EIGENVALUES OF LAPLACIANS ON A CLOSED RIEMANNIAN MANIFOLD AND ITS NETS

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Dedicated to Professor Hideki Ozeki on his sixtieth birthday.

Abstract. We study the relation between the eigenvalues of the Laplacian of a Riemannian manifold and the combinatorial Laplacians of an approximating sequence of nets in the manifold.

1. Introduction and statement of the main Theorem

To recall the definition of the Laplacian of graphs [B], [F], let Γ be a finite, connected graph, $V(\Gamma)$ the set of its vertices, and $E(\Gamma)$ the set of its directed edges. We assume there are no edges joining a vertex with itself and if two distinct vertices x and y are joined by an edge, in which case we denote $x \sim y$, then there are exactly two edges of opposite directions between them. The edge from x to y, if it exists, is denoted by $[x,y]$ or $-[y,x]$.

A *length function*, $l : E(\Gamma) \to \mathbf{R}_+$, is a positive function on $E(\Gamma)$ with $l([x,y]) = l([y,x])$. Then the *weight function* on $V(\Gamma)$, m_l, is given by

$$m_l(x) = \sum_{x\sim y} l([x,y]),$$

where $\sum_{x\sim y}$ means to take the sum over all the vertices y connected to x. We sometimes write m instead of m_l for simplicity. Put

$$L^2(V(\Gamma)) = \{f : V(\Gamma) \to \mathbf{R}\},$$
$$L^2(E(\Gamma)) = \{\phi : E(\Gamma) \to \mathbf{R} \mid \phi(-e) = -\phi(e)\},$$

and define inner products for $f, g \in L^2(V)$ and $\phi, \psi \in L^2(E)$ by

$$(f,g) = \sum_{x\in V(\Gamma)} m(x)f(x)g(x), \quad (\phi,\psi) = \frac{1}{2}\sum_{e\in E(\Gamma)} l(e)\phi(e)\psi(e).$$

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Define an operator $d : L^2(V) \longrightarrow L^2(E)$ by

$$df([x,y]) = \frac{f(x)-f(y)}{l([x,y])} \quad \text{for} \quad f \in L^2(V).$$

The adjoint operator $\delta : L^2(E) \longrightarrow L^2(V)$ is then given by

$$\delta\phi(x) = \frac{1}{m(x)} \sum_{x \sim y} \phi([x,y]) \quad \text{for} \quad \phi \in L^2(E).$$

The definition of the Laplacian of (Γ, l), Δ, is

$$\Delta f = \delta d f.$$

We have

$$(\Delta f, f) = (df, df),$$

and we can rewrite

$$\Delta f(x) = \frac{1}{m(x)} \sum_{x \sim y} \frac{f(x)-f(y)}{l([x,y])}.$$

The smallest eigenvalue $\lambda_0(\Gamma, l)$ for Δ is always 0 and the one dimensional eigenspace for 0 consists of the constant functions, since Γ is connected. We denote the k-th positive eigenvalue of Δ by $\lambda_k(\Gamma, l)$.

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n, \quad \text{where} \quad n = \sharp V - 1.$$

Before discussing the general case for approximating the eigenvalues of Laplacian of a closed Riemannian manifold by graphs, we give a simple example. Let S^1 be the unit circle, and $\lambda_k(S^1)$ denote the k-th eigenvalue of the Laplacian of S^1. Then

$$\{\lambda_k(S^1)\}_{k=1}^{\infty} = \{0, \underbrace{1, 2^2, 3^2, 4^2, \ldots}_{\text{mult. }=2}\}.$$

Let (C_n, l_n) be the circle graph of n-vertices with length function $l_n \equiv 2\pi/n$. We may directly calculate the values for $\lambda_k(C_n, l_n)$, which we denote by $\mathrm{spec}(C_n)$. If n is odd,

$$\mathrm{spec}(C_n) = \left(\frac{n}{2\pi}\right)^2 \times \{0, \underbrace{2(1-\cos\frac{2\pi}{n}), 2(1-\cos\frac{4\pi}{n}), \ldots, 2(1-\cos\frac{n-1}{n}\pi)}_{\text{mult. }=2}\}.$$

If n is even,

$$\mathrm{spec}(C_n) = \left(\frac{n}{2\pi}\right)^2 \times \{0, \underbrace{2(1-\cos\frac{2\pi}{n}), 2(1-\cos\frac{4\pi}{n}), \ldots, 2(1-\cos\frac{n-2}{n}\pi)}_{\text{mult. }=2}, 4\}.$$

Since $\lim_n(\frac{n}{2\pi})^2 2(1-\cos\frac{2k}{n}\pi)=k^2$, we have

$$\lim_{n\to\infty}\lambda_k(C_n)=\lambda_k(S^1),$$

for each k.

To approximate the eigenvalues of the Laplacian of a closed Riemannian manifold M, we take an $\varepsilon-$ *net* in M, which is a graph obtained in the following way for $\varepsilon>0$. A subset of M is called $\varepsilon-$ *separated* if $d_M(x,y)\geq\varepsilon$ for any distinct points x,y of the set. Take a maximal ε -separated subset V in M and join distinct points x and y of V by two directed edges from x to y and from y to x if and only if $d_M(x,y)\leq 3\varepsilon$. The resulting graph is termed an $\varepsilon-$ *net* in M. It is known that a maximal ε-separated set exists for any $\varepsilon>0$ and its graph is connected if M is connected, [K]. It is clear from the construction that an ε-net in M roughly approximates M as a metric space. Moreover, it approximates the eigenvalues of the Laplacian of M in a certain way, which we state as the following theorem.

Theorem. *Let M be a closed Riemannian manifold of dimension d. Take a sequence of 1/n-nets in M, $(\Gamma_n,l_n),1\leq n<\infty$, with length functions $l_n\equiv 1/n$. There exists a constant $C(d)$ depending only on the dimension d, s.t.*

$$\frac{1}{C}\limsup_{n\to\infty}\lambda_k(\Gamma_n,l_n)\leq\lambda_k(M)\leq C\liminf_{n\to\infty}\lambda_k(\Gamma_n,l_n),$$

for any $k\geq 0$. The constants $C(d)$ satisfy $C(d)\leq 2\cdot 50^d$ for any $d\geq 1$.

Though it grows exponentially, the constant $C(d)$ depends only on the dimension of M, but not on other geometry of M, for example, curvatures. The speed of the convergence in the estimate depends on the curvature.

In the theorem, the estimate is satisfied by any sequence of nets. There might exist a constant C', which does not depend even on the dimension of M, s.t. if we take a nice sequence of nets in M, then the estimate of the theorem hold for the sequence and the constant C'. But the author suspects one can find a sequence of nets in M s.t. the eigenvalues of the combinatorial Laplacians of the nets converge to the eigenvalues of Δ_M, i.e., $C=1$ in the theorem.

2. Proof of the Theorem

The proof is an application of the following Lemma (see [B], Chapter 1 of [C]), called the *minimax principle.* In this section, we unambiguously write Γ for $V(\Gamma)$, and $L^2(\Gamma)$ for $L^2(V(\Gamma))$.

Lemma.

$$\lambda_k(\Gamma)(resp.\ \lambda_k(M))=\inf_{\mathcal{F}_{k+1}}\sup_{f\in\mathcal{F}}\frac{(df,df)}{(f,f)}$$

where $\mathcal{F}_{k+1}$ runs over linear subspaces of $L^2(\Gamma)(resp.\ L^2(M))$ of dimension $k+1$.

The expression $(df,df)/(f,f)$ is called the *Rayleigh quotient* of f.

The proof consists of two parts. First, to show $\lambda_k(M) \leq C \liminf_n \lambda_k(\Gamma_n)$, we construct a linear operator

$$S_n : L^2(\Gamma_n) \to C^\infty(M)$$

for each n, s.t. for sufficiently large n,

$$\frac{(dS_n(f), dS_n(f))_M}{(S_n(f), S_n(f))_M} \leq C \frac{(df, df)_{\Gamma_n}}{(f, f)_{\Gamma_n}},$$

for any $f \in L^2(\Gamma_n)$. Next, to show $\limsup_n \lambda_k(\Gamma_n) \leq C\lambda_k(M)$, we construct a linear operator

$$T_n : C^\infty(M) \to L^2(\Gamma_n)$$

for each n with the following property. Let $\mathcal{F}$ be a finite dimensional linear subspace of $C^\infty(M)$, and $\mathcal{F}(1)$ denote the subset $\{f \in \mathcal{F} | (f, f) = 1\}$. Then for any $\varepsilon > 0$, taking sufficiently large n, we have

$$\frac{(dT_n(f), dT_n(f))_{\Gamma_n}}{(T_n(f), T_n(f))_{\Gamma_n}} \leq C \frac{(df, df)_M + \varepsilon}{(f, f)_M - \varepsilon},$$

for any $f \in \mathcal{F}(1)$. From the above two estimates of the Rayleigh quotient, applying the Lemma, we obtain the inequalities in the theorem.

Constants. Here we give several geometric constants which we will use in the proof. For a point $x \in M$, we write the set $\{y \in M | d(x, y) < r\}$ by $B(x, r)$ and denote its volume in M by $\mathrm{vol}(B(x, r))$. It is seen that there exist some positive constants $C_1, C_2, \ldots, C_8$ which depend only on the dimension of M, d, and satisfy the following properties: taking sufficiently large n, we have for any $x_i \in \Gamma_n$,

$$\left\{ \begin{array}{c} C_1 \leq \sharp\{x_j \in \Gamma_n; x_i \sim x_j\} \leq C_2, \\ n^d \mathrm{vol}(B(x_i, \frac{1}{n})) \leq C_3, \\ C_4 \leq n^d \mathrm{vol}(B(x_i, \frac{1}{3n})), \\ C_5 \leq n^d \mathrm{vol}(B(x_i, \frac{1}{2n})) \leq C_6, \\ \mathrm{vol}(B(x_i, \frac{1}{n})) \leq C_7 \mathrm{vol}(B(x_i, \frac{1}{2n})), \\ \sharp(\Gamma_n) \leq C_8 n^d \mathrm{vol}(M). \end{array} \right\}$$

These constants satisfy

$$(*) \qquad \left\{ \begin{array}{c} 2^d \leq C_1, C_2 \leq 7^d, C_3 \leq 2^d, (3/\sqrt{2})^d \leq C_4, \\ (1/\sqrt{2})^d \leq C_5, C_6 \leq 1, C_7 \leq 2^d, C_8 \leq (\sqrt{2})^d. \end{array} \right\}$$

Proof of the Theorem. Fix n and denote $\{x_j\}_{j=1}^{\sharp V(\Gamma_n)} = V(\Gamma_n)$. Take a partition of unity $\{u_{n,j}\}_j$ on M with the following properties.

$$\left\{ \begin{array}{c} \mathrm{supp}(u_{n,j}) \subset B(x_j, \frac{2}{n}) \quad \text{for each} \quad j, \\ u_{n,j} = 1 \quad \text{on} \quad B(x_j, \frac{1}{3n}), \\ (du_{n,j}(x), du_{n,j}(x)) \leq n^2, \quad \text{for any} \quad x \in M. \end{array} \right\}$$

Since $\sum_j u_{n,j} = 1$,

$$\sum_j du_{n,j} = 0. \tag{1}$$

For $x \in M$, if $d(x, x_j) > \frac{2}{n}$, then

$$du_{n,j}(x) = 0. \tag{2}$$

We define a linear operator

$$S_n : L^2(\Gamma_n) \to C^\infty(M)$$

for each n by

$$S_n(f)(x) = \sum_{x_j \in \Gamma_n} f(x_j) u_{n,j}(x)$$

for $f \in L^2(\Gamma_n)$. From the definition of S_n, S_n is injective. Thus, for any linear subspace $\mathcal{F}$ in $L^2(\Gamma_n)$, we have $\dim \mathcal{F} = \dim S_n(\mathcal{F})$.

Claim 1. Taking sufficiently large n,

$$(dS_n(f), dS_n(f))_M \leq \frac{2C_2C_3}{n^{d-1}} (df, df)_{\Gamma_n}$$

for any $f \in L^2(\Gamma_n)$.

Proof of the Claim 1. For each $x \in M$, take $x_k \in \Gamma_n$ with $d(x, x_k) \leq \frac{1}{n}$, and fix it. Then

$$\begin{aligned} dS_n(f)(x) &= \sum_{x_j \in \Gamma_n} f(x_j) du_{n,j}(x) \\ &= \sum_j (f(x_j) - f(x_k)) du_{n,j}(x) + f(x_k) \sum_j du_{n,j}(x), \end{aligned}$$

using (1)

$$= \sum_j (f(x_j) - f(x_k)) du_{n,j}(x),$$

using (2)

$$= \sum_{x_j \in \Gamma_n; d(x,x_j) \leq \frac{2}{n}} (f(x_j) - f(x_k)) du_{n,j}(x).$$

Since $d(x, x_j) \leq \frac{2}{n}$ and $d(x, x_k) \leq \frac{1}{n}$ imply $d(x_j, x_k) \leq \frac{3}{n}$,

$$\begin{aligned} |dS_n(f)(x)| &\leq \sum_{x_j; d(x_j,x_k) \leq \frac{3}{n}} |f(x_j) - f(x_k)||du_{n,j}(x)| \\ &\leq \sum_{x_j; x_j \sim x_k} |f(x_j) - f(x_k)| n. \end{aligned}$$

Thus,

$$\begin{aligned} (dS_n(f)(x), dS_n(f)(x)) &\leq n^2 (\sum_{x_j; x_j \sim x_k} |f(x_j) - f(x_k)|)^2 \\ &\leq n^2 C_2 \sum_{x_j; x_j \sim x_k} (f(x_j) - f(x_k))^2. \end{aligned}$$

Therefore,

$$\begin{aligned} (dS_n(f), dS_n(f)) &\leq n^2 C_2 \sum_{x_k \in \Gamma_n} \{ \sum_{x_j; x_j \sim x_k} (f(x_j) - f(x_k))^2 \mathrm{vol}(B(x_k, \frac{1}{n})) \} \\ &\leq C_2 C_3 \frac{1}{n^{d-1}} \sum_{x_k \in \Gamma_n} \sum_{x_j; x_j \sim x_k} \frac{(f(x_i) - f(x_k))^2}{n} = \frac{2C_2 C_3}{n^{d-1}} (df, df)_{\Gamma_n} \end{aligned}$$

□

Claim 2. For sufficiently large n, we have

$$(S_n(f), S_n(f))_M \geq \frac{C_4}{C_2 n^{d-1}} (f, f)_{\Gamma_n}$$

for any $f \in L^2(\Gamma_n)$.

Proof of the Claim 2.

$$\begin{aligned} (f, f)_{\Gamma_n} &= \sum_{x_j \in \Gamma_n} f^2(x_j) m_{l_n}(x_j) \leq \frac{C_2}{n} \sum_{x_j \in \Gamma_n} f^2(x_j) \\ &\leq \frac{C_2}{n} \frac{n^d}{C_4} \sum_{x_j \in \Gamma_n} f^2(x_j) \mathrm{vol} B(x_j, \frac{1}{3n}) \\ &\leq \frac{C_2}{C_4} n^{d-1} \int_M (S_n(f), S_n(f)) dM = \frac{C_2}{C_4} n^{d-1} (S_n(f), S_n(f))_M \end{aligned}$$

□

From the Claim 1 and the Claim 2, we have the next claim.

Claim 3. For sufficiently large n, we have

$$\frac{(dS_n(f), dS_n(f))_M}{(S_n(f), S_n(f))_M} \leq \frac{2C_2^2 C_3}{C_4} \frac{(df, df)_{\Gamma_n}}{(f, f)_{\Gamma_n}}$$

for any $f \in L^2(\Gamma_n)$.

Using the Claim 3, we can show $\lambda_k(M) \leq \frac{2C_2^2 C_3}{C_4} \liminf_n \lambda_k(\Gamma_n, l_n)$ as follows. From the Lemma, for any $\varepsilon > 0$, we can take a $(k+1)$-dimensional linear subspace $\mathcal{F}$ of $L^2(\Gamma_n)$ such that

$$\sup_{f \in \mathcal{F}} \frac{(df, df)}{(f, f)} \leq \lambda_k(\Gamma_n) + \varepsilon. \tag{3}$$

From the Claim 3, for sufficiently large n, we have

$$\sup_{g \in S_n(\mathcal{F})} \frac{(dg, dg)}{(g, g)} \leq \frac{2C_2^2 C_3}{C_4} \sup_{f \in \mathcal{F}} \frac{(df, df)}{(f, f)}. \tag{4}$$

Since $\dim(\mathcal{F}) = \dim(S_n(\mathcal{F})) = k+1$, we have

$$\lambda_k(M) \leq \sup_{g \in S_n(\mathcal{F})} \frac{(dg, dg)}{(g, g)}, \tag{5}$$

from the Lemma. Combining (3), (4), (5), we have

$$\lambda_k(M) \leq \frac{2C_2^2 C_3}{C_4} (\lambda_k(\Gamma_n) + \varepsilon), \tag{6}$$

for sufficiently large n. Since ε was arbitrary, we have

$$\lambda_k(M) \leq \frac{2C_2^2 C_3}{C_4} \liminf_{n \to \infty} \lambda_k(\Gamma_n, l_n). \tag{7}$$

Next, we define a linear operator

$$T_n : C^\infty(M) \to L^2(\Gamma_n)$$

for each n by

$$T_n(f)(x_i) = \frac{\int_{B(x_i, \frac{1}{n})} f dV}{\mathrm{vol} B(x_i, \frac{1}{n})},$$

at $x_i \in \Gamma_n$ for $f \in C^\infty(M)$.

Let $\mathcal{F}$ be a finite dimensional linear subspace of $C^\infty(M)$. $\mathcal{F}(1)$ is to denote the set $\{f \in \mathcal{F} | (f,f) = 1\}$. Then for any $\varepsilon > 0$, taking sufficiently large n, we have

Claim 4.

$$(f,f)_M \leq \frac{2C_3}{C_1 n^{d-1}} (T_n(f), T_n(f))_{\Gamma_n} + \varepsilon C_7 \mathrm{vol}(M), \quad \text{for any} \quad f \in \mathcal{F}(1).$$

Proof of the Claim 4. Taking n sufficiently large, we have

$$\int_{B(x_i, \frac{1}{n})} (f,f) dV \leq \{2(T_n(f)(x_i))^2 + \varepsilon\} \mathrm{vol} B(x_i, \frac{1}{n}),$$

for any $x_i \in \Gamma_n$ and $f \in \mathcal{F}(1)$ since $\mathcal{F}$ is finite dimensional. Therefore,

$$\begin{aligned}
(f,f)_M &\leq \sum_i \int_{B(x_i,\frac{1}{n})} (f,f) dV \\
&\leq 2 \sum_i (T_n(f)(x_i))^2 \mathrm{vol} B(x_i, \frac{1}{n}) + \varepsilon \sum_i \mathrm{vol} B(x_i, \frac{1}{n}) \\
&\leq \frac{2C_3}{n^d} \sum_i (T_n(f)(x_i))^2 + \varepsilon C_7 \sum_i \mathrm{vol} B(x_i, \frac{1}{2n}) \\
&\leq \frac{2C_3}{C_1 n^{d-1}} \sum_i (T_n(f)(x_i))^2 m_{l_n}(x_i) + \varepsilon C_7 \mathrm{vol}(M) \\
&= \frac{2C_3}{C_1 n^{d-1}} (T_n(f), T_n(f))_{\Gamma_n} + \varepsilon C_7 \mathrm{vol}(M).
\end{aligned}$$

□

Also, for any $\varepsilon > 0$, taking sufficiently large n, we have

Claim 5.

$$(dT_n(f), dT_n(f))_{\Gamma_n} \leq n^{d-1} \left\{ \frac{9C_2}{C_5} (df, df)_M + \varepsilon \frac{9C_2}{2C_5} \mathrm{vol}(M) \right\} \quad \text{for any} \quad f \in \mathcal{F}(1).$$

Proof of the Claim 5. Since $\mathcal{F}$ is finite dimensional, taking n sufficiently large, we have, for any $x_i, x_j \in \Gamma_n$ with $x_i \sim x_j$,

$$(T_n(f)(x_i) - T_n(f)(x_j))^2 \leq \left\{ \frac{2\int_{B(x_i,\frac{1}{2n})} (df, df) dV}{\mathrm{vol} B(x_i, \frac{1}{2n})} + \varepsilon \right\} d^2(x_i, x_j),$$

and since $d^2(x_i, x_j) \leq \frac{9}{n^2}$,

$$\leq \frac{18}{n^2} \frac{n^d}{C_5} \int_{B(x_i, \frac{1}{2n})} (df, df) dV + \frac{9}{n^2} \varepsilon.$$

Therefore,

$$\begin{aligned}(dT_n(f), dT_n(f))_{\Gamma_n} &= \frac{1}{2}\sum_{x_i \sim x_j} (T_n(f)(x_i) - T_n(f)(x_j))^2 n \\ &\leq \frac{9C_2 n^{d-1}}{C_5} \sum_{x_i \in \Gamma_i} \int_{B(x_i, \frac{1}{2n})} (df, df)dV + \frac{9C_2}{2n} \sharp(\Gamma_n)\varepsilon \\ &\leq \frac{9C_2 n^{d-1}}{C_5} \int_M (df, df)dV + \frac{9C_2 n^{d-1}}{2C_5} \mathrm{vol}(M)\varepsilon \\ &= \frac{9C_2 n^{d-1}}{C_5} (df, df)_M + \frac{9C_2 n^{d-1}}{2C_5} \mathrm{vol}(M)\varepsilon.\end{aligned}$$

□

Combining the Claim 4 and the Claim 5, we have the next claim.

Claim 6. Let $\mathcal{F}$ be a finite dimensional linear subspace of $C^\infty(M)$. Then for any sufficiently small $\varepsilon > 0$, taking sufficiently large n, we have

$$\frac{(dT_n(f), dT_n(f))_{\Gamma_n}}{(T_n(f), T_n(f))_{\Gamma_n}} \leq \frac{18C_2C_3}{C_1C_5} \frac{(df, df)_M + \varepsilon}{(f, f)_M - \varepsilon},$$

for any $f \in \mathcal{F}(1)$.

Using the Claim 6, for any sufficiently small $\varepsilon > 0$, taking sufficiently large n, we can show that

$$\lambda_k(\Gamma_n, l_n) \leq \frac{18C_2C_3}{C_1C_5} (\lambda_k(M) + \varepsilon) \tag{8}$$

as we showed (6) from the Claim 3. In this case, we need a more subtle argument since the operator T_n may decrease the dimension of the linear subspace $\mathcal{F}$ of $L^\infty(M)$. But we can always retake $\mathcal{F}$ to satisfy $\dim\mathcal{F} = \dim T_n(\mathcal{F})$ in each step of the argument. Details are left to the reader. From (8), we have

$$\limsup_{n\to\infty} \lambda_k(\Gamma_n, l_n) \leq \frac{18C_2C_3}{C_1C_5} \lambda_k(M). \tag{9}$$

Therefore, taking $C(d) = \max\{\frac{2C_2^2C_3}{C_4}, \frac{18C_2C_3}{C_1C_5}\}$, we have the Theorem from (7) and (9). From (*) , $C(d)$ satisfies $C(d) \leq 2 \cdot 50^d$. □

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