Initial intervals of partial orders
and reverse mathematics

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Initial intervals and ideals

Definition
Let $P$ be a partial order.

- $I \subseteq P$ is an **initial interval** of $P$ if
  \[ \forall x, y \in P(x \leq_P y \land y \in I \implies x \in I); \]
- An initial interval $I$ of $P$ is an **ideal** if
  \[ \forall x, y \in I \exists z \in I(x \leq_P z \land y \leq_P z) \text{ (compatibility)}; \]
- $A \subseteq P$ is an **antichain** if \[ \forall x, y \in A (x = y \lor x \perp y); \]
- $S \subseteq P$ is a **strong antichain** in $P$ if
  \[ \forall x, y \in S(x = y \lor \lnot \exists z \in P x, y \leq_P z); \]
The theorems

Theorem (Bonnet 1975)

A partial order $P$ is FAC (no infinite antichains) if and only if every initial interval of $P$ is a finite union of ideals.

Theorem (Erdös-Tarski 1943)

If a partial order $P$ has no infinite strong antichains, then there are no arbitrarily large strong antichains in $P$.

Theorem (Bonnet 1973)

A countable partial order $P$ has countably many initial intervals iff it is scattered (no copy of $\mathbb{Q}$ in $P$) and FAC.
Results

Theorem

*Over RCA$_0$, the following are pairwise equivalent:*

1. ACA$_0$;
2. *if a partial order is FAC, then every initial interval is a finite union of ideals* (Bonnet 1975);
3. *every partial order with no infinite strong antichains has a finite bound on the size of strong antichains* (Erdös-Tarski 1943).
Results

**Theorem**

*Over $\text{ACA}_0$, the following are equivalent:*

1. $\text{ATR}_0$;
2. *every scattered FAC partial order has countably many initial intervals* (Bonnet 1973).
Results

Theorem

*Provably in WKL₀ but not in RCA₀,*

1. every partial order which is not FAC contains an initial interval which is not a finite union of ideals.

2. every partial order which is not FAC has uncountably many initial intervals.

Theorem (Patey)

*Neither 1. nor 2. is provable in WWKL₀.*
Reversal of Bonnet 1975

Theorem

Over RCA₀, the statement “if a partial order is FAC, then every initial interval is a finite union of ideals” implies ACA₀.

Proof.

Fix a one-to-one function \( f : \mathbb{N} \rightarrow \mathbb{N} \). We build a partial order \( P \) which encodes the range of \( f \). We use false and true stages (\( n \) is false if \( f(i) < f(n) \) for some \( i > n \)). We can arrange false and true stages into an \( \omega + \omega^* \)-chain. We then add an \( \omega \)-chain “above” the false stages. Antichains have at most two elements. Apply the statement. The ideal which contains the \( \omega \)-chain separates false from true stages.
Initial intervals and trees

Let $P$ be a partial order and $\mathcal{I}(P)$ be the class of initial intervals.

- $P$ has countably many initial intervals if there exists \{\(I_n: n \in \mathbb{N}\)\} such that $\forall I \in \mathcal{I}(P) \exists n \in \mathbb{N} \, I = I_n$.
- $P$ has perfectly many initial intervals if there exists a nonempty perfect tree $T \subseteq 2^{<\mathbb{N}}$ such that $[T] \subseteq \mathcal{I}(P)$.

The tree of initial intervals of $P$ is $T(P) \subseteq 2^{<\mathbb{N}}$: 

$\sigma \in T(P)$ iff for all $x, y < |\sigma|$:
- $\sigma(x) = 1$ implies $x \in P$;
- $\sigma(y) = 1$ and $x \leq_P y$ imply $\sigma(x) = 1$. 
Initial intervals and trees

- $P$ has countably many initial intervals iff $T(P)$ has countably many paths (within RCA$_0$).
- $P$ has perfectly many initial intervals iff $T(P)$ contains a perfect subtree (within RCA$_0$).
- “$P$ has uncountably many initial intervals” is $\Sigma^1_1$ (within ATR$_0$ via the Perfect Tree Theorem).
Mathematics of initial intervals

A proof of Bonnet 1973

Theorem

$\text{ATR}_0$ proves that every scattered FAC partial order has countably many initial intervals.

We want to iterate the following lemma:

Lemma (one-step lemma)

$\text{ACA}_0$ proves that if $P$ has perfectly many initial intervals, then there exists $x \in P$ such that either

(1) both $\downarrow x$ and $\uparrow x$ have uncountably many initial intervals, or

(2) $\bot x$ has uncountably many initial intervals.
We aim to define by recursion a pruned tree $T \subseteq \mathbb{N}^{<3}$ labeled with pairs $(x_\sigma, P_\sigma)$ such that

- $P_\sigma \subseteq P$ has uncountably many initial intervals;
- if $\sigma \in T$ splits then
  \begin{enumerate}
  \item $P_{\sigma 0} = P_\sigma \cap \downarrow x_\sigma$ and $P_{\sigma 1} = P_\sigma \cap \uparrow x_\sigma$
  \item if $\sigma \in T$ does not split then
  \begin{align*}
  \text{(2) } P_{\sigma 3} &= P_\sigma \cap \perp x.
  \end{align*}
  \end{enumerate}

This recursion is $\Sigma^1_1$ within ATR$_0$.

We overcome this and make it arithmetical by using countable coded $\omega$-models. The key point is that $T(P_\sigma)$ is indeed computable in $P$. 
The proof in ATR$_0$

**Theorem**

ATR$_0$ proves that for all $X$ there exists a countable coded $\omega$-model $M$ of $\Sigma^1_1$-DC$_0$ such that and $X \in M$ and $M$ satisfies the Perfect Tree Theorem for all trees computable in $X$.

**Proof of Theorem**

1. Assume $P$ has uncountably many initial intervals and let $U$ be a perfect subtree of $T(P)$;
2. Let $M$ be an $\omega$-model as above;
3. Define $T$ by recursion using $M$ as a parameter.
4. At each stage, apply the one-step lemma.
A proof of Bonnet 1975

**Theorem**

WKL$_0$ proves that every partial order which is not FAC contains an initial interval which is not a finite union of ideals.

**Proof.**

Let $P$ be a partial order with an infinite antichain $A$. We define an infinite binary tree such that any path is an initial interval which contains $A$ and no elements above $A$. By WKL$_0$, there is a path and so an initial interval $I$ with this property. $I$ cannot be a finite union of ideals because $A$ is a strong antichain in $I$. By the finite pigeonhole principle, two elements of $A$ would be in the same ideal and so compatible.
Let NCF be the statement “every partial order which is not FAC contains an initial intervals which is not the downward closure of a finite set”.

**Theorem**

NCF *is false in REC.*

**Proof.**

Diagonalize against any $\Phi_e$ computing an initial interval.

To defeat $\Phi_e$, fix $x$ and wait until you see $x \in \Phi_e$ or $x \notin \Phi_e$.

In the first case, build below $x$. In the second case, above $x$. □
Nonimplications by conservativity

NCF is a restricted $\Pi^1_2$ statement and so is not implied by $\Pi^0_1G$. 
Unprovability in WWKL$\_0$

Theorem (Patey)

There exists a computable partial order with a computable antichain such that the class of oracles computing initial intervals which are not the closure of a finite set is null.

Proof.
Diagonalize against any $\Phi^X_e$ computing an initial interval with enough measure.
To defeat $\Phi^X_e$, fix $x$ and wait until you see $\mu(X : x \in \Phi^X_e) \geq 0.49$ or $\mu(X : x \notin \Phi^X_e) \geq 0.49$.
In the first case, build below $x$. In the second, above $x$.  \qed
Corollary

WWKL\(_0\) does not prove NCF.

Proof.

Given \(P\) as above, let \(X\) be a ML random real not in the class. Thus \(X\) computes only “trivial” initial intervals of \(P\). Build an \(\omega\)-model of WWKL\(_0\) below \(X\) as usual. Inside the model, every initial interval of \(P\) is the closure of a finite set. In particular, initial intervals are uniformly computable, and hence \(P\) has countably many initial intervals.

Corollary

WWKL\(_0\) does not prove the “easy” directions of Bonnet 75 and Bonnet 73.
Open questions

Is “every partial order which is not FAC has an initial interval which is not a finite union of ideals” equivalent to WKL₀?

Gregory Igusa showed that any computable instance has a solution which is not a PA degree. On the other hand, WKL₀ is known to be very robust.
Open questions

Is “every partial order which is not FAC has an initial interval which is not a finite union of ideals” equivalent to $\text{WKL}_0$?

Gregory Igusa showed that any computable instance has a solution which is not a PA degree. On the other hand, $\text{WKL}_0$ is known to be very robust.

Is “every partial order which is not FAC has uncountably many initial intervals” is equivalent to $\text{WKL}_0$?
Thanks