Homological Hodge complexes I

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Let (X, H) be a pair consisting of a smooth variety and a normal crossing divisor. We will describe a particular type of Hodge complex that calculates the Hodge structure of the cohomology of the pair. The Hodge complex is explicit in that it only uses only (1) the complex of topological chains, (2) the complex of differential forms on X, possibly with logarithmic singularities, and the maps given by integration. The construction is based on the Cauchy-Stokes formula, a combination of the Cauchy formula and the Stokes formula, and involves the dual of the complex of logarithmic forms, which should be viewed as "currents" where we allow the test forms to have logarithmic singularities.

The comparison to the Hodge complex of Deligne and Beilinson is the main theorem.

Conventions

For a map of complexes $u : K \to L$, its cone shifted by -1 may be expressed by

$$[K \xrightarrow{u} L] \ ,$$

which as a graded module equals $K \oplus L[-1]$. In some cases the same abbreviation denotes the cone of u, without shift; there will be no confusion as we see to it.

1 The Hodge complexes for a smooth complete variety

For a sheaf on X, let $\mathcal{C}^\bullet(\mathcal{F})$ be its canonical resolution (by Godement), [Br], [Go]. For an open set U of X, let $S^\bullet(X)$ be the complex of semi-algebraic singular cochains, and $\mathcal{S}^\bullet$ be its sheafication (the singular cochain sheaf). Let $\mathcal{A}^\bullet_X$ be the sheaf of smooth forms on X.

There is a canonical map $c : A^\bullet(U) \to S^\bullet(U)$ given by

$$\langle c(\varphi), \alpha \rangle = \int_\alpha \varphi$$

for a smooth singular chain α. It induces a quasi-isomorphism of complexes of sheaves $c : \mathcal{A}^\bullet_X \to \mathcal{S}^\bullet_{\mathbb{C}}$.

There are canonical quasi-isomorphisms

$$S^\bullet_{\mathbb{C}}(X) \xrightarrow{\alpha} \Gamma(X, \mathcal{S}^\bullet \otimes_{\mathbb{Q}} \mathcal{A}^\bullet) \xleftarrow{\beta} A^\bullet(X)$$

and canonical quasi-isomorphisms of complexes of sheaves

$$\mathcal{S}^\bullet_{\mathbb{C}} \xrightarrow{\alpha} \mathcal{S}^\bullet \otimes_{\mathbb{Q}} \mathcal{A}^\bullet \xleftarrow{\beta} \mathcal{A}^\bullet .$$

Definition 1. Let $\mathbb{K} = \mathbb{K}(X)$ be the triple of filtered complexes

$$\left[\Gamma(X, \mathcal{C}^\bullet\mathbb{Q}) \longrightarrow \Gamma(X, \mathcal{C}^\bullet\mathcal{A}^\bullet) \longleftarrow \Gamma(X, \mathcal{A}^\bullet)\right] .$$

This is a mixed Hodge complex for X.

Define also a triple $\overline{\mathbb{K}}$ by

$$\left[S^\bullet(X) \xrightarrow{\alpha} \Gamma(X, \mathcal{S}^\bullet \otimes_{\mathbb{Q}} \mathcal{A}^\bullet) \xleftarrow{\beta} A^\bullet(X)\right] . \qquad (i)$$

which is another mixed Hodge complex.

Proposition 2. There is a canonical quasi-isomorphism between $\mathbb{K}(X)$ and $\overline{\mathbb{K}}(X)$.

Proof. Consider the diagram of complexes in which the vertical and horizontal arrows are filtered quasi-isomorphisms.

$$\begin{tikzcd}
\Gamma(\mathcal{C}^\bullet\mathbb{Q}) \arrow[r] \arrow[d] & \Gamma(\mathcal{C}^\bullet\mathcal{A}^\bullet) \arrow[d] & \Gamma(\mathcal{A}^\bullet) \arrow[l] \arrow[d] \\
\Gamma(\mathcal{C}^\bullet\mathcal{S}^\bullet) \arrow[r] & \Gamma(\mathcal{C}^\bullet(\mathcal{S}^\bullet \otimes \mathcal{A}^\bullet)) & \Gamma(\mathcal{S}^\bullet \otimes \mathcal{A}^\bullet) \arrow[l] \\
\Gamma(\mathcal{S}^\bullet) \arrow[r] \arrow[u] & \Gamma(X, \mathcal{S}^\bullet \otimes \mathcal{A}^\bullet) \arrow[u] & \Gamma(\mathcal{A}^\bullet) \arrow[l] \arrow[u] \\
S^\bullet(X) \arrow[r] \arrow[u] & \Gamma(X, \mathcal{S}^\bullet \otimes \mathcal{A}^\bullet) \arrow[l, phantom] \arrow[u, "id"] & A^\bullet(X) \arrow[l] \arrow[u, "id"']
\end{tikzcd}$$

The top and the bottom rows are $\mathbb{K}(X)$ and $\overline{\mathbb{K}}(X)$, respectively. $\square$

2 The comparison in case $H = \emptyset$

Assume that X is triangulated, and a partial ordering on vertices (which is assumed to be a total ordering on each simplex) in the triangulation. One has the complex of simplicial chains $C_\bullet(X)$ and the complex of simplicial cochains $C^\bullet(X)$. Then there is a natural map from $S^\bullet(X)$ to $C^\bullet(X)$.

Also there is a map

$$\kappa : C^\bullet(X) \to C_{2n-\bullet}(X)$$

which sends $u \in C^p(X)$ to $u \cap [X] \in C_{2n-p}(X)$.

Let

$$\lambda = \Phi \circ \kappa \circ (1 \cup c) : \Gamma(X, \mathcal{S}^\bullet \otimes_{\mathbb{Q}} \mathcal{A}^\bullet) \to DA^{2n-\bullet}(X)\,.$$

One has a diagram of homomorphisms of complexes:

$$\begin{tikzcd}
S^\bullet(X) \arrow[r, "\alpha"] \arrow[d, "\kappa"'] & \Gamma(X, \mathcal{S}^\bullet \otimes_{\mathbb{Q}} \mathcal{A}^\bullet) \arrow[d, "\lambda"'] & A^\bullet(X) \arrow[l, "\beta"'] \arrow[d, "id"] \\
C_{2n-\bullet}(X) \arrow[r, "\Phi"] & DA^{2n-\bullet}(X) & A^\bullet(X) \arrow[l, "\mathcal{P}"']
\end{tikzcd}$$

The left square commutes.

We recall a theorem of Guggenheim [Gu]. The proof we give is organized in a slightly different manner so that it will be convenient for us later.

Theorem 3. *There exists a map $\rho : A^\bullet(X) \otimes A^\bullet(X) \to S^\bullet_{\mathbb{C}}(X)$ be a map of degree -1 such that*

$$(d\rho + \rho d))(\psi \otimes \varphi) = c(\psi \wedge \varphi) - c(\psi) \cup c(\varphi)\,.$$

Proof. Let $A^{(2),\bullet}(X) = A^\bullet(X) \otimes A^\bullet(X)$. Denote by m either of the wedge product $A^\bullet(X) \otimes A^\bullet(X) \to A^\bullet(X)$ or the cup product $S^\bullet(X) \otimes S^\bullet(X) \to S^\bullet(X)$.

We show that there is a map $\rho : A^\bullet(X) \otimes A^\bullet(X) \to S^\bullet_{\mathbb{C}}(X)$ of degree -1, functorial in X, such that $d\rho + \rho d = -m(c \otimes c) + cm$. The functoriality means the identity $f^*\rho^m = \rho^m f^*$ for any map $f : Y \to X$.

Write ρ^m for the restriction of ρ to the degree m part: $\rho^m : A^{(2),m}(X) \to S^{m-1}(X)$. The condition required for ρ^m is

$$d\rho^{m-1} + \rho^m d = -m(c \otimes c) + cm\,. \tag{$(*)_{m-1}$}$$

With $\rho^m = 0$ for $m \leq 0$, the identity $(*)_{m-1}$ obviously holds for $m \leq 0$. We will find ρ^m, $m \geq 1$, by induction. Let

$$\theta^{m-1} = -d\rho^{m-1} - m(c \otimes c) + cm : A^{(2),m-1}(X) \to S^{m-1}(X)\,.$$

Observe that $m(c\otimes c) = cm$ on $A^{(2),0}(X)$; hence when $m=1$, $\theta^0 = m(c\otimes c)+cm = 0$. Also note that for a map $f : Y \to X$ the identity $f^*\theta^{m-1} = \theta^{m-1}f^*$ holds. We are to find ρ^m such that $\rho^m d = \theta^{m-1}$ holds; also ρ^m should satisfy $f^*\rho^m = \rho^m f^*$.

When X is one of the models Δ^p, there is a map $S : A^{(2),\bullet}(\Delta^p) \to A^{(2),\bullet}(\Delta^p)$ of degree -1 such that

$$dS + Sd = 1 - r_b^*$$

where $r_b : \Delta^p \to \{b\} \subset \Delta^p$ is the contraction map to a base point b of Δ^p. If $m \geq 2$, then one has $r_b^* = 0$ on $A^{(2),m-1}(\Delta^p)$. Since $\theta^0 = 0$, we have $\theta^{m-1}r_b^* = 0$ for any m, a fact to be used later.

For $a \in A^{(2),m}(X)$, define an element $\rho^m(a) \in S^{m-1}(X)$ as follows. If $v : \Delta^{m-1} \to X$ is a singular simplex, the value of $\rho^m(a)$ at v is given by

$$\langle \rho^m(a), v\rangle = \langle \theta^{m-1} S v^* a, 1_{\Delta^{m-1}}\rangle\,.$$

Here $v^*a \in A^{(2),m}(\Delta^{m-1})$, and the maps S, θ^{m-1} are as in the diagram

$$\begin{tikzcd}
 & S_{\mathbb{C}}^{m-1}(\Delta^{m-1}) \\
A^{(2),m-1}(\Delta^{m-1}) \arrow[ur, "\theta^{m-1}"] \arrow[r, "d"] & A^{(2),m}(\Delta^{m-1}) \arrow[l, bend left, "S"]
\end{tikzcd}$$

(i) One has $f^*\rho^m = \rho^m f^*$ for a map $f : Y \to X$.

Indeed for an $m-1$ simplex w of Y,

$$\begin{aligned}
\langle f^*\rho^m(a), w\rangle &= \langle \rho^m(a), fw\rangle \\
&= \langle \theta^{m-1} S (fw)^* a, 1_{\Delta^{m-1}}\rangle\,,
\end{aligned}$$

while $\langle \rho^m f^*(a), w\rangle = \langle \theta^{m-1} S w^*(f^*a), 1_{\Delta^{m-1}}\rangle$, so they coincide.

(ii) One has $\theta^{m-1}d = 0$.

Both c and m commute with d. Using also $(*)_{m-2}$ we have

$$\begin{aligned}
\theta^{m-1}d &= -d\rho^{m-1}d - m(c\otimes c)d + cmd \\
&= -d\rho^{m-1}d - dm(c\otimes c) + dcm \\
&= d(\rho^{m-1}d - m(c\otimes c) + cm) \\
&= d(d\rho^{m-2}) = 0\,.
\end{aligned}$$

(iii) One has $\theta^{m-1}Sd = \theta^{m-1}$ as maps $A^{(2),m-1}(\Delta^{m-1}) \to S_{\mathbb{C}}^{m-1}(\Delta^{m-1})$.

In the identity

$$\theta^{m-1}Sd = \theta^{m-1}(-dS + 1 - r_b^*)$$

the first term on the right is zero by (ii), and the term $\theta^{m-1}r_b^*$ is also zero as noted before.

(iv) The identity $\rho^m d = \theta^{m-1}$ holds.

Using (iii) and the funtoriality of θ^{m-1} we have:

$$\begin{aligned}
\langle \rho^m d(a), v\rangle &= \langle \theta^{m-1} S v^*(da), 1_{\Delta^{m-1}}\rangle \\
&= \langle \theta^{m-1} S d v^*(a), 1_{\Delta^{m-1}}\rangle \\
&= \langle \theta^{m-1} v^*(a), 1_{\Delta^{m-1}}\rangle && [\text{by (iii)}] \\
&= \langle v^*\theta^{m-1}(a), 1_{\Delta^{m-1}}\rangle && [\text{by } v^*\theta^{m-1} = \theta^{m-1}v^*] \\
&= \langle \theta^{m-1}(a), v\rangle \,.
\end{aligned}$$

□

The next result concerns the indeterminacy of the map ρ.

Proposition 4. *Assume that ρ' is another functorial map satisfying the same property as for ρ. Then there exists a map $\pi : A^\bullet(X) \otimes A^\bullet(X) \to S^\bullet_{\mathbb{C}}(X)$ of degree -2, functorial in X, such that*

$$d\pi + \pi d = \rho - \rho' \,.$$

Proof. Let $\pi^m : A^{(2),m}(X) \to S^{m-2}_{\mathbb{C}}(X)$ be the degree m part of π. Set $\pi^m = 0$ for $m \le 1$. By induction on m we will find π^m such that

$$d\pi^{m-1} + \pi^m d = \rho^{m-1} - \rho'^{m-1} : A^{(2),m-1}(X) \to S^{m-2}_{\mathbb{C}}(X) \qquad (*)_{m-1}$$

holds. For $m \le 1$ it is trivially true. Assuming π^j for $j < m$ have been defined, let

$$\tau^{m-1} = -d\pi^{m-1} + \rho^{m-1} - \rho'^{m-1} \,.$$

It is also functorial in X.

On Δ^{m-2} we take $S : A^{(2),m}(\Delta^{m-2}) \to A^{(2),m-1}(\Delta^{m-2})$ such that $dS + Sd = 1 - r_b^*$. One also has $\tau^{m-1}r_b^* = 0$, since r_b^* is non-zero only in degree 0, and τ^0 is trivially zero.

For any element $a \in A^{(2),m}(X)$ and $v : \Delta^{m-2} \to X$, we have $v^*a \in A^{(2),m}(\Delta^{m-2})$; we set

$$\langle \pi^m(a), v\rangle = \langle \tau^{m-1} S v^* a, 1_{\Delta^{m-2}}\rangle \,,$$

which defines an element $\pi^m(a) \in S^{m-2}_{\mathbb{C}}(X)$. The verification of the following facts are parallel to the previous case, using slightly different hypotheses.

(i) One has $f^*\pi^m = \pi^m f^*$ for a map $f : Y \to X$.

(ii) One has $\tau^{m-1} d = 0$.

This follows by substituting the definition of τ^{m-1}, using the identity $d\rho + \rho d = d\rho' + \rho' d$ and the hypothesis $(*)_{m-2}$.

(iii) One has $\tau^{m-1}Sd = \tau^{m-1}$.

In the identity

$$\tau^{m-1}Sd = \tau^{m-1}(-dS + 1 - r_b^*)$$

we have $\tau^{m-1}r_b^* = 0$, and also $\tau^{m-1}dS = 0$ by (ii).

(iv) The identity $\pi^m d = \tau^{m-1}$ holds.

Using (iii) and the funtoriality of π^{m-1} we have:

$$\begin{aligned}
\langle \pi^m d(a), v\rangle &= \langle \tau^{m-1}Sv^*(da), 1_{\Delta^{m-1}}\rangle \\
&= \langle \tau^{m-1}Sdv^*(a), 1_{\Delta^{m-1}}\rangle \\
&= \langle \tau^{m-1}v^*(a), 1_{\Delta^{m-1}}\rangle && \text{[by (iii)]} \\
&= \langle v^*\tau^{m-1}(a), 1_{\Delta^{m-1}}\rangle && [\text{by } v^*\tau^{m-1} = \tau^{m-1}v^*] \\
&= \langle \tau^{m-1}(a), v\rangle\,.
\end{aligned}$$

□

Define a map $\xi : A^\bullet(X) \to DA^{2n-\bullet}(X)$ of degree -1 by the formula

$$\langle \xi(\psi), \varphi\rangle = \langle \rho(\psi \otimes \varphi), [X]\rangle\,.$$

Proposition 5. *One has*

$$d\xi + \xi d = \lambda \circ \beta - \mathcal{P}\,.$$

Proof. For $\psi, \varphi \in A^\bullet(X)$ of degree adding up to $2n$, one has

$$\begin{aligned}
\langle \xi(d\psi), \varphi\rangle &= \langle \rho(d\psi \otimes \varphi), [X]\rangle\,, \\
\langle \xi(\psi), d\varphi\rangle &= \langle \rho(\psi \otimes d\varphi), [X]\rangle\,,
\end{aligned}$$

thus

$$\begin{aligned}
\langle (\xi d + d\xi)(\psi), \varphi\rangle &= \langle \rho(d\psi \otimes \varphi + \psi \otimes d\varphi), [X]\rangle \\
&= \langle c(\psi \wedge \varphi) - c(\psi) \cup c(\varphi), [X]\rangle\,.
\end{aligned}$$

Also for $\psi \in A^i(X)$ one has $(1 \cup c)\beta(\psi) = c(\psi)$, and

$$\kappa(1 \cup c)\beta(\psi) = \sum_\sigma \sigma'\langle c(\psi), \sigma''\rangle$$

in which σ varies over the $2n$-simplices such that $\sum \sigma = [X]$. Therefore

$$\begin{aligned}
\langle \Phi\kappa(1 \cup c)\beta(\psi), \varphi\rangle &= \sum_\sigma \langle c(\varphi), \sigma'\rangle\langle c(\psi), \sigma''\rangle \\
&= \langle c(\varphi) \cup c(\psi), [X]\rangle\,.
\end{aligned}$$

□

3 The complexes of differential forms with logarithmic poles

For a map of complexes $u : K \to L$, its cone $\mathrm{Cone}(u)$ is a complex such that $\mathrm{Cone}(u) = K[1] \oplus L$ as a graded module. We often write $[K \xrightarrow{u} L]$ for $\mathrm{Cone}(u)[-1]$.

For a filtered complex K and an integer n, $K(n)$ denotes its Tate twist.

(3.1) Let $\Omega_X^\bullet$ be the complex of sheaves of holomorphic differential forms on X, and $\Omega_X^\bullet\langle H\rangle$ the complex of holomorphic forms with logarithmic poles along H. It is well-known that it has the weight filtration $W_\bullet$, [De].

Let $R_H : \Omega_X^\bullet\langle H\rangle \to \Omega_H^\bullet(-1)[-1]$ be the residue map,

$$R_H(\frac{dz_1}{z_1} \wedge \varphi_1) = (2\pi i)\, \varphi_1|_H$$

where φ_1 is a holomorphic form. One has a filtered exact sequence of complexes

$$0 \to \Omega_X^\bullet \to \Omega_X^\bullet\langle H\rangle \xrightarrow{R_H} \Omega_H^\bullet(-1)[-1] \to 0\,.$$

We introduce

$$(\Omega_X^\bullet)_H := \mathrm{Cone}(R_H)[-1] = [\Omega_X^\bullet\langle H\rangle \xrightarrow{R_H} \Omega_H^\bullet(-1)\,[-1]]\ .$$

The inclusion $\Omega_X^\bullet \to (\Omega_X^\bullet)_H$ is a filtered quasi-isomorphism. In the sequel one often omits (-1) in $\Omega_H^\bullet(-1)$, but we remind it when necessary.

(3.2) Denote by $\mathcal{A}_X^\bullet$ the complex of sheaves of smooth differential forms on X. Let $\mathcal{A}_X^{p,q}$ be the sheaf of smooth forms of type (p, q). One has differentials ∂, $\overline{\partial}$ of degree $(1, 0)$ and $(0, 1)$, respectively, so that $\mathcal{A}_X^{\bullet,\bullet}$ is a double complex. The total complex of this is equal to $\mathcal{A}_X^\bullet$.

We have the Dolbeault resolution $\mathcal{O}_X \to \mathcal{A}_X^{0,\bullet}$. This extends a map of complexes $\Omega_X^\bullet \to \mathcal{A}_X^{\bullet,\bullet}$. For each p, the map $\Omega_X^p \to \Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{A}_X^{0,\bullet} = \mathcal{A}^{p,\bullet}$ is induced by the Dolbeault resolution of $\mathcal{O}_X$, so it is also a resolution; hence the map $\Omega_X^\bullet \to \mathcal{A}_X^\bullet$ is a quasi-isomorphism.

For each (p, q), let

$$\mathcal{A}_X^{p,q}\langle H\rangle = \Omega_X^p\langle H\rangle \otimes_{\mathcal{O}_X} \mathcal{A}_X^{0,q}$$

Again one has differentials ∂, $\overline{\partial}$, which makes it a double complex. For each p, the map $\Omega_X^p\langle H\rangle \to \Omega_X^p\langle H\rangle \otimes_{\mathcal{O}_X} \mathcal{A}_X^{0,\bullet}$ is a resolution, and the map $\Omega_X^\bullet\langle H\rangle \to \mathcal{A}_X^\bullet\langle H\rangle$ is a quasi-isomorphism.

By the Malgrange preparation theorem, each term $\mathcal{A}^{0,q}$ is flat over $\mathcal{O}_X$. For an $\mathcal{O}_X$-module $\mathcal{F}$, the canonical map $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}_X^{0,\bullet}$ is therefore a quasi-isomorphism. Also the complex $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}_X^{0,\bullet}$ is exact in $\mathcal{F}$. So if $\mathcal{F}$ has a filtration by $\mathcal{O}_X$-submodules, it induces a filtration on $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}_X^{0,\bullet}$, and the map $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}_X^{0,\bullet}$ is a filtered quasi-isomorphism.

In particular the filtration $W_\bullet$ on $\Omega^p_X\langle H\rangle$ induces a filtration $W_\bullet$ on $\mathcal{A}^p_X\langle H\rangle$, and the map $\Omega^p_X\langle H\rangle \to \mathcal{A}^{p,\bullet}_X\langle H\rangle$ is a filtered quasi-isomorphism; it follows the map $\Omega^\bullet_X\langle H\rangle \to \mathcal{A}^\bullet_X\langle H\rangle$ is also a filtered quasi-isomorphism.

Taking Gr^W_1 of this, we have a quasi-isomorphism $\Omega^\bullet_H \to \Omega^\bullet_H \otimes_{\mathcal{O}_X} \mathcal{A}^{0,\bullet}_X$. There is also an obvious map

$$\Omega^\bullet_H \otimes_{\mathcal{O}_X} \mathcal{A}^{0,\bullet}_X \to \Omega^\bullet_H \otimes_{\mathcal{O}_H} \mathcal{A}^{0,\bullet}_H = \mathcal{A}^\bullet_H$$

The composition of these is the canonical map $\Omega^\bullet_H \to \mathcal{A}^\bullet_H$, which we is a quasi-isomorphism. Thus both maps are quasi-isomorphisms.

There is a commutative diagram of complexes

$$\begin{tikzcd}
0 \arrow[r] & \Omega^\bullet_X \arrow[r] \arrow[d] & \Omega^\bullet_X\langle H\rangle \arrow[r, "R_H"] \arrow[d] & \Omega^\bullet_H(-1)[-1] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{A}^\bullet_X \arrow[r] & \mathcal{A}^\bullet_X\langle H\rangle \arrow[r, "R_H\otimes 1"] \arrow[dr] & \Omega^\bullet_H \otimes_{\mathcal{O}_X} \mathcal{A}^{0,\bullet}_X(-1)[-1] \arrow[r] \arrow[d] & 0 \\
 & & & \mathcal{A}^\bullet_H(-1)[-1] &
\end{tikzcd} \tag{3.2.1}$$

with exact rows where the maps from the first row to the second is of the form $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}^{0,\bullet}_X$ and the the third column is what we mentioned before. The composed map $\mathcal{A}^\bullet_X\langle H\rangle \to \mathcal{A}^\bullet_H(-1)[-1]$ will also be denoted by R_H.

Define

$$(\mathcal{A}^\bullet_X)_H = \mathrm{Cone}(R_H)[-1] = [\mathcal{A}^\bullet_X\langle H\rangle \xrightarrow{R_H} \mathcal{A}^\bullet_H(-1)[-1]]\ ;$$

there is a quasi-isomorphism $(\Omega^\bullet_X)_H \to (\mathcal{A}^\bullet_X)_H$ induced from diagram (3.2.1).

One has a commutative diagram of complexes

$$\begin{tikzcd}
\mathbb{C}_X \arrow[r] & \Omega^\bullet_X \arrow[r] \arrow[d] & \mathcal{A}^\bullet_X \arrow[d] \\
 & (\Omega^\bullet_X)_H \arrow[r] & (\mathcal{A}^\bullet_X)_H
\end{tikzcd}$$

with all arrows quasi-isomorphisms.

(3.3) Each term of $\mathcal{A}^\bullet$ is an $\mathcal{A}^0_X$-module, hence fine, in particular c-soft. The same holds for the complex $(\mathcal{A}^\bullet)_H$. Thus the quasi-isomorphism $\mathcal{A}^\bullet_X \to (\mathcal{A}^\bullet_X)_H$ induces a quasi-isomorphism on global sections

$$\Gamma(X, \mathcal{A}^\bullet_X) \to \Gamma(X, (\mathcal{A}^\bullet_X)_H)\,.$$

Defining

$$A^\bullet(X)_H = \Gamma(X, (\mathcal{A}^\bullet_X)_H)\,,$$

we have a a quasi-isomorphism $A^\bullet(X) \to A^\bullet(X)_H$.

Suppose $Z \subset X$ is a closed set. The quasi-isomorphism $\mathcal{A}_X^\bullet \to (\mathcal{A}_X^\bullet)_H$ restricts to a quasi-isomorphism of sheaves on Z,

$$\mathbb{C}_Z \to \mathcal{A}_X^\bullet|Z \to (\mathcal{A}_X^\bullet)_H|Z$$

where $\mathcal{A}_X^\bullet|Z$ and $(\mathcal{A}_X^\bullet)_H|Z$ consist of c-soft sheaves on Z. Define

$$A^\bullet(Z) = \Gamma(\mathcal{A}_X^\bullet|Z) \quad \text{and} \quad A^\bullet(Z)_H := \Gamma((\mathcal{A}_X^\bullet)_H|Z)\,.$$

We have a quasi-isomorphism $A^\bullet(Z) \to A^\bullet(Z)_H$ and isomorphisms

$$H^p(Z, \mathbb{C}) \cong H^p(A^\bullet(Z)) \cong H^p(A^\bullet(Z)_H)\,.$$

If the set Z is contractible, the map $\pi : Z \to \star$ induces a quasi-isomorphism $\pi^* : \mathbb{C} \to A^\bullet(Z)$. For b a point of Z and $\epsilon_b : \star \to Z$ is the map with image b, the pull-back $\epsilon_b^* : A^\bullet(Z) \to \mathbb{C}$ satisfies $\epsilon_b^* \pi^* = 1$, hence they are homotopy inverse to each other.

If $b \in Z - H$, one extends the above ϵ_b^* to $\epsilon_b^* : A^\bullet(Z)_H \to \mathbb{C}$ in an obvious manner, and one still has $\epsilon_b^* \pi^* = 1$; Thus the maps $\pi^* : \mathbb{C} \to A^\bullet(Z)_H$ and ϵ_b^* are homotopy inverse to each other.

(3.4) **Proposition.** *If Z is contractible and $b \in Z - H$, then the maps*

$$\mathbb{C} \underset{\epsilon_b^*}{\overset{\pi^*}{\rightleftarrows}} A^\bullet(Z)_H$$

are homotopy inverse to each other.

(3.5) Assume again Z is just closed. For any sheaf $\mathcal{F}$ on X, one has

$$\Gamma(\mathcal{F}|Z) = \varinjlim_{V \supset Z} \Gamma(\mathcal{F}|V)$$

in the right hand side of which V varies over the open neighborhoods of Z. (See [Bredon], II, Theorem 9.5.)

In particular, one has

$$\Gamma((\mathcal{A}_X^\bullet)_H|Z) = \varinjlim_{V \supset Z} \Gamma((\mathcal{A}_X^\bullet)_H|V)$$

Thus an element φ of $\Gamma((\mathcal{A}_X)_H^i|Z)$ is represented by an element $\varphi_V \in \Gamma((\mathcal{A}_X)_H^i|V)$, which is a pair

$$(\varphi_{V,X}, \varphi_{V,H})\,, \quad \text{with } \varphi_{V,X} \in \Gamma(\mathcal{A}_X^i\langle H\rangle|V) \text{ and } \varphi_{V,H} \in \Gamma(\mathcal{A}_H^{i-2}|V \cap H)).$$

Assume further that Z is a subcomplex. We have a map of complexes $c : A^\bullet(Z)_H \to C^\bullet(Z)_H$ defined as follows. For an element $\alpha \in C_\bullet(Z)_H$, view it as an element of $C_\bullet(X)_H$, and l$c(\varphi)$ is defined by

$$\langle c(\varphi), \alpha\rangle = \int_\alpha \varphi_{V,X} - \int_{\alpha.H} \varphi_{V,H}\,.$$

We have obviously the restriction maps $i^* : A^\bullet(X)_H \to A^\bullet(Z)_H$ and $i^* : C^\bullet(X)_H \to C^\bullet(Z)_H$. The maps $c : A^\bullet(X)_H \to C^\bullet(X)_H$ and $c : A^\bullet(Z)_H \to C^\bullet(Z)_H$ are compatible via the i^*s.

10. One has the wedge product

$$A^\bullet(X) \otimes A^\bullet(X)_H \to A^\bullet(X)_H$$

defined by

$$\psi \otimes (\varphi_X, \varphi_H) \mapsto (\psi \wedge \varphi_X, \psi \wedge \varphi_H)$$

There is a map of complexes

$$i_* : A^\bullet(H)[-2] \to A^\bullet(X)_H\,, \quad \varphi \mapsto (0, \varphi)$$

and the following diagram commutes.

$$\begin{tikzcd}
A^\bullet(X) \otimes A^\bullet(X)_H \arrow[r, "m"] & A^\bullet(X)_H \\
A^\bullet(X) \otimes A^\bullet(H)[-2] \arrow[r, "m"] \arrow[u, "1\otimes i_*"] & A^\bullet(H)[-2] \arrow[u, "i_*"']
\end{tikzcd}$$

Let X be a smooth complete variety, and $H = \sum_{i=1,\cdots N} H_i$ be a normal crossing divisor on X. For a subset I of $[1, N]$, we set $H_I = \bigcap_{i \in I} H_i$.

The divisor $\widehat{H}_I$ on H_I given by

$$\widehat{H}_I = \sum_{j \notin I} H_j \cap H_I$$

is a normal crossing divisor; we have the complex $\Omega^*_{H_I}\langle \widehat{H}_I \rangle$.

1. The complex $\Omega^*_X\langle H\rangle$ is quipped with the weight filtration $W_\bullet$ as defined in [De].

For a local section φ of $\Omega^*_X\langle H\rangle$ and a component H_1 of H, one has the residue $R_{H_1}(\varphi)$, a local section of $\Omega^*_{H_i}\langle \widehat{H}_i\rangle$; if H_1 is defined locally by $z_1 = 0$ and $\varphi = \dfrac{dz_1}{z_1} \wedge \psi$, then $R_{H_1}(\varphi) = 2\pi i \cdot \psi$. This gives a map of complexes $\Omega^*_X\langle H\rangle \to \Omega^*_{H_i}\langle \widehat{H}_i\rangle[-1]$, namely $dR_{H_1}(\varphi) = -R_{H_1}(d\varphi)$.

Let $e_{-1} = (2\pi i)^{-1}$ be the generator of $\mathbb{C}(-1)$, and consider a slightly modified map map $\Omega^*_X\langle H\rangle \to \Omega^{*-1}_{H_i}\langle \widehat{H}_i\rangle(-1)$, which takes φ to $R_{H_1}(\varphi) \otimes e_{-1}$. This gives us a map of complexes

$$R_{H_i} : \Omega^*_X\langle H\rangle \to \Omega^*_{H_i}\langle \widehat{H}_i\rangle(-1)[-1]\,.$$

One verifies immediately that it takes W_m to $W_m(\Omega^*_{H_i}\langle \widehat{H}_i\rangle(-1)[-1]) = W_{m-1}(\Omega^*_{H_i}\langle \widehat{H}_i\rangle)(-1)[-1]$.

More generally, for I of cardinality m, composing these one has a map (Poincaré residue, [De]) $W_m\Omega^*_X\langle H\rangle \to \Omega^*_{H_I}(-m)[-m]$ which induces an isomorphism

$$\mathrm{Gr}^W_m \Omega^*_X\langle H\rangle \xrightarrow{\sim} \bigoplus_{|I|=m} \Omega^*_{H_I}(-m)[-m]\,. \tag{a}$$

Similarly for each pair of subsets I, J with $I \subset J$, $|J| = |I| + 1$, we have the residue map

$$R^I_J : \Omega^*_{H_I}\langle \widehat{H}_I\rangle \to \Omega^*_{H_J}\langle \widehat{H}_J\rangle(-1)[-1]\,.$$

We thus obtain a complex of mixed Hodge complexes

$$0 \to \Omega^*_X \to \Omega^*_X\langle H\rangle \to \bigoplus_i \Omega^*_{H_i}\langle \widehat{H}_i\rangle(-1)[-1] \to \bigoplus_{I:|I|=2} \Omega^*_{H_I}\langle \widehat{H}_I\rangle(-2)[-2] \to \cdots$$

in which the differentials are the sums of R^I_J. Equip the complex Ω^*_X with the trivial filtration W_{tr}.

Proposition 6. The complex of filtered complexes

$$0 \to \Omega^*_X \to \Omega^*_X\langle H\rangle \to \bigoplus_i \Omega^*_{H_i}\langle \widehat{H}_i\rangle(-1)[-1] \to \bigoplus_{I:|I|=2} \Omega^*_{H_I}\langle \widehat{H}_I\rangle(-2)[-2] \to \cdots$$

is filtered exact.

Proof. Use the isomorphism (a), reduce to exactness of the complex of the following type. Let $I = \{1, \cdots, N\}$ be a finite ordered set, M be a module. Then the complex

$$0 \to M \to \bigoplus_{i \in I} M \to \bigoplus_{i_1 < i_2} M \to \cdots \to M \to 0$$

in which the differentials are the alternating sums of i_M, is acyclic. □

It follows that the map $\Omega^*_X \to (\Omega^*_X)_H$, where $(\Omega^*_X)_H$ is defined to be

$$\mathrm{Tot}([\Omega^*_X\langle H\rangle \to \bigoplus_i \Omega^*_{H_i}\langle \widehat{H}_i\rangle(-1)[-1] \to \bigoplus_{I:|I|=2} \Omega^*_{H_I}\langle \widehat{H}_I\rangle(-2)[-2] \to \cdots])$$

is a filtered quasi-isomorphism.

2. The functor $(-)\otimes_{\mathcal{O}_X} \mathcal{A}^{0,*}_X$ on $\mathcal{O}_X$-modules is exact, since $\mathcal{A}^{0,*}_X$ are flat $\mathcal{O}_X$-modules by a theorem of Malgrange. It follows that for any $\mathcal{O}_X$-module $\mathcal{F}$, the canonical map $\mathcal{F} \to \mathcal{F}\otimes_{\mathcal{O}_X} \mathcal{A}^{0,*}_X$ is a quasi-isomorphism.

In particular, let $\mathcal{A}^{p,*}_X\langle H\rangle = \Omega^p_X\langle H\rangle \otimes_{\mathcal{O}_X} \mathcal{A}^{0,*}$; we obtain the double complex $\mathcal{A}^{**}_X\langle H\rangle$, the total complex of which is written $\mathcal{A}^*_X\langle H\rangle$. The canonical map of complexes

$$\Omega^*_X\langle H\rangle \to \mathcal{A}^*_X\langle H\rangle$$

is a quasi-isomorphism.

Also, the filtration $W_\bullet$ on $\Omega^*_X\langle H\rangle$ induces a filtration $W_\bullet$ on $\mathcal{A}^*_X\langle H\rangle$ and $\mathrm{Gr}^W_m(\mathcal{A}^*_X\langle H\rangle) = (\mathrm{Gr}^W_m \Omega^*_X\langle H\rangle)\otimes_{\mathcal{O}_X} \mathcal{A}^{0,*}_X$.

3. For each I, we have canonical maps

$$\Omega^*_{H_I}\langle \widehat{H_I}\rangle \to \Omega^*_{H_I}\langle \widehat{H_I}\rangle \otimes_{\mathcal{O}_X} \mathcal{A}^{0,*}_X \to \mathcal{A}^*_{H_I}\langle \widehat{H_I}\rangle\,.$$

The first map is a quasi-isomorphism, and the composition of the two maps is also a quasi-isomorphism, hence the second map is also a quasi-isomorphism. We have thus a commutative diagram of complexes

$$\begin{tikzcd}
0 \arrow[r] & \Omega^*_X \arrow[r] \arrow[d] & \Omega^*_X\langle H\rangle \arrow[r] \arrow[d] & \bigoplus_i \Omega^*_{H_i}\langle \widehat{H}_i\rangle(-1)[-1] \arrow[r] \arrow[d] & \bigoplus_{|I|=2} \Omega^*_{H_I}\langle \widehat{H}_I\rangle(-2)[-2] \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \Omega^*_X\otimes_{\mathcal{O}_X}\mathcal{A}^{0,*}_X \arrow[r] \arrow[d] & \Omega^*_X\langle H\rangle\otimes_{\mathcal{O}_X}\mathcal{A}^{0,*}_X \arrow[r] \arrow[d] & \bigoplus_i \Omega^*_{H_i}\langle \widehat{H}_i\rangle(-1)[-1]\otimes_{\mathcal{O}_X}\mathcal{A}^{0,*}_X \arrow[r] \arrow[d] & \bigoplus_{|I|=2} \Omega^*_{H_I}\langle \widehat{H}_I\rangle(-2)[-2]\otimes_{\mathcal{O}_X}\mathcal{A}^{0,*}_X \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{A}^*_X \arrow[r] & \mathcal{A}^*_X\langle H\rangle \arrow[r] & \bigoplus_i \mathcal{A}^*_{H_i}\langle \widehat{H}_i\rangle(-1)[-1] \arrow[r] & \bigoplus_{|I|=2} \mathcal{A}^*_{H_I}\langle \widehat{H}_I\rangle(-2)[-2] \arrow[r] & \cdots
\end{tikzcd}$$

with vertical maps quasi-isomorphisms.

Let $(\mathcal{A}^*_X)_H$ be the total complex

$$\mathrm{Tot}([\mathcal{A}^*_X\langle H\rangle \to \bigoplus_i \mathcal{A}^*_{H_i}\langle \widehat{H}_i\rangle(-1)[-1] \to \bigoplus_{I:|I|=2} \mathcal{A}^*_{H_I}\langle \widehat{H}_I\rangle(-2)[-2] \to \cdots])\,;$$

then there is map $\mathcal{A}^*_X \to (\mathcal{A}^*_X)_H$, and one has a commutative diagram of complexes

$$\begin{tikzcd}
\mathbb{C}_X \arrow[r] & \Omega^\bullet_X \arrow[r] \arrow[d] & \mathcal{A}^\bullet_X \arrow[d] \\
& (\Omega^\bullet_X)_H \arrow[r] & (\mathcal{A}^\bullet_X)_H
\end{tikzcd}$$

In the square on the right, the maps $\Omega^\bullet_X \to \mathcal{A}^\bullet_X$, $(\Omega^\bullet_X)_H \to (\mathcal{A}^\bullet_X)_H$, and $\Omega^\bullet_X \to (\Omega^\bullet_X)_H$ are quasi-isomorphisms, thus so is $\mathcal{A}^\bullet_X \to (\mathcal{A}^\bullet_X)_H$. The map $\mathbb{C}_X \to \Omega^\bullet_X$ being a quasi-isomorphism, all the maps in this diagram are quasi-isomorphisms.

4. Each term in the second row is an $\mathcal{A}_X$-module, thus a fine sheaf on X and in particular a c-soft sheaf on X; the same holds for the terms of the third row. Each vertical map from the a term in the second row to a term in the third row induces a quasi-isomorphism upon applying the functor $\Gamma(X, -)$. Also the second row remains exact.

Therefore we obtained:

Proposition 7. The map $\Omega^*_X\langle H\rangle \to \mathcal{A}^*_X\langle H\rangle$ is a filtered quasi-isomorphism.

For a local section φ of $\Omega^*_X\langle H\rangle$ and a component H_1 of H, one has the residue $R_{H_1}(\varphi)$, a local section of $\Omega^*_{H_i}\langle \widehat{H}_i\rangle$; if H_1 is defined locally by $z_1 = 0$ and $\varphi = \dfrac{dz_1}{z_1} \wedge \psi$, then $R_{H_1}(\varphi) = 2\pi i \cdot \psi$. This gives a map of complexes $\Omega^*_X\langle H\rangle \to \Omega^*_{H_i}\langle \widehat{H}_i\rangle[-1]$, namely $dR_{H_1}(\varphi) = -R_{H_1}(d\varphi)$.

Let $e_{-1} = (2\pi i)^{-1}$ be the generator of $\mathbb{Q}(-1)$, and consider a slightly modified map map $\Omega^*_X\langle H\rangle \to \Omega^{*-1}_{H_i}\langle \widehat{H}_i\rangle(-1)$, which takes φ to $R_{H_1}(\varphi) \otimes e_{-1}$. This gives us a map of complexes

$$R_{H_i} : \Omega^*_X\langle H\rangle \to \Omega^*_{H_i}\langle \widehat{H}_i\rangle(-1)[-1]\,.$$

One verifies immediately that it takes W_m to $W_m(\Omega^*_{H_i}\langle \widehat{H}_i\rangle(-1)[-1]) = W_{m-1}(\Omega^*_{H_i}\langle \widehat{H}_i\rangle)(-1)[-1]$, namely this is a map of mixed Hodge complexes.

By Proposition 7, we obtain

Proposition 8. The map

$$\mathcal{A}^*_X \to \mathrm{Tot}([\Omega^*_X\langle H\rangle \to \bigoplus_i \mathcal{A}^*_{H_i}\langle \widehat{H}_i\rangle(-1)[-1] \to \bigoplus_{I:|I|=2} \mathcal{A}^*_{H_I}\langle \widehat{H}_I\rangle(-2)[-2] \to \cdots])$$

is a filtered quasi-isomorphism.

If we apply the global section functor $\Gamma(X, -)$ to a filtered sheaf $\mathcal{F}$, we get a filtered abelian group. If in addition the graded quotient $Gr^W\mathcal{F}$ is $\Gamma(X, -)$-acyclic, then one has $\mathrm{Gr}^W \Gamma(X, \mathcal{F}) = \Gamma(X, \mathrm{Gr}^W \mathcal{F})$.

In particular, we obtain the filtered complexes

$$\Omega^*(X)\langle H\rangle := \Gamma(X, \Omega^*_X\langle H\rangle)$$

and

$$A^{**}(X)\langle H\rangle := \Gamma(X, \mathcal{A}^{**}_X\langle H\rangle)\,.$$

Since $\mathcal{A}^{**}_X\langle H\rangle$ is a complex of fine sheaves such that $\mathrm{Gr}\,\mathcal{A}^{**}_X\langle H\rangle$ is also fine, one has $R\Gamma(X, \mathcal{A}^{**}_X\langle H\rangle) = A^{**}(X)\langle H\rangle$ as a filtered complex.

Proposition 9. The map of complexes of $\mathbb{C}$-vector spaces

$$A^*(X) \to Tot([A^*(X)\langle H\rangle \to \bigoplus_i A^*(H_i)\langle \widehat{H}_i\rangle(-1)[-1] \to \bigoplus_{I:|I|=2} A^*(H_I)\langle \widehat{H}_I\rangle(-2)[-2] \to \cdots])$$

is a filtered quasi-isomorphism with respect to the filtration W.

4 The complex $C_\bullet(X)_H$ and its dual $C^\bullet(X)_H$

Let H_I be defined as before, and $H(i) = \cup_{|I|=i} H_I$.

We will henceforth assume that the triangulation of X is of the following special type. Assume K_0 be a triangulation of X such that each H_I is a subcomplex, and let K be its first barycentric subdivision. The vertices of K are the barycenters $\hat{\sigma}$ of the simplicies σ of K_0. Give a partial ordering on the vertices of X by: $\hat{\sigma} < \hat{\sigma'}$ iff $\sigma \prec \sigma'$.

The ordering is said to be compatible with H if if v, v' are vertices, $v < v'$ and $v' \in H_I$, then $v \in H_I$; in other words x is more special than v'.

1. An m-simplex σ of X is written $\sigma = v_0 \cdots v_m$, where $v_0 < \cdots < v_m$. If $p \leq m$, the front p-face of σ is the simplex $v_0 \cdots v_p$ (also written σ_p^F) and its back $(m-p)$-face is $v_p \cdots v_m$ (also written σ_{m-p}^B). We will write then

$$\sigma = (v_0 \cdots v_p) \circ (v_p \cdots v_m)$$

the symbol $\circ$ denoting "concatenation" of simplices.

Let $\nu = v_0 \cdots v_p$ be a p-simplex. For an element $\alpha = \sum c_\sigma \cdot \sigma \in C_m(X)$, let

$$\alpha_\nu := \sum_{\sigma_p^F = \nu} c_\sigma \cdot \sigma$$

the sum over those σ with front face ν. Also define

$$(\alpha_\nu)_{m-p}^B := \sum_{\sigma_p^F = \nu} c_\sigma \cdot \sigma_{m-p}^B\,.$$

If we denote by $C_m(X)_\nu$ the submodule consisting of the α with $\alpha = \alpha_\nu$, then clearly we have $C_m(X) = \bigoplus_\nu C_m(X)_\nu$. The homomorphism

$$C_m(X) \to C_p(X) \otimes C_{m-p}(X)$$

which sends a simplex σ to $\sigma_p^F \otimes \sigma_{m-p}^B$ is called the "decatenation" map. Note it takes an element $\alpha \in C_m(X)$ to $\sum_\nu \nu \otimes (\alpha_\nu)_{m-p}^B$.

In particular, if $\nu = v$ is a vertex, $C_m(X)_v$ is the submodule of α which is a sum of simplicies with first vertex v, and $C_m(X) = \bigoplus_\nu C_m(X)_v$.

2. An m-simplex σ is said to be *H-transversal* if $\dim(\sigma \cap H_I) \leq m - 2|I|$ for each I. (Alternatively we say σ satisfies condition (T).)

We shall introduce an equivalent condition. For a positive integer k, let $N(k)$ be the number of vertices v of σ such that $v \in H(k)$. If there is no such vertex, we set $N(k) = -\infty$. Then $\sigma \cap H(k)$ has dimension equal to $N(k) - 1$ (the empty set has dimension $-\infty$ by convention). We have:

Proposition 1. *(1) σ is H-transversal iff $N(k) \le m - 2k + 1$ for $k \ge 1$.*

(2) If $\sigma = v_0 \cdots v_m$, then it is H-transversal iff $v_{m-2k+1} \notin H(k)$ for $k \ge 1$.

Definition 2. We say that an element $\alpha \in C_m(X)$ is *H-transversal* if each σ with $c_\sigma \ne 0$ is H-transversal. It is said to be *H-admissible* if α and $\partial\alpha$ are H-transversal.

Let $C_m(X)_H$ be the submodule of $C_m(X)$ consisting of the H-admissible elements; one has a subcomplex $C_\bullet(X)_H$ of $C_\bullet(X)$.

The following is obvious.

Proposition 3. *For $\alpha = \sum c_\sigma \sigma \in C_m(X)$ be H-admissible, it is necessary and sufficient that the following two conditions are satisfied:*

(i) For each σ with $c_\sigma \ne 0$, σ is H-transversal, and

(ii) For each $(m-1)$-simplex τ which is not H-transversal, the coefficient of τ in $\partial\alpha$ is zero:

$$\operatorname{coeff}(\tau; \partial\alpha) = \sum_{\sigma \succ \tau} c_\sigma [\sigma : \tau] = 0 \qquad (*)$$

where σ varies over the m-simplices having τ as a face, and $[\sigma : \tau]$ is the coefficient of τ in $\partial\sigma$.

Proposition 4. *Let $\sigma = v_0 \cdots v_m$ be an m-simplex, p is an integer with $0 \le p \le m$ and let $\nu = v_0 \cdots v_p$, $s = v_p \cdots v_m$.*

(1) If σ satisfies (T), then ν satisfies (T).

(2) If ν and s satisfy (T), then σ satisfies (T).

Proof. We write N_σ, N_ν and N_s for the function $N(k)$ for σ, ν and s, respectively.

(1) If $v_p \in H(k)$, then $v_i \in H(k)$ for $i \le p$ so we have $N_\sigma(k) = N_s(k) + p$. By hypothesis one has $N_\sigma(k) \le m - 2k + 1$, hence $N_s(k) \le (m-p) - 2k + 1$.

If $v_p \notin H(k)$, then $v_i \notin H(k)$ for $i \ge p$, thus $N_s(k) = -\infty$.

(2) Suppose $v_p \in H(k)$ so that $N_\sigma(k) = N_s(k) + p$. By assumption $N_s(k) \le (m-p) - 2k + 1$, thus $N_\sigma(k) \le m - 2k + 1$.

If $v_p \notin H(k)$, then $N_\sigma(k) = N_\nu(k)$. Since ν satisfies (T) we have $N_\nu(k) \le p - 2k + 1$, hence $N_\sigma(k) \le m - 2k + 1$. $\square$

Proposition 5. *Assume H is smooth. Let ν be a p-simplex with $p \le m - 2$. Then for any $\alpha \in C_m(X)_H$, one has $\alpha_\nu \in C_m(X)_H$.*

Proof. Let τ be an $(m-1)$-simplex which does not satisfy (T), but is a face of an m-simplex σ satisfying (T) and $\sigma_p^F = \nu$. We then claim $\tau_p^F = \nu$. Indeed assume otherwise. One can write $\sigma = v_0 \cdots v_m$ satisfying (T) (namely $v_{m-1} \notin H$) with $\nu = v_p \cdots v_p$, and $\tau = v_0 \cdots \widehat{v_i} \cdots v_p \cdots v_m$ for some $i \leq p$. But then τ satisfies (T), contradicting the assumption.

We now check condition (ii) of Proposition 4. Take an $(m-1)$-simplex τ not satisfying (T); we may assume $\tau_p^F = \nu$ by the previous paragraph. If $\alpha = \sum c(\sigma)\sigma$, then

$$\alpha_\nu = \sum c(\sigma)\sigma$$

the sum over the m-simplices σ that satisfy (T) and $\sigma_p^F = \nu$. So

$$\operatorname{coeff}(\tau; \partial\alpha_\nu) = \sum c(\sigma)[\sigma : \tau] \qquad (\text{sum over } \sigma \text{ satisfying } (T),\ \sigma_p^F = \nu \text{ and } \sigma \supset \tau)\,. \tag{1}$$

We compare this with

$$\operatorname{coeff}(\tau; \partial\alpha) = \sum c(\sigma)[\sigma : \tau] \qquad (\text{sum over } \sigma \text{ satisfying } (T) \text{ and } \sigma \supset \tau)\,. \tag{2}$$

by showing that the sums are over the same set.

For an m-simplex σ satisfying (T) such that $\sigma \supset \tau$, one has $\sigma_p^F = \nu$. Indeed writing $\tau = v_0 \cdots v_{m-1}$, one has $v_{m-2} \in H$. The additional vertex w to make up σ must satisfy $w \notin H$, and then we have $v_{m-2} < w$; since $p \leq m-2$ the front p-face of σ equals ν. This says that in the sum (1) one may let σ vary over the m-simplices satisfying (T), and $\sigma \supset \tau$. Therefore (1) and (2) are equal, but (2) is zero by hypothesis. □

Therefore if $C_m(X)_{H,\nu}$ is defined to be $C_m(X)_H \cap C_m(X)_\nu$, we have

$$C_m(X)_H = \bigoplus_\nu C_m(X)_{H,\nu}\,.$$

As a special case, we have:

Proposition 6. *Let v be a vertex. We have $C_m(X)_H = \bigoplus_v C_m(X)_{H,v}$ where $C_m(X)_{H,v}$ is defined to be $C_m(X)_H \cap C_m(X)_v$. Hence if $C_m(X)^0$ denotes the subcomplex gererated by simplicies disjoint from H, one has*

$$C_m(X)_H = \bigoplus_{v \in H} C_m(X)_{H,v} \oplus C_m(X)^0\,.$$

3. Let H be smooth. For $\alpha = \sum c_\sigma \sigma$ in $C_m(X)_H$, we recall the formula for the intersection

$$\alpha.H = \sum_s \mu(s;\alpha)\, s\,.$$

Here s varies over the $(m-2)$-simplices in H, and the integer $\mu(s;\alpha)$ is given as

$$\mu(s;\alpha) = \langle Th_H, (\alpha_s)_2^B \rangle\,.$$

Here Th_H is a cocycle representing the Thom class of H; α_s is the "s-part" of α, and $(\alpha_s)_2^B$ its back chain.

Proposition 7. *Assume H is smooth. Let ν be a p-simplex with $p \le m-2$. Then for $\alpha \in C_m(X)_H$, one has*

$$(\alpha.H)_\nu = (\alpha_\nu).H\,.$$

Proof. We have

$$\alpha.H = \sum_s \mu(s:\alpha)s$$

the sum over the $(m-2)$-simplices in H, and thus

$$\alpha.H = \sum_s \mu(s;\alpha)s \qquad (\text{sum over } s \text{ contained in } H \text{ such that } s_p^F = \nu)\,.$$

Also we have

$$(\alpha_\nu).H = \sum_s \mu(s;\alpha_\nu)s \qquad (\text{sum over } s \text{ contained in } H \text{ such that } s_p^F = \nu)\,.$$

We are thus reduced to showing, for each $(m-2)$-simplex in H with $s_p^F = \nu$, the equality

$$\mu(s;\alpha) = \mu(s;\alpha_\nu)\,.$$

This follows from the definition of μ and the identity $(\alpha_\nu)_s = \alpha_s$, a consequence of $s_p^F = \nu$. $\square$

Proposition 8. *Let ν be a p-simplex with $p \le m-2$. Let α be an element of $C_m(X)_H$ with $\alpha = \alpha_\nu$.*

(1) We have $\alpha^B_{m-p} \in C_{m-p}(X)_H$.

(2) We have

$$(\alpha^B_{m-p}).H = (\alpha.H)^B_{m-p-2} \quad \textit{in} \quad C_{m-p-2}(H)\,.$$

Proof. (1) Let $\nu = v_0 \cdots v_p$. Write $\alpha = \sum_\sigma c(\sigma)\sigma$, where σ varies over the m-simplicies satistying (T) and $\sigma_p^F = \nu$. Each σ is of the form $\nu \circ s$ where s is an $(m-p)$-simplex. Thus

$$\alpha = \sum c(\nu \circ s)\,(\nu \circ s)$$

where s varies over the $(m-p)$-simplices with initial vertex v_p such that $\nu \circ s$ satisfies (T). We thus get

$$\alpha^B_{m-p} = \sum c(\nu \circ s)\, s\,.$$

Let t be an $(m-p-1)$-simplex not satisfying (T), and we shall examine the coefficient of t in $\partial(\alpha^B_{m-p})$. Suppose first t begins with v_p. Then

$$\operatorname{coeff}\big(t;\partial(\alpha^B_{m-p})\big) = \sum_s c(\nu \circ s)\,[s:t]$$

the sum over s as above, with extra condition $s \supset t$. On the other hand with respect to the $(m-1)$-simplex $\nu \circ t$, one has

$$\mathrm{coeff}(\nu \circ t; \partial\alpha) = \sum_s c(\nu \circ s)\, [\nu \circ s : \nu \circ t]$$

the sum over the same set of s. Since $[\nu \circ s : \nu \circ t] = (-1)^p [s : t]$, we have

$$\mathrm{coeff}\,\big(t; \partial(\alpha^B_{m-p})\big) = (-1)^p \cdot \mathrm{coeff}(\nu \circ t; \partial\alpha)\,.$$

But $\nu \circ t$ does not satisfy (T) by Proposition 4, (1), so the right hand side is zero, hence the left hand side is also zero.

Suppose now t does not satisfy (T), with initial vertex $\neq v_p$. Then for an $(m-p)$-simplex s with initial vertex v_p such that $\nu \circ s$ satisfying (T), it contributes non-trivially to $\mathrm{coeff}\,\big(t; \partial(\alpha^B_{m-p})\big)$ only when $s = v_p t$. But then t satisfies (T) by Proposition 4, (1), contradicting the assumption. Therefore one has $\mathrm{coeff}\,\big(t; \partial(\alpha^B_{m-p})\big) = 0$.

(2) As in the proof of the previous proposition, we have

$$\alpha.H = \sum_s \mu(s;\alpha) s \qquad \text{(sum over } s \text{ contained in } H \text{ such that } s^F_p = \nu)\,,$$

so

$$(\alpha.H)^B_{m-p-2} = \sum_s \mu(s;\alpha) s^B_{m-p-2} \qquad \text{(sum over } (m-2)\text{-simplices } s \text{ contained in } H \text{ such that } s^F_p = \nu)\,.$$

One also has

$$(\alpha^B_{m-p-2}).H = \sum_t \mu(t;\alpha^B_{m-p}) t \qquad \text{(sum over } (m-p-2)\text{-simplices } t \text{ contained in } H \text{ with initial vertex}$$

There is a bijection between the set of $(m-2)$-simplices s contained in H such that $s^F_p = \nu$, and the set of $(m-p-2)$-simplices t contained in H with initial vertex v_p, given as follows:

$$s \mapsto s^B_{m-p-2}\,, \quad \nu \circ t \leftarrow\!\shortmid t\,.$$

Therefore we are reduced to the identity (when s corresponds to t)

$$\mu(s;\alpha) = \mu(t;\alpha^B_{m-p})$$

This immediately follows from the identity $(\alpha_s)^B_2 = \big((\alpha^B_{m-p})_t\big)^B_2$. □

Thanks to the propositions, the decatenation map induces the map

$$C_m(X)_H \to C_p(X) \otimes C_{m-p}(X)_H\,,$$

which sends $\alpha \in C_m(X)_H$ to $\sum \nu \otimes (\alpha_\nu)^B_{m-p}$ (the sum over the p-simplices ν).

Definition 9. The cup product

$$C^p(X) \otimes C^q(X)_H \to C^{p+q}(X)_H$$

is defined by dualizaing the map $C_{p+q}(X)_H \to C_p(X) \otimes C_q(X)_H$. Thus for $u \in C^p(X)$, $v \in C^q(X)_H$ and $\alpha \in C_{p+q}(X)_{H,\nu}$, we have

$$(u \cup v)(\alpha) = u(\nu)v(\alpha'') \,.$$

One has the map of restriction

$$i^* : C_\bullet(X)_H \to C_{\bullet-2}(H) \,.$$

Dualizing gives a map of complexes

$$i_* : C^\bullet(H)[-2] \to C^\bullet(X)_H \,.$$

We have a commutative diagram

$$\begin{tikzcd}
C_m(X)_H \arrow[rr] \arrow[d, "i^*"'] & & C_p(X) \otimes C_{m-p}(X)_H \arrow[d, "1\otimes i^*"] \\
C_{m-2}(H) \arrow[r] & C_p(H) \otimes C_{m-p}(H) \arrow[r, "i_*\otimes 1"'] & C_p(X) \otimes C_{m-p-2}(H)
\end{tikzcd}$$

where the lower horizontal arrow is the composition of the decatenation map with $i_* \otimes 1$. Dualizing gives a commutative diagram

$$\begin{tikzcd}
C^\bullet(X) \otimes C^\bullet(X)_H \arrow[rr, "m"] & & C^\bullet(X)_H \\
C^\bullet(X) \otimes C^\bullet(H)[-2] \arrow[u, "1\otimes i_*"] \arrow[r, "i^*\otimes 1"'] & C^\bullet(X) \otimes C^\bullet(H)[-2] \arrow[r, "m"'] & C^\bullet(H)[-2] \,. \arrow[u, "i_*"']
\end{tikzcd}$$

Refinement of the moving lemma.

1. Let M be a smooth projective complex variety of dimension d, and H a smooth divisor. Given a semi-algebraic triangulation K of M, one has the complex of K-chains $C_*^K(M)$, and the subcomplex

$$C_*^K(M)_H := \{\alpha \in C_*^K(M) \mid \alpha \text{ and } \partial\alpha \text{ meets } H \text{ properly}\}.$$

Passing to the limit over K, we obtain complexes,

$$C_*(M)_H \subset C_*(M)\,.$$

Proposition 10. *The inclusion $C_*(M)_H \subset C_*(M)$ is a homology isomorphism.*

[Proof is omitted here. In the proof, the following fact is used: For each $\alpha \in C_p(M)$, there exists $\mathcal{H}(\alpha) \in C_{p+1}(M)$ and $h(\alpha) \in C_p(M)_H$ such that

$$\partial\mathcal{H}(\alpha) + \mathcal{H}(\partial\alpha) = \alpha - h(\alpha)\,.$$

Further, if $\alpha \in C_p(M)_H$, then $\mathcal{H}(\alpha) \in C_{p+1}(M)_H$ as well.]

If K' is a refinement of K, then one has the subdivision map $\lambda : C_*^K(M) \to C_*^{K'}(M)$ as in [Mu, §17]. Since this map preserves support, it restricts to a map $\lambda : C_*^K(M)_H \to C_*^{K'}(M)_H$.

Proposition 11. *For any triangulation K, the map $\lambda : C_*^K(M)_H \to C_*^{K'}(M)_H$ is injective on homology.*

Proof. It is shown in [Mu, p.97] that if $g : K' \to K$ is a simplicial approximation to the identity of M, then λ and

$$g_\sharp : C_*^{K'}(M) \to C_*^K(M)$$

are homotopy inverse to each other, and also that $g_\sharp \circ \lambda = id$ (see [Mu, p. 100]).

One can take g so that if v is a K'-vertex not contained in H, then $g(v)$ is not contained in H either. Indeed, the K-simplex σ containing v in its interior is not contained in H, and one can take as $g(v)$ one of its vertices not in H. (See [Mu, Lemma 15.1]). If g is so made, intersection property with H of K'-chains, when $g_\sharp$ is applied, gets no worse. In particular, $g_\sharp$ restricts to define

$$g_\sharp : C_*^{K'}(M)_H \to C_*^K(M)_H\,.$$

We also have $g_\sharp \circ \lambda = id$, so the assertion follows. (The homotopy between $\lambda \circ g_\sharp$ may not restrict to homotopy between the complexes $(-)_H$, since it is not carried by a chain in $C_*^K(M)_H$. Thus we fall short of verifying homology isomorphism. □

Proposition 12. *There is a triangulation K such that the inclusion $C_*^K(M)_H \to C_*^K(M)$ is a homology isomorphism.*

Proof. We take a triangulation K. For each cycle $\alpha \in C_p^K(M)$, by what we recalled above, there exists a refinement K' of K, chains $\mathcal{H}(\alpha) \in C_{p+1}^{K'}(M)$ and $h(\alpha) \in C_p^{K'}(M)_H$ such that $\partial\mathcal{H}(\alpha) = \alpha - h(\alpha)$. Thus the image of $[\alpha] \in H_pC_*^K(M)$ in $H_pC_*^{K'}(M)$ comes from $[h(\alpha)]$ $H_pC_*^{K'}(M)_H$.

Since M is compact, the group $H_pC_*^K(M)$ is finitely generated. The above being the case for each of a finite set of generators of $H_pC_*^K(M)$, taking a refinement K' that works for them all, it follows that the map $C_*^{K'}(M)_H \subset C_*^{K'}(M)$ is surjective on homology.

Next, in the commutative diagram

$$\begin{array}{ccc} H_p(C_*^{K'}(M)_H) & \longrightarrow & H_p(C_*^{K'}(M)) \\ \big\downarrow & & \big\downarrow{\scriptstyle\cong} \\ H_p(C_*(M)_H) & \xrightarrow{\cong} & H_p(C_*(M)) \end{array}$$

the lower horizontal arrow is an isomorphism by Proposition 1, the right vertical arrow is obviously an isomorphism. Further the left vertical arrow is an injection by Proposition 2, and the upper horizontal arrow is a surjection as we have just shown. It follows that all the maps are isomorphisms. $\square$

§5. The map $c : A^\bullet(X)_H \to C^\bullet(X)_H$ and its multiplicativity

One has the map $c : A^\bullet(X) \to C^\bullet(X)$ and also the map $c : A^\bullet(X)_H \to C^\bullet(X)_H$ defined by

$$\langle c(\varphi_X, \varphi_H), \alpha \rangle = \int_\alpha \varphi_X + \int_{\alpha.H} \varphi_H \,.$$

The latter is a map of complexes by what we call the Cauchy-Stokes formula:

Theorem 1. Let $\alpha \in C_m(X)_H$. Then one has

$$-\int_{\partial\alpha} \varphi + \int_\alpha d\varphi + \int_{i^*\alpha} R_H(\varphi) = 0\,.$$

This can be shown by considering the Stokes formula applied to the chain α excised near H, and then taking the limit.

Recall the wedge product $A^\bullet(X) \otimes A^\bullet(X)_H \to A^\bullet(X)_H$ and and cup product $\cup : C^\bullet(X) \otimes C^\bullet(X)_H \to C^\bullet(X)_H$ defined before.

Let $A^{(2),\bullet}(X)_H = A^\bullet(X) \otimes A^\bullet(X)_H$ and $C^{(2),\bullet}(X)_H = C^\bullet(X) \otimes C^\bullet(X)_H$. There are products $m : A^{(2),\bullet}(X)_H \to A^\bullet(X)_H$ and $m : C^{(2),\bullet}(X)_H \to C^\bullet(X)_H$.

For a vertex v in H. let $M_v = \bar{D}(v)$ be the corresponding dual cell. The collection of these dual cells together with the simplices $s = \Delta^p$ disjoint from H will play the role of "models". We will write M for one of these M_v or s. Write $i : M \to X$ for the inclusion in either case. There are restriction maps $i^* : A^\bullet(X)_H \to A^\bullet(M)_H$ and $i^* : C^\bullet(X)_H \to C^\bullet(M)_H$, as well as $i^* : A^{(2),\bullet}(X)_H \to A^{(2),\bullet}(M)_H$ and $i^* : C^{(2),\bullet}(X)_H \to C^{(2),\bullet}(M)_H$.

Proposition 2. There exist a map of degree -1

$$\rho_X : A^\bullet(X) \otimes A^\bullet(X)_H \to C^\bullet_{\mathbb{C}}(X)_H$$

and a map of degree -1

$$\rho_M : A^\bullet(M) \otimes A^\bullet(M)_H \to C^\bullet_{\mathbb{C}}(M)_H$$

for each model M, which satisfy the identities $d\rho + \rho d = -m(c \otimes c) + cm$ and $i^*\rho_X = \rho_M i^*$ for $i : M \to X$.

Proof. If ρ^m is the restriction of ρ to $A^{(2),m}(X)_H$ (or $A^{(2),m}(M)_H$), we need the condition

$$d\rho^{m-1} + \rho^m d = -m(c \otimes c) + cm\,. \qquad (*)_{m-1}$$

on X or M. If $\rho^m = 0$ for $m \le 0$, then $(*)_{m-1}$ holds for $m \le 0$. Assuming ρ^j for $j < m$ have been found, let

$$\theta^{m-1} = -d\rho^{m-1} - m(c \otimes c) + cm : A^{(2),m-1}(X) \to C^{m-1}_{\mathbb{C}}(X)$$

and similarly on M. Note that $i^*\theta^{m-1} = \theta^{m-1} i^*$ holds.

If we choose a base point $b \in M - H$, there is a map $S : A^\bullet(M)_H \to A^\bullet(M)_H$ of degree -1 such that $dS + Sd = 1 - r_b^*$. There is a similar homotopy for the complex $A^\bullet(M)$, hence there is an induced map $S : A^{(2),\bullet}(M)_H \to A^{(2),\bullet}(M)_H$ of degree -1 satisfying $dS + Sd = 1 - r_b^*$. Since r_b^* is zero in degrees $\neq 0$, and since $\theta^0 = -m(c \otimes c) + cm = 0$, one has $\theta^{m-1} r_b^* = 0$ for all m.

We now produce a map $\rho^m : A^{(2),m}(X)_H \to C_{\mathbb{C}}^{m-1}(X)_H$. Let $a \in A^{(2),m}(X)_H$. If v is a vertex in H, $i : M = M_v \to X$ the inclusion, one has $i^* a \in A^{(2),m}(M)_H$. For $\alpha \in C_{m-1}(X)_{H,v}$, recalling that $\alpha \in C_{m-1}(M_v)_H$ we let

$$\langle \rho^m(a), \alpha \rangle = \langle \theta^{m-1} S i^* a, \alpha \rangle \in \mathbb{C}\,.$$

For a simplex $v : s = \Delta^{m-1} \to X$ disjoint from H, let $\langle \rho^m(a), v \rangle = \langle \theta^{m-1} S v^* a, 1_s \rangle$. Since

$$C_{m-1}(X)_H = \bigoplus C_{m-1}(X)_{H,v} \oplus C_{m-1}(X)^0$$

this defines an element $\rho^m(a) \in C_{\mathbb{C}}^{m-1}(X)_H$.

If in this argument X is replaced with a “model” M, one obtains a map $\rho^m : A^{(2),m}(M)_H \to C_{\mathbb{C}}^{m-1}(M)_H$.

(i) $i^* \rho^m = \rho^m i^*$ for a map $i : M \to X$.

If $v' > v$ is another vertex in H, and $\alpha \in C_{m-1}(X)_{H,v'}$, then for the inclusion $i : M_v \to X$ we have

$$\begin{aligned} \langle i^* \rho^m(a), \alpha \rangle &= \langle \rho^m(a), \alpha \rangle \\ &= \langle \theta^{m-1} S i'^* a, \alpha \rangle \end{aligned}$$

where $i' : M_{v'} \to X$. The map i' factors as $M_{v'} \xrightarrow{k} M_v \xrightarrow{i'} X$, and one has

$$\langle \rho^m(i^* a), \alpha \rangle = \langle \theta^{m-1} S(k^* i^* a), \alpha \rangle\,.$$

The two thus coincide. The verification for $i : s \to X$ is obvious.

(ii) $\theta^{m-1} d = 0$. (Follows from $(*)_{m-2}$.)

(iii) $\theta^{m-1} S d = \theta^{m-1}$ on M. (Follows from (ii) and the fact $\theta^{m-1} r_b^* = 0$.)

(iv) One has $\rho^m d = \theta^{m-1}$ on X and M.

For $\alpha \in C_{m-1}(X)_{H,v}$, we have

$$\begin{aligned} \langle \rho^m d(a), \alpha \rangle &= \langle \theta^{m-1} S i^*(da), \alpha \rangle \\ &= \langle \theta^{m-1} i^*(a), \alpha \rangle && [\text{by (iii)}] \\ &= \langle i^* \theta^{m-1}(a), \alpha \rangle && [\text{by } i^* \theta^{m-1} = \theta^{m-1} i^*] \\ &= \langle \theta^{m-1}(a), \alpha \rangle\,. \end{aligned}$$

For $s \in C_{m-1}(X)$, the same reasoning holds. If X is replaced with M, the same argument holds. $\square$

Proposition 3. *Assume that ρ' is another functorial map satisfying the same property as for ρ. Then there exists a map $\pi : A^{(2),\bullet}(X) \to C^{\bullet}_{\mathbb{C}}(X)$ of degree -2, and a similar map on M, satisfying*

$$d\pi + \pi d = \rho - \rho'$$

on X (and on M), and the identity $i^*\pi = \pi i^*$.

We need a variant of Proposition 2. Consider now the failure of commutativitiy of the diagram

$$\begin{tikzcd} A^\bullet(X)\otimes A^\bullet(H)[-2] \arrow[r, "m(i^*\otimes 1)"] \arrow[d, "c\otimes c\, i_*"'] & A^\bullet(H)[-2] \arrow[d, "c\, i_*"] \\ C^\bullet(X)\otimes C^\bullet(X)_H \arrow[r, "m"'] & C^\bullet(X)_H\,. \end{tikzcd} \qquad (**)$$

in which the top horizontal map is the composition $A^\bullet(X)\otimes A^\bullet(H)[-2]\xrightarrow{i^*\otimes 1}A^\bullet(H)\otimes A^\bullet(H)[-2]\xrightarrow{m}A^\bullet(H)[-2]$, and the $c\,i_*$ is the composition $A^\bullet(H)[-2]\xrightarrow{i_*}A^\bullet(X)_H\xrightarrow{c}C^\bullet(X)_H$.

As in Proposition 2 also consider the diagram accompanying it, obtained by replacing X with a "model" M, and H with $M\cap H$:

$$\begin{tikzcd} A^\bullet(M)\otimes A^\bullet(M\cap H)[-2] \arrow[r, "m(i^*\otimes 1)"] \arrow[d, "c\otimes c\, i_*"'] & A^\bullet(M\cap H)[-2] \arrow[d, "c\, i_*"] \\ C^\bullet(M)\otimes C^\bullet(M)_H \arrow[r, "m"'] & C^\bullet(M)_H\,. \end{tikzcd}$$

Note that if $M=s$ is a simplex disjoint from H, then $M\cap H$ is empty and $A^\bullet(M\cap H)=0$.

Proposition 4. *There exist a map of degree -1*

$$\rho_X : A^\bullet(X)\otimes A^\bullet(H)[-2]\to C^\bullet_{\mathbb{C}}(X)_H$$

and a map of degree -1

$$\rho_M : A^\bullet(M)\otimes A^\bullet(M\cap H)[-2]\to C^\bullet_{\mathbb{C}}(M)_H$$

for each model M, which satisfy the identities $d\rho+\rho d=-m(c\otimes c)+cm$ on X and M, and $i^*\rho_X=\rho_M i^*$ for $i: M\to X$.

Further, if (ρ'_X,ρ'_M) is another collection of maps satisfying the same property, there exists a map of degree -2,

$$\nu_X : A^\bullet(X)\otimes A^\bullet(H)[-2]\to C^\bullet(X)_H$$

and a map of degree -2

$$\nu_M : A^\bullet(M)\otimes A^\bullet(M\cap H)[-2]\to C^\bullet(M)_H$$

for each M, satisfying

$$d\nu+\nu d=\rho-\rho'\,,$$

and $i^*\nu=\nu i^*$ for $i: M\to X$.

The proof of this is parallel to that for Proposition 2, with some differences as we point out. The identity $m(c\otimes c)=cm$ in degree 0 for the previous proposition must be replaced with:

Lemma 5. *The following diagram commutes:*

$$\begin{tikzcd} A^0(X) \otimes (A^\bullet(H)[-2])^2 \arrow[r, "m"] \arrow[d, "c\otimes c"'] & (A^\bullet(H)[-2])^2 \arrow[d, "c"] \\ C^0(X) \otimes C^2(X)_H \arrow[r, "m"'] & C^2(X)_H\,. \end{tikzcd}$$

Proof. For $f \otimes g \in A^0(X) \otimes A^0(H)$, one must show $c(i_*(f|_H \cdot g)) = c(f) \cup c(i_* g)$.

For an element $\alpha \in C_2(X)_{H,v}$, with v a vertex in H, we have

$$\langle c(i_*(f|_H \cdot g)), \alpha\rangle = \int_{\alpha.H} f|_H \cdot g\,;$$

if $\alpha.H = mv$, then the right hand side equals $m(f \cdot g)(v)$. On the other hand,

$$\langle c(f) \cup c(i_* g), \alpha\rangle = f(v) \int_{\alpha.H} g = f(v) \cdot mg(v)\,,$$

so the two coincide.

As for a simplex $s \in C_2(X)$ disjoint from H, both cocycles obviously take the value zero. $\square$

Proof of Proposition 4. We take the proof of Proposition 2 and repeat it with changes as follows.

• Let ρ^m be the restriction of ρ to the degree m part of the complex $A^\bullet(X) \otimes A^\bullet(H)[-2]$. We set $\rho^m = 0$ for $m \leq 2$.

Let $m > 2$ and proceed to find ρ^m. Defining θ^{m-1} as before. We have $\theta^2 = 0$ by the above lemma, and $r_b^* = 0$ in degree $\neq 2$, thus $\theta^{m-1} r_b^* = 0$ in all degrees.

• If $M = M_v = \overline{D}(v)$, then $M \cap H = \overline{D}_H(v)$, the dual cell of v in the simplicial complex H. Since $\overline{D}_H(v)$ is contractible, there exists a map S from $A^\bullet(M \cap H)$ to itself satisfying $dS + Sd = 1 - r_b^*$, with $b \in M \cap H$. It follows that there is a map S of degree -1 from $A^\bullet(M) \otimes A^\bullet(M \cap H)$ to itself with the property $dS + Sd = 1 - r_b^*$.

• In defining $\rho^m(a)$ for an element a of degree m in $A^\bullet(X) \otimes A^\bullet(H)[-2]$, one has $\langle \rho^m(a), s\rangle = 0$ for simplices s disjoint from H. $\square$

Our goal is Theorem 9. We first note the facts (projection formulas):

Lemma 6. *The following digram commutes.*

$$\begin{tikzcd} A(X) \otimes A(H)[-2] \arrow[r, "i^*\otimes 1"] \arrow[d, "1\otimes i_*"'] & A(H) \otimes A(H)[-2] \arrow[r, "m"] & A(H)[-2] \arrow[d, "i_*"] \\ A(X) \otimes A(X)_H \arrow[rr, "m"] & & A(X)_H \end{tikzcd}$$

Lemma 7. *The following digram commutes.*

$$\begin{tikzcd}
C(X)\otimes C(H)[-2] \arrow[r, "i^*\otimes 1"] \arrow[d, "1\otimes i_*"'] & C(H)\otimes C(H)[-2] \arrow[r, "m"] & C(H)[-2] \arrow[d, "i_*"] \\
C(X)\otimes C(X)_H \arrow[rr, "m"] & & C(X)_H
\end{tikzcd}$$

The map $c\, i_* m(i^*\otimes 1) : A^\bullet(X)\otimes A^\bullet(H)[-2] \to C^\bullet(X)_H$ appearing in the square $(**)$ is equal to $cm(1\otimes i_*)$ by the projection formula for $A^\bullet(X)$, and also to $i_* cm(i^*\otimes 1)$ by $ci_* = i_* c$. The other map in the same diagram $m(c\otimes c\, i_*)$ is equal to $m(c\otimes c)(1\otimes i_*)$ clearly, and also

$$m(c\otimes c\, i_*) = m(1\otimes i_*)(c\otimes c) = i_* m(i^*\otimes 1)(c\otimes c) = i_* m(c\otimes c)(i^*\otimes 1)$$

using the projection formula for $C^\bullet(X)$ and $ci^* = i^* c$.

Let now ρ_X be a homotopy as in Proposition 2; similarly let $\rho_H : A(H)\otimes A(H)[-2] \to C(H)[-2]$ be a map such that

$$d\rho_H + \rho_H d = -m(c\otimes c) + cm : A(H)\otimes A(H)[-2] \to C(H)[-2]\,.$$

The element $\rho_X(1\otimes i_*)$ gives a homotopy between the maps

$$\rho_X(1\otimes i_*) : m(c\otimes c)(1\otimes i_*) \simeq cm(1\otimes i_*)$$

and $i_*\rho_H(i^*\otimes 1)$ gives a homotopy

$$i_*\rho_H(i^*\otimes 1) : i_* m(c\otimes c)(i^*\otimes 1) \simeq i_* cm(i^*\otimes 1)\,.$$

But we know that the source and the target for the maps are the same, thus by the latter half of Proposition 4 there is a map ν of degree -2 giving homotopy

$$\nu : \rho_X(1\otimes i_*) \simeq i_*\rho_H(i^*\otimes 1)\,.$$

Proposition 8. *Let*

$$\begin{tikzcd}
K \arrow[r, "u"] \arrow[d, "f"'] & L \arrow[d, "g"] \\
K' \arrow[r, "u'"'] & L'
\end{tikzcd}$$

be a commutative diagram of complexes. Assume there exists a map $\xi : K \to L$ (resp. $\xi' : K' \to L'$) of degree -1 such that $u = d\xi + \xi d$ (resp. $u' = d\xi' + \xi' d$). Assume also there exists a map $\nu : K \to L'$ of degree -2 such that

$$g\xi - \xi' f = d\nu + \nu d\,.$$

Then the map

$$(u, u') : C_f \to C_g$$

is homotopic to zero.

One has the map

$$\mathcal{P} : A(X, H) \to D(A(X)\langle\!\langle H\rangle\!\rangle)\,.$$

Let $\mathcal{Q}_X : A(X, H) \to D(A(X)\langle\!\langle H\rangle\!\rangle)$ be the composition of the maps

$$A(X, H) \xrightarrow{c} C^\bullet(X, H)_H \xrightarrow{\kappa} C_{2n-2-\bullet}(X \mid H) \xrightarrow{\Phi} D(A(X)\langle\!\langle H\rangle\!\rangle)\,.$$

We apply the above to the diagram

$$\begin{tikzcd}
A^\bullet(X) \arrow[r, "\mathcal{P}_X - \mathcal{Q}_X"] \arrow[d, "i^*"'] & D(A^\bullet(X)_H) \arrow[d, "i^*"] \\
A^\bullet(H) \arrow[r, "\mathcal{P}_H - \mathcal{Q}_H"'] & D(A^\bullet(H)[-2])
\end{tikzcd}$$

and the maps ρ_X, ρ_H and ν. We obtain:

Theorem 9. There exists a map $\xi : A(X)_H \to D(A(X)\langle\!\langle H\rangle\!\rangle)$ of degree -1 such that

$$d\xi + \xi d = \mathcal{P} - \mathcal{Q}\,.$$

§6. The explicit complex $\mathbb{E}(X,H)$

Let H be a smooth divisor on X. One has a map

$$\Phi : C_*(X) \to D(A(X)_H)$$

given by

$$\langle \Phi(\alpha), (\varphi_X, \varphi_H)\rangle = \int_\alpha \varphi_X + \int_{\alpha.H} \varphi_H \,.$$

Similarly one has $\Phi : C_*(H) \to D(A(H))$. The inclusion $A(H)[-2] \to A(X)_H$ induces a surjection $i^* : D(A(X)_H) \to D(A(H)[-2])$. The following square commutes:

$$\begin{array}{ccc} C_*(X)_H & \xrightarrow{\Phi} & D(A(X)_H) \\ {\scriptstyle i^*}\big\downarrow & & \big\downarrow{\scriptstyle i^*} \\ C_{*-2}(H) & \xrightarrow{\Phi} & D(A(H)[-2]) \end{array}$$

There is the map $\mathcal{P} : A^\bullet(X) \to D(A(X)_H)$ given by

$$\langle \mathcal{P}(\omega), (\varphi_X, \varphi_H)\rangle = \int_X \omega \wedge \varphi_X + \int_H (\omega|_H) \wedge \varphi_H \,.$$

One also has a similar map $\mathcal{P} : A^\bullet(H) \to D(A(H))$. The following square commutes:

$$\begin{array}{ccc} A^\bullet(X)_H & \xrightarrow{\mathcal{P}} & D(A(X)_H) \\ {\scriptstyle i^*}\big\downarrow & & \big\downarrow{\scriptstyle i^*} \\ A^\bullet(H) & \xrightarrow{\mathcal{P}} & D(A(H)[-2]) \end{array}$$

The commutative diagram of complexes

$$\begin{array}{ccccc} C_*(X)_H & \xrightarrow{\Phi} & D(A(X)_H) & \xleftarrow{\mathcal{P}} & A^\bullet(X)_H \\ {\scriptstyle i^*}\big\downarrow & & {\scriptstyle i^*}\big\downarrow & & \big\downarrow{\scriptstyle i^*} \\ C_{*-2}(H) & \xrightarrow{\Phi} & D(A(H)[-2]) & \xleftarrow{\mathcal{P}} & A^\bullet(H) \end{array}$$

gives a Hodge complex; it may be abbreviated to

$$[C_*(X|H) \xrightarrow{\Phi} D(A(X)\langle\!\langle H\rangle\!\rangle) \xleftarrow{\mathcal{P}} A^\bullet(X,H)] \,.$$

By means of the canonical map $s^* : D(A(X)\langle\!\langle H\rangle\!\rangle) \to D(A(X)\langle H\rangle)$ we obtain another Hodge complex

$$[C_*(X|H) \xrightarrow{\Phi} D(A(X)\langle H\rangle) \xleftarrow{\mathcal{P}} A^\bullet(X,H)] \,.$$

This is the explicit Hodge complex $\mathbb{E}(X,H)$.

Definition 1. The complex

$$\big[\Gamma(X, \mathcal{C}^\bullet\mathbb{Q}) \to \Gamma(H, \mathcal{C}^\bullet\mathbb{Q})\big] = \mathrm{Cone}(i^*)[-1]$$

will be abbreviated to $\Gamma(X,H;\mathcal{C}^\bullet\mathbb{Q})$. Similarly one defines the complexes $\Gamma(X,H;\mathcal{C}^\bullet\mathcal{A}^\bullet)$ and $\Gamma(X,H;\mathcal{A}^\bullet)$. There are maps among these complexes

$$\big[\Gamma(X,H;\mathcal{C}^\bullet\mathbb{Q}) \longrightarrow \Gamma(X,H;\mathcal{C}^\bullet\mathcal{A}^\bullet) \longleftarrow \Gamma(X,H;\mathcal{A}^\bullet)\big] \,.$$

This triple gives a Hodge complex, denoted $\mathbb{K}(X,H)$.

Similarly we have a triple of complexes

$$\left[S^\bullet(X,H) \xrightarrow{\alpha} (\mathcal{S}^\bullet \otimes \mathcal{A}^\bullet)(X,H) \xleftarrow{\beta} A^\bullet(X,H)\right].$$

This gives a Hodge complex denoted $\overline{\mathbb{K}}(X,H)$.

One shows that there is a quasi-isomorphism between the Hodge complexes $\mathbb{K}(X,H)$ and $\overline{\mathbb{K}}(X,H)$.

Proposition 2. There exists a quasi-isomorphism between the Hodge complexes $\mathbb{K}(X,H)$ and $\mathbb{E}(X,H)$.

In the following diagram

$$\begin{tikzcd}
S^\bullet(X,H) \arrow[r, "\alpha"] \arrow[d, "\kappa"'] & (\mathcal{S}^\bullet \otimes \mathcal{A}^\bullet)(X,H) \arrow[d, "\lambda"'] & A^\bullet(X,H) \arrow[l, "\beta"'] \arrow[d, "id"] \\
C_{2n-\bullet}(X|H) \arrow[r, "\Phi"] & D(A^{2n-\bullet}(X)\langle\langle H\rangle\rangle) & A^\bullet(X,H) \arrow[l, "\mathcal{P}"']
\end{tikzcd}$$

the left square commutes and the right square commutes up to homotopy.

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