JSJ-DECOMPOSITIONS OF FINITELY PRESENTED GROUPS AND COMPLEXES OF GROUPS

KOJI FUJIWARA AND PANOS PAPASOGLU

Dedicated to Professor David Epstein for his sixtieth birthday

Abstract. A JSJ-splitting of a group G over a certain class of subgroups is a graph of groups decomposition of G which describes all possible decompositions of G as an amalgamated product or an HNN extension over subgroups lying in the given class. Such decompositions originated in 3-manifold topology. In this paper we generalize the JSJ-splitting constructions of Sela, Rips-Sela and Dunwoody-Sageev and we construct a JSJ-splitting for any finitely presented group with respect to the class of all slender subgroups along which the group splits. Our approach relies on Haefliger's theory of group actions on CAT(0) spaces.

1. Introduction

The type of graph of groups decompositions that we will consider in this paper has its origin in 3-dimensional topology. Waldhausen in [W] defined the characteristic submanifold of a 3-manifold M and used it in order to understand exotic homotopy equivalences of 3-manifolds (i.e., homotopy equivalences that are not homotopic to homeomorphisms). Here is a (weak) version of the characteristic submanifold theory used by Waldhausen that is of interest to us: Let M be a closed, irreducible, orientable 3-manifold. Then there is a finite collection of embedded 2-sided incompressible tori such that each piece obtained by cutting M along this collection of tori is either a Seifert fibered space or atoroidal and acylindrical. Furthermore every embedded incompressible torus of M is either homotopic to one of the cutting tori or can be isotoped into a Seifert fibered space piece. We note that embedded incompressible tori of M correspond to splittings of the fundamental group of M over abelian subgroups of rank 2. So from the algebraic point of view we have a 'description' of all splittings of $\pi_1(M)$ over abelian groups of

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rank 2. Waldhausen in [W] did not give a proof of this theorem; it was proven later independently by Jaco-Shalen [JS] and Johannson [J] (this explains the term JSJ-decomposition).

We recall that by Grushko's theorem every finitely generated group G can be decomposed as a free product of finitely many indecomposable factors. Now if G has no $\mathbb{Z}$ factors any other free decomposition of G is simply a product of a rearrangement of conjugates of these indecomposable factors. One can see JSJ-decomposition as a generalization of this description for splittings of groups over certain classes of subgroups.

We recall that a group is termed *small* if it has no free subgroups of rank 2. Our paper deals with splittings over slender groups which are a subclass of small groups. We recall that a finitely generated group G is *slender* if every subgroup of G when it acts on a tree either leaves an infinite line invariant or it fixes a point. It turns out that a group is slender if and only if all its subgroups are finitely generated (see [DS]). For example finitely generated nilpotent groups are slender.

To put our results on JSJ-decompositions in perspective we note that Dunwoody has shown that if G is a finitely presented group then if Γ is a graph of groups decomposition of G with corresponding G-tree T_Γ then there is a G-tree T' and a G-equivariant map $\alpha : T' \to T_\Gamma$ such that T'/G has at most $\delta(G)$ essential vertices (see [BF], lemma 1). We recall that a vertex in a graph of groups is not essential if it is adjacent to exactly two edges and both edges and the vertex are labelled by the same group. In other words one can obtain all graph of groups decompositions of G by 'folding' from some graph of group decompositions which have less than $\delta(G)$ vertices.

We remark that in general there is no bound on the number of vertices of the graph of groups decompositions that one obtains after folding. However in the special case of decompositions with small edge groups Bestvina and Feighn ([BF]. See Thm 5.3 in this paper) have strengthened this result showing that every reduced decomposition Γ of a finitely presented group G with small edge groups has at most $\gamma(G)$ vertices. Essentially they showed that in the case of small splittings the number of 'foldings' that keep the edge groups small is bounded. The JSJ decomposition that we present here complements the previous results as it gives a description of a set of decompositions with slender edge groups from which we can obtain any other decomposition by 'foldings'. Roughly this set is obtained as follows: we start with the JSJ decomposition and then we refine it by picking for each enclosing group some splittings that correspond to disjoint simple closed curves on the underlying surface. Of course there are infinitely many such possible refinements but they are completely described by the 'surfaces' that correspond to the enclosing groups.

Sela in [S] was the first to introduce the notion of a JSJ-decomposition for a generic class of groups, namely for hyperbolic groups. Sela's JSJ-decomposition of hyperbolic groups describes all splittings of a hyperbolic group over infinite cyclic subgroups and was used to study the group of automorphisms of a hyperbolic group. Sela's result was subsequently generalized by Rips and Sela ([RS]) to all finitely presented groups. Dunwoody and Sageev ([DS]) generalized this result further and produced a JSJ-decomposition which describes all splittings of a finitely presented group over slender groups under the assumption that the group does not split over groups 'smaller' than the ones considered. Bowditch in [B] gives a different way of constructing the JSJ-decomposition of a hyperbolic group using the boundary of the group. In particular this shows that the JSJ-decomposition is invariant under quasi-isometries.

In this paper we produce for every finitely presented group G a JSJ-decomposition of G that describes all splittings of G over all its slender subgroups.

Our approach to JSJ-decompositions differs from that of [S],[RS] and of [DS] in that we use neither $\mathbb{R}$-trees nor presentation complexes. We use instead Haefliger's theory of complexes of groups and actions on products of trees. To see how this can be useful in studying splittings of groups consider the following simple example: Let G be the free abelian group on two generators a, b. Then G splits as an HNN extension over infinitely many of its cyclic subgroups. Consider now two HNN decompositions of G, namely the HNN decomposition of G over $\langle a\rangle$ and over $\langle b\rangle$. The trees corresponding to these decompositions are infinite linear trees. Consider now the diagonal action of G on the product of these two trees. The quotient is a torus. Every splitting of G is now represented in this quotient by a simple closed curve. We see therefore how we can arrive at a description of infinitely many splittings by considering an action on a product of trees corresponding to two splittings.

Before stating our results we give a brief description of our terminology: Let T_A, T_B be Bass-Serre trees for one edge splittings of a group G over subgroups A, B. We say that the splitting over A is elliptic with respect to the splitting over B if A fixes a vertex of T_B. If the splitting over A is not elliptic with respect to the splitting over B we say that it is hyperbolic. We say that the pair of two splittings is hyperbolic-hyperbolic if they are hyperbolic with respect to each other. We define similarly elliptic-elliptic etc (see def.2.1). If a splitting over a slender group A is not hyperbolic-elliptic with respect to any other splitting over a slender group then we say it is minimal. Finally we use the term enclosing group (def.4.5) for what Rips-Sela call quadratically hanging group and Dunwoody-Sageev call hanging $\mathcal{K}$-by-orbifold group.

This paper is organized as follows. In section 2 we prove some preliminary results and recall basic definitions from [RS]. In section 3 we introduce the notion of 'minimality' of splittings and prove several technical lemmas about minimal splittings that are used in the sequel. In section 4 we apply Haefliger's theory to produce 'enclosing groups' for pairs of hyperbolic-hyperbolic minimal splittings. Proposition 4.7 is the main step in our construction of JSJ decompositions. It says that we can always find a graph decomposition that contains both splittings of a given pair of splittings. We note that, although in our main theorem we consider only finitely presented groups, proposition 4.7 is valid for groups that are only finitely generated. Moreover proposition 4.7 holds also for pairs of hyperbolic-hyperbolic splittings over small groups.

In section 5 using the same machinery as in section 4 we show that there is a graph of groups that 'contains' all splittings from a family of hyperbolic-hyperbolic minimal splittings (proposition 5.4). Using this we describe a refinement process that produces the JSJ-decomposition of a finitely presented group over all its slender subgroups. Because of the accessibility results of Bestvina-Feighn ([BF]. See Thm 5.3), there is an upper bound on the complexity of graph decompositions that appear in the refinement process, therefore this process must terminate. The terminal graph decomposition must "contain" all minimal splittings.

The graph decomposition has special vertex groups (maybe none) which are called maximal enclosing groups with adjacent edge groups to be peripheral (see Def 4.5). Each of them is an extension of the orbifold fundamental group of some compact 2-orbifold with boundary (maybe empty) by a slender group, F. Examples are surface groups (F is trivial) and the fundamental group of a Seifert space ($F \simeq \mathbb{Z}$), which is a 3-manifold. We produce a graph decomposition using minimal splittings (see Def 3.1) of G. See Def 2.1 for the definition of the type of a pair of splittings, namely, hyperbolic-hyperbolic, elliptic-elliptic.

Theorem 5.13. *Let G be a finitely presented group. Then there exists a graph decomposition, Γ, of G such that*

1. *all edge groups are slender.*

2. *Each edge of Γ gives a minimal splitting of G along a slender group. This splitting is elliptic-elliptic with respect to any minimal splitting of G along a slender subgroup.*
3. *Each maximal enclosing group of G is a conjugate of some vertex group of Γ, which we call a (maximal) enclosing vertex group. The edge group of an edge adjacent to the vertex of a maximal enclosing vertex group is a peripheral subgroup of the enclosing group.*
4. *Let $G = A *_C B$ or $A*_C$ be a minimal splitting along a slender group C, and T_C its Bass-Serre tree.*
   (a) *If it is elliptic-elliptic with respect to all minimal splittings of G along slender groups, then all vertex groups of Γ are elliptic on T_C.*
   (b) *If it is hyperbolic-hyperbolic with respect to some minimal splitting of G along a slender group, then there is an enclosing vertex group, S, of Γ which contains a conjugate of C, which is unique among enclosing vertex groups of Γ. S is also the only one among enclosing vertex groups which is hyperbolic on T_C. There exist a base 2-orbifold, Σ, for S and an essential simple closed curve or a segment on Σ whose fundamental group (in the sense of complex of groups) is a conjugate of C.*
   All vertex groups except for S of Γ are elliptic on T_C.
   In particular, there is a graph decomposition, $\mathcal{S}$, of S whose edge groups are in conjugates of C, which we can substitute for S in Γ such that all vertex groups of the resulting refinement of Γ are elliptic on T_C.

Although we produce Γ, called a JSJ-decomposition, using only minimal splittings, it turns out that it is also good for non-minimal splittings.

Theorem 5.15. *Let G be a finitely presented group, and Γ a graph decomposition we obtain in Theorem 5.13. Let $G = A *_C B, A*_C$ be a splitting along a slender group C, and T_C its Bass-Serre tree.*

1. *If the group C is elliptic with respect to any minimal splitting of G along a slender group, then all vertex groups of Γ are elliptic on T_C.*
2. *Suppose the group C is hyperbolic with respect to some minimal splitting of G along a slender group. Then*
   (a) *all non-enclosing vertex groups of Γ are elliptic on T_C.*
   (b) *For each enclosing vertex group, V, of Γ, there is a graph decomposition of V, $\mathcal{V}$, whose edge groups are in conjugates of C, which we can substitute for V in Γ such that if we substitute*

for all enclosing vertex groups of Γ *then all vertex groups of the resulting refinement of* Γ *are elliptic on* T_C.

The first version of this paper is written in 1998. Since then a very important application of JSJ-decompositions is found by Z.Sela on Tarski's conjecture on the equivalence of the elementary theory of $\mathbb{F}_2, \mathbb{F}_3$ (see [S1] and the following papers of Sela on this). He uses JSJ-decompositions along abelian subgroups. We note also that the question of 'uniqueness' of JSJ-splittings has been treated in [Fo]. We would like to thank M.Bestvina, M.Feighn, V.Guirardel, B.Leeb, M.Sageev, Z.Sela and G.A.Swarup for discussions related to this work. We would like to thank A. Haefliger for his interest in this work and many suggestions that improved the exposition. Finally we would like to thank the referee for detailed suggestions which we found very helpful.

2. Pairs of splittings

In this section we recall and generalize notation from [RS].

Definition 2.1 (types of a pair). Let $A_1 \star_{C_1} B_1$ (or $A_1 \star_{C_1}$), $A_2 \star_{C_2} B_2$ (or $A_2 \star_{C_2}$) be two splittings of a finitely generated group G with corresponding Bass-Serre trees T_1, T_2. We say that the first splitting is hyperbolic with respect to the second if there is $c_1 \in C_1$ acting as a hyperbolic element on T_2. We say that the first splitting is elliptic with respect to the second if C_1 fixes a point of T_2. We say that this pair of splittings is hyperbolic-hyperbolic if each splitting is hyperbolic with respect to the other. Similarly we define what it means for a pair of splittings to be elliptic-elliptic, elliptic-hyperbolic and hyperbolic-elliptic.

It is often useful to keep in mind the 'geometric' meaning of this definition: Consider for example a closed surface. Splittings of its fundamental group over $\mathbb{Z}$ correspond to simple closed curves on the surface. Two splittings are hyperbolic-hyperbolic if their corresponding curves intersect and elliptic-elliptic otherwise. Consider now a punctured surface and two splittings of its fundamental group: one corresponding to a simple closed curve (a splitting over $\mathbb{Z}$) and a free splitting corresponding to an arc having its endpoints on the puncture such that the two curves intersect at one point. This pair of splittings is hyperbolic-elliptic.

Proposition 2.2. *Let* $A_1 \star_{C_1} B_1$ *(or* $A_1 \star_{C_1}$*),* $A_2 \star_{C_2} B_2$ *(or* $A_2 \star_{C_2}$*) be two splittings of a group* G *with corresponding Bass-Serre trees* T_1*,* T_2*. Suppose that there is no splitting of* G *of the form* $A \star_C B$ *or* $A \star_C$ *with*

C an infinite index subgroup of C_1 or of C_2. Then this pair of splittings is either hyperbolic-hyperbolic or elliptic-elliptic.

Proof. We treat first the amalgamated product case. Let T_1, T_2 be the Bass-Serre trees of the two splittings $A_1 \star_{C_1} B_1$, $A_2 \star_{C_2} B_2$. Suppose that C_1 does not fix any vertex of T_2 and that C_2 does fix a vertex of T_1. Without loss of generality we can assume that C_2 fixes the vertex stabilized by A_1. Consider the actions of A_2, B_2 on T_1. Suppose that both A_2, B_2 fix a vertex. If they fix different vertices then C_2 fixes an edge, so it is a finite index subgroup of a conjugate of C_1. But then C_1 can not be hyperbolic with respect to $A_2 \star_{C_2} B_2$. On the other hand it is not possible that they fix the same vertex since A_2, B_2 generate G. So at least one of them, say A_2, does not fix a vertex. But then the action of A_2 on T_1 induces a splitting of A_2 over a group C which is an infinite index subgroup of C_1. Since C_2 is contained in a vertex group of this splitting we obtain a splitting of G over C which is a contradiction.

We consider now the case one of the splittings is an HNN-extension: say we have the splittings $A_1 \star_{C_1} B_1$, $A_2 \star_{C_2}$ with Bass-Serre trees T_1, T_2. Assume C_1 is hyperbolic on T_2 and C_2 elliptic on T_1. Again it is not possible that A_2 fix a vertex of T_1. Indeed $C_2 = A_2 \cap tA_2t^{-1}$ and if A_2 fixes a vertex C_2 is contained in a conjugate of C_1 which is impossible (note that t can not fix the same vertex as A_2). We can therefore obtain a splitting of G over an infinite index subgroup of C_1 which is a contradiction. If C_1 is elliptic on T_2 and C_2 hyperbolic on T_1 we argue as in the first case. The case where both splitting are HNN extension is treated similarly. □

Remark 2.3. In the proof of the Proposition 2.2 one shows in fact that if $A_2 \star_{C_2} B_2$ is elliptic with respect to $A_1 \star_{C_1} B_1$ then either $A_1 \star_{C_1} B_1$ is elliptic too, or there is a splitting of G over a subgroup of infinite index of C_1.

3. Minimal splittings

Definition 3.1 (Minimal splittings)**.** We call a splitting $A \star_C B$ (or $A \star_C$) of a group G minimal if it is not hyperbolic-elliptic with respect to any other splitting of G over a slender subgroup.

Remark 3.2. Remark 2.3 implies that if G splits over C but does not split over an infinite index subgroup of C then the splitting of G over C is minimal. There are examples of minimal and non-minimal splittings over a common subgroup. For example let H be a group which does not split and let $G = \mathbb{Z}^2 * H$. If a, b are generators of $\mathbb{Z}^2$ the splitting of $\mathbb{Z}^2$ over $\langle a\rangle$ induces a minimal splitting of G over $\langle a\rangle$. On the other hand the splitting of G given by $G = \mathbb{Z}^2 *_{\langle a\rangle} (H * \langle a\rangle)$ is not minimal. Indeed it is hyperbolic-elliptic with respect to the splitting of G over $\langle b\rangle$ which is induced from the splitting of $\mathbb{Z}^2$ over $\langle b\rangle$.

We collect results on minimal splittings we need. We first show the following:

Lemma 3.3. *Suppose that a group G splits over the slender groups C_1, C_2 and $K \subset C_2$. Assume moreover that the splittings over C_1, C_2 are hyperbolic-hyperbolic, the splitting over C_1 is minimal and that G admits an action on a tree T such that C_2 acts hyperbolically and K fixes a vertex. Then the splittings over C_1 and K are not hyperbolic-hyperbolic.*

Proof. We will prove this by contradiction. Let T_1, T_2, T_3 be, respectively, the Bass-Serre trees of the splittings over C_1, C_2, K. Without loss of generality we can assume the axes of C_2, K when acting on T_1 contain an edge stabilized by C_1. Let $t \in K \subset C_2$ be an element acting hyperbolically on T_1. Similarly let $u \in C_1$ be an element acting hyperbolically on T_2, T_3 and $y \in C_2$ be an element acting hyperbolically on T. We distinguish 2 cases:
Case 1: y acts elliptically on T_1. Then either y fixes the axis of translation of C_2 or it acts on it by a reflection (in the dihedral action case). In both cases $y^2 \in C_1$. Since $y^2 \notin K$ we have that y^2 acts hyperbolically on T_3. Therefore there are $m, n \in \mathbb{Z}$ such that $y^m u^n$ fixes an edge of T_3 and $y^m u^n \in C_1 \cap zKz^{-1}$, hence $y^m u^n \in C_1 \cap xC_2x^{-1}$. This is clearly impossible since y is elliptic when acting on T_2 while u is hyperbolic.
Case 2: y acts hyperbolically on T_1. Without loss of generality we assume that t fixes the axis of translation of y on T. Indeed if this is not so we can replace t by t^2. Since both t and y act hyperbolically on T_1 there are $m, n \in \mathbb{Z}$ such that $t^m y^n \in C_1$. On the other hand $t^m y^n$ acts hyperbolically on T since t fixes the axis of translation of y. So $t^m y^n$ does not lie in a conjugate of K. For the same reason $(t^m y^n)^2$ does not lie in a conjugate of K. We consider now the action of C_1 on T_3. If $t^m y^n$ is elliptic then $(t^m y^n)^2$ fix the axis of translation of C_1 and therefore lies in a conjugate of K, which is impossible as we noted above. Therefore both $t^m y^n$ and u acts hyperbolically on T_3. We infer that there are $p, q \in \mathbb{Z}$ such that $(t^m y^n)^p u^q$ lies in a conjugate of K. Therefore this element fixes the translation axis of C_1 when acting on T_2. This is however impossible since $t^m y^n \in C_1 \cap C_2$ so it fixes the axis while u acts hyperbolically on T_2. This finishes the proof of the lemma. □

Using lemma 3.3, we show the following.

Proposition 3.4 (dual-minimality)**.** *Let $A_1 \star_{C_1} B_1$ (or $A_1 \star_{C_1}$) be a minimal splitting of G over a slender group C_1. Suppose that $A_1 \star_{C_1} B_1$ ($A_1 \star_{C_1}$)is hyperbolic-hyperbolic with respect to another splitting of G, $A_2 \star_{C_2} B_2$ (or $A_2 \star_{C_2}$), where C_2 is slender. Then $A_2 \star_{C_2} B_2$ (or $A_2 \star_{C_2}$) is also minimal.*

Proof. We denote the Bass-Serre trees for the splittings over C_1, C_2 by T_1, T_2 respectively.

Suppose that the splitting over C_2 is not minimal; then it is hyperbolic-elliptic with respect to another splitting over a slender subgroups C_3. We distinguish 2 cases:

1st case: The splitting over C_3 is an amalgamated product, say $A_3 \star_{C_3} B_3$.

We let A_3, B_3 act on T_2 and we get graph of groups decompositions for A_3, B_3, say Γ_1, Γ_2. Since C_3 is elliptic when acting on T_2 we can refine $A_3 \star_{C_3} B_3$ by replacing A_3, B_3 by Γ_1, Γ_2. We collapse then the edge labelled by C_3 and we obtain a new graph of groups decomposition that we call Γ. We note that all vertex groups of Γ fix vertices of T_2. This implies that C_1 is not contained in a conjugate of a vertex group of Γ. Therefore we can collapse all edges of Γ except one and obtain a splitting over a subgroup K of C_2 such that C_1 is hyperbolic with respect to this splitting. Since C_1 is minimal the pair of splittings over C_1, K is hyperbolic-hyperbolic. This however contradicts lemma 3.3 since K fixes a vertex of the Bass-Serre tree of $A_3 \star_{C_3} B_3$ while C_2 acts hyperbolically on this tree.

2nd case: The splitting over C_3 is an HNN-extension, say $A_3 \star_{C_3}$.

The argument is similar in this case but a bit more delicate. We let A_3 act on T_2 and we obtain a graph decomposition for A_3, say Γ_1. Since C_3 fixes a vertex of T_2 we can refine $A_3 \star_{C_3}$ by replacing A_3 by Γ_1. Let e be the edge of $A_3 \star_{C_3}$. If e stays a loop in Γ_1 we argue as in the amalgamated product case. After we collapse e in Γ_1 we obtain a graph of groups such that C_1 is not contained in the conjugate of any vertex group. We arrive then at a contradiction as before.

Assume now that e connects to two different vertices in Γ_1. Let V, U be the vertex groups of these vertices. Clearly C_2 is not contained in a conjugate of either V or U. We remark that the vertex we get after collapsing e in Γ_1 is labelled by $\langle V_1, V_2 \rangle$. By Bass-Serre theory C_2 is not contained in a conjugate of $\langle V_1, V_2 \rangle$ either. Let's call Γ the graph of groups obtained after collapsing e. Let T_Γ be its Bass-Serre tree. Clearly C_2 does not fix any vertex of T_Γ. It follows that we can collapse all edges of Γ except one and obtain an elliptic-hyperbolic splitting with respect to C_2. What we have gained is that this splitting is over a subgroup of C_2, say K. Of course if this splitting is an amalgam we are done by case 1 so we assume it is an HNN-extension $A\star_K$. Let's call e_1 the edge of this HNN-extension and let T_K be its Bass-Serre tree. If C_1 acts hyperbolically on T then we are done as before by lemma 3.3 since C_2 is hyperbolic on T and K elliptic. Otherwise we let A act on T_2 and we refine $A\star_K$ as before. Let's call Γ' the graph of groups obtained. If e_1 stays a loop in Γ' we are done as before. Otherwise by collapsing e_1 we obtain a graph of groups such that no vertex group contains a conjugate of C_1. We can collapse this graph further to a one edge splitting over, say $K_1 < C_2$ such that its vertex groups do not contain a conjugate of C_1. Since the splitting over C_1 is minimal this new splitting is hyperbolic-hyperbolic with respect to the splitting over C_1. Moreover K_1 fixes a vertex of T while C_2 acts hyperbolically on T. This contradicts lemma 3.3. □

We prove an accessibility result for minimal splittings.

Proposition 3.5 (accessibility of minimal splittings)**.** *Let G be a finitely generated group. There is no infinite sequence of splittings of G of the form $A_n\star_{C_n}B_n$ or of the form $A_n\star_{C_n}$ where C_{n+1} is a subgroup of C_n, C_1 is a (finitely generated) slender group, such that C_n acts hyperbolically on some G-tree T_n while C_{n+1} fixes a vertex of T_n.*

Proof. We define a sequence of homomorphisms f_n from C_1 to $\mathbb{Z}/2\mathbb{Z}$ as follows: Consider the graph of groups corresponding to the action of C_1 on T_n. The fundamental group of this graph of groups is C_1. If the underlying graph of this graph of groups is a circle, map this group to $\mathbb{Z}/2\mathbb{Z}$ by mapping all vertex groups to 0 and the single loop of the graph to 1. If C_1 acts by a dihedral type action on T_n, map C_1 to $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ in the obvious way and then map $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$ so that C_{n+1} is mapped to 0. This is possible since C_{n+1} is elliptic. By construction $f_{n+1}(C_{n+1}) = \mathbb{Z}/2\mathbb{Z}$. It follows that the map $\Phi_n : C_1 \to (\mathbb{Z}/2\mathbb{Z})^n$ is onto for every n. This contradicts the fact that C_1 is finitely generated. □

Each edge, e, of a graph decomposition, Γ, of a group G gives (rise to) a splitting of G along the edge group of e, E, by collapsing all edges of Γ but e. We do this often in the paper. To state the main result of this section, we give one definition.

Definition 3.6 (Refinement)**.** Let Γ be a graph of groups decomposition of G. We say that Γ' is a refinement of Γ if each vertex group of Γ' is contained in a conjugate of a vertex group of Γ. We say that Γ' is a proper refinement of Γ if Γ' is a refinement of Γ and Γ is not a refinement of Γ'.

Let Γ be a graph of groups and let V be a vertex group of Γ, which admits a graph of groups decomposition Δ such that a conjugate of each edge group adjacent to V in Γ is contained in a vertex group of Δ. Then one can obtain a refinement of Γ by replacing V by Δ. In the refinement, each edge, e, in Γ adjacent to the vertex for V is connected to a vertex of Δ whose vertex group contains a conjugate of the edge group of e. The monomorphism from the edge group of e to the vertex group of Δ is equal to the corresponding monomorphism in Γ modified by conjugation.

This is a special type of refinement used often in this paper. We say that we *substitute* Δ for V in Γ.

Proposition 3.7 (Modification to minimal splittings)**.** *Let G be a finitely presented group. Suppose that Γ is a graph of groups decomposition of G with slender edge groups. Then there is a graph of groups decomposition of G, Γ', which is a refinement of Γ such that all edges of Γ' give rise to minimal splittings of G. All edge groups of Γ' are subgroups of edge groups of Γ.*

Proof. We give two proofs of this proposition. We think that in the first one the idea is more transparent, which uses actions on product of trees. Since we use terminologies and ideas from the part we construct "enclosing groups" in Proposition 4.7, one should read the first proof after reading that part. The second proof uses only classical Bass-Serre theory and might be more palatable to readers not accustomed to Haefliger's theory.
1st proof. We define a process to produce a sequence of refinements of Γ which we can continue as long as an edge of a graph decomposition in the sequence gives a non-minimal splitting, and then show that it must terminate in a finite step.

If every edge of Γ gives a minimal splitting, nothing to do, so suppose there is an edge, e, of Γ with the edge group E which gives a splitting which is is hyperbolic-elliptic with respect to, say, $G = P *_R Q$ or $P*_R$. We consider the action of G on the product of trees T_Γ, T_R where T_Γ is the Bass-Serre tree of Γ and T_R the tree of the splitting over R. We consider the diagonal action of G on $T_\Gamma \times T_R$, then produce a G-invariant subcomplex of $T_\Gamma \times T_R$, Y, such that Y/G is compact as we will do in the proof of Proposition 4.7. Y/G has a structure of a complex of groups whose fundamental group is G. In the construction of Y/G, we give priority (see Remark 4.9) to the decomposition Γ over the splitting of G along R, so that we can recover Γ from Y/G by collapsing the complex of groups obtained. To fix ideas we think of T_Γ as horizontal (see the paragraph after Lemma 4.1). Since the slender group E acts hyperbolically on T_R, there is a line, l_E, in T_R which is invariant by E. Then, l_E/E is either a segment, when the action of E is dihedral, or else, a circle. Put $c_E = l_E/E$, which is the core for E. There is a map from $c_E \times [0, 1]$ to Y/G, and let's call the image, b_E, the band for E. b_E is a union of finite squares. For other edges, e_i, of Γ than e with edge groups E_i, we have similar objects, b_{E_i}, which can be a segment, when the action of E_i on T_R is elliptic. As in Proposition 4.7, the squares in Y/G is exactly the union of the squares contained in b_E and other b_{E_i}'s. Note that there is at least one square in Y/G, which is contained in b_E.

On the other hand since R is elliptic on T_Γ, T_Γ/R is a tree, so that the intersection of T_Γ/R (note that this is naturally embedded in $T_\Gamma \times T_R$) and Y/G is a forest. Therefore, we can remove squares from Y/G without changing the fundamental group, which is G. We then collapse all vertical edges which are left. In this way, we obtain a graph decomposition of G, which we denote Γ_1. By construction, all vertex groups of Γ_1 are elliptic on T_Γ, so that Γ_1 is a refinement of Γ. Also edge groups of Γ_1 are subgroups of edge groups of Γ. There is no edge group of Γ_1 which is hyperbolic-elliptic with respect to the splitting over R. If all edge of Γ_1 gives a minimal splitting of G, then Γ_1 is a desired one. If not, we apply the same process to Γ_1, and obtain a refinement, Γ_2. But this process must terminate by Prop 3.5 and Theorem 5.3, which gives a desired one.

2nd proof. If all edges of Γ correspond to minimal splittings then there is nothing to prove. Assume therefore that an edge e of Γ corresponds to a splitting which is not minimal. We will construct a finite sequence of refinements of Γ such that the last term of the sequence is Γ'. Let's say that e is labelled by the slender group E. Let $A *_E B$ (or $A*_E$) be the decomposition of G obtained by collapsing all edges of Γ except e. Since this splitting is not minimal it is hyperbolic-elliptic with respect to another splitting of G over a slender group, say $P *_R Q$ (or $P*_R$).

Let T_E, T_R be the Bass-Serre tree of the splittings of G over E, R and let T_Γ be the Bass-Serre tree corresponding to Γ. We distinguish two cases:

Case 1: R is contained in a conjugate of E.

In this case we let P, Q act on T_Γ. We obtain graph of groups decompositions of P, Q and we refine $P *_R Q$ by substituting P, Q by these graphs of groups decompositions. In this way we obtain a graph of groups decomposition Γ_1. If some edges of Γ_1 (which are not loops) are labelled by the same group as an adjacent vertex we collapse them. For simplicity we still call Γ_1 the graph of groups obtained after this collapsing. We remark that all edge groups of Γ_1 fix a vertex of the tree of the splitting $P *_R Q$. We argue in the same way if the spitting over R is an HNN-extension.

Case 2: R is not contained in any conjugate of E.

We let P, Q act on T_E and we obtain graph of groups decompositions of these groups. We refine $P *_R Q$ as before by substituting P, Q by the graph of groups obtained. We note that by our hypothesis in case 2 P, Q do not fix both vertices of T_E. Let's call Δ the graph of groups obtained in this way. Since C fixes a vertex of T_E we can assume without loss of generality that $C \subset P$. We collapse the edge of Δ labelled by R and we obtain a graph of groups Δ_1. We note now that if E acts hyperbolically on the Bass-Serre tree of Δ_1 we are in the case 1 (i.e., we have a pair of hyperbolic-elliptic splittings where the second splitting is obtained by appropriately collapsing all edges of Δ_1 except one). So we can refine Γ and obtain a decomposition Γ_1 as in case 1. We suppose now that this is not the case. Let's denote by P' the group of the vertex obtained after collapsing the edge labelled by R. We let P' act on T_Γ and we obtain a graph of groups decomposition of P', say Δ_2. We note that the vertex obtained after this collapsing is now labelled by a subgroup of P, say P'. We let P' act on T_E and we obtain a graph of groups decomposition of P', say Δ_2. If every edge group of Δ_1 acts elliptically on T_R then we let all other vertices of Δ_1 act on T_Γ and we substitute all these vertices in Δ_1 by the graphs obtained. We also substitute P' by Δ_2. We call the graph of groups obtained in this way Γ_1.

Finally we explain what we do if some edge of Δ_2 acts hyperbolically on T_R. We note that P' splits over R, indeed P' corresponds to a one-edge subgraph of Δ. Abusing notation we call still T_R the tree of the splitting of P' over R. We now repeat with P' the procedure applied to G. We note that we are necessarily in case 2 as R can not be contained in a conjugate of an edge group of Δ_2. As before we either obtain a refinement of Δ_2 such that all edge groups of Δ_2 act elliptically on T_R or we obtain a non-trivial decomposition of P', say Δ', such that an edge of Δ' is labelled by R and the following holds: If we collapse the edge of Δ' labelled by R we obtain a vertex P" which has the same property as E'. Namely if Δ_3 is the decomposition of P" obtained by acting on T_Γ then some edge group of Δ_3 acts hyperbolically on T_R. If we denote by Δ_1' the decomposition of P' obtained after the collapsing we remark that we can substitute P' by Δ_1' in Δ_1 and obtain a decomposition of G with more edges than Δ_1. Now we repeat the same procedure to P". By Theorem 5.3, this process terminates and produces a refinement of Γ which we call Γ_1.

By the argument above we obtain in both cases a graph of groups decomposition of G Γ_1 which has the following properties:
1) Γ_1 is a refinement of Γ and
2) There is an action of G on a tree T such that some edge group of Γ act on T hyperbolically while all edge groups of Γ_1 act on T elliptically.

Now we repeat the same procedure to Γ_1 and we obtain a graph of groups Γ_2 etc. One sees that this procedure will terminate using Prop 3.5 and Theorem 5.3. The last step of this procedure produces a decomposition Γ' as required by this proposition. □

One finds an argument similar to the 1st proof, using product of trees and retraction in the paper [DF].

4. Enclosing groups for a pair of hyperbolic-hyperbolic splittings

4.1. **Product of trees and core.** Producing a graph of groups which 'contains' a given pair of hyperbolic-hyperbolic splittings along slender groups is the main step in the construction of a JSJ-decomposition of a group. This step explains also what type of groups should appear as vertex groups in a JSJ-decomposition of a group. We produce such a graph of groups in proposition 4.7.

We recall here the definitions of a complex of groups and the fundamental group of such a complex. They were first given in the case of simplicial complexes in [H] and then generalized to polyhedral complexes in [BH]. Here we will give the definition only in the case of 2-dimensional complexes. We recommend [BH] ch.III.C for a more extensive treatment.

Let X be a polyhedral complex of dimension less or equal to 2.

We associate to X an oriented graph as follows: The vertex set $V(X)$ is the set of n-cells of X (where $n = 0, 1, 2$). The set of oriented edges $E(X)$ is the set $E(X) = \{(\tau, \sigma)\}$ where σ is an n-cell of X and τ is a face of σ. If $e \in E(X)$, $e = (\tau, \sigma)$, we define the original vertex of e, $i(e)$ to be τ and the terminal vertex of e, $t(e)$ to be σ.

If $a, b \in E(X)$ are such that $i(a) = t(b)$ we define the composition ab of a, b to be the edge $ab = (i(b), t(a))$. If $t(b) = i(a)$ for edges a, b we say that these edges are composable. We remark that the set $E(X)$ is in fact the set of edges of the barycentric subdivision of X. Geometrically one represents the edge $e = (\tau, \sigma)$ by an edge joining the barycenter of τ to the barycenter of σ. Also $V(X)$ can be identified with the set of vertices of the barycentric subdivision of X, to a cell σ there corresponds a vertex of the barycentric subdivision, the barycenter of σ.

A complex of groups $G(X) = (X, G_\sigma, \psi_a, g_{a,b})$ with underlying complex X is given by the following data:
1. For each n-cell of X, σ we are given a group G_σ.
2. If a is an edge in $E(X)$ with $i(a) = \sigma$, $t(a) = \tau$ we are given an injective homomorphism $\psi_a : G_\sigma \to G_\tau$.
3. If a, b are composable edges we are given an element $g_{a,b} \in E_{t(a)}$ such that

$$g_{a,b}\psi_{ab}g_{a,b}^{-1} = \psi_a\psi_b$$

We remark that when $\dim(X) = 1$, $G(X)$ is simply a graph of groups. In fact in Haefliger's setup loops are not allowed so to represent a graph of groups with underlying graph Γ one eliminates loops by passing to the barycentric subdivision of Γ. In this case there are no composable edges so condition 3 is void.

We define the fundamental group of a complex of groups $\pi_1(G(X), \sigma_0)$ as follows:

Let $E^{\pm}(X)$ be the set of symbols a^+, a^- where $a \in E(X)$. Let T be a maximal tree of the graph $(V(X), E(X))$ defined above. $\pi_1(G(X), \sigma_0)$ is the group with generating set:

$$\coprod G_\sigma,\ \sigma \in V(X),\ \coprod E^{\pm}(X)$$

and set of relations:

$relations\ \ of\ \ G_\sigma$, $(a^+)^{-1} = a^-$, $(a^-)^{-1} = a^+$, $(\forall a \in E(X))$

$a^+b^+ = g_{a,b}(ab)^+, \forall a, b \in E(X)$, $\psi_a(g) = a^+ g a^-$, $\forall g \in G_{i(a)}$, $a^+ = 1, \forall a \in T$

It is shown in [BH] that this group does not depend up to isomorphism on the choice of maximal tree T and its elements can be represented by 'homotopy classes' of loops in a similar way as for graphs of groups.

It will be useful for us to define barycentric subdivisions of complexes of groups $G(X)$. This will be an operation that leaves the fundamental group of the complex of groups unchanged but substitutes the underlying complex X with its barycentric subdivision X'. We explain this first in the case of graphs of groups. If $G(X)$ is a graph of groups then we have a group G_v for each vertex v of the barycentric subdivision of X. If v is the barycenter of an edge e, G_v is by definition in Haefliger's notation the group associated to the edge e, G_e. Now if v is a vertex of the second barycentric subdivision then v lies in some edge e of X so we define $G_v = G_e$. The oriented edges $E(X)$ are of two types:

1) an edge e from a barycenter v of the second barycentric subdivision to a barycenter w of the first barycentric subdivision. If this case the map $\psi_e : G_v \to G_w$ is the identity.

2) an edge e from a barycenter v of the second barycentric subdivision to a vertex w of X. In this case v is the barycenter of an edge a of the first barycentric subdivision and G_v is isomorphic to $G_{i(a)}$. We define then $\psi_e : G_v \to G_w$ to be ψ_e.

Let's call $G(X')$ the graph of groups obtained by this operation. It is clear that the fundamental group of $G(X')$ is isomorphic to the fundamental group of $G(X)$.

Let now X be a 2-dimensional complex and $G(X)$ a complex of groups with underlying complex X. Let X' be the barycentric subdivision of X. We associate to X' a graph $((V(X'), E(X'))$ as we did for X. Now we explain what are the groups and maps associated to $(V(X'), E(X'))$.

In order to describe the groups associated to $V(X')$ it is convenient to recall the geometric representation of $V(X), E(X)$.

The vertices of $V(X)$ correspond geometrically to barycenters of n-cells of X, i.e. they are just the vertices of the barycentric subdivision of X. Similarly the edges $E(X)$ are the edges of the barycentric subdivision and the orientation of an edge is from the barycenter of a face of X to a vertex of X.

$V(X')$ analogously can be identified with the set of vertices of the second barycentric subdivision of X and the edges $E(X')$ with the edge set of the second barycentric subdivision of X. All 2-cells of the barycentric subdivision of X are 2-simplices.

If v is a vertex of $V(X')$ which is the barycenter of the 2-simplex σ then there is a single 2-cell τ of X containing σ. If w is the barycenter of τ we define $G_v = G_w$. If v is a vertex of $V(X')$ which is the barycenter of an edge a we define $G_v = G_{i(a)}$.

We explain now what are the homomorphisms corresponding to $E(X')$. If a is an edge of $E(X')$ then there are two cases:

1) $i(a)$ is the barycenter of a 2-simplex σ of X'. Then if $t(a)$ is the barycenter of a 2-cell τ of X by definition $G_{i(a)} = G_{t(a)}$ and we define ψ_a to be the identity. Otherwise if w is the barycenter of the 2-cell of X containing σ we have that $G_{i(a)} = G_w$ and there is an edge e in $E(X)$ from w to $t(a)$. We define then ψ_a to be ψ_e.

2) $i(a)$ is the barycenter of an edge e of X'. If $t(a) = i(e)$ we define ψ_a to be the identity. Otherwise $t(a) = t(e)$ and we define ψ_a to be ψ_e.

It remains to define the 'twisting elements' for pairs of composable edges of $G(X')$. We remark that if a', b' are composable edges of X' then either $\psi_{b'} = id$ and $\psi_{a'b'} = \psi_{a'}$ in which case we define $g_{a',b'} = e$

or there are composable edges a, b of $G(X)$ and $\psi_{a'} = \psi_a, \psi_{b'} = \psi_b$. In this case we define $g_{a',b'} = g_{a,b}$.

One can see using presentations that the fundamental group of $G(X')$ is isomorphic to the fundamental group of $G(X)$. We explain this in detail now. It might be helpful for the reader to draw the barycentric subdivision of a 2-simplex while following our argument.

Let T be the maximal tree that we pick for the presentation of the fundamental group of $G(X)$. We will choose a maximal tree for $G(X')$ that contains T. We focus now on the generators and relators added by a subdivision of a 2-simplex of X, σ. σ in X has three edges which correspond to generators. After subdivision we obtain 4 new vertex groups and 11 edges (6 of which are subdivisions of old edges). 4 of the edges correspond ,by definition, to the identity homomorphism from old vertex groups to the four new vertex groups. To obtain T' We add these 4 edges to T (some of course might already be contained in T). The relations $\psi_a(g) = a^+ g a^-$ for these 4 edges together with $a^+ = 1, \, for \, a^+ \in T'$ show that the new vertex groups do not add any new generators. Let's call the 4 edges we added to T, a_1, a_2, a_3, a_4. We remark that 2 more edges, say b_1, b_2 correspond by definition to the identity map and the relations $a^+ b^+ = g_{a,b}(ab)^+$ show that these two new generators are also trivial (the corresponding $g_{a,b}$'s here are trivial as the maps that we compose are identity maps). We define now a homomorphism from the fundamental group of $G(X)$ to the fundamental group of $G(X')$ (the presentations given with respect to T, T' respectively). We focus again on the generators of the 2-simplex σ. Vertex groups G_τ are mapped by the identity map to themselves. Each edge of σ is subdivided in two edges, one of which we added to T'. We map each edge to the edge of the subdivision that we did not add to T'. By the definition of $G(X')$ all relators are satisfied so we have a homomorphism. It remains to see that it is onto. As we remarked before 6 of the new edges are trivial in the group. The other ones can be obtained by successive compositions of the edges contained in the image (together with edges that are trivial). The relations $a^+ b^+ = g_{a,b}(ab)^+$ for composable edges show that all generators corresponding to edges are contained in the image. It is clear that the homomorphism we defined is also 1-1. So it is an isomorphism.

We return now to our treatment of pairs of splittings.

Let $A_1 \star_{C_1} B_1$ (or $A_1 \star_{C_1}$), $A_2 \star_{C_2} B_2$ (or $A_2 \star_{C_2}$) be a pair of hyperbolic-hyperbolic splittings of a group G with corresponding Bass-Serre trees T_1, T_2. We consider the diagonal action of G on $Y = T_1 \times T_2$ given by

$$g(t_1, t_2) = (gt_1, gt_2)$$

where t_1, t_2 are vertices of, respectively, T_1 and T_2 and $g \in G$. We consider the quotient complex of groups in the sense of Haefliger . If X is the quotient complex Y/G we denote the quotient complex of groups by $G(X)$.

We give now a detailed description of $G(X)$. We assume for notational simplicity that the two splittings are $A_1 \star_{C_1} B_1$ and $A_2 \star_{C_2} B_2$ (i.e., they are both amalgamated products). One has similar descriptions in the other two cases. In the following, if, for example, the splitting along C_1 is an HNN-extension, namely, $G = A_1 *_{C_1}$, then one should just disregard B_1, $\mathcal{B}_1$, etc. When there are essential differences in the HNN- case we will explain the changes.

Let $\mathcal{A}_1 = T_2/A_1$, $\mathcal{B}_1 = T_2/B_1$, $\mathcal{A}_2 = T_1/A_2$, $\mathcal{B}_2 = T_1/B_2$, $\Gamma_1 = T_2/C_1$, $\Gamma_2 = T_1/C_2$ be the quotient graphs of the actions of A_1, B_1, C_1 on T_2 and of A_2, B_2, C_2 on T_1. Let $A_1(\mathcal{A}_1), B_1(\mathcal{B}_1), C_1(\Gamma_1)$, $A_2(\mathcal{A}_2), B_2(\mathcal{B}_2), C_2(\Gamma_2)$ be the corresponding Bass-Serre graphs of groups. We note now that if e is an edge of X which lifts to an edge of T_1 in Y then the subgraph of the barycentric subdivision of X perpendicular to the midpoint of this edge is isomorphic to Γ_1, and if we consider it as a graph of groups using the groups assigned to the vertices and edges by $G(X)$ then we get a graph of groups isomorphic to $C_1(\Gamma_1)$. We identify therefore this one-dimensional subcomplex of $G(X)$ with $C_1(\Gamma_1)$ and in a similar way we define a subcomplex of $G(X)$ isomorphic to $C_2(\Gamma_2)$ and we call it $C_2(\Gamma_2)$.

We have the following:

Lemma 4.1 (Van-Kampen theorem)**.** *Let Γ be a connected 1-subcomplex of the barycentric subdivision of X separating locally X in two pieces. We consider Γ as a graph of groups where the groups of the 0 and 1 cells of this graph are induced by $G(X)$. Let C be the image of the fundamental group of this graph into the fundamental group of $G(X)$. Then the fundamental group of $G(X)$ splits over C.*

Proof. It follows easily from the presentation of the fundamental group of $G(X)$ given in [H] or in [BH]. A detailed explanation is given in [BH], ch. III, 3.11 (5), p. 552. □

Since $Y = T_1 \times T_2$ is a product we sometimes use terms "perpendicular" and "parallel" for certain one-dimensional subsets in Y. Formally speaking, let p_1, p_2 be the natural projections of Y to T_1, T_2. Let e be an edge of T_1 and v a vertex of T_2. For a point $x \in (e \times v) \subset Y$, we say that the set $p_1^{-1}(p_1(x))$ is perpendicular to $(e \times v)$ at x.

For convenience we say that the T_2-direction is 'vertical' and the T_1 direction is 'horizontal'.

More formally a set of the form $p_2^{-1}(p_2(x))$ ($x \in Y$) is 'horizontal'. We also say that two vertical sets are "parallel". In the same way, all "horizontal sets", which are of the form $p_2^{-1}(p_2(x))$, are parallel to each other. Also, those terms make sense for the quotient Y/G since the action of G is diagonal, so we may use those terms for the quotient as well.

Definition 4.2 (Core subgraph)**.** Let A be a finitely generated group acting on a tree T. Let $\mathcal{A} = T/A$ be the quotient graph and let $A(\mathcal{A})$ be the corresponding graph of groups. Let T' be a minimal invariant subtree for the action of A on T. We define the core of $A(\mathcal{A})$ to be the subgraph of groups of $A(\mathcal{A})$ corresponding to T'/A.

Note that the core of $A(\mathcal{A})$ is a finite graph. If A does not fix a point of T this subgraph is unique. Otherwise it is equal to a single point whose stabilizer group is A. In what follows we will assume that C_1, C_2 are slender groups. Therefore the core of $C_1(\Gamma_1)$ (resp. $C_2(\Gamma_2)$) is a circle unless C_1 (resp. C_2) acts on T_2 (resp. T_1) by a dihedral action in which case the core is a segment (which might contain more than one edge).

We give now an informal description of the quotient complex of groups $G(X)$. This description is not used in the sequel but we hope it will help the reader gain some intuition for $G(X)$.

We consider the graphs of groups $A_1(\mathcal{A}_1), B_1(\mathcal{B}_1), C_1(\Gamma_1)$. (Disregard $B_1(\mathcal{B}_1)$ if the splitting along C_1 is an HNN-extension. In what follows, this kind of trivial modification should be made). There are graph morphisms from Γ_1 to $\mathcal{A}_1, \mathcal{B}_1$ coming from the inclusion of C_1 into A_1, B_1. (If HNN, both morphisms are to $\mathcal{A}_1$). We consider the complex $[0,1] \times \Gamma_1$. We glue $0 \times \Gamma_1$ to $\mathcal{A}_1$ using the morphism from Γ_1 to $\mathcal{A}_1$ and $1 \times \Gamma_1$ to $\mathcal{B}_1$ using the morphism from Γ_1 to $\mathcal{B}_1$. The complex we get this way is equal to X. The vertex groups are the vertex groups of $A_1(\mathcal{A}_1), B_1(\mathcal{B}_1)$. There are two kinds of edges: the (vertical) edges of $A_1(\mathcal{A}_1), B_1(\mathcal{B}_1)$ and the (horizontal) edges of the form $[0,1] \times v$ where v is a vertex of Γ_1. The groups of the edges of the first type are given by $A_1(\mathcal{A}_1), B_1(\mathcal{B}_1)$. The group of an edge $[0,1] \times v$ is the group of v in $C_1(\Gamma_1)$.

Finally the group of a 2-cell $[0,1] \times e$ is just the group of e where e is an edge of Γ_1.

Proposition 4.3 (Core subcomplex)**.** *There is a finite subcomplex Z of X such that the fundamental group of $G(Z)$ is equal to the fundamental group of $G(X)$.*

Proof. We will show that there is a subcomplex $\tilde{Z}$, of $T_1 \times T_2$ which is invariant under the action of G and such that the quotient complex of groups corresponding to the action of G on $\tilde{Z}$ is finite. We denote this quotient complex by $G(Z)$. Clearly the fundamental group of $G(Z)$ is equal to the fundamental group of $G(X)$.

We describe now how can one find such a complex $\tilde{Z}$. Let $e = [a, b]$ be an edge of T_1 stabilized by C_1 and let $p_1 : T_1 \times T_2 \to T_1$ the natural projection. $p_1^{-1}(e)$ is equal to $T_2 \times [a, b]$ and C_1 acts on T_2 leaving invariant a line l, because C_1 is slender. A_1 acts on $p_1^{-1}(a)$ and B_1 acts on $p_1^{-1}(b)$. Let S_1, S_2 be, respectively, minimal invariant subtrees of $p_1^{-1}(a)$, $p_1^{-1}(b)$ for these actions. We note that $l \subset S_1, S_2$ since C_1 is contained in both A_1, B_1. We can take then $\tilde{Z} = G(S_1 \cup \{l \times [0, 1]\} \cup S_2)$.

The construction is similar if the splitting over C_1 is an HNN-extension ($G = A_1 \star_{C_1}$); we simply take $\tilde{Z} = G(S_1 \cup \{l \times [0, 1]\})$ is this case.

Note that $\{l \times (0, 1)\}/C_1$ embeds in Z. If the action of C_1 on the line l is not dihedral, then it is an (open) annulus and if the action is dihedral then it is a rectangle. In Z, some identification may happen at $\{(l \times \{0\}) \cup (l \times \{1\})\}/C_1$, so that, for example, $\{l \times [0, 1]\}/C_1$ can be a closed surface in Z.

It is easy to see that $Z = \tilde{Z}/G$ is a finite complex. Vertices of Z are in 1-1 correspondence with the union of vertices of $S_1/A_1 \cup S_2/B_1$ and the latter set is finite. Edges of Z correspond to edges of $S_1/A_1 \cup S_2/B_1$ and vertices of l/C_1 while 2-cells are in 1-1 correspondence with edges of l/C_1. □

One can define Z also using our previous description of $G(X)$:
We take Z to be the union of the cores of $A_1(\mathcal{A}_1), B_1(\mathcal{B}_1)$ and $\{$core of $C_1(\Gamma_1) \times [0, 1]\}$. We have then that the fundamental group of $G(Z)$ is $A_1 \star_{C_1} B_1$. To see this, consider the (vertical) graph perpendicular to the midpoint of an edge of the form $v \times [0, 1]$, $(v \in \Gamma_1)$. The fundamental group of the graph is C_1. This graph separates Z in two pieces. The fundamental groups of these pieces are A_1, B_1. So from Lemma 4.1 (Van-Kampen theorem) we conclude that the fundamental group of $G(Z)$ is $A_1 \star_{C_1} B_1$.

4.2. **Enclosing groups.**

Definition 4.4 (a set of hyperbolic-hyperbolic splittings)**.** A set, I, of splittings of G over slender subgroups is called a set of hyperbolic-hyperbolic splittings if for any two splittings in I there is a sequence of splittings in I of the form $A_i \star_{C_i} B_i$ or $A_i \star_{C_i}$, $i = 1, ..., n$, such that the first and the last splitting of the sequence are the given splittings and any two successive splittings of the sequence are hyperbolic-hyperbolic.

We remark that a pair of splittings is hyperbolic-hyperbolic (def. 2.1) if and only if the set containing the 2 splittings is a set of hyperbolic-hyperbolic splittings. This follows from prop. 3.4.

Definition 4.5 (Enclosing graph decomposition)**.** Let I be a set of hyperbolic-hyperbolic minimal splittings of a group G along slender groups. An enclosing group of I, denoted by $S(I)$, is a subgroup in G, which is a vertex group of some graph decomposition of G with the following properties:

1. There is a graph of groups decomposition, Γ, of G with a vertex, v, such that $S(I)$ is the vertex group of v, all edges are adjacent to v and their stabilizers are slender and peripheral subgroups of $S(I)$ (see below for the definition of peripheral subgroups). $S(I)$ contains conjugates of all the edge groups of the splittings of G contained in I. Each edge of Γ gives a splitting which is elliptic with respect to all splittings in I. Γ is called an enclosing graph decomposition.
2. (rigidity) Suppose Γ' is a graph decomposition of G such that any edge group is slender and gives a splitting which is elliptic-elliptic to any of the splittings in I. Then $S(I)$ is a subgroup of a conjugate of a vertex group of Γ'.
3. $S(I)$ is an extension of the (orbifold) fundamental group of a compact 2-orbifold, Σ, by a group F which is a normal subgroup of some edge group of a splitting contained in I. We say that Σ is a base orbifold of $S(I)$, and F is the fiber group. A subgroup of a group in $S(I)$ which is the induced extension of the (orbifold) fundamental group of $\partial\Sigma$ by F is called a peripheral (or boundary) subgroup. We also consider subgroups of F and of induced extensions of the (finite cyclic) groups of the singular points of Σ, to be peripheral subgroups as well.
4. Each edge of Γ gives a minimal splitting of G.

Remark 4.6.

1. Peripheral subgroups are always proper subgroups of infinite index of an enclosing group.
2. An enclosing groups is not slender except when its base 2-orbifold has a fundamental group isomorphic to $\mathbb{Z} \times \mathbb{Z}$ or $(\mathbb{Z}_2 * \mathbb{Z}_2) \times \mathbb{Z}$,(i.e. the orbifold is a torus or an annulus whose two boundary circles are of cone points of index 2). Those two cases are the only tricky ones that an enclosing group may have more than one "seifert structure", i.e., the structure of the extension migth be not unique. For example, $\mathbb{Z}^3$ has more than one structures of an extension of $\mathbb{Z}^2$ by $\mathbb{Z}$.

4.3. **Producing enclosing group.** We will show that an enclosing graph decomposition with an enclosing vertex group $S(I)$ exists (in fact we construct it) for I given. We start with the simplest case where there are only two splittings in I. As a first step, in the following proposition using products of trees, we produce a graph decomposition Γ with a vertex group which has properties (1),(2),(3) in Def 4.5. Later we will show that one can also ensure that Γ satisfies (4) as well.

One may wonder what happens if we use A_2, B_2, C_2 instead of A_1, B_1, C_1 to construct Z in Prop 4.3. In fact, if both splittings are minimal, then we get the same finite complex. This is the idea behind the next proposition.

Proposition 4.7 (Enclosing groups for a pair of splittings)**.** *Let $A_1 \star_{C_1} B_1$ (or $A_1 \star_{C_1}$) and $A_2 \star_{C_2} B_2$ (or $A_2 \star_{C_2}$) be a pair of hyperbolic-hyperbolic splittings of a finitely generated group G over two slender groups C_1, C_2. Suppose that both splittings are minimal. Then there is a graph decomposition of G with a vertex group which has the properties 1, 2, 3 in Def 4.5 for C_1, C_2.*

Remark 4.8. If G does not split over a subgroup of infinite index in C_1, C_2 then the two splittings along C_1, C_2 are minimal (see remark 2.3). This hypothesis is used in [RS] and [DS] instead of minimality.

We first construct a graph decomposition of G and show that it is the desired one for Prop 4.7 later. Let T_i be the Bass-Serre tree of the splitting over C_i. Consider the diagonal action of G on $Y = T_1 \times T_2$. Let $G(X)$ be the quotient complex in the sense of Haefliger. Consider the finite subcomplex of $G(X)$, $G(Z)$, constructed in proposition 4.3. Let e be an edge of $A_1(\mathcal{A}_1)$ lying in the core of $C_1(\Gamma_1) \times [0,1]$. In other words $e \in \mathcal{A}_1 \cap \{\Gamma_1 \times \{0\}\}$. Consider the (horizontal) graph in $G(Z)$ perpendicular to e at its midpoint. In other words this is the maximal connected graph passing through the midpoint of e whose lift to $T_1 \times T_2$ is parallel to T_1. The fundamental group of the graph of groups corresponding to this graph is a subgroup of a conjugate of C_2. Since both the assumptions and the conclusions of Prop 4.7 do not change if we take conjugates in G of the splittings along C_1, C_2, without loss of generality (by substituting the splitting along C_2 by a conjugate in G) we can assume that this graph is a subgraph of $C_2(\Gamma_2)$.
claim. Consider the squares intersecting the core of $C_2(\Gamma_2)$. Then, this set of squares contains all squares of $G(Z)$.
To argue by contradiction, we distinguish two cases regarding the set $Z \cap (T_1/C_2 \times \{1/2\})$. We naturally identify the core of $C_2(\Gamma_2)$ with a subgraph in $T_1/C_2 \times \{1/2\}$.

(i) If Z does not contain the core of $C_2(\Gamma_2)$ then, by Lemma 4.1(Van-Kampen), G splits over an infinite index subgroup of C_2. Moreover, $A_1 \star_{C_1} B_1$ (or $A_1 *_{C_1}$) is hyperbolic-elliptic with respect to this new splitting contradicting our hypothesis. We explain this in more detail. The type of the splitting over C_1, i.e., either an amalgamation or HNN extension, does not make difference in this discussion. On the other hand, we may need to make a minor change according to the type of the splitting along C_2, which we pay attention to. What requires more attention, because the topology of the base 2-orbifold of $S(I)$ becomes different, is the type of the action of C_2 on T_1, i.e., dihedral or not, although the type of the action of C_1 on T_2 is not important once the subcomplex Z is constructed.

Let's first assume that the splitting along C_2 is not dihedral. Let $l_2 \subset T_1$ be the invariant line of the action by C_2. The core of $C_2(\Gamma_2)$ is l_2/C_2, which is a circle by the assumption we made. The circle l_2/C_2 is a retract of a graph T_1/C_2, so that $T_1/C_2 - l_2/C_2$ is a forest, i.e., each connected component is a tree. Therefore, if Z does not contain the core of $T_1/C_2 \times \{1/2\}$, then $Z \cap (T_1/C_2 \times \{1/2\})$ is a forest. Let $U_1, \cdots, U_n$ be the connected components of the forest. Then, if we cut Z along each U_i, and apply Lemma 4.1, we get a splitting of G along the fundamental group (in the sense of graph of groups) of U_i, K_i, which is a subgroup of infinite index in C_2.

By construction, K_i is contained in C_1. Therefore this splitting along K_i is elliptic with respect to $A_1 *_{C_1} B_1$ (or $A_1 *_{C_1}$). Moreover, C_1 is hyperbolic to at least one of the splittings along K_i's. This is because if not, then C_1 is contained in a conjugate of A_2 or B_2 (or A_2 in the case that the splitting along C_2 is an HNN-extension), which is impossible, since C_1 is hyperbolic with respect to $A_2 *_{C_2} B_2$(or$A_2 *_{C_2}$). The last claim does not require the theory of complex of groups, but just Bass-Serre theory; since each U_i is a tree, U_i contains a vertex, u_i, whose vertex group is K_i. Let $e_i = u_i \times [0,1] \subset Z$. If we delete all (open) squares and edges parallel to e_i (except for e_i) in $U_i \times [0,1]$ from Z, the fundamental group (in the sense of Haefliger) does not change, and also the edge e_i gives the splitting of G along K_i. If we do the same thing for all U_i's, the subcomplex of Z we obtain is indeed a graph, Λ, where the edges e_i's are parallel to each other, and no other edges are parallel to them. Note that all of those other edges are the ones which were in the graph $\mathcal{A}_2 \cup \mathcal{B}_2 \subset Z$ (or just $\mathcal{A}_2$ in the case of HNN-extension). Therefore, if C_1 is elliptic with respect to all the splittings along K_i, it means that C_1 is conjugated to the fundamental group (in the sense of Bass-Serre) of a connected component of $\Lambda - \cup_i U_i$, which is a subgraph of either $\mathcal{A}_2$ or $\mathcal{B}_2$ (or just $\mathcal{A}_2$ in the case of HNN-extension). This means that C_1 is a conjugate of a subgroup of either A_2 or B_2 (or A_2 in the case of HNN-extension). This is what we want.

We are still left with the case that C_2 is dihedral on T_1. The argument only requires a notational change. $T_1/C_2 \times \{1/2\}$ is a forest and we look at each connected component, and appropriately delete all squares and some edges from Z without changing the fundamental group, which is G, and get a graph of groups at the end as before. We omit details.

We remark that our argument does not change if the action of C_1 on T_2 is dihedral or not. So we treated all possibilities in terms of the type of the splittings along C_1, C_2 and also the type of the actions of C_1, C_2.

(ii) If on the other hand $Z \cap (T_1/C_2 \times \{1/2\})$ is bigger than the core of $C_2(\Gamma_2)$ we can delete from $G(Z)$ the 2-cells (i.e., squares) containing edges of this graph which do not belong to the core of $C_2(\Gamma_2)$ without altering the fundamental group. To explain the reason, let's first suppose that the action of C_2 on T_1 is not dihedral. Then the core is topologically a circle, c_2. The connected component, U, of the finite graph $Z \cap (T_1/C_2 \times \{1/2\})$ which contains the core is topologically the circle with some trees attached. The fundamental group (in the sense of graph of groups) of not only the circle c_2 , but also the graph U is C_2.Therefore, one can remove those trees from U without changing the fundamental group, which is C_2.

In Z, $U \times (0,1)$ embeds, and one can remove the part $(U - c_2) \times (0,1)$ from Z without changing its fundamental group. One can see this using the presentation of the fundamental group of $G(Z)$. For the reader's convenience we give also an argument using the action of G on $\tilde{Z}$. Let $p_2 : \tilde{Z} \to T_2$ be the natural projection from $\tilde{Z}$ to T_2. if e is an edge of T_2, $p_2^{-1}(e)$ is a connected set of the form $L_e \times e$. Let $Stab(e)$ be the stabilizer of e in T_2 (which is a conjugate of C_2) and l_e the line invariant under $Stab(e)$ on T_1. Then by the discussion above L_e contains l_e and is connected. We will show that $L_e = l_e$. Indeed if not we consider the subcomplex of $\tilde{Z}$ obtained by the union of $l_e \times e$ over all edges $e \in T_2$ with $p_2^{-1}(v)$ over all vertices $v \in T_2$. Let's call this complex $\tilde{Z}_1$. It is clear that $\tilde{Z}_1$ is connected, simply connected and invariant under the action of G. The quotient complex of groups $G(Z_1)$ is a subcomplex of G. If for some e, $L_e \neq l_e$ $G(Z_1)$ is properly contained in $G(Z)$. By the preceding discussion it follows that the splittings over C_2, C_1 are hyperbolic-elliptic, a contradiction.

In the case when the action of C_2 on T_1 is dihedral, then the core is a segment, and $Z \cap (T_1/C_2 \times \{1/2\})$ is a graph which is the segment with some finite trees attached. In this case one can delete the squares which contain those trees from Z without changing the fundamental group of Z, which is G.

But then the complex obtained, after deleting those unnecessary squares, does not contain the core of $C_1(\Gamma_1)$, because there are no other squares in Z than the ones which contains the core of $C_1(\Gamma_1)$, which implies that G splits over an infinite index subgroup of C_1, and this new splitting is elliptic-hyperbolic with respect to $A_2 \star_{C_2} B_2$ (or $A_2 *_{C_2}$), which is a contradiction. The last part follows from the same consideration as the last part of the case (i), so we omit the details. We showed the claim.

From this claim it follows that if we apply the same construction of Z in Prop 4.3 using A_2, B_2 instead of A_1, B_1, the resulting complex contains the same set of squares.

We describe the topology of Z. If none of C_1, C_2 acts as a dihedral group the above implies that every edge in Z which is a side of a 2-cell lies on exactly two 2-cells. Therefore the link of every vertex of Z is a union of disjoint circles and points. It then follows that the union of 2-cells in Z is topologically a closed surface with, possibly, some (vertex) points identified. Z is this 2-dimensional object with some graphs attached at vertices; if one deletes from Z those graphs including attaching vertices and identified vertices, one obtains a compact surface with punctures.

If at least one of C_1, C_2 acts as a dihedral group then Z is topologically a compact surface with boundary with, possibly, some points identified and some graph attached. Therefore the links of vertex points on this surface are disjoint unions of circles, segments and points. The boundary components come from the dihedral action(s), and there are at most 4 connected components. To see this, suppose that only C_1 is dihedral on T_2, and let l_1 be its invariant line. Then the rectangle $l_1/C_1 \times (0,1)$ embeds in the surface. Let u_1, u_2 be the boundary points of the segment l_1/C_1. Then the edges $u_1 \times [0,1], u_2 \times [0,1]$ are exactly the boundary of the surface. Note that $u_1 \times (0,1)$ embeds, but possibly, $u_1 \times [0,1]$ may become a circle in Z. $u_1 \times [0,1]$ and $u_2 \times [0,1]$ may become one circle in Z as well. Therefore, the surface has at most two boundary components in this case. If C_2 is dihedral on T_1 as well, then there are two more edges which are on the boundary, so that there are at most 4 boundary components.

Remark 4.9 (Priority among splittings)**.** Let Z be the complex we constructed in the proof of Prop. 4.3. Let l_1 be the invariant line in T_2 by C_1 and $c_1 = l_1/C_1$. We may call c_1 the core of C_1. If the action of C_1 is dihedral, then c_1 is a segment, or else, a circle. The core c_1 embeds in Z, and if we cut Z along c_1 we get (not a conjugate, but exactly) the splitting $A_1 *_{C_1} B_1$ (or $A_1 *_{C_1}$). Similarly, let l_2 be the invariant line in T_1 by C_2, and $c_2 = l_2/C_2$. As before, this core c_2 is either a segment or a circle, and embeds in Z. Cutting Z along c_2 , we get a splitting of G along a conjugate of C_2. But, unlike the splitting along C_1, this splitting may be different from the original splitting along C_2. This point becomes important later, that we can keep at least one splitting unchanged (along C_1 in this case) in Z, because we gave priority to the splitting along C_1 over C_2 when we constructed Z.

However, it is true that if G does not split along a subgroup in C_1 of infinite index, then the new splitting along C_2 obtained by cutting Z along c_2 is the same (up to conjugation) as the original one. It is because that under this assumption, Z does not have any graphs attached, and it is just a squared complex.

Although the new splitting along C_2 may be different from the original one, it is hyperbolic-hyperbolic with respect to the splitting along C_1. It follows from Lemma 3.4 that the new splitting along C_2 is minimal.

We now explain how to obtain the desired graph of groups decomposition of G. First, let the group $S = S(C_1, C_2)$ be the subgroup of the fundamental group of $G(Z)$ corresponding to the subcomplex of $G(Z)$ which is the union of the cores of $C_1(\Gamma_1)$ and $C_2(\Gamma_2)$, namely, S is the image in G of the fundamental group (in the sense of a graph of groups) of this union. Here we use Haefliger's notation; the cores of Γ_1, Γ_2 are contained in the barycentric subdivision of Z which is used in the definition of its fundamental group.

Using lemma 4.1(Van-Kampen theorem) we show that $G(Z)$ is the fundamental group of a graph of groups, which we call Γ. The vertices of this graph are as follows: there is a vertex for each connected component of Z minus the cores of $C_1(\Gamma_1)$ and $C_2(\Gamma_2)$. The vertex group is the fundamental group of the component in Haefliger's sense. We remark that each such component contains exactly one vertex group of the 'surface' piece of Z with, possibly, a graph attached at the vertex. The fundamental group of the component is then the fundamental group of the graph of groups of the attached graph (and is equal to the group of the vertex if there is no graph attached).

There is also a vertex with group $S(C_1, C_2)$. There is an edge for each component of the intersection between the union of the cores of $C_1(\Gamma_1)$, $C_2(\Gamma_2)$ and each vertex component.

Note that such an intersection is topologically a circle or a segment (this happens only when at least one of the actions of C_i is dihedral).

As this intersection is a subgraph of the union of the cores of $C_1(\Gamma_1)$, $C_2(\Gamma_2)$ there is a group associated to it, namely the image of the fundamental group of this subgraph in $G(Z)$. Note that the graph of groups that we described here is a graph of groups in a generalized sense, i.e., the edge groups do not necessarily inject into the vertex groups. Note that every vertex group except S injects in G.

To understand the group S better, let U be the union of the cores of $C_1(\Gamma_1)$, $C_2(\Gamma_2)$, which is a graph in Z. If we consider a small closed neighborhood, $\bar{U}$, of U in Z, it is a compact surface with boundary in general. We may consider the graph, P, corresponding to an edge, e, of Γ as a subset in the boundary of $\bar{U}$, which is either a circle or a segment. Let $F < G$ be the stabilizer of a square in Z. Then, the fundamental group, in the sense of Haefliger, of P is an extension of $\mathbb{Z}$ (when P is a circle) or $\mathbb{Z}_2 * \mathbb{Z}_2$ (when P is a segment) by F. Since the group F is a subgroup of G, the image of the fundamental group of P in G is an extension of (1) $\mathbb{Z}$, (1') $\mathbb{Z}_n$, (1") the trivial group; (2) $\mathbb{Z}_2 * \mathbb{Z}_2$, (2') $\mathbb{Z}_2$, or (2") a finite dihedral group of order $2n$ by F. As a consequence, the image of the fundamental group of U (as well as $\bar{U}$) in G, which is S by our definition, is an extension of the orbifold fundamental group of a 2-orbifold, Σ, by F such that Σ is obtained from the compact surface $\bar{U}$ by adding to each P (1) nothing, (1') a disk with a cone point of index n at the center, (1") a disk; (2) nothing, (2') a half disk such that the diameter consists of cone points of index 2 (in other words, we just collapse the segment P to a point), or (2") a half disk such that the diameter consists of cone points of index 2 except for the center whose index is $2n$. The orbifold fundamental group of this 2-orbifold is S. Note that Σ is no more embedded in Z, but the surface $\bar{U}$ is a subsurface of Σ.

Remark 4.10. By our construction of Z, U, Σ, there is a simple closed curve or a segment (the core) on Σ which corresponds to each of C_1, C_2. If we cut Σ along it, we obtain a splitting of S along C_1 or C_2, respectively, which also gives a splitting of G, as we do by cutting Z. Although one of them may be different from the original one, both of them are minimal (use Prop 3.4).

4.4. **Proof of Prop 4.7.**

Proof. We will show that the graph decomposition Γ with $S(C_1, C_2)$ we constructed satisfies the properties 1,2,3 in Def 4.5. In fact $S(C_1, C_2)$ is an enclosing group for C_1, C_2 although we may need to modify Γ so that the property (4) holds as well. We will discuss this point later.

(3) is clear. By construction, S is an extension of the orbifold fundamental group of the compact 2-orbifold Σ by a group F, which is

the edge stabilizer subgroup in C_1 when it acts on the tree T_2, hence a normal subgroup of C_1. F is slender since it is a subgroup of a slender group C_1.

(1). Let v be the vertex of Γ whose vertex group is S. By our construction, all edges are adjacent to v. The edge group, E, of an edge is slender since there is the following exact sequence; $1 \to F \to E \to Z \to 1$ such that the group Z is either the trivial group, the fundamental group of one of the singular points of Σ, so that isomorphic to $\mathbb{Z}_n$, or a subgroup of the (orbifold) fundamental group of $\partial\Sigma$, so that isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$. In any case, E is slender and a peripheral subgroup of S. Clearly S contains conjugates of C_1, C_2, because Σ contains the graph U, which is the union of the cores for C_1, C_2. By construction of Γ, all vertex groups except for S are elliptic on both T_1 and T_2, so that all edge groups of Γ are elliptic on T_1, T_2 since they are subgroups of vertex groups. We showed (1).

To prove that enclosing groups are 'rigid', namely the property (2) in Def 4.5, we recall some results from Bass-Serre theory.

Proposition 4.11 (Cor 2 in §6.5 of [Se])**.** *Suppose G acts on a tree. Assume G is generated by $s_1, \cdots, s_l$ and all s_i and $s_i s_j (i \neq j)$ are elliptic on the tree. Then G is elliptic.*

Let c be a simple closed curve on Z which avoids vertices of Z. Using repeated barycentric subdivisions of $G(Z)$ we see that c is homotopic in $Z - Z^{(0)}$ to a curve lying in the 1-skeleton of the iterated barycentric subdivision. Let's assume then that c is a curve lying in the 1-skeleton of an iterated barycentric subdivision. Then cutting Z along c, we get a splitting of G along the fundamental group (in the sense of graph of groups) of c, (4.1). If the fundamental group of c is not contained to a vertex group, then the splitting induced by c is hyperbolic-hyperbolic with respect to either the splitting along C_1 or C_2, so that in particular, it is non trivial and a minimal splitting. We call such simple closed curves c *essential*

In the case Z has a boundary (i.e., there exists an edge which is contained in only one square), let c be an embedded segment whose boundary points are in the boundary of Z. Cutting Z along c, one also obtains a splitting of G along the fundamental group of c. If the fundamental group of c is not contained to the fundamental group of ∂Z then this splitting is hyperbolic-hyperbolic with respect to at least one of the splittings along C_1 and C_2, so it is non trivial and minimal. We call such segments c *essential.* We remark that essential simple closed curves and essential segments correspond to subgroups of the fundamental group of Σ.

If $\partial\Sigma$ contains segments of singular points of index two (reflection points) we denote this set by $(\partial\Sigma; 2)$.

Corollary 4.12. *If all splittings over the slender groups which are represented by essential simple closed curves on Σ and essential embedded segments are elliptic on Γ, then $S = S(C_1, C_2)$ is elliptic on Γ.*

Proof. We explain how to choose a finite set of generators of S so that we can apply Prop 4.11. First choose a finite set of generators f_i of F (F is slender, so that finitely generated).

Let's first assume that $(\partial\Sigma; 2) = \emptyset$. Then, one can choose a set of non-boundary simple closed curves $c_1, \cdots, c_l$ on Σ so that all $c_i c_j (i \neq j)$ are also represented by simple closed curves and that the elements corresponding to c_i generate the fundamental group of Σ. Each c_i or $c_i c_j (i \neq j)$ represents a slender subgroup in G with the fiber group F, which gives a splitting of S. By assumption all of those splittings are elliptic on Γ. Therefore we apply Prop 4.11 to S with the generating set of $\{f_i, c_j\}$ and conclude that S is elliptic on Γ.

In the case $(\partial\Sigma; 2) \neq \emptyset$, we need extra elements represented by embedded segments $(s, \partial s) \subset (\Sigma, (\partial\Sigma; 2))$. Put an order to the connected components of $(\Sigma, (\partial\Sigma; 2))$, and take a finite set of embedded segments, s_i, such that any adjacent (in the order) pair of components of $(\partial\Sigma; 2)$ is joined by a segment. Then the set $\{f_i, c_j, s_k\}$ generates a subgroup $S' < S$ of finite index. By Prop 4.11, S' is elliptic on Γ, so that so is S. $\square$(Cor 4.12).

We now show that enclosing groups are 'rigid'.

Lemma 4.13 (Rigidity)**.** *Let $A_1 \star_{C_1} B_1$ (or $A_1 \star_{C_1}$) and $A_2 \star_{C_2} B_2$ (or $A_2 \star_{C_2}$) be as in proposition 4.7 and let $S = S(C_1, C_2)$ be the group constructed above. Suppose Γ' is a graph decomposition of G such that any edge group is slender and elliptic to both of the splittings over C_1, C_2. Then S is a subgroup of a conjugate of a vertex group of Γ'.*

Proof. Γ denotes the graph of groups decomposition we constructed with S as a vertex group. As we pointed out in Remark 4.9, although the splitting of G over C_2 which we obtain by cutting Σ along the core curve for C_2 may be different from the original one, this splitting is minimal, because it is hyperbolic-hyperbolic to the (original) splitting over C_1, (see Prop. 3.4).

Let T, T' be the Bass-Serre trees of Γ, Γ'. Our goal is to show that S is elliptic on T'. Let $c \subset \Sigma$ be an essential simple closed curve or $(s, \partial s) \subset (\Sigma, (\partial\Sigma; 2))$ an embedded essential segment, and $C < S$ the group represented by it. If we show that C is elliptic on T', then Prop 2.9 implies that S is elliptic on T'. The splitting of G over C by cutting

Σ along c or s is minimal by Prop. 3.4 since it is hyperbolic-hyperbolic to one of the minimal splittings over C_1, C_2.

Let e be an edge of Γ' with edge group, E. Since the subgroup E is elliptic with respect to the splittings over C_1, C_2 (i.e., elliptic on the both trees for the two splittings), it fixes a vertex of $T_1 \times T_2$. Therefore it is contained in a conjugate of a vertex group of $G(Z)$, which is not S. It follows that the group E is elliptic with respect to the splitting of G over C. Since the splitting along C is minimal, by Prop 3.4, it is elliptic-elliptic with respect to the splitting of G over E obtained from Γ' by collapsing all edges but e. Since e was arbitrary, the subgroup C is elliptic on T'. □(lemma 4.13).

Lemma 4.13 implies (2) in Def 4.5. We have verified the items (1),(2),(3) in Def 4.5 for $S(C_1, C_2)$ which finishes the proof of Prop 4.7. □(Prop 4.7).

Remark 4.14 (Maximal peripheral subgroup)**.** Note that the subset of $\partial\Sigma$ which is produced by the cutting of Z is exactly $\partial\Sigma -$interior of$(\partial\Sigma; 2)$. Let c be a connected component of this subset, and E the corresponding (peripheral) subgroup of S. Let's call such peripheral subgroup of S *maximal*. E is an edge group of Γ by our construction, so that any maximal peripheral subgroup of S is an edge group of Γ. For example when the fiber group F is trivial, Σ is a 2-manifold with boundary. Then the infinite cyclic subgroup in $S = \pi_1(\Sigma)$ corresponding to each boundary component of Σ is an edge group of Γ. In this sense, Σ does not have any free boundary points.

4.5. **Producing enclosing graph decomposition.** We now discuss the property (4) of Def 4.5. In general the edges of Γ we obtained in Prop 4.7 may give non-minimal splittings. See the example.

However, by applying Prop 3.7 to Γ, there is a refinement (see Def 3.6) of Γ such that all edges give minimal splittings of G. We then verify that the refinement satisfies all the properties of Def 4.5, most importantly, S remains a vertex group, and is the enclosing group of the decomposition we get.

Example. This example is suggested by V.Guirardel to us. We thank him. Let $G = \mathbb{Z}^3 * A$ such that A is a non-trivial group. Fix free abelian generators a_1, a_2, a_3 of $\mathbb{Z}^3$. Write $\mathbb{Z}^3$ as an HNN-extension $\mathbb{Z}^2 *_{\mathbb{Z}^2}$ such that $\mathbb{Z}^2 = \langle a_2, a_3 \rangle$ and the stable letter is a_1. This extends to an HNN-extension $G = (\mathbb{Z}^2 * A) *_{\mathbb{Z}^2}$ over $\mathbb{Z}^2 = \langle a_2, a_3 \rangle$. Let T_1 be the Bass-Serre tree of this splitting. We abuse the notation and call the splitting T_1 as well. Similarly, we obtain HNN-extensions T_2 and T_3: $G = (\mathbb{Z}^2 * A) *_{\mathbb{Z}^2}$ over $\mathbb{Z}^2 = \langle a_1, a_3 \rangle$, and $\langle a_1, a_2 \rangle$, with Bass-Serre trees T_2, T_3. For $i \neq j$, the pair of splittings T_i, T_j is hyperbolic-hyperbolic. Each splitting T_i

is minimal, because if $G = (\mathbb{Z}^2 * A)*_{\mathbb{Z}^2}$ was not minimal, then the corresponding HNN-extension $\mathbb{Z}^3 = \mathbb{Z}^2 *_{\mathbb{Z}^2}$ would give (for example, by taking product of trees) a splitting of $\mathbb{Z}^3$ over $\mathbb{Z}$ or the trivial group, which is impossible. We obtain a graph decomposition for the pair T_1, T_2 by taking product of trees: $G = \mathbb{Z}^3 *_{\mathbb{Z}} (\mathbb{Z} * A)$ such that this is an amalgamation over $\langle a_3 \rangle$ with two vertex groups $\mathbb{Z}^3$ and $\langle a_3 \rangle * A$. Let's call this decomposition, and its Bass-Serre tree T. The vertex group $\mathbb{Z}^3$ is the enclosing group such that the base is a torus with the fundamental group $\langle a_1, a_2 \rangle$ and the fiber group is $\langle a_3 \rangle$. This splitting along $\langle a_3 \rangle$ is not minimal, because it is hyperbolic-elliptic to T_3. We now demonstrate how to handle this problem using Prop 3.7. Following the first proof of Prop 3.7, we refine T such that all edge gives a minimal splitting. Take product of trees of T, T_3. The core is topologically an annulus, which contains one square, and three edges, where two of them are vertical, and they are loops. Since A fixes a vertex of $T \times T_3$, A is a vertex group of the core. We remove the (open) square, and also one vertical loop, whose edge group is trivial, appropriately without changing the fundamental group. We obtain a graph decomposition with two edges, $G = A * \mathbb{Z}^2 *_{\mathbb{Z}^2}$ such that both $\mathbb{Z}^2$ are $\langle a_1, a_2 \rangle$. Next, collapse the other vertical loop, whose edge group is $\langle a_1, a_2 \rangle$, in the graph decomposition. We are left with the horizontal edge, whose edge group is trivial, and obtain $G = \mathbb{Z}^3 * A$, which is a refinement of T. This is an enclosing decomposition for T_1, T_2 with the enclosing vertex group $\mathbb{Z}^3$.

Lemma 4.15 (Refinement of Γ)**.** *There is a refinement, Γ', of Γ such that*

1. *each edge of Γ' gives a minimal splitting of G,*
2. *each edge group of Γ' is a subgroup of some edge group of Γ,*
3. *$S = S(C_1, C_2)$ remains a vertex group of Γ',*
4. *each edge group is a peripheral subgroup of S.*

Proof. Apply Prop 3.7 to Γ and obtain a refinement Γ' such that each edge of Γ' gives a minimal splitting of G. Each edge group, E, of Γ' is a subgroup of some edge group of Γ. Therefore, by Prop 4.7, E is elliptic to both splittings of G along C_1, C_2. Since both of the splitting along C_1, C_2 are minimal, each of them is elliptic-elliptic with respect to the splitting over E. Therefore by Lemma 4.13, $S = S(C_1, C_2)$ is a subgroup of a conjugate of some vertex group, V, of Γ'. But since Γ' is a refinement of Γ and S is a vertex group of Γ, S is a conjugate of V. E is a peripheral subgroup of S since it is a subgroup of a peripheral subgroup. ∎

We collapse all edges in Γ' which are not adjacent to the vertex whose vertex group is S, and still call it Γ'. Then by Prop 4.7 and Lemma 4.15, Γ' is an enclosing graph decomposition with an enclosing vertex group $S(C_1, C_2)$ for the splittings along C_1, C_2. We have shown the following.

Proposition 4.16 (Enclosing decomposition for a pair of splittings). *Let $A_1 \star_{C_1} B_1$ (or $A_1 \star_{C_1}$) and $A_2 \star_{C_2} B_2$ (or $A_2 \star_{C_2}$) be a pair of hyperbolic-hyperbolic splittings of a finitely generated group G over two slender groups C_1, C_2. Suppose that both splittings are minimal. Then an enclosing graph decomposition of G exists for those two splittings.*

5. JSJ-decomposition

5.1. **Dealing with a third splitting.** Let G be a finitely presented group. We want to show that an enclosing graph decomposition exists for a set, I, of hyperbolic-hyperbolic minimal splittings of G along slender groups. We already know this when I contains only two elements by Prop 4.16. We now discuss the case when I has three elements.

Proposition 5.1 (Enclosing group). *Let I be a set of hyperbolic-hyperbolic splittings (Def 4.4) of a finitely generated group G. Suppose all of them are minimal splittings. Suppose that I consists of three splittings. Then an enclosing graph decomposition of G exists for I.*

Proof. Suppose that the three splittings in I are along C_1, C_2, C_3. We may assume that the pair of the splittings along C_1, C_2, and also the pair for C_2, C_3 are hyperbolic-hyperbolic. Apply Prop 4.7 to the first pair, and obtain an enclosing graph decomposition, Γ, with the vertex group $S = S(C_1, C_2)$. We remark that $S(C_1, C_2)$ depends on the two splittings, not only the two subgroups. Note that by cutting the 2-orbifold Σ for S along a simple closed curve or a segment corresponding to each of C_1, C_2, we obtain a minimal splitting of G along C_1, and C_2, respectively. Although this splitting along C_2 is possibly different from the original one, it is still a minimal splitting so it is hyperbolic-hyperbolic with respect to the splitting along C_3.

Let's assume first that the group C_3 is elliptic with respect to Γ. Then C_3 is a subgroup of a conjugate of S. This is because if C_3 was a subgroup of a conjugate of a vertex group of Γ which is not S, then the group C_3 is elliptic with respect to both of the (new) splittings of G along C_1, C_2 which we obtain by cutting Σ. It then follows that the splitting along C_3 would be elliptic-elliptic with respect to both of the original splittings of G along C_1, C_2, which is a contradiction. Let Γ' be a refinement of Γ which we obtain by Prop 4.16, which is an enclosing graph decomposition for the splittings along C_1, C_2. We claim that Γ' with an enclosing vertex group S is an enclosing decomposition for the three splittings. First, the properties 2,3,4 are clear. To verify (the non-trivial part of) the property 1, let e be an edge of Γ' with edge group, E. We want to show that the group E is elliptic with respect to the splitting in I along, C_3. We know that $C_3 < S$ by our assumption. Since the group S is elliptic with respect to the splitting of G along E which the edge e gives, so is C_3. Since both of the splittings along C_3 and E are minimal, it follows that the group E is elliptic with respect to the splitting along C_3. This proves property 1.

We treat now the case that C_3 is hyperbolic with respect to Γ. This is the essential case. Let T_Γ, T_3 be, respectively, the Bass-Serre trees of Γ and the splitting over C_3. Since the splitting along C_3 is minimal, there is at least one edge, e, of Γ such that the splitting of G which the edge e gives, along the edge group, E, is hyperbolic-hyperbolic with respect to the splitting along C_3. The group E acts hyperbolically on T_3. F denotes the fiber group of $S(C_1, C_2)$. We have the following lemma:

Lemma 5.2 (Elliptic fiber)**.** *Letting E act on T_3, we obtain a presentation*

$$E = \langle t, F | tFt^{-1} = \alpha(F) \rangle,$$

where $\alpha \in \mathrm{Aut}(F)$ or

$$E = L *_F M,$$

where $[L : F] = [M : F] = 2$.

Proof. We first show that F is elliptic on T_3. To argue by contradiction, assume that there is $a \in F$ acting hyperbolically on T_3. Then the pairs C_1, C_3 and C_2, C_3 are hyperbolic-hyperbolic. Let F_1 be the fiber of the enclosing group $S(C_1, C_3)$ corresponding to the pair C_1, C_3 and let F_2 be the fiber of $S(C_2, C_3)$. We claim that there is $w_1 \in F_1$ which does not lie in any conjugate of F. Indeed F_1 is contained in a conjugate of C_1. C_1 acts on T_2 hyperbolically preserving an axis which is stabilized by F. If F_1 contains an element, w_1, that acts hyperbolically on T_2 then w_1 does not lie in any conjugate of F. Otherwise F_1 fixes an axis and it is contained in a conjugate of F. In this case consider the actions of C_1 on T_2 and T_3. By passing, if necessary (in the dihedral action case), to a subgroup of index 2 we can assume that C_1 is generated by $\langle t, F \rangle$ where t acts hyperbolically on T_2. Similarly C_1 is generated by some x acting hyperbolically on T_3 and a conjugate of F_1 which is contained in F. Since C_1 acts hyperbolically on T_2 x acts hyperbolically on T_2 and $x = tf$ where $f \in F$. Since a acts hyperbolically on T_3 we have

$a = x^k f'$ with $f' \in F$.Then $t^{-k}a$ acts elliptically on T_3, therefore it lies in F. But this is a contradiction since $t \notin F$.

Now if $b \in C_3$ either $b \in F_2$ or $b^k w_1^n \in F_2$. This is because if $b \notin F_2$, w_1, b act both as hyperbolic elements on T_2 and they fix the same axis (since $w_1, b \in C_3$ and C_3 is slender). But then $b^k \in S(C_1, C_2)$. Since the translation length of any hyperbolic element of C_3, for its action on T_2, is a multiple of a fixed number we can pick the same k for all $b \in C_3$. So one has $C_3^k \subset S(C_1, C_2)$. Therefore if we consider the graph of groups corresponding to $S(C_1, C_2)$ and its Bass-Serre tree then C_3 fixes a vertex of this tree. Therefore either it fixes the vertex stabilized by $S(C_1, C_2)$ or C_3^k is contained in the edge stabilizer of an edge adjacent to the vertex stabilized by $S(C_1, C_2)$. But in the first case we have that $C_3 \subset S(C_1, C_2)$ and in the second it is impossible that C_3 is hyperbolic-hyperbolic with respect to, say, C_1. We conclude that there is no $a \in F$ acting hyperbolically on T_3. Therefore F is elliptic on T_3.

On the other hand the splitting over C_3 is hyperbolic-hyperbolic with respect one of the splittings used to construct Γ. Let's say that it is hyperbolic-hyperbolic with respect to the splitting over C_1. Since $F \subset C_1$ and F fixes an axis of T_3 a conjugate of F is contained in C_3. Therefore since the splitting over C_3 is hyperbolic-hyperbolic with respect to the splitting over E, E contains a conjugate of F. Moreover this conjugate of F is an infinite index subgroup of E. This clearly implies that

$$E = \langle t, F | tFt^{-1} = \alpha(F) \rangle$$

where α is an automorphism of F or that

$$E = L *_F M,$$

where $[L : F] = [M : F] = 2$. □(Lemma 5.2).

Let $\{e_i\}$be the collection of the edges of Γ whose edge groups, E_i, are hyperbolic on T_3, and $\{d_j\}, \{D_j\}$ the collections of the rest of the edges and their edge groups. Let T_{E_i} be the Bass-Serre tree of the splitting of G we obtain by collapsing all edges of Γ but e_i. The group C_3 is hyperbolic on T_{E_i} by the way we took e_i, and the splitting along C_3 is minimal. Consider the diagonal action of G on $T_3 \times T_\Gamma$. In the same way as in the proof of Proposition 4.3, we can show that there is a subcomplex $\tilde{Z}$ of $T_3 \times T_\Gamma$ which is invariant by G such that $\tilde{Z}/G$ is finite. We explain this in detail: Let S be the enclosing group of Γ and let T_S be the minimal invariant subtree of T_3 for the action of S. For each $E_i \subset S$ let l_i be the invariant line for the action of E_i on T_3. Finally for each D_j we pick a vertex v_j on T_3 fixed by D_j. Let $\tilde{e}_i$ be a lifting of e_i to $T_1 \times T_2$ with an endpoint on l_i and $\tilde{d}_j$ a lifting of d_j on

$T_1 \times T_2$ with an endpoint on v_j. Let

$$Z_1 = T_S \cup_i (l_i \times \tilde{e}_i) \cup_j (\tilde{d}_j)$$

where the union is over all the $\tilde{e}_i's, \tilde{d}_j's$. We take then Z to be the complex obtained by the translates GZ_1.

As before, we give a description of Z using gluings of graphs. Let $\{k_i\}$ be the vertices of Γ other than the one for S, and $\{K_i\}$ their vertex groups. Let $T_3/S = \mathcal{S}, T_3/E_i = \mathcal{E}_i, T_3/D_i = \mathcal{D}_i, T_3/K_i = \mathcal{K}_i$ be the quotient graph of groups. Since the action of E_i on T_3 is hyperbolic and E_i is slender, there is an invariant line l_i in T_3 by E_i and the core of $\mathcal{E}_i$ is $c_i = l_i/E_i$, which is topologically a segment if the action of E_i on T_3 is dihedral, or else a circle. A core of $\mathcal{D}_i$ is a vertex since the action is elliptic. Let's denote a core of a graph of groups, $\mathcal{A}$, as $\mathrm{co}(\mathcal{A})$. A core complex, Z, of the diagonal action of G on $T_\Gamma \times T_3$ is given as follows:

$$Z = \mathrm{co}(\mathcal{S}) \bigcup \cup_i \mathrm{co}(\mathcal{K}_i) \bigcup \cup_i (\mathrm{co}(\mathcal{E}_i) \times [0,1]) \bigcup \cup_i (\mathrm{co}(\mathcal{D}_i) \times [0,1]).$$

Note that each $\mathrm{co}(\mathcal{E}_i) \times [0,1]$ and $\mathrm{co}(\mathcal{D}_i) \times [0,1]$ is attached to $\mathrm{co}(\mathcal{S}) \bigcup \cup_i \mathrm{co}(\mathcal{K}_i)$ by the graph morphism induced by the homomorphism of each of E_i, D_j to S and to K_k given in Γ.

Z is a finite complex, and the fundamental group in the sense of complexes of groups (let's call such fundamental group *H-fundamental group* in this proof) is G. Let $\mathcal{C}_3 = T_\Gamma/C_3$. Since C_3 is slender, and the action of C_3 on T_Γ is hyperbolic, there is an invariant line l in T_Γ. Since the splittings of G along E_i, C_3 are minimal, we can conclude, as in Prop. 4.7, that $\cup_i(\mathrm{co}(\mathcal{E}_i) \times [0,1]) = \mathrm{co}(\mathcal{C}_3) \times [0,1]$ in $(T_\Gamma \times T_3)/G$. Although l/C_3 is embedded in Z, which locally separates Z, the splitting of G along C_3 which we obtain by cutting Z along l/C_3 might be different from the original splitting along C_3.

Consider the following subcomplex, W, of Z,

$$W = \mathrm{co}(\mathcal{S}) \bigcup \cup_i (\mathrm{co}(\mathcal{E}_i) \times [0,1]) \bigcup \cup_i (\mathrm{co}(\mathcal{D}_i) \times [0,1]).$$

Let $\{p_j\}$ be the set of vertices in W which are not contained in $\mathrm{co}(\mathcal{S})$. Let m_j be the link of p_j in W, which we denote by $Lk(p_j, W)$. Since each $\mathrm{co}(\mathcal{D}_i)$ is a point, if p_j is in $\cup_i(\mathrm{co}(\mathcal{D}_i) \times [0,1])$, then m_j is a point, whose fundamental group (in the sense of graph of groups) is one of the D_i's (the group corresponding to the edge which contains p_j). The point m_j locally separates W, and also Z. If the vertex p_j is in $\cup_i(\mathrm{co}(\mathcal{E}_i) \times [0,1])$, then the link m_j is the finite union of circles and segments, such that each of them locally separates W, and also Z.

If we cut Z along the union of those links $\cup_j Lk(p_j, W)$, we obtain a graph decomposition of G by Lemma 4.1 such that edge groups are the image in G of the H-fundamental groups of connected components of $\cup_j Lk(p_j, W)$. Let V be the connected component of $W - \cup_j Lk(p_j, W)$ which contains $\mathrm{co}(\mathcal{S})$. The image in G of the H-fundamental group of V contains $S = S(C_1, C_2)$. Let's denote it by S'. We claim that S' is an extension of the fundamental group of some 2-orbifold, Σ', by F, the fiber group of S such that $\Sigma \subset \Sigma'$. To see it, let $U = \cup_i(\mathrm{co}(\mathcal{E}_i) \times [0, 1])$, which is a squared surface possibly with some vertices identified. Note $U \subset W$. Let $\{q_i\}$ be the vertices in $U \cap \mathrm{co}(\mathcal{S})$. Define $l_i = Lk(q_i, U)$ for each i. Note that each $p_i \in U$ and also $m_i \subset U$. If we cut U along $\cup_i l_i$ and $\cup_i m_i$, we obtain a graph decomposition along slender groups, which are the image (in G) of the H-fundamental group of l_i's and m_i's. Let $U' \subset U$ be the connected component of $U - (\cup_i l_i \bigcup \cup_i m_i)$ which does not contain any of p_i, q_i, i.e., the vertices of U. We know that U' is a surface with boundary. Also $U' \subset V$. Cutting V along $\cup_i l_i$, where U' is one of the connected component after the cutting, we obtain a graph decomposition of S' along the slender groups corresponding to l_i's. The vertex group, S_0, corresponding to U' is an extension of the fundamental group of some 2-orbifold, Σ_0, by F by our construction. Σ_0 is obtained from the 2-manifold U' attaching a disks or a half disk with cone points appropriately each time if the fundamental group of m_i does not inject to G (cf. we did the same thing when we constructed Σ for S previously). A vertex group other than S_0 is not only a subgroup of S, but also it corresponds to a suborbifold in Σ, the 2-orbifold for S. To see it, consider a small neighborhood, $\overline{\mathrm{co}(\mathcal{S})}$, of $\mathrm{co}(\mathcal{S})$ in W. To be concrete, for example, we take a barycentric subdivision of W and collect all cells which intersect $\mathrm{co}(\mathcal{S})$. The H-fundamental group of $\overline{\mathrm{co}(\mathcal{S})}$ is S. One can consider that $\mathrm{co}(\mathcal{S})$ is a deformation retract of $\overline{\mathrm{co}(\mathcal{S})}$. We may assume that each l_i is in $\overline{\mathrm{co}(\mathcal{S})}$. Cutting $\overline{\mathrm{co}(\mathcal{S})}$ along $\cup_i l_i$, we obtain a graph decomposition of S along slender groups. This decomposition is realized by cutting Σ along simple closed curves and segments. (Consider the quotients by F of the H-fundamental groups of $\overline{\mathrm{co}(\mathcal{S})}$ and l_i's and obtain a decomposition of the orbifold fundamental group of Σ along slender groups, and reduce the argument to surface topology. Note that all maximal peripheral subgroups of S are elliptic with respect to the graph decomposition, cf. Rem 4.14, so that Σ does not have any free boundary points in the decomposition, and the H-fundamental group of l_i injects in G). Let S_i be the H-fundamental group of the connected component of $\overline{\mathrm{co}(\mathcal{S})} - \cup_i l_i$ which contains q_i. Note that this is identical to the connected component of $V - \cup_i l_i$ which contains q_i. Let $\Sigma_i \subset \Sigma$ be the sub-orbifold such that S_i is the

H-fundamental group of Σ_i (each Σ_i is a connected component of Σ after the cutting we obtained in the above). Then, S_i is an extension of the orbifold fundamental group of Σ_i by F. Since the graph decomposition of S' we obtained by cutting V along $\cup_i l_i$ has vertex groups S_i's (corresponding to q_i's) and S_0, with edge groups corresponding to the H-fundamental groups of l_i's, and S_0 is also an extension of the orbifold fundamental group of Σ_0 by F, we conclude that S' is an extension of the orbifold fundamental group of a 2-orbifold, Σ', by F such that Σ' is the union of Σ_i's and Σ_0 pasted along l_i's. By construction, $\Sigma \subset \Sigma'$.

Let Γ' be the graph decomposition of G obtained by cutting Z along $\cup_j Lk(p_j, W)$, with a vertex group S'. We first show that Γ' satisfies the properties 1,2,3 of Def 4.5 for the three splittings (cf. Prop 4.7). Then we apply Prop 3.7 to Γ' and obtain a refinement, Γ'', such that each edge of Γ'' gives a minimal splitting of G. We will show that Γ'' satisfies all the properties of Def 4.5 so that it is an enclosing graph decomposition for the three splittings, with an enclosing vertex group S'. The argument is similar to Prop 4.16.

The group S' has the property 3 by the construction. Regarding the property 1 of Γ', it is clear that S' contains some conjugates of C_1, C_2, C_3.

To verify the property 2(rigidity) of S', let Λ be a graph decomposition of G such that the splitting of G which any edge of Λ gives is elliptic-elliptic with respect to any of the three splittings. We argue in the same way as in the proof of Prop 4.7 to show that S' is elliptic on the Bass-Serre tree of Λ, T_Λ. Let c be either an essential simple closed curve on Σ' or an essential embedded segment on $(\Sigma', (\partial\Sigma'; 2))$. Let $C < S'$ be the fundamental group for c. Cutting Σ' along c, we obtain a splitting of G along C. This splitting is minimal by Prop 3.4. Let T_C be its Bass-Serre tree. For our purpose, by Cor 4.12, it suffices for us to show that the group C is elliptic on T_Λ to conclude that so is S'. Let e be an edge of Λ, which gives a splitting of G along its edge group, E. To conclude C is elliptic on T_Λ, we will show that C is elliptic with respect to the splitting along E. Since the group E is elliptic with respect to the (original) splittings of G along C_1, C_2, it is elliptic on $T_{C_1} \times T_{C_2}$, i.e., E fixes a vertex. Therefore E is elliptic on T_Γ, which is the Bass-Serre tree of the enclosing graph decomposition, Γ, we constructed for the splittings along C_1, C_2. Moreover, we know that E is not in a conjugate of S (cf. the proof of Prop 4.7). Since the group E is elliptic on T_{C_3} as well by our assumption, it fixes a vertex when it acts on $T_\Gamma \times T_{C_3}$. It follows that E is elliptic on $T_{\Gamma'}$, the Bass-Serre tree for Γ', by the way we constructed it. Therefore E is in a conjugate of a vertex group of Γ', which is not S'. This implies that the group E

is elliptic on T_C. Since the splitting along C is minimal, we find that the pair of splittings along C and E is elliptic-elliptic. But, the edge e was an arbitrary edge of Λ, so that the group C is elliptic on T_Λ. We showed the property 2 for S'.

So far, we have shown that the graph decomposition Γ' with a vertex group S' satisfies the properties 2,3 and a part of the property 1. As we obtain Prop 4.7 from Prop 4.16 for a pair of splittings, we apply Prop 3.7 to Γ' and obtain a graph decomposition Γ'' such that each edge gives a minimal splitting. We now claim that Γ'' has S' as an enclosing vertex group and satisfies all the properties to be an enclosing decomposition for the three splittings. The argument is same as when we show Prop 4.16 from Prop 4.7, so we omit some details. By construction, Γ'' has the property 4. S' is a vertex group of Γ'' because the edge groups of Γ'' are in edge groups of Γ' and the rigidity of S'. Therefore, Γ'' with S' satisfies the properties 2, 3, and the property 1 except for the last item, which we did not verify for Γ'.

To verify the rest of the property 1 for Γ'', let e be an edge of Γ'' with the edge group, E. Let T_{C_i} be the Bass-Serre tree of the (original) splitting of G along C_i, $i = 1, 2, 3$. We want to show that the edge group E is elliptic on all T_{C_i}. The splitting of G along E which the edge e gives is minimal. Let T_E be the Bass-Serre tree of this splitting. By the property 2 (rigidity) of S', S' is elliptic on T_E, so that the subgroups C_i are elliptic as well. It follows that the group E is elliptic on all T_{C_i} because the splitting along E is minimal. This is what we want. We showed all the properties for Γ'' with S', so that the proof of Prop 5.1 is complete. □(Prop 5.1)

5.2. **Maximal enclosing decompositions.** Following the previous subsection, we produce an enclosing graph decomposition of a set, I, of hyperbolic-hyperbolic minimal splittings of G along slender subgroups. We put an order to the elements in I such that if I_i denotes the set of the first i elements, then each I_i is a set of hyperbolic-hyperbolic splittings. Then we produce a sequence of graph decompositions, Γ_i, of G such that Γ_i is an enclosing graph decomposition for I_i with enclosing vertex group S_i. We already explained how to construct Γ_2, then Γ_3 using it.In the same way as we produce Γ_3 from Γ_2 fromthe splitting along C_3, we produce Γ_{i+1} from Γ_i. Note that Γ_{i+1} is identical to Γ_i if the edge group, C_{i+1}, of the $(i+1)$-th splitting is contained in a conjugate of S_i.

Although Γ_i is an infinite sequence in general, there exists a number N such that Γ_i is identical if $i \geq N$ by the following result. We recall that a graph of groups, whose fundamental group is G, is *reduced* if its Bass-Serre tree does not contain any proper subtree which is G-invariant, and the vertex group of any vertex of the graph of valence 2 properly contains the edge groups of the associated edges.

Theorem 5.3 (Bestvina-Feighn accessibility [BF]). *Let G be a finitely presented group. Then there exists a number $\gamma(G)$ such that if Γ is a reduced graph of groups with fundamental group isomorphic to G, and small edge groups, then the number of vertices of Γ is at most $\gamma(G)$.*

Let Σ_i be the 2-orbifold for the enclosing vertex group S_i. Then, $\Sigma_i \subset \Sigma_{i+1}$ as 2-orbifolds. Any system, $\mathcal{F}$, of disjoint essential simple closed curves on Σ_i and essential segments on $(\Sigma_i, (\partial\Sigma_i; 2))$ such that any two of them are not homotopic to each other gives a reduced graph decomposition of not only S_i but also G. Because the number of the connected components of $\Sigma_i \backslash \mathcal{F}$ are bounded by $\gamma(G)$ by Theorem 5.3, there exists N such that Σ_i is constant if $i \geq N$. This implies, by the way we constructed $\{\Gamma_i\}$, Γ_i is also constant if $i \geq N$. Γ_N is an enclosing graph decomposition for I. We have shown the following.

Proposition 5.4. *Let G be a finitely presented group. Let I be a set of hyperbolic-hyperbolic minimal splittings of G along slender subgroups. Then, an enclosing graph decomposition of G, Γ_I, exists for I.*

If a set of hyperbolic-hyperbolic minimal splitting along slender groups, I, is maximal, we call Γ_I maximal. A maximal enclosing graph decomposition has the following property.

Lemma 5.5 (Maximal enclosing graph decomposition). *Let G be a finitely generated group. Let Γ be a maximal enclosing decomposition of G. Suppose $A *_C B, A*_C$ is a minimal splitting of G along a slender subgroup C. Then the splitting along C is elliptic-elliptic with respect to the splitting of G which each edge of Γ gives. In particular, the group C is elliptic on T_Γ, the Bass-Serre tree for Γ.*

Remark 5.6. Although the existence of maximal enclosing group is guaranteed only for a finitely presented group, the lemma is true if a maximal enclosing group exists for a finitely generated group.

Proof. Suppose not. Then there exists an edge, e, of Γ with edge group, E, such that the minimal splitting of G the edge e gives is hyperbolic-hyperbolic with respect to the splitting along C. Suppose Γ is an enclosing decomposition for a maximal set I. Since $E < S$, S is hyperbolic on T_C, the Bass-Serre tree for the splitting along C. Then by the rigidity (Def 4.5) of S, there is a splitting in I which is hyperbolic-hyperbolic with respect to the splitting along C. Let I' be the union of I and the splitting along C, which is a set of hyperbolic-hyperbolic splittings. If we produce an enclosing decomposition for I' using Γ and the decomposition along C, we obtain a graph decomposition with a different enclosing vertex group (namely, the 2-orbifold is larger) from Γ, because the group C is hyperbolic with respect to the splitting along E, which is impossible since Γ is maximal. The last claim is clear from Bass-Serre theory. □

Proposition 5.7 (Rigidity of maximal enclosing group)**.** *Let G be a finitely generated group. Let Γ, Γ' be maximal enclosing decompositions of G with enclosing groups S, S'. Then the group S is elliptic on $T_{\Gamma'}$, the Bass-Serre tree of Γ', so that S is a subgroup of a conjugate of a vertex group of Γ'. If S' is a subgroup of a conjugate of S, then it is a conjugate of S.*

Proof. Let I, I' be maximal sets of hyperbolic-hyperbolic minimal splittings of G for Γ, Γ'. We may assume $I \neq I'$. Since they are maximal, if they have a common splitting, then $I = I'$, so that $I \cap I' = \emptyset$. Also, a pair consisting of any splitting in I and any splitting in I' is elliptic-elliptic. Moreover, there is no (minimal) splitting of G along a slender subgroup which is hyperbolic-hyperbolic with respect to some splittings in both of I, I', because, then such splitting and $I \cup I'$ would violate the maximality of I.

Let Σ be the 2-orbifold for the enclosing group S. Let d be a simple closed curve on Σ or a segment on $(\Sigma, (\partial\Sigma; 2))$ which is essential, then cutting Σ along d, we obtain a splitting of G along the group, D, which is the fundamental group of d. D is slender, and the splitting of G along D is minimal since it is hyperbolic-hyperbolic with respect to one of the minimal splittings in I (Prop 3.4). Therefore, the splitting along D is elliptic-elliptic to all splittings in I'.

By Cor 4.12, it suffices for us to show that the group D is elliptic on $T_{\Gamma'}$, the Bass-Serre tree of Γ' to show that S is elliptic on it. Let e be an edge of Γ', with edge group E. Collapsing all edges of Γ' except e, we obtain a splitting of G along E, which is minimal. Let T_E be the Bass-Serre tree of this splitting. Then, it is enough for us to show that the group D is elliptic on T_E. Since the splitting along E is minimal, it suffices to show that the group E is elliptic on T_D, the Bass-Serre tree for the splitting along D. Since $E < S'$, it is enough if we show that S' is elliptic on T_D. By the property 2 (rigidity) of S', it suffices to show that the splitting along D is elliptic-elliptic with respect to all splittings in I', which we already know. We have shown that S is elliptic on T_{Γ}.

By the same argument, S' is elliptic on T_{Γ}. Suppose S is in a conjugate of S', i.e., $S < gS'g^{-1}, g \in G$. Then S' is also in a conjugate of

S, since, otherwise, S' is in a conjugate of a vertex group of Γ which is not S. Then this vertex group contain a conjugate of S, which is impossible since all edge of Γ which is adjacent to the vertex whose vertex group is a conjugate of S has an edge group which is a proper subgroup of the conjugate of S. Suppose $S' < hSh^{-1}, h \in G$. Therefore, $S < ghS(gh)^{-1}$. This implies that $gh \in S$, and $S = ghS(gh)^{-1}$, so that $S = gS'g^{-1}$. □

5.3. **JSJ-decomposition for hyperbolic-hyperbolic minimal splittings.** Using maximal enclosing graph decompositions, we produce a graph decomposition of G, Λ, which "contains" all maximal enclosing groups. Λ will deal with all minimal splittings of G along slender subgroups which are hyperbolic-hyperbolic with respect to some (minimal) splittings along slender subgroups.

Consider all maximal enclosing decompositions, Γ_i, of G with enclosing groups, S_i. Let T_i be the Bass-Serre tree of Γ_i. We construct a sequence of refinements $\{\Lambda_i\}$ such that $\Gamma_1 = \Lambda_1$. We then show that after a finite step, the graph decompositions stay the same. We denote the decomposition obtained after this step by Λ.

We put $\Lambda_1 = \Gamma_1$. We consider now Γ_2. By Prop 5.7, S_2 is elliptic on T_1. If S_2 is a subgroup of a conjugate of S_1, then we do nothing and put $\Lambda_2 = \Gamma_1$. If S_2 is conjugate into a vertex group, A, of Γ_1 which is not S_1, then we let A act on T_2 and obtain a refinement, Γ_2', of Γ_1. Namely, let $\mathcal{A}$ be the graph decomposition of A we get. We substitute $\mathcal{A}$ to the vertex, a, for A in Γ_1 (see Def 3.6 and the following remarks). We can do this since each edge group, E, of Γ_1 is elliptic on T_2 (because E is a subgroup of S_1, which is elliptic on T_2), so that E is a subgroup of a conjugate of a vertex group of $\mathcal{A}$. Note that all edge groups of Γ_2' are slender since they are subgroups of conjugates of edge groups of Γ_1, Γ_2. Γ_2' has conjugates of S_1, S_2 as vertex groups. Also they are peripheral subgroups of either S_1 or S_2.

Each edge, e, of Γ_2' gives a minimal splitting of G along its edge group, E, which is a subgroup of a conjugate of the edge group, E', of an edge, e', of $\Gamma_i, (i = 1 \,\text{or}\, 2)$. We show this by contradiction: suppose that the edge e gives a non-minimal splitting, which is hyperbolic-elliptic with respect to a splitting $G = P *_D Q, (\text{or} P*_D)$ such that D is slender. Let T_D be its Bass-Serre tree. Since the group E is hyperbolic on T_D, so is E'. Because the splitting of G along E', the one e' gives, is minimal, it is hyperbolic-hyperbolic with respect to $G = P *_D Q, (\text{or} P*_D)$. By Prop 3.4, the splitting along D is minimal. On the other hand, by Lemma 5.5, the group D is elliptic on T_{Γ_i}, the Bass-Serre tree of Γ_i

since it is maximal. It follows that the group D is elliptic on $T_{E'}$, the Bass-Serre tree of the splitting along E', a contradiction.

We collapse all edges of this decomposition which are not adjacent to the vertices with vertex groups S_1, S_2. If the resulting graph decomposition is not reduced (cf. Theorem 5.3) at some vertex, then we collapse one of the associated two edges, appropriately, to make it reduced. Note that it is reduced at a vertex whose vertex group is an enclosing group since all edge groups are proper subgroups at the vertex of an enclosing group. We denote the resulting reduced graph decomposition by Λ_2. We remark that Λ_2 is a refinement of Λ_1.

By our construction, all edge groups of Λ_2 are conjugates of edge groups of Γ_1, Γ_2, and each edge of Λ_2 is connected to the vertex of an enclosing group. This is not obvious, we recall that we only know that edge groups of Γ_1 are elliptic on T_1. When we substitute $\mathcal{A}$ for A, some of these edge groups are connected to (a conjugate of) S_2. We have to show that these edge groups are peripheral in gS_2g^{-1}. To see it, let E be an edge group, and suppose that $E < S_2$, where we assume that $g = 1$ for notational simplicity. (In general, just take conjugates by g appropriately in the following argument). We will show E is peripheral in S_2. Note that this is the only essential case since E can be only peripheral in S_1 because we don't do anything around the vertex for S_1 when we construct Λ_2. Also we may assume $E < S_1$. Let Σ_2 be the 2-orbifold for S_2, and d an essential simple closed curve/segment on it with the group D represented by d. Cutting Σ_2 along d, we obtain a splitting of G along D, with Bass-Serre tree T_D. It suffices to show that the group E is elliptic with respect to T_D to conclude that E is peripheral in S_2. Suppose not. Then, the splittings of G along E and D are hyperbolic-hyperbolic since both of them are minimal. Then S_1 is hyperbolic on T_D since $E < S_1$. Let Σ_1 be the 2-orbifold for S_1. It follows from Cor 4.12 that there exists an essential simple closed curve or a segment, d', on Σ_1 such that cutting Σ_1 along d' gives a splitting of G along the group for d', D', such that the splittings along D and D' are hyperbolic-hyperbolic. Then, the set $I_1 \cup I_2$ with the two splittings along D, D' is a set of hyperbolic-hyperbolic, which is impossible because the enclosing group for this set must be strictly bigger than S_1, and S_2 as well, which is impossible since they are maximal. We have show that all edge groups of Λ_2 are peripheral subgroups of enclosing vertex groups.

We continue similarly and obtain a sequence of reduced graph decompositions of G; $\Lambda_1, \Lambda_2, \Lambda_3, \cdots$. Namely, we first show that the enclosing group S_3 is elliptic with respect to Λ_2, using the maximality of S_i and rigidity. If S_3 is a subgroup of a conjugate of S_1 or S_2, then

Λ_3 is Λ_2. Otherwise, there exists a vertex group, A_2, of Λ_2 which is different from S_1, S_2 and contains a conjugate of S_3. We let A_2 act on the Bass-Serre tree of Γ_3 and obtain a graph decomposition, $\mathcal{A}_2$, of A_2. We then substitute $\mathcal{A}_2$ to the vertex for A_2 in Λ_2, which is Γ'_3. We show that all edges of Γ'_3 give minimal splittings of G along slender subgroups. We then collapse all edges which are not adjacent to the vertices with the vertex groups conjugates of S_1, S_2, S_3, and also collapse edges appropriately at non-reduced vertices, to obtain a reduced graph decomposition, Λ_3. The edge groups of Λ_3 which are adjacent to some conjugates of S_i are the conjugates of peripheral subgroups of S_i. In this way, we obtain Λ_{n+1} from Λ_n using Γ_{n+1}. This is a sequence of refinements.

We claim that there exists a number N such that if $n \geq N$ then $\Lambda_{n+1} = \Lambda_n$. Indeed, if not, then the number of vertices in Λ_n whose vertex groups are enclosing groups S_i tends to infinity as n goes to infinity. This is impossible since the number of the vertices of Λ_n is at most $\gamma(G)$ by Theorem 5.3. Note that slender groups are small. Let's denote Λ_N by Λ, and state some of the properties we have shown as follows.

Proposition 5.8 (JSJ-decomposition for hyp-hyp minimal splittings along slender groups)**.** *Let G be a finitely presented group. Then there exists a reduced graph decomposition, Λ, with the following properties:*

1. *all edge groups are slender.*
2. *Each edge of Λ gives a minimal splitting of G along a slender group. This splitting is elliptic-elliptic with respect to any minimal splitting of G along a slender subgroup.*
3. *Each maximal enclosing group of G is a conjugate of some vertex group of Λ, which we call a (maximal) enclosing vertex group. The edge group of any edge adjacent to the vertex of a maximal enclosing vertex group is a peripheral subgroup of the enclosing group.*
4. *Each edge of Λ is adjacent to some vertex group whose vertex group is a maximal enclosing group.*
5. *Let $G = A *_C B$ or $A*_C$ be a minimal splitting of G along a slender subgroup C, and T_C its Bass-Serre tree.*
   (a) *If it is hyperbolic-hyperbolic with respect to a minimal splitting of G along a slender subgroup, then*
      (i) *a conjugate of C is a subgroup of a unique enclosing vertex group, S, of Λ. S is also the only one among enclosing vertex groups which is hyperbolic on T_C. There exists a base 2-orbifold, Σ, for S and an essential simple closed*

curve or a segment on Σ whose fundamental group (in the sense of complex of groups) is a conjugate of C.

(ii) *Moreover, if G does not split along a group which is a subgroup of C of infinite index, then all non-enclosing vertex groups of Λ are elliptic on T_C.*

(b) *If it is elliptic-elliptic with respect to any minimal splitting of G along a slender subgroup, then all vertex groups of Λ which are maximal enclosing groups are elliptic on T_C.*

Proof. By the previous discussion we know that properties 1,3,4, and a part of property 2 hold. Let's show the rest of the property 2. To argue by contradiction, suppose that the edge, e, of Λ gives a minimal splitting along the edge group, E, which is hyperbolic-hyperbolic with respect to a minimal splitting of G along a slender subgroup, C. But then C would be contained in an enclosing vertex group of some graph decomposition Γ_i and it would not be a peripheral group in Γ_i, a contradiction

We show now (5-a). There is a maximal enclosing group which contains a conjugate of C, such that C is the fundamental group of an essential simple closed curve or a segment of the 2-orbifold for the enclosing group. This is because we start with the splitting along C to construct the enclosing group. By the construction of Λ, this enclosing group is a conjugate of some vertex group, S, of Λ. To argue by contradiction, suppose there is another enclosing vertex group which contains a conjugate of C. Then, by Bass-Serre theory, there must be an edge associated to each of those two vertices whose edge group contains a conjugate of C. But the edge and its edge group has the property 2, which contradicts the assumption on the splitting along C that it is hyperbolic-hyperbolic. One can show that all enclosing vertex groups of Λ except S are hyperbolic on T_C using Cor 4.12, and we omit details since similar arguments appeared repeatedly.

To show the last claim, suppose that there is a non-enclosing vertex group, V, of Λ which is hyperbolic on T_C. Letting V act on T_C, we obtain a graph decomposition of V, which we can substitute for V in Λ. All edge groups of this graph decomposition are conjugates of subgroups of C, which have to be of finite index by our additional assumption. Since a conjugate of C is contained in S, which is different from V, by Bass-Serre theory, there must be an edge in Λ adjacent to the vertex for S whose edge group, E, is a conjugate of a subgroup of C of finite index. But the edge and its edge group E, so that the group C as well, satisfies the property 2, which contradicts the assumption on C that it is hyperbolic-hyperbolic.

We show (5-b). Let S be a maximal enclosing vertex group of Λ. Let Γ be a maximal enclosing decomposition which has S as the maximal enclosing group. Let Σ be the 2-orbifold for S, and c an essential simple closed curve or an essential segment on Σ. Cutting Σ along d, we obtain a splitting of G along the slender group, D, which corresponds to d. This splitting is minimal by Prop 3.4. By the assumption on the splitting along C, the group D is elliptic on T_C. It follows from Cor 4.12 that S is elliptic on T_C. □

5.4. **Elliptic-Elliptic splittings.** Let $G = A_n *_{C_n} B_n$ (or $A_n *_{C_n}$) be all minimal splittings of G along slender groups, C_n, which are elliptic-elliptic with respect to any minimal splitting of G along slender subgroups. To deal with them as well, we refine Λ which we obtained in Prop 5.8. As we constructed a sequence of refinements $\{\Lambda_n\}$ to obtain Λ, we construct a sequence, $\{\Delta_n\}$, of refinements of Λ using the sequence of splittings along C_n. Then we show that after a finite step the sequence stabilizes in some sense, again by Theorem 5.3, and obtain the desired graph decomposition of G, Δ.

We explain how to refine Λ in the first step. Let $G = A *_C B$ or $A*_C$ be a minimal splitting along a slender group C which is elliptic-elliptic with respect to any minimal splitting of G along a slender group. By Prop 5.8, all enclosing vertex groups and all edge groups of Λ are elliptic on T_C, the Bass-Serre tree of the splitting along C. Let U be a vertex group of Λ which is not an enclosing vertex group. Letting U act on T_C, we obtain a graph decomposition of U, $\mathcal{U}$, which may be a trivial decomposition. Substituting $\mathcal{U}$ for the vertex for U in Λ, which one can do since all edge groups of Λ are elliptic on T_C, we obtain a refinement of Λ. We do this to all non-enclosing vertex groups of Λ. Then we apply Prop 3.7 to this graph decomposition, and obtain a further refinement of Λ, which we denote Δ_1, such that each edge of Δ_1 gives a minimal splitting of G along a slender group. By construction, all vertex groups of Δ_1 are elliptic on T_C. Although when we apply Prop 3.7, a vertex group may become smaller, all enclosing vertex groups of Λ stay as vertex groups in Δ_1. We see this by an argument similar to the one in the proof of Prop 5.8. We omit details, but just remark that all edge groups of Δ_1 are edge groups of Λ or subgroups of conjugates of C.

Note that Δ_1 might not be reduced. This may cause a problem when we want to apply Theorem 5.3 later. To handle this problem, if there is a vertex of Δ_1 of valence two such that one of the two edge group is same as the vertex group, we collapse that edge. Note that the other edge group is properly contained in the vertex group in our case. We do this to all such vertices of Δ_1 at one time, and obtain a reduced decomposition, which we keep denoting Δ_1. In general, we can obtain a reduced decomposition from a non-reduced decomposition in this way. We call the inverse of this operation an *elementary unfolding*. By definition, a composition of elementary unfoldings is an elementary unfolding. If we obtain a graph decomposition, Γ', by an elementary folding from Γ, we may say Γ' is an elementary unfolding of Γ.

Example 5.9 (Elementary unfolding)**.** Let Γ be a graph decomposition of G which is $G = A *_P B *_Q C$ and suppose B has a graph decomposition $\mathcal{B}$ which is $P *_{P'} B' *_{Q'} Q$. Then one can substitute $\mathcal{B}$ to the vertex of B in Γ and obtains a new graph decomposition Γ', which is an elementary unfolding. One can substitute $\mathcal{B}$ to Γ because each edge group adjacent to the vertex for B in Γ (P, Q in this case) is a subgroup of some vertex group of $\mathcal{B}$. If we refine Γ using $\mathcal{B}$, we obtain Γ': $G = A *_P P *_{P'} B' *_{Q'} Q *_Q C$. Although Γ' has two more vertices than Γ , Γ' is not reduced. And if we collapse edges of Γ' to obtain a reduced decomposition we get $G = A *_{P'} B' *_{Q'} C$, which has the same number of vertices as Γ.

As this example shows Theorem 5.3 can not control a sequence of reduced graph decomposition which is obtained by elementary unfoldings. But we have another accessibility result to control this, which we prove later.

As we said, we now produce a sequence of refinements Δ_n using the splittings of G along C_n. We may assume that the first splitting is the splitting along C, with which we already constructed Δ_1. We now refine Δ_1 using the splitting along C_2. Same as Λ, all edge groups and all enclosing vertex groups of Δ_1 are elliptic on T_2, the Bass-Serre tree of the splitting along C_2. As before, we let a non-enclosing vertex group of Δ_1, U, act on T_2 and obtain a graph decomposition of U, which we substitute for the vertex labelled by U in Δ_1. We do this for all non-enclosing vertex groups of Δ_1, then we apply Prop 3.7. If the resulting graph decomposition is an elementary unfolding of Δ_1, then we put $\Delta_2 = \Delta_1$. Otherwise, if the graph decomposition is not reduced, then we collapse one edge at a vertex where it is not reduced, and obtain a reduced graph decomposition of G, which we denote Δ_2. Δ_2 has following properties:

1. Δ_2 is a refinement of Δ_1. Δ_2 is identical to Δ_1, or has more vertices.
2. Each edge of Δ_2 gives a minimal splitting of G along a slender group. The edge group is a subgroup of a conjugate of either an edge group of Δ_1 or C_2.

3. Each maximal enclosing group is a conjugate of some vertex group of Δ_2.
4. After, if necessary, performing an elementary unfolding to Δ_2, each vertex group is elliptic on T_2, and also T_1, the Bass-Serre tree of the splitting along C_1.

We repeat the same process; refine Δ_n using the splitting along C_n to Δ_{n+1}. Because of the property 1 in the above list, by Theorem 5.3, there exists a number, N, such that if $n \geq N$, then Δ_{n+1} is equal to or an elementary unfolding of Δ_n. Let's denote Δ_N by Δ. We state some properties of Δ.

Proposition 5.10 (JSJ decomposition for minimal splittings with elementary unfoldings)**.** *Let G be a finitely presented group. Let $G = A_n *_{C_n} B_n$ (or $A_n *_{C_n}$) be all minimal splittings of G along slender groups, C_n, which are elliptic-elliptic with respect to any minimal splittings of G along slender subgroups. Let T_n be their Bass-Serre trees. Then there exists a graph decomposition, Δ, of G such that*

1,2,3. *Same as the properties 1,2,3 of Prop 5.8.*

4. *For each n, there exists an elementary unfolding of Δ such that each vertex group is elliptic on T_n.*
5. *Let $G = A *_C B$ or $G = A*_C$ be a minimal splitting along a slender group C which is hyperbolic-hyperbolic with respect to some minimal splitting along a slender group, and T_C its Bass-Serre tree. Then,*
   (a) *same as 5(a)i of Prop 5.8.*
   (b) *There exists an elementary unfolding of Δ, at non-enclosing vertex groups, such that all vertex groups in the elementary unfolding except for S are elliptic on T_C, the Bass-Serre tree of the splitting along C.*

Remark 5.11. In fact we do not need an elementary unfolding in the properties 4 and 5 in the proposition, if we construct a more refined Δ. We show this in Theorem 5.13.

Proof. We already know 1,2,3,4 from the way we constructed Δ. Also the property 5(a) is immediate from Prop 5.8. To show 5(b), let U be a non-enclosing vertex group of Δ which is not elliptic on T_C. If such vertex does not exist, we are done. As usual, letting U act on T_C, we obtain a graph decomposition, $\mathcal{U}$, of U, then we substitute this for U in Δ to obtain a refinement, Δ', of Δ whose edges give minimal splittings, after we apply Prop 3.7 if necessary. But this resulting decomposition has to be an elementary unfolding of $\Delta = \Delta_N$, because otherwise we must have refined Δ_N further when we constructed Δ. □

5.5. **Elementary unfolding and accessibility.** As we said in the remark after Prop 5.10, we do not need elementary unfoldings. But as we saw in Example 5.9, Theorem 5.3 can not control a sequence of elementary unfoldings because they are not reduced. We prove another accessibility result. This result was suggested to us by Bestvina. The argument is similar to the one used by Swarup in a proof of Dunwoody's accessibility result ([Sw]).

Proposition 5.12 (Intersection accessibility)**.** *Let G be a finitely presented group. Suppose Γ_i is a sequence of graph decompositions of G such that all edge groups are slender. Suppose for any i, Γ_{i+1} is obtained from Γ_i by an elementary unfolding. Then there is a graph decomposition Γ of G with all edge groups slender such that for any i, Γ is a refinement of Γ_i.*

Proof. We define a partial order on the set of graph of groups decompositions of G. We say that $\Gamma < \Lambda$ if all vertex groups of Γ act elliptically on the Bass-Serre tree corresponding to Λ, in other words, Γ is a refinement of Λ. We have $\Gamma_{i+1} < \Gamma_i$ for all i.

We can apply Dunwoody's tracks technique to obtain a graph of groups decomposition Γ such that $\Gamma < \Gamma_i$ for all i. We describe briefly how this is done: Let K be a presentation complex for G. Without loss of generality we assume that K corresponds to a triangular presentation. Let T_i be the Bass-Serre tree of Γ_i. As we noted earlier there are maps $\phi_i : T_{i+1} \to T_i$ obtained by collapsing some edges. We choose a sequence of points (x_i) such that x_i is a midpoint of an edge and $\phi(x_{i+1}) = x_i$.

We will define maps $\alpha_i : K \to T_i$. Each oriented edge of K corresponds to a generator of G. Given an edge e corresponding to an element $g \in G$ we map it by a linear map to the geodesic joining x_i to gx_i. We extend linearly this map to the 2-skeleton of K. A track is a preimage of a vertex of T_i under this map. We note that the tracks we obtain from T_i are a subset of the tracks obtained from T_{i+1} (or to be more formal each track obtained from T_i is 'parallel' to a track obtained from T_{i+1}). We remark that for each i the tracks obtained from α_i give rise to a decomposition Γ'_i of G. Γ_i is obtained from Γ'_i by subdivisions and foldings.

Since G is finitely presented, so that K is compact, there is a $\lambda(G)$ such that there are at most $\lambda(G)$ non-parallel tracks we conclude that there is an n such that each track obtained from T_k $(k > n)$ is parallel to a track obtained from T_n. We can then take as Γ_∞ the graph of groups decomposition corresponding to the tracks obtained from T_n. It follows that $\Gamma_i > \Gamma_\infty$. Put $\Gamma = \Gamma_\infty$. ∎

5.6. **JSJ-decomposition along slender groups.** We state one of our main theorems.

Theorem 5.13 (JSJ-decomposition for minimal splittings along slender groups). *Let G be a finitely presented group. Then there exists a graph decomposition, Γ, of G such that*

1,2,3. *same as the properties 1,2,3 of Prop 5.8.*

4. *Let $G = A *_C B$ or $A*_C$ be a minimal splitting along a slender group C, and T_C its Bass-Serre tree.*
   (a) *If it is elliptic-elliptic with respect to all minimal splittings of G along slender groups, then all vertex groups of Γ are elliptic on T_C.*
   (b) *If it is hyperbolic-hyperbolic with respect to some minimal splitting of G along a slender group, then there is an enclosing vertex group, S, of Γ which contains a conjugate of C and the property 5(a)i of Prop 5.8 holds for S. All vertex groups except for S of Γ are elliptic on T_C.*
   In particular, there is a graph decomposition, $\mathcal{S}$, of S whose edge groups are in conjugates of C, which we can substitute for S in Γ such that all vertex groups of the resulting refinement of Γ are elliptic on T_C.

Proof. Let Δ be the graph decomposition of G which we have constructed for Prop 5.10. We will obtain Γ as a refinement of Δ at non-enclosing vertex groups. Let $G = A_n *_{C_n} B_n$ (or $A_n*_{C_n}$) be all minimal splittings of G along slender groups, C_n, which are elliptic-elliptic with respect to any minimal splitting of G along slender subgroups, and T_n their Bass-Serre trees. We have defined a process to refine a graph decomposition using this collection to obtain a sequence $\{\Delta_n\}$ for Prop 5.10. We apply nearly the same process to Δ again using the splittings along C_n, and produce a sequence $\{\Gamma_n\}$. The only difference is that we do not make a graph decomposition reduced in each step. Let T_n be the Bass-Serre tree of the splitting along C_n. To start with, put $\Gamma_0 = \Delta$. Letting all vertex groups act on T_1, we obtain graph decompositions, then substitute them for the corresponding vertex groups in Γ_0, which is Γ_1. Γ_1 is an elementary unfolding of Γ_0, because otherwise, we must have refined Δ farther in the proof of Prop 5.10. Note that Γ_1 is not reduced, but we do not collapse any edges. We repeat the same process; we let all vertex groups of Γ_1 act on T_2, substitute those graph decompositions for the corresponding vertex groups in Γ_1. The resulting non-reduced graph decomposition is Γ_2, and so on. In this way, we obtain a sequence of graph decompositions Γ_n such that Γ_{n+1} is an elementary unfolding of Γ_n. We remark that in each step enclosing vertex groups stay unchanged since they are elliptic on all T_n. Note that each Γ_n satisfies the properties 1,2,3 and 5(a)i of Prop 5.8.

Suppose that there exists N such that for any $n \geq N$, $\Gamma_n = \Gamma_{n+1}$. Then Γ_N satisfies the properties 4 and 5(b) of Prop 5.10 as well without elementary unfoldings, so that the property 4 of the theorem follows. Putting $\Gamma = \Gamma_N$, we obtain a desired Γ.

If such N does not exist, then we apply Prop 5.12 to our sequence and obtain a graph decomposition which is smaller (or equal to), for the order defined in Prop 5.12, than all Γ_n. Let's take a minimal element, Γ, with respect to our order. Such a decomposition exists by Zorn's lemma. Γ is the decomposition that we look for, because if we apply the process to refine Γ using the sequence of decompositions along C_n as before, nothing happens, because Γ is minimal in our order. It follows that Γ satisfies all the properties. □

We call a graph decomposition of G we obtain in Theorem 5.13 a *JSJ decomposition of G for splittings along slender groups*. We will prove that Γ has the properties stated in this theorem not only for minimal splittings of G along slender groups, but also non-minimal splittings as well in Theorem 5.15.

Corollary 5.14 (Uniqueness of JSJ decomposition)**.** *Let G be a finitely presented group. Suppose a graph decomposition, Γ, of G satisfies the properties 2 and 4(a) of Theorem 5.13.*

1. *Suppose Γ' is a graph decomposition of G which satisfies the properties 2 and 4(a) of Theorem 5.13. Then all vertex groups of Γ' are elliptic on the Bass-Serre tree for Γ.*
2. *Γ satisfies the property 3 of Theorem 5.13.*
3. *Γ satisfies the property 4(b) if G does not split along an infinite index subgroup of C.*

Proof. 1. Let T be the Bass-Serre tree for Γ. Let V be a vertex group of Γ'. Let e be an edge of Γ with edge group E, and T_e the Bass-Serre tree of the splitting of G along E which the edge e gives. To show V is elliptic on T, it suffices to show that it is elliptic on T_e for all e. This splitting along the slender group E is minimal, and elliptic-elliptic with respect to any minimal splitting of G along a slender group by the property 2 of Γ. Therefore, by the property 4(a) of Γ', all vertex groups of Γ' are elliptic on T_e. In particular V is elliptic on T_e, so it is elliptic on T.

2. Let T be the Bass-Serre tree of Γ. Let S be the maximal enclosing vertex group in a maximal enclosing decomposition, Λ, of G. We first show that S is elliptic on T. To show it, as usual, we use Cor 4.12. Let

Σ be the 2-orbifold for S, and s an essential simple closed curve or a segment on Σ. Cutting Σ along s, we obtain a splitting of not only S but also G along the slender group, C, represented by s. This splitting is minimal by Prop 3.4. By Cor 4.12, it suffices to show that the group C is elliptic on T. By the property 1 of Γ, the pair of the splittings of G along E (from the previous paragraph) and C is elliptic-elliptic. Therefore, the group C is elliptic on T_E, so that it is elliptic on T as well since the edge e was arbitrary.

We already know that S is in a conjugate of a vertex group, V, of Γ. We want to show that indeed S is a conjugate of V. Let T_Λ be the Bass-Serre tree of the maximal enclosing decomposition Λ with S. It suffices to show that V is elliptic on T_Λ to conclude that S is a conjugate of V, because S is the only vertex group of Λ which can contain a conjugate of V. This is because all edge groups adjacent to S are peripheral subgroups, so they are proper subgroups of S. Let d be an edge of Λ with the edge group D. The splitting of G along D which the edge d gives is minimal since Λ is an enclosing decomposition, and elliptic-elliptic with respect to any minimal splitting along a slender group since Λ is maximal. By the property 3 (a) of Γ, all vertex groups of Γ are elliptic on T_D, the Bass-Serre tree of the splitting along D. Since the edge d was arbitrary, all vertex groups of Γ are elliptic on T_Λ, in particular, so is V.

3. Let I be a maximal set of hyperbolic-hyperbolic minimal splittings of G along slender groups which contains the splitting along C. Let Λ be a maximal enclosing decomposition of G for I with enclosing vertex group S. Let Σ be the 2-orbifold for S. We can assume that there is an essential simple closed curve or a segment, s, on Σ such that by cutting Σ along s we obtain a splitting of G along C. This is because when we construct Λ for I using a sequence of graph decomposition, we can start with the splitting along C. Although we do not know in general if this splitting is the same as the one we are given, it is the case under our extra assumption. By the property 3 of Γ, a conjugate of S is a vertex group of Γ, which therefore contains a conjugate of C. No other vertex group of Γ contains a conjugate of C because if it did, then an edge group of Γ has to contain a conjugate of C, which is a contradiction since the splitting along C is hyperbolic-hyperbolic while Γ has property 2.

Let w be the vertex of Γ with the vertex group, W, which is a conjugate of S. Let v be a vertex of Γ with vertex group, V, such that $v \neq w$. We want to show that V is elliptic on T_C, the Bass-Serre tree of the splitting $G = A *_C B$, or $A *_C$, which we know is obtained by cutting Σ along s. We first claim that V is elliptic on T_Λ. This is because each edge of Λ gives a minimal splitting which is elliptic-elliptic since Λ is maximal, so that V is elliptic on T_Λ by the property 4(a). (Use it to each edge decomposition of Λ). Therefore V is in a conjugate of some vertex group, U, of Λ. If U is not S, we are done, because then U is elliptic on T_C. We have used that the original splitting along C is identical to the one we obtain by cutting along s. Suppose $U = S$, then V is in a conjugate of W. By Bass-Serre theory, this means that there is an edge in Γ adjacent to v whose edge group is V. It then follows from the property 2 for Γ that the edge group V is elliptic on T_Λ. The proof is complete. □

As we said, Γ indeed can deal with non-minimal splittings of G along slender groups as well.

Theorem 5.15 (JSJ decomposition for splittings along slender groups)**.** *Let G be a finitely presented group, and let Γ be the graph decomposition we obtain in Theorem 5.13. Let $G = A *_C B, A*_C$ be a splitting along a slender group C, and T_C its Bass-Serre tree.*

1. *If the group C is elliptic with respect to any minimal splitting of G along a slender group, then all vertex groups of Γ are elliptic on T_C.*
2. *Suppose the group C is hyperbolic with respect to some minimal splitting of G along a slender group. Then*
   (a) *all non-enclosing vertex groups of Γ are elliptic on T_C.*
   (b) *For each enclosing vertex group, V, of Γ, there is a graph decomposition of V, $\mathcal{V}$, whose edge groups are in conjugates of C, which we can substitute for V in Γ such that if we substitute for all enclosing vertex groups of Γ then all vertex groups of the resulting refinement of Γ are elliptic on T_C.*

Proof. 1. If the splitting along C is minimal, then nothing to prove (Theorem 5.13). Suppose not. Apply Prop 3.7 to the splitting and obtain a refinement, Λ, such that each edge, e, of Λ gives a minimal splitting of G along a slender group, E, which is a subgroup of a conjugate of C. Let $G = P *_E Q$ (or $P*_E$) be the splitting along E which the edge e gives. Let T_E be its Bass-Serre tree. Let T_Λ be the Bass-Serre tree of Λ. We want to prove that each vertex group, V, of Γ is elliptic on T_Λ, which implies that V is elliptic on T_C, since Λ is a refinement of the splitting along C. By Bass-Serre theory, it suffices to prove that V is elliptic on T_E. By our assumption, the group C is elliptic with respect to any minimal splitting of G along a slender group, so that so is E since it is a subgroup of a conjugate of C. Therefore, the minimal splitting $G = P *_E Q$ (or $P*_E$) is elliptic-elliptic with respect to any minimal splitting of G along a slender group, so that by the property

4(a), Theorem 5.13 of Γ, all vertex groups of Γ, in particular V, are elliptic on T_E.

2. If the given splitting along C is minimal, then nothing to prove because we have 4(b), Theorem 5.13. Suppose it is not minimal, and apply Prop 3.7 to obtain a refinement, Λ such that each edge gives a minimal splitting of G. All edge groups of Λ are in conjugates of C. Let T_Λ be its Bass-Serre tree. Then all non-enclosing vertex groups of Γ are elliptic on T_Λ. The argument is similar to the case 1 above and the proof for 3(b), Theorem 5.13. We omit details. Let V be an enclosing vertex group of Γ. Letting V act on T_Λ, we obtain a graph decomposition, $\mathcal{V}$, of V such that all edge groups are in conjugates of C. Because each edge of Λ gives a minimal splitting of G along a slender group, by the property 1, Theorem 5.13 for Γ, all edge groups of Γ are elliptic on T_Λ. Therefore we can substitute $\mathcal{V}$ for V in Γ. If we substitute for all enclosing vertex groups in this way, we obtain the desired refinement of Γ. $\square$

One can interpret our theorems using the language of foldings. What we show is that if Γ is the JSJ-decomposition of a finitely presented group and $A *_C B$ (or $A*_C$) is a splitting of G over a slender group C with Bass-Serre tree T_C then we can obtain a graph decomposition Γ' from Γ such that all vertex groups of Γ' act elliptically on T_C. Let's call T the Bass-Serre tree of Γ'. Since all vertex groups of Γ' fix vertices of T_C we can define a G-equivariant simplicial map f from a subdivision of T to T_C. To see this pick a tree $S \subset T$ such that the projection from S to Γ is bijective on vertices. If $v \in S^0$ pick a vertex $u \in T_C$ such that $Stab(v) \subset Stab(u)$. Define then $f(v) = u$. We extend this to the edges of S by sending the edge joining two vertices to the geodesic joining their images in T_C. This can be made simplicial by subdividing the edge. Finally extend this map equivariantly on T. It follows that the splitting over C can be obtained from Γ by first passing to Γ' and then performing a finite sequence of subdivisions and foldings. In this sense Γ 'encodes' all slender splittings of G.

6. Final remarks

For producing the JSJ-decomposition we did not put any restriction on G; in particular we did not assume that G does not split over groups ‘smaller’ than the class considered. One of the difficulties in this is that there is no natural ‘order’ on the set of slender groups. Otherwise one could work inductively starting from the ‘smallest’ ones. This is the reason we introduced the notion of minimal splittings.

We remark that the situation is simpler if one restricts one's attention to polycyclic groups as one can 'order' them.

It is a natural question whether there is a JSJ-decomposition over small groups. Our results (in particular proposition 4.7) might prove useful in this direction. The main difficulty for generalizing it to an arbitrary number of small splittings to produce a JSJ-decomposition over small groups is that the edge groups of the decomposition in prop. 4.7 are not small in general. So one can not apply induction in the case of small splittings.

We note however that if a JSJ-decomposition over small groups exist its edge groups are not small. We illustrate this by the following example.

Example 6.1. Let's denote by A the group given by $A = \langle a, t, s | tat^{-1} = a^2, sas^{-1} = a^2 \rangle$ and let H be an unsplittable group containing F_2, e.g. $SL_3(\mathbb{Z})$.

Let's consider a complex of groups $G(X)$ with underlying complex a sphere obtained by gluing two squares along their boundary. We label all 4 vertices (0-cells) by $A \times H$. We label 2-cells and 1-cells by an infinite cyclic group $\langle c \rangle$. In one of the squares all maps from the group of a 2-cell to a group of a 1-cell are isomorphisms while all maps to the group of 0-cell send c to a.

We describe now the maps in the second square τ. Let e_1, e_2 be two adjacent edges (1-cells) of the square. Let a_{12} be the common vertex of e_1, e_2 and let a_1 be the other vertex of e_1 and a_2 the other vertex of e_2. Let finally b be the fourth vertex of the square.

The monomorphisms $\psi_1 : G_\tau \to G_{e_1}, \psi_2 : G_\tau \to G_{e_2}$ are given by $c \to c^2$. All other maps are defined as in the first square. To satisfy condition 3 of the definition of a complex of groups (see subsec. 4.1) we define the 'twisting' element $g_{e,f}$ for two composable edges e, f as follows:

For each vertex of the square there are two pairs of composable edges from the barycenter of τ to the vertex. For the vertex b we put $g_{e,f} = 1$ for the first pair and $g_{e,f} = t^{-1}s$ for the second pair. We remark that $t^{-1}s$ commutes with a so condition 3 is satisfied. For the vertex a_1 for the pair of composable edges that corresponds to an isomorphism from G_τ to G_{a_1} we put $g_{e,f} = 1$ and for the other pair we put $g_{e,f} = t$. Similarly for the vertex a_2 for the pair of composable edges that corresponds to an isomorphism from G_τ to G_{a_2} we put $g_{e,f} = 1$ and for the other pair we put $g_{e,f} = t$. Finally for the vertex a_{12} for one pair we put $g_{e,f} = t$ and for the other $g_{e,f} = s$.

It is now a straightforward computation to see that this complex is developable. In fact to show this here it is enough to show that links of vertices do not contain simple closed curves of length 2. Let

v be a vertex of X. The link of v, $Lk\,v$ has as set of vertices the pairs $(g\psi_a(G_{i(e)}), e)$ where $g \in G_v$ and $e \in E(X)$ with $t(e) = v$. The set of edges of the barycentric subdivision of $Lk\,v$ is the set of triples $(g\psi_{ef}(G_{i(f)}), e, f)$ where e, f are composable edges in $E(X)$ with $t(e) = v$. The initial and terminal vertices of an edge are given by:

$$i(g\psi_{ef}(G_{i(f)}), e, f) = (g\psi_{ef}(G_{i(f)}), ef)$$
$$i(g\psi_{ef}(G_{i(f)}), e, f) = (gg_{e,f}^{-1}\psi_e(G_{i(e)}), e)$$

Our choice of twisting elements now insures that there are no curves of length 2 in the link. To see this notice that if e.g. one assigns $g_{e,f} = 1$ to all pairs of composable edges in $E(X)$ with $t(e) = b$ then condition 3 of the definition of the complex of groups is satisfied but now the link has a simple closed curve of length 2. Our choice of non-trivial twisting element insures that there are no length 2 curves in the link.

Let's denote by G the fundamental group of $G(X)$. We remark that the two simple closed curves perpendicular at the midpoints of e_1, e_2 give rise to two small splittings of G over $BS(1,2) = \langle x, y | xyx^{-1} = y^2 \rangle$. Note also that this pair of splittings is hyperbolic-hyperbolic.

We claim that this complex gives the JSJ-decomposition of G over small groups. Let $\tilde{X}$ be a complex on which G acts with quotient complex of groups $G(X)$. Let T be the Bass-Serre tree of a splitting of G over a small group C. Then all vertex stabilizers of $\tilde{X}$ fix vertices of T. This implies that there is a G -equivariant map $f : \tilde{X} \to T$. The preimage of a midpoint of an edge of T is a graph (a tree) in $\tilde{X}$ projecting to an essential simple closed curve on X corresponding to a splitting of G over a conjugate of C. In other words this complex gives us a JSJ-decomposition for G. Now one of the edge groups of this decomposition has the presentation $\langle s, a | sa^2s^{-1} = a^2 \rangle$ (it is the edge corresponding to the vertex lying in both e_1, e_2), which clearly is not a small group. We remark that 2 edge groups are labelled by $BS(1,2)$ and one by $\mathbb{Z} \times \mathbb{Z}$, so they are small.

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E-mail address, Koji Fujiwara: `fujiwara@math.tohoku.ac.jp`
E-mail address, Panos Papasoglu: `panos@math.uoa.gr`

(Koji Fujiwara) Mathematical Institute, Tohoku University. Sendai, 980-8578, Japan

(Panos Papasoglu) Mathematics Department, University of Athens, Athens 157 84, Greece