A generalization of the Keller—Segel system to higher dimensions from a structural viewpoint

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The Kelller-Segel system

$$(\mathrm{KS}) \begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) \\ v_t = \Delta v - v + u \end{cases}$$

Keller-Segel (1970)

Chemotaxis:

- Cells produce a signal substance.
- Cells move towards higher concentrations of a signal substance.

Purpose of my study

There are many variants:

logistic source, sensitivity function, nonlinear diffusion, nonlinear production term, coupled with fluid equation, two species, et al.

Construct a generalization of the Keller-Segel system from a structural view point!

Problem

Consider two-chemical substances chemotaxis system:

$$(\mathbf{P}) \begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) & \text{ in } \Omega \times (0, \infty), \\ v_t = \Delta v - v + w & \text{ in } \Omega \times (0, \infty), \\ w_t = \Delta w - w + u & \text{ in } \Omega \times (0, \infty), \end{cases}$$

Background

Attraction-repulsion chemotaxis system:

$$(ext{ATC}) \left\{ egin{array}{ll} u_t = \Delta u - \chi
abla \cdot (u
abla z) + \xi
abla \cdot (u
abla w) \ z_t = \Delta z - eta z + lpha u \ w_t = \Delta w - \delta w + \gamma u \end{array}
ight.$$

Luca et al. (2003)

If
$$\chi = \xi$$
, $\alpha = \gamma = 1$, $\beta = 1$, $\delta = 2$, the triplet
($u, z - w, w$) corresponds with solutions to (P).

Review: (KS)

$$(\mathrm{KS})' \begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) \\ v_t = \Delta v + u \end{cases}$$

• Mass conservation:

$$\|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \quad t > 0$$

• Scaling invariance

$$egin{aligned} &u_\lambda(x,t):=\lambda^2 u(\lambda x,\,\lambda^2 t)\ &v_\lambda(x,t):=v(\lambda x,\,\lambda^2 t) \end{aligned}$$

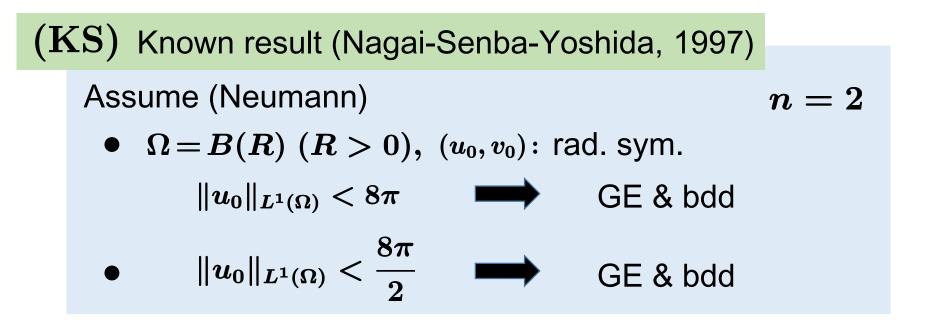
$$\|u_\lambda(t)\|_{L^1}=\lambda^2\int u(x,t)\lambda^{-n}\,dx=\lambda^{2-n}\|u_0(t)\|_{L^1}$$

Lyapunov functional

$$egin{aligned} &\mathcal{KS} ig) \ &rac{d}{dt}\mathcal{F}(u,v)+\mathcal{D}(u,v)=0, \ &\mathcal{F}(u,v)=\int_\Omega u\log u-\int_\Omega uv+rac{1}{2}\int_\Omega v^2+rac{1}{2}\int_\Omega |
abla v|^2, \ &\mathcal{D}(u,v)=\int_\Omega u|
abla(\log u-v)|^2. \end{aligned}$$

(KS) Trudinger-Moser type inequality (Chang-Yang, 1988)

$$egin{aligned} &\log\left(\int_{\Omega}e^{|v|}
ight) \leq rac{1}{2\cdot M_{TM}}\|
abla v\|_{L^{2}(\Omega)}^{2}+C(\|v\|_{L^{1}(\Omega)}) & v\in H^{1}(\Omega), \ &M_{TM}=&\left\{egin{aligned} &8\pi & ext{for rad. function}\ &rac{8\pi}{2} & ext{for arbitrary function} & n=2 \end{array}
ight. \end{aligned}$$



Remarks

- n = 1 GE & bdd for all initial data. Osaki-Yagi (2001)
- n=2 There is an unbounded solution with

$$\|u_0\|_{L^1(\Omega)} \ge 8\pi ext{ or } rac{8\pi}{2}.$$
 Horstmann (2002) etc.

Problem

$$(\mathbf{P}) \begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) & \text{ in } \Omega \times (0, \infty), \\ v_t = \Delta v - v + w & \text{ in } \Omega \times (0, \infty), \\ w_t = \Delta w - w + u & \text{ in } \Omega \times (0, \infty), \end{cases}$$

- $\Omega \subset \mathbb{R}^4$: bounded domain with smooth $\partial \Omega$,
- One of the following boundary conditions:

(Neumann)
$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0$$
 on $\partial \Omega \times (0, \infty)$,
(Mix) $\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} = v = w = 0$ on $\partial \Omega \times (0, \infty)$,
 $u(\cdot, 0) = u_0, v(\cdot, 0) = v_0, w(\cdot, 0) = w_0$ in Ω .
(Nonnegative smooth data) 9/23

Observation

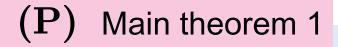
$$(\mathbf{P})' \begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) \\ v_t = \Delta v + w \\ w_t = \Delta w + u \end{cases}$$

• Mass conservation:

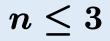
 $\|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \quad t > 0$

• Scaling invariance

$$egin{aligned} & \left\{egin{aligned} & u_\lambda(x,t):=\lambda^4 u(\lambda x,\,\lambda^2 t)\ & v_\lambda(x,t):=v(\lambda x,\,\lambda^2 t)\ & w_\lambda(x,t):=\lambda^2 w(\lambda x,\,\lambda^2 t)\ & \|u_\lambda(t)\|_{L^1}=\lambda^4\int u(x,t)\lambda^{-n}\,dx=\lambda^{4-n}\|u_0(t)\|_{L^1} \end{aligned}
ight.$$



Assume (Neumann) or (Mix)

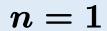


GE & bdd for all initial data.

Strategy of Proof: smoothing effect of semigroup.

(KS) Known result (Osaki-Yagi, 2001)

Assume (Neumann)



GE & bdd for all initial data.

Assume (Neumann)

$$n=4$$

n=2

•
$$\Omega = B(R) \ (R > 0), \ (u_0, v_0, w_0)$$
: rad. sym.

$$\|u_0\|_{L^1(\Omega)} < (8\pi)^2 \quad \longrightarrow \quad \mathsf{GE \& bdd}$$

(KS) Known result (Nagai-Senba-Yoshida, 1997)

Assume (Neumann)

• $\Omega = B(R) \ (R > 0), \ (u_0, v_0): \text{ rad. sym.}$

$$\|u_0\|_{L^1(\Omega)} < 8\pi$$
 \implies GE & bdd
 $\|u_0\|_{L^1(\Omega)} < rac{8\pi}{2}$ \implies GE & bdd

(P) Main theorem 3
Assume (Mix).

$$n = 4$$

$$\|u_0\|_{L^1(\Omega)} < (8\pi)^2 \longrightarrow GE \& bdd$$
(KS) Known result (Nagai-Senba-Yoshida, 1997)
Assume (Neumann)

$$\Omega = B(R) \ (R > 0), \ (u_0, v_0) : rad. sym.$$

$$\|u_0\|_{L^1(\Omega)} < 8\pi \qquad \longrightarrow GE \& bdd$$

$$\|u_0\|_{L^1(\Omega)} < \frac{8\pi}{2} \qquad \longrightarrow GE \& bdd$$

(P) VS.(KS) (Lyapunov functional)

$$egin{aligned} egin{aligned} &egin{aligned} &rac{d}{dt}\mathcal{F}(u,v)+\mathcal{D}(u,v)=0,\ &\mathcal{F}(u,v)=\int_{\Omega}u\log u-\int_{\Omega}uv+rac{1}{2}\int_{\Omega}|v_t|^2+rac{1}{2}\int_{\Omega}|(-\Delta+1)v|^2,\ &\mathcal{D}(u,v)=2\int_{\Omega}(|v_t|^2+|
abla v_t|^2)+\int_{\Omega}u|
abla(\log u-v)|^2. \end{aligned}$$

$$egin{aligned} &(\mathrm{KS})\ &rac{d}{dt}\mathcal{F}(u,v)+\mathcal{D}(u,v)=0,\ &\mathcal{F}(u,v)=\int_{\Omega}u\log u-\int_{\Omega}uv+rac{1}{2}\int_{\Omega}v^{2}+rac{1}{2}\int_{\Omega}|
abla v|^{2},\ &\mathcal{D}(u,v)=\int_{\Omega}u|
abla(\log u-v)|^{2}. \end{aligned}$$

14/23

(P) VS.(KS) (Key inequality)

 (\mathbf{P}) Adams type inequality (Ruf-Sani, 2013 and Tarsi, 2012)

$$\log\left(\int_{\Omega} e^{|v|}\right) \leq \frac{1}{2 \cdot (8\pi)^2} \|(-\Delta + 1)v\|_{L^2(\Omega)}^2 + C,$$

for $v \in H^2_0(\Omega)$ or radial functions $v \in H^2(\Omega)$. n=4

(KS) Trudinger-Moser type inequality (Chang-Yang, 1988)

$$\log\left(\int_{\Omega} e^{|v|}
ight) \leq rac{1}{2 \cdot M_{TM}} \|
abla v\|_{L^{2}(\Omega)}^{2} + C(\|v\|_{L^{1}(\Omega)}) \qquad v \in H^{1}(\Omega),$$
 $M_{TM} = egin{cases} 8\pi & ext{for rad. function} \ rac{8\pi}{2} & ext{for arbitrary function} \end{cases} n = 2$

Regularity estimate

Testing the first equation by pu^{p-1} (p > 1):

$$rac{d}{dt}\int_{\Omega}u^p=-rac{4(p-1)}{p}\int_{\Omega}|
abla u^rac{p}{2}|^2+p(p-1)\int_{\Omega}u^{p-1}
abla u\cdot
abla v.$$

Since Young's inequality implies

$$p(p-1)\int_\Omega u^{p-1}
abla u\cdot
abla v\leq rac{2(p-1)}{p}\int_\Omega |
abla u^rac{p}{2}|^2+rac{p(p-1)}{2}\int_\Omega u^p|
abla v|^2,$$

It follows that

$$rac{d}{dt}\int_{\Omega}u^p+rac{2(p-1)}{p}\int_{\Omega}|
abla u^rac{p}{2}|^2\leq rac{p(p-1)}{2}\int_{\Omega}u^p|
abla v|^2.$$

16/23

In the study of (KS)

$$rac{d}{dt}\int_{\Omega}u^p+rac{2(p-1)}{p}\int_{\Omega}|
abla u^rac{p}{2}|^2\leq rac{p(p-1)}{2}\int_{\Omega}u^p|
abla v|^2.$$

Maximal regularity theory implies

$$|\int_0^T \int_\Omega u^p |
abla v|^2 \leq C \int_0^T \int_\Omega u^{p+1}.$$

Apply the Sobolev inequality in 2D $\|f\|_{L^2} \leq K \|f\|_{W^{1,1}}$:

$$\int_\Omega u^{p+1} \leq C \int_\Omega u \cdot \int_\Omega |
abla u^{rac{p}{2}}|^2 + C(\|u_0\|_{L^1}).$$

Combine the above with the next inequality:

$$\int_{\Omega} u \leq \frac{1}{\log s} \int_{\Omega} (u \log u + e^{-1}) + C(s).$$

New approach

$$rac{d}{dt}\int_{\Omega}u^p+rac{2(p-1)}{p}\int_{\Omega}|
abla u^rac{p}{2}|^2\leq rac{p(p-1)}{2}\int_{\Omega}u^p|
abla v|^2.$$

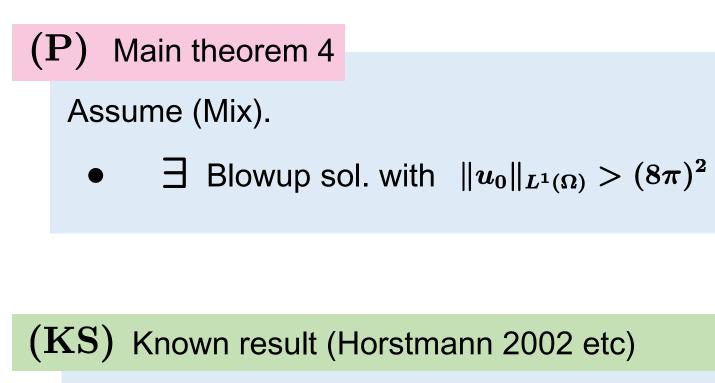
Apply the Cauchy-Schwarz inequality:

$$rac{d}{dt}\int_\Omega u^p+rac{2(p-1)}{p}\int_\Omega |
abla u^rac{p}{2}|^2\leq rac{p(p-1)}{2}\left(\int_\Omega u^{2p}
ight)^rac{1}{2}\left(\int_\Omega |
abla v|^4
ight)^rac{1}{2}.$$

By the Sobolev inequality $\|f\|_{L^4} \leq K \|f\|_{H^1}$:

$$rac{d}{dt}\int_{\Omega}u^p+rac{2(p-1)}{p}\int_{\Omega}|
abla u^{rac{p}{2}}|^2\leq C\left(\int_{\Omega}|
abla u^{rac{p}{2}}|^2+\int_{\Omega}u^p
ight)\left(\int_{\Omega}|
abla v|^4
ight)^{rac{1}{2}}.$$

We consider local-in-space estimates!



Assume (Neumann)

- $\Omega = B(R) \ (R > 0), \ (u_0, v_0): \text{ rad. sym.}$
 - \exists Blowup sol. with $||u_0||_{L^1(\Omega)} > 8\pi$
- \exists Blowup sol. with $||u_0||_{L^1(\Omega)} > \frac{8\pi}{2}$

n = 4

n=2

Outline of proof (use the method in Horstmann (2002))

Consider the stationary problem:

$$\left\{egin{array}{ll} 0=\Delta u-
abla\cdot(u
abla v)& ext{ in }\Omega,\ 0=\Delta v-v+w& ext{ in }\Omega,\ 0=\Delta w-w+u& ext{ in }\Omega,\ rac{\partial u}{\partial
u}=rac{\partial v}{\partial
u}=rac{\partial w}{\partial
u}=rac{\partial w}{\partial
u}=0& ext{ on }\partial\Omega. \end{array}
ight.$$

$$egin{aligned} 0 &= & \Delta u -
abla \cdot (u
abla v) \ &= & \nabla \cdot u
abla (\log u - v) \end{aligned} egin{aligned} egin{aligned} egin{aligned} egin{aligned} &= & & U = C e^v \ &= & & & U = C e^v \end{aligned}$$

Outline of Proof

Putting $\Lambda = \|u_0\|_{L^1(\Omega)}$, the system is written as

$$(S) \begin{cases} (-\Delta + 1)^2 v = \frac{\Lambda}{\int_{\Omega} e^v} e^v & \text{in } \Omega, \\ u = \frac{\Lambda}{\int_{\Omega} e^v} e^v & \text{in } \Omega, \\ w = -\Delta v + v & \text{in } \Omega, \\ v = \Delta v = 0 & \text{on } \partial \Omega. \end{cases}$$

 $\mathcal{S}(\Lambda):=\{(u,v,w)\,|\,(u,v,w) ext{ satisfies } (\mathbf{S}) ext{ with } \|u\|_{L^1(\Omega)}=\Lambda\}$

21/23

Outline of Proof

 $\begin{aligned} & \text{Proposition} \\ & \Lambda \in (0,\infty) \setminus \{(8\pi)^2\} \mathbb{N} \\ & \implies \text{There exists } C > 0 \text{ such that} \\ & \sup\{\|(u,v,w)\|_{L^{\infty}(\Omega)} \mid (u,v,w) \in \mathcal{S}(\Lambda)\} \leq C, \\ & F_*(\Lambda) := \inf\{\mathcal{F}(u,v,w) \mid (u,v,w) \in \mathcal{S}(\Lambda)\} \geq -C. \end{aligned}$

We construct nonnegative functions (u_0, v_0, w_0) such that

 $\mathcal{F}(u_0,v_0,w_0) < F_*(\Lambda) ext{ and } \Lambda \in ((8\pi)^2,\infty) \setminus \{(8\pi)^2\}\mathbb{N}.$

Summary

- We consider two-chemical substances chemotaxis system.
- We construct Lyapunov functional.
- We use the Adams type inequality (instead of T-M ineq.).
- The critical constant changes to $(8\pi)^2$ from 8π .

Our system is regarded as a generalization of the Keller-Segel system from a structural view point!