \in F(X^M)$ by

$$f(M) = \sum f(\alpha_1, M_1) \otimes \cdots \otimes f(\alpha_c, M_c) \ ,$$

the sum over $M_1, \cdots, M_c$ and $\alpha_i : M_i \to \{1,2\}$ as above. Using the compatibility of $\sigma$ and $\varphi$ with the additivity, one shows:

**Proposition.** *(1) The set of elements $f(M) \in F(X^M)$ is $\sigma$-consistent, namely it satisfies $\sigma_k(f(M)) = f(M_{\le k}) \otimes f(M_{\ge k})$ in $F(M_{\le k}) \otimes F(M_{\ge k})$. Further $(f(M))$ is the unique set of $\sigma$-consistent elements such that, for each $M$, $f(M)$ projects to $(f(\alpha, M))$ under the projection $F(M) \to \bigoplus_\alpha F(X^{\alpha,M})$.*

*We call $f$ the* $\sigma$-consistent prolongation *of $f$.*

*(2) Assume in addition* $\deg_1 f(\alpha, M) = 0$ *and the sum $(f(\alpha, M)) \in \bigoplus F(X^{\alpha,M})$ is $\delta$-closed. Then $(f(M)) \in \bigoplus F(X^M)$ is $\delta$-closed.*

(3.8) *Variant for $\mathbb{F}(K, L)$.* Assume given two sequences of objects $K_a = (K_a^m)$ for $a = 1, 2$. Let $K$ be the sequence of objects given by $K^m = K_1^m \oplus K_2^m$. Assume for each free sequence $M$ and a function $\alpha : M \to \{1,2\}$, given an element $f(\alpha, M) \in F(K^{\alpha,M})$ of $\deg_1 = 0$, such that $(f(\alpha, M))$ is $\sigma$-consistent and $\delta$-closed. Then the $\sigma$-consistent prolongation $f(M) \in F(K^M)$ as in the previous subsection is again of first degree zero and $\delta$-closed; thus the data $(K^m; f(M))$ gives us a $C$-diagram.

We need a variant of the $\sigma$-consistent prolongation for the complex $\mathbb{F}(K, L)$. As in the previous subsection, one has

$$F(K^M; L^N) = \bigoplus F(K^{\alpha_1,M_1}) \otimes F(K^{\alpha_2,M_2}) \otimes \cdots \otimes F(K^{\alpha_r,M_r}; L^N)$$

the sum is over segmentations $M_1, \cdots, M_r$ of $M$ and functions $\alpha_i : M_i \to \{1,2\}$ taking distinct values at the intersections of the subintervals. For convenience let us call the summand $\bigoplus F(K^{\alpha,M}; L^N)$ the *primary part.*

Assume for each free double sequence $(M;N)$ and a function $\alpha : M \to \{1,2\}$, given an element $u(\alpha, M; N) \in F(K^{\alpha,M}; L)$ satisfying $\sigma$-consistency: for each $k \in M$,

$$\sigma_k(u(\alpha, M; N)\,) = f(\alpha_{\leq k}, M_{\leq k}) \otimes u(M_{\geq k}; N) \ ;$$

for $k \in N$,

$$\sigma_k(u(\alpha, M; N)\,) = u(M; N_{\leq k}) \otimes f_L(N_{\geq k}) \ .$$

Then we define $u(M;N) \in F(K^M; L^N)$ by a formula analogous to the one for $f(M)$ in the previous subsection.

(3.8.1) **Proposition.** *The set of elements $u(M;N) \in F(K^M; L^N)$ is $\sigma$-consistent. It is the unique $\sigma$-consistent set of elements projecting to $u(\alpha, M; N)$.*

The elements $u(\alpha, M; N)$ are called the *primary part* of $u(M;N)$.

Recall $\mathbb{F}(K, L) = \Phi[\bigoplus_{(M;N)} F(K^M; L^N)]$ is the subcomplex consisting of the $\sigma$-consistent elements. Likewise let $\Phi[\bigoplus_{(\alpha,M;N)} F(K^{\alpha,M}; L^N)]$ denote the subcomplex of $\sigma$-consistent elements in $\bigoplus_{(\alpha,M;N)} F(K^{\alpha,M}; L^N)$. (The notation is legitimate: there is indeed a double complex whose $\Phi$-part coincides with this.) Let

$$\mathcal{P} : \Phi[\bigoplus_{(\alpha,M;N)} F(K^{\alpha,M}; L^N)] \to \Phi[\bigoplus_{(M;N)} F(K^M; L^N)]$$

be the map given by the $\sigma$-consistent prolongation. It is an isomorphism of modules. By the additivity of $\varphi$, it follows:

(3.8.2) **Proposition.** *The map $\mathcal{P}$ is an isomorphism of complexes.*

This enables us to reduce the study of $u(M;N)$ to that of the primary parts $u(\alpha, M; N)$. For example, if $\deg_1 u(\alpha, M; N) = 0$ and $(u(\alpha, M; N)\,) \in \bigoplus F(K^{\alpha,M}; L^N)$ is $\delta$-closed, then $(u(M;N)\,) \in \bigoplus F(K^M; L^N)$ is also $\delta$-closed.

*Variants.* It is obvious how to generalize the above to the situation where

(i) one has a direct sum decomposition of $K$ into more than two objects;

(ii) the $L$ has a direct sum decomposition;

(iii) one considers the complex $\mathbb{G}(K_1, \cdots, K_n)$ with $n \geq 3$ in place of $\mathbb{G}(K, L)$.

(3.9) *Direct sum and additivity.* As a special case, assume we have two $C$-diagrams $K_a = (K_a^m; f_a(M))$ for $a = 1, 2$. Let $f(\alpha, M) \in F(K^{\alpha,M})$ be given as $f(\underline{1}, M) = f_1(M)$, $f(\underline{2}, M) = f_2(M)$, and zero otherwise. Here $\underline{1}$ is the function with constant value 1. Then the resulting $C$-diagram $K = (K^m; f(M))$ is by definition the direct sum of $K_1$ and $K_2$. The next proposition shows this is a categorical direct sum.

**Proposition.** *We have*

$$\mathbb{F}(K_1 \oplus K_2, L) = \mathbb{F}(K_1, L) \oplus \mathbb{F}(K_2, L) \ .$$

*Similarly $\mathbb{F}(K, L)$ is additive in $L$.*

*Proof.* Under the more general assumption of (3.8), we have an isomorphism between $\Phi[\bigoplus_{(\alpha,M;N)} F(K^{\alpha,M};L^N)]$ and $\Phi[\bigoplus_{(M;N)} F(K^M;L^N)]$. In the present case the source coincides with $\Phi[\bigoplus F(K_2^M;L^N)] \oplus \Phi[\bigoplus F(K_1^M;L^N)]$.

Generalizing the above, one can show $\mathbb{F}(K_1,\cdots,K_n)$ is additive in each variable in the sense of §1. For this we first show, by a similar argument, that $\mathbb{G}(K_1,\cdots,K_n|\Sigma)$ satisfies the partial additivity: Assume $K_i = L_i \oplus M_i$. If $i \in \Sigma$,

$$\mathbb{G}(K_1,\cdots,K_n|\Sigma) = \mathbb{G}(K_1,\cdots,L_i,\cdots,K_n|\Sigma) \oplus \mathbb{G}(K_1,\cdots,M_i,\cdots,K_n|\Sigma) \ ,$$

and if $i \notin \Sigma$

$$\begin{aligned} & \mathbb{G}(K_1,\cdots,K_n|\Sigma) \\ = \ & \mathbb{G}(K_1,\cdots,L_i,\cdots,K_n|\Sigma) \oplus \mathbb{G}(K_1,\cdots,M_i,\cdots,K_n|\Sigma) \\ & \oplus\mathbb{G}(K_1,\cdots,L_i|\Sigma_1) \otimes \mathbb{G}(M_i,\cdots,K_n|\Sigma_2) \oplus \mathbb{G}(K_1,\cdots,M_i|\Sigma_1) \otimes \mathbb{G}(L_i,\cdots,K_n|\Sigma_2) \end{aligned}$$

where $\Sigma_1, \Sigma_2$ is the partition of $\Sigma$ by $i$. Further $\rho_i$ is zero on the last two summands.

In case $n = 3$ and $\Sigma = \{2\}$, here is the outline of the proof. With the notation of (2.9), we have obviously

$$\Phi[(\oplus F(A_1)\,)\tilde{\otimes}(\oplus F(A_2)\,)] = \Phi[(\oplus F(A_1)\,) \otimes (\oplus F(A_2)\,)] \cap [(\oplus F(A_1)\,)\tilde{\otimes}(\oplus F(A_2)\,)] \ .$$

Since $\Phi[(\oplus F(A_1)\,) \otimes (\oplus F(A_2)\,)] \cong \Phi[(\oplus F(A_1)\,)] \otimes \Phi[(\oplus F(A_2)\,)]$ we have only to identify $\Phi[(\oplus F(A_1)\,)]$, which can be done as in the above proposition.

It follows $\mathbb{F}(K_1,\cdots,K_n)$ satisfies the additivity. Compare the argument with that in (1.7).

(3.10) *Shifting functor.* For an increasing sequence $M = (m_1,\cdots,m_\mu)$, let $M[1] = (m_1 + 1,\cdots,m_\mu + 1)$. For a sequence $\mathbb{M} = (M|M')$ as in §2, let

$$\mathbb{M}[1] = (M[1]|M'[1]) \ .$$

Likewise for a double sequence $(\mathbb{M};\mathbb{N})$, another double sequence $(\mathbb{M}[1];\mathbb{N}[1])$ is defined.

Let $K = (K^m; f(M))$ be a $C$-diagram. Define another $C$-diagram

$$K[1] = (K[1]^m; (f[1])(M)\,)$$

by $(K[1])^m = K^{m+1}$ and $(f[1])(M) = f(M[1]) \in F(M[1])$ (namely $(f[1])(m_1,\cdots,m_\mu) = f(m_1+1,\cdots,m_\mu+1)\,$).

Let $K$ and $L$ be $C$-diagrams. Recall

$$H^{\bullet\bullet}(K,L) = \bigoplus_{(\mathbb{M};\mathbb{N})} F(K^{\mathbb{M}};L^{\mathbb{N}}) = \bigoplus F(\mathbb{M};\mathbb{N}) \ .$$

So an element $u$ has $(\mathbb{M};\mathbb{N})$-component $u(\mathbb{M};\mathbb{N}) \in F(\mathbb{M};\mathbb{N})$. Similarly

$$H^{\bullet\bullet}(K[1],L[1]) = \bigoplus_{(\mathbb{M};\mathbb{N})} F(\mathbb{M}[1];\mathbb{N}[1]) \ .$$

Define a map

$$(-)[1] : H^{\bullet\bullet}(K,L) \to H^{\bullet\bullet}(K[1],L[1])$$

by sending $u = (u(\mathbb{M};\mathbb{N})\,)$ to $u[1]$ with $(u[1])(\mathbb{M};\mathbb{N}) = u(\mathbb{M}[1];\mathbb{N}[1]) \in F(\mathbb{M}[1],\mathbb{N}[1])$. This map preserves $|u|$, $\gamma(u)$ and $\tau(u)$, so it preserves $\deg_1$ and $\deg_2$; it is also compatible with the maps $\delta$ and $d'$. Thus shift is an isomorphism of complexes (indeed an isomorphism of the underlying "double" complexes). We change the sign and let

$$\mathrm{shift} : H^{\bullet\bullet}(K,L) \to H^{\bullet\bullet}(K[1],L[1])$$

be the map given by by $u \mapsto -u[1]$. Taking $\Phi$ we get an isomorphism of complexes shift : $\mathbb{G}(K,L) \to \mathbb{G}(K[1],L[1])$.

Similarly for $n$ $C$-diagrams $K_1,\cdots,K_n$, one defines the isomorphism of complexes

$$\mathrm{shift} : \mathbb{G}(K_1,\cdots,K_n) \to \mathbb{G}(K_1[1],\cdots,K_n[1])$$

by which sends $u$ to $(-1)^{n-1}u[1]$. For $\Sigma$ a subset of $(1,n)$ with cardinality $r-1$, let $I_1,\cdots,I_r$ be the corresponding segmentation of $[1,n]$, and let

$$\mathrm{shift} : \mathbb{G}(K_1,\cdots,K_n|\Sigma) \to \mathbb{G}(K_1[1],\cdots,K_n[1]|\Sigma)$$

be given by

$$u = u_1\otimes\cdots\otimes u_r \mapsto (-1)^{|I_1|-1}u[1]\otimes\cdots\otimes(-1)^{|I_r|-1}u_r[1]$$

*Claim.* *The map* shift *commutes with* $\rho_m$ *and* $\Pi_k$.

*Proof.* This follows from the definitions. To verify the commutativity with $\Pi_k$, assume for example $\Sigma = \emptyset$ and $u \in F(A)$, where $A = ((M_1|M_1');\cdots;(M_n|M_n'))$ is a multi-sequence on $[1,n]$, and $\mathbb{M}_k = (M_k|\emptyset)$ with $M_k = \{j\}$; then both $\mathrm{shift}\circ\Pi_k(u)$ and $\Pi_k\circ\mathrm{shift}(u)$ are equal to $(-1)^j\varphi_j(u)$. Note we need to need a minus sign for $u[1]$ for the commutativity with $\Pi_k$.

Thus passing to the bar complexes one obtains an isomorphism

$$\mathrm{shift} : \mathbb{F}(K_1,\cdots,K_n|S) \to \mathbb{F}(K_1[1],\cdots,K_n[1]|S)$$

compatible with the maps $\sigma_{S\,S'}$ and $\varphi_K$. Thus $K \mapsto K[1]$, $u \mapsto \mathrm{shift}(u)$ give an auto-equivalence of the quasi DG category $\mathcal{C}^\Delta$.

If $u : K \to L$ is a morphism represented by $u = (\mathbf{u}(M;N)) \in \mathbb{F}(K,L)^0$, then $u[1] : K[1] \to L[1]$ is the morphism represented by $-u[1] = (-u(M[1];N[1])\,) \in \mathbb{F}(K[1],L[1])^0$.

(3.11) *The cone of a morphism.* Let $u : (K^m; f(M)\,) \to (L^m; g(M)\,)$ be a morphism. Take a representative $\underline{u} = (\underline{u}(M;N)\,)$ in $\mathbb{F}(K,L)^0$.

The *cone* of $u$ is the object $C_u = (C_u^m; h(M)\,)$, where

$$C_u^m = K^{m+1} \bigoplus L^m \ ,$$

and the elements

$$h(m_1,\cdots,m_\mu) \in F(C_u^{m_1},\cdots,C_u^{m_\mu})^{-\gamma(m_1,\cdots,m_\mu)}$$

are to be specified. For $m_1 < \cdots < m_\mu$, the complex $F(C_u^{m_1},\cdots,C_u^{m_\mu})$ contains as a direct summand

$$F(K^{m_1+1},\cdots,K^{m_\mu+1}) \oplus \bigoplus_{r=1,\cdots,\mu-1} F(K^{m_1+1},\cdots,K^{m_r+1};L^{m_{r+1}},\cdots,L^{m_\mu}) \oplus F(L^{m_1},\cdots,L^{m_\mu})\ .$$

The inclusion is compatible with the differentials $\partial$, $\varphi_k$, and preserves $|u|$ and $\gamma(u)$.

Now let $h(m_1, \cdots, m_\mu)$ be the element in this subcomplex consisting of:

$$f(m_1+1, \cdots, m_\mu+1) \in F(K^{m_1+1}, \cdots, K^{m_\mu+1})^{-\gamma(m_1,\cdots,m_\mu)} ,$$

$$u(m_1+1, \cdots, m_r+1; m_{r+1}, \cdots, m_\mu) \in F(K^{m_1+1}, \cdots, K^{m_r+1}; L^{m_{r+1}}, \cdots, L^{m_\mu})^{-\gamma(m_1,\cdots,m_\mu)}$$

for $1 \le r \le \mu - 1$, and

$$g(m_1, \cdots, m_\mu) \in F(L^{m_1}, \cdots, L^{m_\mu})^{-\gamma(m_1,\cdots,m_\mu)} .$$

One verifies the condition

$$\partial h(m_1, \cdots, m_\mu) + \sum \boldsymbol{\varphi}_k(h(m_1, \cdots, m_t, k, m_{t+1}, \cdots, m_\mu)) = 0 .$$

To be concise, $h(M) \in F(C_u^M)$ consists of $f(M[1]) \in F(K^{M[1]})$, $\underline{u}(M'[1]; M'') \in F(K^{M'[1]}; L^{M''})$ for partitions $M = M' \cup M''$, and $g(M) \in F(L^M)$. The following matrix expression is useful:

$$h(M) = \begin{array}{cc} & \begin{array}{cc} K[1] & L \end{array} \\ \begin{array}{c} K[1] \\ L \end{array} & \begin{bmatrix} f(M[1]) & 0 \\ u(M'[1]; M'') & g(M) \end{bmatrix} \end{array}$$

where $M', M''$ are obtained from $M$ by partition.

Referring to (3.7) for notation, with respect to the decomposition $C_u = K[1] \oplus L$, for a set $M$ and a function $\alpha : M \to \{1, 2\}$ there corresponds an element $h(\alpha, M) \in F(C_u^{\alpha,M})$; only for the functions $\alpha = (1, \cdots, 1, 2, \cdots, 2)$ the corresponding elements are non-zero, and given as above. We now apply the process of $\sigma$-consistent prolongation to obtain elements in $F(C_u^M)$, still denoted $h(M)$. This gives us a $C$-diagram $(C_u : h(M))$, the cone of $u$.

There are canonical morphisms $\alpha(u) : L \to C_u$ and $\beta(u) : C_u \to K[1]$ defined as follows. The $\alpha(u)$ is the morphism represented by the elements

$$\underline{\alpha(u)}(m_1, \cdots, m_\mu; n_1, \cdots, n_\nu) = \tilde{g}(m_1, \cdots, m_\mu; n_1, \cdots, n_\nu)$$

in $\bigoplus F(L^{m_1}, \cdots, L^{m_\mu}; L^{n_1}, \cdots, L^{n_\nu}) \subset \mathbb{F}(L; C_u)$. Recall $\tilde{g}$ is given by

$$\begin{aligned} &\tilde{g}(m_1, \cdots, m_\mu; n_1, \cdots, n_\nu) \\ = &\begin{cases} (-1)^{n_1} g(m_1, \cdots, m_\mu; n_1, \cdots, n_\nu) & \text{if } m_\mu = n_1 \\ 0 & \text{if } m_\nu < n_1 \ . \end{cases} \end{aligned}$$

Clearly $\underline{\alpha(u)} \in \mathbb{F}(L, C_u)^0$, and $\delta$-closed, thus representing a morphism. In short one may write $\underline{\alpha}(M; N) = \tilde{g}(M; N)$, or more precisely

$$\underline{\alpha}(M; N) = \begin{array}{cc} & L \\ \begin{array}{c} K[1] \\ L \end{array} & \begin{bmatrix} 0 \\ \tilde{g}(M; N) \end{bmatrix} \end{array} .$$

The $\beta(u)$ is represented by

$$\underline{\beta}(m_1, \cdots, m_\mu; n_1, \cdots, n_\nu)$$

in $\bigoplus F(K[1]^{m_1}, \cdots, K[1]^{m_\mu}; K[1]^{n_1}, \cdots, K[1]^{n_\nu}) \subset \mathbb{F}(C_u; K[1])$ given by

$$\begin{aligned} &\underline{\beta}(m_1, \cdots, m_\mu; n_1, \cdots, n_\nu) \\ =& \begin{cases} (-1)^{n_1} f(m_1+1, \cdots, m_\mu+1, n_1+1, \cdots, n_\nu+1) & \text{if } m_\mu = n_1 \\ 0 & \text{if } m_\mu < n_1 \ . \end{cases} \end{aligned}$$

In other words,

$$\underline{\beta}(M;N) = \begin{array}{ccc} & K[1] & L \\ K[1] & [\ (f[1])\tilde{}(M;N) & 0]\,. \end{array}$$

It is obvious that the composition of $\alpha$ and $\beta$ is zero. One thus has a triangle

$$K \xrightarrow{u} L \to C_u \xrightarrow{[1]} \ .$$

Such a triangle is called a standard distinguished triangle. We declare the *distinguished triangles* to be the ones isomorphic to the standard ones.

The verification of the axioms of triangulated category is parallel to the case of the homotopy category of complexes; see e.g. [10], §1.4 for a detailed exposition. The arguments are similar to the DG case, done in [6], II in detail. The above definitions of $h(M)$, $\alpha$, $\beta$ are motivated by the DG case.

Two of the axioms are non-trivial. We will only illustrate the proof of one of the axioms.

(3.12) **Proposition.** *There exists an isomorphism $\phi : K[1] \to C_{\alpha(u)}$ such that the following diagram commutes:*

$$\begin{tikzcd} L \arrow[r, "\alpha(u)"] \arrow[d, "id"'] & C_u \arrow[r, "\beta(u)"] \arrow[d, "id"'] & K[1] \arrow[r, "-u[1]"] \arrow[d, "\phi"'] & L[1] \arrow[d, "id"'] \\ L \arrow[r, "\alpha(u)"] & C_u \arrow[r, "\alpha(\alpha(u))"] & C_{\alpha(u)} \arrow[r, "\beta(\alpha(u))"] & L[1]\ . \end{tikzcd}$$

*Proof.* First note that $C^m_{\alpha(u)} = L^{m+1} \oplus K^{m+1} \oplus L^m$, and the structural elements $k(M) \in F(C^M_{\alpha(u)})$ are given as follows:

$$k(M) = \begin{array}{cc} & \begin{array}{ccc} L[1] & K[1] & L \end{array} \\ \begin{array}{c} L[1] \\ K[1] \\ L \end{array} & \begin{bmatrix} g(M[1]) & 0 & 0 \\ 0 & f(M[1]) & 0 \\ g(M'[1]_\vartriangle M'') & \underline{u}(M'[1]; M'') & g(M) \end{bmatrix} \end{array}$$

Here we let $_\vartriangle$ mean, if $\mathrm{tm}(M') + 1 = \mathrm{in}(M'') = \ell$, taking the diagonal extension at $K^{\ell+1}$ and putting the sign $(-1)^\ell$ (*not* $(-1)^{\ell+1}$) . These elements are the primary ones, and one must take the $\sigma$-consistent prolongation with respect to the decomposition of $C_{\alpha(u)}$ into three summands.

Define morphisms $\psi : C_{\alpha(u)} \to K[1]$ and $\phi : K[1] \to C_{\alpha(u)}$ as follows. The $\psi$ is represented by

$$\underline{\psi}(M;N) = \begin{array}{cccc} & L[1] & K[1] & L \\ K[1] & [\ 0 & (f[1])\tilde{}(M;N) & 0] \end{array} \in F(C^M_{\alpha(u)}; K[1]^N)\ .$$

One verifies $\underline{\psi}$ is in $\mathbb{F}$ and $\delta$-closed, so represents a morphism.

The $\phi$ is given by the elements $\underline{\phi}(M;N) \in F(K[1]^M; C^N_{\alpha(u)})$, each $\underline{\phi}(M;N)$ consisting of (a)-(c):

(a) the elements $\underline{u}(M[1]; N'[1]_\vartriangle N'') \in F(K[1]^M; L[1]^{N'}, L^{N''})$ for partitions $N = N' \amalg N''$ such that $N' \neq \emptyset$ and $\operatorname{tm} N' + 1 = \operatorname{in} N'' = \ell$ ($N''$ is allowed to be empty). Here $_\vartriangle$ indicates diagonal extension at $L^\ell$ and putting sign $(-1)^\ell$. The element thus obtained

$$\underline{u}(M[1]; N'[1]_\vartriangle N'') \in F(K^{M[1]}; L^{N'[1] \cup N''})$$

can be viewed as in $F(K[1]^M; L[1]^{N'}, L^{N''})$, which is part of $F(K[1]^M; C^N_{\alpha(u)})$.

(b) the elements $\underline{u}(M[1]_\vartriangle S[1]; N'') \in F(K[1]^M; K[1]^S, L^{N''})$ for partitions $N = S \amalg N''$ such that $S \neq \emptyset$ and $\operatorname{tm} M = \operatorname{in} S = \ell$ ($N''$ may be empty). Here $_\vartriangle$ means diagonal extension at $K^{\ell+1}$ and putting sign $(-1)^\ell$ (*not* $(-1)^{\ell+1}$) . One must view the element in $F(K^{M[1] \cup S[1]}; L^{N''})$ as in $F(K[1]^M; K[1]^S, L^{N''})$, which is part of $F(K[1]^M; C^N_{\alpha(u)})$.

(c) the elements $(f[1])\tilde{\ }(M;N) \in F(K[1]^M; K[1]^N)$.

These primary elements are $\sigma$-consistent and $\delta$-closed; taking the $\sigma$-consistent prolongation we obtain elements, still denoted $\underline{\phi}(M;N) \in F(K[1]^M; C^N_{\alpha(u)})$, that are $\sigma$-consistent and $\delta$-closed.

One can verify

(i) $\alpha(\alpha(u)) \cdot \psi = \beta(u)$.

(ii) $\phi \cdot \beta(\alpha(u)) = -u[1]$.

(iii) $\phi \cdot \psi = id$.

(iv) $\psi \cdot \phi = id$.

We write out the proof of (i) and (ii) only. $\alpha(\alpha(u))$ is represented by $\underline{\alpha(\alpha)}(M;N) \in F(C^M_u; C^N_{\alpha(u)})$ given as

$$\underline{\alpha(\alpha)}(M;N) = \begin{array}{c c} & \begin{array}{cc} K[1] & L \end{array} \\ \begin{array}{c} L[1] \\ K[1] \\ L \end{array} & \begin{bmatrix} 0 & 0 \\ f(M[1]_\vartriangle N[1]) & 0 \\ * & g(M_\vartriangle N) \end{bmatrix} \end{array}$$

(we will not need the precise form of the component $*$). So the only non-zero component of $\underline{\alpha(\alpha)} \otimes \underline{\psi} \in \mathbb{G}(C_u, C_{\alpha(u)}) \tilde{\otimes} \mathbb{G}(C_{\alpha(u)}, K[1])$ is in $\mathbb{G}(K[1], K[1]) \tilde{\otimes} \mathbb{G}(K[1], K[1])$, which is

$$\sum f(M[1]_\vartriangle N[1]) \otimes f(\bar{N}[1]_\vartriangle R[1]) \ ,$$

where $(M;N)$ and $(\bar{N};R)$ vary over free double sequences. Hence

$$\rho(\underline{\alpha(\alpha)} \otimes \underline{\psi}) = \sum_{\operatorname{tm} N = \operatorname{in} \bar{N}} f(M[1]_\vartriangle N[1]) \otimes f(\bar{N}[1]_\vartriangle R[1]) \ .$$

Let $\underline{W} \in \mathbb{G}(C_u, C_{\alpha(u)}, K[1], \emptyset)$ be the element with primary parts

$$\underline{W}(M;N;R) = f(M[1]_\vartriangle N[1]_\vartriangle R[1]) \in F(K[1]^M; K[1]^N; K[1]^R) \ .$$

One then has $\boldsymbol{\sigma}_2(\underline{W}) = \rho(\underline{\alpha(\alpha)} \otimes \underline{\psi})$ (it is enough to check this for primary parts). Thus $\alpha(\alpha(u)) \cdot \psi$ is represented by $\overline{\Pi_2(\underline{W})}$. The only non-zero terms come from $\underline{W}(M;N;R)$ with $|N| = 1$ and $\operatorname{tm} M = \operatorname{in} N = \ell$, so we have

$$\begin{aligned} \Pi_2(\underline{W})(M;R) &= (-1)^\ell \varphi_{K^{\ell+1}}(f(M[1]_\vartriangle N[1]_\vartriangle R[1])) \\ &= f(M[1]_\vartriangle R[1]) \ . \end{aligned}$$

This represents $\beta(u)$.

To prove (ii), the map $\beta(\alpha)$ is represented by elements

$$\underline{\beta(\alpha)}(M;N) = \begin{array}{c|ccc} & L[1] & K[1] & L \\ K[1] & [\ (g[1])\tilde{}(M;N) & 0 & 0] \end{array} \in F(C^M_{\alpha(u)}; K[1]^N) \ .$$

Note $(g[1])\tilde{}(M;N) = g(M[1]_\vartriangle N[1])$. So $\underline{\phi} \otimes \underline{\beta(\alpha)} \in \mathbb{G}(K[1], C_{\alpha(u)}) \tilde{\otimes} \mathbb{G}(C_{\alpha(u)}, L[1])$ has non-zero components only in $F(K[1]^M; L[1]^N) \otimes F(L[1]^{\bar{N}}; L[1]^R)$, which are

$$\sum \underline{u}(M[1]; N[1]) \otimes g(\bar{N}[1]; R[1])$$

the sum over free double sequences $(M;N)$ and $(\bar{N};R)$. So $\rho(\underline{\phi} \otimes \underline{\beta(\alpha)})$ is the sum of terms with restriction $\operatorname{tm} N = \operatorname{in} \bar{N}$. Let $\underline{W} \in \mathbb{G}(K[1], L[1], L[1], \emptyset)$ be the element with components

$$\underline{W}(M;N;R) = \underline{u}(M[1]; N[1]_\vartriangle R[1])$$

the meaning of $_\vartriangle$ being as above. Then one has $\boldsymbol{\sigma}_2(\underline{W}) = \rho(\underline{\phi} \otimes \underline{\beta(\alpha)})$. Further one shows $\Pi_2(\underline{W})(M;R)) = \underline{u}(M[1]; R[1])$ as before. So $\phi \cdot \beta(\alpha)$ is represented by this. On the other hand, recall $u[1]$ is represented by$-\underline{u}(M[1]; N[1])$, hence $\phi \cdot \beta(\alpha) = -u[1]$.

# 4 The triangulated category of motives over a variety

(4.1) Let $S$ be a quasi-projective variety over a field $k$. We take the quasi DG category of symbols $Symb(S)$, recalled in (1.7), and apply the results of the previous sections. We obtain the quasi DG category $Symb(S)^\Delta$, and then its homotopy category $Ho(Symb(S)^\Delta)$.

**Definition.** We set $\mathcal{D}(S) = Ho(Symb(S)^\Delta)$. This is a triangulated category. We call this the *triangulated category of mixed motives over $S$.*

The next theorem follows from (1.7) and Main Theorem.

(4.2) **Theorem.** *For $X$ in* (Smooth/$k$, Proj/$S$) *and $r \in \mathbb{Z}$, there corresponds an object $h(X/S)(r) := (X/S, r)[-2r]$ in $\mathcal{D}(S)$. For two such objects we have*

$$\operatorname{Hom}_{\mathcal{D}(S)}(h(X/S)(r)[2r], h(Y/S)(s)[2s-n]) = \operatorname{CH}_{\dim Y - s + r}(X \times_S Y, n)$$

*the right hand side being the higher Chow group of the fiber product $X \times_S Y$.*

*There is a functor*

$$h : (\mathrm{Smooth}/k\,, \mathrm{Proj}/S)^{opp} \to \mathcal{D}(S)$$

*that sends* $X$ *to* $h(X/S)$*, and a map* $f : X \to Y$ *to the class of its graph* $[\Gamma_f] \in \mathrm{CH}_{\dim X}(Y \times_S X)$.

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