New Lyapunov-like functional of 1D quasilinear Keller-Segel system and its application

Kentarou Fujie (Tokyo University of Science)

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Joint work with
Bartosz Bieganowski (Nicolaus Copernicus University),
Tomasz Cieślak (Polish Academy of Sciences),
Takasi Senba (Fukuoka University).

Problem

Consider the following 1D quasilinear fully parabolic Keller–Segel system (KS):

$$
\text{(KS)} \qquad \begin{cases} \partial_t u = \partial_x \left(a(u)\partial_x u - u\partial_x v \right) & \text{in } (0,T) \times (0,1), \\ \partial_t v = \partial_x^2 v - v + u & \text{in } (0,T) \times (0,1), \\ \partial_x u = \partial_x v = 0 & \text{in } (0,T) \times \{0,1\}, \\ u(0,\cdot) = u_0, \quad v(0,\cdot) = v_0, & \text{in } (0,1), \end{cases}
$$

where

- $a \in C^2(0,\infty) \cap C[0,\infty)$ is a positive function (Typical example: $a(u) = (1+u)^p$ $(p \in \mathbb{R})$),
- $0 \le u_0 \in C^1[0,1]$, $0 \le v_0 \in C^1[0,1]$.

Known: Local existence of classical solution, positivity, mass conservation.

Known results ($n \geq 2$) $\quad u_t = \nabla \cdot (a(u)\nabla u - u\nabla v)$

Case of $a(u) = (1+u)^p$ ($p \in \mathbb{R}$):

subcritical $p > 1 - \frac{2}{n}$	critical $p = 1 - \frac{2}{n}$		supercritical $p < 1 - \frac{2}{n}$
	small data	large data	
GE&bdd [1]	GE&bdd [2, 3]	NGE* [4]	NGE [5]

GE&bdd : global-in-time solution & $\|u(t)\|_{L^\infty}$ is uniformly bounded,

NGE : a finite-time blowup solution exists

[1]: T. Senba, T. Suzuki: Abstr. Appl. Anal. (2006)
[2]: T. Nagai, T. Senba, K. Yoshida: Funkc. Ekvacioj, Ser. Int. (1997)
[3]: A. Blanchet, Ph. Laurençot: Comm. Partial Differential Equations (2013)
[4]: Ph. Laurençot, N. Mizoguchi: Ann. Inst. H. Poincaré Anal. Non Linéaire (2017)
[5]: T. Cieślak, C. Stinner: J. Differential Equations (2012)

Known results

In the one-dimensional setting:

$p > -1$	$p = -1$		$p < -1$
	small data	large data	
GE&bdd [6]	GE&bdd [6]		NGE* [7]

GE&bdd : global-in-time solution & $\|u(t)\|_{L^\infty}$ is uniformly bounded,

NGE : a finite-time blowup solution exists

[6]: J. Burczak, T. Cieślak, C. Morales-Rodrigo: Nonlinear Anal. (2012)
[7]: T. Cieślak, Ph. Laurençot: Ann. Inst. H. Poincaré Anal. Non Linéaire (2010)

Main result

Theorem (Cieślak-F. (Proc. AMS 2018), Bieganowski-Cieślak-F.-Senba (ArXiv))

$a(u) = 1/(1+u)$. (KS) *has a unique classical positive solution* (u, v), *which exists globally-in-time. Moreover the solution remains uniformly bounded:*

$$\sup_{t>0} \|u(t)\|_{L^\infty(0,1)} < \infty.$$

$p > -1$	$p = -1$		$p < -1$
	small data	large data	
GE&bdd [6]	GE&bdd [6]	**GE&bdd**	NGE [7]

Remark

No mass critical phenomenon occurs in the one-dimensional setting.

Main result (general nonlinearity a)

Theorem (Cieślak-F. (Proc. AMS 2018), Bieganowski-Cieślak-F.-Senba (ArXiv))

Assume that the positive function $a \in C[0,\infty) \cap C^2(0,\infty)$ satisfies:

(A1) $\exists \alpha > 0 \; \forall s \geq 0; \; sa(s) \leq \alpha,$

(A2) $\displaystyle\int_1^\infty a(s)\,ds = \infty$, *i.e.* $a \notin L^1(1,\infty)$.

Then (KS) *has a unique positive classical solution* (u,v), *which exists globally. Moreover the solution remains bounded:* $\sup_{t>0} \|u(t)\|_{L^\infty(0,1)} < \infty$.

Remark

Typical example:

$$a(u) = (1+u)^{-1}(\log(1+u))^{-\alpha}, \quad \alpha \in [0,1].$$

Remark on Lyapunov functional

The following functional $L(u, v) := \int_\Omega b(u) - \int_\Omega uv + 1/2 \|v\|^2_{H^1(\Omega)}$ satisfies

$$\frac{d}{dt} L(u(t), v(t)) + \int_\Omega v_t^2 + \int_\Omega u |\nabla (b'(u) - v)|^2 = 0,$$

where $b \in C^2(0, \infty)$ is such that $b''(r) = \frac{a(r)}{r}$ and $b(1) = b'(1) = 0$.

Example $n = 2, a(u) = 1$: $L(u, v) = \int_\Omega u \log u - \int_\Omega uv + 1/2 \|v\|^2_{H^1(\Omega)}$

$\Longrightarrow$ The Trudinger-Moser inequality: $\|u_0\|_{L^1} < 4\pi \to$ GE&bdd.

If $n = 1, a(u) = 1/u$: $L(u, v) = \int_0^1 (u - \log u - 1) - \int_0^1 uv + 1/2 \|v\|^2_{H^1(0,1)}$

$\Longrightarrow v \in L^\infty(0, \infty; H^1(0, 1))$ & $v_t \in L^2(0, \infty; L^2(0, 1))$.

Lyapunov functional has poor information

Motivation 1

T. Cieślak, Ph. Laurençot (DCDS 2010) considered :

$$\begin{cases} \partial_t u = \partial_x \left(a(u)\partial_x u - u\partial_x v \right) & \text{in } (0,T)\times(0,1), \\ 0 = \partial_x^2 v - M + u, \ \ \int_0^1 v\,dx = 0, & \text{in } (0,T)\times(0,1), \end{cases}$$

where $M = \int_0^1 u_0$.

Introduced the peculiar change of variables:
$U(t,x) := \int_0^x u(t,z)\,dz$ $(t \in [0,T), x \in [0,1])$. For any $t \in [0,T)$, write the inverse function of $x \mapsto U(t,x); [0,1] \to [0,M]$ as $y \mapsto F(t,y)$. Then the function

$$f(t,y) := \partial_y F(t,y)$$

satisfies the equation: $\partial_t f = \partial_y^2 \Psi(f) - 1 + Mf$ $(\Psi = \Psi(a))$.

This parabolic equation has a Lyapunov functional $\Rightarrow$ GE&bdd

Motivation 2

the Keller–Segel system	$\Longrightarrow$	the simplified system
$\Downarrow$		$\Downarrow$
$\Downarrow$		some parabolic equation
$\Downarrow$	correspond?	$\Downarrow$
? ? ? ?	$\Longleftarrow$	Lyapunov functional

hidden & new mathematical structure?

NEW mathematical structure (Lyapunov-like functional)

(KS) satisfies the following

$$\frac{d}{dt}\mathcal{F}(u(t)) + \mathcal{D}(u(t), v(t)) = \mathcal{G}(u(t), v(t)),$$

where,

$$\begin{aligned}
\mathcal{F}(u(t)) &:= \frac{1}{2}\int_0^1 \frac{(a(u))^2}{u}|\partial_x u|^2 - \int_0^1 u \int_1^u a(r)\,dr, \\
\mathcal{D}(u(t), v(t)) &:= \int_0^1 ua(u)\left|\partial_x\left(\frac{a(u)}{u}\partial_x u\right) - \partial_x^2 v + \frac{(v+\partial_t v)}{2}\right|^2, \\
\mathcal{G}(u(t), v(t)) &:= \int_0^1 ua(u)\cdot\frac{(v+\partial_t v)^2}{4}.
\end{aligned}$$

Derivation of Lyapunov-like functional

Lemma

Let $\phi \in C^3(0,1)$*. Then the following identity holds:*

$$\phi \partial_x \mathcal{M}(\phi) = \partial_x \left(\phi a(\phi) \partial_x \left(\frac{a(\phi)}{\phi} \partial_x \phi \right) \right),$$

where

$$\mathcal{M}(\phi) := \frac{a(\phi)a'(\phi)}{\phi} |\partial_x \phi|^2 - \frac{(a(\phi))^2}{2\phi^2} |\partial_x \phi|^2 + \frac{(a(\phi))^2}{\phi} \partial_x^2 \phi.$$

Multiplying the first equation of (KS) by $\mathcal{M}(u)$ and integrating over $(0,1)$:

$$\begin{aligned}\int_0^1 \partial_t u \mathcal{M}(u) &= \int_0^1 \partial_x \left(a(u)\partial_x u - u\partial_x v\right) \mathcal{M}(u) \\ &= \int_0^1 \partial_x \left(u \left(\frac{a(u)}{u} \partial_x u - \partial_x v \right)\right) \mathcal{M}(u).\end{aligned}$$

By the integration by parts and Lemma, it follows that

$$\begin{aligned}\int_0^1 \partial_t u \mathcal{M}(u) &= -\int_0^1 \left(\frac{a(u)}{u}\partial_x u - \partial_x v \right) \cdot u\partial_x \mathcal{M}(u) \\ &= -\int_0^1 \left(\frac{a(u)}{u}\partial_x u - \partial_x v \right) \cdot \partial_x \left(u a(u) \partial_x \left(\frac{a(u)}{u} \partial_x u \right)\right) \\ &= \int_0^1 \partial_x \left(\frac{a(u)}{u}\partial_x u - \partial_x v \right) \cdot \left(u a(u) \partial_x \left(\frac{a(u)}{u} \partial_x u \right)\right) \\ &= \int_0^1 u a(u) \left| \partial_x \left(\frac{a(u)}{u} \partial_x u \right) \right|^2 - \int_0^1 u a(u) \partial_x^2 v \cdot \partial_x \left(\frac{a(u)}{u} \partial_x u \right).\end{aligned}$$

NEW mathematical structure (Lyapunov-like functional)

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$$\frac{d}{dt}\mathcal{F}(u(t)) + \mathcal{D}(u(t), v(t)) = \mathcal{G}(u(t), v(t)),$$

where,

$$\begin{aligned}
\mathcal{F}(u(t)) \quad &:= \quad \frac{1}{2}\int_0^1 \frac{(a(u))^2}{u}|\partial_x u|^2 - \int_0^1 u \int_1^u a(r)\,dr, \\
\mathcal{D}(u(t), v(t)) \quad &:= \quad \int_0^1 ua(u)\left|\partial_x\left(\frac{a(u)}{u}\partial_x u\right) - \partial_x^2 v + \frac{(v+\partial_t v)}{2}\right|^2, \\
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\end{aligned}$$

AIM: Control the growth term $\mathcal{G}(u(t), v(t))$

Sketch of Proof: Global existence

Growth term

$$\mathcal{G}(u(t), v(t)) = \int_0^1 \frac{ua(u)(v + \partial_t v)^2}{4}$$

Recall: $ua(u) \leq C$ & $v \in L^\infty(0,\infty; H^1(0,1))$ & $\partial_t v \in L^2(0,\infty; L^2(0,1))$ from Classical Lyapunov functional. We have for $T > t_0 > 0$,

$$\mathcal{F}(u(T)) \leq \mathcal{F}(u(t_0)) + \int_{t_0}^T \mathcal{G}(u(t), v(t))\, dt \leq C(T).$$

Sketch of Proof: Global existence

Step 1: $\mathcal{F}(u(t)) = \frac{1}{2}\int_0^1 \frac{(a(u))^2}{u}|\partial_x u|^2 - \int_0^1 u \int_1^u a(r)\,dr \leq C(T)$

Step 2: Sobolev's embedding & the Cauchy–Schwarz inequality imply

$$-\int_0^1 u \int_1^u a(r)\,dr\,dx \geq -c\|u_0\|_{L^1(0,1)}^2 - \|u_0\|_{L^1(0,1)}^{3/2}\left(\int_0^1 \frac{(a(u))^2}{u}|\partial_x u|^2\,dx\right)^{1/2}.$$

$$\Longrightarrow \int_0^1 \frac{(a(u))^2}{u}|\partial_x u|^2 \leq C(T)$$

New!! Improved Regularity estimate

Step 2: $\displaystyle\int_0^1 \frac{(a(u))^2}{u}|\partial_x u|^2 \le C(T)$

Step 3: The Cauchy-Schwarz inequality implies

$$\int_0^1 \left| \left(\int_1^u a(s)ds \right)_x \right| dx \le \left(\int_0^1 \frac{a^2(u)|\partial_x u|^2}{u}\, dx \right)^{1/2} \left(\int_0^1 u\, dx \right)^{1/2},$$

Step 2 and mass conservation infer $\| \int_1^{u(t,\cdot)} a(s)ds \|_{W^{1,1}(0,1)} \le C$. From the Sobolev embedding we see

$$\left| \int_1^{u(t,x)} a(s)\, ds \right| \le C \qquad \text{for all } t > 0,\ x \in \Omega.$$

Combining the above with $a \notin L^1(1,\infty)$, we attain estimates on $\|u\|_{L^\infty(0,1)}$.

How to get time-independent estimate?

$$\frac{d}{dt}\mathcal{F}(u(t)) \leq \mathcal{G}(u(t), v(t)) = \int_0^1 ua(u) \cdot \frac{(v + \partial_t v)^2}{4}.$$

We know $ua(u) \leq C$ & $v \in L^\infty(0, \infty; H^1(0, 1))$ & $\partial_t v \in L^2(0, \infty; L^2(0, 1))$.

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Idea: add $\mathcal{F}(u)$

$$\frac{d}{dt}\mathcal{F}(u) + \mathcal{F}(u) \le \mathcal{G}(u, v) + \mathcal{F}(u).$$

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Then we have

$$\mathcal{F}(u(t)) \leq \mathcal{F}(u(t_0)) + \int_{t_0}^t e^{s-t} \int_0^1 \frac{ua(u)(v + \partial_t v)^2}{4}\, dxds + \int_{t_0}^t e^{s-t}\mathcal{F}(u)\, ds.$$

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Recall the classical Lyapunov functional:

$$\frac{d}{dt}L(u(t),v(t)) + \int_\Omega v_t^2 + \int_\Omega u|\nabla(b'(u)-v)|^2 = 0,$$

where $b \in C^2(0,\infty)$ is such that $b''(r) = \frac{a(r)}{r}$ and $b(1) = b'(1) = 0$.

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Discussion: What we did? What is it?

Lyapunov-like functional:

$$\frac{d}{dt}\mathcal{F}(u(t)) + \mathcal{D}(u(t), v(t)) = \mathcal{G}(u(t), v(t)),$$

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The classical Lyapunov functional:

$$\frac{d}{dt}L(u(t), v(t)) + \int_\Omega v_t^2 + \int_\Omega u|\nabla(b'(u) - v)|^2 = 0.$$

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We calculated the SECOND derivative of the Lyapunov functional!

Thank you for your kind attention!