REVERSE MATHEMATICS AND INITIAL INTERVALS

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Abstract. In this paper we study the reverse mathematics of two theorems by Bonnet about partial orders. These results concern the structure and cardinality of the collection of the initial intervals. The first theorem states that a partial order has no infinite antichains if and only if its initial intervals are finite unions of ideals. The second one asserts that a countable partial order is scattered and does not contain infinite antichains if and only if it has countably many initial intervals. We show that the left to right directions of these theorems are equivalent with \textit{ACA}_0 and \textit{ATR}_0, respectively. On the other hand, the opposite directions are both provable in \textit{WKL}_0, but not in \textit{RCA}_0. We also prove the equivalence with \textit{ACA}_0 of the following result of Erdős and Tarski: a partial order with no infinite strong antichains has no arbitrarily large finite strong antichains.

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1. Introduction

In this paper we study from the viewpoint of reverse mathematics some theorems dealing with the structure and the size of the collection of initial intervals (also called downward closed subsets) in a partial order. Recall that an ideal is an initial
interval such that every pair of elements is compatible (i.e. has a common upper bound) in the interval.

The first result is a characterization of partial orders with no infinite antichains in terms of the decomposition of initial intervals into union of ideals. It is due to Bonnet [Bon75, Lemma 2] and can be found in Fraïssé’s monograph [Fra00, §4.7.2]:

**Theorem 1.1.** A partial order has no infinite antichains if and only if every initial interval is a finite union of ideals.

In [PS06] Theorem 1.1 is attributed to Erdős and Tarski because its ‘hard’ (left to right) direction can be deduced quite easily from the following result, which is part of [ET43, Theorem 1]:

**Theorem 1.2.** If a partial order has no infinite strong antichains then it has no arbitrarily large finite strong antichains.

Here, by strong antichain we mean a set of pairwise incompatible (and not only incomparable, as in antichain) elements. (Notice that Erdős and Tarski work with what we would call filters and final intervals.)

An intermediate step between Theorems 1.2 and 1.1 is the following characterization of partial orders with no infinite strong antichains:

**Theorem 1.3.** A partial order has no infinite strong antichains if and only if it is finite union of ideals.

Our proof of Lemma 4.2 shows how to deduce the left to right direction of Theorem 1.3 from Theorem 1.2.

In [Bon75] Theorem 1.1 is a step in the proof of the following result, which is also featured in Fraïssé’s monograph [Fra00, §6.7]:

**Theorem 1.4.** If an infinite partial order \( P \) is scattered (i.e. there is no embedding of the rationals into \( P \)) and has no infinite antichains, then the set of initial intervals of \( P \) has the same cardinality of \( P \).

The converse of Theorem 1.4 is in general false, but it holds when \(|P| < 2^\omega\), and in particular when \( P \) is countable:

**Theorem 1.5.** A countable partial order is scattered and has no infinite antichains if and only if it has countably many initial intervals.

The program of reverse mathematics ([Sim09] is the basic reference) gauges the strength of mathematical theorems by means of the subsystems of second order arithmetic necessary for their proofs. This approach allows only the study of statements about countable (or countably coded) objects. We therefore study the strength of Theorem 1.5 and of the restrictions of Theorems 1.1, 1.2 and 1.3 to countable partial orders. We notice that [ET43, Bon75, Fra00] put no restriction on the cardinality of the partial order and therefore often use set-theoretic techniques which are not available in (subsystems of) second order arithmetic. On the other hand we can always assume that the partial orders are defined on a subset of the set of the natural numbers, and this is on occasion helpful.

Since Theorems 1.1, 1.3, and 1.5 are equivalences, we study separately the two implications, which turn out to have different axiomatic strengths. In particular, the ‘easy’ (right to left) directions of Theorems 1.1 and 1.5 are quite interesting from the viewpoint of reverse mathematics and we are not able to settle the problem of establishing their strength, leaving open the possibility that they have strength intermediate between \( \text{RCA}_0 \) and \( \text{WKL}_0 \).
We assume familiarity with the ‘big five’ of reverse mathematics, namely, in order of increasing strength, $\text{RCA}_0$, $\text{WKL}_0$, $\text{ACA}_0$, $\text{ATR}_0$, and $\Pi^1_1$-$\text{CA}_0$.

We now state our main results and at the same time describe the organization of the paper. In section 2, we establish our notation and terminology and recall some basic results. In 3, we prove a couple of technical lemmas that are useful later on.

In Section 4, we consider Theorem 1.2 and the right to left directions of Theorems 1.1, 1.3, and 1.5. Subsection 4.1 culminates in Theorem 4.5 where we prove, over $\text{RCA}_0$, the equivalence of $\text{ACA}_0$ with each of the three statements:

- in a countable partial order with no infinite antichains every initial interval is a finite union of ideals;
- in a countable partial order with no infinite strong antichains there is a bound on the size of the strong antichains;
- every countable partial order with no infinite strong antichains is a finite union of ideals.

In subsection 4.2, we show that the statement

- every countable partial order which is scattered and has no infinite antichains has countably many initial intervals.

is equivalent to $\text{ATR}_0$ over $\text{ACA}_0$ (Theorem 4.8). To obtain the reversal we slightly modify a proof in [Clo89].

In section 5, we deal with the right to left directions of Theorems 1.1, 1.3, and 1.5, i.e. with the the statements:

- if every initial interval of a countable partial order is a finite union of ideals, then the partial order has no infinite antichains;
- if a countable partial order is a finite union of ideals then it has no infinite strong antichains;
- if a countable partial order has countably many initial intervals, then it has no infinite antichains;
- if a countable partial order has countably many initial intervals, then it is scattered.

The obvious proofs of these statements go through in $\text{ACA}_0$, but we show that they are all provable in weaker systems. In fact $\text{RCA}_0$ proves the second and fourth statement (Lemma 5.1 and Theorem 5.2). On the other hand, the first and third statement are both provable in $\text{WKL}_0$ (Theorems 5.5 and 5.10) and fail in the $\omega$-model of computable sets and hence cannot be proved in $\text{RCA}_0$ (Theorems 5.12 and 5.13). Our results thus do not completely determine the strength of these two statements.

In Section 6, we briefly discuss the open problems left by our results.

2. Terminology, Notation, and Basic Facts

All definitions in this section are made in $\text{RCA}_0$.

2.1. Finite sequences and trees. We typically use $\sigma$ and $\tau$ to denote finite sequences of natural numbers, that is elements of $\mathbb{N}^{<\mathbb{N}}$. Often they belong to $2^{<\mathbb{N}}$, i.e. they are binary, and in one occasion to $3^{<\mathbb{N}}$, i.e. they are ternary. Let $|\sigma|$ be the length of $\sigma$ and list it as $\langle \sigma(0), \ldots, \sigma(|\sigma| - 1) \rangle$. In particular $\langle \rangle$ is the unique sequence of length 0. We write $\sigma \sqsubseteq \tau$ to mean that $\sigma$ is an initial segment of $\tau$, while $\sigma \vartriangleleft \tau$ denotes the concatenation of $\sigma$ and $\tau$. By $\sigma \upharpoonright k$ we mean the initial segment of $\sigma$ of length $k$ and similarly, when $f$ is a function, $f \upharpoonright k$ is the finite sequence $\langle f(0), \ldots, f(k - 1) \rangle$.

A tree $T$ is a set of finite sequences such that $\tau \in T$ and $\sigma \sqsubseteq \tau$ imply $\sigma \in T$. A tree is pruned if it contains no endnodes, i.e. ($\forall \sigma \in T)(\exists \tau \in T)\sigma \sqsubset \tau$. A path in $T$
is a function $f$ such that for all $n$ the finite sequence $f \upharpoonright n$ belongs to $T$. We write $[T]$ to denote the collection of all paths in $T$: $[T]$ does not formally exist in second order arithmetic, but $f \in [T]$ is a convenient shorthand.

A tree $T$ is perfect if for all $\sigma \in T$ there exist $\tau_0, \tau_1 \in T$ such that $\sigma \subseteq \tau_0, \tau_1$ and neither $\tau_0 \subseteq \tau_1$ nor $\tau_1 \subseteq \tau_0$ hold. A tree $T$ has countably many paths if there exists a sequence $\{f_n : n \in \mathbb{N}\}$ (coded by a single set) such that for every $f \in [T]$ we have $f = f_n$ for some $n \in \mathbb{N}$. If $T$ does not have countably many paths then we say that it has uncountably many paths.

By [Sim09, Theorem V.5.5] $\text{ATR}_0$ is equivalent to the perfect tree theorem:

**Theorem 2.1 (ACA$_0$).** The following are equivalent:

1. $\text{ATR}_0$;
2. every tree with uncountably many paths contains a perfect subtree.

2.2. Partial orders. Within $\text{RCA}_0$ saying that $(P, \leq)$ is a partial order means that $P \subseteq \mathbb{N}$ and $\leq \subseteq P \times P$ is reflexive, antisymmetric and transitive. As usual, we use $\prec$ to denote the strict order. From now on we refer to $(P, \leq)$ simply as $P$. When we deal with several partial orders at the same time, we use subscripts as in $\leq_P$ to distinguish between the relations.

Finite partial orders can easily be studied in $\text{RCA}_0$ and hence, whenever it is convenient and without further notice, we assume that $P$ is infinite.

We say that $x, y \in P$ are comparable if $x \leq y$ or $y \leq x$. A partial order $P$ is a linear order if all its elements are pairwise comparable. A linear order $P$ is dense if for all $x, y \in P$ such that $x \prec y$ there exists $z \in P$ with $x \prec z \prec y$. If $x$ and $y$ are incomparable we write $x \perp y$.

A subset $D \subseteq P$ is an antichain if all its elements are pairwise incomparable, i.e.

$$(\forall x, y \in D)(x \neq y \implies x \perp y).$$

We say that $x, y \in P$ are compatible in $P$ if there is $z \in P$ such that $x \leq z$ and $y \leq z$. Notice that two elements of $P$ might be compatible in $P$ but not in some $X \subseteq P$ to which they belong.

A subset $S \subseteq P$ is a strong antichain in $P$ if its elements are pairwise incompatible in $P$, i.e.

$$(\forall x, y \in S)(\exists z \in P)(x, y \leq z \implies x = y).$$

A subset $I \subseteq P$ is an initial interval of $P$ if

$$(\forall x, y \in P)(x \leq y \land y \in I \implies x \in I).$$

An initial interval $A$ of $P$ is an ideal if every two elements of $A$ are compatible in $A$, i.e.

$$(\forall x, y \in A)(\exists z \in A)(x \leq z \land y \leq z).$$

If $x \in P$ we let $P_{\perp x} = \{y \in P : x \perp y\}$ and define the upper and lower cones determined by $x$ setting

$${P_{\geq x} = \{y \in P : x \geq y\} \text{ and } P_{\leq x} = \{y \in P : y \leq x\}}.$$

$P_{\geq x}$ and $P_{\leq x}$ are defined in the obvious way. If $X \subseteq P$ we write $\downarrow X$ for the downward closure of $X$, i.e. $\bigcup_{x \in X} P_{\leq x}$. Notice that for $\downarrow X$ to exist as a set we need $\text{ACA}_0$.

2.3. Well-partial orders, scattered partial orders and lexicographic sums.

A partial order $P$ is well-founded if $P$ contains no infinite descending sequence, i.e. no function $f : \mathbb{N} \to P$ such that $f(i) \succ f(j)$ for all $i < j$. A well-founded linear order is a well-order.

A partial order $P$ is a well-partial order if for every function $f : \mathbb{N} \to P$ there exist $i < j$ such that $f(i) \preceq f(j)$. There are many classically equivalent definitions of
well-partial order. In particular a well-partial order is a well-founded partial order with no infinite antichains. For a reverse mathematics study of these equivalences we refer to [CMS04]. For our purposes, it is enough to know that all these equivalences are provable in ACA₀ and that RCA₀ suffices to show that every well-partial order is well-founded and has no infinite antichains.

The Kleene-Brouwer order on finite sequences is the linear order defined by $\sigma \leq_{KB} \tau$ if either $\tau \sqsubseteq \sigma$ or $\sigma(i) < \tau(i)$ for the least $i$ such that $\sigma(i) \neq \tau(i)$. One of the main features of $\leq_{KB}$ is that, provably in ACA₀, its restriction to a tree $T$ is a well-order if and only if $T$ has no paths ([Sim09, Lemma V.1.3]).

An embedding of a partial order $Q$ into a partial order $P$ is a function $f : Q \to P$ such that for all $x, y \in Q$ we have $x \preceq_Q y$ if and only if $f(x) \preceq_P f(y)$. A partial order $P$ is scattered if there is no embedding of $Q$ (the order of the rationals) into $P$.

**Lemma 2.2 (RCA₀).** A partial order is scattered if and only if it does not contain any dense linear order.

**Proof.** The left to right is immediate because RCA₀ suffices to carry out the usual back-and-forth argument. For the other direction, given an embedding $f : Q \to P$ by recursion we can find $D \subseteq Q$ dense such that $f$ restricted to $D$ is strictly increasing with respect to the ordering of the natural numbers. Thus the range of $f$ restricted to $D$ is a dense linear order.

If $P$ is a partial order and $\{P_x : x \in P\}$ is a sequence of partial orders indexed by $P$ we define the lexicographic sum of the $P_x$ along $P$, denoted by $\sum_{x \in P} P_x$, to be the partial order on the set $Q = \{(x, y) : x \in P \land y \in P_x\}$ defined by $(x, y) \preceq_Q (x', y') \iff x \preceq_P x' \land (x = x' \land y \preceq_{P_x} y')$.

**Lemma 2.3 (RCA₀).** The lexicographic sum of scattered partial orders along a scattered partial order is scattered.

**Proof.** Let $Q = \sum_{x \in P} P_x$ be a lexicographic sum and suppose that $Q$ is not scattered. Fix an embedding $f : Q \to Q$.

First suppose that for some $a <_Q b$ and $x \in P$ we have $f(a) = (x, y)$ and $f(b) = (x, y')$. Then, the composition of $f$ with the projection on the second coordinate is an embedding of the rational interval $(a, b)_Q$ into $P_x$. Since $Q$ embeds into its open intervals, $P_x$ is not scattered.

Otherwise, composing $f$ with the projection on the first coordinate, we obtain an embedding of $Q$ into $P$, and $P$ is not scattered.

2.4. The set of initial intervals. We denote by $\mathcal{I}(P)$ the collection of initial intervals of the partial order $P$. In second order arithmetic, $\mathcal{I}(P)$ does not formally exist, and $I \in \mathcal{I}(P)$ is a shorthand for the formula “$I$ is an initial interval of $P$”. To study Theorem [13] we need to discuss the cardinality of $\mathcal{I}(P)$.

We say that the partial order $P$ has countably many initial intervals if there exists a sequence $\{I_n : n \in \mathbb{N}\}$ such that for every $I \in \mathcal{I}(P)$ we have $I = I_n$ for some $n \in \mathbb{N}$. Otherwise, we say that $P$ has uncountably many initial intervals.

Within ACA₀ we can prove that if $P$ has countably many initial intervals there exists a sequence $\{I_n : n \in \mathbb{N}\}$ such that $I \in \mathcal{I}(P)$ if and only if there exists $n \in \mathbb{N}$ such that $I = I_n$. In this case we write $\mathcal{I}(P) = \{I_n : n \in \mathbb{N}\}$.

The partial order $P$ has perfectly many initial intervals if there exists a nonempty perfect tree $T \subseteq 2^{<\mathbb{N}}$ such that $[T] \subseteq \mathcal{I}(P)$, that is, for all $f \in [T]$, the set $\{x \in \mathbb{N} : f(x) = 1\} \in \mathcal{I}(P)$.

A useful tool for studying the notions we just defined is the tree of finite approximations of initial intervals of the partial order $P$. We define the tree $T(P) \subseteq 2^{<\mathbb{N}}$ by letting $\sigma \in T(P)$ if and only if for all $x, y < |\sigma|$:
\[ \sigma(x) = 1 \text{ implies } x \in P; \]
\[ \sigma(y) = 1 \text{ and } x \preceq y \text{ imply } \sigma(x) = 1. \]

Notice that \( T(P) \) is a pruned tree and that the paths in \( T(P) \) are exactly the characteristic functions of the initial intervals of \( P \). From the latter observation we easily obtain:

**Lemma 2.4 (RCA\(_0\)).** Let \( P \) be a partial order.

(i) \( P \) has countably many initial intervals if and only if \( T(P) \) has countably many paths;

(ii) \( P \) has perfectly many initial intervals if and only if \( T(P) \) contains a perfect subtree.

In particular, the formula “\( P \) has perfectly many initial intervals” is provably \( \Sigma^1_1 \) within \( \text{RCA}_0 \). Moreover a straightforward diagonal argument shows in \( \text{RCA}_0 \) that a nonempty perfect tree has uncountably many paths. Therefore we have that \( \text{RCA}_0 \) proves that a partial order with perfectly many initial intervals has uncountably many initial intervals. This implies that the formula “\( P \) has uncountably many initial intervals” is provably \( \Sigma^1_1 \) within \( \text{RCA}_0 \).

In connection with this recall the following result due to Peter Clote \[\text{Clo89}\]:

**Theorem 2.5 (ACA\(_0\)).** The following are equivalent:

1. \( \text{ATR}_0 \);
2. any countable linear order has countably many or perfectly many initial intervals;
3. any scattered linear order has countably many initial intervals.

Clote actually states the equivalence of \( \text{ATR}_0 \) only with (2), but his proofs yield also the equivalence with (3).

2.5. **The system \( \text{ATR}_X^0 \).** Recall that, by \[\text{Sim09}, \text{Theorem VIII.3.15}\], \( \text{ATR}_0 \) is equivalent over \( \text{ACA}_0 \) to the statement

\[ (\forall X)(\forall a \in O^X)(H^X_a \text{ exists}) \]

where \( O^X \) is the collection of (indices for) \( X \)-computable ordinals and \( H^X_a \) codes the iteration of the jump along \( a \) starting from \( X \). This naturally leads to consider lightface versions of \( \text{ATR}_0 \), as in \[\text{Tan89}, \text{Tan90}, \text{and Mar91}\]. Here we make explicit mention of the set parameter we use (rather then deal only with the parameterless case and then invoke relativization) and let \( \text{ATR}_X^0 \) be \( \text{ACA}_0 \) plus the formula \( (\forall a \in O^X)(H^X_a \text{ exists}) \). In \( \text{ATR}_X^0 \) one can prove arithmetical transfinite recursion along any \( X \)-computable well-order.

By checking the proof of the forward direction of Theorem 2.1 one readily realizes that \( \text{ATR}_X^0 \) proves the perfect tree theorem for \( X \)-computable trees:

**Theorem 2.6 (ATR\(_X^0\)).** Every \( X \)-computable tree with uncountably many paths contains a perfect subtree.

The following is \[\text{Sim09}, \text{Lemma VIII.4.19}\]:

**Theorem 2.7 (ATR\(_X^0\)).** There exists a countable coded \( \omega \)-model \( M \) such that \( X \in M \) and \( M \) satisfies \( \Sigma^1_1-\text{DC}_0 \).

We will use the following corollary:

**Corollary 2.8 (ATR\(_0\)).** For all \( X \) and \( Y \) there exists a countable coded \( \omega \)-model \( M \) such that \( X,Y \in M \), and \( M \) satisfies both \( \Sigma^1_1-\text{DC}_0 \) and \( \text{ATR}^X_0 \).
Proof. We argue in \( \text{ATR}_0 \) and let \( X \) and \( Y \) be given. By \( \Sigma_1^1 \text{-AC}_0 \), which is a consequence of \( \text{ATR}_0 \), the main axiom of \( \text{ATR}_0 \), is equivalent to a \( \Sigma_1^1 \) formula \((\exists Z)\varphi(Z, X)\) with \( \varphi \) arithmetic. This formula is true in \( \text{ATR}_0 \), and hence we can fix \( Z \) such that \( \varphi(Z, X) \). By Theorem 2.7 there exists a countable coded \( \omega \)-model \( M \) of \( \Sigma_1^1 \text{-DC}_0 \) such that \( X \uplus Y \uplus Z \in M \). In particular, \( X, Y \in M \) and, as \( Z \in M \) and \( M \) is a model of \( \Sigma_1^1 \text{-DC}_0 \) (hence also of \( \Sigma_1^1 \text{-AC}_0 \)), \( M \) satisfies \( \text{ATR}_0^X \). \( \square \)

3. Initial interval separation and essential unions

In this section we prove two technical results that are useful in the remainder of the paper.

3.1. Initial interval separation. Our first result is a new equivalent of \( \text{WKL}_0 \), inspired by the usual \( \Sigma_1^0 \) separation ([Smi09 Lemma IV.4.4]) but producing separating sets which are also initial intervals.

Lemma 3.1. Over \( \text{RCA}_0 \), the following are equivalent:

1. \( \text{WKL}_0 \);
2. \( \Sigma_1^0 \) initial interval separation. Let \( P \) be a partial order and \( \varphi(x) \), \( \psi(x) \) be \( \Sigma_1^0 \) formulas with one distinguished free number variable.
   If \((\forall x, y \in P)(\varphi(x) \land \psi(y) \implies y \not\equiv x)\), then there exists an initial interval \( I \) of \( P \) such that
   \((\forall x \in P)(\varphi(x) \implies x \in I) \land (\psi(x) \implies x \not\in I)).\)
3. Initial interval separation. Let \( P \) be a partial order and suppose \( A, B \subseteq P \) are such that \((\forall x \in A)(\forall y \in B)y \not\equiv x \). Then there exists an initial interval \( I \) of \( P \) such that \( A \subseteq I \) and \( B \cap I = \emptyset \).

Proof. We first assume \( \text{WKL}_0 \) and prove (2). Fix the partial order \( P \) and let \( \varphi(x) \equiv (\exists m)\varphi_0(x, m) \) and \( \psi(n) \equiv (\exists m)\psi_0(x, m) \) be \( \Sigma_1^0 \) formulas with \( \varphi_0 \) and \( \psi_0 \). Assume \((\forall x, y \in P)(\varphi(x) \land \psi(y) \implies y \not\equiv x)\).

Form the binary tree \( T \subseteq 2^{\mathbb{N}} \) by letting \( \sigma \in T \) if and only if \( \sigma \in T(P) \) and for all \( x, y < |\sigma|\):

(i) \((\exists m < |\sigma|) \varphi_0(x, m) \implies \sigma(x) = 1, \) and
(ii) \((\exists m < |\sigma|) \psi_0(x, m) \implies \sigma(x) = 0.\)

To see that \( T \) is infinite, we show that for every \( k \in \mathbb{N} \) there exists \( \sigma \in T \) with \(|\sigma| = k \). Given \( k \) let

\[ \sigma(x) = \begin{cases} 1 & \iff x \in P \land (\exists y, m < k)(\varphi_0(y, m) \land x \not\equiv y) \\ 0 & \iff x \not\in P \land (\exists y, m < k)(\varphi_0(y, m) \land x \not\equiv y) \end{cases} \]

for all \( x < k \). It is easy to verify that \( \sigma \in T \). By weak König’s lemma, \( T \) has a path \( f \). By \( \Sigma_1^0 \) comprehension, let \( I = \{ x : f(x) = 1 \} \). It is straightforward to see that \( I \) is as desired.

(3) is the special case of (2) obtained by considering the \( \Sigma_0^0 \), and hence \( \Sigma_1^1 \), formulas \( x \in A \) and \( x \in B \).

It remains to prove \( (3) \implies (1) \). It suffices to derive in \( \text{RCA}_0 \) from (3) the existence of a set separating the disjoint ranges of two one-to-one functions ([Smi09 Lemma IV.4.4]). Let \( f, g : \mathbb{N} \to \mathbb{N} \) be one-to-one functions such that \((\forall n, m \in \mathbb{N})f(n) \neq g(m)\). Define a partial order on \( P = \{ a_n, b_n, c_n : n \in \mathbb{N} \} \) by letting \( c_n \preceq a_m \) if and only if \( f(m) = n \), \( b_m \preceq c_n \) if and only if \( g(m) = n \), and adding no other comparabilities. Let \( A = \{ a_n : n \in \mathbb{N} \} \) and \( B = \{ b_n : n \in \mathbb{N} \} \), so that \((\forall x \in A)(\forall y \in B)y \not\equiv x \). By (3) there exists an initial interval \( I \) of \( P \) such that \( A \subseteq I \) and \( B \cap I = \emptyset \). It is easy to check that \( \{ n : c_n \in I \} \) separates the range of \( f \) from the range of \( g \). \( \square \)
3.2. Essential unions of sets. Our second result deals with finite union of sets and will be applied to finite unions of ideals.

**Definition 3.2 (RCA₀).** Let $I \subseteq \mathbb{N}$. A family of sets $\{A_i : i \in I\}$ is essential if

$$\left( \forall n \in I \right) (A_i \not\subseteq \bigcup_{j \in I, j \neq i} A_j).$$

The union of such a family is called an essential union.

Not every family of sets can be made essential without losing elements from the union. The simplest example is a sequence $\{A_n : n \in \mathbb{N}\}$ of sets such that $A_n \subseteq A_{n+1}$ for every $n$. However the following shows that, provably in RCA₀, every finite family of sets can be made essential.

**Lemma 3.3 (RCA₀).** For every family of sets $\{A_i : i \in F\}$ with $F$ finite there exists $I \subseteq F$ such that $\{A_i : i \in I\}$ is essential and

$$\bigcup_{i \in F} A_i = \bigcup_{i \in I} A_i.$$

**Proof.** By bounded $\Pi^0_1$ comprehension, available in RCA₀ by [Sim09, Theorem II.3.9], we can form the set

$$X = \{(i, J) : i \in F \land J \subseteq F \land A_i \subseteq \bigcup_{j \in J} A_j\}.$$  

We fix $k \notin F$ and let $F' = F \cup \{k\}$. To define $I$ we are going to define by primitive recursion a function $f : \mathbb{N} \to F'$. The idea is that if some $i \in F$ belongs to the range of $f$ then $i \notin I$. It is convenient to use a set-theoretic notation and write $f[n]$ for the set $\{i : \exists m < n \ f(m) = i\}$.

We let $f(n)$ to be the least $i \in F \setminus f[n]$ such that $(i, F \setminus (f[n] \cup \{i\})) \in X$ if $f[n] \subseteq F$ and there exists such an $i$, $f(n) = k$ otherwise.

There exists $n$ such that $f(n) = k$. If not, $f$ would be a one-to-one function from $\mathbb{N}$ to $F$, since $f(n) \in F'$ implies $f(n) \notin f[n]$.

Let $n_0$ be least such that $f(n_0) = k$ and let $I = F \setminus f[n_0]$. Now, the family $\{A_i : i \in I\}$ is clearly essential.

It remains to see that

$$\bigcup_{i \in F} A_i = \bigcup_{i \in I} A_i.$$  

To do this, we prove by $\Pi^0_1$-induction that for all $n \leq n_0$

$$\bigcup_{i \in F} A_i = \bigcup_{i \in F \setminus f[n]} A_i.$$  

When $n = 0$ there is nothing to prove. Suppose this equality holds for $n < n_0$ and check that

$$\bigcup_{i \in F \setminus f[n]} A_i = \bigcup_{i \in F \setminus f[n+1]} A_i.$$  

Since $n < n_0$ we have $f(n) \in F$, and thus by definition $A_{f(n)} \subseteq \bigcup_{i \in F \setminus f[n+1]} A_i$, so that the equality holds.

\[\square\]

4. The left to right directions

In this section we study Theorem 1.2 and the left to right directions of Theorems 1.5 and 1.3. It turns out that the left to right direction of Theorem 1.5 is equivalent to $\text{ATR}_0$ and the other statements are equivalent to $\text{ACA}_0$. 


4.1. Equivalences with ACA₀. We consider the following equivalence, which includes Theorems 1.2 and 1.3

Theorem 4.1. Let $P$ be a partial order. Then the following are equivalent:

1. $P$ is a finite union of ideals;
2. There is a finite bound on the size of the strong antichains in $P$;
3. $P$ has no infinite strong antichains.

We notice that (1) $\implies$ (2) and (2) $\implies$ (3) are easily provable in RCA₀. We show that (2) $\implies$ (1) and (3) $\implies$ (2) are provable in ACA₀.

We start with implication (2) $\implies$ (1).

Lemma 4.2 (ACA₀). Let $P$ be a partial order with no arbitrarily large finite strong antichains. Then $P$ is a finite union of ideals.

Proof. Let $\ell \in \mathbb{N}$ be the maximum size of a strong antichain in $P$ and let $S$ be a strong antichain of size $\ell$. For every $z \in S$ define by arithmetical comprehension

$$A_z = \{ x \in P : x \text{ and } z \text{ are compatible} \} .$$

Since $S$ is maximal with respect to inclusion it is immediate that $P = \bigcup_{z \in S} A_z$ and it suffices to show that each $A_z$ is an ideal.

Fix $z \in S$ and $x, y \in A_z$. Let $x_0, y_0$ be such that $x \preceq x_0, y \preceq y_0$, and $z \preceq x_0, y_0$. It suffices to show that $x_0$ and $y_0$ are compatible in $A_z$. If this is not the case, $x_0$ and $y_0$ are incompatible also in $P$ (because $P_{x_0} \subseteq P_{x_0} \subseteq A_z$). Moreover for each $w \in S \setminus \{ z \}$ each of $x_0$ and $y_0$ is incompatible with $w$ in $P$ because $z$ and $w$ are incompatible in $P$. Thus $(S \setminus \{ z \}) \cup \{ x_0, y_0 \}$ is a strong antichain of size $\ell + 1$, a contradiction. □

To obtain (3) $\implies$ (2) of Theorem 4.1 we are going to use the existence of maximal (with respect to inclusion) strong antichains. We first show that this statement is equivalent to ACA₀.

Lemma 4.3. Over RCA₀, the following are equivalent:

1. ACA₀;
2. Every strong antichain in a partial order extends to a maximal strong antichain;
3. Every partial order contains a maximal strong antichain.

Proof. We show (1) $\implies$ (2). Let $P$ be a partial order and $S \subseteq P$ a strong antichain. By primitive recursion, we define a maximal strong antichain $T \supseteq S$ in $P$. Suppose we have defined $T_y = T \cap \{ x \in P : x < y \}$. Then $y \in T$ if and only if $S \cup T_y \cup \{ y \}$ is a strong antichain.

Implication (2) $\implies$ (3) is trivial. To show (3) $\implies$ (1), we argue in RCA₀ and derive from (3) the existence of the range of any one-to-one function. Given $f : \mathbb{N} \to \mathbb{N}$ one-to-one consider $P = \{ a_n, b_n, c_n : n \in \mathbb{N} \}$. For all $n, m \in \mathbb{N}$ let $a_n \preceq c_m$ if and only if $b_n \preceq c_m$ if and only if $f(m) = n$, and add no other comparabilities. By (3), let $S \subseteq P$ be a maximal strong antichain. Then, $n$ belongs to the range of $f$ if and only if $a_n \notin S \lor b_n \notin S$. Thus the range of $f$ exists by $\Sigma^0_0$ comprehension. □

The following is implication (3) $\implies$ (2) of Theorem 4.1, i.e. our formalization of the left to right direction of Theorem 1.2.

Lemma 4.4 (ACA₀). Let $P$ be a partial order with no infinite strong antichains. Then there are no arbitrarily large finite strong antichains in $P$. 


Proof. Suppose for a contradiction that \( P \) has arbitrarily large finite strong antichains but no infinite strong antichains. We define by recursion a sequence of elements \((x_n, y_n) \in P^2\).

Let \((x_0, y_0)\) be a pair such that \(x_0\) and \(y_0\) are incompatible in \( P \) and \( P_{\geq x_0} \) contains arbitrarily large finite strong antichains. Suppose we have defined \( x_n \) and \( y_n \). Using arithmetical comprehension, search for a pair \((x_{n+1}, y_{n+1})\) such that \(x_n \preceq x_{n+1}, y_{n+1}\), \(x_{n+1}\) and \(y_{n+1}\) are incompatible in \( P \), and \( P_{\geq x_{n+1}} \) contains arbitrarily large finite strong antichains.

To show that the recursion never stops assume that \( U \subseteq P \) is a final interval with arbitrarily large finite strong antichains (\( U = P \) at stage 0, \( U = P_{\geq x_0} \) at stage \( n + 1 \)). By Lemma \([1,3]\) there exists a maximal strong antichain \( S \subseteq U \) with at least two elements. By hypothesis, \( S \) is finite and we apply the following claim:

Claim. There exists \( x \in S \) such that \( P_{\geq x} \) contains arbitrarily large finite strong antichains.

Proof of claim. Let \( n = |S| \). We first show that for every \( k \geq 1 \) there exists \( u \in S \) such that \( P_{\geq u} \) contains a strong antichain of size \( k \).

Given \( k \geq 1 \), let \( T \) be a strong antichain of size \( n \cdot k \). Since \( S \) is maximal, every element \( y \in T \) is compatible with some element of \( S \). For any \( y \in T \) let \( (u(y), v(y)) \) be the least pair such that \( u(y) \in S \) and \( u(y), v(y) \) are incompatible in \( P \). Then \( \{v(y) : y \in T\} \) is again a strong antichain of size \( n \cdot k \). As \( y \mapsto u(y) \) defines a function from \( T \) to \( S \), it easily follows that for some \( u \in S \) the upper cone \( P_{\geq u} \) contains at least \( k \) elements of the form \( v(y) \) with \( y \in T \).

Now, for all \( k \geq 1 \), let \( u_k \in S \) be such that \( P_{\geq u_k} \) contains a strong antichain of size \( k \). Since \( S \) is finite, by the infinite pigeonhole principle (which is provable in \( \text{ACA}_0 \)), there exists \( x \in S \) such that \( x = u_k \) for infinitely many \( k \). The upper cone \( P_{\geq x} \) thus contains arbitrarily large finite strong antichains.

In particular, \( x_n \preceq y_m \) for all \( n < m \) and \( x_n \) and \( y_n \) are incompatible in \( P \). It follows that \( y_n \) is incompatible with \( y_m \) for all \( n < m \). Then \( \{y_n : n \in \mathbb{N}\} \) is an infinite strong antichain, for the desired contradiction.

The following Theorem shows that our use of \( \text{ACA}_0 \) in several of the preceding Lemmas is necessary and establish the reverse mathematics results about Theorem \([1,2]\) and the left to right directions of Theorems \([1,1]\) and \([1,3]\) (these are respectively conditions (3), (5), and (4) in the statement of the Theorem). We also show that apparently weaker statements, such as the restriction of Theorems \([1,1]\) and \([1,3]\) to well-partial orders, require \( \text{ACA}_0 \).

**Theorem 4.5.** Over \( \text{RCA}_0 \), the following are pairwise equivalent:

1. \( \text{ACA}_0 \);
2. every partial order with no arbitrarily large finite strong antichains is a finite union of ideals;
3. every partial order with no infinite strong antichains does not contain arbitrarily large finite strong antichains;
4. every partial order with no infinite strong antichains is a finite union of ideals;
5. if a partial order has no infinite antichains then every initial interval is a finite union of ideals;
6. every well-partial order is a finite union of ideals.

Proof. \((1) \implies (2)\) is Lemma \([1,2]\) and \((1) \implies (3)\) is Lemma \([1,4]\). The combination of Lemma \([1,3]\) and Lemma \([1,2]\) shows \((1) \implies (4)\). Since a strong antichain in a subset of a partial order is an antichain, \((4) \implies (5)\) holds. For \((5) \implies (6)\), recall that, provably in \( \text{RCA}_0 \), a well-partial order has no infinite antichains.
It remains to show that each of (2), (3) and (6) implies ACA₀. Reasoning in RCA₀ fix a one-to-one function \( f : \mathbb{N} \to \mathbb{N} \). In each case we build a suitable partial order \( P \) which encodes the range of \( f \).

We start with (2) \( \implies \) (1). Let \( P = \{a_n, b_n : n \in \mathbb{N}\} \cup \{c\} \). We define a partial order on \( P \) by letting:

(i) \( a_n \preceq c \) for all \( n \);
(ii) \( b_n \preceq b_m \) for \( n \leq m \);
(iii) \( a_n \preceq b_m \) if and only if \((\exists i < m)f(i) = n\);

and adding no other comparabilities. It is easy to verify that every strong antichain in \( P \) has at most 2 elements. By (2) \( P \) is a finite union of ideals \( A₀, \ldots, A_n \). By Lemma [13] we may assume that this union is essential. Let us assume \( b_0 \in A₀ \).

By \( \Sigma^1_0 \)-induction (actually \( \Sigma^0_0 \)) we prove that \((\forall m)(b_m \in A₀) \). The base step is obviously true. Suppose \( b_m \in A₀ \) and \( b_{m+1} \not\in A₀ \). Then \( A₀ = \{x \in P : x \not\in b_m\} \) (because every element \( \succ b_m \) is \( \preceq b_{m+1} \)). Suppose \( b_{m+1} \in A₁ \). Then \( A₀ \subseteq A₁ \) and the decomposition is not essential, a contradiction. Therefore, \( A₀ \) contains all the \( b_m \)'s. Now, it is straightforward to see that \((\exists m)f(m) = n \) if and only if \( a_n \in A₀ \), so that the range of \( f \) can be defined by \( \Delta^0_1 \) comprehension.

To prove (3) \( \implies \) (1) we exploit the notion of false and true stage. Recall that \( n \in \mathbb{N} \) is said to be a false stage for \( f \) (or simply false) if \( f(k) < f(n) \) for some \( k > n \) and true otherwise. We may assume to have infinitely many false stages, since otherwise the range of \( f \) exists by \( \Delta^0_1 \) comprehension. On the other hand, there are always infinitely many true stages (i.e. for every \( m \) there exists \( n > m \) which is true), because otherwise we can build an infinite descending sequence of natural numbers.

Let \( P = \{a_n, b_n : n \in \mathbb{N}\} \) and define

(i) \( b_n \preceq b_m \) for all \( n < m \);
(ii) \( a_n \preceq b_m \) if and only if \( f(k) < f(n) \) for some \( k \) with \( n < k \leq m \) (i.e. if at stage \( m \) we know that \( n \) is false);

and there are no other comparabilities.

Notice that the \( b_n \)'s and the \( a_n \)'s with \( n \) false are pairwise compatible in \( P \). Therefore every strong infinite antichain in \( P \) consists of infinitely many \( a_n \)'s with \( n \) true and at most one \( b_n \) or \( a_n \) with \( n \) false. Possibly removing that single element we have an infinite set of true stages. From this in RCA₀ we can obtain a strictly increasing enumeration of true stages \( i \mapsto n_i \). Since \((\exists n)f(n) = m \) if and only if \((\exists n \leq n_m)f(n) = m \), the range of \( f \) exists by \( \Delta^0_1 \) comprehension. Thus the existence of an infinite strong antichain in \( P \) implies the existence of the range of \( f \) in RCA₀.

To apply (3) and conclude the proof we need to show that \( P \) contains arbitrarily large finite strong antichains. To do this apparently we need \( \Sigma^0_2 \)-induction (which is not available in RCA₀) to show that for all \( k \) there exists \( k \) distinct true stages.

To remedy this problem (with the same trick used for this purpose in [MSH Lemma 4.2]) we replace each \( a_n \) with \( n + 1 \) distinct elements. Thus we set \( P' = \{a'_n, b_n : n \in \mathbb{N}, i \leq n\} \) and substitute (ii) with \( a'_n \preceq b_n \) if and only if \( f(k) < f(n) \) for some \( k \) with \( n < k \leq m \). Then also the existence of an infinite strong antichain in \( P' \) suffices to define the range of \( f \) in RCA₀. However the existence of arbitrarily large finite strong antichains in \( P' \) of the form \( \{a'_i : i \leq n\} \) follows immediately from the existence of infinitely many true stages.

We now show (6) \( \implies \) (1). We again use false and true stages and as before we assume to have infinitely many false stages. The idea for \( P \) is to combine a linear order \( P₀ = \{a_n : n \in N\} \) of order type \( \omega + \omega^* \) with a linear order \( P₁ = \{b_n : n \in \mathbb{N}\} \) of order type \( \omega \). The false and true stages give rise respectively to the \( \omega \) and \( \omega^* \)
part of $P_0$, and every false stage will be below some element of $P_1$. We proceed as follows.

Let $P = \{a_n, b_n : n \in \mathbb{N}\}$. For $n \leq m$, set

(i) $a_n \preceq a_m$ if $f(k) < f(n)$ for some $n < k \leq m$ (i.e. if at stage $m$ we know that $n$ is false);

(ii) $a_m \preceq a_n$ if $f(k) > f(n)$ for all $n < k \leq m$ (i.e. if at stage $m$ we believe $n$ to be true).

When condition (i) holds, we also put $a_n \preceq b_m$. Then we linearly order the $b_m$'s by putting $b_i \preceq b_j$ if and only if $i \leq j$. There are no other comparabilities.

It is not difficult to verify that $P$ is a partial order with no infinite antichains. We start with a technical Theorem 1.5, i.e. the statement every countable scattered partial order with no infinite antichains has countably many initial intervals. We proceed as before. The range of $\tau$ is a finite union of ideals.

First assume that $P$ is a perfect tree. Then it is not difficult to show that $P_0$ has order type $\omega + \omega^*$. We now assume that $P$ is a well-partial order. By definition, there exists $g : \mathbb{N} \to P$ such that $i < j$ implies $g(i) \neq g(j)$. As for every false $n$ there are only finitely many $x \in P$ such that $a_n \not\preceq x$, we must have $g(i) \neq a_n$ for all $i$ and for all false $n$. We may assume that $g(i) \neq b_n$ for all $i, n$, since there are finitely many $b_m$ such that $b_n \not\preceq b_m$. We thus have $g(i) = a_n$, with $i_n$ true for all $i$. Since $a_m \not\preceq a_n$ and $n < m$ imply $n$ false, the map $i \mapsto i_n$ is a strictly increasing enumeration of true stages. As before, the range of $f$ exists by $\Delta^1_0$ comprehension.

We now assume that $P$ is a well-partial order. Apply (5), so that $P = \bigcup\{A_i : i < k\}$ is a finite union of ideals. By Lemma 3.3 we may assume that the union is essential so that there exists an ideal, say $A_0$, that contains all the $b_m$'s.

We claim that $n$ is false if and only if $a_n \in A_0$. To see this, let $n$ be false. Thus $a_n \preceq b_m$ for some $m$, and hence $a_n \in A_0$. Conversely, if $a_n \in A_0$ then it is compatible with, for instance, $b_0$, and yet again it is $\preceq b_m$ for some $m$. Hence, the set of true stages is $\{n : a_n \not\in A_0\}$, and the conclusion follows as before. □

4.2. Equivalences with $\text{ATR}_0$. We now consider the left to right direction of Theorem 1.5, i.e. the statement every countable scattered partial order with no infinite antichains has countably many initial intervals. We start with a technical Lemma:

Lemma 4.6 ($\text{ACA}_0$). If a partial order $P$ has perfectly many initial intervals, then there exists $x \in P$ such that either

(i) $P_{\preceq x}$ has uncountably many initial intervals, or

(ii) both $P_{\preceq x}$ and $P_{\succ x}$ have uncountably many initial intervals.

Proof. Let $P$ be a partial order with perfectly many initial intervals. Let $T \subseteq T(P)$ be a perfect tree.

We first show that there exist $x \in P$ such that both

$$\{I \in \mathcal{I}(P) : x \notin I\} \text{ and } \{I \in \mathcal{I}(P) : x \in I\}$$

are uncountable. Let $\tau \in T$ be such that both $\tau_0 = \tau^-(0)$ and $\tau_1 = \tau^-(1)$ belong to $T$. Let $x = \vert \tau \vert$ and notice that $x \in P$. For $i < 2$ define $T_i = \{\sigma \in T : \sigma \not\subseteq \tau_i \lor \tau_i \not\subseteq \sigma\}$. The trees $T_0$ and $T_1$ are perfect and witness the fact that the two collections of initial intervals are uncountable.

Now, suppose that condition (i) fails and let $\mathcal{I}(P_{\preceq x}) = \{J_n : n \in \mathbb{N}\}$. We aim to show that (ii) holds.

Suppose for a contradiction that $P_{\preceq x}$ has countably many initial intervals and let $\mathcal{I}(P_{\preceq x}) = \{I_n : n \in \mathbb{N}\}$. Then it is not difficult to show that

$$\{I \in \mathcal{I}(P) : x \notin I\} = \{I_n \cup \downarrow J_m : n, m \in \mathbb{N}\}.$$

This contradicts the fact that $\{I \in \mathcal{I}(P) : x \notin I\}$ is uncountable.
Similarly, suppose that $P_{\sigma,x}$ has countably many initial intervals and let $I(P_{\sigma,x}) = \{I_n : n \in \mathbb{N}\}$. Then, it is not difficult to show that
\[\{I \in I(P) : x \in I\} = \{\downarrow I_n \cup J_m : n, m \in \mathbb{N} \land I_n \neq \emptyset\}\]
This contradicts the fact that $\{I \in I(P) : x \in I\}$ is uncountable. Therefore, condition (ii) holds.

Theorem 4.7 (ATR$_0$). Every countable scattered partial order with no infinite antichains has countably many initial intervals.

Proof. Let $P$ be a countable partial order with uncountably many initial intervals.

Let $\text{Fin}(P)$ the set of (codes for) finite subsets of $P$. For all $F, G, H \in \text{Fin}(P)$, let
\[P_{F,G,H} = \bigcap_{x \in F} P_{<x} \cap \bigcap_{x \in G} P_{\sigma,x} \cap \bigcap_{x \in H} P_{\perp x}\]
We want to define a pruned tree $T \subseteq 3^{<\mathbb{N}}$ and a function $f : T \rightarrow \text{Fin}(P)^3$ such that the following hold (where $f(\sigma) = (F_\sigma, G_\sigma, H_\sigma)$ and $P_\sigma = P_{f(\sigma)}$):

(i) $f(\langle \rangle) = (\emptyset, \emptyset, \emptyset)$;
(ii) for all $\sigma \in T$, $\sigma^- \langle 0 \rangle \in T$ if and only if $\sigma^- \langle 1 \rangle \in T$ and only if $\sigma^- \langle 2 \rangle \notin T$ (in other words there are two possibilities: either exactly $\sigma^- \langle 0 \rangle$ and $\sigma^- \langle 1 \rangle$ belong to $T$, or only $\sigma^- \langle 2 \rangle \notin T$);
(iii) if $\sigma^- \langle 0 \rangle \in T$, then $f(\sigma^- \langle 0 \rangle) = (F_\sigma \cup \{x\}, G_\sigma, H_\sigma)$ and $f(\sigma^- \langle 1 \rangle) = (F_\sigma, G_\sigma \cup \{x\}, H_\sigma)$ for some $x \in P_\sigma$;
(iv) if $\sigma^- \langle 2 \rangle \notin T$, then $f(\sigma^- \langle 2 \rangle) = (F_\sigma, G_\sigma, H_\sigma \cup \{x\})$ for some $x \in P_\sigma$.

We first show that if there exist $T$ and $f$ as above, then $P$ is not scattered or it contains an infinite antichain.

First suppose there exists a path $g \in [T]$ such that $g(n) = 2$ for infinitely many $n$. Then let
\[D = \bigcup_{n \in \mathbb{N}} H_g \upharpoonright n\]
It is easy to check, using (iv) and the definition of $P_{F,G,H}$, that $D$ is an infinite antichain.

If there are no paths $g \in [T]$ such that $g(n) = 2$ for infinitely many $n$ then it is easy to see, using (ii), that $T$ is perfect. For all $\sigma^- \langle 0 \rangle \in T$, let $x_\sigma$ be the unique element of $F_{\sigma^- \langle 0 \rangle} \setminus F_\sigma$. We claim that
\[Q = \{x_\sigma : \sigma^- \langle 0 \rangle \in T\}\]
is a dense linear order in $P$.

We first note that $x_\sigma \neq x_\tau$ for $\sigma, \tau \in T$ with $\sigma \neq \tau$. Now fix distinct $x_\sigma, x_\tau \in Q$ with the goal of showing that they are comparable in $P$ and that there exists an element of $Q$ strictly between them. First assume that $\sigma$ and $\tau$ are comparable as sequences, let us say $\sigma \sqsubseteq \tau$. Then, using (iii), $x_\tau \prec x_\sigma$ if $\sigma^- \langle 0 \rangle \sqsubseteq \tau$ and $x_\sigma \prec x_\tau$ if $\sigma^- \langle 1 \rangle \sqsubseteq \tau$. Suppose $x_\sigma \prec x_\tau$ (the other case is similar) and let $\eta \in T$ so that $\tau^- \langle 1 \rangle \sqsubseteq \eta$ and $\eta^- \langle 0 \rangle \in T$. Then $x_\tau \prec x_\eta \prec x_\sigma$ by (iii). Suppose now that $\sigma$ and $\tau$ are not one initial segment of the other. We may assume that $\eta^- \langle 0 \rangle \sqsubseteq \sigma$ and $\eta^- \langle 1 \rangle \sqsubseteq \tau$ for some $\eta$. Then $x_\sigma \sqsubseteq x_\eta \sqsubseteq x_\tau$.

It remains to show that we can define $T$ and $f$ satisfying (i)–(iv).

By Theorem 2.16, $P$ has perfectly many initial intervals. Let $U$ be a perfect subtree of $T(P)$. By Corollary 2.8, there exists a countable coded $\omega$-model $M$ such that $P, U \in M$ and $M$ satisfies $\Sigma^1_1$-$\text{DC}_0$ and ATR$_0^P$.

We recursively define $T$ and $f$ using $M$ as a parameter. Let $\langle \rangle \in T$ and $f(\langle \rangle) = (\emptyset, \emptyset, \emptyset)$ as required by (i). Note that $M$ satisfies "$T(P_\emptyset)$ contains a perfect subtree". Let $\sigma \in T$ and assume by arithmetical induction that $M$ satisfies "$T(P_\sigma)$ contains...
a perfect subtree”. Since $M$ is a model of $\text{ACA}_0$, by Lemma 4.6 applied to $P_\sigma$, there exists $x \in P_\sigma$ such that either

(a) $M$ satisfies “$T(P_\sigma \cap P_{\perp x})$ has uncountably many paths”, or
(b) $M$ satisfies “both $T(P_\sigma \cap P_{\perp x})$ and $T(P_\sigma \cap P_{\perp x})$ have uncountably many paths”.

Search the least $x$ with this arithmetical property. If (a) holds (and we can check this arithmetically outside $M$), use $\text{ATR}_0^\mathrm{P}$ within $M$ to apply Theorem 2.6 to the $P$-computable tree $T(P_\sigma \cap P_{\perp x})$. We obtain that $M$ satisfies “$T(P_\sigma \cap P_{\perp x})$ contains a perfect subtree”. Thus, let $\sigma(2) = (F_\sigma, G_\sigma, H_\sigma \cup \{x\})$. If (b) holds, then arguing analogously we obtain that $M$ satisfies “both $T(P_\sigma \cap P_{\perp x})$ and $T(P_\sigma \cap P_{\perp x})$ contain perfect subtrees”. Thus let $\sigma(0), \sigma(1) \in T$ and set

$$f(\sigma(0)) = (F_\sigma \cup \{x\}, G_\sigma, H_\sigma) \text{ and } f(\sigma(1)) = (F_\sigma, G_\sigma \cup \{x\}, H_\sigma).$$

In any case, (ii)-(iv) are satisfied and the induction hypothesis that $M$ satisfies “$T(P_\sigma)$ contains a perfect subtree” is preserved. \hfill \Box

**Theorem 4.8.** Over $\text{ACA}_0$, the following are equivalent:

1. $\text{ATR}_0$;
2. every countable scattered partial order with no infinite antichains has countably many initial intervals;
3. every countable scattered linear order has countably many initial intervals.

**Proof.** (1) $\implies$ (2) is Theorem 4.7 and (2) $\implies$ (3) is immediate. We show (3) $\implies$ (1) by essentially repeating the proof of [Clo89, Theorem 18].

Assume $\text{ACA}_0$. We wish to prove $\text{ATR}_0$. By [Sim09, Theorem V.5.2], $\text{ATR}_0$ is equivalent (over $\text{RCA}_0$) to the statement asserting that for every sequence of trees $\{T_i : i \in \mathbb{N}\}$ such that every $T_i$ has at most one path, there exists the set $\{i \in \mathbb{N} : [T_i] \neq \emptyset\}$. So let $\{T_i : i \in \mathbb{N}\}$ be such a sequence. Let us order each $T_i$ with the Kleene-Brouwer order $\leq_{KB}$ and define the linear order $L = \bigcup_{i \in \mathbb{N}} T_i$.

We aim to show that $L$ is scattered. By Lemma 2.3, it suffices to prove that every $T_i$ is scattered. To this end, we show that if a tree $T$ has at most one path then the Kleene-Brouwer order on $T$ is of the form

$$X + \sum_{n \in \omega} Y_n,$$

where $X$ and the $Y_n$ are (possibly empty) well-orders. Applying Lemma 2.3 again, we obtain that $T$ is scattered.

If $T$ has no path, then $\text{ACA}_0$ proves that $\leq_{KB}$ well-orders $T$, and hence we can take $X = T$ and the $Y_n$’s empty. Now let $f$ be the unique path of $T$. Let $X = \{\sigma \in T : (\forall n) \sigma <_{KB} f \upharpoonright n\}$ and $Y_n = \{\sigma \in T : f \upharpoonright n + 1 <_{KB} \sigma \leq_{KB} f \upharpoonright n\}$, for all $n \in \mathbb{N}$. It is straightforward to see that (*) holds. We now claim that $X$ is a well-order. Suppose not, and let $(\sigma_n)_{n \in \mathbb{N}}$ be an infinite descending sequence in $X$. Form the tree $T_0 = \{\sigma \in T : (\exists n) \sigma \subseteq \sigma_n\}$. Then $T_0$ is not well-founded and so it has a path. As $T_0$ is a subtree of $T$, this path must be $f$. Let $i \in \mathbb{N}$ be such that $\sigma_0 \upharpoonright i = f \upharpoonright i$ and $\sigma(i) < f(i)$ (such an $i$ exists because $\sigma_0 \in X$). On the other hand, $f \upharpoonright i + 1 \in T_0$, and thus $f \upharpoonright i + 1 \subseteq \sigma_n$ for some $n \in \mathbb{N}$. It follows that $\sigma_0 <_{KB} \sigma_n$, a contradiction. The proof that each $Y_n$ is a well-order is easier and we omit it.

Apply (3) to $L$ and let $I(L) = \{I_n : n \in \mathbb{N}\}$. It is easy to check that $I_1$ has a path if and only if

$$(\exists n)\left(\bigcup_{j < i} T_j \subseteq I_n \land T_i \not\subseteq I_n \land L \setminus I_n \text{ has no least element}\right).$$

Therefore, the set $\{i \in \mathbb{N} : [T_i] \neq \emptyset\}$ exists by arithmetical comprehension. \hfill \Box
It is worth noticing that a natural weakening of condition (3) of Theorem 4.8 is provable in RCA₀:

**Lemma 4.9 (RCA₀).** Every linear order with perfectly many initial intervals is not scattered.

**Proof.** Let \( L \) be a linear order and \( T \subseteq T(L) \) be a perfect tree. Define

\[
Q = \{ x \in L : (\exists \sigma \in T)(|\sigma| = x \land \sigma^- (0); \sigma^- (1) \in T) \}.
\]

The argument showing that \( Q \) is a dense subset of \( L \) is similar to the one in the proof of Theorem 4.7. \( \Box \)

5. The right to left directions

In this section we study the right to left directions of Theorems 1.1, 1.3, and 1.5. The right to left direction of Theorem 1.5 naturally splits into two statements with the same hypothesis (the existence of countably many initial intervals) and different conclusions (the partial order is scattered and the partial order has no infinite antichains). We have thus four different statements altogether. All these statements have simple proofs in ACA₀, but it turns out that each of them can be proved in a properly weaker system.

5.1. Proofs in RCA₀. We start with a simple observation about the right to left direction of Theorem 1.3

**Lemma 5.1 (RCA₀).** Every partial order which is a finite union of ideals has no infinite strong antichains.

**Proof.** Since an ideal does not contain incompatible elements, if the partial order is the union of \( k \) ideals we have even a finite bound on the size of strong antichains. \( \Box \)

Another statement that can be proved in RCA₀ is the following half of the right to left direction of Theorem 1.5

**Theorem 5.2 (RCA₀).** Every partial order with countably many initial intervals is scattered.

**Proof.** We show that if \( P \) is not scattered, then \( P \) has perfectly many initial intervals. By Lemma 2.2 we may assume that \( P \) contains a dense linear order \( Q \).

We define by recursion an embedding \( f : 2^{< \omega} \to T(P) \). Thus \( T₀ = \{ \tau \in T(P) : (\exists \sigma \in 2^{< \omega}) \tau \subseteq f(\sigma) \} \) is a perfect subtree of \( T(P) \). Since \( \tau \in T₀ \) if and only if \((\exists \sigma \in 2^{< \omega})(|\sigma| = |\tau| \land \tau \subseteq f(\sigma))\), \( T₀ \) exists in RCA₀.

We say that \( x \in P \) is free for \( \tau \in T(P) \) if

\[
(\forall y < |\tau|)((\tau(y) = 1 \implies x \notin y) \land (\tau(y) = 0 \implies y \notin x)).
\]

In other words, \( x \) is free for \( \tau \) if and only if there exist \( \tau_0, \tau_1 \in T(P) \) with \( \tau \subset \tau_i \) and \( \tau_i(x) = i \). Since \( T(P) \) is a pruned tree this means that there exist two initial intervals of \( P \) whose characteristic function extends \( \tau \), one containing \( x \) and the other avoiding \( x \).

Let \( f(\langle \rangle) = \langle \rangle \). Suppose we have defined \( f(\sigma) = \tau \). Assume by \( \Sigma^0_1 \) induction that \( Q \) contains at least two (and hence infinitely many) elements that are free for \( \tau \). Then search for \( a \prec b \prec c \in Q \) that are free for \( \tau \). We will define \( \tau_0, \tau_1 \in T(P) \) which are extensions of \( \tau \) with \(|\tau_i| = b + 1\) and \( \tau_i(b) = i \). Thus \( \tau_0 \) and \( \tau_1 \) are incompatible and we can let \( f(\sigma^- (i)) = \tau_i \).

We show how to define \( \tau_0 \) (to define \( \tau_1 \) replace \( a \) with \( b \) and \( b \) with \( c \)). Since \( \{ x \in P : x < b \} \) is finite, we can find \( a', b' \in Q \) with \( a \prec a' \prec b' \prec b \) such that \( a', b' > b \), and for no \( x \in P \) with \( x < b \) we have \( a' \prec x \prec b' \). Let \( x < |\tau_0| \). If \( x \notin P \) or \( x \in P \) and \( x \) is not free for \( \tau \), \( \tau_0(x) \) is determined by the fact that we want \( \tau_0 \).
to extend $\tau$ and belong to $T(P)$ (in the first case $\tau_0(x) = 0$; in the second case if for some $y < |\tau|$ with $\tau(y) = 1$ we have $x \preceq y$ we let $\tau_0(x) = 1$, while if for some $y < |\tau|$ with $\tau(y) = 0$ we have $y \preceq x$ we let $\tau_0(x) = 0$). Last, if $x$ is free for $\tau$ we define $\tau_0(x)$ according to the following cases:

(i) if $x < b'$, let $\tau_0(x) = 1$;
(ii) if $x > a'$, let $\tau_0(x) = 0$;
(iii) otherwise, let $\tau_0(x) = 0$.

It is not difficult to check that $\tau_0$ extends $\tau$, $\tau_0(b) = 0$ and both $a'$ and $b'$ are free for $\tau$, preserving the induction hypothesis. □

5.2. Proofs in $\textsf{WKL}_0$. We now look at the right to left direction of Theorem 1.1 which states that every partial order with an infinite antichain contains an initial interval that cannot be written as a finite union of ideals. The proof can be carried out very easily in $\textsf{ACA}_0$; just take the downward closure of the given antichain. We improve this upper bound by showing that $\textsf{WKL}_0$ suffices. We first point out that $\textsf{RCA}_0$ proves a particular instance of the statement.

Lemma 5.3 ($\textsf{RCA}_0$). Let $P$ be a partial order with a maximal (with respect to inclusion) infinite antichain. Then there exists an initial interval that is not a finite union of ideals.

Proof. Let $D$ be a maximal infinite antichain of $P$. The maximality of $D$ implies that for all $x \in P$ we have

$$(\exists d \in D)x \preceq d \iff \neg(\exists d \in D)d < x.$$ 

Therefore the downward closure of $D$ is $\Delta^0_1$ definable and thus exists in $\textsf{RCA}_0$. Letting $I = \{x \in P : (\exists d \in D)x \preceq d\}$ we obtain an initial interval which is not finite union of ideals, since distinct elements of $D$ are incompatible in $I$. □

To use Lemma 5.3 in the general case we need to extend an infinite antichain to a maximal one. While it is easy to show that $\textsf{RCA}_0$ proves the existence of maximal antichains in any partial order, the statement that every antichain is contained in a maximal antichain is equivalent to $\textsf{ACA}_0$, and thus does not help in our case.

Lemma 5.4. Over $\textsf{RCA}_0$, the following are equivalent:

1. $\textsf{ACA}_0$;
2. every antichain in a partial order extends to a maximal antichain.

Proof. We first show (1) $\implies$ (2). Let $P$ be a partial order and $D \subseteq P$ be an antichain. By recursion we define $f : \mathbb{N} \rightarrow \{0, 1\}$ by letting $f(x) = 1$ if and only if $D \cup \{y < x : f(y) = 1\} \cup \{x\}$ is an antichain in $P$. Then $E = \{x : f(x) = 1\}$ is a maximal antichain with $D \subseteq E$.

For the reversal argue in $\textsf{RCA}_0$ and fix a one-to-one function $f : \mathbb{N} \rightarrow \mathbb{N}$. Let $P = \{a_n, b_n : n \in \mathbb{N}\}$ and define the partial order by letting $b_m \leq a_n$ if and only if $f(m) = n$, and adding no other comparabilities. Then apply (2) to the antichain $D = \{b_m : m \in \mathbb{N}\}$ and obtain a maximal antichain $E$ such that $D \subseteq E$. It is immediate that $(\exists m)f(m) = n$ if and only if $a_n \notin E$, so that in $\textsf{RCA}_0$ we can prove the existence of the range of $f$. □

We now show how to prove the right to left direction of Theorem 1.1 in $\textsf{WKL}_0$.

Theorem 5.5 ($\textsf{WKL}_0$). Every partial order with an infinite antichain contains an initial interval that is not finite union of ideals.

Proof. We reason in $\textsf{WKL}_0$. Let $P$ be a partial order such that every initial interval is a finite union of ideals. Suppose towards a contradiction that there exists an infinite antichain $D$. 

Let \( \varphi(x) \) and \( \psi(x) \) be the \( \Sigma^0_1 \) formulas \( x \in D \) and \( (\exists y)(y \in D \land y \prec x) \) respectively. It is obvious that \( (\forall x,y \in P)(\varphi(x) \land \psi(y) \implies y \not\prec x) \). By \( \Sigma^0_1 \) initial interval separation (Lemma 3.1), there exists an initial interval \( I \subseteq P \) such that
\[
(\forall x \in P)((\varphi(x) \implies x \in I) \land (\psi(x) \implies x \not\in I)).
\]
Therefore, \( I \) contains \( D \) and no element above any element of \( D \). To see that \( I \) cannot be the union of finitely many ideals notice that distinct \( x, x' \in D \) cannot belong to the same ideal \( A \subseteq I \), for otherwise there would be \( z \in I \) such that \( x, x' \preceq z \), which implies \( \psi(z) \).

We do not know whether the statement of Lemma 5.6 implies WKL\(_0\). Notice however that the proof above uses the existence of an initial interval \( I \) containing the infinite antichain \( D \) and with no elements above any element of \( D \). We now show that even the existence of an initial interval \( I \) containing infinitely many elements of the antichain \( D \) and no elements above \( D \) is actually equivalent to WKL\(_0\). Therefore a proof of the right to left direction of Theorem 1.5 in a system weaker than WKL\(_0\) must avoid using such an \( I \).

Lemma 5.6. Over RCA\(_0\), the following are equivalent:

1. WKL\(_0\);
2. if a partial order \( P \) contains an infinite antichain \( D \), then \( P \) has an initial interval \( I \) such that \( D \subseteq I \) and \( (\forall x \in D)(\forall y \in I)x \not\prec y \);
3. if a partial order \( P \) contains an infinite antichain \( D \), then \( P \) has an initial interval \( I \) such that \( I \cap D \) is infinite and \( (\forall x \in D)(\forall y \in I)x \not\prec y \).

Proof. We just need to show \( (3) \implies (1) \). If \( f, g: \mathbb{N} \to \mathbb{N} \) are one-to-one functions with \( (\forall n, m \in \mathbb{N})f(n) \neq g(m) \), let \( P = \{a_n, b_n: n \in \mathbb{N}\} \) the partial order defined by letting

1. \( a_n \leq_P b_{g(n)} \);
2. \( b_k \leq_P a_n \) if \( (\exists i < n)(i < g(n) \land f(i) = k) \), i.e. \( k \) enters the range of \( f \) before stage \( \min\{n, g(n)\} \);
3. \( b_k \leq_P b_m \) if \( (\exists i < m)f(i) = k \), i.e. \( k \) enters the range of \( f \) before stage \( m \), and adding no other comparabilities. Notice that \( P \) is indeed a partial order (in particular transitivity holds). The set \( D = \{a_n: n \in \mathbb{N}\} \) is an infinite antichain. Applying (3) to \( D \), we obtain an initial interval \( I \) of \( P \) which contains infinitely many elements of \( D \) and no elements above \( D \). It is easy to show that \( \{k \in \mathbb{N}: b_k \in I\} \) separates the range of \( f \) from the range of \( g \).

We notice that a possible weakening of statement (2) of the Lemma above, i.e. “if a partial order \( P \) contains an infinite antichain \( D \), then there exists an initial interval \( I \) such that \( D \subseteq I \) and \( (\forall y \in I)(\exists x \in D)x \not\prec y \)”, even though it does not imply the statement of Theorem 5.5 is still equivalent to WKL\(_0\) (the reversal uses the partial order of the proof above equipped with the inverse order).

Our next goal is to show that WKL\(_0\) suffices to prove the half of the right to left direction of Theorem 1.5 that was not proved in RCA\(_0\) in Theorem 5.2. In other words, we study the statement that every partial order with countably many initial intervals has no infinite antichains. We start with a simple observation:

Lemma 5.7 (RCA\(_0\)). An infinite antichain has perfectly many initial intervals.

Proof. If \( P \) is an antichain then the tree \( T(P) \) consists of all \( \sigma \in 2^{<\mathbb{N}} \) such that \( x \notin P \) implies \( \sigma(x) = 0 \). If \( P \) is infinite it is immediate that this tree is perfect and thus Lemma 2.4 implies that \( P \) has perfectly many initial intervals.
Since we need to study partial orders with infinite antichains we are lead to consider the relationship between initial intervals of partial orders contained one into the other.

**Lemma 5.8.** Over RCA₀, the following are equivalent:

1. WKL₀;
2. Let Q and P be partial orders and f be an embedding of Q into P. Then \( \mathcal{I}(Q) = \{ f^{-1}(J) : J \in \mathcal{I}(P) \} \);
3. Let Q be a subset of a partial order P. Then \( \mathcal{I}(Q) = \{ J \cap Q : J \in \mathcal{I}(P) \} \).

**Proof.** We start with (1) \( \implies \) (2). Let f: Q \rightarrow P be an embedding. It is easy to check that if J \( \in \mathcal{I}(P) \) then \( f^{-1}(J) \in \mathcal{I}(Q) \), so that the right to left inclusion is established even in RCA₀.

For the other inclusion fix I \( \in \mathcal{I}(Q) \). Let \( \varphi(x) \) and \( \psi(x) \) be the \( \Sigma^0_1 \) formulas \((\exists y \in Q)(y \in I \land x = f(y))\) and \((\exists y \in Q)(y \notin I \land x = f(y))\) respectively. Since f is an embedding and I is an initial interval, we have

\[
(\forall x, y \in P)(\varphi(x) \land \psi(y) \implies y \notin f \ x).
\]

Apply \( \Sigma^0_1 \) initial interval separation (Lemma 3.1) to get J \( \in \mathcal{I}(P) \) such that f(I) \( \subseteq J \) and \( J \cap f(Q \setminus f(\cdot)) = \emptyset \). It is immediate that \( I = f^{-1}(J) \).

Since the implication (2) \( \implies \) (3) is obvious, it remains to show (3) \( \implies \) (1).

Instead of WKL₀, we prove statement (3) of Lemma 3.1 i.e. initial interval separation. Let P be a partial order and A, B \( \subseteq P \) such that \((\forall x \in A)(\forall y \in B) y \notin \ x \). Let Q = A \cup B \( \subseteq P \) and notice that A \( \in \mathcal{I}(Q) \). By (3) there exists J \( \in \mathcal{I}(P) \) such that A = J \( \cap Q \). It is easy to see that A \( \subseteq J \) and J \( \cap B = \emptyset \), completing the proof. \( \square \)

Notice that the obvious proof of the nontrivial direction of (2), namely given I \( \in \mathcal{I}(Q) \) let J be the downward closure of f(I), uses arithmetical comprehension.

**Corollary 5.9 (WKL₀).** Let P and Q be partial orders such that Q embeds into P. If P has countably many initial intervals, then Q does.

**Proof.** Fix an embedding f: Q \rightarrow P. Let \( \{ J_n : n \in \mathbb{N} \} \) be such that for all J \( \in \mathcal{I}(P) \) there exists n with J = J_n. For every n let I_n = f^{-1}(J_n), which exists in RCA₀. Then, by Lemma 5.8 for all I \( \in \mathcal{I}(Q) \) there exists n with I = I_n, showing that Q has countably many initial intervals. \( \square \)

We can now prove in WKL₀ the part of the right to left direction of Theorem 1.5 we are interested in.

**Theorem 5.10 (WKL₀).** Every partial order with countably many initial intervals has no infinite antichains.

**Proof.** Immediate from Lemma 5.7 and Corollary 5.9. \( \square \)

### 5.3. Unprovability in RCA₀

In this section we show that RCA₀ does not suffice to prove the statements shown in the previous section to be provable in WKL₀.

A single construction actually works for both statements.

**Lemma 5.11.** There exists a computable partial order P with an infinite computable antichain such that any computable initial interval of P is the downward closure of a finite subset of P.

Before proving Lemma 5.11 we show how to deduce from it the unprovability results.
Theorem 5.12. RCA₀ does not prove that every partial order such that all its initial intervals are finite union of ideals has no infinite antichains.

Proof. It suffices to show that the statement fails in REC, the \( \omega \)-model of computable sets. Let \( P \) the computable partial order of Lemma 5.11 and let \( I \) be a computable initial interval of \( P \). Let \( F \) be a finite set such that \( P = \downarrow F \). Then \( I = \bigcup_{x \in F} P_{\geq x} \) and each \( P_{\geq x} \) is a computable ideal.

Thus all initial intervals of \( P \) which belong to REC are finite union of ideals also belonging to REC. On the other hand, \( P \) has an infinite antichain in REC, showing the failure of the statement.

\( \square \)

Theorem 5.13. RCA₀ does not prove that every partial order with countably many initial intervals has no infinite antichains.

Proof. We again show that the statement fails in REC, and again use the computable partial order \( P \) of Lemma 5.11. Since the downward closures of finite subsets of \( P \) are uniformly computable, there exists a set \( \{I_n: n \in N\} \) in REC which lists all computable initial intervals of \( P \). Therefore REC satisfies that \( P \) has countably many initial intervals. Since \( P \) has an infinite antichain in REC, the statement fails.

\( \square \)

Proof of Lemma 5.11. We build \( P \) by a finite injury priority argument. We let \( P = \{x_n, y_n: n \in \omega\} \) and ensure the existence of an infinite computable antichain by making the \( x_n \)'s pairwise incomparable.

We further make sure that, for all \( e \in \omega \), \( P \) meets the requirements:

\[ \exists z \left( \Phi_e(z) = 1 \implies (\forall z \in P) z \leq y \right) \land \left( \Phi_e(y) = 0 \implies (\forall z \in P) y \leq z \right) \]

Here, as usual, \( \Phi_e \) is the function computed by the Turing machine of index \( e \) and \( \forall^\infty \) means ‘for all but finitely many’.

We first show that meeting all the requirements implies that \( P \) satisfies the statement of the Lemma. If \( I \) is a computable initial interval of \( P \) with characteristic function \( \Phi_e \), fix \( y \) given by \( R_e \). We must have \( \Phi_e(y) \in \{0,1\} \). If \( \Phi_e(y) = 0 \) then, by \( R_e \), \( (\forall z \in P) y \leq z \). As \( y \notin I \), this implies that \( I \) is finite and hence \( I = \downarrow I \) is the downward closure of a finite set. If \( \Phi_e(y) = 1 \), then by \( R_e \) we have \( (\forall z \in P) z \leq y \). Thus \( P \setminus P_y \) and hence \( I \setminus P_y \) are finite. As \( y \in I \), \( I = \downarrow I \) \( \cup \{y\} \cup (I \setminus P_y) \) is the downward closure of a finite set.

Our strategy for meeting a single requirement \( R_e \) consists in fixing a witness \( y_n \) and waiting for a stage \( s + 1 \) such that

\[ \Phi_{e,s}(y_n) \in \{0,1\} \]

If this never happens, \( R_e \) is satisfied. If \( \Phi_{e,s}(y_n) = 0 \), we put every \( x_m \) and \( y_m \) with \( m > s \) above \( y_n \). If \( \Phi_{e,s}(y_n) = 1 \), we put every \( x_m \) and \( y_m \) with \( m > s \) below \( y_n \).

In this way \( R_e \) is obviously satisfied.

To meet all the requirements, the priority order is \( R_0, R_1, R_2, \ldots \) At every stage \( s \), we associate \( n_{e,s} \) to \( R_e \) and mark the requirements by a \( \{0,1\} \)-valued function \( r(e,s) \) such that \( r(e,s) = 0 \) if and only if \( R_e \) might require attention at stage \( s \).

Stage \( s = 0 \). For all \( e \), \( n_{e,0} = e \) and \( r(e,0) = 0 \).

Stage \( s + 1 \). We say that \( R_e \) requires attention at stage \( s + 1 \) if \( e \leq s \), \( n_{e,s} \leq s \), \( r(e,s) = 0 \) and \( \Phi_{e,s}(y_{n_{e,s}}) \in \{0,1\} \). If no \( R_e \) requires attention, then let \( n_{i,s+1} = n_{i,s} \) and \( r(i,s+1) = r(i,s) \) for all \( i \). Otherwise, let \( e \) be least such that \( R_e \) requires attention. Then \( R_e \) receives attention at stage \( s + 1 \) and \( n = n_{e,s} \) is activated and declared low if \( \Phi_{e,s}(y_n) = 0 \), high if \( \Phi_{e,s}(y_n) = 1 \). Let \( n_{e,s+1} = n_{e,s} \) and \( r(e,s+1) = 1 \). For \( i < e \), \( n_{i,s+1} = n_{i,s} \) and \( r(i,s+1) = r(i,s) \). For \( i > e \), \( n_{i,s+1} = s + i - e \) and \( r(i,s+1) = 0 \).
The following two properties are easily seen to hold:

1. every \( n \) is activated at most once;
2. if \( n \) is activated at stage \( s \), then no \( m \) such that \( n < m < s \) is activated after \( s \).

We define \( n \geq m \) by stipulating that for all \( n < m \):

(i) \( x_n \) is incomparable with both \( x_m \) and \( y_m \);
(ii) \( y_n \leq (\geq) x_m, y_m \) if and only if \( n \) is activated at some stage \( s \) such that \( n < s \leq m \), is declared low (high) and no \( k < n \) is activated at any stage \( t \) such that \( s < t \leq m \).

When (ii) occurs, it follows by (2) that no \( k < n \) is activated at any stage \( t \) such that \( n < t \leq m \).

**Claim 1.** \( P \) is a partial order.

**Proof of claim.** We use \( z_n \) to denote one of \( x_n \) and \( y_n \).

To show antisymmetry, suppose for a contradiction that \( z_n \preceq z_m \) and \( z_m \preceq z_n \) with \( n < m \). By (i) \( z_n \) must be \( y_n \). Since \( n \) can be activated only once, it follows that \( n \) is activated at some stage \( s \) with \( n < s \leq m \) and, by (ii), is declared both low and high, a contradiction.

To check transitivity, let \( z_n < z_m < z_p \). Notice that \( n, m, \) and \( p \) are all distinct.

We consider the following cases:

(a) \( n < m, p \). Then \( z_n = y_p \) and \( n \) is activated and declared low at some stage \( s \) such that \( n < s \leq m \). It is easy to verify that no \( k < n \) is activated at any stage \( t \) such that \( n < t \leq p \), and thus \( y_n \leq z_p \).

(b) \( m < n, p \). Then \( z_m = y_n \) and \( m \) is declared both high and low, contradiction.

(c) \( p < n, m \). Then \( z_p = y_m \) and \( p \) is activated and declared high at some stage \( s \) such that \( p < s \leq m \). As in case (a), it is easy to check that no \( k < p \) is activated at any stage \( t \) such that \( p < t \leq n \), and so \( z_n \preceq z_p \).

**Claim 2.** Every \( R_e \) receives attention at most finitely often and is satisfied.

**Proof of claim.** As usual, the proof is by induction on \( e \). Let \( s \) be the least such that no \( R_i \) with \( i < e \) receives attention after \( s \). Let \( n = n_{e,s} \). Then \( n = n_{e,t} \) for all \( t \geq s \), because a witness for a requirement changes only when a stronger priority requirement receives attention. If \( \Phi_e(y_n) \notin \{0, 1\} \), \( R_e \) is clearly satisfied.

Suppose that \( \Phi_e(y_n) = 0 \) (case 1 is similar) and let \( t \) be minimal such that \( t \geq s \) and \( \Phi_{e,t}(y_n) = 0 \). Then \( R_e \) receives attention at stage \( t + 1 \), \( n \) is activated and declared low and no \( m < n \) will be activated after stage \( t + 1 \) (because \( n_{i,u} > n \) for all \( i > e \) and \( u > t \)). Then \( y_n \preceq x_m, y_m \) for all \( m > t \) and so \( R_e \) is satisfied.

Claim 2 completes the proof of the Lemma.

**6. Open problems**

The results of Sections 5.2 and 5.3 leave open the status of the right to left directions of Theorems 1.1 and 1.5. Each of the statements (1) “every partial order with an infinite antichain contains an initial interval which is not finite union of ideals” and (2) “every partial order with an infinite antichain has uncountably many initial intervals” can be either equivalent to \( \text{WKL}_0 \) or of strength strictly between \( \text{RCA}_0 \) and \( \text{WWKL} \). The latter case would be quite interesting, since the only mathematical statements with this intermediate strength are those from measure theory that are equivalent to the system \( \text{WWKL}_0 \). Another principle which is intermediate between \( \text{RCA}_0 \) and \( \text{WWKL}_0 \) has been isolated by Flood (12), who called it \( \text{RKL} \) (\( \text{RKL} \) has the interesting feature of following from each of \( \text{WKL}_0 \) and
Ramsey’s theorems for pairs). We have not been able to find any relation between our statements and RKL.

With regard to the first problem, Gregory Igusa showed that there cannot be a uniform proof of WKL_0 from (1), meaning that for no computable partial order P there is a Turing functional \( \Phi \) such that whenever \( I \) is an initial interval of \( P \) which is not a finite union of ideals, \( \Phi^I \) is of PA-degree. However Igusa’s result does not rule out the possibility that (1) implies WKL_0: e.g. it might exist a proof using twice the statement, the second time using it on a partial order built from the initial interval obtained by the first application.

Again, Ludovic Patey [BPS13] was able to improve our result and show that WWKL_0 does not imply (1) by modifying the proof of Lemma 5.11. We refer to [BPS13] for this and other several non-implications involving our statement (there denoted by NCF).

**References**


