A generalization of the Keller—Segel system to higher dimensions from a structural viewpoint

Joint work with
Takasi Senba (Fukuoka University)

Kentarou FUJIE (Tokyo University of Science)

Equadiff 2017 (MS13)

The Kellller-Segel system

$$
(\mathbf{KS})\begin{cases} \boldsymbol{u_t = \Delta u - \nabla \cdot (u \nabla v)} \\ \boldsymbol{v_t = \Delta v - v + u} \end{cases}
$$

Keller-Segel (1970)

Chemotaxis:

- Cells produce a signal substance.
- Cells move towards higher concentrations of a signal substance.

Purpose of my study

There are many variants:
logistic source, sensitivity function, nonlinear diffusion, nonlinear production term, coupled with fluid equation, two species, et al.

Construct a generalization of the Keller-Segel system from a structural view point!

Problem

Consider two-chemical substances chemotaxis system:

$$
\text{(P)} \begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) & \text{in } \Omega \times (0,\infty), \\ v_t = \Delta v - v + w & \text{in } \Omega \times (0,\infty), \\ w_t = \Delta w - w + u & \text{in } \Omega \times (0,\infty), \end{cases}
$$

Background

Attraction-repulsion chemotaxis system:

$$(\mathbf{ATC})\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla z) + \xi \nabla \cdot (u \nabla w) \\ z_t = \Delta z - \beta z + \alpha u \\ w_t = \Delta w - \delta w + \gamma u \end{cases}$$

Luca et al. (2003)

If $\chi = \xi$, $\alpha = \gamma = 1$, $\beta = 1$, $\delta = 2$, the triplet $(u, z - w, w)$ corresponds with solutions to (P).

Review: $(\mathbf{KS})$

$$(\mathbf{KS})' \begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) \\ v_t = \Delta v + u \end{cases}$$

- Mass conservation:

$$\|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \quad t > 0$$

- Scaling invariance

$$\begin{cases} u_\lambda(x,t) := \lambda^2 u(\lambda x,\, \lambda^2 t) \\ v_\lambda(x,t) := v(\lambda x,\, \lambda^2 t) \end{cases}$$

$$\|u_\lambda(t)\|_{L^1} = \lambda^2 \int u(x,t)\lambda^{-n}\, dx = \lambda^{2-n}\|u_0(t)\|_{L^1}$$

Lyapunov functional

(KS)

$$\frac{d}{dt}\mathcal{F}(u,v) + \mathcal{D}(u,v) = 0,$$

$$\mathcal{F}(u,v) = \int_\Omega u \log u - \int_\Omega uv + \frac{1}{2}\int_\Omega v^2 + \frac{1}{2}\int_\Omega |\nabla v|^2,$$

$$\mathcal{D}(u,v) = \int_\Omega u|\nabla(\log u - v)|^2.$$

(KS) Trudinger-Moser type inequality (Chang-Yang, 1988)

$$\log\left(\int_\Omega e^{|v|}\right) \leq \frac{1}{2\cdot M_{TM}}\|\nabla v\|^2_{L^2(\Omega)} + C(\|v\|_{L^1(\Omega)}) \qquad v \in H^1(\Omega),$$

$$M_{TM} = \begin{cases} 8\pi & \text{for rad. function} \\ \frac{8\pi}{2} & \text{for arbitrary function} \end{cases} \qquad n = 2$$

(KS) Known result (Nagai-Senba-Yoshida, 1997)

Assume (Neumann) $n = 2$

- $\Omega = B(R)$ $(R > 0)$, (u_0, v_0): rad. sym.

  $\|u_0\|_{L^1(\Omega)} < 8\pi$ ➡ GE & bdd

- $\|u_0\|_{L^1(\Omega)} < \dfrac{8\pi}{2}$ ➡ GE & bdd

Remarks

- $n = 1$ GE & bdd for all initial data. Osaki-Yagi (2001)

- $n = 2$ There is an unbounded solution with

  $$\|u_0\|_{L^1(\Omega)} \geq 8\pi \text{ or } \frac{8\pi}{2}.$$

  Horstmann (2002) etc.

Problem

$$(\mathrm{P})\ \begin{cases} u_t = \Delta u - \nabla\cdot(u\nabla v) & \text{in } \Omega\times(0,\infty), \\ v_t = \Delta v - v + w & \text{in } \Omega\times(0,\infty), \\ w_t = \Delta w - w + u & \text{in } \Omega\times(0,\infty), \end{cases}$$

- $\Omega \subset \mathbb{R}^4$: bounded domain with smooth $\partial\Omega$,
- One of the following boundary conditions:

(Neumann) $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0$ on $\partial\Omega\times(0,\infty)$,

(Mix) $\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = v = w = 0$ on $\partial\Omega\times(0,\infty)$,

- $u(\cdot,0) = u_0,\ v(\cdot,0) = v_0,\ w(\cdot,0) = w_0$ in Ω.

(Nonnegative smooth data)

Observation

$$
(\mathrm{P})' \begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) \\ v_t = \Delta v + w \\ w_t = \Delta w + u \end{cases}
$$

- Mass conservation:

$$
\|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \quad t > 0
$$

- Scaling invariance

$$
\begin{cases} u_\lambda(x,t) := \lambda^4 u(\lambda x,\, \lambda^2 t) \\ v_\lambda(x,t) := v(\lambda x,\, \lambda^2 t) \\ w_\lambda(x,t) := \lambda^2 w(\lambda x,\, \lambda^2 t) \end{cases}
$$

$$
\|u_\lambda(t)\|_{L^1} = \lambda^4 \int u(x,t) \lambda^{-n} \, dx = \lambda^{4-n} \|u_0(t)\|_{L^1}
$$

$\mathbf{(P)}$ Main theorem 1

Assume (Neumann) or (Mix) $\quad n \leq 3$

- GE & bdd for all initial data.

Strategy of Proof: smoothing effect of semigroup.

$\mathbf{(KS)}$ Known result (Osaki-Yagi, 2001)

Assume (Neumann) $\quad n = 1$

- GE & bdd for all initial data.

$(\mathbf{P})$ Main theorem 2

Assume (Neumann) $\quad n = 4$

- $\Omega = B(R)\ (R > 0)$, (u_0, v_0, w_0): rad. sym.

  $\|u_0\|_{L^1(\Omega)} < (8\pi)^2 \Longrightarrow$ GE & bdd

$(\mathbf{KS})$ Known result (Nagai-Senba-Yoshida, 1997)

Assume (Neumann) $\quad n = 2$

- $\Omega = B(R)\ (R > 0)$, (u_0, v_0): rad. sym.

  $\|u_0\|_{L^1(\Omega)} < 8\pi \Longrightarrow$ GE & bdd

- $\|u_0\|_{L^1(\Omega)} < \frac{8\pi}{2} \Longrightarrow$ GE & bdd

$(\mathbf{P})$ Main theorem 3

Assume (Mix). $n = 4$

- $\|u_0\|_{L^1(\Omega)} < (8\pi)^2$ ➡ GE & bdd

$(\mathbf{KS})$ Known result (Nagai-Senba-Yoshida, 1997)

Assume (Neumann) $n = 2$

- $\Omega = B(R)$ $(R > 0)$, (u_0, v_0): rad. sym.

  $\|u_0\|_{L^1(\Omega)} < 8\pi$ ➡ GE & bdd

- $\|u_0\|_{L^1(\Omega)} < \frac{8\pi}{2}$ ➡ GE & bdd

(P) vs. (KS) (Lyapunov functional)

(P)

$$\frac{d}{dt}\mathcal{F}(u,v)+\mathcal{D}(u,v)=0,$$

$$\mathcal{F}(u,v)=\int_\Omega u\log u-\int_\Omega uv+\frac{1}{2}\int_\Omega |v_t|^2+\frac{1}{2}\int_\Omega |(-\Delta+1)v|^2,$$

$$\mathcal{D}(u,v)=2\int_\Omega(|v_t|^2+|\nabla v_t|^2)+\int_\Omega u|\nabla(\log u-v)|^2.$$

(KS)

$$\frac{d}{dt}\mathcal{F}(u,v)+\mathcal{D}(u,v)=0,$$

$$\mathcal{F}(u,v)=\int_\Omega u\log u-\int_\Omega uv+\frac{1}{2}\int_\Omega v^2+\frac{1}{2}\int_\Omega |\nabla v|^2,$$

$$\mathcal{D}(u,v)=\int_\Omega u|\nabla(\log u-v)|^2.$$

$(\mathbf{P})$ vs. $(\mathbf{KS})$ (Key inequality)

$(\mathbf{P})$ Adams type inequality (Ruf-Sani, 2013 and Tarsi, 2012)

$$\log\left(\int_\Omega e^{|v|}\right) \le \frac{1}{2\cdot(8\pi)^2}\|(-\Delta+1)v\|^2_{L^2(\Omega)} + C,$$

for $v \in H^2_0(\Omega)$ or radial functions $v \in H^2(\Omega)$. $n = 4$

$(\mathbf{KS})$ Trudinger-Moser type inequality (Chang-Yang, 1988)

$$\log\left(\int_\Omega e^{|v|}\right) \le \frac{1}{2\cdot M_{TM}}\|\nabla v\|^2_{L^2(\Omega)} + C(\|v\|_{L^1(\Omega)}) \qquad v \in H^1(\Omega),$$

$$M_{TM} = \begin{cases} 8\pi & \text{for rad. function} \\ \dfrac{8\pi}{2} & \text{for arbitrary function} \end{cases}$$

$n = 2$

Regularity estimate

Testing the first equation by pu^{p-1} $(p>1)$:

$$\frac{d}{dt}\int_\Omega u^p = -\frac{4(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + p(p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla v.$$

Since Young's inequality implies

$$p(p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla v \le \frac{2(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \frac{p(p-1)}{2}\int_\Omega u^p|\nabla v|^2,$$

It follows that

$$\frac{d}{dt}\int_\Omega u^p + \frac{2(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \le \frac{p(p-1)}{2}\int_\Omega u^p|\nabla v|^2.$$

In the study of (KS)

$$\frac{d}{dt}\int_\Omega u^p + \frac{2(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \le \frac{p(p-1)}{2}\int_\Omega u^p|\nabla v|^2.$$

Maximal regularity theory implies $\displaystyle\int_0^T\int_\Omega u^p|\nabla v|^2 \le C\int_0^T\int_\Omega u^{p+1}.$

Apply the Sobolev inequality in 2D $\|f\|_{L^2} \le K\|f\|_{W^{1,1}}$:

$$\int_\Omega u^{p+1} \le C\int_\Omega u \cdot \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + C(\|u_0\|_{L^1}).$$

Combine the above with the next inequality:

$$\int_\Omega u \le \frac{1}{\log s}\int_\Omega (u\log u + e^{-1}) + C(s).$$

New approach

$$\frac{d}{dt}\int_\Omega u^p + \frac{2(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \le \frac{p(p-1)}{2}\int_\Omega u^p|\nabla v|^2.$$

Apply the Cauchy-Schwarz inequality:

$$\frac{d}{dt}\int_\Omega u^p + \frac{2(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \le \frac{p(p-1)}{2}\left(\int_\Omega u^{2p}\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla v|^4\right)^{\frac{1}{2}}.$$

By the Sobolev inequality $\|f\|_{L^4} \le K\|f\|_{H^1}$:

$$\frac{d}{dt}\int_\Omega u^p + \frac{2(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \le C\left(\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \int_\Omega u^p\right)\left(\int_\Omega |\nabla v|^4\right)^{\frac{1}{2}}.$$

We consider local-in-space estimates!

$(\mathbf{P})$ Main theorem 4

Assume (Mix). $\quad n = 4$

- $\exists$ Blowup sol. with $\|u_0\|_{L^1(\Omega)} > (8\pi)^2$

$(\mathbf{KS})$ Known result (Horstmann 2002 etc)

Assume (Neumann) $\quad n = 2$

- $\Omega = B(R)$ $(R > 0)$, (u_0, v_0): rad. sym.

  $\exists$ Blowup sol. with $\|u_0\|_{L^1(\Omega)} > 8\pi$

- $\exists$ Blowup sol. with $\|u_0\|_{L^1(\Omega)} > \dfrac{8\pi}{2}$

Outline of proof (use the method in Horstmann (2002))

Consider the stationary problem:

$$\begin{cases} 0 = \Delta u - \nabla \cdot (u \nabla v) & \text{in } \Omega, \\ 0 = \Delta v - v + w & \text{in } \Omega, \\ 0 = \Delta w - w + u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

$$\begin{aligned} 0 &= \Delta u - \nabla \cdot (u \nabla v) \\ &= \nabla \cdot u \nabla (\log u - v) \end{aligned} \quad \Longrightarrow \quad u = Ce^{v}$$

Outline of Proof

Putting $\boldsymbol{\Lambda} = \|\boldsymbol{u_0}\|_{L^1(\Omega)}$, the system is written as

$$
(\mathrm{S}) \begin{cases} (-\boldsymbol{\Delta} + 1)^2 \boldsymbol{v} = \dfrac{\boldsymbol{\Lambda}}{\int_{\Omega} e^{v}} e^{v} & \textbf{in } \boldsymbol{\Omega}, \\ \boldsymbol{u} = \dfrac{\boldsymbol{\Lambda}}{\int_{\Omega} e^{v}} e^{v} & \textbf{in } \boldsymbol{\Omega}, \\ \boldsymbol{w} = -\boldsymbol{\Delta v} + \boldsymbol{v} & \textbf{in } \boldsymbol{\Omega}, \\ \boldsymbol{v} = \boldsymbol{\Delta v} = 0 & \textbf{on } \boldsymbol{\partial\Omega}. \end{cases}
$$

$$
\mathcal{S}(\Lambda) := \{(u, v, w) \mid (u, v, w) \text{ satisfies } (\mathrm{S}) \text{ with } \|u\|_{L^1(\Omega)} = \Lambda\}
$$

Outline of Proof

Proposition

$$\boldsymbol{\Lambda} \in (\mathbf{0}, \infty) \setminus \{(\mathbf{8}\boldsymbol{\pi})^{\mathbf{2}}\}\mathbb{N}$$

➡ There exists $C > 0$ such that

$$\sup\{\|(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})\|_{L^\infty(\Omega)} \,|\, (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{\mathcal{S}}(\boldsymbol{\Lambda})\} \le \boldsymbol{C},$$

$$\boldsymbol{F}_*(\boldsymbol{\Lambda}) := \inf\{\boldsymbol{\mathcal{F}}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \,|\, (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{\mathcal{S}}(\boldsymbol{\Lambda})\} \ge -\boldsymbol{C}.$$

We construct nonnegative functions $(\boldsymbol{u}_0, \boldsymbol{v}_0, \boldsymbol{w}_0)$ such that

$\boldsymbol{\mathcal{F}}(\boldsymbol{u}_0, \boldsymbol{v}_0, \boldsymbol{w}_0) < \boldsymbol{F}_*(\boldsymbol{\Lambda})$ and $\boldsymbol{\Lambda} \in ((\mathbf{8}\boldsymbol{\pi})^{\mathbf{2}}, \infty) \setminus \{(\mathbf{8}\boldsymbol{\pi})^{\mathbf{2}}\}\mathbb{N}.$

Summary

- We consider two-chemical substances chemotaxis system.
- We construct Lyapunov functional.
- We use the Adams type inequality (instead of T-M ineq.).
- The critical constant changes to $(8\pi)^2$ from 8π.

Our system is regarded as a generalization of the Keller-Segel system from a structural view point!