Stability and instability of group invariant asymptotic profiles for fast diffusion equations

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Abstract

This paper is concerned with group invariant solutions for fast diffusion equations in symmetric domains. It is first proved that the group invariance of weak solutions is inherited from initial data. After briefly reviewing previous results on asymptotic profiles of vanishing solutions and their stability, the notions of stability and instability of group invariant profiles are introduced under a similarly invariant class of perturbations, and moreover, some stability criteria are exhibited and applied to an annular domain case.

Keywords: fast diffusion equation; group invariance; asymptotic profile; stability analysis.

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1 Introduction

In this paper, we are concerned with the Cauchy-Dirichlet problem for the fast diffusion equation,

$$\partial_t \left(|u|^{m-2}u\right) = \Delta u \quad \text{in } \Omega \times (0,\infty), \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{1.2}$$

$$u(\cdot,0) = u_0 \quad \text{in } \Omega, \tag{1.3}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $m > 2$, $\partial_t = \partial/\partial t$, $u_0 \in H^1_0(\Omega)$ and Δ stands for the N-dimensional Laplacian. By putting $w = |u|^{m-2}u$, Equation (1.1) can be rewritten in a usual form of *fast diffusion equation,*

$$\partial_t w = \Delta \left(|w|^{r-2}w\right) \qquad \text{in } \Omega \times (0,\infty) \tag{1.4}$$

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with the exponent $r = m/(m-1) < 2$. Fast diffusion equations arise in the studies of plasma physics (see [6]), kinetic theory of gases, solid state physics and so on. It is well known that every solution of (1.1)–(1.3) vanishes in finite time for the case that $2 < m \leq 2^* := 2N/(N-2)_+$ (see [18], [5]). Moreover, for the case that $2 < m < 2^*$, Berryman and Holland [7] studied asymptotic profiles of solutions as well as the explicit rate of the extinction of solutions.

Now, let us address ourselves to the stability and instability of asymptotic profiles. Namely, our question is the following: For any initial data $u_0 \in H^1_0(\Omega)$ sufficiently close to an asymptotic profile ϕ, does the asymptotic profile of the unique solution $u = u(x,t)$ for (1.1)–(1.3) also coincide with ϕ or not ? In [7] and [15], the stability of the unique positive asymptotic profile is discussed for nonnegative initial data in some special cases (e.g., $N = 1$). Recently, in [8], further detailed behaviors of nonnegative solutions near the extinction time are investigated. Moreover, in [3], the notions of stability and instability of asymptotic profiles are precisely defined for (possibly) sign-changing initial data, and furthermore, some criteria for the stability and instability are presented under $2 < m < 2^*$. Furthermore, they are applied to several concrete cases of the domain Ω (e.g., ball domains) and the exponent m. However, there are still cases (e.g., annular domain case) which do not fall within the scope of the criteria.

In this paper, we treat symmetric domain cases and discuss the stability and instability of group invariant asymptotic profiles. More precisely, for a subgroup G of $O(N)$ and a G-invariant domain Ω, we only deal with G-invariant (e.g., radial) initial data and solutions of (1.1)–(1.3). Furthermore, the stability and instability of profiles are also discussed only under G-invariant perturbations.

In the next section, we prove the G-invariance of weak solutions for parabolic problems such as (1.1)–(1.3) with G-invariant initial data and domains. This issue would be obvious in strong formulations, where one can directly calculate the change of variables. However, one should pay careful attention in weak formulations of parabolic problems such as nonlinear diffusion equations because of the lack of pointwise representation of the time-derivative of solution in a dual space $H^{-1}(\Omega) = (H^1_0(\Omega))^*$. In Section 3, we first briefly review previous studies, particularly [3], on asymptotic profiles of vanishing solutions for fast diffusion equations and their stability. We next define the notions of stability and instability of G-invariant asymptotic profiles under G-invariant perturbations, and then, stability criteria will be presented for them under $2 < m < 2^*$. Finally, we discuss applications of the stability criteria to some cases (e.g., the annular domain case) which do not fall within the scope of the criteria in [3].

Notation. Let $H^1_0(\Omega)$ be the closure of $C^\infty_0(\Omega)$ in the usual Sobolev space

$H^1(\Omega) = W^{1,2}(\Omega)$. Let us denote by $\|\cdot\|_m$ the usual norm of $L^m(\Omega)$-space, and moreover, $\|\cdot\|_{1,2} := \|\nabla\cdot\|_2$ stands for the norm of $H^1_0(\Omega)$. For a function $u = u(x,t) : \Omega \times (0,\infty) \to \mathbb{R}$, we often write $u(t) := u(\cdot,t)$, which is a function from Ω into $\mathbb{R}$, for a fixed time $t > 0$.

2 Group invariance of weak solutions for parabolic problems

In this section, we shall prove that the group invariance of weak solutions for parabolic problems such as (1.1)–(1.3) is inherited from initial data and domains. More precisely, let G be a subgroup of $O(N)$ and let Ω be a G-*invariant domain* of $\mathbb{R}^N$, i.e., $g(\Omega) = \Omega$ for any $g \in G$, with smooth boundary $\partial\Omega$. Here let us treat

$$\partial_t \left(|u|^{m-2}u\right) = \Delta u + \lambda |u|^{m-2}u \quad \text{in } \Omega \times (0,\infty), \tag{2.1}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{2.2}$$

$$u(\cdot,0) = u_0 \quad \text{in } \Omega \tag{2.3}$$

with

$$\lambda \in \mathbb{R}, \quad 1 < m < \infty \quad \text{and} \quad u_0 \in H^1_0(\Omega) \cap L^m(\Omega)$$

(as an independent interest, we also treat $1 < m < 2$ and $\lambda \in \mathbb{R}$). We shall prove that u is G-*invariant*, i.e., $u(g^{-1}x,t) = u(x,t)$ for all $g \in G$, provided that u_0 is G-invariant. This fact can be easily checked for strong solutions by directly calculating the change of variables. As for weak formulations of differential equations, one should more carefully treat this issue. There are many papers on this topic for elliptic problems. However, there seems to be very little contribution to weak formulations for parabolic problems (see [4]).

We start with the definition of weak solutions for (2.1)–(2.3) by setting

$$X := H^1_0(\Omega) \cap L^m(\Omega),$$

which coincides with $H^1_0(\Omega)$, provided that $1 < m \le 2^*$.

Definition 2.1 (Solution of (2.1)–(2.3)). *A function $u : \Omega \times (0,\infty) \to \mathbb{R}$ is said to be a (weak) solution of (2.1)–(2.3), if the following conditions are all satisfied:*

- *$u \in C([0,\infty); X)$ and $|u|^{m-2}u \in C^1([0,\infty); X^*)$.*
- *For all $t \in (0,\infty)$ and $\psi \in C^\infty_0(\Omega)$,*

$$\left\langle \frac{d}{dt}\left(|u|^{m-2}u\right)(t), \psi \right\rangle_X + \int_\Omega \nabla u(x,t) \cdot \nabla \psi(x)dx$$
$$= \lambda \int_\Omega \left(|u|^{m-2}u\right)(x,t)\psi(x)dx.$$

- $u(\cdot,t) \to u_0$ *strongly in* X *as* $t \to +0$.

Hence the weak formulation of (2.1)–(2.3) stated above can be written as an evolution equation for $u(t) := u(\cdot,t)$ in X^*,

$$\frac{d}{dt}\left(|u|^{m-2}u\right)(t) - \Delta u(t) = \lambda\left(|u|^{m-2}u\right)(t) \text{ in } X^*, \quad t > 0, \quad u(0) = u_0.$$

Then for any $u_0 \in X$, the problem (2.1)–(2.3) admits a unique solution (see, e.g., [9], [20, 21] and also [2]).

Remark 2.2. In case $m > 2$, where (2.1) is the fast diffusion equation, every sign-definite solution becomes a classical solution (see [12]). However, sign-changing solutions should be treated in the weak formulation, because the transformed equation from (2.1) in a similar way to (1.4) has a singularity when $u(x,t) = 0$. In case $m < 2$, where (2.1) is the porous medium equation, the weak formulation is essentially required for sign-definite solutions as well as for sign-changing solutions because of the lack of regularity of solution.

Let G be a subgroup of $O(N)$ whose elements leave Ω invariant. For $g \in G$ and a function $u : \Omega \to \mathbb{R}$, we define a function $gu : \Omega \to \mathbb{R}$ by

$$(gu)(x) := u(g^{-1}x) \quad \text{for } x \in \Omega.$$

Then X and X^* become Banach G-spaces. More precisely, we have a representation π_X of G over X given by

$$\pi_X(g)u := gu \quad \text{for } u \in X \text{ and } g \in G,$$

where $\pi_X(g)$ is a bounded linear operator in X. Moreover, define a representation π_{X^*} of G over X^* by

$$\langle \pi_{X^*}(g)f, u\rangle_X := \langle f, \pi_X(g^{-1})u\rangle_X \quad \text{for } u \in X,\ f \in X^* \text{ and } g \in G.$$

The following facts are well known in the variational analysis of elliptic problems. For the convenience of the reader, we briefly give a proof.

Proposition 2.3 (G-equivariance of $-\Delta u$ and $|u|^{m-2}u$). *Define operators* $A, B : X \to X^*$ *by*

$$A(u) := -\Delta u, \quad B(u) = |u|^{m-2}u \quad \textit{for } u \in X.$$

Then A *and* B *are* G*-equivariant, i.e.,*

$$\pi_{X^*}(g)\left(A(u)\right) = A\left(\pi_X(g)u\right), \quad \pi_{X^*}(g)\left(B(u)\right) = B\left(\pi_X(g)u\right)$$

for all $u \in X$ *and* $g \in G$.

Proof. It is well known that $A = \phi_A'$ and $B = \phi_B'$, where $\phi_A, \phi_B : X \to [0, \infty)$ are given by

$$\phi_A(u) := \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx, \quad \phi_B(u) := \frac{1}{m}\int_\Omega |u(x)|^m dx \quad \text{ for } \ u \in X.$$

Moreover, ϕ_A and ϕ_B are G-invariant, i.e., $\phi_A(gu) = \phi_A(u)$ for all $u \in X$ and $g \in G$. Hence ϕ_A' and ϕ_B' are G-equivariant from the general fact that the derivative of G-invariant functional is G-equivariant. Indeed, for an G-invariant Gâteaux differentiable functional $\phi : X \to \mathbb{R}$, the Gâteaux differential ϕ' of ϕ satisfies

$$\begin{aligned}
\langle \phi'(\pi_X(g)u), e\rangle_X &= \lim_{h\to 0} \frac{\phi(\pi_X(g)u + he) - \phi(\pi_X(g)u)}{h} \\
&= \lim_{h\to 0} \frac{\phi(u + h\pi_X(g^{-1})e) - \phi(u)}{h} \\
&= \langle \phi'(u), \pi_X(g^{-1})e\rangle_X \\
&= \langle \pi_{X^*}(g)\phi'(u), e\rangle_X \quad \text{ for any } e, u \in X \ \text{ and } \ g \in G,
\end{aligned}$$

which implies $\phi'(\pi_X(g)u) = \pi_{X^*}(g)\phi'(u)$ for all $u \in X$ and $g \in G$. One can also obtain a similar conclusion for Fréchet differentials as well. □

A tiny novelty of this section is the following proposition, where the G-equivariance of the time-differential operator is shown in a space of vector functions with values in X^*. A similar attempt has been done for a Gel'fand triplet setting in [4], where a parabolic version of the so-called "principle of symmetric criticality" is established.

To this end, we work on a large space, $\mathcal{H} := L^2(0, T; X^*)$. Then the representation $\pi_\mathcal{H}$ of G over $\mathcal{H}$ is given by

$$(\pi_\mathcal{H}(g)u)(t) = \pi_{X^*}(g)u(t) \quad \text{ for } \ t \in (0, T), \ u \in \mathcal{H} \ \text{ and } \ g \in G.$$

Moreover, we define the time-differential operator,

$$\frac{d}{dt} : \mathcal{H} \to \mathcal{H}$$

with the domain

$$D(d/dt) := \{u \in \mathcal{H} \colon du/dt \in \mathcal{H}\} = W^{1,2}(0, T; X^*).$$

PROPOSITION 2.4 (G-equivariance of d/dt). *The differential operator d/dt is G-equivariant, i.e.,*

$$\pi_\mathcal{H}(g)\frac{du}{dt} = \frac{d}{dt}(\pi_\mathcal{H}(g)u) \quad \textit{for all } \ u \in D(d/dt) \ \textit{ and } \ g \in G,$$

which is equivalently rewritten as

$$\pi_{X^*}(g)\frac{du}{dt}(t) = \frac{d}{dt}\left(\pi_{X^*}(g)u(t)\right) \quad \text{for all } \ u \in D(d/dt) \ \text{ and } \ g \in G$$

for a.e. $t \in (0,T)$.

Proof. For $u \in D(d/dt)$, $g \in G$, $\eta \in C_0^\infty(0,T)$ and $e \in X$, it follows that

$$\begin{aligned}
\left\langle \int_0^T \left(\pi_{\mathcal{H}}(g)\frac{du}{dt}\right)(t)\ \eta(t)\ dt,\ e \right\rangle_X &= \int_0^T \left\langle \pi_{X^*}(g)\frac{du}{dt}(t),\ e \right\rangle_X \eta(t)\ dt \\
&= \int_0^T \left\langle \frac{du}{dt}(t),\ \pi_X(g^{-1})e \right\rangle_X \eta(t)\ dt \\
&= \left\langle \int_0^T \frac{du}{dt}(t)\eta(t)\ dt,\ \pi_X(g^{-1})e \right\rangle_X \\
&= \left\langle -\int_0^T u(t)\frac{d\eta}{dt}(t)\ dt,\ \pi_X(g^{-1})e \right\rangle_X \\
&= -\int_0^T \langle u(t),\ \pi_X(g^{-1})e\rangle_X \frac{d\eta}{dt}(t)\ dt \\
&= -\int_0^T \langle \pi_{X^*}(g)u(t),\ e\rangle_X \frac{d\eta}{dt}(t)\ dt \\
&= \left\langle -\int_0^T \left(\pi_{\mathcal{H}}(g)u\right)(t)\frac{d\eta}{dt}(t)\ dt,\ e \right\rangle_X .
\end{aligned}$$

Thus we have

$$\int_0^T \left(\pi_{\mathcal{H}}(g)\frac{du}{dt}\right)(t)\eta(t)\ dt = -\int_0^T \left(\pi_{\mathcal{H}}(g)u\right)(t)\frac{d\eta}{dt}(t)\ dt \ \text{ in } X^*,$$

which implies

$$\pi_{\mathcal{H}}(g)\frac{du}{dt} = \frac{d}{dt}\left(\pi_{\mathcal{H}}(g)u\right)$$

in the sense of distribution. Hence d/dt is G-equivariant in $\mathcal{H}$. □

Combining all these facts, we are now in position to prove the following theorem.

THEOREM 2.5 (G-invariance of weak solutions). *Let G be a subgroup of $O(N)$ and let Ω be a G-invariant bounded domain of $\mathbb{R}^N$ with smooth boundary. Let $u = u(x,t)$ be a weak solution of* (2.1), (2.2). *Then so is $gu := u(g^{-1}x,t)$ for any $g \in G$.*

In addition, if the initial data u_0 is G-invariant, then so is the unique weak solution of (2.1)–(2.3).

Proof. Let $u = u(x,t)$ be a weak solution of (2.1), (2.2) and put $w(x,t) = u(g^{-1}x,t)$. Then, $w(t) = \pi_X(g)u(t)$. Then for each $\phi \in X$, it follows that

$$\left\langle \frac{d}{dt}B(u(t)) + A(u(t)) - \lambda B(u(t)),\ \pi_X(g^{-1})\phi \right\rangle_X = 0.$$

Then by Propositions 2.3 and 2.4, we have

$$\begin{aligned}
&\left\langle \frac{d}{dt}B(u(t)) + A(u(t)) - \lambda B(u(t)),\ \pi_X(g^{-1})\phi \right\rangle_X \\
&= \left\langle \pi_{X^*}(g)\left(\frac{d}{dt}B(u(t))\right) + \pi_{X^*}(g)A(u(t)) - \lambda\pi_{X^*}(g)B(u(t)),\ \phi \right\rangle_X \\
&= \left\langle \frac{d}{dt}\Big(\pi_{X^*}(g)B(u(t))\Big) + A(w(t)) - \lambda B(w(t)),\ \phi \right\rangle_X \\
&= \left\langle \frac{d}{dt}B(w(t)) + A(w(t)) - \lambda B(w(t)),\ \phi \right\rangle_X .
\end{aligned}$$

Therefore w also solves (2.1), (2.2).

In addition, if the initial data u_0 is G-invariant, all the solutions $w(x,t) = u(g^{-1}x,t)$ for any $g \in G$ solve (2.1)–(2.3) with the same data u_0. Therefore from the uniqueness of weak solution, w coincides with u for all $g \in G$. Consequently, the unique solution u is G-invariant. □

3 Stability analysis of group invariant asymptotic profiles

This section is devoted to a stability analysis of asymptotic profiles invariant under symmetry group for vanishing solutions of (1.1)–(1.3). Throughout this section, we assume that

$$2 < m < 2^* := \begin{cases} 2N/(N-2) & \text{if } N \geq 3, \\ \infty & \text{if } N = 1,2 \end{cases} \qquad \text{and} \qquad u_0 \in H^1_0(\Omega) \tag{3.1}$$

(then $H^1_0(\Omega)$ is compactly embedded in $L^m(\Omega)$). In Subsection 3.1, we overview the stability analysis of asymptotic profiles for fast diffusion equations so far. In Subsection 3.2, we define the notions of stability and instability of asymptotic profiles under group invariant perturbations and present some stability criteria. Moreover, this stability criteria will be proved in the following two subsections. The contents in these subsections would be similar to those in [3], even though the setting under consideration here is not covered. However, results in Subsection 3.5 to be obtained by applying these criteria would be noteworthy, because they enable us to discuss the asymptotic stability of radial profiles under radial perturbations as well as to investigate further information on the instability of sign-changing profiles.

3.1 Asymptotic profiles for fast diffusion equations

In this subsection, we briefly review previous results on asymptotic profiles for fast diffusion equations. The finite-time extinction of solutions for fast diffusion equations is first proved by Sabinina [18] (for $N = 1$), and then, generalized by Bénilan and Crandall [5]. We denote by $t_*(u_0)$ the extinction time of the unique solution u of (1.1)–(1.3) for the initial data u_0. Berryman and Holland [7] obtained an optimal rate of the finite-time extinction for each solution u of (1.1)–(1.3),

$$c(t_* - t)_+^{1/(m-2)} \leq \|u(t)\|_{1,2} \leq c^{-1}(t_* - t)_+^{1/(m-2)}$$

with the extinction time t_* of u and a positive constant $c > 0$. Moreover, they showed the existence of asymptotic profiles

$$\phi(x) := \lim_{t_n \nearrow t_*} (t_* - t_n)_+^{-1/(m-2)} u(x, t_n) \quad \text{in } H_0^1(\Omega)$$

with some sequence $t_n \nearrow t_*$ for positive classical solutions.

In order to characterize ϕ, let us apply the following transformation:

$$v(x, s) := (t_* - t)^{-1/(m-2)} u(x, t) \quad \text{and} \quad s := \log(t_*/(t_* - t)) \geq 0. \tag{3.2}$$

Then s tends to infinity as $t \nearrow t_*$. Moreover, the asymptotic profile $\phi = \phi(x)$ of $u = u(x, t)$ is reformulated as

$$\phi(x) := \lim_{s_n \nearrow \infty} v(x, s_n) \ \text{ in } H_0^1(\Omega) \ \text{ with } \ s_n := \log(t_*/(t_* - t_n)) \to \infty.$$

Furthermore, the Cauchy-Dirichlet problem (1.1)–(1.3) for $u = u(x, t)$ is rewritten as the following rescaled problem:

$$\partial_s \left(|v|^{m-2} v\right) = \Delta v + \lambda_m |v|^{m-2} v \quad \text{in } \Omega \times (0, \infty), \tag{3.3}$$

$$v = 0 \quad \text{on } \partial\Omega \times (0, \infty), \tag{3.4}$$

$$v(\cdot, 0) = v_0 \quad \text{in } \Omega, \tag{3.5}$$

where the initial data v_0 and the constant λ_m are given by

$$v_0 = t_*(u_0)^{-1/(m-2)} u_0 \quad \text{and} \quad \lambda_m = (m-1)/(m-2) > 0. \tag{3.6}$$

Then (3.3)–(3.5) can be regarded as a generalized gradient system,

$$\frac{d}{ds} |v|^{m-2} v(s) = -J'(v(s)) \quad \text{for } s > 0,$$

where $J : H_0^1(\Omega) \to \mathbb{R}$ is given by

$$J(w) := \frac{1}{2} \int_\Omega |\nabla w(x)|^2 dx - \frac{\lambda_m}{m} \int_\Omega |w(x)|^m dx \quad \text{for } w \in H_0^1(\Omega),$$

and moreover, the function $s \mapsto J(v(s))$ is nonincreasing. One can prove the following theorem (see [7], [15], [19], [8], [3]):

Theorem 3.1 (Existence of asymptotic profiles and their characterization). *For any sequence $s_n \to \infty$, there exist a subsequence (n') of (n) and $\phi \in H^1_0(\Omega) \setminus \{0\}$ such that $v(s_{n'}) \to \phi$ strongly in $H^1_0(\Omega)$. Moreover, ϕ is a nontrivial stationary solution of (3.3)–(3.5), that is, ϕ solves the Dirichlet problem,*

$$-\Delta\phi = \lambda_m |\phi|^{m-2}\phi \quad in \ \ \Omega, \qquad \phi = 0 \quad on \ \ \partial\Omega. \tag{3.7}$$

The Dirichlet problem (3.7) is an Euler-Lagrange equation for J.

Remark 3.2. (i) If ϕ is a nontrivial solution of (3.7), then the function $U(x,t) = (1-t)_+^{1/(m-2)}\phi(x)$ solves (1.1)–(1.3) with $U(x,0) = \phi(x)$. Hence $t_*(\phi) = 1$ and the profile of $U(x,t)$ coincides with $\phi(x)$.

(ii) Hence, by Theorem 3.1, the set of all asymptotic profiles of solutions for (1.1)–(1.3) coincides with the set of all nontrivial solutions of (3.7). Obviously, they also coincide with the set of all nontrivial critical points of J. We shall denote these sets by $\mathcal{S}$.

(iii) Due to [13], the asymptotic profile is uniquely determined for each nonnegative data $u_0 \geq 0$.

In [7], [15], [19], [8], the stability of positive profiles is discussed for nonnegative solutions. However, until the work in [3], sign-changing profiles had not been treated, and the stability of positive profiles had not been discussed under a wider class of perturbations which allow sign-changing initial data.

In [3], the notions of stability and instability of asymptotic profiles of solutions for (1.1)–(1.3) were first precisely defined for possibly sign-changing solutions by introducing a set,

$$\mathcal{X} := \left\{ t_*(u_0)^{-1/(m-2)} u_0 \colon u_0 \in H^1_0(\Omega) \setminus \{0\} \right\},$$

which coincides with the level set $\{v_0 \in H^1_0(\Omega) \colon t_*(v_0) = 1\}$ of the functional $t_* : H^1_0(\Omega) \to \mathbb{R}$. Here we note

Lemma 3.3 (Property of $\mathcal{X}$, [3]). *Let v be a solution of (3.3)–(3.5) for an initial data v_0.*

(i) *If $v_0 \in \mathcal{X}$, then $v(s) \in \mathcal{X}$ for all $s \geq 0$.*

(ii) *If $v_0 \in \mathcal{X}$, then for any $s_n \to \infty$, up to a subsequence, $v(s_n) \to \phi$ for some $\phi \in \mathcal{S}$ (by Theorem 3.1).*

(iii) *It follows that $\mathcal{S} \subset \mathcal{X}$.*

Moreover, the following criteria for the stability and instability of profiles were presented:

- Each least energy solution ϕ of (3.7) is (resp., asymptotically) stable in the sense of asymptotic profiles, if ϕ is isolated from the other least energy (resp., sign-definite) solutions.

- All the sign-changing solutions are not asymptotically stable profiles. Moreover, they are unstable, if they are isolated from the other profiles with lower energies.

As a by-product of [3], the whole of the energy space $H_0^1(\Omega)$ of initial data is completely classified in terms of large-time behaviors of solutions for (3.3)–(3.5). In particular, the set $\mathcal{X}$ turns out to be a separatrix between stable and unstable sets (cf. see [14] for a semilinear heat equation).

However, the criteria stated above do not cover all the situations. Indeed, in case Ω is a thin annulus, there exists a positive radial profile ϕ_1 which may not take the least energy among $\mathcal{S}$.

In the following subsections, we introduce the notions of stability and instability of G-invariant profiles under similarly invariant perturbations for a subgroup G of $O(N)$ and slightly modify the argument of [3] to obtain stability criteria for G-invariant profiles. As a typical application of the criteria, we shall discuss the stability of the unique positive radial profile in the annulus case under $O(N)$-invariant perturbations.

3.2 Stability and instability of G-invariant profiles

Let G be a subgroup of $O(N)$ and let Ω be a G-invariant domain of $\mathbb{R}^N$ with smooth boundary. Assume (3.1) and denote the space of G-invariant functions of class $H_0^1(\Omega)$ by

$$H^1_{0,G}(\Omega) := \{u \in H^1_0(\Omega) \colon gu = u \quad \text{for all} \quad g \in G\}.$$

Each asymptotic profile lying on $H^1_{0,G}(\Omega)$ is called *a G-invariant asymptotic profile*. Let us introduce the notions of stability and instability of G-invariant asymptotic profiles of solutions for (1.1)–(1.3) under G-invariant perturbations. To this end, we first introduce the set,

$$\mathcal{X}_G := \left\{ t_*(u_0)^{-1/(m-2)} u_0 \colon u_0 \in H^1_{0,G}(\Omega) \setminus \{0\} \right\} = \mathcal{X} \cap H^1_{0,G}(\Omega).$$

Then we define:

Definition 3.4 (Stability and instability of profiles under G-invariant perturbations). *Let $\phi \in H^1_{0,G}(\Omega)$ be an asymptotic profile of vanishing solutions for* (1.1)–(1.3).

(i) *ϕ is said to be stable under G-invariant perturbations, if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that any solution v of* (3.3)–(3.5) *satisfies*

$$v(0) \in \mathcal{X}_G \cap B_{H_0^1}(\phi;\delta) \quad \Rightarrow \quad \sup_{s\in[0,\infty)} \|v(s)-\phi\|_{1,2} < \varepsilon,$$

where $B_{H_0^1}(\phi;\delta) := \{w \in H_0^1(\Omega) \colon \|\phi - w\|_{1,2} < \delta\}$.

(ii) *ϕ is said to be unstable under G-invariant perturbations, if ϕ is not stable under G-invariant perturbations.*

(iii) *ϕ is said to be asymptotically stable under G-invariant perturbations, if ϕ is stable under G-invariant perturbations, and moreover, there exists $\delta_0 > 0$ such that any solution v of* (3.3)–(3.5) *satisfies*

$$v(0) \in \mathcal{X}_G \cap B_{H_0^1}(\phi;\delta_0) \quad \Rightarrow \quad \lim_{s\nearrow\infty} \|v(s)-\phi\|_{1,2} = 0.$$

Remark 3.5. Apparently, $\mathcal{X}_G$ is a subset of $\mathcal{X}$. Hence if an asymptotic profile ϕ is (asymptotically) stable in the sense of [3], then so is it under G-invariant perturbations. On the other hand, if ϕ is unstable or not asymptotically stable under G-invariant perturbations, then so is ϕ without restriction of perturbation.

To state our stability criteria under G-invariant perturbations, we set up notation. Let $\mathcal{S}_G = \mathcal{S} \cap H_{0,G}^1(\Omega)$, which is the set of all G-invariant nontrivial solutions. A function $\phi \in \mathcal{S}_G$ is called *least energy G-invariant solution* if ϕ attains the infimum of J over $\mathcal{S}_G$. Then our criteria read as follows.

Theorem 3.6 (Stability of G-invariant profiles). *Assume* (3.1). *Let $\phi \in H_{0,G}^1(\Omega)$ be a least energy G-invariant solution of* (3.7). *Then it follows that*

(i) *ϕ is a stable profile under G-invariant perturbations, if ϕ is isolated in $H_0^1(\Omega)$ from the other least energy G-invariant solutions.*

(ii) *ϕ is an asymptotically stable profile under G-invariant perturbations, if ϕ is isolated in $H_0^1(\Omega)$ from the other sign-definite G-invariant solutions.*

Theorem 3.7 (Instability of G-invariant profiles). *Assume* (3.1). *Let $\phi \in H_{0,G}^1(\Omega)$ be a sign-changing G-invariant solution of* (3.7). *Then it follows that*

(i) *ϕ is not an asymptotically stable profile under G-invariant perturbations.*

(ii) *ϕ is an unstable profile under G-invariant perturbations, if ϕ is isolated in $H_0^1(\Omega)$ from any $\psi \in \mathcal{S}_G$ satisfying $J(\psi) < J(\phi)$.*

3.3 Proof of Theorem 3.6

We first prepare a couple of lemmas.

LEMMA 3.8 (Properties of $\mathcal{X}_G$). *Let v be a solution of* (3.3)–(3.5) *for an initial data v_0.*

(i) *If $v_0 \in \mathcal{X}_G$, then $v(s) \in \mathcal{X}_G$ for all $s \geq 0$.*

(ii) *If $v_0 \in \mathcal{X}_G$, then for any $s_n \to \infty$, up to a subsequence, $v(s_n) \to \phi$ for some $\phi \in \mathcal{S}_G$.*

(iii) *It holds that $\mathcal{S}_G \subset \mathcal{X}_G$.*

Proof. Combining Theorem 2.5 with (i) of Lemma 3.3, we have (i). Let $v_0 \in \mathcal{X}_G$. By Theorem 3.1, there exist $s_n \to \infty$ and $\phi \in \mathcal{S}$ such that $v(s_n) \to \phi$ strongly in $H^1_0(\Omega)$. Since $v(s_n) \in H^1_{0,G}(\Omega)$ by (i) and $H^1_{0,G}(\Omega)$ is closed, ϕ is G-invariant. Thus (ii) holds. Recall $\mathcal{X}_G = \mathcal{X} \cap H^1_{0,G}(\Omega)$, $\mathcal{S}_G = \mathcal{S} \cap H^1_{0,G}(\Omega)$ and (iii) of Lemma 3.3 to obtain (iii). □

LEMMA 3.9 (Weak closedness of $\mathcal{X}_G$). *If $u_n \in \mathcal{X}_G$ and $u_n \to u$ weakly in $H^1_0(\Omega)$, then $u \in \mathcal{X}_G$.*

Proof. The (sequentially) weak closedness of $\mathcal{X}$ is proved in [3]. Moreover, $H^1_{0,G}(\Omega)$ is also weakly closed, and hence, so is $\mathcal{X}_G = \mathcal{X} \cap H^1_{0,G}(\Omega)$. □

LEMMA 3.10 (Variational feature of $\mathcal{X}_G$). *Let $d_1 = \inf_{\mathcal{S}_G} J$. Then*

$$\mathcal{X}_G \subset [d_1 \leq J] := \left\{ v_0 \in H^1_0(\Omega) \colon d_1 \leq J(v_0) \right\}.$$

Moreover, if $v_0 \in \mathcal{X}_G$ and $J(v_0) = d_1$, then $J'(v_0) = 0$.

Proof. Let $v_0 \in \mathcal{X}_G$ and let $v(s)$ be a solution of (3.3)–(3.5) with $v(0) = v_0$. Then by (ii) of Lemma 3.8 there exist $s_n \to \infty$ and $\phi \in \mathcal{S}_G$ such that $v(s_n) \to \phi$ strongly in $H^1_0(\Omega)$. From the nonincrease of $J(v(\cdot))$, we deduce that

$$J(v_0) \geq J(v(s)) \geq J(\phi) \geq d_1 = \inf_{\mathcal{X}_G} J.$$

Hence $d_1 \leq J(v_0)$.

If $v_0 \in \mathcal{X}_G$ and $J(v_0) = d_1$, then $J(v_0) = \min_{\mathcal{X}_G} J$. Hence $v(s) \equiv v_0$. □

Denote by $\mathcal{LES}_G$ the set of all least energy G-invariant solutions of (3.7). Let us assume that

$$B_{H^1_0}(\phi; r) \cap \mathcal{LES}_G = \{\phi\} \tag{3.8}$$

with some $r > 0$. Here we write $B_{H^1_0}(\phi; r) := \{w \in H^1_0(\Omega) \colon \|w - \phi\|_{1,2} < r\}$.

Claim 3.1. *For any $\varepsilon \in (0, r)$, it holds that*

$$c := \inf\{J(v) \colon v \in \mathcal{X}_G,\ \|v - \phi\|_{1,2} = \varepsilon\} > d_1.$$

Assume on the contrary that $c = d_1$, i.e., there exists $v_n \in \mathcal{X}_G$ such that

$$\|v_n - \phi\|_{1,2} = \varepsilon \ \text{ and } \ J(v_n) \to d_1.$$

Since $m < 2^*$, it entails that, up to a subsequence,

$$v_n \to v_\infty \quad \text{weakly in } H_0^1(\Omega) \ \text{ and strongly in } L^m(\Omega).$$

By Lemmas 3.9 and 3.10, we obtain

$$v_\infty \in \mathcal{X}_G, \quad \text{and hence,} \quad d_1 \le J(v_\infty).$$

Therefore it follows that

$$\begin{aligned} \frac{1}{2}\|v_n\|_{1,2}^2 &= J(v_n) + \frac{\lambda_m}{m}\|v_n\|_m^m \\ &\to d_1 + \frac{\lambda_m}{m}\|v_\infty\|_m^m \le J(v_\infty) + \frac{\lambda_m}{m}\|v_\infty\|_m^m = \frac{1}{2}\|v_\infty\|_{1,2}^2. \end{aligned}$$

By using the weak lower semicontinuity,

$$\liminf_{n\to\infty} \|v_n\|_{1,2} \ge \|v_\infty\|_{1,2},$$

and the uniform convexity of $\|\cdot\|_{1,2}$, we deduce that $v_n \to v_\infty$ strongly in $H_0^1(\Omega)$. Hence $\|v_\infty - \phi\|_{1,2} = \varepsilon$ and $J(v_\infty) = d_1$. Thus $v_\infty \in \mathcal{LES}_G$ by Lemma 3.10. However, the fact that $\|v_\infty - \phi\|_{1,2} = \varepsilon < r$ contradicts (3.8). □

Let $\varepsilon \in (0, r)$ be arbitrarily given. Choose $\delta \in (0, \varepsilon)$ so small that

$$J(v) < c \quad \text{for all } \ v \in B_{H_0^1}(\phi; \delta).$$

Here it is possible, because $c > d_1 = J(\phi)$ by Claim 3.1, and J is continuous in $H_0^1(\Omega)$. For any $v_0 \in \mathcal{X}_G \cap B_{H_0^1}(\phi; \delta)$, let $v(s)$ be a solution of (3.3)–(3.5). Then $v(s) \in \mathcal{X}_G$ for $s \ge 0$ by (i) of Lemma 3.8.

Claim 3.2. *For any $s \ge 0$, $v(s) \in B_{H_0^1}(\phi; \varepsilon)$, and hence ϕ is stable.*

Assume on the contrary that $v(s_0) \in \partial B_{H_0^1}(\phi; \varepsilon)$ at some $s_0 > 0$. By the definition of c, it holds that $c \le J(v(s_0))$. However, it contradicts the fact that $J(v(s_0)) \le J(v_0) < c$. Thus $v(s) \in B_{H_0^1}(\phi; \varepsilon)$ for all $s \ge 0$.

Moreover, if ϕ is isolated in $H_0^1(\Omega)$ from all sign-definite G-invariant solutions of (3.7), then one can prove that $v(s_n)$ converges strongly in $H_0^1(\Omega)$ to ϕ along any sequence $s_n \to \infty$ (see [3] for more details).

3.4 Proof of Theorem 3.7

Let ϕ be a sign-changing G-invariant solution of (3.7) (hence ϕ admits more than two nodal domains).

We first prove (i). Let D be a nodal domain of ϕ and define

$$\phi_\mu(x) := \begin{cases} \mu\phi(x) & \text{if } x \in D, \\ \phi(x) & \text{if } x \in \Omega \setminus D \end{cases} \qquad \text{for } \mu \geq 0$$

(Note: ϕ_μ might not belong to $\mathcal{X}$). Then one can observe that

- ϕ_μ is G-invariant,
- $\phi_\mu \to \phi$ strongly in $H_0^1(\Omega)$ as $\mu \to 1$,
- if $\mu \neq 1$, then $J(c\phi_\mu) < J(\phi)$ for any $c \geq 0$.

Moreover, we set

$$u_{0,\mu} := \phi_\mu, \quad \tau_\mu := t_*(u_{0,\mu}), \quad v_{0,\mu} := \tau_\mu^{-1/(m-2)} u_{0,\mu} \in \mathcal{X}_G.$$

As in [3], it then follows that

- $\tau_\mu \to t_*(\phi) = 1$ and $v_{0,\mu} \to \phi$ strongly in $H_0^1(\Omega)$ as $\mu \to 1$,
- if $\mu \neq 1$, then $J(v_{0,\mu}) < J(\phi)$.

Hence the solution $v_\mu(s)$ of (3.3)–(3.5) with $v_\mu(0) = v_{0,\mu}$ never converges to ϕ as $s \to \infty$. Therefore ϕ is not an asymptotically stable profile under G-invariant perturbations.

Let us move on to (ii). Here we further assume that

$$\overline{B_{H_0^1}(\phi; R)} \cap \{\psi \in \mathcal{S}_G \colon J(\psi) < J(\phi)\} = \emptyset \tag{3.9}$$

with some $R > 0$.

Claim 3.3. *If $\mu \neq 1$, then $v_\mu(s) \notin \overline{B_{H_0^1}(\phi; R)}$ for any $s \gg 1$.*

Assume on the contrary that $v_\mu(s_n) \in \overline{B_{H_0^1}(\phi; R)}$ with some sequence $s_n \to \infty$. Then by (ii) of Lemma 3.8, we deduce that, up to a subsequence,

$$v_\mu(s_n) \to \psi \quad \text{strongly in } H_0^1(\Omega)$$

with some $\psi \in \overline{B_{H_0^1}(\phi; R)} \cap \mathcal{S}_G$. Moreover, we have

$$J(\psi) \leq J(v_{0,\mu}) < J(\phi),$$

which contradicts (3.9). Thus ϕ is an unstable profile.

3.5 Applications of stability criteria

We first apply the preceding stability criteria to the case that Ω is an annular domain given by

$$\Omega := \{x \in \mathbb{R}^N : a < |x| < b\}$$

with constants $0 < a < b$. Then it is known that (3.7) admits a unique positive radial solution ϕ_1 (see [17]) and an arbitrary number of positive nonradial solutions by properly choosing a, b. Particularly, least energy solutions of (3.7) are nonradial, provided that $(b-a)/a \ll 1$ (see [11], [16], [10]). Hence the unique positive radial solution ϕ_1 is out of the scope of the stability criteria proposed in [3]. It is also known that (3.7) admits infinitely many radial sign-changing solutions under (3.1) (see also [17]) and all the sign-changing solutions turn out to be not asymptotically stable under a wider class of perturbations by [3].

Corollary 3.11. *Let Ω be an annular domain in $\mathbb{R}^N$ and assume* (3.1)*. Then the sign-definite radial solutions $\pm\phi_1$ are asymptotically stable profiles under $O(N)$-invariant perturbations. Furthermore, the other radial solutions of* (3.7) *are not asymptotically stable profiles under $O(N)$-invariant perturbations.*

Proof. Let $G = O(N)$. Since ϕ_1 is the least energy G-invariant solution, we conclude by Theorem 3.6 that $\pm\phi_1$ are asymptotically stable profiles under G-invariant perturbations. The other radial solutions are sign-changing, so by Theorem 3.7 they are not asymptotically stable under $O(N)$-invariant perturbations. □

Remark 3.12. The frame of stability analysis for positive radial profiles in annular domains without restriction of perturbation will be discussed in a forthcoming joint paper with Ryuji Kajikiya.

We next give a corollary for general G-invariant domains. Here we call ϕ *least energy sign-changing G-invariant solution* if ϕ is a sign-changing G-invariant solution of (3.7) and takes the least energy among sign-changing G-invariant solutions. Such a least energy sign-changing G-invariant solution always exists for any subgroup $G \subset O(N)$ under (3.1).

Corollary 3.13. *Let Ω be a G-invariant domain of $\mathbb{R}^N$ with smooth boundary and assume* (3.1)*. Then least energy sign-changing G-invariant solutions of* (3.7) *are unstable asymptotic profiles under G-invariant perturbations.*

Proof. As in [3], one can prove that every least energy sign-changing G-invariant solution ϕ is distinct from all the G-invariant nontrivial solutions taking lower energies (i.e., sign-definite G-invariant solutions) by maximum principle. Thus ϕ is unstable under G-invariant perturbations by Theorem 3.7. □

This fact is new also for the stability analysis as in [3]. When Ω is an annulus, least energy sign-changing solutions are nonradial by [1] and unstable in the sense of asymptotic profiles for (1.1)–(1.3) by [3]. By Remark 3.5, this corollary further assures that least energy sign-changing G-invariant solutions are also unstable profiles for any subgroup G of $O(N)$.

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