Relative algebraic correspondences and mixed motivic sheaves

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Abstract

We introduce the notion of a *quasi DG category*, and give a procedure to construct a triangulated category associated to it. Then we apply it to the construction of the triangulated category of mixed motivic sheaves over a base variety.

Introduction. We will introduce the notion of a *quasi DG category*, generalizing that of a DG category. To a quasi DG category satisfying certain additional conditions, we associate another quasi DG category, the quasi DG category of *C-diagrams*. We then show the homotopy category of the quasi DG category of C-diagrams has the structure of a triangulated category (see §1).

The main example of a quasi DG category comes from algebraic geometry, as explained in §2. We establish a theory of complexes of *relative correspondences*; it generalizes the theory of complexes of correspondences of smooth projective varieties, as developed in [6] . The class of smooth quasi-projective varieties equipped with projective maps to a fixed quasi-projective variety S, and the complexes of relative correspondences between them constitute a quasi DG category, denoted $Symb(S)$.

We apply the above procedure to $Symb(S)$ to obtain $\mathcal{D}(S)$, the triangulated category of mixed motives over S. If the base variety is the Spec of the ground field, this coincides with the triangulated category of motives as in [6] .

The full details of this article will appear elsewhere (see [8] for §2, [9] for §1).

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Notation and conventions. (a) A double complex $A = (A^{i,j}; d', d'')$ is a bi-graded abelian group with differentials d' of degree $(1,0)$, d'' of degree $(0,1)$, satisfying $d'd'' + d''d' = 0$. Its total complex $\mathrm{Tot}(A)$ is the complex with $\mathrm{Tot}(A)^k = \bigoplus_{i+j=k} A^{i,j}$ and the differential $d = d' + d''$.

Let (A, d_A) and (B, d_B) be complexes. Then the tensor product complex $A \otimes B$ is the graded abelian group with $(A \otimes B)^n = \oplus_{i+j=n} A^i \otimes B^j$, and with differential d given by

$$d(x \otimes y) = (-1)^{\deg y} dx \otimes y + x \otimes dy \ .$$

Note this differs from the usual convention. Alternatively one obtains the same complex by viewing $A \otimes B$ as a double complex with differentials $(-1)^j d \otimes 1$ and $1 \otimes d$ and taking its total complex.

More generally for $n \geq 2$ one has the notion of n-tuple complex. An n-tuple complex is a $\mathbb{Z}^n$-graded abelian group $A^{i_1, \cdots, i_n}$ with differentials $d_1, \cdots, d_n$, d_k raising i_k by 1, such that for $k \neq \ell$, $d_k d_\ell + d_\ell d_k = 0$. A single complex $\mathrm{Tot}(A)$, called the total complex, is defined. For n complexes $A_1^\bullet, \cdots, A_n^\bullet$, the tensor product $A_1^\bullet \otimes \cdots \otimes A_n^\bullet$ is an n-tuple complex; one can take its total complex as well.

(b) Let I be a non-empty finite totally ordered set (we will simply say a finite ordered set), so $I = \{i_1, \cdots, i_n\}, i_1 < \cdots < i_n$, where $n = |I|$. Let $\mathrm{in}(I) = i_1$, $\mathrm{tm}(I) = i_n$, and $\mathring{I} = I - \{\mathrm{in}(I), \mathrm{tm}(I)\}$. For example, for a positive integer n, $I = [1, n] = \{1, \cdots, n\}$ is finite ordered set. In this case, if $n \geq 2$, $\mathring{I} = (1, n) := \{2, \cdots, n-1\}$. If $I = \{i_1, \cdots, i_n\}$, a subset I' of the form $[i_a, i_b] = \{i_a, \cdots, i_b\}$ is called a *sub-interval.*

Given a subset of $\mathring{I}$, $\Sigma = \{i_1, \cdots, i_{a-1}\}$, where $i_1 < i_2 < \cdots < i_{a-1}$, one has a decomposition of I into the sub-intervals $I_1, \cdots, I_a$, where $I_k = [i_{k-1}, i_k]$, with $i_0 = i_1$, $i_a = i_n$. Thus the sub-intervals satisfy $I_k \cap I_{k+1} = \{i_k\}$ for $k = 1, \cdots, a-1$. The sequence $I_1, \cdots, I_a$ is called the *segmentation* of I corresponding to Σ.

§1. Quasi DG categories and triangulated categories.

The notion of a quasi DG category is a generalization of that of a DG category. Recall that a DG category is an additive category $\mathcal{C}$, such that for a pair of objects X, Y the group of homomorphisms $F(X,Y)$ has the structure of a complex, and the composition $F(X,Y) \otimes F(Y,Z) \to F(X,Z)$ is a map of complexes.

(1.1) **Definition.** A *quasi DG category* $\mathcal{C}$ consists of data (i)-(iii), satisfying the conditions (1)-(5). When necessary we will also impose additional structure (iv),(v), satisfying (6)-(11).

(i) *The class of objects* $Ob(\mathcal{C})$. There is a distinguished object O, called the zero object. There is direct sum of objects $X \oplus Y$, and one has $(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z)$.

(ii) *Multiple complexes* $F(X_1, \cdots, X_n)$. For each sequence of objects $X_1, \cdots X_n$ $(n \geq 2)$, a complex of free abelian groups $F(X_1, \cdots, X_n)$.

For a subset $S \subset (1, n)$, let $I_1, \cdots, I_a$ be the segmentation of $I = [1, n]$ corresponding to S, and $F(X_1, \cdots, X_n \rceil S) := F(I_1) \otimes \cdots \otimes F(I_a)$; this is an a-tuple complex. More generally, for a finite ordered set I with cardinality ≥ 2 and a sequence of objects $(X_i)_{i \in I}$, one has $F(I) = F(I; X)$ and $F(I \rceil S) = F(I \rceil S; X)$.

(iii) *Multiple complexes* $F(X_1, \cdots, X_n|S)$ *and maps* ι_S, $\sigma_{S\,S'}$ *and* φ_K.

(1) We require given a quasi-isomorphic multiple subcomplex of free abelian groups

$$\iota_S : F(X_1, \cdots, X_n|S) \hookrightarrow F(X_1, \cdots, X_n \rceil S) \ .$$

We assume $F(X_1, \cdots, X_n|\emptyset) = F(X_1, \cdots, X_n)$. The $F(X_1, \cdots, X_n|S)$ is additive in each variable, namely the following properties are satisfied: If a variable $X_i = O$, then it is zero. If $X_1 = Y_1 \oplus Z_1$, then one has a direct sum decomposition of complexes

$$\begin{aligned} &F(Y_1 \oplus Z_1, X_2, \cdots, X_n|S) \\ &= F(Y_1, \cdots, X_n|S) \oplus F(Z_1, \cdots, X_n|S) \ . \end{aligned}$$

The same for X_n. If $1 < i < n$ and $X_i = Y_i \oplus Z_i$, then there is a direct sum decomposition of complexes

$$\begin{aligned} &F(X_1, \cdots, X_{i-1}, Y_i \oplus Z_i, X_{i+1}, \cdots, X_n|S) \\ &= F(X_1, \cdots, Y_i, \cdots, X_n|S) \\ &\oplus F(X_1, \cdots, Z_i, \cdots, X_n|S) \\ &\oplus F(X_1, \cdots, Y_i|S_1) \otimes F(Z_i, \cdots, X_n|S_2) \\ &\oplus F(X_1, \cdots, Z_i|S_1) \otimes F(Y_i, \cdots, X_n|S_2) \end{aligned}$$

where S_1, S_2 is the partition of S by i, namely $S_1 = S \cap (1, i)$, $S_2 = S \cap (i, n)$. We often refer to the last two terms as the *cross terms.* (Note the complex

$F(X_1, \cdots, X_n \top S)$ is additive in this sense.) The inclusion ι_S is compatible with the additivity.

For a subset $T \subset S$, if $I_1, \cdots, I_c$ is the segmentation corresponding to T, and $S_i = S \cap \overset{\circ}{I}_i$, one requires there is an inclusion of multiple complexes

$$F(I|S) \subset F(I_1|S_1) \otimes \cdots \otimes F(I_c|S_c) \tag{1.1.1}$$

where the latter group is viewed as a subcomplex of $F(I \top S)$ by the tensor product of the inclusions $\iota_{S_i} : F(I_i|S_i) \hookrightarrow F(I_i \top S_i)$.

(2) For $S \subset S'$ given a surjective quasi-isomorphism of multiple complexes

$$\sigma_{S\,S'} : F(X_1, \cdots, X_n|S) \to F(X_1, \cdots, X_n|S') \ .$$

For $S \subset S' \subset S''$, $\sigma_{S\,S''} = \sigma_{S'\,S''}\sigma_{S\,S'}$. The $\sigma_{SS'}(X_1, \cdots, X_n)$ is additive in each variable, namely if $X_i = Y_i \oplus Z_i$, then $\sigma_{SS'}(X_1, \cdots, X_n)$ is the direct sum of the maps $\sigma_{SS'}(X_1, \cdots, Y_i, \cdots, X_n)$, $\sigma_{SS'}(X_1, \cdots, Z_i, \cdots, X_n)$, and the maps

$$\begin{aligned} \sigma_{S_1 S_1'} \otimes \sigma_{S_2 S_2'} &: F(X_1, \cdots, Y_i|S_1) \otimes F(Z_i, \cdots, X_n|S_2) \\ &\to F(X_1, \cdots, Y_i|S_1') \otimes F(Z_i, \cdots, X_n|S_2') \ , \end{aligned}$$

$$\begin{aligned} \sigma_{S_1 S_1'} \otimes \sigma_{S_2 S_2'} &: F(X_1, \cdots, Z_i|S_1) \otimes F(Y_i, \cdots, X_n|S_2) \\ &\to F(X_1, \cdots, Z_i|S_1') \otimes F(Y_i, \cdots, X_n|S_2') \ , \end{aligned}$$

on the cross terms.

The σ is assumed compatible with the inclusion in (1.1.1): If $S \subset S'$ and $S_i' = S' \cap \overset{\circ}{I}_i$ the following commutes:

$$\begin{array}{ccc} F(I|S) & \hookrightarrow & F(I_1|S_1) \otimes \cdots \otimes F(I_1|S_1) \\ {\scriptstyle \sigma_{SS'}} \big\downarrow & & \big\downarrow {\scriptstyle \otimes \sigma_{S_i\,S_i'}} \\ F(I|S') & \hookrightarrow & F(I_1|S_1') \otimes \cdots \otimes F(I_1|S_1') \ . \end{array}$$

We write $\sigma_S = \sigma_{\emptyset\,S} : F(I) \to F(I|S)$. The composition of σ_S and ι_S is denoted $\tau_S : F(I) \to F(I \top S)$.

(3) For $K = \{k_1, \cdots, k_b\} \subset (1, n)$ disjoint from S, a map of multiple complexes

$$\begin{aligned} &\varphi_K : F(X_1, \cdots, X_n|S) \\ &\to F(X_1, \cdots, \widehat{X_{k_1}}, \cdots, \widehat{X_{k_b}}, \cdots, X_n|S) \ . \end{aligned}$$

If $K = K' \amalg K''$ then $\varphi_K = \varphi_{K''}\varphi_{K'} : F(I|S) \to F(I-K|S)$. The φ_K is additive in each variable: If $X_i = Y_i \oplus Z_i$, then $\varphi_K(X_1, \cdots, X_n)$ is the sum of $\varphi_K(X_1, \cdots, Y_i, \cdots, X_n)$, $\varphi_K(X_1, \cdots, Z_i, \cdots, X_n)$, and, if $i \notin K$, the maps

$$\varphi_{K_1} \otimes \varphi_{K_2} \text{ on } F(X_1, \cdots, Y_i \intercal S_1) \otimes F(Z_i, \cdots, X_n \intercal S_2) \ ,$$

$$\varphi_{K_1} \otimes \varphi_{K_2} \text{ on } F(X_1, \cdots, Z_i \intercal S_1) \otimes F(Y_i, \cdots, X_n \intercal S_2)$$

on the cross terms (S_1, S_2 is the partition of S by i, and K_1, K_2 is the partition of K by i), and if $i \in K$, the zero maps on the cross terms.

φ_K is assumed to be compatible with the inclusion in (1.1.1): With the same notation as above and $K_i = K \cap I_i$, the following commutes:

$$\begin{tikzcd}
F(I|S) \arrow[r, hook] \arrow[d, "\varphi_K"'] & F(I_1|S_1) \otimes \cdots \otimes F(I_c|S_c) \arrow[d, "\otimes \varphi_{K_i}"] \\
F(I-K|S) \arrow[r, hook] & F(I_1 - K_1|S_1) \otimes \cdots \otimes F(I_c - K_c|S_c) \ .
\end{tikzcd}$$

If K and S' are disjoint the following commutes:

$$\begin{tikzcd}
F(I|S) \arrow[r, "\varphi_K"] \arrow[d, "\sigma_{SS'}"'] & F(I-K|S) \arrow[d, "\sigma_{SS'}"] \\
F(I|S') \arrow[r, "\varphi_K"] & F(I-K|S') \ .
\end{tikzcd}$$

(4) (acyclicity of σ) For disjoint subsets R, J of $\overset{\circ}{I}$ with $|J| \neq \emptyset$, consider the following sequence of complexes, where the maps are alternating sums of σ, and S varies over subsets of J:

$$F(I|R) \xrightarrow{\ \sigma\ } \bigoplus_{\substack{|S|=1 \\ S \subset J}} F(I|R \cup S)$$
$$\xrightarrow{\ \sigma\ } \bigoplus_{\substack{|S|=2 \\ S \subset J}} F(I|R \cup S) \to \cdots \to F(I|R \cup J) \to 0.$$

Then the sequence is exact.

(5) (existence of the identity in the ring $H^0F(X,X)$) Before stating the condition, note there are composition maps for $H^0F(X,Y)$ defined as follows. For three objects X, Y and Z, let

$$\psi_Y : F(X,Y) \otimes F(Y,Z) \to F(X,Z)$$

be the map in the derived category defined as the composition $\varphi_Y \circ (\sigma_Y)^{-1}$ where the maps are as in

$$F(X,Y) \otimes F(Y,Z) \xleftarrow{\sigma_Y} F(X,Y,Z) \xrightarrow{\varphi_Y} F(X,Z) \ .$$

The map ψ_Y is verified to be associative, namely the following commutes in the derived category:

$$\begin{array}{ccc} F(X,Y) \otimes F(Y,Z) \otimes F(Z,W) & \xrightarrow{\psi_Y \otimes id} & F(X,Z) \otimes F(Z,W) \\ {\scriptstyle id \otimes \psi_Z}\big\downarrow & & \big\downarrow{\scriptstyle \psi_Z} \\ F(X,Y) \otimes F(Y,W) & \xrightarrow{\psi_Y} & F(X,W) \ . \end{array}$$

Let $H^0F(X,Y)$ be the 0-th cohomology of $F(X,Y)$. ψ_Y induces a map

$$\psi_Y : H^0F(X,Y) \otimes H^0F(Y,Z) \to H^0F(X,Z) \ ,$$

which is associative. If $u \in H^0F(X,Y)$, $v \in H^0F(Y,Z)$, we write $u \cdot v$ for $\psi_Y(u \otimes v)$.

We now require: For each X there is an element $1_X \in H^0F(X,X)$ such that $1_X \cdot u = u$ for any $u \in H^0F(X,Y)$ and $u \cdot 1_X = u$ for $u \in H^0F(Y,X)$.

(iv) *Diagonal elements and diagonal extension.*

(6) For each irreducible object X and a constant sequence of objects $i \mapsto X_i = X$ on a finite ordered set I with $|I| \geq 2$, there is a distinguished element, called the *diagonal element*

$$\mathbf{\Delta}_X(I) \in F(I) = F(X, \cdots, X)$$

of degree zero and coboundary zero. In particular for $|I| = 2$ we write $\Delta_X = \mathbf{\Delta}_X(I) \in F(X,X)$. One requires:

(6-1) If $S \subset \overset{\circ}{I}$, and $I_1, \cdots, I_c$ the corresponding segmentation, one has

$$\tau_S(\mathbf{\Delta}_X(I)\,) = \mathbf{\Delta}_X(I_1) \otimes \cdots \otimes \mathbf{\Delta}_X(I_c)$$

in $F(I \rceil S) = F(I_1) \otimes \cdots \otimes F(I_c)$.

(6-2) For $K \subset \overset{\circ}{I}$, $\varphi_K(\mathbf{\Delta}_X(I)\,) = \mathbf{\Delta}_X(I - K)$.

(7) Let I be a finite ordered set, $k \in I$, $m \geq 2$, and $I^{\sim}$ be the finite ordered set obtained by replacing k by a finite ordered set with m elements $\{k_1, \cdots, k_m\}$. If $I = [1,n]$, $I^{\sim}$ is $\{1, \cdots, k-1, k_1, \cdots, k_m, k+1, \cdots, n\}$.

There is given a map of complexes, called the *diagonal extension*,

$$\mathrm{diag}(I,\tilde{I}) : F(I) \to F(\tilde{I})$$

subject to the following conditions (for simplicity assume $I = [1,n]$):

(7-1) If $k' \neq k$, $\varphi_{k'}\,\mathrm{diag}(I,\tilde{I}) = \mathrm{diag}(I-\{k'\},\tilde{I}-\{k'\})\varphi_{k'}$, namely the following square commutes:

$$\begin{tikzcd}
F(I) \arrow[r, "\mathrm{diag}(I,\tilde{I})"] \arrow[d, "\varphi_{k'}"'] & F(\tilde{I}) \arrow[d, "\varphi_{k'}"] \\
F(I-\{k\}) \arrow[r, "\mathrm{diag}(I-\{k'\},\tilde{I}-\{k'\})"] & F(\tilde{I}-\{k'\})\ .
\end{tikzcd}$$

If $\ell \in \{k_1,\cdots,k_m\}$, $\varphi_\ell\,\mathrm{diag}(I,\tilde{I}) = \mathrm{diag}(I,\tilde{I}-\{\ell\})$. If $m=2$ the right side is the identity.

(7-2) If $k=n$, $\ell \in \{n_1,\cdots,n_m\}$, let I_1', I'' be the segmentation of $\tilde{I}$ by ℓ. Then the following diagram commutes:

$$\begin{tikzcd}
F(I) \arrow[r, "\mathrm{diag}(I,\tilde{I})"] \arrow[d, "\mathrm{diag}(I,I_1')"'] & F(\tilde{I}) \arrow[d, "\tau_\ell"] \\
F(I_1') \arrow[r] & F(I_1')\otimes F(I'')\ .
\end{tikzcd}$$

The lower horizontal map is $u \mapsto u\otimes \mathbf{\Delta}(I'')$. Note I'' parametrizes a constant sequence of objects, so one has $\mathbf{\Delta}(I'') \in F(I'')$. Similarly in case $k=1$, $\ell \in \{1_1,\cdots,1_m\}$.

If $1<k<n$ and $\ell \in \{k_1,\cdots,k_m\}$, let I_1, I_2 be the segmentation of I by k, and I_1', I_2' of $\tilde{I}$ by ℓ. One then has a commutative diagram:

$$\begin{tikzcd}
F(I) \arrow[r, "\mathrm{diag}(I,\tilde{I})"] \arrow[d, "\tau_k"'] & F(\tilde{I}) \arrow[d, "\tau_\ell"] \\
F(I_1)\otimes F(I_2) \arrow[r] & F(I_1')\otimes F(I_2')\ ,
\end{tikzcd}$$

where the lower horizontal arrow is $\mathrm{diag}(I_1,I_1')\otimes\mathrm{diag}(I_2,I_2')$.

Remark. From (6) and (7) it follows that $[\Delta_X] \in H^0F(X,X)$ is the identity in the sense of (5). Indeed the following stronger property is satisfied for the maps $\psi_Y : H^mF(X,Y)\otimes H^nF(Y,Z) \to H^{m+n}F(X,Z)$ for $m,n\in\mathbb{Z}$, defined in a similar manner as in (5) above.

(5)' For each $u \in H^nF(X,Y)$, $n\in\mathbb{Z}$, one has $1_X\cdot u = u$. Similarly for $u\in H^nF(Y,X)$, $u\cdot 1_X = u$.

(v) *The set of generators, notion of proper intersection, and distinguished subcomplexes with respect to constraints.*

(8)(the generating set) For a sequence X on I, the complex $F(I) = F(I;X)$ is degree-wise $\mathbb{Z}$-free on a given set of generators $\mathcal{S}_F(I) = \mathcal{S}_F(I;X)$. More precisely $\mathcal{S}_F(I) = \amalg_{p\in\mathbb{Z}}\mathcal{S}_F(I)^p$, where $\mathcal{S}_F(I)^p$ generates $F(I)^p$. This set is compatible with direct sum in each variable: Assume for an element $k \in I$ one has $X_k = Y_k \oplus Z_k$; let X'_i (resp. X''_i be the sequence such that $X'_i = X_i$ for $i \neq k$, and $X'_k = Y_k$ (resp. $X''_i = X_i$ for $i \neq k$, and $X''_k = Z_k$). Then $\mathcal{S}_F(I;X) = \mathcal{S}_F(I;X') \amalg \mathcal{S}_F(I;X'')$.

(9) (notion of proper intersection.) Let I be a finite ordered set, $I_1, \cdots, I_r$ be almost disjoint sub-intervals of I, namely one has $\mathrm{tm}(I_i) \leq \mathrm{in}(I_{i+1})$ for each i. Assume given a sequence of objects X_i on I. Let $\alpha_i \in \mathcal{S}_F(I_i)$ be a set of elements where i varies over a subset A of $\{1, \cdots, r\}$. We are given the condition whether the set $\{\alpha_i\}$ is *properly intersecting*. The following condition is to be satisfied.

- If $\{\alpha_i |\ i \in A\}$ is properly intersecting, for any subset B of A, $\{\alpha_i |\ i \in B\}$ is properly intersecting.
- Let A and A' be subsets of $\{1, \cdots, r\}$ such that $\mathrm{tm}(A) < \mathrm{in}(A')$. If $\{\alpha_i |\ i \in A\}$ and $\{\alpha_i |\ i \in A'\}$ are both properly intersecting sets, the union $\{\alpha_i |\ i \in A \cup A'\}$ is also properly intersecting.
- If $\{\alpha_1, \cdots, \alpha_r\}$ is properly intersecting, then for any i, writing $\partial\alpha_i = \sum c_{i\nu}\beta_\nu$ with $\beta_\nu \in \mathcal{S}_F(I_i)$, each set
$$\{\alpha_1, \cdots, \alpha_{i-1}, \beta_\nu, \alpha_{i+1}, \cdots, \alpha_r\}$$
is properly intersecting.
- The condition of proper intersection is compatible with direct sum in each variable. To be precise, under the same assumption as in (8), for a set of elements $\alpha_i \in \mathcal{S}_F(I_i;X')$ for $i = 1, \cdots, r$, the set $\{\alpha_i \in \mathcal{S}_F(I_i;X')\}_i$ is properly intersecting if and only if the set $\{\alpha_i \in \mathcal{S}_F(I_i;X)\}_i$ is properly intersecting.

Remark. For I_i almost disjoint and elements $\alpha_i \in F(I_i)$, one defines $\{\alpha_i \in F(I_i) | i \in A\}$ to be properly intersecting if the following holds. Write $\alpha_i = \sum c_{i\nu}\alpha_{i\,\nu}$ with $\alpha_{i\,\nu} \in \mathcal{S}_F(I_i)$, then for any choice of ν_i for $i \in A$, the set $\{\alpha_{i\,\nu_i} | i \in A\}$ is properly intersecting.

Further, if $S_i \subset \overset{\circ}{I_i}$, one can define the condition of proper intersection for $\{\alpha_i \in F(I_i|S_i) | i \in A\}$ by writing each α_i as a sum of tensors of elements in the generating set.

(10) (description of $F(I|S)$) When $I_1, \cdots, I_r$ is a segmentation of I, namely when $\mathrm{in}(I_1) = \mathrm{in}(I)$, $\mathrm{tm}(I_i) = \mathrm{in}(I_{i+1})$ and $\mathrm{tm}(I_r) = \mathrm{tm}(I)$, the subcomplex of $F(I_1) \otimes \cdot \otimes F(I_r)$ generated by $\alpha_1 \otimes \cdots \otimes \alpha_r$ with $\{\alpha_i\}$ properly intersecting is denoted by $F(I_1) \hat{\otimes} \cdots \hat{\otimes} F(I_r)$. If $S \subset \overset{\circ}{I}$ is the subset corresponding to the segmentation, this subcomplex coincides with $F(I|S)$.

(11)(distinguished subcomplexes) Let I be a finite ordered set, $L_1, \cdots, L_r$ be almost disjoint sub-intervals such that $\cup L_i = I$; equivalently, $\mathrm{in}(L_1) = \mathrm{in}(I)$, $\mathrm{tm}(L_i) = \mathrm{in}(L_{i+1})$ or $\mathrm{tm}(L_i) + 1 = \mathrm{in}(L_{i+1})$, and $\mathrm{tm}(L_r) = \mathrm{tm}(I)$. Assume given a sequence of objects X_i on I. Let *Dist* be the smallest class of subcomplexes of $F(L_1) \otimes \cdots \otimes F(L_r)$ satisfying the conditions below. It is then required that each subcomplex *Dist* is a quasi-isomorphic subcomplex.

(11-1) A subcomplex obtained as follows is in *Dist*. Let $I_1, \cdots, I_c$ be a set of almost disjoint sub-intervals of I with union I, that is coarser than $L_1, \cdots, L_r$; let $S_i \subset \overset{\circ}{I_i}$ such that the segmentations of I_i by S_i, when combined for all i, give precisely the L_i's. Let $I \hookrightarrow \mathbb{I}$ be an inclusion into a finite ordered set $\mathbb{I}$ such that the image of each I_a is a sub-interval. Assume given an extension of X to $\mathbb{I}$. Let $J_1, \cdots, J_s \subset \mathbb{I}$ be sub-intervals of $\mathbb{I}$ such that the set $\{I_i, J_j\}_{i,j}$ is almost disjoint, and $f_j \in F(J_j|T_j)$, $j = 1, \cdots, s$ be a properly intersecting set. Then one defines the subcomplex

$$[F(I_1|S_1) \otimes \cdots \otimes F(I_c|S_c)]_{\mathbb{I};f} \ ,$$

as the one generated by $\alpha_1 \otimes \cdots \otimes \alpha_c$, $\alpha_i \in F(I_i|S_i)$, such that the set $\{\alpha_1, \cdots, \alpha_c, f_j \,(j = 1, \cdots, s)\,\}$ is properly intersecting. We require it is in *Dist*.

The data consisting of $I \hookrightarrow \mathbb{I}$, X on $\mathbb{I}$, $J_i \subset \mathbb{I}$, and $f_j \in F(J_j|T_j)$ is called a *constraint*, and the corresponding subcomplex the distinguished subcomplex for the constraint.

(11-2) Tensor product of subcomplexes in *Dist* is again in *Dist*. For this to make sense, note complexes of the form $F(L_1) \otimes \cdots \otimes F(I_r)$ are closed under tensor products: If I' is another finite ordered set and $L'_1, \cdots, L'_s$ are almost disjoint sub-intervals with union I', then the tensor product

$$F(L_1) \otimes \cdots \otimes F(I_r) \otimes F(L'_1) \otimes \cdots \otimes F(I'_s)$$

is associated with the ordered set $I \amalg I'$ and almost disjoint sub-intervals $(L_1, \cdots, L_r, L'_1, \cdots, L'_s)$.

(11-3) A finite intersection of subcomplexes in *Dist* is again in *Dist*.

(1.2) **Definition.** To a quasi DG category $\mathcal{C}$ one can associate an additive category, called its *homotopy category*, denoted by $Ho(\mathcal{C})$. Objects of $Ho(\mathcal{C})$ are the same as the objects of $\mathcal{C}$, and $\mathrm{Hom}(X, Y) := H^0F(X, Y)$. Composition of arrows is induced from ψ_Y as in (5) above. The object O is the zero object, and the direct sum $X \oplus Y$ is the direct sum in the categorical sense. 1_X gives the identity $X \to X$.

(1.3) **Definition.** Let $\mathcal{C}$ be a quasi DG category. A C-diagram in $\mathcal{C}^\Delta$ is an object of the form $K = (K^m; f(m_1, \cdots, m_\mu))$, where (K^m) is a sequence of objects of $\mathcal{C}$ indexed by $m \in \mathbb{Z}$, almost all of which are zero, and

$$f(m_1, \cdots, m_\mu) \in F(K^{m_1}, \cdots, K^{m_\mu})^{-(m_\mu - m_1 - \mu + 1)}$$

is a collection of elements indexed by sequences $(m_1 < m_2 < \cdots < m_\mu)$ with $\mu \geq 2$. We require the following conditions:

(i) For each $j = 2, \cdots, \mu - 1$

$$\tau_{K^{m_j}}(f(m_1, \cdots, m_\mu)) = f(m_1, \cdots, m_j) \otimes f(m_j, \cdots, m_\mu)$$

in $F(K^{m_1}, \cdots, K^{m_j}) \otimes F(K^{m_j}, \cdots, K^{m_\mu})$.

(ii) For each $(m_1, \cdots, m_\mu)$, one has

$$\partial f(m_1, \cdots, m_\mu) + \sum_t \sum_k (-1)^{m_\mu + \mu + k + t} \varphi_{K^{m_k}}(f(m_1, \cdots, m_t, k, m_{t+1}, \cdots, m_\mu)) = 0$$

(the sum is over t with $1 \leq t < \mu$, and k with $m_t < k < m_{t+1}$).

For an object X in $\mathcal{C}$ and $n \in \mathbb{Z}$, there is a C-diagram K with $K^n = X$, $K^m = 0$ if $m \neq n$, and $f(M) = 0$ for all $M = (m_1, \ldots, m_\mu)$. We write $X[-n]$ for this.

(1.4)**Theorem.** *Let $\mathcal{C}$ be a quasi DG category satisfying the extra conditions (iv),(v) of Definition (1.1). There is a quasi DG category $\mathcal{C}^\Delta$ satisfying the following properties:*

(i) The objects are the C-diagrams in $\mathcal{C}$.

(ii) For a sequence of C-diagrams $K_1, \ldots, K_n$ with $n \geq 2$, as part of the structure of a quasi DG category, one has the corresponding complex of abelian groups $\mathbb{F}(K_1, \ldots, K_n)$, and the maps ι, σ, and φ. This complex has the following description if $n = 2$ and the diagrams K_1, K_2 are "objects of $\mathcal{C}$ with shifts": For a pair of objects X, Y in $\mathcal{C}$, and $m, n \in \mathbb{Z}$, and the corresponding C-diagrams $X[m]$, $Y[n]$, one has a canonical isomorphism of complexes

$$\mathbb{F}(X[m], Y[n]) = F(X, Y)[n-m] \ .$$

In particular, in the homotopy category $Ho(\mathcal{C}^\Delta)$ of $\mathcal{C}^\Delta$, one has

$$\mathrm{Hom}_{Ho(\mathcal{C}^\Delta)}(X[m], Y[n]) = H^{n-m} F(X, Y) \ .$$

Further, the map

$$\psi_Y : H^m F(X, Y) \otimes H^n F(Y, Z) \to H^{m+n} F(X, Z)$$

for $m, n \in \mathbb{Z}$, defined using the maps σ, φ and $F(X, Y, Z)$ (see the remark just before (v) in (1.1)) coincides with the map

$$\begin{aligned} &\psi_Y : H^0 \mathbb{F}(X, Y[m]) \otimes H^0 \mathbb{F}(Y[m], Z[m+n]) \\ &\to H^0 \mathbb{F}(X, Z[m+n]) \end{aligned}$$

defined similarly using the maps σ, φ and $\mathbb{F}(X, Y[m], Z[m+n])$, via the isomorphisms $H^m F(X, Y) = H^0 \mathbb{F}(X, Y[m])$, etc.

(iii) The homotopy category $Ho(\mathcal{C}^\Delta)$ of $\mathcal{C}^\Delta$ has the structure of a triangulated category.

For the proof, we must define the complexes $\mathbb{F}(K_1, \cdots, K_n)$ for a sequence of C-diagrams, together with maps σ and φ, satisfying the condition (ii) of the theorem, and the axioms (i)-(iii) of a quasi DG category. We then proceed to show that the homotopy category of $\mathcal{C}^\Delta$ is triangulated. If $\mathcal{C}$ is a DG category, there is a procedure to construct a triangulated category, as in [6] and [10] . The present construction may be viewed as its generalization.

§2. The quasi DG category of smooth varieties over a base.

We consider quasi-projective varieties over a field k. We refer the reader to [1] , [2] , [3] for the definition of the cycle complexes and the higher Chow groups of quasi-projective varieties. We will use the integral cubical version, as in [3] . Thus to a quasi-projective variety X over k and $s \in \mathbb{Z}$, there corresponds the cycle complex $\mathcal{Z}_s(X, \cdot)$; the group $\mathcal{Z}_s(X, n)$ is a quotient of the free abelian group of algebraic cycles on $X \times \square^n$ of dimension $s + n$, meeting faces properly. (See [3] for the precise definition, where the indexing is by codimension.) The variety X need not be assumed equi-dimensional when we use the indexing by "dimension" instead of codimension. The higher Chow groups are the homology groups of this complex: $\mathrm{CH}_s(X, n) = H_n \mathcal{Z}_s(X, \cdot)$.

Let S be a quasi-projective variety. Let $(\mathrm{Smooth}/k\,, \mathrm{Proj}/S)$ be the category of smooth varieties X equipped with projective maps to S. A *symbol* over S is an object the form

$$\bigoplus_{\alpha \in A} (X_\alpha/S, r_\alpha)$$

where X_α is a collection of objects in $(\mathrm{Smooth}/k\,, \mathrm{Proj}/S)$ indexed by a finite set A, and $r_\alpha \in \mathbb{Z}$.

(2.1) **Theorem.** *There is a quasi DG category satisfying the conditions (iv), (v), denoted Symb(S), with the following properties:*

(i) The objects are the symbols over S.

(ii) For a sequence of symbols $K_1, \dots, K_n$ with $n \geq 2$, as part of the structure of a quasi DG category, one has the corresponding complex of abelian groups $F(K_1, \dots, K_n)$, and the maps ι, σ, and φ. When the symbols are of the form $K_i = (X_i/S, r_i)$, the corresponding complex $F(K_1, \dots, K_n)$ is quasi-isomorphic to

$$\mathcal{Z}_{d_1}(X_1 \times_S X_2) \otimes \cdots \otimes \mathcal{Z}_{d_{n-1}}(X_{n-1} \times_S X_n),$$

with $d_i = \dim X_{i+1} - r_{i+1} + r_i$, the tensor product of the cycle complexes of the fiber products $X_i \times_S X_{i+1}$.

We consider *Symb*$(S)^\Delta$, the quasi DG category of C-diagrams in *Symb*(S), and then take its homotopy category. The resulting category is denoted $\mathcal{D}(S)$, and called the *triangulated category of mixed motives over S*. The next theorem follows from (1.3) and (2.1).

(2.2) **Theorem.** *For X in* $(\mathrm{Smooth}/k\,, \mathrm{Proj}/S)$ *and $r \in \mathbb{Z}$, there corresponds an object $h(X/S)(r) := (X/S, r)[-2r]$ in $\mathcal{D}(S)$. For two such objects we have*

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{D}(S)}(h(X/S)(r)[2r], h(Y/S)(s)[2s-n]) \\ &= \mathrm{CH}_{\dim Y - s + r}(X \times_S Y, n)\end{aligned}$$

the right hand side being the higher Chow group of the fiber product $X \times_S Y$.

There is a functor

$$h : (\mathrm{Smooth}/k\,, \mathrm{Proj}/S)^{opp} \to \mathcal{D}(S)$$

that sends X to $h(X/S)$, and a map $f : X \to Y$ to the class of its graph $[\Gamma_f] \in \mathrm{CH}_{\dim X}(Y \times_S X)$.

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