The explicit Hodge complexes I

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Let (X, H) be a pair consisting of a smooth variety and a normal crossing divisor. We will describe a particular type of Hodge complex that calculates the Hodge structure of the cohomology of the pair. The Hodge complex is explicit in that it only uses only (1) the complex of topological chains, (2) the complex of differential forms on X, possibly with logarithmic singularities, and the maps given by integration. The construction is based on the Cauchy-Stokes formula, a combination of the Cauchy formula and the Stokes formula, and involves the dual of the complex of logarithmic forms, which should be viewed as "currents" where we allow the test forms to have logarithmic singularities.

The comparison to the Hodge complex of Deligne and Beilinson is the main theorem.

1 Complexes of topological chains and integration

(1.1) Let Λ be the ring $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{C}$. We consider complexes of Λ-modules, and complexes of sheaves of Λ-modules on a topological space X. For a sheaf $\mathcal{F}$ of Λ-modules and an open set U of X, the module $\mathcal{F}(U)$ may be also written $\Gamma(U, \mathcal{F})$ or $\Gamma(\mathcal{F}|U)$.

For a sheaf on X, let $\mathsf{C}^\bullet(\mathcal{F})$ be its canonical resolution by Godement, [Br], [Go].

(1.2) Assume that X is locally compact Haudorff space. We take $\Lambda = \mathbb{Z}$ for exposition, but one may take any principal ideal domain (in particular a field) for Λ. For a complex of c-soft sheaves of $\mathbb{Z}$-modules, we recall the definition of its dual.

For a complex $K^\bullet$ of $\mathbb{Z}$-modules, its dual $D(K^\bullet)$ is defined by

$$D(K^\bullet) = \mathrm{Hom}(K^\bullet, I^\bullet)\,,$$

where $I^\bullet$ is the complex $[\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}]$ concentrated in degrees 0 and 1. For $f \in \mathrm{Hom}(K^\bullet, I^\bullet)$ and $x \in K^\bullet$, df is defined by the formula

$$(df)(x) = (-1)^{|f|+1} f(dx) + d(f(x))$$

(where $|f|$ denotes the degree of f).

For $\mathcal{L}^\bullet$ a complex of c-soft sheaves on X, its dual $\mathcal{D}(\mathcal{L}^\bullet)$ is the complex of flabby sheaves given by

$$U \mapsto \mathrm{Hom}(\Gamma_c(U, \mathcal{L}^\bullet), I^\bullet) = D\Gamma_c(U, \mathcal{L}^\bullet)\,,$$

see [Br-V, §2].

(1.3) Next assume X is a locally compact Hausdorff topological space, which satisfies the second axiom of countability, and which is locally contractible, and $\dim_{\mathbb{Z}} X < \infty$. (See [Br, II-§16] for the notion of cohomological dimension.)

Let $S_\bullet(X)$ (resp. $S^\bullet(X)$) be the chain complex (resp. cochain complex) of singular chains (resp. singular cochains) on X. The boundary map of $S_\bullet(X)$ is written ∂ and the coboundary map of $S^\bullet(X)$ is written d. The coefficient ring for these is $\mathbb{Z}$.

The homology of $S_\bullet(X)$ is canonically isomorphic to the singular homology $H_\bullet(X;\mathbb{Z})$, and the cohomology of $S^\bullet(X)$ is isomorphic to the singular cohomology $H^\bullet(X;\mathbb{Z})$.

One also has the chain complex of *locally finite* singular chains of X, denoted by $\tilde{S}_\bullet(X)$. Its homology is identified with the locally finite singular homology $H^{lf}_\bullet(X;\mathbb{Z})$.

We define a subcomplex $\check{S}^\bullet(X)$ of $S^\bullet(X)$ by

$$\check{S}^p(X) = \varinjlim S^p(X, X-K)\,.$$

where K varies over the compact subsets of X. We denote the cohomology of $\check{S}^\bullet(X)$ by $H^\bullet_c(X)$.

For an open set U of X, let $S^\bullet(U)$ be the complex of singular cochains, and $\mathcal{S}^\bullet$ be its sheafication (the singular cochain sheaf). There is a canonical map

$$\theta : S^\bullet(U) \to \Gamma(U, \mathcal{S}^\bullet)$$

which is a surjective quasi-isomorphism.

(1.4) The assignment $U \mapsto S^p(U)$, where $S^p(U)$ is the group of singular p-cochains on U, gives a presheaf on X, and let $\mathcal{S}^p$ be the associated sheaf. Thus we have the differential sheaf $\mathcal{S}^\bullet$, called the singular cochain sheaf on X ([Br- Chap. I, §7]).

One verifies that $\mathcal{S}^\bullet$ is a resolution of the constant sheaf $\mathbb{Z}$ on X by flabby sheaves. As a consequence there is a canonical identification

$$H_p(X;\mathbb{Z}) = H_p\Gamma(X, \mathcal{D}(\mathcal{S}^\bullet))$$

between the Borel-Moore homology of X with coefficient $\mathbb{Z}$, and the homology of the complex $\Gamma(X, \mathcal{D}(\mathcal{S}^\bullet))$.

The map θ gives by restriction the map

$$\theta : \check{S}^\bullet(U) \to \Gamma_c(U, \mathcal{S}^\bullet)$$

which is a surjective and quasi-isomorphism.

Passing to the dual we have a quasi-morphism

$$\theta' : \Gamma(U, \mathcal{D}(\mathcal{S}^\bullet)) \to D(\check{S}^\bullet(U))\,.$$

We define a map $\xi : \tilde{S}_\bullet(X) \to \mathrm{Hom}(\check{S}^\bullet(X), \mathbb{Z}) \subset D\check{S}^\bullet(X)$ as follows. Let $\alpha \in \tilde{S}_m(X)$. For $u \in \check{S}^m(X)$, let K be compact such that $u \in S^m(X, X-K)$, write $\alpha = \alpha' + \alpha''$ with $\alpha' \in S_m(X)$, $\alpha'' \in \tilde{S}_m(X-K)$, and define $\xi(\alpha) \in \mathrm{Hom}(\check{S}^m(X), \mathbb{Z})$ by

$$\langle \xi(\alpha), u \rangle = (-1)^m \langle u, \alpha' \rangle \,.$$

This is well-defined independent of the choice of K and the decomposition of α; one also verifies that it gives a map of complexes.

The maps

$$\tilde{S}_\bullet(X) \xrightarrow{\xi} D\check{S}^\bullet(X) \xleftarrow{\theta'} \Gamma(X, \mathcal{D}(\mathcal{S}^\bullet))$$

turn out to be quasi-isomorphisms. They induce isomorphisms

$$H_p^{lf}(X) \cong H_p D\check{S}^\bullet(X) \cong H_p(X)\,.$$

This is the *canonical identification* of the locally finite singular homology and the Borel-Moore homology.

(1.5) Let now X be a real analytic manifold (satisfying the second axiom of countablity). Instead of continuous singular chains one may consider subanalytic chains; one has the resulting complex of subanalytic singular chains $S_\bullet^{\mathrm{an}}(X)$. There are also the complex of smooth singular cochains $S_{\mathrm{an}}^\bullet(X)$, the complex of locally finite smooth chains $\widetilde{S}_\bullet^{\mathrm{an}}(X)$, the complex $\check{S}_{\mathrm{an}}^\bullet(X)$ of smooth cochains which vanish outside the complement of a compact set.

The canonical maps between smooth and continuous theories, $S_\bullet^{\mathrm{an}}(X) \to S_\bullet(X)$ and $S^\bullet(X) \to S_{\mathrm{an}}^\bullet(X)$, are quasi-isomorphisms; the same holds for $\widetilde{S}_\bullet(X)$ and $\check{S}^\bullet(X)$ and their smooth counterparts.

One has facts and results for these complexes parallel to those for continuous chains and cochains.

- Let $U \mapsto S_{\mathrm{an}}^p(U)$ be the presheaf of smooth singular cochains, and $\mathcal{S}_{\mathrm{an}}^p$ be its sheafication. There is a canonical map $\theta : S_{\mathrm{an}}^\bullet(U) \to \Gamma(U, \mathcal{S}_{\mathrm{an}}^\bullet)$ which is a surjective qusi-isomorphism.
- The complex $\mathcal{S}_{\mathrm{an}}^\bullet$ is resolution of the sheaf $\mathbb{Z}$ by flabby sheaves. Thus its dual $\mathcal{D}(\mathcal{S}_{\mathrm{an}}^\bullet)$ serves to calculate the Borel-Moore homology of X.
- One has a surjective map $\theta : \check{S}_{\mathrm{an}}^\bullet(U) \to \Gamma_c(U, \mathcal{S}_{\mathrm{an}}^\bullet)$ which is a surjective qusi-isomorphism. Its dual $\theta' : \Gamma(U, \mathcal{D}(\mathcal{S}_{\mathrm{an}}^\bullet)) \to D(\check{S}_{\mathrm{an}}^\bullet(U))$ is a quasi-isomorphism.
- There is a map $\xi : \tilde{S}_\bullet^{\mathrm{an}}(X) \to D\check{S}_{\mathrm{an}}^\bullet(X)$, and the maps

$$\tilde{S}_\bullet^{\mathrm{an}}(X) \xrightarrow{\xi} D\check{S}_{\mathrm{an}}^\bullet(X) \xleftarrow{\theta'} \Gamma(X, \mathcal{D}(\mathcal{S}_{\mathrm{an}}^\bullet))$$

are quasi-isomorphisms.

- There is a canonical map of complexes $\mathcal{S}^\bullet \to \mathcal{S}_{\mathrm{an}}^\bullet$, and the induced maps make the diagram

$$\begin{array}{ccccc}
\tilde{S}_\bullet^{\mathrm{an}}(X) & \xrightarrow{\xi} & D\check{S}_{\mathrm{an}}^\bullet(X) & \xleftarrow{\theta'} & \Gamma(X, \mathcal{D}(\mathcal{S}_{\mathrm{an}}^\bullet)) \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{S}_\bullet(X) & \xrightarrow{\xi} & D\check{S}^\bullet(X) & \xleftarrow{\theta'} & \Gamma(X, \mathcal{D}(\mathcal{S}^\bullet))
\end{array}$$

commute.

In the sequel the subanalytic theories play major roles. For the subanalytic theories we drop the sub/super-script an and denote them $S_\bullet$, $S^\bullet$, $\mathcal{S}^\bullet$, etc.

(1.6) Assume that X is an abstract simplicial complex which is assumed to be locally finite, countable, and of finite dimension. The geometric realization $|X|$ will be also written X.

By $C_\bullet(X)$ we denote the complex of *ordered* simplicial chains, and $C^\bullet(X)$ its dual (see [Mu, p. 76], [Sp. p.170]). Recall that $C_m(X)$ is the free abelian group generated by $(v_0, \cdots, v_m)$, where $v_0, \cdots, v_m$ are vertices of X (repetition allowed) spanning a simplex of dimension $\leq m$. Let $\tilde{C}_\bullet(X)$ be the complex of (locally finite) infinite ordered simplicial chains. We let $\check{C}^\bullet(X) \subset C^\bullet(X)$ denote the subcomplex of cochains u satisfying the following condition: there exists a finite subcomplex outside which u vanishes.

The homology of the complex $C_\bullet(X)$ (resp. $\tilde{C}_\bullet(X)$) is the simplicial homology (resp. locally finite simplicial homology) of X. When convenient, we write $H_m^{\mathrm{simp}}(X)$ for $H_m(\tilde{C}_\bullet(X))$.

(1.7) In what follows we assume that X is a real analytic manifold, equipped with a subanalytic triangulation. There are natural maps of complexes, which are known to be quasi-isomorphisms, $C_\bullet(X) \to S_\bullet(X)$, $\tilde{C}_\bullet(X) \to \tilde{S}_\bullet(X)$ and $S^\bullet(X) \to C^\bullet(X)$, $\check{S}^\bullet(X) \to \check{C}^\bullet(X)$.

There is a map of complexes $\xi : \tilde{C}_\bullet(X) \to D\check{C}^\bullet(X)$ defined as the map ξ for singular theory. The diagram

$$\begin{array}{ccc} \tilde{C}_\bullet(X) & \xrightarrow{\xi} & D\check{C}^\bullet(X) \\ \downarrow & & \downarrow \\ \tilde{S}_\bullet(X) & \xrightarrow{\xi} & D\check{S}^\bullet(X) \end{array}$$

commutes.

In each of the chain complex theories $S_\bullet(X)$, $S_\bullet(X)$, $\mathcal{S}^\bullet$ and $C_\bullet(X)$, We show that there is cap product operation, and they are naturally compatible with each other.

For the subanalytic singular theory there is the cap product pairing

$$\cap : \tilde{S}_\bullet(X) \otimes S^\bullet(X) \to \tilde{S}_\bullet(X)\,.$$

By definition it sends $\alpha \otimes u \in \tilde{S}_m(X) \otimes S^p(X)$ with $\alpha = \sum a_\sigma \sigma \in \tilde{S}_m(X)$, $u \in S^p(X)$ to

$$\alpha \cap u = \sum a_\sigma(\sigma \cap u)\,,$$

where for s simplex $\sigma = [v_0, \cdots, v_m]$, one has $\sigma \cap u := u([v_0, \cdots, v_p])[v_p, \cdots, v_m]$.

The same formula defines its smooth variant $\cap : \tilde{S}_\bullet(X) \otimes S^\bullet(X) \to \tilde{S}_\bullet(X)$, and simplicial variant $\tilde{C}_\bullet(X) \otimes C^\bullet(X) \to \tilde{C}_\bullet(X)$.

Similarly one has a pairing

$$\cap : \mathcal{D}(\mathcal{S}) \otimes \mathcal{S}^\bullet \to \mathcal{D}(\mathcal{S})\,;$$

it sends $f \cap s$ for f of degree m and s of degree p is defined by

$$\langle f \cap s, t\rangle = (-1)^{mp}\langle f, s \cup t\rangle\,.$$

By the same formula one defines a map

$$\cap : D\check{S}^\bullet(X) \otimes S^\bullet(X) \to D\check{S}^\bullet(X)\,.$$

These pairing are related via the commutative diagrams

$$\begin{tikzcd}
\tilde{C}_\bullet(X) \otimes & C^\bullet(X) \arrow[r, "\cap"] & \tilde{C}_\bullet(X) \arrow[d] \\
\tilde{S}_\bullet(X) \otimes \arrow[from=u, d] & S^\bullet(X) \arrow[u] \arrow[r, "\cap"] & \tilde{S}_\bullet(X)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Gamma(X, \mathcal{D}(\mathcal{S}^\bullet)) \otimes \arrow[d, "\theta'"'] & \Gamma(X, \mathcal{S}^\bullet) \arrow[r, "\cap"] & \Gamma(X, \mathcal{D}(\mathcal{S}^\bullet)) \arrow[d, "\theta'"] \\
D\check{S}^\bullet(X) \otimes & S^\bullet(X) \arrow[u, "\theta"'] \arrow[r, "\cap"] & D\check{S}^\bullet(X) \\
\tilde{S}_\bullet(X) \otimes \arrow[u, "\xi"] & S^\bullet(X) \arrow[u, equal] \arrow[r, "\cap"] & \tilde{S}_\bullet(X)\,. \arrow[u, "\xi"']
\end{tikzcd}$$

(1.8) Assume X is an oriented real analytic manifold of real dimension m, equipped with a triangulation. The quasi-isomorphisms

$$\tilde{C}_\bullet(X) \to \widetilde{S}_\bullet(X) \to D\check{S}(X) \leftarrow \Gamma(X, \mathcal{D}(\mathcal{S}))$$

induces isomorphisms

$$H_m\tilde{C}_\bullet(X) \cong H_m\widetilde{S}_\bullet(X) \cong H_m\Gamma(X, \mathcal{D}(\mathcal{S})) = H_m(X)\,.$$

The fundamental class $\Gamma \in H_m(X)$ of X may be represented by cycles of dimension m, namely by

$$\gamma^{\mathrm{simp}} \in \tilde{C}_m(X), \quad \gamma \in \widetilde{S}_m(X) \quad \text{and} \quad \underline{\gamma} \in \Gamma(X, \mathcal{D}(\mathcal{S})^{-m})\,.$$

Note we may (and will) take $\gamma = \gamma^{\mathrm{simp}}$.

We obtain the map of complexes

$$\gamma^{\mathrm{simp}} \cap (-) : C^\bullet(X) \longrightarrow \tilde{C}_\bullet(X)[-m]\,,$$

and similarly $\gamma \cap (-) : S^\bullet(X) \longrightarrow \tilde{S}_\bullet(X)[-m]$, and

$$\underline{\gamma} \cap (-) : \Gamma(X, \mathcal{S}^\bullet) \longrightarrow \Gamma(X, \mathcal{D}(\mathcal{S}^\bullet))[-m]\,.$$

When convenient these maps will be all denoted by the symbol κ.

The square diagram

$$\begin{array}{ccc} C^\bullet(X) & \xrightarrow{\gamma^{\text{simp}}\cap} & \widetilde{C}_\bullet(X) \\ \uparrow & & \downarrow \\ S^\bullet(X) & \xrightarrow{\gamma\cap} & \widetilde{S}_\bullet(X) \end{array}$$

commutes since we took $\gamma = \gamma^{\text{simp}}$. Also in the diagram

$$\begin{array}{ccc} S^\bullet(X) & \xrightarrow{\gamma\cap} & \widetilde{S}_\bullet(X)[-m] \\ \| & & \downarrow{\scriptstyle\xi} \\ S^\bullet(X) & \xrightarrow{\xi(\gamma)\cap} & D\check{S}^\bullet(X)[-m] \\ {\scriptstyle\theta}\downarrow & & \uparrow{\scriptstyle\theta'} \\ \Gamma(X,\mathcal{S}^\bullet) & \xrightarrow{\gamma\cap} & \Gamma(X,\mathcal{D}(\mathcal{S}))[-m] \end{array}$$

the upper square commutes and the lower square commutes up to homotopy.

We have therefore a digram, omitting the shifts $[-m]$,

$$\begin{array}{ccccc} C^\bullet(X) & \longleftarrow & S^\bullet(X) & \xrightarrow{\theta} & \Gamma(\mathcal{S}^\bullet) \\ {\scriptstyle\kappa}\downarrow & & {\scriptstyle\kappa}\downarrow & & \\ \widetilde{C}_\bullet(X) & \longrightarrow & \widetilde{S}_\bullet(X) & & \downarrow{\scriptstyle\kappa} \\ {\scriptstyle\xi}\downarrow & & {\scriptstyle\xi}\downarrow & & \\ D\check{C}^\bullet(X) & \longrightarrow & D\check{S}^\bullet(X) & \xleftarrow[\theta']{} & \Gamma(X,\mathcal{D}(\mathcal{S})) \end{array}$$

where the two squares on the left commutes, and the right square commutes up to homotopy.

(1.9) Let $\mathcal{A}^\bullet = \mathcal{A}^\bullet_X$ be the sheaf of smooth forms on X. For each open set U of X, set $S^\bullet(U)_{\mathbb{C}} = S^\bullet(U) \otimes \mathbb{C}$. There is a canonical map $c : \Gamma(U,\mathcal{A}^\bullet) \to S^\bullet_{\mathbb{C}}(U)$ given by

$$\langle c(\varphi), \alpha \rangle = \int_\alpha \varphi$$

for a subanalytic singular chain α. Composing with the map θ, we get the map of complexes $c : \Gamma(U,\mathcal{A}^\bullet) \to \Gamma(U,\mathcal{S}^\bullet)$, giving a quasi-isomorphism of complexes of sheaves $c : \mathcal{A}^\bullet_X \to \mathcal{S}^\bullet_{\mathbb{C}}$.

The map $c : \Gamma(U,\mathcal{A}^\bullet) \to S^\bullet_{\mathbb{C}}(U)$ gives by restriction $c : \Gamma_c(U,\mathcal{A}^\bullet) \to \check{S}^\bullet_{\mathbb{C}}(U)$. We also have the map $\theta : \check{S}^\bullet(U) \hookrightarrow \Gamma_c(U,\mathcal{S}^\bullet)$ for which we may take $\otimes\mathbb{C}$; we have thus maps

$$\Gamma_c(U,\mathcal{A}^\bullet) \xrightarrow{c} \check{S}^\bullet_{\mathbb{C}}(U) \xrightarrow{\theta} \Gamma_c(U,\mathcal{S}^\bullet_{\mathbb{C}})\,.$$

Passing to duals we have maps

$$\Gamma(U,\mathcal{D}(\mathcal{S}^\bullet)) \xrightarrow{\theta'} D\check{S}^\bullet(U) \xrightarrow{c'} \Gamma(U,\mathcal{D}(\mathcal{A}^\bullet))$$

(for the first two terms we need not take $\otimes\mathbb{C}$), and the composition will also be denoted c'.

Composition of the maps $\xi : \tilde{C}_\bullet(U) \to D\check{S}^\bullet(U)$ and $c' : D\check{S}^\bullet(U) \to \Gamma(U, \mathcal{D}(\mathcal{A}^\bullet))$ is denoted Φ:

$$\Phi = c' \circ \xi : \tilde{C}_\bullet(U) \to \Gamma(U, \mathcal{D}(\mathcal{A}^\bullet)) \,.$$

(1.10) Let $\mathcal{S}^\bullet \otimes_{\mathbb{Q}} \mathcal{A}^\bullet = \mathcal{S}^\bullet_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{A}^\bullet$ be the tensor product of the complexes of sheaves. There are maps

$$\mathcal{S}^\bullet \to \mathcal{S}^\bullet_{\mathbb{C}} = \mathcal{S}^\bullet_{\mathbb{C}} \otimes \mathbb{C} \to \mathcal{S}^\bullet_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{A}^\bullet \,;$$

the latter is $1 \otimes \epsilon$ where $\epsilon : \mathbb{C} \to \mathcal{A}^\bullet$ being the augmentation. By composition one has the map $\alpha : \mathcal{S}^\bullet \to \mathcal{S}^\bullet \otimes_{\mathbb{Q}} \mathcal{A}^\bullet$. Note that the induced map $\mathcal{S}^\bullet_{\mathbb{C}} \to \mathcal{S}^\bullet \otimes_{\mathbb{Q}} \mathcal{A}^\bullet$ which is a quasi-isomorphism.

The map $1 \cup c : \mathcal{S}^\bullet \otimes \mathcal{A}^\bullet \to \mathcal{S}^\bullet_{\mathbb{C}}$ is a retraction to $\alpha : \mathcal{S}^\bullet_{\mathbb{C}} \to \mathcal{S}^\bullet \otimes \mathcal{A}^\bullet$. Let

$$\lambda = \Phi \circ \kappa \circ (1 \cup c) : \mathcal{S}^\bullet \otimes \mathcal{A}^\bullet \to \mathcal{D}(\mathcal{A})[-m] \,.$$

Note then that $\lambda \circ \alpha = c' \circ \kappa$.

Therefore we obtain a diagram

$$\begin{array}{ccccc}
S^\bullet(X) & \xrightarrow{\theta} & \Gamma(X, \mathcal{S}^\bullet) & \xrightarrow{\alpha} & \Gamma(X, \mathcal{S}^\bullet \otimes_{\mathbb{Q}} \mathcal{A}^\bullet) \\
\| & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \lambda} \\
S^\bullet(X) & \xrightarrow{\kappa\theta} & \Gamma(X, \mathcal{D}(\mathcal{S}^\bullet))[-m] & \xrightarrow{c'} & \Gamma(X, \mathcal{D}(\mathcal{A}^\bullet))[-m] \\
\downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \theta'} & & \| \\
C_{m-\bullet}(X) & \xrightarrow{\xi} & D\check{S}^\bullet(X)[-m] & \xrightarrow{c'} & \Gamma(X, \mathcal{D}(\mathcal{A}^\bullet))[-m]
\end{array}$$

where all the squares commute except the lower left one, which commutes up to homotopy.

(1.11) **Definition.** Let $\mathbb{K} = \mathbb{K}(X)$ be the triple of filtered complexes

$$\left[\Gamma(X, \mathsf{C}^\bullet\mathbb{Q}) \longrightarrow \Gamma(X, \mathsf{C}^\bullet\mathcal{A}^\bullet) \longleftarrow \Gamma(X, \mathcal{A}^\bullet)\right] .$$

This is a mixed Hodge complex for X (take the trivial weight filtration). It is the same as the one in [De], except we have used the complex $\mathcal{A}^\bullet$ instead of $\Omega^\bullet$. We will call it the cohomological Hodge complex.

Consider also a triple $\overline{\mathbb{K}}(X)$ given by

$$\left[\Gamma(X, \mathcal{S}^\bullet) \xrightarrow{\alpha} \Gamma(X, \mathcal{S}^\bullet \otimes_{\mathbb{Q}} \mathcal{A}^\bullet) \xleftarrow{\beta} \Gamma(X, \mathcal{A}^\bullet)\right] .$$

which is another mixed Hodge complex.

(1.12) There is a canonical quasi-isomorphism between $\mathbb{K}(X)$ and $\overline{\mathbb{K}}(X)$.

Proof. Recall that for a complex of sheaves $\mathcal{F}$ there is a canonical map $\mathcal{F} \to \mathsf{C}^\bullet\mathcal{F}$ to its Godement resoution; also if a map of complexes $\mathcal{F} \to \mathcal{G}$ induces a map $\mathsf{C}^\bullet\mathcal{F} \to \mathsf{C}^\bullet\mathcal{G}$ by functoriality. Consider the following commutative diagram of complexes, with arrows of the kind just mentioned; the vertical and horizontal arrows are quasi-isomorphisms.

$$\begin{tikzcd}
\Gamma(X, \mathsf{C}^\bullet\mathbb{Q}) \arrow[r] \arrow[d] & \Gamma(X, \mathsf{C}^\bullet\mathcal{A}^\bullet) \arrow[d] & \Gamma(X, \mathcal{A}^\bullet) \arrow[l] \arrow[d] \\
\Gamma(X, \mathsf{C}^\bullet\mathcal{S}^\bullet) \arrow[r, "\alpha"] & \Gamma(X, \mathsf{C}^\bullet(\mathcal{S}^\bullet \otimes \mathcal{A}^\bullet)) & \Gamma(X, \mathcal{S}^\bullet \otimes \mathcal{A}^\bullet) \arrow[l, "\beta"'] \\
\Gamma(X, \mathcal{S}^\bullet) \arrow[r] \arrow[u] & \Gamma(X, \mathcal{S}^\bullet \otimes \mathcal{A}^\bullet) \arrow[u] & \Gamma(X, \mathcal{A}^\bullet) \arrow[l] \arrow[u]
\end{tikzcd}$$

The top and the bottom rows are $\mathbb{K}(X)$ and $\overline{\mathbb{K}}(X)$, respectively. $\square$

2 The comparison in case $H = \emptyset$

(2.1) Assume that X is triangulated, and given a partial ordering on vertices (which is assumed to be a total ordering on each simplex) in the triangulation. One has the complex of simplicial chains $C_\bullet(X)$ and the complex of simplicial cochains $C^\bullet(X)$. Then there is a natural map of complexes $S^\bullet(X)$ to $C^\bullet(X)$.

One has the map of complexes $\kappa : S^\bullet(X) \to C_{2n-\bullet}(X)$ which sends $u \in S^p(X)$ to $(-1)^{s(p)}[X] \cap u \in C_{2n-p}(X)$.

In general, if $K^\bullet$ is a cohomological complex of $\mathbb{C}$-vector spaces, we define its dual $D(K^\bullet)$ to be the complex with $D(K)^p = \mathrm{Hom}(K^{-p}, \mathbb{C})$ and differential $f \mapsto f \circ d$.

Let $\Phi : C_\bullet(X) \to D(A^\bullet(X))$ be the map of complexes which sends sends $\alpha \in C_p(X)$ to $\int_\alpha(-)$. Let

$$\lambda = \Phi \circ \kappa \circ (1 \cup c) : \Gamma(X, \mathcal{S}^\bullet \otimes_{\mathbb{Q}} \mathcal{A}^\bullet) \to DA^{2n-\bullet}(X)\,.$$

Also, let $\mathcal{P} : A^\bullet(X) \to D(A^\bullet(X)[2n])$ be the map which sends ψ to $\mathcal{P}(\psi)$, where

$$\langle \mathcal{P}(\psi), \varphi \rangle := (-1)^{s(p)} \int_X \psi \wedge \varphi$$

for ψ of degree p, φ of degree $2n-p$, where $s(p) := (-1)^{\frac{p(p+1)}{2}}$. Then $\mathcal{P}$ is a map of complexes. By Poincaré duality this is a quasi-isomorphism.

One has a diagram of homomorphisms of complexes:

$$\begin{tikzcd}
S^\bullet(X) \arrow[r, "\alpha"] \arrow[d, "\kappa"'] & \Gamma(X, \mathcal{S}^\bullet \otimes_{\mathbb{Q}} \mathcal{A}^\bullet) \arrow[d, "\lambda"'] & A^\bullet(X) \arrow[l, "\beta"'] \arrow[d, "id"] \\
C_{2n-\bullet}(X) \arrow[r, "\Phi"] & DA^{2n-\bullet}(X) & A^\bullet(X) \arrow[l, "\mathcal{P}"']
\end{tikzcd}$$

The left square commutes.

We give a variant of a theorem of Guggenheim [Gu].

Let X be a real analytic manifold equipped with a semi-analytic triangulation K. Consider the category $\mathcal{C}at(X)$ associated with X. The objects are subspaces X, and simplices σ of K; the arrows are, besides the identities, inclusions between the subspaces, $\iota_\sigma : \sigma \to X$ and $\iota_{\sigma,\sigma'} : \sigma \to \sigma'$. The complexes $A^\bullet$ and $C^\bullet$ are contravariant functors on this category, and the map $c : A^\bullet \to C^\bullet$ is a natural transformation.

(2.2) **Proposition.** *There exist maps $\rho_M : A^\bullet(M) \otimes A^\bullet(M) \to C^\bullet_{\mathbb{C}}(M)$ of degree -1, one for each object M of $\mathcal{C}at(X)$, that are functorial in M, and subject to the identities*

$$(d\rho + \rho d))(\psi \otimes \varphi) = c(\psi \wedge \varphi) - c(\psi) \cup c(\varphi)\,.$$

Proof. Functoriality means that for each arrow $\iota : N \to M$ in $\mathcal{C}at(X)$ one has the identity

$$\iota^* \rho_M = \rho_N \iota^* .$$

Let $A^{(2),\bullet}(M) = A^\bullet(M) \otimes A^\bullet(M)$. Denote by m either of the wedge product $A^\bullet(M) \otimes A^\bullet(M) \to A^\bullet(M)$ or the cup product $C^\bullet(M) \otimes C^\bullet(M) \to C^\bullet(M)$.

Write ρ^m for the restriction of ρ to the degree m part: $\rho^m : A^{(2),m}(M) \to C_{\mathbb{C}}^{m-1}(M)$. The condition required for ρ^m is

$$d\rho^{m-1} + \rho^m d = -m(c \otimes c) + cm . \qquad (*)_{m-1}$$

With $\rho^m = 0$ for $m \leq 0$, the identity $(*)_{m-1}$ obviously holds for $m \leq 0$. We will find ρ^m, $m \geq 1$, by induction. Let

$$\theta_M^{m-1} = d\rho^{m-1} - m(c \otimes c) + cm : A^{(2),m-1}(M) \to C_{\mathbb{C}}^{m-1}(M) .$$

Observe that $m(c \otimes c) = cm$ on $A^{(2),0}(M)$; hence when $m = 1$, $\theta^0 = -m(c \otimes c) + cm = 0$. Also note that for an arrow $\iota : N \to M$ the identity $\iota^* \theta^{m-1} = \theta^{m-1} \iota^*$ holds. We are to find ρ^m such that $\rho^m d = \theta^{m-1}$ holds; also ρ^m should satisfy $\iota^* \rho^m = \rho^m \iota^*$.

When M is a simplex $\sigma = \Delta^p$, there is a map $S : A^{(2),\bullet}(\Delta^p) \to A^{(2),\bullet}(\Delta^p)$ of degree -1 such that

$$dS + Sd = 1 - r_b^*$$

where $r_b : \Delta^p \to \{b\} \subset \Delta^p$ is the contraction map to a base point b of Δ^p.

We have

$$\theta^{m-1} r_b^* = 0$$

for any m. Indeed for $m = 1$, $\theta^0 = 0$; if $m \geq 2$, then one has $r_b^* = 0$ on $A^{(2),m-1}(\Delta^p)$.

For $a \in A^{(2),m}(M)$, define an element $\rho_M^m(a) \in C_{\mathbb{C}}^{m-1}(M)$ as follows. Let τ be an $(m-1)$-simplex of M, and $\iota_\tau : \tau \to M$ the inclusion. The value of $\rho^m(a)$ at τ is given by

$$\langle \rho_M^m(a), \tau \rangle = \langle \theta_\tau^{m-1} S \iota_\tau^* a, \tau \rangle \in \mathbb{C} .$$

Here $\iota_\tau^* a \in A^{(2),m}(\Delta^{m-1})$, and the maps S, θ^{m-1} are as in the diagram

$$\begin{tikzcd}
C_{\mathbb{C}}^{m-1}(\Delta^{m-1}) & \\
A^{(2),m-1}(\Delta^{m-1}) \arrow[u, "\theta^{m-1}"] \arrow[r, "d"] & A^{(2),m}(\Delta^{m-1}) . \arrow[l, bend left, "S"]
\end{tikzcd}$$

(i) For an arrow $\iota : N \to M$ in $\mathcal{C}at(X)$ one has $\iota^* \rho_M^m = \rho_N^m \iota^*$.

Indeed for an $(m-1)$ simplex τ of N,

$$\begin{aligned}
\langle \iota^* \rho_M^m(a), \tau \rangle &= \langle \rho_M^m(a), \tau \rangle \\
&= \langle \theta_\tau^{m-1} S \iota_\tau^* a, \tau \rangle ,
\end{aligned}$$

which equals $\langle \rho_N^m(\iota^* a), \tau\rangle$.

(ii) One has $\theta^{m-1} d = 0$.

Both c and m commute with d. Using also $(*)_{m-2}$ we have

$$\begin{aligned}
\theta^{m-1} d &= -d\rho^{m-1} d - m(c \otimes c) d + cmd \\
&= -d\rho^{m-1} d - dm(c \otimes c) + dcm \\
&= d(-\rho^{m-1} d - m(c \otimes c) + cm) \\
&= d(d\rho^{m-2}) = 0\,.
\end{aligned}$$

(iii) One has $\theta^{m-1} Sd = \theta^{m-1}$ as maps $A^{(2),m-1}(M) \to C_{\mathbb{C}}^{m-1}(M)$.

In the identity

$$\theta^{m-1} Sd = \theta^{m-1}(-dS + 1 - r_b^*)$$

the first term on the right is zero by (ii), and the term $\theta^{m-1} r_b^*$ is also zero as noted before.

(iv) The identity $\rho_M^m d = \theta_M^{m-1}$ holds.

For an $(m-1)$ simplex τ we have:

$$\begin{aligned}
\langle \rho^m d(a), \tau\rangle &= \langle \theta_\tau^{m-1} S \iota_\tau^*(da), \tau\rangle \\
&= \langle \theta_\tau^{m-1} Sd(\iota_\tau^* a), \tau\rangle \\
&= \langle \theta_\tau^{m-1}(\iota_\tau^* a), \tau\rangle && \text{[by (iii)]} \\
&= \theta_M^{m-1}(a)\,.
\end{aligned}$$

□

The next result concerns the indeterminacy of the map ρ. If ρ' is another map of degree -1 satisfying the requirement of the previous proposition, one has in particular the identity $d\rho + \rho d = d\rho' + \rho' d$.

(2.3) **Proposition.** *Let ρ and ρ' be functorial maps $A^\bullet(X) \otimes A^\bullet(X) \to C_{\mathbb{C}}^\bullet(X)$ of degree -1 such that $d\rho + \rho d = d\rho' + \rho' d$. Then there exists a map $\pi : A^\bullet(X) \otimes A^\bullet(X) \to C_{\mathbb{C}}^\bullet(X)$ of degree -2, functorial in X, such that*

$$d\pi + \pi d = \rho - \rho'\,.$$

Proof. Let $\pi^m : A^{(2),m}(X) \to C_{\mathbb{C}}^{m-2}(X)$ be the degree m part of π. Set $\pi^m = 0$ for $m \leq 1$. By induction on m we will find π^m such that

$$d\pi^{m-1} + \pi^m d = \rho^{m-1} - {\rho'}^{\,m-1} : A^{(2),m-1}(X) \to C_{\mathbb{C}}^{m-2}(X) \qquad (*)_{m-1}$$

holds. For $m \leq 1$ it is trivially true. Assuming π^j for $j < m$ have been defined, let

$$\tau^{m-1} = -d\pi^{m-1} + \rho^{m-1} - \rho'^{m-1}\,.$$

The map τ^{m-1} is also functorial in X.

On Δ^{m-2} we take $S : A^{(2),m}(\Delta^{m-2}) \to A^{(2),m-1}(\Delta^{m-2})$ such that $dS + Sd = 1 - r_b^*$. One also has $\tau^{m-1} r_b^* = 0$, since r_b^* is non-zero only in degree 0, and τ^0 is trivially zero.

For any element $a \in A^{(2),m}(X)$ and $v : \Delta^{m-2} \to X$, we have $v^* a \in A^{(2),m}(\Delta^{m-2})$; we set

$$\langle \pi^m(a), v\rangle = \langle \tau^{m-1} S v^* a, 1_{\Delta^{m-2}}\rangle\,,$$

which defines an element $\pi^m(a) \in C_{\mathbb{C}}^{m-2}(X)$. The verification of the following facts are parallel to the previous case, using slightly different hypotheses.

(i) One has $f^*\pi^m = \pi^m f^*$ for a map $f : Y \to X$.

(ii) One has $\tau^{m-1} d = 0$.

This follows by substituting the definition of τ^{m-1}, using the identity $d\rho + \rho d = d\rho' + \rho' d$ and the hypothesis $(*)_{m-2}$.

(iii) One has $\tau^{m-1} S d = \tau^{m-1}$.

In the identity

$$\tau^{m-1} S d = \tau^{m-1}(-dS + 1 - r_b^*)$$

we have $\tau^{m-1} r_b^* = 0$, and also $\tau^{m-1} dS = 0$ by (ii).

(iv) The identity $\pi^m d = \tau^{m-1}$ holds.

Using (iii) and the funtoriality of π^{m-1} we have:

$$\begin{aligned}
\langle \pi^m d(a), v\rangle &= \langle \tau^{m-1} S v^*(da), 1_{\Delta^{m-1}}\rangle \\
&= \langle \tau^{m-1} S d v^*(a), 1_{\Delta^{m-1}}\rangle \\
&= \langle \tau^{m-1} v^*(a), 1_{\Delta^{m-1}}\rangle && \text{[by (iii)]} \\
&= \langle v^* \tau^{m-1}(a), 1_{\Delta^{m-1}}\rangle && \text{[by } v^*\tau^{m-1} = \tau^{m-1} v^*\text{]} \\
&= \langle \tau^{m-1}(a), v\rangle\,.
\end{aligned}$$

$\square$

(2.4) **Sign conventions for the Hom complex.** For a pair of complexes K, L, let D be the differential of the complex $\mathrm{Hom}^\bullet(K, L)$ given by $u \mapsto u \circ d_K + (-1)^{k+1} d_L \circ u$ for u of degree k. In particular for a complex K of $\mathbb{C}$-vector spaces, the complex $\mathrm{Hom}(K, \mathbb{C})$ coincides with $D(K)$ in (2.1).

Another choice of the differential, which we will use only for auxiliary purposes, is the map δ which sends $u \in \mathrm{Hom}(K^p, L^q)$ to $u \circ d_K + (-1)^p d_L \circ u$; this is obtained by viewing $\mathrm{Hom}^\bullet(K, L)$ as a double complex and taking its total complex.

There is an isomorphism of complexes

$$\iota : (\mathrm{Hom}^\bullet(K, L), D) \to (\mathrm{Hom}^\bullet(K, L), \delta)$$

that takes $f \in \mathrm{Hom}(K^p, L^q)$ to $(-1)^{s(q)} f$. (Here $s(p) := (-1)^{\frac{p(p+1)}{2}}$.)

(2.5) **Tensor and Hom adjunction.** If M is another complex, this sign convention applies to the complex $\mathrm{Hom}^\bullet(K \otimes L, M)$, as well as to $\mathrm{Hom}^\bullet(K, \mathrm{Hom}^\bullet(L, M))$. There is an isomorphism of complexes

$$\mathrm{Hom}^\bullet(K \otimes L, M) \to \mathrm{Hom}^\bullet(K, \mathrm{Hom}^\bullet(L, M))$$

after the signs are correctly chosen. Indeed for $u : K \otimes L \to M$ of degree k, let $u' : K \to \mathrm{Hom}(L, M)$ be the map given by

$$\langle u'(x), y \rangle = u(x \otimes y)\,.$$

If x is of degree p, modify this by sign and let $\tilde{u} : K \to \mathrm{Hom}^\bullet(L, M)$ be the map given by

$$\tilde{u}(x) = (-1)^{s(p+k)} u'(x) \in \mathrm{Hom}^{p+k}(L, M)\,.$$

One verifies that this gives an isomorphism of complexes, functorial in each of K, L and M.

The sign $(-1)^{s(p+k)}$ may be obtained as follows. There is an obvious isomorphism of triple complexes $\mathrm{Hom}^\bullet(K \otimes L, M) \to \mathrm{Hom}^\bullet(K, \mathrm{Hom}^\bullet(L, M))$. It induces an isomorphism of the total complexes. The total complex of the left has differential δ. The total complex on the right has differential given as follows: take the complex $(\mathrm{Hom}(L, M), \delta)$, and then take the differential δ for $\mathrm{Hom}(K. \mathrm{Hom}(L, N)$. One may then apply the isomorphism ι in (2.4) once to the left total complex, and twice on the right total complex, resulting in the sign as asserted.

We are particularly interested in the case where a map $M \to \mathbb{C}[-m]$ is given, with m an integer. In that case replacing M with $\mathbb{C}[-m]$, one is considering the isomorphism of complexes

$$\mathrm{Hom}(K \otimes L, \mathbb{C}[-m]) \to \mathrm{Hom}(K, \mathrm{Hom}(L, \mathbb{C}[-m]))\,.$$

For the target complex, there are isomorphisms of complexes

$$\mathrm{Hom}(L, \mathbb{C}[-m]) \xrightarrow{\sim} \mathrm{Hom}(L[m], \mathbb{C}) = \mathrm{Hom}(L, \mathbb{C})[-m]\,.$$

The first isomorphism sends $f \in \mathrm{Hom}^k(L, \mathbb{C}[-m]) = \mathrm{Hom}^{k-m}(L, \mathbb{C})$ to $(-1)^{mk} f \in \mathrm{Hom}^k(L[m], \mathbb{C}) = \mathrm{Hom}^{k-m}(L, \mathbb{C})$, and the second isomorphism sends $f \in \mathrm{Hom}^k(L[m], \mathbb{C}) = \mathrm{Hom}^{k-m}(L, \mathbb{C})$ to itself. If m is even, then one can write $\mathrm{Hom}(L, \mathbb{C}[-m]) = \mathrm{Hom}(L[m], \mathbb{C}) = D(L[m])$.

We summarize below what we have for the particular case $M = \mathbb{C}[-m]$ with m even.

(2.6) **Proposition.** *Let K, L be complexes and n be an integer. There is an isomorphism of complexes*

$$\operatorname{Hom}(K \otimes L, \mathbb{C}[-2n]) \to \operatorname{Hom}(K, D(L[2n]))$$

which sends u to $\tilde{u}$. One has

$$\widetilde{D(u)} = D(\tilde{u})\,.$$

If u is of degree k, then $\tilde{u}$ is also of degree k, so that $D(\tilde{u}) = \tilde{u} \circ d + (-1)^{k+1} d \circ \tilde{u}$.

(2.7) We apply this to the case $K = L = A^\bullet(X)$ and $M = S^\bullet_{\mathbb{C}}$; there is the trace map $S^\bullet_{\mathbb{C}} \to \mathbb{C}[-2n]$, which sends $u \in S^{2n}_{\mathbb{C}}$ to $\langle u, [X]\rangle$.

A map of degree k

$$u : A^\bullet(X) \otimes A^\bullet(X) \to C^\bullet_{\mathbb{C}}$$

induces a map $\tilde{u} : A(X)^\bullet \to DA^{2n-\bullet}(X)$, also of degree k; for $\psi \in A^p(X)$ one has

$$\tilde{u}(\psi)(\varphi) = (-1)^{s(p+k)} \langle u(\psi \otimes \varphi), [X]\rangle\,.$$

The map $m(c \otimes c) : A^\bullet(X) \otimes A^\bullet(X) \to S_{\mathbb{C}}{}^\bullet$ is of degree zero. One has

$$\langle (m(c\otimes c))^\sim(\psi), \varphi\rangle = (-1)^{s(p)} \langle c(\psi) \cup c(\varphi), [X]\rangle\,.$$

This coincides with $\langle \Phi\kappa c(\psi), \varphi\rangle$. Since $\lambda \circ \beta = \Phi\kappa c$, we have $(m(c \otimes c))^\sim = \lambda \circ \beta$.

Also $cm : A^\bullet(X) \otimes A^\bullet(X) \to S_{\mathbb{C}}{}^\bullet$ is of degree zero, and

$$\langle (cm)^\sim(\psi), \varphi\rangle = (-1)^{s(p)} \langle c(\psi \wedge \varphi), [X]\rangle = (-1)^{s(p)} \int_X \psi \wedge \varphi\,.$$

This coincides with $\langle \mathcal{P}(\psi), \varphi\rangle$, so we have $(cm)^\sim = \mathcal{P}$.

(2.8) **Proposition.** *One has*

$$d\widetilde{\rho} + \widetilde{\rho} d = -\lambda \circ \beta + \mathcal{P}\,.$$

Proof. The map ρ satisfies $D\rho = -m(c \otimes c) + cm$. Apply the map $u \mapsto \tilde{u}$ to both sides and use (2.6) to obtain the assertion. $\square$

3 Complexes of differential forms with logarithmic poles

(3.1) Let Λ be a field and we consider complexes $K^\bullet$ of Λ-vector spaces. For $k \in \mathbb{Z}$, we set $\Lambda(k) = \Lambda \cdot (2\pi i)^{-k}$, the one dimensional Λ-vector space with basis $(2\pi i)^{-k}$, with filtration W such that $\mathrm{Gr}_m^W = 0$ for $m \neq -2k$. When $\Lambda = \mathbb{C}$, $\mathbb{C}(k)$ has also a decreasing filtration $F^\bullet$ such that $\mathrm{Gr}_F^p = 0$ for $p \neq -k$.

Let $K = (K_\mathbb{Q}, K'_\mathbb{C}, K_\mathbb{C})$ be a $\mathbb{Q}$-mixed Hodge complex (see [Be]). Recall that $K_\mathbb{Q}$ is a complex of $\mathbb{Q}$-vector spaces equipped with a (finite increasing) filtration $W_\bullet$, $K'_\mathbb{C}$ is a complex of $\mathbb{C}$-vector spaces equipped with a filtration $W_\bullet$, and $K_\mathbb{C}$ is a complex of $\mathbb{C}$-vector spaces equipped with a filtration $W_\bullet$ and a finite decreasing filtration $F^\bullet$. There are comparison maps $(K_\mathbb{Q}, W_\bullet) \otimes \mathbb{C} \to (K'_\mathbb{C}, W_\bullet)$ and $(K_\mathbb{C}, W_\bullet) \to (K'_\mathbb{C}, W_\bullet)$.

An example of a mixed Hodge complex is the triple $(\mathbb{Q}(k), \mathbb{C}(k), \mathbb{C}(k))$ with the obvious maps.

Given a mixed Hodge complex define the shift $K[1]$ by

$$W_m(K_\mathbb{Q}[1]) = (W_{m-1}K_\mathbb{Q})[1]\,,$$

similarly for $K'_\mathbb{C}$, $K_\mathbb{C}$, and

$$F^p(K_\mathbb{C}[1]) = (F^pK_\mathbb{C})[1]\,.$$

Then $K[1]$ is a mixed Hodge complex.

If K and L are mixed Hodge complexes, their tensor product $K \otimes L$, equipped with the tensor product of the weight and Hodge filtrations, is a mixed Hodge complex. In particular, $K(i) = K \otimes \mathbb{Q}(i)$ is a mixed Hodge complex.

The dual DK of K is the mixed Hodge complex defined by

$$W_mD(K) = W_{m+2}(K/W_{-1-m}K)$$

and

$$F^pD(K_\mathbb{C}) = D(K/F^{1-p}K_\mathbb{C})\,.$$

Let

$$K = \left[\to K^i \to K^{i+1} \to \cdots\right]$$

be a finite complex of mixed Hodge complexes. Consider the total complex $\mathrm{Tot}(K_\mathbb{Q})$, which is $\bigoplus_i K^i_\mathbb{Q}[-i]$ as a graded module; equip it with the filtration $W_\bullet$ given by

$$W_m \,\mathrm{Tot}(K_\mathbb{Q}) = \bigoplus_i (W_{m+i}K^i_\mathbb{Q})[-i]\,.$$

Similarly the total complex $\mathrm{Tot}(K'_\mathbb{C})$ is equipped with the filtration $W_\bullet$. The total complex $\mathrm{Tot}(K_\mathbb{C})$ is also equipped with $W_\bullet$, but in addition there is a filtration $F^\bullet$ given by and

$$F^p \,\mathrm{Tot}(K_\mathbb{C}) = \bigoplus_i (F^pK^i_\mathbb{C})[-i]\,.$$

Then $(\mathrm{Tot}(K); W, F)$ is a mixed Hodge complex. In particular, if $u : K \to L$ is a morphism of mixed Hodge complexes, its cone C_u, with the filtrations

$$W_m C_u = (W_{m-1}K)[1] \oplus W_m L$$

and

$$F^p C_u = (F^p K)[1] \oplus F^p L\,,$$

is a mixed Hodge complex.

(3.2) Let X a smooth complete variety, and H is a simple normal crossing divisor on X; we assume that the irreducible components $H_1, \cdots, H_r$ of H are totally ordered.

For a subset I of $\{1, \cdots, r\}$, set $H_I = \cap_{i\in I} H_i$ and $H_\emptyset = X$. If $J \supset I$, there is an inclusion $H_J \to H_I$.

If I and J is a pair of the subsets $\{1, \cdots, r\}$ with $J \supset I$ and $|J| = |I| + 1$, we will write $J \rhd I$. For a subset I of $\{1, \cdots, r\}$, we set

$$H_I = \bigcap_{i\in I} H_i\,,$$

and $H_\emptyset = X$. H_I is a non-singular variety. Also let

$$\widehat{H_I} = \sum_J H_J\,,$$

where J varies over the subsets with $J \rhd I$; it is a normal crossing divisor on H_I. One thus has a smooth pair $(H_I, \widehat{H_I})$.

For an $a \geq 0$, set

$$H^{(a)} = \coprod_{|I|=a} (H_I, \widehat{H_I})$$

the disjoint union of the smooth pairs $(H_I, \widehat{H_I})$.

If $k = 0, \cdots, a$, the sum of the inclusions $H_{i_1\cdots i_a} \to H_{i_1\cdots\widehat{i_k},\cdots i_a}$ gives a map $d_k : H^{(a)} \to H^{(a-1)}$.

(3.3) For some of the facts in this subsection details may be found in [De], II, §3.

Let $\Omega^\bullet_X\langle H\rangle$ be the logarithmic de Rham complex of (X, H), namely the sheaf of holomorphic forms on X with logarithmic singularities along H. There is an increasing filtration $W_\bullet$ (the weight filtration) on this complex , and a decreasing filtration $F^\bullet$ (the Hodge filtration).

For each i one has the Poincaré residue map

$$Res_{H_i} : \Omega^\bullet_X\langle H\rangle \to \Omega^\bullet_{H_i}\langle\widehat{H_i}\rangle[-1]\,.$$

More generally, for I with $|I| = a$, a map

$$Res_{H_I} = R_I : \Omega^\bullet_X\langle H\rangle \to \Omega^\bullet_{H_I}\langle\widehat{H_I}\rangle[-a]\,,$$

It restricts to a map $W_a\Omega^\bullet_X\langle H\rangle \to \Omega^\bullet_{H_I}[-a]$ and induces an isomorphism of complexes

$$\mathrm{Gr}^W_a\,\Omega^\bullet_X\langle H\rangle \to \bigoplus_{|I|=a}\Omega^\bullet_{H_I}[-a]\,. \tag{3.3.1}$$

Similarly for a pair (I,J) with $I\subset J$, one has

$$Res_{I,J}:\Omega^\bullet_{H_I}\langle\widehat{H_I}\rangle[-|I|]\to\Omega^\bullet_{H_J}\langle\widehat{H_J}\rangle[-|J|]\,.$$

Since we have assumed the irreducible components of H are totally ordered, there is no ambiguity of signs. One also has the Hodge filtration $F^\bullet$ on the complex $\Omega^\bullet_X\langle H\rangle$.

Let $j:U=X-H\to X$ be the inclusion. One has inclusions of filtered complexes (recall there is the canonical filtration $\tau=\tau_\leq$ on a complex in a abelian category)

$$(\Omega^\bullet_X\langle H\rangle,W)\leftarrow(\Omega^\bullet_X\langle H\rangle,\tau)\to(j_*\Omega^\bullet_U,\tau)\,. \tag{3.3.2}$$

A fact of fundamental importance ([De], (3.18)) states that both maps are filtered quasi-isomorphisms.

There is also a filtration $F^\bullet$ on $\Omega^\bullet_X\langle H\rangle$ that extends the $F^\bullet$ on $\Omega^\bullet_X$, so the complex $\Omega^\bullet_X\langle H\rangle$ is bifiltered with W,F.

(3.4) *The complex* $\Omega^\bullet(X)_H$. The de Rham complex $\Omega^\bullet_X$ on X is also denoted $\Omega^\bullet(X)$; similarly $\Omega^\bullet_X\langle H\rangle$ is also written $\Omega^\bullet(X)\langle H\rangle$.

For a closed subset Z of X, $\Omega^\bullet_X|Z$ denotes the restriction of $\Omega^\bullet_X$ to Z, often viewed as a complex of sheaves on X. If Z is a smooth subvariety, there is the complex $\Omega^\bullet_Z$ of forms on Z. The induced map $\Omega^\bullet_X|Z\to\Omega^\bullet_Z$ is a quasi-isomorphism of sheaves on Z.

We modify the maps Res_I in the previous section as follows. Take I and let $|I|=a$. We define a map R_{H_I} (also written R_I) by

$$R_{H_I}=Res_{H_I}\otimes e_{-a}:\Omega^\bullet_X\langle H\rangle\to\Omega^\bullet_{H_I}\langle\widehat{H_I}\rangle(-a)[-a]$$

which sends φ to $Res_{H_I}(\varphi)\cdot e_{-a}$. With the filtration $W_\bullet$ on the target defined as in (3.1) satisfies $W_m(\Omega^\bullet_{H_I}\langle\widehat{H_I}\rangle(-a)[-a])=(W_{m-a}\Omega^\bullet_{H_I}\langle\widehat{H_I}\rangle)(-a)[-a]$, so the map R_{H_I} respects the filtrations $W_\bullet$. Also it respects the filtrations $F^\bullet$.

Similarly for subsets $I\subset J$ one has the map of complexes

$$R_{I,J}:\Omega^\bullet(H_I)\langle\widehat{H_I}\rangle(-|I|)[-|I|]\to\Omega^\bullet(H_J)\langle\widehat{H_J}\rangle(-|J|)[-|J|]$$

is bifiltered. To be precise, $R_{I,J}$ is a map of complexes of sheaves on H_I, where the target complex is identified with its direct image under the inclusion $i:H_J\to H_I$.

We will often drop the Tate twists for simplicity. If $J=I\cup\{j\}$, one may write R_j for $R_{I,J}$. We have identities (twists and shifts are omitted)

$$R_I=R_{i_a}\cdots R_{i_1}\quad\text{if } I=(i_1,\cdots,i_a),$$

$$R_iR_j = -R_jR_i : \Omega^\bullet(H_I)\langle\widehat{H_I}\rangle \to \Omega^\bullet(H_K)\langle\widehat{H_K}\rangle$$

if $K = I \cup \{i,j\}$, and

$$R_{I,J}R_I = (-1)^{a+k}R_J : \Omega^\bullet(X)\langle H\rangle \to \Omega^\bullet(H_J)\langle\widehat{H_J}\rangle$$

if $I = (i_1, \cdots, i_a)$ and $J = (i_1, \cdots, i_k, j, i_{k+1}, \cdots, i_a)$.

For $a \geq 0$ consider the sum $\bigoplus_{|I|=a} \Omega^\bullet(H_I)\langle\widehat{H_I}\rangle$ which is a complex on X, and let

$$r : \bigoplus_{|I|=a} \Omega^\bullet(H_I)\langle\widehat{H_I}\rangle[-|I|] \to \bigoplus_{|J|=a+1} \Omega^\bullet(H_J)\langle\widehat{H_J}\rangle[-|J|]$$

be the sum of the maps $R_{I,J}$. Then $r \circ r = 0$, so that we have a complex of bifiltered complexes (on X)

$$0 \to \Omega^\bullet(X)\langle H\rangle \xrightarrow{\ r\ } \bigoplus_i \Omega^\bullet(H_i)\langle\widehat{H_i}\rangle[-1] \xrightarrow{\ r\ } \bigoplus_{|I|=2} \Omega^\bullet(H_I)\langle\widehat{H_I}\rangle[-2] \to \cdots$$

(the term $\Omega^\bullet(X)\langle H\rangle$ is placed in degree 0).

Noting the identity

$$\Omega^\bullet(H^{(a)})\langle\widehat{H^{(a)}}\rangle = \bigoplus_{|I|=a} \Omega^\bullet(H_I)\langle\widehat{H_I}\rangle\,,$$

the above may be written

$$0 \to \Omega^\bullet(X)\langle H\rangle \xrightarrow{\ r\ } \Omega^\bullet(H^{(1)})\langle\widehat{H^{(1)}}\rangle[-1] \xrightarrow{\ r\ } \Omega^\bullet(H^{(2)})\langle\widehat{H^{(2)}}\rangle[-2] \xrightarrow{\ r\ } \cdots.$$

This double complex, as well as its total complex (which is a filtered complex), will be denoted $\Omega^\bullet(X)_H$ or $(\Omega^\bullet_X)_H$. In the proposition below, we consider the double complex obtained by adding the complex $\Omega^\bullet(X)$ to the left.

(3.5) **Proposition.** *The complex of bifiltered complexes*

$$0 \to \Omega^\bullet(X) \to \Omega^\bullet(X)\langle H\rangle \to \Omega^\bullet(H^{(1)})\langle\widehat{H^{(1)}}\rangle(-1)[-1] \to \Omega^\bullet(H^{(2)})\langle\widehat{H^{(2)}}\rangle(-2)[-2] \to \cdots$$

is bifiltered exact. Therefore the natural map $\Omega^\bullet(X) \to \Omega^\bullet(X)_H$ is a filtered quasi-isomorphism.

Proof. One verifies using (3.3.1) that each weight graded piece $\mathrm{Gr}^W_m \mathrm{Gr}^p_F$ of the sequence is exact. $\square$

(3.6) *The complexes $\mathcal{A}^\bullet(X)\langle H\rangle$ and $\mathcal{A}^\bullet(X)_H$.* Denote by $\mathcal{A}^\bullet_X$ the complex of sheaves of smooth differential forms on X. Let $\mathcal{A}^{p,q}_X$ be the sheaf of smooth forms of type (p,q). One has differentials ∂, $\overline{\partial}$ of degree $(1,0)$ and $(0,1)$, respectively, so that $\mathcal{A}^{\bullet,\bullet}_X$ is a double complex. The total complex of this is equal to $\mathcal{A}^\bullet_X$. There is a filtration $F^\bullet$ on the double complex $\mathcal{A}^{\bullet,\bullet}_X$.

We have the Dolbeault resolution $\mathcal{O}_X \to \mathcal{A}_X^{0,\bullet}$. This extends a map of complexes $\Omega_X^\bullet \to \mathcal{A}_X^{\bullet,\bullet}$. For each p, the map $\Omega_X^p \to \Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{A}_X^{0,\bullet} = \mathcal{A}^{p,\bullet}$ is induced by the Dolbeault resolution of $\mathcal{O}_X$, so it is also a resolution; hence the map $(\Omega_X^\bullet, F) \to (\mathcal{A}_X^\bullet, F)$ is a filtered quasi-isomorphism.

For each (p,q), let

$$\mathcal{A}_X^{p,q}\langle H\rangle = \Omega_X^p\langle H\rangle \otimes_{\mathcal{O}_X} \mathcal{A}_X^{0,q}$$

Again one has differentials ∂, $\overline{\partial}$, which makes it a double complex. For each p, the map $\Omega_X^p\langle H\rangle \to \Omega_X^p\langle H\rangle \otimes_{\mathcal{O}_X} \mathcal{A}_X^{0,\bullet}$ is a resolution, and the map $(\Omega_X^\bullet\langle H\rangle, F) \to (\mathcal{A}_X^\bullet\langle H\rangle, F)$ is a filtered quasi-isomorphism.

By the Malgrange preparation theorem, each term $\mathcal{A}^{0,q}$ is flat over $\mathcal{O}_X$. For an $\mathcal{O}_X$-module $\mathcal{F}$, the canonical map $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}_X^{0,\bullet}$ is therefore a quasi-isomorphism. Also the complex $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}_X^{0,\bullet}$ is exact in $\mathcal{F}$. So if $\mathcal{F}$ has a filtration by $\mathcal{O}_X$-submodules, it induces a filtration on $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}_X^{0,\bullet}$, and the map $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}_X^{0,\bullet}$ is a filtered quasi-isomorphism.

In particular the filtration $W_\bullet$ on $\Omega_X^p\langle H\rangle$ induces a filtration $W_\bullet$ on $\mathcal{A}_X^p\langle H\rangle$, and the map $\Omega_X^p\langle H\rangle \to \mathcal{A}_X^{p,\bullet}\langle H\rangle$ is a filtered quasi-isomorphism; it follows the map $(\Omega_X^\bullet\langle H\rangle; W, F) \to (\mathcal{A}_X^\bullet\langle H\rangle; W, F)$ is a bifiltered quasi-isomorphism. Note also that $\mathrm{Gr}^W \mathrm{Gr}_F \mathcal{A}_X^\bullet\langle H\rangle$ is $\Gamma(X,-)$-acyclic.

We will write $\mathcal{A}^\bullet(X)\langle H\rangle$ for $\mathcal{A}_X^\bullet\langle H\rangle$, as we did for the holomorphic de Rham complex.

As in the holomorphic case we have the residue maps

$$Res_{H_I} : \mathcal{A}^\bullet(X)\langle H\rangle \to \mathcal{A}^\bullet(H_I)\langle \widehat{H_I}\rangle[-|I|]$$

and more generally for $I \subset J$ one has $Res_{I,J} : \mathcal{A}^\bullet(H_I)\langle \widehat{H_I}\rangle[-|I|] \to \mathcal{A}^\bullet(H_J)\langle \widehat{H_J}\rangle[-|J|]$.

One may modify them, as in the holomorphic case, to the maps of bifiltered complexes

$$R_{H_I} : \mathcal{A}^\bullet(X)\langle H\rangle \to \mathcal{A}^\bullet(H_I)\langle \widehat{H_I}\rangle(-|I|)[-|I|]$$

and

$$R_{I,J} : \mathcal{A}^\bullet(H_I)\langle \widehat{H_I}\rangle(-|I|)[-|I|] \to \mathcal{A}^\bullet(H_J)\langle \widehat{H_J}\rangle(-|J|)[-|J|]\,.$$

The same facts hold for these maps as for the holomorphic ones; as a consequence the same procedure as before gives us a bifiltered double complex

$$0 \to \mathcal{A}^\bullet(X)\langle H\rangle \xrightarrow{r} \mathcal{A}^\bullet(H^{(1)})\langle \widehat{H^{(1)}}\rangle(-1)[-1] \xrightarrow{r} \mathcal{A}^\bullet(H^{(2)})\langle \widehat{H^{(2)}}\rangle(-2)[-2] \xrightarrow{r} \cdots\,.$$

We denote this by $\mathcal{A}^\bullet(X)_H$ or $(\mathcal{A}_X^\bullet)_H$.

There is a natural map of bifiltered double complexes

$$\begin{tikzcd}
0 \arrow[r] & \Omega^\bullet(X) \arrow[r] \arrow[d] & \Omega^\bullet(X)\langle H\rangle \arrow[r, "r"] \arrow[d] & \Omega^\bullet(H^{(1)})\langle H^{(1)}\rangle(-1)[-1] \arrow[r, "r"] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{A}^\bullet(X) \arrow[r] & \mathcal{A}^\bullet(X)\langle H\rangle \arrow[r, "r"] & \mathcal{A}^\bullet(H^{(1)})\langle H^{(1)}\rangle(-1)[-1] \arrow[r, "r"] & \cdots\,.
\end{tikzcd}$$

Each vertical map is a bifiltered quasi-isomorphism. Hence the map $\Omega^\bullet(X)_H \to \mathcal{A}^\bullet(X)_H$ is also a bifiltered quasi-isomorphism. Also, the total complex of the second row is acyclic.

(3.7) **Proposition.** *One has a commutative diagram of bifiltered complexes*

$$\begin{tikzcd} \mathbb{C}_X \arrow[r] & \Omega(X)^\bullet \arrow[r] \arrow[d] & \mathcal{A}(X)^\bullet \arrow[d] \\ & \Omega(X)^\bullet_H \arrow[r] & \mathcal{A}(X)^\bullet_H \end{tikzcd}$$

where all the arrows bifiltered quasi-isomorphisms.

(3.8) Each term of $\mathcal{A}^\bullet$ is an $\mathcal{A}^0_X$-module, hence fine, in particular c-soft. The same holds for the complex $(\mathcal{A}^\bullet)_H$. Thus the quasi-isomorphism $\mathcal{A}^\bullet_X \to (\mathcal{A}^\bullet_X)_H$ induces a quasi-isomorphism on global sections

$$\Gamma(X, \mathcal{A}^\bullet_X) \to \Gamma(X, (\mathcal{A}^\bullet_X)_H)\,.$$

Set $A^\bullet(X) = \Gamma(X, \mathcal{A}^\bullet_X)$ and

$$A^\bullet(X)_H = \Gamma(X, (\mathcal{A}^\bullet_X)_H)\,,$$

so that we have a quasi-isomorphism $A^\bullet(X) \to A^\bullet(X)_H$.

Suppose $Z \subset X$ is a closed set. The quasi-isomorphism $\mathcal{A}^\bullet_X \to (\mathcal{A}^\bullet_X)_H$ restricts to a quasi-isomorphism of sheaves on Z,

$$\mathbb{C}_Z \to \mathcal{A}^\bullet_X|Z \to (\mathcal{A}^\bullet_X)_H|Z$$

where $\mathcal{A}^\bullet_X|Z$ and $(\mathcal{A}^\bullet_X)_H|Z$ consist of c-soft sheaves on Z. Define

$$A^\bullet(Z) = \Gamma(\mathcal{A}^\bullet_X|Z) \quad \text{and} \quad A^\bullet(Z)_H := \Gamma((\mathcal{A}^\bullet_X)_H|Z)\,.$$

We have a quasi-isomorphism $A^\bullet(Z) \to A^\bullet(Z)_H$ and isomorphisms

$$H^p(Z, \mathbb{C}) \cong H^p(A^\bullet(Z)) \cong H^p(A^\bullet(Z)_H)\,.$$

The restriction map $\mathcal{A}^\bullet(X) \to \mathcal{A}^\bullet(X)|Z$ induces by taking global section a map of complexes $i^* : A^\bullet(X) \to A^\bullet(Z)$. Similarly the map $\mathcal{A}^\bullet(X)_H \to \mathcal{A}^\bullet(X)_H|Z$ induces a map $i^* : A^\bullet(X)_H \to A^\bullet(Z)_H$.

Suppose that the set Z is contractible. Then the map $\pi : Z \to \star$ induces a quasi-isomorphism $\pi^* : \mathbb{C} \to A^\bullet(Z)$. For b a point of Z and $\epsilon_b : \star \to Z$ is the map with image b, the pull-back $\epsilon_b^* : A^\bullet(Z) \to \mathbb{C}$ satisfies $\epsilon_b^*\pi^* = 1$, hence they are homotopy inverse to each other.

If $b \in Z - H$, one extends the above ϵ_b^* to $\epsilon_b^* : A^\bullet(Z)_H \to \mathbb{C}$ in an obvious manner, and one still has $\epsilon_b^*\pi^* = 1$; Thus the maps $\pi^* : \mathbb{C} \to A^\bullet(Z)_H$ and ϵ_b^* are homotopy inverse to each other. We state this as a record:

(3.9) **Proposition.** *If Z is contractible and $b \in Z - H$, then the maps*

$$\mathbb{C} \underset{\epsilon_b^*}{\overset{\pi^*}{\rightleftarrows}} A^\bullet(Z)_H$$

are homotopy inverse to each other.

(3.10) For the complexes $C^\bullet(X)_H$ and $C_\bullet(X)_H$, see a later section.

Recall we have the map of integration $c : A^\bullet(X) \to C^\bullet(X)$. Extending this we define a map of complexes $c : A^\bullet(X)_H \to C^\bullet(X)_H$.

Let $\varphi = (\varphi_{H_I}) \in A^i(X)_H$, where $\varphi_{H_I} \in A^{i-2|I|}(H_I)\langle\widehat{H_I}\rangle$. For each $\alpha \in C_i(X)_H$, define $c(\varphi) \in C^\bullet(X)_H$ by setting

$$\langle c(\varphi), \alpha\rangle = \sum_I \int_{\alpha.H_I} \varphi_{H_I}\,.$$

It follows from the Cauchy-Stokes formula that c is a map of complexes.

We have a variant of this for $Z = \bar{D}_v$, the dual cell of a vertex $v \in H$. There is a map of complexes $c : A^\bullet(Z)_H \to C^\bullet(Z)_H$ defined below.

Recall that any sheaf $\mathcal{F}$ on X and a closed set Z in X, one has

$$\Gamma(\mathcal{F}|Z) = \varinjlim_{V \supset Z} \Gamma(\mathcal{F}|V)$$

in the right hand side of which V varies over the open neighborhoods of Z. (See [Bredon], II, Theorem 9.5.)

Thus an element φ of $A^i(Z)_H$ is represented by a set of elements $(\varphi_{H_I \cap V})_I$, where V is an open set containing Z, and

$$\varphi_{H_I \cap V} \in \Gamma(\mathcal{A}^{i-2|I|}(H_I)\langle\widehat{H_I}\rangle | V \cap H_I)\,.$$

For $\alpha \in C_i(Z)_H$, $c(\varphi)$ is defined by

$$\langle c(\varphi), \alpha\rangle = \sum_I \int_{\alpha.H_I} \varphi_{H_I \cap V}\,.$$

The resulting map is one of complexes.

The restriction maps $i^* : A^\bullet(X)_H \to A^\bullet(Z)_H$ and $i^* : C^\bullet(X)_H \to C^\bullet(Z)_H$ are compatible with the maps c, namely the following diagram commutes:

$$\begin{array}{ccc} A^\bullet(X)_H & \xrightarrow{c} & C^\bullet(X)_H \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i^*} \\ A^\bullet(Z)_H & \xrightarrow{c} & C^\bullet(Z)_H\,. \end{array}$$

(3.11) One has the wedge product

$$A^\bullet(X) \otimes A^\bullet(X)_H \to A^\bullet(X)_H$$

which sends $\psi \otimes \varphi$ with $\varphi = (\varphi_I)$ to $\psi \wedge \varphi$ with components

$$\psi \wedge \varphi_{H_I} .$$

One verifies this gives a map of complexes.

There is a map of complexes

$$i_* : A^\bullet(H)[-2] \to A^\bullet(X)_H , \quad \varphi \mapsto (0, \varphi)$$

and the following diagram commutes.

$$\begin{tikzcd}
A^\bullet(X) \otimes A^\bullet(X)_H \arrow[r, "m"] & A^\bullet(X)_H \\
A^\bullet(X) \otimes A^\bullet(H)[-2] \arrow[r, "m"] \arrow[u, "1\otimes i_*"] & A^\bullet(H)[-2] \arrow[u, "i_*"']
\end{tikzcd}$$

4 The complex $C_\bullet(X)_H$ and its dual $C^\bullet(X)_H$

(4.1) Let H_I be defined as before, and for integers $i \geq 0$, set $H(i) = \cup_{|I|=i} H_I$. This gives a decreasing sequence of closed sets $X = H(0) \supset H(1) \supset \cdots$.

We will henceforth assume that the triangulation K of X satisfies the following condition. There is given a partial ordering on the set of vertices of K, which is a total ordering when restricted to the set of vertices of each simplex. The ordering is assumed to be *compatible with* H in the sense that if v, v' are vertices, $v < v'$ and $v' \in H_I$, then $v \in H_I$; in other words $v < v'$ implies that v' is more general than v with respect to H.

For example such an ordering is obtained as follows. Assume K_0 is a triangulation of X such that each H_I is a subcomplex, and let K be its first barycentric subdivision. The vertices of K are the barycenters $\hat{\sigma}$ of the simplicies σ of K_0. Give a partial ordering on the vertices of X by: $\hat{\sigma} < \hat{\sigma'}$ if and only if $\sigma \prec \sigma'$ (notation meaning that σ and σ' are simplicies of K_0 and σ is a proper face of σ').

The chain complex of oriented simplices will be denoted $C_\bullet(X)$, $C_\bullet(K)$, or sometimes $C^K_\bullet(X)$. An m-simplex σ of X is written $\sigma = v_0 \cdots v_m$, where $v_0 < \cdots < v_m$. If $p \leq m$, the front p-face of σ is the simplex $v_0 \cdots v_p$ (also written σ^F_p) and its back $(m-p)$-face is $v_p \cdots v_m$ (also written σ^B_{m-p}).

Let $\nu = v_0 \cdots v_p$ be a p-simplex. For an element $\alpha = \sum c_\sigma \sigma \in C_m(X)$, define another element $\alpha_\nu \in C_m(X)$ by

$$\alpha_\nu = \sum_{\sigma^F_p = \nu} c_\sigma \sigma \,,$$

the sum over those σ with front face ν. Also for the α define an element $\alpha^B_{m-p} \in C_{m-p}(X)$ by

$$\alpha^B_{m-p} = \sum_\sigma c_\sigma \, \sigma^B_{m-p} \,,$$

the "back $(m-p)$ chain" of α. The homomorphism

$$\mathfrak{d} : C_m(X) \to \bigoplus_p C_p(X) \otimes C_{m-p}(X)$$

which sends a simplex σ to $\sum \sigma^F_p \otimes \sigma^B_{m-p}$ is called the "decatenation" map, or the Alexander-Whitney map. One readily verifies that it is a map of complexes, and also that it is coassociative. It takes an element $\alpha \in C_m(X)$ to $\sum_p \sum_\nu \nu \otimes (\alpha_\nu)^B_{m-p}$, in which $0 \leq p \leq m$ and ν varies over the p-simplices of X.

If we denote by $C_m(X)_\nu$ the submodule consisting of the α with $\alpha = \alpha_\nu$, then clearly we have $C_m(X) = \bigoplus_\nu C_m(X)_\nu$.

In particular, if $\nu = v$ is a vertex, $C_m(X)_v$ is the submodule of α which is a sum of simplicies with first vertex v, and $C_m(X) = \bigoplus_v C_m(X)_v$.

(4.2) Let $h = H_I$ with $|I| = r \geq 1$. An m-simplex σ is said to be *h-transversal* if $\dim(\sigma \cap h) \leq m - 2r$.

We introduce equivalent conditions. Let $N_\sigma(h)$ be the number of vertices of σ that are contained in h (set $N_\sigma(h) = -\infty$ if there is no such vertex). The following are equivalent:

(i) σ is h-transversal.

(ii) If $\sigma = v_0 \cdots v_m$, then $v_{m-2r+1} \notin h$.

(iii) $N_\sigma(h) \leq m - 2r + 1$.

(4.3) **Definition.** $h = H_I$. An element $\alpha = \sum c_\sigma \sigma \in C_m(X)$ is *h-transversal* if each σ with $c_\sigma \neq 0$ is h-transversal. It is said to be *h-admissible* if α and $\partial\alpha$ are h-transversal.

Let $C_m(X)_h$ be the submodule of $C_m(X)$ consisting of the h-admissible elements; one obtains a subcomplex $C_\bullet(X)_h$ of $C_\bullet(X)$.

The following is obvious.

(4.4) **Proposition.** *For $\alpha = \sum c_\sigma \sigma \in C_m(X)$ be h-admissible, it is necessary and sufficient that the following two conditions are satisfied:*

(i) If $c_\sigma \neq 0$, then the simplex σ is h-transversal, and

(ii) For each $(m-1)$-simplex τ which is not *h-transversal, the coefficient of τ in $\partial\alpha$ is zero:*

$$\mathrm{coeff}(\tau; \partial\alpha) = \sum_{\sigma \succ \tau} c_\sigma [\sigma : \tau] = 0 \qquad (*)$$

where σ varies over the m-simplices having τ as a face, and $[\sigma : \tau]$ is the coefficient of τ in $\partial\sigma$.

(4.5) **Proposition.** *Let $\sigma = v_0 \cdots v_m$ be an m-simplex, p is an integer with $p \leq m$ and let $s = \sigma^B_{m-p} = v_p \cdots v_m$. If σ is h-transversal, then s is also h-transversal.*

Proof. If $v_p \in h$, then $v_i \in h$ for $i \leq p$ so we have $N_\sigma(h) = N_s(h) + p$. By hypothesis $N_\sigma(h) \leq m - 2r + 1$, hence $N_s(h) \leq (m-p) - 2r + 1$.

If $v_p \notin h$, then $v_i \notin h$ for $i \geq p$, thus $N_s(h) = -\infty$. □

(4.6) **Definition.** For the normal crossing divisor H, $\alpha \in C_m(X)$ is said to be H-transversal (resp. H-admissible) if α is H_I-transversal (resp. H_I-admissible) for each H_I.

In the previous propositions one may replace h-admissibility with H-admissibility (or h-transversality with H-transversality).

(4.7) **Lemma.** *Let $\sigma = v_0 \cdots v_m$ be an m-simplex satisfying H-transversality. Suppose $\sigma \cap H(r+1)$ is empty. Let $\tau = v_0 \cdots \widehat{v_i} \cdots v_m$ be an $(m-1)$-face of σ.*

(1) If $i \leq m - 2r$, then τ is H-transversal.

(2) If $i > m - 2r$, then $\sigma^F_{m-2r} = \tau^F_{m-2r}$.

Proof. An m-simplex $\sigma = v_0 \cdots v_m$ disjoint from $H(r+1)$ is H-transversal iff $v_{m-2k+1} \notin H(k)$ for $1 \le k \le r$. If σ satisfies satisfies this, then $\tau = \partial_i \sigma$ with $i \le m - 2r$ satisfies a similar condition, namely (1) holds. (2) is obvious. $\square$

(4.8) **Proposition.** *Suppose $\alpha \in C_m(X)_H$ such that $|\alpha| \cap H(r+1) = \emptyset$, and ν is a p-simplex with $p \le m - 2r$. Then $\alpha_\nu \in C_m(X)_H$.*

Proof. Let $\alpha = \sum c_\sigma \sigma$, where σ varies over the H-transversal m-simplices. Then

$$\alpha_\nu = \sum c_\sigma \, \sigma \qquad \text{(the sum over the } m\text{-simplices } \sigma,\ H\text{-transversal, and } \sigma_p^F = \nu)\,.$$

To verify that α_ν is H-admissible, we verify the condition (ii) of Proposition (4.4). Take an $(m-1)$-simplex τ that is not H-transversal, and we must show that

$$\operatorname{coeff}(\tau; \partial\alpha_\nu) = \sum c_\sigma[\sigma : \tau] \qquad \text{(the sum over } \sigma,\ H\text{-transversal, } \sigma_p^F = \nu \text{ and } \sigma \succ \tau) \tag{4.8.1}$$

is zero.

On the other hand from hypothesis

$$0 = \operatorname{coeff}(\tau; \partial\alpha) = \sum c_\sigma[\sigma : \tau] \qquad \text{(the sum over } \sigma,\ H\text{-transversal, and } \sigma \succ \tau)\,. \tag{4.8.2}$$

For each σ appearing in (4.8.2), if $\tau = \partial_i \sigma$, one must have $i > m - 2r$ by (1) of (4.7). Thus $\sigma_{m-2r}^F = \tau_{m-2r}^F$ by (2) of (4.7), and since $p \le m - 2r$ we have $\sigma_p^F = \tau_p^F = \nu$. Therefore the sums in the right hand sides of (4.8.1) and (4.8.2) are over the same set of σ's, so they are equal. $\square$

(4.9) Let $h = H_I$ be of codimension r. We define a map of complexes

$$i^* : C_\bullet(X)_h \to C_{\bullet-2}(h)$$

which sends an element α to $i^*\alpha = \alpha.h$; intuitively $\alpha.h$ is the intersection of α with h with adequate multiplicities.

An element $\alpha \in C_m(X)$ determines its support $|\alpha|$, which is a subcomplex of K, and α determines a cycle in $C_\bullet(|\alpha|, |\partial\alpha|)$, so one has its homology class $[\alpha] \in H_m(|\alpha|, |\partial\alpha|)$.

Let $u_h \in H^{2r}_h(X)$ be the Thom class of h in X. Suppose $\alpha \in C_m(X)_H$. Cap product with u_h gives us a map

$$\cap u_h : H_m(|\alpha|, |\partial\alpha|) \to H_{m-2r}(|\alpha| \cap h, |\partial\alpha| \cap h)\,.$$

and there is a natural inclusion

$$H_{m-2r}(|\alpha| \cap h, |\partial\alpha| \cap h) \hookrightarrow C_{m-2r}(|\alpha| \cap h)$$

due to the fact that $|\alpha| \cap h$ has dimension $\leq m-2$ and $|\partial\alpha| \cap h$ has $\leq m-3$. The image of $[\alpha] \cap u_h$ in $C_{m-2r}(h)$ is denoted by $\alpha.h$ or $i^*\alpha$.

We have the following properties:

(1) (additivity) For $\alpha, \alpha' \in C_m(X)_h$, one has $(\alpha+\alpha').h = \alpha.h + \alpha'.h$.

(2) (boundary) One has $\partial(\alpha.h) = (\partial\alpha).h$.

(3) If $h' = H_J$ meets h transversally, then one has $(\alpha.h).(h \cap h') = \alpha.(h \cap h')$.

These follow from the known properties of cap product (see [Iv], p. 378-379).

We write

$$\alpha.h = \sum \mu_h(s;\alpha)s$$

the sum over the $(m-2r)$-simplicies s in h, where $\mu_h(s;\alpha) \in \mathbb{Z}$ is the coefficient of s in $\alpha.h$. We will usually write $\mu(s;\alpha)$, dropping the subscript h.

(4.10) Let $h = H_I$ be of codimension r. Given an element $\alpha \in C_m(K)_h$, write it as $\alpha = \sum c_\sigma \sigma$ where σ are h-transversal m-simplices. For a simplex s of dimension $\leq m-2r$ in h (possibly $s = \emptyset$), define

$$\alpha(s) = \sum c_\sigma \sigma \qquad \text{(sum over the } \sigma \text{ with } \sigma \cap h = s) \ .$$

Then each $\alpha(s)$ is in $C_m(K)_h$ and we have

$$\alpha = \sum \alpha(s)$$

the sum over s in h. It is obvious that $\mu(s;\alpha)$ is carried by $\alpha(s)$ with $\dim s = m-2r$ alone. More precisely,

(1) For each s of dimension $m-2r$, one has

$$\alpha(s).h = \mu(s;\alpha(s))s \quad \text{and} \quad \mu(s;\alpha) = \mu(s;\alpha(s))\,.$$

(2) For each s of dimension $< m-2r$, one has $\alpha(s).h = 0$.

The study of the intersection $\alpha.h$ is thus reduced to that of $\alpha(s).h$ for $\dim s = m - 2r$. Then the support $|\alpha(s)|$ of $\alpha(s)$ is an m-dimensional subcomplex of K. We shall study such subcomplexes in a more general framework.

(4.11) **Definition.** Suppose m is a non-negative integer, and s be a simplex in h of dimension $\ell \leq m - 2r$ (thus s is empty if $m \leq 1$). A finite subcomplex K' of K is an m-dimensional subcomplex *centered at the simplex* s if the following conditions hold:

(i) One has $|K'| \cap h = s$.

(ii) The maximal simplices of K' are all of dimension m. Each maximal simplex σ meets h exactly in s: $\sigma \cap h = s$.

Of course, for $\alpha(s)$ in the previous subsection, $K' = |\alpha(s)|$ is centered at s.

(4.12) Suppose K' is an m-dimensional subcomplex centered at an $(m-2r)$-simplex s in h. The subcomplex of $C_\bullet(K')$ consisting of h-admissible chains is denoted by $C_\bullet(K')_h$, namely

$$C_\bullet(K')_h = C_\bullet(K') \cap C_\bullet(K)_h \,.$$

Let L' be the subcomplex of K' consisting of simplices not containing s (we say L' is the subcomplex associated with K').

One may reformulate the construction of i^* in terms of the pair of simplicial complexes (K', L'): Each element $\beta \in C_m(K')_h$ satisfies $\partial\beta \in C_{m-1}(L')$, hence determines a cycle in $C_m(K', L')$. Let $[\beta] \in H_m(K', L')$ be its homology class.

Let $u_h \in H_h^{2r}(X)$ be the Thom class of h. Consider the cap product

$$\cap u_h : H_m(K', L') \to H_{m-2r}(K' \cap h, L' \cap h)$$

where $K' \cap h$, say, means the subcomplex K' induces on h. There is inclusion $H_{m-2r}(K' \cap h, L' \cap h) \to C_{m-2r}^{K'}(h)$ since $K' \cap h$ has dimension $m - 2r$ and $L' \cap h$ has dimension $\leq m - 2r - 1$. The image of $[\beta]$ under the composition is denoted by $i^*\beta \in C_{m-2r}(h)$.

That this agrees with the first construction of i^* follows from the following fact, the verification of which is left to the reader.

Claim. If $\alpha \in C_m(K)_H$, $\dim s = m - 2r$ and $K' = |\alpha(s)|$, then L' coincides with $|\partial(\alpha(s))|$.

(4.13) With K' and L' as above, let $s = v_0 \cdots v_{m-2r}$, and let $\bar{K}'$ (resp. $\bar{L}'$) be the subcomplex of K consisting of the simplices meeting h at most in v_{m-2r} (resp. not meeting h). Then $\bar{K}'$ is a simplicial subcomplex of dimension $2r$ centered at the vertex v_{m-2r}, and $\bar{L}'$ is the associated subcomplex to it.

There is an isomorphism of complexes

$$\mathfrak{b} : C_i(K', L') \to C_{i-m+2r}(\bar{K}', \bar{L}') \,.$$

which takes an i-simplex ρ to its back $(i-m+2r)$-face ρ^B_{i-m+2r}. We have an induced isomorphism of homology

$$\mathfrak{b} : H_i(K', L') \to H_{i-m+2r}(\bar{K}', \bar{L}')$$

This is compatible with cap product with the Thom class u_h. Hence for any $\beta \in C_m(K')_h$ and its image $\bar{\beta} \in C_{2r}(\bar{K}')_{v_{m-2r}}$ one has

$$\mu(s; \beta) = \mu(v_{m-2r}; \bar{\beta})\,.$$

Consequently we have

$$\mu(s; \alpha(s)) = \mu(v_{m-2r}; \overline{\alpha(s)}) \tag{4.13.1}$$

where $\overline{\alpha(s)}$ is the back 2-chain of $\alpha(s)$.

(4.14) **Proposition.** *Let ν be any p-simplex in X, where $0 \le p \le m$. Then for $\alpha \in C_m(X)_H$, one has*

$$(\alpha_\nu)^B_{m-p} \in C_{m-p}(X)_H\,.$$

We remark that α_ν need not be in $C_m(X)_H$.

Proof. Write $\alpha = \sum c_\sigma \sigma$, a sum over m-simplices σ that are H-transversal. Then

$$\alpha_\nu = \sum c_\sigma \sigma \qquad \text{(the sum over } \sigma\text{, } H\text{-transversal, satisfying } \sigma^F_p = \nu)\,.$$

Thus

$$(\alpha_\nu)^B_{m-p} = \sum_s (\sum_\sigma c_\sigma)\, s\,,$$

where s varies over the $(m-p)$-simplices that are H-transversal, and σ varies over the m-simplices, H-transversal, satisfying $\sigma^F_p = \nu$, $\sigma^B_{m-p} = s$. To show this is H-admissible, we verify condition (ii) of (4.4).

For an $(m-p-1)$-simplex t that is *not* H-transversal, the number $\mathrm{coeff}(t; \partial((\alpha_\nu)^B_{m-p}))$ equals

$$\sum_s (\sum_\sigma c_\sigma)\,[s:t] \tag{4.14.1}$$

where s varies over the $(m-p)$-simplices that are H-transversal and $s \succ t$, and σ varies over the m-simplices, H-transversal, satisfying $\sigma^F_p = \nu$, $\sigma^B_{m-p} = s$. Also $[s:t]$ means the coefficient of t in ∂s.

For each such σ, there is an $(m-1)$-face τ of σ such that $\tau^F_p = \nu$, $\tau^B_{m-p-1} = t$; note τ is not H-transversal by (4.5). Thus the above can be written

$$\sum_\tau \sum_s (\sum_\sigma c_\sigma)\,[s:t] \tag{4.14.2}$$

where

τ varies over the $(m-1)$-simplices, not H-transversal, $\tau^F_p = \nu$, $\tau^B_{m-p-1} = t$;
s varies over the $(m-p)$-simplices, H-transversal, $s \succ t$;

σ varies over the m-simplices, H-transversal, $\sigma_p^F = \nu$, $\sigma_{m-p}^B = s$.

Now let $v = \nu_0^B$ be the last vertex of ν.

(i) If the first vertex of t is not equal to v, then there exists no σ as above, since if one existed, then t is the back $(m-p-1)$-face of σ; but σ is H-transversal, a contradiction to (4.5). Thus the sum (4.14.2) equals zero in this case.

(ii) Suppose the first vertex of t is v. Then for each σ we have $\sigma_p^F = \nu$. Indeed assume $\tau = \partial_i \sigma$ where $0 \leq i \leq m$. If $i \leq p$, then t is the back $(m-p-1)$-face of σ, which contradicts the H-transversality of σ. Hence $i \geq p+1$, and it follows $\sigma_p^F = \tau_p^F = \nu$.

For each τ in the sum, since α is H-admissible we have

$$0 = \operatorname{coeff}(\tau; \partial\alpha) = \sum c_\sigma [\sigma : \tau]$$

in which σ varies over the m-simplices, H-transversal, and $\sigma \succ \tau$. By what we just noted one may also assume $\sigma_p^F = \nu$. If t is the back $(m-p-1)$-face of τ, then $s \succ t$ and one has the equality $[\nu \circ s : \nu \circ t] = (-1)^p [s : t]$, thus

$$0 = \sum_s \sum_\sigma c_\sigma \, [s : t]$$

the sum over the $(m-p)$-simplices s, H-transversal, $s \succ t$, and the sum over σ with $\sigma \succ \tau$, $\sigma_p^F = \nu$, $\sigma_{m-p}^F = s$. Taking the sum of these over τ to obtain that (4.14.2) equals zero. □

As a consequence of (4.14) the decatenation map restricts to a map of complexes

$$\mathfrak{d} : C_m(X)_h \to \bigoplus_p C_p(X) \otimes C_{m-p}(X)_h \, .$$

(4.15) **Proposition.** *Let ν be a p-simplex contained in h, with $p \leq m - 2r$. For α an element of $C_m(X)_h$ one has*

$$\alpha_\nu . h = (\alpha . h)_\nu \, .$$

Proof. One has

$$(\alpha . h)_\nu = \sum \mu(s; \alpha(s)) \, s \, ,$$

the sum over the $(m-2r)$-simplices s contained in h satisfying the condition: $s_p^F = \nu$, and there exists an m-simplex σ, h-transversal, such that $\sigma \cap h = \sigma_{m-2r}^F = \nu$. Also we have

$$(\alpha_\nu).h = \sum \mu(s; \alpha_\nu(s)) \, s \, ;$$

the sum is over the $(m-2r)$-simplices s in h satisfying the condition: there exists an m-simplex σ which is h-transversal, such that $\sigma_p^F = \nu$, and $\sigma \cap h = \sigma_{m-2r}^F = \nu$.

One sees that the two set of s are identical; in addition one verifies that when $s_p^F = \nu$ we have

$$\alpha(s) = \alpha(s)_\nu \, .$$

□

(4.16) **Proposition.** *Let ν be a p-simplex with $p \le m-2r$. For α an element of $C_m(X)_h$ one has*
$$((\alpha.h)_\nu)^B_{m-p-2r} = (\alpha_\nu)^B_{m-p}.h \quad \text{in} \quad C_{m-p-2r}(h)\,.$$

Proof. We may assume $\alpha = \alpha_\nu$. The identity follows from
$$\mu(s;\alpha) = \mu(s^B_{m-p-2r};\alpha^B_{m-p})\,.$$

To show this note
$$\mu(s;\alpha) = \mu(s;\alpha(s))\,,$$
$$\mu(s^B_{m-p-2r};\alpha^B_{m-p}) = \mu(s^B_{m-p-2r};\alpha^B_{m-p}(s^B_{m-p-2r})) = \mu(s^B_{m-p-2r};(\alpha(s))^B_{m-p})\,.$$
Both are equal to $\mu(v_{m-2r};\overline{\alpha(s)})$ by (4.13). □

Consider the diagram of complexes
$$\begin{tikzcd}
C_m(X)_h \arrow[rr, "\mathfrak{d}"] \arrow[d, "i^*"'] & & C_p(X)\otimes C_{m-p}(X)_h \arrow[d, "1\otimes i^*"] \\
C_{m-2}(h) \arrow[r, "\mathfrak{d}"'] & C_p(h)\otimes C_{m-p}(h) \arrow[r, "i_*\otimes 1"'] & C_p(X)\otimes C_{m-p-2}(h)
\end{tikzcd}$$
where the lower horizontal arrow is the composition of the decatenation map with $i_*\otimes 1$. This diagram commutes. Indeed for $\alpha \in C_m(X)_h$, look at the terms in the expression
$$\mathfrak{d}(\alpha) = \sum \nu \otimes (\alpha_\nu)^B_{m-p}\,.$$
If ν is not contained in h, then $(\alpha_\nu)^B_{m-p}$ has support disjoint from h, hence $i^*((\alpha_\nu)^B_{m-p}) = 0$. Now Proposition (4.16) implies the assertion.

(4.17) **Definition.** The cup product
$$m : C^p(X)\otimes C^q(X)_h \to C^{p+q}(X)_h$$
is defined by dualizing the map $C_{p+q}(X)_h \to C_p(X)\otimes C_q(X)_h$. Thus for $u \in C^p(X)$, $v \in C^q(X)_h$ and $\alpha \in C_{p+q}(X)_h$, we have
$$(u\cup v)(\alpha) = \sum_\nu u(\nu)v((\alpha_\nu)^B_{m-p})\,.$$

Dualizing the map of restriction $i^* : C_\bullet(X)_h \to C_{\bullet-2r}(h)$ gives a map of complexes
$$i_* : C^\bullet(h)[-2r] \to C^\bullet(X)_h\,.$$
We have a commutative diagram
$$\begin{tikzcd}
C^\bullet(X)\otimes C^\bullet(X)_h \arrow[rr, "m"] & & C^\bullet(X)_h \\
C^\bullet(X)\otimes C^\bullet(h)[-2r] \arrow[u, "1\otimes i_*"] \arrow[r, "i^*\otimes 1"'] & C^\bullet(X)\otimes C^\bullet(h)[-2r] \arrow[r, "m"'] & C^\bullet(h)[-2r]\,. \arrow[u, "i_*"']
\end{tikzcd}$$

(4.18) Let $h = H_I$ be of codimension r in X. For a semi-analytic map $f : \Delta^m \to X$, we say f is *transversal to* h if $f^{-1}(h) = \Delta^\ell$, where $\ell \leq m - 2r$ and Δ^ℓ is the front ℓ-face of Δ^m.

If $\gamma_f \subset \Delta^m \times X$ is the graph of f, and $f' : \Delta^\ell \to h$ is the restriction of f to Δ^ℓ, $\gamma_{f'} \subset \Delta^ell \times h$ its graph, then $\gamma_f \cap \Delta^m \times h = \gamma_{f'}$.

If ω is a local section of $\mathcal{A}(X)\langle H \rangle$ then the integral $\int_{\gamma_f} p_X^* \omega$ locally converges.

(4.19) **Definition.** Let $\alpha = \sum c_f f$ be an element of $S_m^{an}(X)$. It is said to be *H-transversal* if each f with $c_f \neq 0$ is H-transversal. It is *H-admissible* if α and $\partial\alpha$ are H-transversal.

Let $S_m(X)_H$ be the subgroup of $S_m(X)$ consisting of H-admissible elements. Note $S_\bullet(X)_H$ is a subcomplex of $S_\bullet(X)$.

The inclusion $S_\bullet(X)_H \to S_\bullet(X)$ is a quasi-isomorphism.

(4.20) Let N be large enough ($N > 2 \dim X$). For $f \in S_m(X)$ with $m \leq N$, its graph γ_f may be viewed as a semi-analytic m-simplex of $\Delta^N \times X$. The map γ gives an injective map of complexes

$$\gamma : S_\bullet(X) \to C_\bullet(\Delta^N \times X)\,.$$

(one considers both complexes in degrees $\leq N$). We denote the image of γ by $C_\bullet^{graph}(\Delta^N \times X)$, consisting of the *graphic elements* of $C_\bullet(\Delta^N \times X)$.

There is a map of complexes

$$i^* : C_\bullet(\Delta^N \times X)_{\Delta^N \times H} \to C_{\bullet-2}(\Delta^N \times H)\,,$$

and it takes the graphic elements of the former to graphic elements of the latter. Therefore it induces a map of complexes

$$i^* : S_\bullet(X) \to S_{\bullet-2}(H)\,.$$

(4.21) The facts from (4.4) through (4.17) we have shown for $C_*(X)$ carry over to $S_*(X)$. The proofs are mostly easy modifications. In particular we have the map of complexes $i^* : S_\bullet(X)_H \to S_{\bullet-2}(H)$, the product $S^\bullet(X) \otimes S^\bullet(X)_H \to S^\bullet(X)_H$. They are subject to the same compatibilities as for $C_\bullet(X)$.

Refinement of the moving lemma.

1. Let M be a smooth projective complex variety of dimension d, and H a smooth divisor. Given a semi-algebraic triangulation K of M, one has the complex of K-chains $C_*^K(M)$, and the subcomplex

$$C_*^K(M)_H := \{\alpha \in C_*^K(M) \mid \alpha \text{ and } \partial\alpha \text{ meets } H \text{ properly}\}.$$

Passing to the limit over K, we obtain complexes,

$$C_*(M)_H \subset C_*(M)\,.$$

Proposition 1. *The inclusion $C_*(M)_H \subset C_*(M)$ is a homology isomorphism.*

[Proof is omitted here. In the proof, the following fact is used: For each $\alpha \in C_p(M)$, there exists $\mathcal{H}(\alpha) \in C_{p+1}(M)$ and $h(\alpha) \in C_p(M)_H$ such that

$$\partial\mathcal{H}(\alpha) + \mathcal{H}(\partial\alpha) = \alpha - h(\alpha)\,.$$

Further, if $\alpha \in C_p(M)_H$, then $\mathcal{H}(\alpha) \in C_{p+1}(M)_H$ as well.]

If K' is a refinement of K, then one has the subdivision map $\lambda : C_*^K(M) \to C_*^{K'}(M)$ as in [Mu, §17]. Since this map preserves support, it restricts to a map $\lambda : C_*^K(M)_H \to C_*^{K'}(M)_H$.

Proposition 2. *For any triangulation K, the map $\lambda : C_*^K(M)_H \to C_*^{K'}(M)_H$ is injective on homology.*

Proof. It is shown in [Mu, p.97] that if $g : K' \to K$ is a simplicial approximation to the identity of M, then λ and

$$g_\sharp : C_*^{K'}(M) \to C_*^K(M)$$

are homotopy inverse to each other, and also that $g_\sharp \circ \lambda = id$ (see [Mu, p. 100]).

One can take g so that if v is a K'-vertex not contained in H, then $g(v)$ is not contained in H either. Indeed, the K-simplex σ containing v in its interior is not contained in H, and one can take as $g(v)$ one of its vertices not in H. (See [Mu, Lemma 15.1]). If g is so made, intersection property with H of K'-chains, when $g_\sharp$ is applied, gets no worse. In particular, $g_\sharp$ restricts to define

$$g_\sharp : C_*^{K'}(M)_H \to C_*^K(M)_H\,.$$

We also have $g_\sharp \circ \lambda = id$, so the assertion follows. (The homotopy between $\lambda \circ g_\sharp$ may not restrict to homotopy between the complexes $(-)_H$, since it is not carried by a chain in $C_*^K(M)_H$. Thus we fall short of verifying homology isomorphism. □

Proposition 3. *There is a triangulation K such that the inclusion $C_*^K(M)_H \to C_*^K(M)$ is a homology isomorphism.*

Proof. We take a triangulation K. For each cycle $\alpha \in C_p^K(M)$, by what we recalled above, there exists a refinement K' of K, chains $\mathcal{H}(\alpha) \in C_{p+1}^{K'}(M)$ and $h(\alpha) \in C_p^{K'}(M)_H$ such that $\partial\mathcal{H}(\alpha) = \alpha - h(\alpha)$. Thus the image of $[\alpha] \in H_pC_*^K(M)$ in $H_pC_*^{K'}(M)$ comes from $[h(\alpha)]$ $H_pC_*^{K'}(M)_H$.

Since M is compact, the group $H_pC_*^K(M)$ is finitely generated. The above being the case for each of a finite set of generators of $H_pC_*^K(M)$, taking a refinement K' that works for them all, it follows that the map $C_*^{K'}(M)_H \subset C_*^{K'}(M)$ is surjective on homology.

Next, in the commutative diagram

$$\begin{tikzcd}
H_p(C_*^{K'}(M)_H) \arrow[r] \arrow[d] & H_p(C_*^{K'}(M)) \arrow[d, "\cong"'] \\
H_p(C_*(M)_H) \arrow[r, "\cong"] & H_p(C_*(M))
\end{tikzcd}$$

the lower horizontal arrow is an isomorphism by Proposition 1, the right vertical arrow is obviously an isomorphism. Further the left vertical arrow is an injection by Proposition 2, and the upper horizontal arrow is a surjection as we have just shown. It follows that all the maps are isomorphisms. $\square$

§5. The map $c : A^\bullet(X)_H \to C^\bullet(X)_H$ and its multiplicativity

One has the map $c : A^\bullet(X) \to C^\bullet(X)$ and also the map $c : A^\bullet(X)_H \to C^\bullet(X)_H$ defined by

$$\langle c(\varphi_X, \varphi_H), \alpha\rangle = \int_\alpha \varphi_X + \int_{\alpha.H} \varphi_H \,.$$

The latter is a map of complexes by the Cauchy-Stokes formula:

Theorem 1. Let $\alpha \in C_m(X)_H$. Then one has

$$-\int_{\partial\alpha} \varphi + \int_\alpha d\varphi + \int_{i^*\alpha} R_H(\varphi) = 0\,.$$

Recall the wedge product $A^\bullet(X) \otimes A^\bullet(X)_H \to A^\bullet(X)_H$ and and cup product $\cup : C^\bullet(X) \otimes C^\bullet(X)_H \to C^\bullet(X)_H$ defined before.

Let $A^{(2),\bullet}(X)_H = A^\bullet(X) \otimes A^\bullet(X)_H$ and $C^{(2),\bullet}(X)_H = C^\bullet(X) \otimes C^\bullet(X)_H$. There are products $m : A^{(2),\bullet}(X)_H \to A^\bullet(X)_H$ and $m : C^{(2),\bullet}(X)_H \to C^\bullet(X)_H$.

For a vertex v in H. let $M_v = \bar{D}(v)$ be the corresponding dual cell. The collection of these dual cells together with the simplices $s = \Delta^p$ disjoint from H will play the role of "models". We will write M for one of these M_v or s. Write $i : M \to X$ for the inclusion in either case. There are restriction maps $i^* : A^\bullet(X)_H \to A^\bullet(M)_H$ and $i^* : C^\bullet(X)_H \to C^\bullet(M)_H$, as well as $i^* : A^{(2),\bullet}(X)_H \to A^{(2),\bullet}(M)_H$ and $i^* : C^{(2),\bullet}(X)_H \to C^{(2),\bullet}(M)_H$.

Proposition 2. There exist a map of degree -1

$$\rho_X : A^\bullet(X) \otimes A^\bullet(X)_H \to C^\bullet_{\mathbb{C}}(X)_H$$

and a map of degree -1

$$\rho_M : A^\bullet(M) \otimes A^\bullet(M)_H \to C^\bullet_{\mathbb{C}}(M)_H$$

for each model M, which satisfy the identities $d\rho + \rho d = -m(c \otimes c) + cm$ and $i^*\rho_X = \rho_M i^*$ for $i : M \to X$.

Proof. If ρ^m is the restriction of ρ to $A^{(2),m}(X)_H$ (or $A^{(2),m}(M)_H$), we need the condition

$$d\rho^{m-1} + \rho^m d = -m(c \otimes c) + cm\,. \qquad (*)_{m-1}$$

on X or M. If $\rho^m = 0$ for $m \le 0$, then $(*)_{m-1}$ holds for $m \le 0$. Assuming ρ^j for $j < m$ have been found, let

$$\theta^{m-1} = -d\rho^{m-1} - m(c \otimes c) + cm : A^{(2),m-1}(X) \to C^{m-1}_{\mathbb{C}}(X)$$

and similarly on M. Note that $i^*\theta^{m-1} = \theta^{m-1} i^*$ holds.

If we choose a base point $b \in M - H$, there is a map $S : A^\bullet(M)_H \to A^\bullet(M)_H$ of degree -1 such that $dS + Sd = 1 - r_b^*$. There is a similar homotopy for the complex $A^\bullet(M)$, hence there is an induced map $S : A^{(2),\bullet}(M)_H \to A^{(2),\bullet}(M)_H$ of degree -1 satisfying $dS + Sd = 1 - r_b^*$. Since r_b^* is zero in degrees $\neq 0$, and since $\theta^0 = -m(c \otimes c) + cm = 0$, one has $\theta^{m-1} r_b^* = 0$ for all m.

We now produce a map $\rho^m : A^{(2),m}(X)_H \to C_{\mathbb{C}}^{m-1}(X)_H$. Let $a \in A^{(2),m}(X)_H$. If v is a vertex in H, $i : M = M_v \to X$ the inclusion, one has $i^* a \in A^{(2),m}(M)_H$. For $\alpha \in C_{m-1}(X)_{H,v}$, recalling that $\alpha \in C_{m-1}(M_v)_H$ we let

$$\langle \rho^m(a), \alpha \rangle = \langle \theta^{m-1} S i^* a, \alpha \rangle \in \mathbb{C}\,.$$

For a simplex $v : s = \Delta^{m-1} \to X$ disjoint from H, let $\langle \rho^m(a), v \rangle = \langle \theta^{m-1} S v^* a, 1_s \rangle$. Since

$$C_{m-1}(X)_H = \bigoplus C_{m-1}(X)_{H,v} \oplus C_{m-1}(X)^0$$

this defines an element $\rho^m(a) \in C_{\mathbb{C}}^{m-1}(X)_H$.

If in this argument X is replaced with a "model" M, one obtains a map $\rho^m : A^{(2),m}(M)_H \to C_{\mathbb{C}}^{m-1}(M)_H$.

(i) $i^* \rho^m = \rho^m i^*$ for a map $i : M \to X$.

If $v' > v$ is another vertex in H, and $\alpha \in C_{m-1}(X)_{H,v'}$, then for the inclusion $i : M_v \to X$ we have

$$\begin{aligned} \langle i^* \rho^m(a), \alpha \rangle &= \langle \rho^m(a), \alpha \rangle \\ &= \langle \theta^{m-1} S i'^* a, \alpha \rangle \end{aligned}$$

where $i' : M_{v'} \to X$. The map i' factors as $M_{v'} \xrightarrow{k} M_v \xrightarrow{i'} X$, and one has

$$\langle \rho^m(i^* a), \alpha \rangle = \langle \theta^{m-1} S(k^* i^* a), \alpha \rangle\,.$$

The two thus coincide. The verification for $i : s \to X$ is obvious.

(ii) $\theta^{m-1} d = 0$. (Follows from $(*)_{m-2}$.)

(iii) $\theta^{m-1} S d = \theta^{m-1}$ on M. (Follows from (ii) and the fact $\theta^{m-1} r_b^* = 0$.)

(iv) One has $\rho^m d = \theta^{m-1}$ on X and M.

For $\alpha \in C_{m-1}(X)_{H,v}$, we have

$$\begin{aligned} \langle \rho^m d(a), \alpha \rangle &= \langle \theta^{m-1} S i^*(da), \alpha \rangle \\ &= \langle \theta^{m-1} i^*(a), \alpha \rangle && \text{[by (iii)]} \\ &= \langle i^* \theta^{m-1}(a), \alpha \rangle && [\text{by } i^* \theta^{m-1} = \theta^{m-1} i^*] \\ &= \langle \theta^{m-1}(a), \alpha \rangle\,. \end{aligned}$$

For $s \in C_{m-1}(X)$, the same reasoning holds. If X is replaced with M, the same argument holds. $\square$

Proposition 3. *Assume that ρ' is another functorial map satisfying the same property as for ρ. Then there exists a map $\pi : A^{(2),\bullet}(X) \to C^{\bullet}_{\mathbb{C}}(X)$ of degree -2, and a similar map on M, satisfying*

$$d\pi + \pi d = \rho - \rho'$$

on X (and on M), and the identity $i^*\pi = \pi i^*$.

We need a variant of Proposition 2. Consider now the failure of commutativitiy of the diagram

$$\begin{tikzcd} A^\bullet(X) \otimes A^\bullet(H)[-2] \arrow[r, "m(i^*\otimes 1)"] \arrow[d, "c\otimes c\,i_*"'] & A^\bullet(H)[-2] \arrow[d, "c\,i_*"] \\ C^\bullet(X) \otimes C^\bullet(X)_H \arrow[r, "m"'] & C^\bullet(X)_H\,. \end{tikzcd} \qquad (**)$$

in which the top horizontal map is the composition $A^\bullet(X)\otimes A^\bullet(H)[-2] \xrightarrow{i^*\otimes 1} A^\bullet(H)\otimes A^\bullet(H)[-2] \xrightarrow{m} A^\bullet(H)[-2]$, and the $c\,i_*$ is the composition $A^\bullet(H)[-2] \xrightarrow{i_*} A^\bullet(X)_H \xrightarrow{c} C^\bullet(X)_H$.

As in Proposition 2 also consider the diagram accompanying it, obtained by replacing X with a "model" M, and H with $M\cap H$:

$$\begin{tikzcd} A^\bullet(M) \otimes A^\bullet(M\cap H)[-2] \arrow[r, "m(i^*\otimes 1)"] \arrow[d, "c\otimes c\,i_*"'] & A^\bullet(M\cap H)[-2] \arrow[d, "c\,i_*"] \\ C^\bullet(M) \otimes C^\bullet(M)_H \arrow[r, "m"'] & C^\bullet(M)_H\,. \end{tikzcd}$$

Note that if $M = s$ is a simplex disjoint from H, then $M \cap H$ is empty and $A^\bullet(M\cap H) = 0$.

Proposition 4. *There exist a map of degree -1*

$$\rho_X : A^\bullet(X) \otimes A^\bullet(H)[-2] \to C^\bullet_{\mathbb{C}}(X)_H$$

and a map of degree -1

$$\rho_M : A^\bullet(M) \otimes A^\bullet(M\cap H)[-2] \to C^\bullet_{\mathbb{C}}(M)_H$$

for each model M, which satisfy the identities $d\rho + \rho d = -m(c\otimes c) + cm$ on X and M, and $i^*\rho_X = \rho_M i^*$ for $i : M \to X$.

Further, if (ρ'_X, ρ'_M) is another collection of maps satisfying the same property, there exists a map of degree -2,

$$\nu_X : A^\bullet(X) \otimes A^\bullet(H)[-2] \to C^\bullet(X)_H$$

and a map of degree -2

$$\nu_M : A^\bullet(M) \otimes A^\bullet(M\cap H)[-2] \to C^\bullet(M)_H$$

for each M, satisfying

$$d\nu + \nu d = \rho - \rho'\,,$$

and $i^*\nu = \nu i^*$ for $i : M \to X$.

The proof of this is parallel to that for Proposition 2, with some differences as we point out. The identity $m(c\otimes c) = cm$ in degree 0 for the previous proposition must be replaced with:

Lemma 5. *The following diagram commutes:*

$$\begin{tikzcd}
A^0(X) \otimes (A^\bullet(H)[-2])^2 \arrow[r, "m"] \arrow[d, "c\otimes c"'] & (A^\bullet(H)[-2])^2 \arrow[d, "c"] \\
C^0(X) \otimes C^2(X)_H \arrow[r, "m"'] & C^2(X)_H\,.
\end{tikzcd}$$

Proof. For $f \otimes g \in A^0(X) \otimes A^0(H)$, one must show $c(i_*(f|_H \cdot g)) = c(f) \cup c(i_* g)$.

For an element $\alpha \in C_2(X)_{H,v}$, with v a vertex in H, we have

$$\langle c(i_*(f|_H \cdot g)), \alpha \rangle = \int_{\alpha.H} f|_H \cdot g\,;$$

if $\alpha.H = mv$, then the right hand side equals $m(f \cdot g)(v)$. On the other hand,

$$\langle c(f) \cup c(i_* g), \alpha \rangle = f(v) \int_{\alpha.H} g = f(v) \cdot mg(v)\,,$$

so the two coincide.

As for a simplex $s \in C_2(X)$ disjoint from H, both cocycles obviously take the value zero. $\square$

Proof of Proposition 4. We take the proof of Proposition 2 and repeat it with changes as follows.

• Let ρ^m be the restriction of ρ to the degree m part of the complex $A^\bullet(X) \otimes A^\bullet(H)[-2]$. We set $\rho^m = 0$ for $m \leq 2$.

Let $m > 2$ and proceed to find ρ^m. Defining θ^{m-1} as before. We have $\theta^2 = 0$ by the above lemma, and $r_b^* = 0$ in degree $\neq 2$, thus $\theta^{m-1} r_b^* = 0$ in all degrees.

• If $M = M_v = \overline{D}(v)$, then $M \cap H = \overline{D}_H(v)$, the dual cell of v in the simplicial complex H. Since $\overline{D}_H(v)$ is contractible, there exists a map S from $A^\bullet(M \cap H)$ to itself satisfying $dS + Sd = 1 - r_b^*$, with $b \in M \cap H$. It follows that there is a map S of degree -1 from $A^\bullet(M) \otimes A^\bullet(M \cap H)$ to itself with the property $dS + Sd = 1 - r_b^*$.

• In defining $\rho^m(a)$ for an element a of degree m in $A^\bullet(X) \otimes A^\bullet(H)[-2]$, one has $\langle \rho^m(a), s \rangle = 0$ for simplices s disjoint from H. $\square$

Our goal is Theorem 9. We first note the facts (projection formulas):

Lemma 6. *The following digram commutes.*

$$\begin{tikzcd}
A(X) \otimes A(H)[-2] \arrow[r, "i^*\otimes 1"] \arrow[d, "1\otimes i_*"'] & A(H) \otimes A(H)[-2] \arrow[r, "m"] & A(H)[-2] \arrow[d, "i_*"] \\
A(X) \otimes A(X)_H \arrow[rr, "m"] & & A(X)_H
\end{tikzcd}$$

Lemma 7. *The following digram commutes.*

$$\begin{tikzcd}
C(X)\otimes C(H)[-2] \arrow[r, "i^*\otimes 1"] \arrow[d, "1\otimes i_*"'] & C(H)\otimes C(H)[-2] \arrow[r, "m"] & C(H)[-2] \arrow[d, "i_*"] \\
C(X)\otimes C(X)_H \arrow[rr, "m"] & & C(X)_H
\end{tikzcd}$$

The map $c\,i_*m(i^*\otimes 1) : A^\bullet(X)\otimes A^\bullet(H)[-2] \to C^\bullet(X)_H$ appearing in the square $(**)$ is equal to $cm(1\otimes i_*)$ by the projection formula for $A^\bullet(X)$, and also to $i_*cm(i^*\otimes 1)$ by $ci_* = i_*c$. The other map in the same diagram $m(c\otimes c\,i_*)$ is equal to $m(c\otimes c)(1\otimes i_*)$ clearly, and also

$$m(c\otimes c\,i_*) = m(1\otimes i_*)(c\otimes c) = i_*m(i^*\otimes 1)(c\otimes c) = i_*m(c\otimes c)(i^*\otimes 1)$$

using the projection formula for $C^\bullet(X)$ and $ci^* = i^*c$.

Let now ρ_X be a homotopy as in Proposition 2; similarly let $\rho_H : A(H)\otimes A(H)[-2] \to C(H)[-2]$ be a map such that

$$d\rho_H + \rho_H d = -m(c\otimes c) + cm : A(H)\otimes A(H)[-2] \to C(H)[-2]\,.$$

The element $\rho_X(1\otimes i_*)$ gives a homotopy between the maps

$$\rho_X(1\otimes i_*) : m(c\otimes c)(1\otimes i_*) \simeq cm(1\otimes i_*)$$

and $i_*\rho_H(i^*\otimes 1)$ gives a homotopy

$$i_*\rho_H(i^*\otimes 1) : i_*m(c\otimes c)(i^*\otimes 1) \simeq i_*cm(i^*\otimes 1)\,.$$

But we know that the source and the target for the maps are the same, thus by the latter half of Proposition 4 there is a map ν of degree -2 giving homotopy

$$\nu : \rho_X(1\otimes i_*) \simeq i_*\rho_H(i^*\otimes 1)\,.$$

Proposition 8. *Let*

$$\begin{tikzcd}
K \arrow[r, "u"] \arrow[d, "f"'] & L \arrow[d, "g"] \\
K' \arrow[r, "u'"'] & L'
\end{tikzcd}$$

be a commutative diagram of complexes. Assume there exists a map $\xi : K \to L$ (resp. $\xi' : K' \to L'$) of degree -1 such that $u = d\xi + \xi d$ (resp. $u' = d\xi' + \xi' d$). Assume also there exists a map $\nu : K \to L'$ of degree -2 such that

$$g\xi - \xi' f = d\nu + \nu d\,.$$

Then the map

$$(u, u') : C_f \to C_g$$

is homotopic to zero.

One has the map

$$\mathcal{P} : A(X, H) \to D(A(X)\langle\langle H\rangle\rangle)\,.$$

Let $\mathcal{Q}_X : A(X, H) \to D(A(X)\langle\langle H\rangle\rangle)$ be the composition of the maps

$$A(X, H) \xrightarrow{c} C^\bullet(X, H)_H \xrightarrow{\kappa} C_{2n-2-\bullet}(X \mid H) \xrightarrow{\Phi} D(A(X)\langle\langle H\rangle\rangle)\,.$$

We apply the above to the diagram

$$\begin{tikzcd} A^\bullet(X) \arrow[r, "\mathcal{P}_X - \mathcal{Q}_X"] \arrow[d, "i^*"'] & D(A^\bullet(X)_H) \arrow[d, "i^*"] \\ A^\bullet(H) \arrow[r, "\mathcal{P}_H - \mathcal{Q}_H"'] & D(A^\bullet(H)[-2]) \end{tikzcd}$$

and the maps ρ_X, ρ_H and ν. We obtain:

Theorem 9. There exists a map $\xi : A(X)_H \to D(A(X)\langle\langle H\rangle\rangle)$ of degree -1 such that

$$d\xi + \xi d = \mathcal{P} - \mathcal{Q}\,.$$

§6. The explicit complex $\mathbb{E}(X,H)$

Let H be a smooth divisor on X. One has a map

$$\Phi : C_*(X) \to D(A(X)_H)$$

given by

$$\langle \Phi(\alpha), (\varphi_X, \varphi_H)\rangle = \int_\alpha \varphi_X + \int_{\alpha.H} \varphi_H\,.$$

Similarly one has $\Phi : C_*(H) \to D(A(H))$. The inclusion $A(H)[-2] \to A(X)_H$ induces a surjection $i^* : D(A(X)_H) \to D(A(H)[-2])$. The following square commutes:

$$\begin{array}{ccc} C_*(X)_H & \xrightarrow{\Phi} & D(A(X)_H) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i^*} \\ C_{*-2}(H) & \xrightarrow{\Phi} & D(A(H)[-2]) \end{array}$$

There is the map $\mathcal{P} : A^\bullet(X) \to D(A(X)_H)$ given by

$$\langle \mathcal{P}(\omega), (\varphi_X, \varphi_H)\rangle = \int_X \omega \wedge \varphi_X + \int_H (\omega|_H) \wedge \varphi_H\,.$$

One also has a similar map $\mathcal{P} : A^\bullet(H) \to D(A(H))$. The following square commutes:

$$\begin{array}{ccc} A^\bullet(X)_H & \xrightarrow{\mathcal{P}} & D(A(X)_H) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i^*} \\ A^\bullet(H) & \xrightarrow{\mathcal{P}} & D(A(H)[-2]) \end{array}$$

The commutative diagram of complexes

$$\begin{array}{ccccc} C_*(X)_H & \xrightarrow{\Phi} & D(A(X)_H) & \xleftarrow{\mathcal{P}} & A^\bullet(X)_H \\ {\scriptstyle i^*}\downarrow & & {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i^*} \\ C_{*-2}(H) & \xrightarrow{\Phi} & D(A(H)[-2]) & \xleftarrow{\mathcal{P}} & A^\bullet(H) \end{array}$$

gives a Hodge complex; it may be abbreviated to

$$[C_*(X|H) \xrightarrow{\Phi} D(A(X)\langle\langle H\rangle\rangle) \xleftarrow{\mathcal{P}} A^\bullet(X,H)]\,.$$

By means of the canonical map $s^* : D(A(X)\langle\langle H\rangle\rangle) \to D(A(X)\langle H\rangle)$ we obtain another Hodge complex

$$[C_*(X|H) \xrightarrow{\Phi} D(A(X)\langle H\rangle) \xleftarrow{\mathcal{P}} A^\bullet(X,H)]\,.$$

This is the explicit Hodge complex $\mathbb{E}(X,H)$.

Definition 1. The complex

$$\big[\Gamma(X, \mathcal{C}^\bullet\mathbb{Q}) \to \Gamma(H, \mathcal{C}^\bullet\mathbb{Q})\big] = \mathrm{Cone}(i^*)[-1]$$

will be abbreviated to $\Gamma(X, H; \mathcal{C}^\bullet\mathbb{Q})$. Similarly one defines the complexes $\Gamma(X, H; \mathcal{C}^\bullet\mathcal{A}^\bullet)$ and $\Gamma(X, H; \mathcal{A}^\bullet)$. There are maps among these complexes

$$\big[\Gamma(X, H; \mathcal{C}^\bullet\mathbb{Q}) \longrightarrow \Gamma(X, H; \mathcal{C}^\bullet\mathcal{A}^\bullet) \longleftarrow \Gamma(X, H; \mathcal{A}^\bullet)\big]\,.$$

This triple gives a Hodge complex, denoted $\mathbb{K}(X,H)$.

Similarly we have a triple of complexes

$$\left[S^\bullet(X,H) \xrightarrow{\alpha} (\mathcal{S}^\bullet \otimes \mathcal{A}^\bullet)(X,H) \xleftarrow{\beta} A^\bullet(X,H)\right].$$

This gives a Hodge complex denoted $\overline{\mathbb{K}}(X,H)$.

One shows that there is a quasi-isomorphism between the Hodge complexes $\mathbb{K}(X,H)$ and $\overline{\mathbb{K}}(X,H)$.

Proposition 2. There exists a quasi-isomorphism between the Hodge complexes $\mathbb{K}(X,H)$ and $\mathbb{E}(X,H)$.

In the following diagram

$$\begin{tikzcd}
S^\bullet(X,H) \arrow[r, "\alpha"] \arrow[d, "\kappa"'] & (\mathcal{S}^\bullet \otimes \mathcal{A}^\bullet)(X,H) \arrow[d, "\lambda"'] & A^\bullet(X,H) \arrow[l, "\beta"'] \arrow[d, "id"] \\
C_{2n-\bullet}(X|H) \arrow[r, "\Phi"] & D(A^{2n-\bullet}(X)\langle\langle H\rangle\rangle) & A^\bullet(X,H) \arrow[l, "\mathcal{P}"']
\end{tikzcd}$$

the left square commutes and the right square commutes up to homotopy.

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