An introduction to Reverse Mathematics

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The idea is to develop mathematics in a formal setting, and calibrate the strength of mathematical theorems on the base of what we need (and also what is needed) to prove them. The setting is given by 2nd order arithmetic.

Given a mathematical theorem, we find a subsystem $T$ of 2nd order arithmetic in which we can prove it, and we try to prove the axioms of $T$ from the theorem over some base theory (the reversal).
The full system

We have a language that extends the language of Peano arithmetic, and where we can say, for instance, \( n \in X \), where \( n \) is a natural number variable, while \( X \) is a set (of natural numbers) variable.

2nd order arithmetic is simply a theory in that language. More precisely, it’s Peano with the induction scheme replaced by the induction axiom

\[
0 \in X \land (\forall n)(n \in X \rightarrow n + 1 \in X) \rightarrow (\forall n)(n \in X),
\]

and with the following scheme of comprehension

\[
(\exists X)(\forall n)(n \in X \leftrightarrow \varphi(n)).
\]
The base system

By restricting comprehension to $\Delta^0_1$ formulas, and by taking $\Sigma^0_1$ induction scheme instead of induction axiom, we have the base theory of Reverse Mathematics, called $RCA_0$. 
The systems of 2nd order arithmetic

The big five

1. $\text{RCA}_0$
The systems of 2nd order arithmetic

Overview

Partial orderings and Reverse Mathematics

The big five

1. \( \text{RCA}_0 \)
2. \( \text{WKL}_0 \): \( \text{RCA}_0 \) plus Weak Konig’s Lemma. \( \text{WKL} \) says that every infinite tree \( T \subseteq 2^{<\mathbb{N}} \) has a path.
The systems of 2nd order arithmetic

The big five

1. $RCA_0$
2. $WKL_0$: $RCA_0$ plus Weak Konig’s Lemma. $WKL$ says that every infinite tree $T \subseteq 2^{<\mathbb{N}}$ has a path.
3. $ACA_0$: 2nd order arithmetic with comprehension restricted to arithmetical formulas only.
The big five

1. $\text{RCA}_0$
2. $\text{WKL}_0$: $\text{RCA}_0$ plus Weak Konig’s Lemma. $\text{WKL}$ says that every infinite tree $T \subseteq 2^{<\mathbb{N}}$ has a path.
3. $\text{ACA}_0$: 2nd order arithmetic with comprehension restricted to arithmetical formulas only.
4. $\text{ATR}_0$: $\text{ACA}_0$ plus “arithmetical recursion along well orderings”.
5. $\Pi^1_1$-$\text{CA}_0$: 2nd order arithmetic with comprehension for $\Pi^1_1$ formulas.

Other systems are considered too.
Every system is strictly weaker than the next one, but

- $WKL_0$ has the same first order part of $RCA_0$ (i.e. $\Sigma^0_1$-PA), and the same proof-theoretic ordinal $\omega^\omega$.
- $ACA_0$ is conservative over Peano, and they have the same ordinal $\varepsilon_0$.
- $|ATR_0| = \Gamma_0$, the Feferman-Shutte ordinal.
- $|\Pi^1_1-CA_0|$ is very very big....it’s the Bachmann-Howard ordinal.
$RCA_0$

$RCA_0$ corresponds to a formalization of recursive or computable mathematics.
In $RCA_0$ the finite sets of natural numbers can be encoded as single numbers (following Godel), a key result in proving primitive recursion.
Within $RCA_0$ we give the majority of the definitions we need. For instance, we define the real numbers. We also prove some theorems: for example we prove the Baire category theorem for $\mathbb{R}^k$, i.e. “every countable sequence of dense open sets in $\mathbb{R}^k$ has nonempty intersection”. But many theorems are not provable. A simple example is given by the Bolzano/Weierstrass theorem “every bounded sequence of real numbers has a convergent subsequence”.
Reverse Mathematics for $WKL_0$

- Every covering of $[0, 1]$ by a sequence of open intervals has a finite subcovering (Heine/Borel covering lemma).
- Every continuous real-valued function on $[0, 1]$ (or on any compact metric space) has a supremum.
- Every consistent countable set of sentences in the predicate calculus has a countable model (Gödel’s completeness theorem).
- A countable set of sentences is satisfiable iff it is finitely satisfiable (Gödel’s compactness theorem).
- Every countable commutative ring has a prime ideal.
- Every (uniformly) continuous function $f : [0, 1]^n \to [0, 1]^n$ has a fixed point.
Reverse Mathematics for $ACA_0$

- Every bounded sequence of reals numbers has a convergent subsequence (Bolzano/Weierstrass theorem).
- Every bounded equicontinuous sequence of real-valued functions on a bounded interval has a uniformly convergent subsequence (Ascoli lemma)
- Every countable commutative ring has a maximal ideal.
- Every countable vector space has a basis.
- Every countable field of char 0 has a transcendence basis.
- Every infinite, finitely branching tree has a path (Konig’s lemma).
- Every function $f: [\mathbb{N}]^3 \to \{0, 1, \ldots, k - 1\}$ has an homogeneous set (Ramsey’s theorem).
Reverse Mathematics for $\text{ATR}_0$

- Any two countable well-orderings are comparable.
- Any two countable reduced Abelian $p$-groups which have the same Ulm invariants are isomorphic (Ulm’s theorem).
- Every uncountable analytic set has a perfect subset (Perfect Set theorem).
- Any two disjoint analytic sets can be separated by a Borel set (Lusin’s separation theorem)
- Every clopen (or open) game in $\mathbb{N}^\mathbb{N}$ is determined.
- Every clopen (or open) subset of $[\mathbb{N}]^\mathbb{N}$ has the Ramsey property.
Reverse Mathematics for $\Pi^1_1-CA_0$

- Every closed subset of $\mathbb{N}^{\mathbb{N}}$ (or of any complete separable metric space) is the union of a countable set and a perfect set (Cantor-Bendixson theorem).
- Every countable Abelian group is the direct sum of a divisible group and a reduced group.
- Every difference of two open sets in the Baire space $\mathbb{N}^{\mathbb{N}}$ is determined.
- Every $G_\delta$ set in $[\mathbb{N}]^{\mathbb{N}}$ has the Ramsey property.
- For every Borel (or coanalytic, or $F_\sigma$) equivalence relation with uncountably many equivalence classes, there exists a perfect set of inequivalent elements (Silver’s theorem).
Beyond the big five

A few examples:

- A theorem at level $\Pi^1_2$ over $\Pi^1_1$ (Mummert/Simpson)
  “Let $P$ be a partial ordering. Then the topological space
  $\{\text{maximal filters on } P\}$ is completely metrizable iff it’s
  regular”.

- Ramsey’s theorem for pairs is below $ACA_0$. It was known that
  $RT(2) \not\subseteq WKL_0$. Recently, it has been shown that
  $WKL_0 \not\subseteq RT(2)$ (Jiayi Liu).
WQO and BQO theory

A quasi-ordering (reflexive and transitive) $Q$ is a well quasi-ordering (wqo) if it contains no descending infinite sequences and no infinite antichains. Equivalently, for any $\omega$-sequence $\langle x_i : i < \omega \rangle \in Q^\omega$ there exist $i < j$ such that $x_i \leq x_j$.

A better quasi-order (bqo) is something better! The notion was introduced by Nash-Williams to show that some quasi-orderings were actually wqo. The definition uses barriers, particular subsets $B$ of $[\omega]<\omega$, where a relation $\triangleleft$ is defined. Once we have barriers, we say that $Q$ is bqo if for any map $\langle x_s : s \in B \rangle \in Q^B$ there exist $s \triangleleft t$ such that $x_s \leq x_t$. 
Reverse Mathematics of some basic results

The following theorems state the closure of being wqo (bqo) under some finitary or infinitary operation.

- $Q$ wqo implies $Q^{<\omega}$ wqo (Higman’s theorem). We have $\text{RCA}_0 \vdash \text{HT} \iff \text{ACA}_0$.
- $Q$ bqo implies $Q^{<\omega}$ bqo (generalized Higman’s theorem).
- $T = \{\text{finite trees}\}$ wqo (Kruskal’s theorem).
- $Q$ wqo implies $T(Q)$ wqo (generalized Kruskal’s theorem). We have $ATR_0 \not\vdash \text{GKT}$.
- $Q$ bqo implies \{countable sequences of elements of $Q$\} bqo (Nash-Williams theorem). We have $\Pi^1_1\text{-CA}_0 \vdash \text{NWT}$, $ATR_0 \vdash \text{NWT} \iff \text{GHT}$, and $\text{RCA}_0 \vdash \text{NWT} \rightarrow ATR_0$. 
The Fraisse’s conjecture

• $\mathcal{L} = \{\text{countable linear orderings}\}$ wqo (Fraisse’s conjecture).
• $Q$ bqo implies $\mathcal{L}(Q)$ bqo (Laver’s theorem)

The usual proofs of these theorems can be carried out in $\Pi^1_2$-$CA_0$. Moreover, since in general $\Pi^1_2$ statements consistent with $\text{ATR}_0$ do not imply $\Pi^1_1$-$CA_0$, and $FC$ is such a statement, this consideration also applies to $FC$.

The Reverse Mathematics status of $FC$ is still unknown.
A theorem on the ideals of a partial order

The theorem says that “every well partial order \( P \) is a finite union of ideals, and the decomposition is essential”.

We show that this theorem is equivalent to \( ACA_0 \) over \( RCA_0 \).
The systems of 2nd order arithmetic

Overview

Partial orderings and Reverse Mathematics

The proof in $\text{ACA}_0$

- Let $P = \{x_n : n \in \mathbb{N}\}$ be infinite. Note that in $\text{RCA}_0$ infinite sets can be enumerated in strictly increasing order.
- We recursively define $a_{i,n} \in P$ and $w_n \in \mathbb{N}$ for all $n$ and for all $i < w_n$. $w_0 = 0$. Suppose to have $a_{i,n}$ for $i < w_n$ and look at $x_n$....board....
- $I = \{i : (\exists n)(i < w_n)\}$. It’s easy to see that for $i \in I$ the set $A_i = \{p \in P : (\exists n)(p \leq a_{i,n})\}$ is an ideal, and that $P$ is the union of these ideals.
- To simplify, define $b_{i,m}$ for $i \in I$....board....
- Suppose $I = \mathbb{N}$ towards a contradiction.
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Partial orderings and Reverse Mathematics

The proof in ACA₀

- Define \( l : [\mathbb{N}]^2 \to \mathbb{N} \) such that

  \[ l(i, j) = (\text{the least } m)(b_{i,m} \& b_{j,0} \text{ are incompatible}). \]

- Given \( i \), the function \( j \to l(i, j) \) assumes only finitely many values (by using \( P \text{ wpo} \)). Thus define

  \[ g(i) = (\text{the greatest } m)(m = l(i, j) \text{ for some } j > i). \]

- Verify that the \( b_{i,g(i)} \)'s form an infinite antichain.
- Notice that in any case the decomposition is essential.
The reversal

- We reason in $RCA_0$. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a one-to-one function. We will use true and false stages.
- Define $P$ as follows.....board....
- $P$ has no infinite antichains. By the excluded middle we have two cases:
  1. $P$ has a descending $\omega$-sequence.
  2. $P$ has no descending $\omega$-sequences.
- In the first case if $g : \mathbb{N} \rightarrow P$ is such a sequence then $g(i) = a_{n_i}$ with $n_i$ true stage, and $n \mapsto n_i$ is a strictly increasing enumeration of true stages.
The reversal

• In the second case we have that $P$ is a wpo. Apply the theorem, and let $\{A_i : i < k\}$ be an essential finite decomposition of $P$ into ideals.

• Take an ideal, let’s say $A_0$, that contains $b_0$ (there is only one, actually). Prove by induction that $(\forall k)(b_k \in A_0)$. We have that $n$ is a false stage iff $a_n \in A_0$. 