

Chow cohomology groups of algebraic surfaces

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Abstract

For an algebraic surface with isolated singularities we consider its higher Chow group and Chow cohomology (the latter defined by the author). We study the canonical map from the Chow cohomology to the higher Chow group by relating it to the canonical map from the higher Chow group to Chow cohomology of the exceptional divisor of a desingularization of the surface.

Introduction. For a quasi-projective variety S over a field, S. Bloch defined its *higher Chow groups* $\mathrm{CH}^r(S, n)$ as the homology of a certain complex $\mathcal{Z}^r(S, \cdot)$ called the cycle complex, [Bl 1], [Bl 2], [Bl 3]. One may view this as a Borel-Moore homology theory; for example it is covariantly functorial for proper maps, and contravariantly functorial for open immersions (more generally for flat maps).

For a quasi-projective variety S over a field of characteristic zero, using resolution of singularities and Bloch's cycle complexes, we defined the *Chow cohomology groups* $\mathrm{CHC}^r(S, n)$ (see [Ha 2] for details). To briefly recall it, we take a cubical hyperresolution $X_\bullet \rightarrow S$, which gives a strict truncated simplicial scheme; it consists of smooth varieties X_a for $0 \leq a \leq N$ with some N , and the face maps $d_i : X_n \rightarrow X_{n-1}$, $i = 0, \dots, n$, satisfying the usual identities. (It differs from a simplicial scheme in that there are only face maps and no degeneracies, and there are only finite many terms.)

We then form the double complex

$$\mathcal{Z}^r(X_0, \cdot) \xrightarrow{d^*} \mathcal{Z}^r(X_1, \cdot) \xrightarrow{d^*} \cdots \xrightarrow{d^*} \mathcal{Z}^r(X_N, \cdot)$$

where the a -th column is the cycle complex of X_a , and the horizontal differentials $d^* : \mathcal{Z}^r(X_a, \cdot) \rightarrow \mathcal{Z}^r(X_{a+1}, \cdot)$ are the alternating sums of the pull-backs d_i^* by the face maps. (Strictly speaking one must take appropriate quasi-isomorphic subcomplexes for d^* to be defined, see §1.) The total complex of this double complex is denoted $\mathcal{Z}^r(X_\bullet, \cdot)^*$, and called the *cohomological cycle complex* of S . Then $\mathrm{CHC}^r(S, n)$ is by definition the $(-n)$ -th cohomology of $\mathcal{Z}^r(X_\bullet, \cdot)^*$.

It is proven in [Ha 2] that $\mathrm{CHC}^r(S, n)$ is well-defined up to canonical isomorphism, independent of the choice of a hyperresolution, and that the association $S \mapsto \mathrm{CHC}^r(S, n)$ is contravariantly functorial for all maps. The condition the characteristic being zero is unnecessary if $\dim S \leq 2$, since its desingularization (and thus its cubical hyperresolution) exist. In that case we exhibit the cubical hyperresolution and the cohomological cycle complex of S explicitly in §§1 and 2. (The relationship of our Chow cohomology to the motivic cohomology of [FV] will be discussed in a separate paper.)

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If $d = \dim S$, there is a canonical map $\mathrm{CHC}^r(S, n) \rightarrow \mathrm{CH}_{d-r}(S, n)$; it is an isomorphism if S is smooth. This is analogous to the following situation in topology. For a good topological space S , there is a canonical map from cohomology to Borel-Moore homology, $H^i(S) \rightarrow H_{2\dim S - i}^{BM}(S)$, induced by the cap product by the fundamental class. The purpose of this paper is to study this map (tensored with \mathbb{Q}) for a surface S with isolated singularities. We take a desingularization $p : X \rightarrow S$ so that the exceptional locus E is a divisor with normal crossings. We show in (3.3):

Theorem. *Let S be a quasi-projective surface with isolated singularities over an algebraically closed field of characteristic zero. Then the canonical map $\mathrm{CHC}^r(S, n)_{\mathbb{Q}} \rightarrow \mathrm{CH}_{2-r}(S, n)_{\mathbb{Q}}$ is an isomorphism for all r, n if and only if E is a rational tree.*

The paper is organized as follows. In §1, we give some calculations of the Chow cohomology of curves, in particular of normal crossing divisors on a smooth surface. In §2, we describe the Chow cohomology and homology of a surface S with isolated singularities in terms of the Chow cohomology and homology of its desingularization X and the exceptional divisor E . We prove the above theorem in §3; the canonical map from the Chow cohomology to homology of S is studied by relating it to the canonical map from the Chow homology to cohomology of the exceptional divisor.

We point out some prior work on the subject. A. Vistoli studied varieties for which the operational Chow groups (as in [Fu]) are isomorphic to the Chow groups, [Vi]. Subsequently S.-I. Kimura worked on operational Chow groups and hyper-envelopes, [Ki]. The main difference from the present work is that they take the operational theory while we consider the Chow cohomology. But the reader also observes similarities to our approach: we use desingularizations and hyperresolutions instead of hyper-envelopes.

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1 Chow cohomology of curves

We consider quasi-projective varieties over a field k . We refer to [Bl 1], [Bl 2] and [Bl 3] for the details of the theory of higher Chow groups. Here we recall the basic properties.

(1) Let $\square^1 = \mathbb{P}_k^1 - \{1\}$ and $\square^n = (\square^1)^n$ with coordinates (x_1, \dots, x_n) . Faces of \square^n are intersections of codimension one faces, and the latter are divisors of the form $\square_{i,a}^{n-1} = \{x_i = a\}$ where $a = 0$ or ∞ . A face of dimension m is canonically isomorphic to \square^m .

Let X be an equi-dimensional variety (or a scheme). Let $\mathcal{Z}^r(X \times \square^n)$ be the free abelian group on the set of codimension r irreducible subvarieties of $X \times \square^n$ meeting each $X \times \text{face}$ properly. An element of $\mathcal{Z}^r(X \times \square^n)$ is called an *admissible* cycle. The inclusions of codimension one faces $\delta_{i,a} : \square_{i,a}^{n-1} \hookrightarrow \square^n$ induce the map

$$\partial = \sum (-1)^i (\delta_{i,0}^* - \delta_{i,\infty}^*) : \mathcal{Z}^r(X \times \square^n) \rightarrow \mathcal{Z}^r(X \times \square^{n-1}) .$$

One has $\partial \circ \partial = 0$. Let $\pi_i : X \times \square^n \rightarrow X \times \square^{n-1}$, $i = 1, \dots, n$ be the projections, and $\pi_i^* : \mathcal{Z}^r(X \times \square^{n-1}) \rightarrow \mathcal{Z}^r(X \times \square^n)$ be the pull-backs. Let $\mathcal{Z}^r(X, n)$ be the quotient of $\mathcal{Z}^r(X \times \square^n)$ by the sum of the images of π_i^* . Thus an element of $\mathcal{Z}^r(X, n)$ is represented uniquely by a cycle

whose irreducible components are non-degenerate (not a pull-back by π_i). The map ∂ induces a map $\partial : \mathcal{Z}^r(X, n) \rightarrow \mathcal{Z}^r(X, n-1)$, and $\partial \circ \partial = 0$. The complex $\mathcal{Z}^r(X, \cdot)$ thus defined is the *cycle complex* of X in codimension r . The higher Chow groups are the homology groups of this complex:

$$\mathrm{CH}^r(X, n) = H_n \mathcal{Z}^r(X, \cdot) .$$

Note $\mathrm{CH}^r(X, 0) = \mathrm{CH}^r(X)$, the Chow group of X . In this paper we would rather use the indexing by dimensions: for $s \in \mathbb{Z}$, $\mathcal{Z}_s(X, \cdot) = \mathcal{Z}^{\dim X - r}(X, \cdot)$, and $\mathrm{CH}_s(X, n)$ is the homology group.

(2) For a proper map $f : X \rightarrow Y$ of k -schemes, the push-forward $f_* : \mathcal{Z}_s(X, \cdot) \rightarrow \mathcal{Z}_s(Y, \cdot)$, hence also $f_* : \mathrm{CH}_s(X, n) \rightarrow \mathrm{CH}_s(Y, n)$ is defined.

(3) For a flat map $f : X \rightarrow Y$ of relative equi-dimension d , the pull-backs $f^* : \mathcal{Z}_s(Y, \cdot) \rightarrow \mathcal{Z}_{s+d}(X, \cdot)$ and $f^* : \mathrm{CH}_s(Y, n) \rightarrow \mathrm{CH}_{s+d}(X, n)$ are defined. If $f : X \rightarrow Y$ be a map where Y is smooth and X equi-dimensional, there is a map $f^* : \mathrm{CH}^r(Y, n) \rightarrow \mathrm{CH}^r(X, n)$. In fact there is a quasi-isomorphic subcomplex $\mathcal{Z}^r(Y, \cdot)'$ of $\mathcal{Z}^r(Y, \cdot)$ on which $f^* : \mathcal{Z}^r(Y, \cdot)' \rightarrow \mathcal{Z}^r(X, \cdot)$ is defined.

(4) If X is smooth quasi-projective and equi-dimensional, one has the intersection product $\mathrm{CH}_s(X, n) \otimes \mathrm{CH}_t(X, m) \rightarrow \mathrm{CH}_{s+t-\dim X}(X, n+m)$.

(5) Projection formula.

(6) Projective bundle formula.

(7) Localization sequence. If X is a quasi-projective variety and U is an open set, letting $Z = X - U$, one has an exact sequence of complexes $0 \rightarrow \mathcal{Z}_s(Z, \cdot) \rightarrow \mathcal{Z}_s(X, \cdot) \rightarrow \mathcal{Z}_s(U, \cdot)$. The localization theorem [Bl 2] asserts that the induced map $\mathcal{Z}_s(X, \cdot)/\mathcal{Z}_s(Z, \cdot) \rightarrow \mathcal{Z}_s(U, \cdot)$ is a quasi-isomorphism.

(8) The self-intersection formula [Ha 3, §1]. X be a smooth quasi-projective variety and $Y \subset X$ a smooth closed subvariety of codimension d , $i : Y \rightarrow X$ the closed immersion, and $N = N_Y X$ the normal bundle. Then

$$i^* i_*(y) = c_d(N) \cdot y$$

for $y \in \mathrm{CH}_s(Y, n)$.

In this paper we need this only for X smooth projective; in that case this can also be deduced from the formula $i^* \circ i_* = c_d(N)$ in the correspondence ring $\mathrm{CH}^{\dim Y}(Y \times Y)$ (see [Ma]).

Recall the definition of Chow cohomology groups for curves. For a quasi-projective curve over k , not necessarily irreducible, let $p : \tilde{C} \rightarrow C$ be the normalization of C , $\Sigma \subset C$ the singular locus, and $\tilde{\Sigma} = p^{-1}(\Sigma)$ (with reduced scheme structure). One has a commutative square

$$\begin{array}{ccc} \tilde{\Sigma} & \xrightarrow{k} & \tilde{C} \\ q \downarrow & & \downarrow p \\ \Sigma & \xrightarrow{i} & C \end{array}$$

where i is the embedding and k, q are the induced maps. This gives a cubical hyperresolution of C . Let

$$\mathcal{Z}^r(C, \cdot)^* := \mathrm{Cone}[\mathcal{Z}^r(\tilde{C}, \cdot) \oplus \mathcal{Z}^r(\Sigma, \cdot) \xrightarrow{k^* - q^*} \mathcal{Z}^r(\tilde{\Sigma}, \cdot)][-1] .$$

To be precise one must replace $\mathcal{Z}^r(\tilde{C}, \cdot)$ with a quasi-isomorphic subcomplex so that the pull-back $k^* : \mathcal{Z}^r(\tilde{C}, \cdot) \rightarrow \mathcal{Z}^r(\tilde{\Sigma}, \cdot)$ is defined. The cycle complex is a homological complex, which

can be viewed as a cohomological complex in the usual manner. So $\mathcal{Z}^r(C, \cdot)^*$ is a cohomological complex. We let

$$\mathrm{CHC}^r(C, n) = H^{-n} \mathcal{Z}^r(C, \cdot)^* .$$

We also write $\mathrm{CHC}^r(C) = \mathrm{CHC}^r(C, 0)$. There is a long exact sequence

$$\begin{array}{ccccccc} \rightarrow & \mathrm{CHC}^r(C, n) & \rightarrow & \mathrm{CH}^r(\tilde{C}, n) \oplus \mathrm{CH}^r(\Sigma, n) & \rightarrow & \mathrm{CH}^r(\tilde{\Sigma}, n) & \\ \rightarrow & \mathrm{CHC}^r(C, n-1) & \rightarrow & \cdots & & & \end{array}$$

For $r = 1$, the following can be shown using the fact that if X is smooth $\mathrm{CH}^1(X, n) = 0$ for $n \neq 0, 1$, and $\mathrm{CH}^1(X, 1) = \Gamma(X, \mathcal{O}_X^*)$ (see [Bl 1]).

(1.1) Proposition. *Let C be a quasi-projective curve over k . Then $\mathrm{CHC}^1(C, n) = 0$ for $n \neq 0, 1$, and there is an exact sequence*

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{CHC}^1(C, 1) & \longrightarrow & \Gamma(\tilde{C}, \mathcal{O}_{\tilde{C}}^*) \oplus \Gamma(\Sigma, \mathcal{O}_{\Sigma}^*) & \longrightarrow & \Gamma(\tilde{\Sigma}, \mathcal{O}_{\tilde{\Sigma}}^*) & \\ \longrightarrow & \mathrm{CHC}^1(C) & \longrightarrow & \mathrm{CH}^1(\tilde{C}) & \longrightarrow & 0 & \end{array}$$

In the rest of this section we assume k is algebraically closed. Then for C projective and irreducible, the result is much simpler:

(1.2) Proposition. *Let C be an irreducible projective curve over an algebraically closed field k . Then $\mathrm{CHC}^1(C, n) = 0$ for $n \neq 0, 1$, $\mathrm{CHC}^1(C, 1) = k^*$, and there is an exact sequence*

$$0 \rightarrow \bigoplus_{P \in \Sigma} \left(\bigoplus_{Q \rightarrow P} k^* \right) / k^* \rightarrow \mathrm{CHC}^1(C) \rightarrow \mathrm{CH}^1(\tilde{C}) \rightarrow 0 .$$

(For each P in Σ , one takes the direct sum of copies of k^* , one for each $Q \in \tilde{C}$ over P , and mod out by the subgroup k^* embedded diagonally in the sum.)

Let S be a smooth quasi-projective surface on a field k , and E be a connected normal crossing divisor, with each irreducible component projective, on S . Let $\{E_i\}_{i=1, \dots, N}$ be the irreducible components of E , and $E_{ij} = E_i \cap E_j$ for $i < j$ be the points of intersection. Set $E^{(0)} := \coprod E_i$, $E^{(1)} := \coprod_{i < j} E_{ij}$. Let

$$\delta_i : E^{(1)} \rightarrow E^{(0)} , \quad i = 0, 1$$

be the maps which restrict to the inclusions $E_{ij} \rightarrow E_j$, $E_{ij} \rightarrow E_i$, respectively. There is a natural map $a : E^{(0)} \rightarrow E$. The cohomological cycle complex of E is quasi-isomorphic to the complex

$$\mathcal{Z}^r(E, \cdot)^* = \mathrm{Cone}[\mathcal{Z}^r(E^{(0)}, \cdot) \xrightarrow{\delta^*} \mathcal{Z}^r(E^{(1)}, \cdot)][-1]$$

where $\delta^* = \delta_0^* - \delta_1^*$.

The *dual graph* Γ of E consists of the vertices corresponding to the components E_i , and the edges corresponding to E_{ij} . Since the vertices are ordered, the edges are oriented. Also, Γ is connected since E is connected.

The chain complex of Γ is the two term homological complex

$$C_\bullet(\Gamma) = [C_1(\Gamma) \xrightarrow{\partial} C_0(\Gamma)]$$

where $C_0(\Gamma)$ (resp. $C_1(\Gamma)$) is a free \mathbb{Z} -module with basis $\{E_i\}$ (resp. $\{E_{ij}\}$), and ∂ sends E_{ij} to $E_j - E_i$. The homology of this complex is denoted by $H_*(\Gamma)$. One has $H_0(\Gamma) = \mathbb{Z}$ since Γ is connected, and $H_1(\Gamma)$ is a finitely generated free abelian group.

The cochain complex $C^\bullet(\Gamma)$ is the two term complex $d : C^0(\Gamma) \rightarrow C^1(\Gamma)$ where $C^i(\Gamma) = \text{Hom}(C_i(\Gamma), \mathbb{Z})$ and d is the dual of ∂ with minus sign. (The change of sign is made so that the natural evaluation map is a map of complexes.) So $C^0(\Gamma)$ (resp. $C^1(\Gamma)$) is the free \mathbb{Z} -module with basis $\{e_i\}$ dual to $\{E_i\}$ (resp. with basis $\{e_{ij}\}$ dual to $\{E_{ij}\}$), and

$$d(e_i) = \sum_{i < j} e_{ij} - \sum_{m < i} e_{mi} .$$

By definition $H^*(\Gamma)$ is the cohomology of this complex. We have $H^0(\Gamma) = \mathbb{Z}$ and $H^1(\Gamma) = \text{Hom}(H_1(\Gamma), \mathbb{Z})$ by the universal coefficient theorem.

We say E is a *rational tree* if each E_i is isomorphic to \mathbb{P}^1 and $H_1(\Gamma) = 0$ (equivalently, $H^1(\Gamma) = 0$).

(1.3) Proposition. *For a connected normal crossing divisor E as above, we have $\text{CHC}^1(E, n) = 0$ if $n \neq 0, 1$, $\text{CHC}^1(E, 1) = k^*$, and there is an exact sequence*

$$0 \rightarrow H^1(\Gamma) \otimes k^* \rightarrow \text{CHC}^1(E) \xrightarrow{a^*} \text{CH}^1(E^{(0)}) \rightarrow 0 .$$

Proof. Recall $\text{CHC}^1(E, n)$ is the homology of the complex

$$\text{Cone}[\mathcal{Z}^1(E^{(0)}, \cdot) \xrightarrow{\delta^*} \mathcal{Z}^1(E^{(1)}, \cdot)][-1] .$$

One thus has $\text{CHC}^1(E, n) = 0$ for $n \neq 0, 1$, and an exact sequence

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{CHC}^1(E, 1) & \rightarrow & \text{CH}^1(E^{(0)}, 1) & \xrightarrow{\delta^*} & \text{CH}^1(E^{(1)}, 1) \\ & & \rightarrow & \text{CHC}^1(E, 0) & \rightarrow & \text{CH}^1(E^{(0)}, 0) & \longrightarrow & 0 \end{array} .$$

We have $\text{CH}^1(E^{(0)}, 1) = C^0(\Gamma) \otimes k^*$, $\text{CH}^1(E^{(1)}, 1) = C^1(\Gamma) \otimes k^*$, and δ^* is identified with $d \otimes id : C^0(\Gamma) \otimes k^* \rightarrow C^1(\Gamma) \otimes k^*$. The result now follows.

The group $\text{CHC}^r(E, n)$ for ≥ 2 does not allow a simple description in general. But if E is a normal crossing divisor consisting of rational curves, it can be calculated as follows. Before stating it, we define the *reduced* Chow cohomology of E .

For a connected normal crossing divisor on a smooth surface, let $\pi : E \rightarrow pt = \text{Spec } k$ be the structure map, and

$$\tilde{\mathcal{Z}}^r(E, \cdot)^* = \text{Cone}[\pi^* : \mathcal{Z}^r(pt, \cdot) \rightarrow \mathcal{Z}^r(E, \cdot)^*] .$$

Its $(-n)$ -th cohomology is denoted $\widetilde{\text{CHC}}^r(E, n)$.

There is an exact sequence

$$\begin{array}{ccccccc} \longrightarrow & \text{CH}^r(pt, n) & \xrightarrow{\pi^*} & \text{CHC}^r(E, n) & \longrightarrow & \widetilde{\text{CHC}}^r(E, n) \\ \longrightarrow & \text{CH}^r(pt, n-1) & \longrightarrow & \dots & & & \end{array}$$

For any closed point x of E , π^* splits by x^* , so the above long exact sequence splits to short exact sequences.

Let $\tilde{\pi} : E^{(0)} \rightarrow \coprod_N pt$ be the map to the disjoint union of N copies of pt , which restricts to the structure map on each component E_i . It induces the map $\tilde{\pi}_* : \mathrm{CH}^r(E^{(0)}, n) \rightarrow \mathrm{CH}^{r-1}(pt, n)^{\oplus N}$.

Consider the map $\pi^* : \mathcal{Z}^r(pt, \cdot) \rightarrow \mathcal{Z}^r(E^{(0)}, \cdot)$, and take its cone, and let its $(-n)$ -th cohomology group be $\mathrm{CH}^r(E^{(0)} \rightarrow pt, n)$. (A possible notation $\widetilde{\mathrm{CH}}^r(E^{(0)}, n)$ is avoided here, since one may confuse it with $\oplus \widetilde{\mathrm{CH}}^r(E_i, n)$. Also see the remark below.) The commutative diagram

$$\begin{array}{ccc} \mathcal{Z}^r(pt, \cdot) & \xrightarrow{\pi^*} & \mathcal{Z}^r(E^{(0)}, \cdot) \\ \downarrow & & \downarrow \tilde{\pi}_* \\ 0 & \longrightarrow & \mathcal{Z}^{r-1}(pt, \cdot)^{\oplus N} \end{array}$$

defines a map $\tilde{\pi}_* : \mathrm{CH}^r(E^{(0)} \rightarrow pt, n) \rightarrow \mathrm{CH}^{r-1}(pt, n)^{\oplus N}$. From the definitions there is a commutative square

$$\begin{array}{ccc} \mathrm{CHC}^r(E, n) & \xrightarrow{a^*} & \mathrm{CH}^r(E^{(0)}, n) \\ \downarrow & & \downarrow \\ \widetilde{\mathrm{CHC}}^r(E, n) & \longrightarrow & \mathrm{CH}^r(E^{(0)} \rightarrow pt, n) . \end{array}$$

So by composition one defines a map, still called $\tilde{\pi}_*$, from any one of the four groups to $\mathrm{CH}^{r-1}(pt, n)^{\oplus N}$.

Remark. For a not necessarily connected normal crossing divisor E , let E_λ be its connected components, define $\tilde{\mathcal{Z}}^r(E, \cdot)^* = \oplus_\lambda \tilde{\mathcal{Z}}^r(E_\lambda, \cdot)^*$, and let $\widetilde{\mathrm{CHC}}^r(E, n) = \oplus \widetilde{\mathrm{CHC}}^r(E_\lambda, n)$ be its cohomology.

(1.4) **Proposition.** *If E is a connected normal crossing divisor with each $E_i \cong \mathbb{P}^1$, one has an exact sequence (N is the number of irreducible components of E)*

$$0 \rightarrow H^1(\Gamma) \otimes \mathrm{CH}^r(pt, n+1) \rightarrow \widetilde{\mathrm{CHC}}^r(E, n) \rightarrow \mathrm{CH}^{r-1}(pt, n)^{\oplus N} \rightarrow 0 .$$

where the surjection is the map $\tilde{\pi}_*$. In particular if E is a rational tree, $\widetilde{\mathrm{CHC}}^r(E, n) \cong \mathrm{CH}^{r-1}(pt, n)^{\oplus N}$.

Proof. We have an exact sequence

$$\begin{array}{ccccccc} \longrightarrow & \mathrm{CHC}^r(E, n) & \longrightarrow & \mathrm{CH}^r(E^{(0)}, n) & \xrightarrow{\delta^*} & \mathrm{CH}^r(E^{(1)}, n) & \\ \longrightarrow & \mathrm{CHC}^r(E, n-1) & \longrightarrow & \dots & & & \end{array}$$

Since $\mathrm{CH}^r(\mathbb{P}^1, n) = \mathrm{CH}^r(pt, n) \oplus \mathrm{CH}^{r-1}(pt, n)$, one has

$$\mathrm{CH}^r(E^{(0)}, n) = C^0(\Gamma) \otimes (\mathrm{CH}^r(pt, n) \oplus \mathrm{CH}^{r-1}(pt, n)) ,$$

$\mathrm{CH}^r(E^{(1)}, n) = C^1(\Gamma) \otimes \mathrm{CH}^r(pt, n)$; δ^* is zero on $C^0(\Gamma) \otimes \mathrm{CH}^{r-1}(pt, n)$, and coincides with $d \otimes 1$ on $C^0(\Gamma) \otimes \mathrm{CH}^r(pt, n)$. So we have an exact sequence

$$0 \rightarrow H^1(\Gamma) \otimes \mathrm{CH}^r(pt, n+1) \rightarrow \mathrm{CHC}^r(E, n) \rightarrow \mathrm{CH}^r(pt, n) \oplus \mathrm{CH}^{r-1}(pt, n)^{\oplus N} \rightarrow 0 ,$$

proving the proposition.

We will state a result parallel to (1.4) for Chow groups. First we discuss the reduced Chow group. Let

$$\tilde{\mathcal{Z}}_s(E, \cdot) = \mathrm{Cone}[\pi_* : \mathcal{Z}_s(E, \cdot) \rightarrow \mathcal{Z}_s(pt, \cdot)][-1]$$

and $\widetilde{\text{CH}}_s(E, n)$ be its cohomology. If E is not connected, let E_λ be its connected components and set

$$\widetilde{\mathcal{Z}}_s(E, \cdot) = \bigoplus_\lambda \widetilde{\mathcal{Z}}_s(E_\lambda, \cdot).$$

There is a long exact sequence

$$\begin{array}{ccccccc} \longrightarrow & \widetilde{\text{CH}}_s(E, n) & \longrightarrow & \text{CH}_s(E, n) & \xrightarrow{\pi_*} & \text{CH}_s(pt, n) & \\ \longrightarrow & \widetilde{\text{CH}}_s(E, n-1) & \longrightarrow & \dots & & & \end{array}$$

For any closed point x of E , π_* splits by x_* , so the above long exact sequence splits to short exact sequences.

Consider the map $\pi_* : \mathcal{Z}_s(E^{(0)}, \cdot) \rightarrow \mathcal{Z}_s(pt, \cdot)$; take its cone, shift by -1 , and denote its $(-n)$ -th cohomology by $\text{CH}_s(E^{(0)} \rightarrow pt, n)$. (One should distinguish it from $\text{CH}^{1-s}(E^{(0)} \rightarrow pt, n)$ defined earlier.) The commutative diagram

$$\begin{array}{ccc} \mathcal{Z}_s(E^{(0)}, \cdot) & \xrightarrow{\pi_*} & \mathcal{Z}_s(pt, \cdot) \\ \tilde{\pi}^* \uparrow & & \uparrow \\ \mathcal{Z}_{s-1}(pt, \cdot)^{\oplus N} & \longrightarrow & 0 \end{array}$$

induces a map $\tilde{\pi}^* : \text{CH}_{s-1}(pt, n)^{\oplus N} \rightarrow \text{CH}_s(E^{(0)} \rightarrow pt, n)$. There is a commutative square

$$\begin{array}{ccc} \text{CH}_s(E^{(0)}, n) & \xrightarrow{a_*} & \text{CH}_s(E, n) \\ \uparrow & & \uparrow \\ \text{CH}_s(E^{(0)} \rightarrow pt, n) & \longrightarrow & \widetilde{\text{CH}}_s(E, n). \end{array}$$

One thus has a map, still denoted $\tilde{\pi}^*$, from $\text{CH}_{s-1}(pt, n)^{\oplus N}$ to any one of the four groups. The proof of the following is similar to that of (1.4).

(1.5) Proposition. *Let E be a connected normal crossing divisor with each $E_i \cong \mathbb{P}^1$. One has an exact sequence*

$$0 \rightarrow \text{CH}_{s-1}(pt, n)^{\oplus N} \rightarrow \widetilde{\text{CH}}_s(E, n) \rightarrow \text{CH}_s(pt, n-1) \otimes H_1(\Gamma) \rightarrow 0$$

where the injection is the map $\tilde{\pi}^*$. If E is a rational tree, $\widetilde{\text{CH}}_s(E, n) \cong \text{CH}_{s-1}(pt, n)^{\oplus N}$.

2 Chow cohomology and homology of surfaces

Let now S be an irreducible quasi-projective surface over a field k with at most isolated singularities; so the singular locus Σ consists of finite number of points. Let $p : X \rightarrow S$ be a desingularization such that the inverse image (with reduced scheme structure) $E = p^{-1}(\Sigma)$ is a normal crossing divisor. We have a Cartesian diagram

$$\begin{array}{ccc} E & \xrightarrow{k} & X \\ q \downarrow & & \downarrow p \\ \Sigma & \xrightarrow{i} & S \end{array}$$

with i, k the inclusions and q the induced map. This is the first step of a cubical hyperresolution of S .

Let

$$\mathcal{Z}^r(S, \cdot)^* := \text{Cone}[k^* - q^* : \mathcal{Z}^r(X, \cdot) \oplus \mathcal{Z}^r(\Sigma, \cdot) \rightarrow \mathcal{Z}(E, \cdot)^*][-1]$$

with $\mathcal{Z}^r(E, \cdot)^*$ defined before. By definition $\text{CHC}^r(S, n)$ is the $(-n)$ -th cohomology of this complex. In [Ha 2] we showed it is independent of the choice of a desingularization, and contravariantly functorial for all maps. Note $\text{CHC}^r(S, n) = 0$ for $n \leq -2$, but $\text{CHC}^r(S, -1)$ may not be zero.

There is a canonical map $\text{CHC}^r(S, n) \rightarrow \text{CH}_{2-r}(S, n)$ defined as the composition of the maps

$$\text{CHC}^r(S, n) \xrightarrow{p^*} \text{CH}^r(X, n) = \text{CH}_{2-r}(X, n) \xrightarrow{p_*} \text{CH}_{2-r}(S, n) .$$

One easily shows it is independent of the choice of $p : X \rightarrow S$.

In the rest of this section k is algebraically closed.

(2.1) Proposition. *We have a long exact sequence*

$$\begin{array}{ccccccc} \longrightarrow & \text{CHC}^r(S, n) & \xrightarrow{p^*} & \text{CH}^r(X, n) & \xrightarrow{k^*} & \widetilde{\text{CHC}}^r(E, n) & \\ \longrightarrow & \text{CHC}^r(S, n-1) & \longrightarrow & \cdots & & & \end{array}$$

For $r = 1$, one has $\text{CHC}^1(S, n) = 0$ for $n \neq -1, 0, 1$, $\text{CHC}^1(S, 1) = k^*$, and there is an exact sequence

$$0 \rightarrow \text{CHC}^1(S) \xrightarrow{p^*} \text{CH}^1(X) \xrightarrow{k^*} \text{CHC}^1(E) \rightarrow \text{CHC}^1(S, -1) \rightarrow 0 .$$

Proof. The complex $\mathcal{Z}^r(S, \cdot)^*$ is quasi-isomorphic to

$$\text{Cone}[\mathcal{Z}^r(X, \cdot) \xrightarrow{k^*} \frac{\mathcal{Z}^r(E, \cdot)^*}{\mathcal{Z}^r(\Sigma, \cdot)}][-1] .$$

The long exact sequence hence follows. If $r = 1$, from (1.2) one has additional information on $\widetilde{\text{CHC}}^1(E, n)$: $\widetilde{\text{CHC}}^1(E, n) = 0$ if $n \neq 0$ and $\widetilde{\text{CHC}}^1(E) = \text{CHC}^1(E)$. The assertion follows from this.

One can study the Chow homology of S using the resolution in a similar manner. As observed in [Ha 2, §2], from the localization theorem [Bl 3] it follows that if $X_\bullet \rightarrow S$ is a cubical hyperresolution, then the complex $\mathcal{Z}_s(X_\bullet, \cdot)_*$ defined as the total complex of the double complex

$$\mathcal{Z}_s(X_N, \cdot) \xrightarrow{d_*} \cdots \xrightarrow{d_*} \mathcal{Z}_s(X_1, \cdot) \xrightarrow{d_*} \mathcal{Z}_s(X_0, \cdot)$$

is quasi-isomorphic to $\mathcal{Z}_s(S, \cdot)$.

So in our case the complex $\mathcal{Z}_s(S, \cdot)$ is quasi-isomorphic to

$$\text{Cone}[(k_*, q_*) : \mathcal{Z}_s(E, \cdot) \rightarrow \mathcal{Z}_s(X, \cdot) \oplus \mathcal{Z}_s(\Sigma, \cdot)]$$

and thus also to

$$\text{Cone}[k_* : \tilde{\mathcal{Z}}_s(E, \cdot) \rightarrow \mathcal{Z}_s(X, \cdot)] ,$$

since $\tilde{\mathcal{Z}}_s(E, \cdot) = \text{Cone}[q_* : \mathcal{Z}_s(E, \cdot) \rightarrow \mathcal{Z}_s(\Sigma, \cdot)][-1]$ according to the definition.

(2.2) **Proposition.** *We have a long exact sequence*

$$\begin{array}{ccccccc} \longrightarrow & \widetilde{\text{CH}}_s(E, n) & \xrightarrow{k_*} & \text{CH}_s(X, n) & \xrightarrow{p_*} & \text{CH}_s(S, n) & \\ \longrightarrow & \widetilde{\text{CH}}_s(E, n-1) & \longrightarrow & \cdots & & & \end{array}$$

If $s = 1$, one has $\text{CH}_1(S, n) = 0$ if $n \neq 0, 1$, $\text{CH}_1(S, 1) = k^*$, and there is an exact sequence

$$0 \rightarrow \text{CH}_1(E) \xrightarrow{k_*} \text{CH}_1(X) \xrightarrow{p_*} \text{CH}_1(S) \rightarrow 0,$$

where $\text{CH}_1(E) = \bigoplus \mathbb{Z}[E_i]$.

Proof. The exact sequence follows from the definition. If $s = 1$, we have $\widetilde{\text{CH}}_1(E, n) = 0$ for $n \neq 0$ and $\widetilde{\text{CH}}_1(E) = \text{CH}_1(E)$, which is free with generators E_i . We have an exact sequence

$$\begin{array}{ccccccc} \longrightarrow & 0 & \longrightarrow & \text{CH}_1(X, 1) & \xrightarrow{p_*} & \text{CH}_1(S, 1) & \\ \longrightarrow & \text{CH}_1(E) & \longrightarrow & \text{CH}_1(X) & \longrightarrow & \text{CH}_1(S) & \rightarrow 0 \end{array}$$

where $\text{CH}_1(X, 1) = k^*$. The map $\text{CH}_1(E) \rightarrow \text{CH}_1(X)$ is injective since the intersection matrix $(E_i \cdot E_j)$ is negative definite, [Mu]. (In [Mu] normal surface singularities are considered, but the same continues to hold for isolated surface singularities.) Indeed if $\sum a_i E_i = 0$ in $\text{CH}_1(X)$, then the intersection number $(\sum a_i E_i)^2 = 0$, so all a_i are zero. The assertion now follows.

3 The map $\text{CHC}^r(S, n) \rightarrow \text{CH}_{2-r}(S, n)$

In this section assume k is algebraically closed (although one may take k to be an arbitrary field until (3.3)).

We will consider the higher Chow groups and Chow cohomology groups tensored with \mathbb{Q} ; for simplicity we will write $\text{CH}_s(S, n)$ for $\text{CH}_s(S, n)_{\mathbb{Q}}$, $\text{CHC}_s(S, n)$ for $\text{CHC}_s(S, n)_{\mathbb{Q}}$, and so on.

Under the same assumption as in the previous section, we study the maps

$$\text{CH}_s(E, n) \xrightarrow{k_*} \text{CH}_s(X, n) = \text{CH}^{2-s}(X, n) \xrightarrow{k^*} \text{CHC}^{2-s}(E, n).$$

For this goal, let X be a smooth quasi-projective surface and E, F be smooth projective irreducible curves on X ; we assume either E and F meet transversally in a point or $E = F$. We have a Cartesian square

$$\begin{array}{ccc} E \cap F & \xrightarrow{i'} & F \\ j' \downarrow & & \downarrow j \\ E & \xrightarrow{i} & X \end{array}$$

For each r, n one has the map $i^* j_* : \text{CH}^r(F, n) \rightarrow \text{CH}^{r+1}(E, n)$. In case $E \neq F$ this map is induced by the element $\delta_*(E \cap F) \in \text{CH}^2(F \times E)$; here $\delta : E \cap F \rightarrow F \times E$ is the natural inclusion. Indeed for $\alpha \in \text{CH}^r(F)$, we have

$$\begin{aligned} (\delta_*(E \cap F))_* \alpha &= p_{1*}[(E \times \alpha) \cdot \delta_*(E \cap F)] \\ &= p_{1*} \delta_* \delta^*(E \times \alpha) \\ &= j'_* i'^* \alpha \end{aligned}$$

which coincides with $i^*j_*\alpha$. (The first equality holds by definition, the second by the projection formula and the third by $p_1\delta = j'$ and $p_2\delta = i'$.)

If $E = F$,

$$i^*i_*\alpha = \alpha \cdot c_1(\mathcal{O}_E(E))$$

by the self-intersection formula for higher Chow groups, as recalled in §1. So $i^*i_* : \mathrm{CH}^r(E, n) \rightarrow \mathrm{CH}^{r+1}(E, n)$ is induced by $\delta_*(c_1(\mathcal{O}_E(E)))$.

Consider the composition of maps (π_F and π_E are the structure maps)

$$\mathrm{CH}^r(pt, n) \xrightarrow{\pi_F^*} \mathrm{CH}^r(F, n) \xrightarrow{i^*j_*} \mathrm{CH}^{r+1}(E, n) \xrightarrow{(\pi_E)_*} \mathrm{CH}^r(pt, n) .$$

It is the multiplication by the integer $(E \cdot F)$. Indeed it is induced by the correspondence

$$(\pi_F \times \pi_E)_*\delta_*(E \cap F) = (E \cdot F) \in \mathrm{CH}_0(pt) = \mathbb{Z}$$

in case $E \neq F$ and by $(\pi_F \times \pi_E)_*\delta_*(c_1(\mathcal{O}_E(E))) = (E \cdot E)$ in case $E = F$. (If $E \neq F$, the number $(E \cdot F) = 1$ since k is algebraically closed. But as we mentioned at the beginning of this section, one may assume k arbitrary, in which case $(E \cdot F)$ is a positive integer.)

We go back to the assumption of §2. For the normal crossing divisor E , we studied its Chow cohomology and homology in §1. Let $a : E^{(0)} \rightarrow E$ be the canonical map, and $\tilde{k} = k \circ a : E^{(0)} \rightarrow X$. From the above argument we have:

(3.1) **Lemma.** *The following square commutes:*

$$\begin{array}{ccc} \mathrm{CH}^r(E^{(0)}, n) & \xrightarrow{\tilde{k}^*\tilde{k}_*} & \mathrm{CH}^{r+1}(E^{(0)}, n) \\ \tilde{\pi}^* \uparrow & & \downarrow \tilde{\pi}_* \\ \mathrm{CH}^r(pt, n)^{\oplus N} & \longrightarrow & \mathrm{CH}^r(pt, n)^{\oplus N} . \end{array}$$

Here the lower horizontal map is given by the matrix $(E_i \cdot E_j)$, which is a \mathbb{Q} -isomorphism.

It follows the commutativity of the square

$$\begin{array}{ccc} \mathrm{CH}_{1-r}(E, n) & \xrightarrow{k^*k_*} & \mathrm{CHC}^{r+1}(E, n) \\ \tilde{\pi}^* \uparrow & & \downarrow \tilde{\pi}_* \\ \mathrm{CH}^r(pt, n)^{\oplus N} & \longrightarrow & \mathrm{CH}^r(pt, n)^{\oplus N} . \end{array}$$

One still has a commutative square if one replaces $\mathrm{CH}_{1-r}(E, n)$ with $\widetilde{\mathrm{CH}}_{1-r}(E, n)$, and $\mathrm{CHC}^{r+1}(E, n)$ with $\widetilde{\mathrm{CHC}}^{r+1}(E, n)$.

(3.2) **Lemma.** *Assume E is a rational tree. (1) The map $k^* : \mathrm{CH}^r(S, n) \rightarrow \widetilde{\mathrm{CHC}}^r(E, n)$ is surjective. So the long exact sequence of (2.1) splits into short exact sequences.*

(2) *The map $k_* : \widetilde{\mathrm{CH}}_s(E, n) \rightarrow \mathrm{CH}_s(X, n)$ is injective; the long exact sequence of (2.2) splits into short exact sequences.*

Proof. (1) Since $\tilde{\pi}_* : \widetilde{\mathrm{CHC}}^r(E, n) \rightarrow \mathrm{CH}^{r-1}(pt, n)^{\oplus N}$ is an isomorphism by (1.4), it is enough to show that the composition $\tilde{\pi}_*k^* : \mathrm{CH}^r(X, n) \rightarrow \mathrm{CHC}^r(E, n) \rightarrow \mathrm{CH}^{r-1}(pt, n)^{\oplus N}$ is surjective. But that is a consequence of Lemma (3.1). The proof of (2) is similar.

(3.3) **Theorem.** *Assume k is an algebraically closed field of characteristic zero, and let S , X , and E be as in §2. Then the following conditions are equivalent:*

- (1) *The canonical map $\mathrm{CHC}^r(S, n) \rightarrow \mathrm{CH}_{2-r}(S, n)$ is a (\mathbb{Q}) -isomorphism for any r, n .*
- (2) *The canonical map $\mathrm{CHC}^1(S, n) \rightarrow \mathrm{CH}_1(S, n)$ is (\mathbb{Q}) -isomorphism for any n .*
- (3) *E is rational tree.*

Proof. (1) clearly implies (2). We first show that (3) implies (1). If E is a rational tree, by (3.2) there are exact sequences

$$0 \rightarrow \mathrm{CHC}^r(S, n) \xrightarrow{p^*} \mathrm{CH}^r(X, n) \xrightarrow{k^*} \widetilde{\mathrm{CHC}}^r(E, n) \rightarrow 0$$

and

$$0 \rightarrow \widetilde{\mathrm{CH}}_{2-r}(E, n) \xrightarrow{k_*} \mathrm{CH}_{2-r}(X, n) \xrightarrow{p_*} \mathrm{CH}_{2-r}(S, n) \rightarrow 0$$

where the middle terms are equal.

In the commutative diagram of (3.1),

$$\begin{array}{ccc} \widetilde{\mathrm{CH}}_s(E, n) & \xrightarrow{k^*k_*} & \widetilde{\mathrm{CHC}}^{2-s}(E, n) \\ \pi^* \uparrow & & \downarrow \pi_* \\ \mathrm{CH}_{s-1}(pt, n)^{\oplus N} & \longrightarrow & \mathrm{CH}_{s-1}(pt, n)^{\oplus N} \end{array}$$

the lower horizontal map is an (\mathbb{Q}) -isomorphism, and the vertical maps are isomorphisms by (1.4) and (1.5). Thus the upper horizontal map is a (\mathbb{Q}) -isomorphism. It follows that the map $p_*p^* : \mathrm{CHC}^r(S, n) \rightarrow \mathrm{CH}_{2-r}(S, n)$ is an isomorphism.

We next show (2) implies (3). Since $\mathrm{CH}_1(S, -1) = 0$, one has $\mathrm{CHC}^1(S, -1) = 0$ and an exact sequence

$$0 \rightarrow \mathrm{CHC}^1(S) \xrightarrow{p^*} \mathrm{CH}^1(X) \xrightarrow{k^*} \mathrm{CHC}^1(E) \rightarrow 0 .$$

On the other hand by (2.2) there is an exact sequence

$$0 \rightarrow \mathrm{CH}_1(E) \xrightarrow{k_*} \mathrm{CH}_1(X) \xrightarrow{p_*} \mathrm{CH}_1(S) \rightarrow 0 .$$

Since p_*p^* is an isomorphism, $k^*k_* : \mathrm{CH}_1(E) \rightarrow \mathrm{CHC}^1(E)$ is an isomorphism.

The group $\mathrm{CHC}^1(E)$ has a filtration as follows. By (1.3), there is a surjection $p^* : \mathrm{CHC}^1(E) \rightarrow \mathrm{CH}^1(E^{(0)})$ with kernel $(H^1(\Gamma) \otimes k^*)_{\mathbb{Q}}$. Also the degree map $\mathrm{deg} : \mathrm{CH}^1(E^{(0)}) \rightarrow \mathbb{Q}^N$ is a surjection with kernel $\bigoplus \mathrm{Pic}^0(E_i)_{\mathbb{Q}}$. But the composition

$$\mathrm{CH}_1(E) \xrightarrow{k^*k_*} \mathrm{CHC}^1(E) \xrightarrow{p^*} \mathrm{CH}^1(E^{(0)}) \xrightarrow{\mathrm{deg}} \mathbb{Q}^N$$

is an isomorphism. Thus for k^*k_* be an isomorphism, it is necessary that $(H^1(\Gamma) \otimes k^*)_{\mathbb{Q}} = 0$ and $\mathrm{Pic}^0(E_i)_{\mathbb{Q}} = 0$ for each i . Since the characteristic of k is zero, k^* has a non-zero free subgroup, so the first condition implies $H^1(\Gamma) = 0$. The condition $\mathrm{Pic}^0(E_i)_{\mathbb{Q}} = 0$ implies $E_i \cong \mathbb{P}^1$. This concludes the proof.

Remark. (1) The assumption on the characteristic is necessary, since if k is the algebraic closure of a finite field, then the group $k^* \otimes \mathbb{Q}$ is zero.

(2) If k is the complex number field, the condition (3) is equivalent to the link of each singular point of S being a rational homology sphere.

(3) There is a ‘‘motivic’’ version of the theorem where the Chow groups are replaced with mixed motives. We will take this up elsewhere.

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