(An)isohedrality and skeleton of spherical monohedral polygonal tilings

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1 Introduction

By Euler formula, the tile of spherical monohedral polygonal tiling is a 3-gon, a 4-gon or a 5-gon. Once all spherical monohedral polygonal tilings are classified, all spherical *anisohedral* polygons will be enumerated (Section 2) and the "reverse problem" of Grünbaum-Shephard's theorem [4] on spherical isohedral tilings will be solved (Section 3).

2 Anisohedral spherical tiles

We say a tiling on a constant curvature space is *isohedral*, if the symmetry group transitively acts on the tiles. We say a tile is *anisohedral*, if copies of it admit a monohedral tilings but no isohedral tiling. In his eighteenth problem, Hilbert posed three problems. The second problem of the three is "are there any anisohedral tile of the Euclidean space $\mathbb{R}^3$?". As for the problem with $\mathbb{R}^3$ replaced by the sphere $\{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$, the answer is yes. According to [5], the set of anisohedral, spherical 3-gons consists of an infinite series of isosceles 3-gons AI_n ($5 \neq n \geq 4$), and an infinite series of right scalene 3-gons ARS_m ($m \geq 4$), which is the bisection of AI_m. AI_n is determined by the three inner angles $\left(\pi\left(\frac{1}{2} - \frac{1}{2n}\right), \pi\left(\frac{1}{2} - \frac{1}{2n}\right), \frac{2\pi}{n}\right)$. This is an application of classification of all spherical tilings by congruent 3-gons [6]. For spherical monohedral, non-isohedral tilings by anisohedral triangles, see Figure 1.

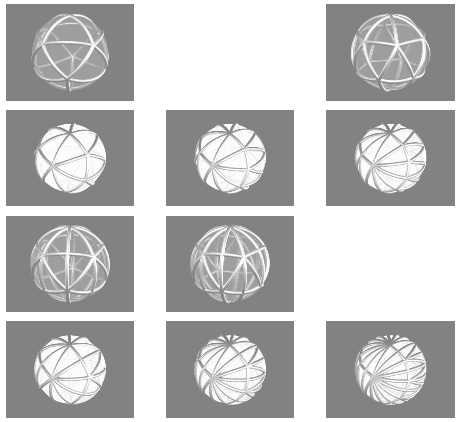

Figure 1: The spherical monohedral tilings by the anisohedral isosceles 3-gons AI_n are exactly two series. The first few tilings of the first (second, resp.) series are the first line (the second line, resp.). The spherical monohedral tilings by the anisohedral right scalene 3-gons ARS_n are exactly two series. The first few of the first (the second, resp.) series are the third line (the fourth line, resp.).

3 Skeletons and isohedrality

A *semi-regular dual* is a polytope P such that the symmetry group of P transitively acts on the faces but not the vertices. The set of semi-regular duals consists of an infinite series of 3-gon-faced ones (i.e., n-gonal bipyramids [1] ($3 \leq n \neq 4$)), an infinite series of 4-gon-faced ones (i.e., n-gonal trapezohedra [2] ($4 \leq$

[1] The dual of n-gonal prisms.
[2] The dual of n-gonal antiprisms.

n)), 7 3-gon-faced ones, 4 4-gon-faced ones, and 2 5-gon-faced ones. The last 13 are called *Archimedean duals* (or *Catalan solids*).

Grünbaum-Shephard [4] proved that the skeleton of a spherical isohedral tiling is exactly that of a Platonic solid or the dual of a semi-regular polytope. In [1], we prove that there is, however, a spherical *non-isohedral* tiling $\mathcal{A}$ by 12 congruent *concave* 4-gons such that the skeleton of $\mathcal{A}$ is that of 6-gonal trapezohedron (Movies of $\mathcal{A}$ spinning around the three 2-fold rotation axes are available at the website `http://www.math.tohoku.ac.jp/akama/stcq`). *Is $\mathcal{A}$ the only spherical monohedral, non-isohedral tiling such that the skeleton is that of a Platonic solid or that of a semi-regular dual?* This "skeleton-isohedrality conjecture" is partially answered:

Theorem 1 *1. Any spherical monohedral polygonal tiling with a Platonic skeleton is isohedral [Y. A. and M. YAN].*

2. Any spherical monohedral 3-gon-faced tiling with a semi-regular dual skeleton is isohedral.

3. Let P be a 4-gon-faced semi-regular dual or the "14-th" Archimedean dual [3]. If a spherical tiling $\mathcal{T}$ by congruent convex *4-gons is topologically P, $\mathcal{T}$ is isohedral. Unless P is the deltoidal icositetrahedron, $\mathcal{T}$ is the central projection of P. [Y. A., N. van Cleemput and Y. Sakano]*

Proof. (1) By spherical isoperimetric inequality and spherical trigonometry and [3]. (2) From [6]. (3) For an infinite series of trapezohedra P, by case-by-case analysis with forbidden substructures. For the other P, by massive computation [7] of linear programming with forbidden substructures [2] for convex tiles, for all angle-assignments to all perfect face-matchings. For example, the deltoidal hexecontahedron, a 4-gon-faced Archimedean dual, has 6066688 face-matchings, and $6066688 \cdot 2^{60}$ angle-assignments to the skeleton. With forbidden substructures [2] for convex tiles, the number of angle-assignments to check reduces to 144. Linear programming finds no solvable, non-trivial angle-assignment. □

[3]The polytope P consisting of 24 congruent kites such that P is not a semi-regular dual but the face figures are all the same. The 24 deltoidal tiles of the "14-th" Archimedean dual organize the central projection of the deltoidal icositetrahedron, an Archimedean dual.

There is no non-trivial spherical tiling by 30 congruent possibly *concave* 4-gons such that the skeleton is the rhombic triacontahedron. The rhombic triacontahedron, a 4-gon-faced Archimedean dual, has 2^{11} face matchings, $2^{11} \cdot 2^{30}$ angle-assignments to the skeleton. No non-trivial angle-assignment is solvable. Theorem 1 (3) and this were computed by a PC cluster at the Mathematical Institute of Tohoku University and verified by a cluster computer at the computer center of Ghent University. We should find forbidden substructures for possibly concave tiles. We will discuss miscellaneous conjectures on degree, and relation to polycycle (due to M. Deza and Detour). The author thanks G. Brinkmann, N. van Cleemput, B. McKay, M. Deza, and M. Yan.

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