Doubly nonlinear evolution equations with non-monotone perturbations

Goro Akagi[*1]

[1] Department of Machinery and Control Systems, School of Systems Engineering, Shibaura Institute of Technology, 307 Fukasaku, Minuma-ku, Saitama-shi, Saitama 330-8570, Japan.

The local (in time) existence of strong solutions to Cauchy problems for doubly nonlinear abstract evolution equations with non-monotone perturbations in reflexive Banach spaces is proved under appropriate assumptions, which allow the case where solutions of the corresponding unperturbed problem may not be unique. To prove the existence, a couple of approximate problems are introduced and delicate limiting procedures are discussed by using various tools from convex analysis and the Kakutani-Ky Fan fixed point theorem. Furthermore, an application of the preceding abstract theory to a nonlinear PDE is also given.

1 Introduction

Let V and V^* be a reflexive Banach space and its dual space, respectively, and let H be a Hilbert space whose dual space H^* is identified with itself such that

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^* \tag{1}$$

with continuous and densely defined canonical injections. Let φ and ψ^t be proper lower semi-continuous functions from V into $(-\infty, \infty]$, and let $\partial_V \varphi, \partial_V \psi^t : V \to 2^{V^*}$ be subdifferential operators of φ and ψ^t, respectively, for each $t \in [0, T]$. This talk deals with the existence of strong solutions for the following Cauchy problem:

$$\text{(CP)} \quad \begin{cases} \partial_V \psi^t(u'(t)) + \partial_V \varphi(u(t)) + B(t, u(t)) \ni f(t) \ \text{ in } V^*, \quad 0 < t < T, \\ u(0) = u_0, \end{cases}$$

where B denotes an operator from $(0, T) \times V$ into 2^{V^*} and $f : (0, T) \to V^*$ and $u_0 \in D(\varphi) := \{u \in V; \varphi(u) < \infty\}$ are given data.

For the unperturbed problem ($B \equiv 0$), Colli [4] provided sufficient conditions for the existence of strong solutions to (CP) with $\psi^t \equiv \psi$, and moreover, his results were extended to non-autonomous cases in [2]. As for perturbation problems ($B \not\equiv 0$), Aso, Frémond and Kenmochi [3] proved the existence of time-global strong solutions for (CP) with $\partial_V \psi^t(u'(t))$ replaced by $A(u(t), u'(t))$, where A is an operator from $H \times H \to 2^H$, in the Hilbert space setting, i.e., $V = V^* = H$. Ôtani [5] established an abstract theory on the existence of strong solutions for (CP) with $\partial_V \psi^t(u'(t))$ replaced by $u'(t)$ in the Hilbert space setting. His results were applied to various nonlinear PDEs (e.g., quasilinear reaction-diffusion equation, Navier-Stokes equation).

In this study, we attempt to prove the existence of time-local strong solutions for (CP) by imposing appropriate conditions (the coerciveness, the boundedness and the t-smoothness of $\partial_V \psi^t$, the precompactness of sub-level sets of φ and the boundedness, the compactness and the measurability of B) on the non-monotone operator B as well as the functionals φ, ψ^t. We also emphasize that our abstract result is established in reflexive Banach space setting, and it covers the case where solutions may blow up in finite time.

As a typical example of nonlinear PDEs which fall within our abstract theory, we deal with the following initial-boundary value problem (IBVP):

$$\begin{aligned} &|u_t|^{p-2} u_t(x,t) - \operatorname{div}(|\nabla u|^{m-2} \nabla u)(x,t) - |u|^{q-2} u(x,t) = f(x,t), \quad (x,t) \in \Omega \times (0,T), \\ &u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \qquad u(x,0) = u_0(x), \quad x \in \Omega, \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $1 < p, m, q < \infty$, and $f : \Omega \times (0, T) \to \mathbb{R}$, $u_0 : \Omega \to \mathbb{R}$ are given.

2 Main result

Before describing our main result, we introduce assumptions on ψ^t, φ and B. Let $p \in (1, \infty)$ and $T > 0$ be fixed.

* Corresponding author: e-mail: g-akagi@sic.shibaura-it.ac.jp, Phone: +81 48 683 2020 Fax: +81 48 687 5197

(A1) There exist positive constants C_i $(i = 1, 2, 3, 4)$ such that
$C_1|u|_V^p \leq \psi^t(u) + C_2$ for all $t \in [0, T]$ and $u \in D(\psi^t)$,
$|\eta|_{V^*}^{p'} \leq C_3\psi^t(u) + C_4$ for all $t \in [0, T]$ and $[u, \eta] \in \partial_V \psi^t$.

(A2) There exist a constant $\delta > 0$ such that for all $t_0 \in [0, T]$ and $v_0 \in D(\psi^{t_0})$, we can take a function $u : I_\delta(t_0) := [t_0 - \delta, t_0 + \delta] \to V$ satisfying
$|u(t) - v_0|_V \leq |\alpha(t) - \alpha(t_0)|\ell_0(|\psi^{t_0}(v_0)| + |v_0|_V)$,
$\psi^t(u(t)) \leq \psi^{t_0}(v_0) + |\beta(t) - \beta(t_0)|)\ell_0(|\psi^{t_0}(v_0)| + |v_0|_V)$ for all $t \in I_\delta(t_0)$
with $\alpha, \beta \in C([0, T])$ and a non-decreasing function ℓ_0 in $\mathbb{R}$.

(A3) There exist a Banach space X and a non-decreasing function ℓ_1 in $\mathbb{R}$ such that X is compactly embedded in V and $|u|_X \leq \ell_1([\varphi(u)]_+ + |u|_H)$ for all $u \in D(\partial_V \varphi)$, where $[s]_+ := \max\{s, 0\}$.

(A4) $D(\partial_V \varphi) \subset D(B(t, \cdot))$ for a.e. $t \in (0, T)$. For all $\varepsilon > 0$, there exist a constant $C_\varepsilon \geq 0$ and a non-decreasing function ℓ_2 in $\mathbb{R}$ independent of ε such that
$|g|_{V^*}^{p'} \leq \varepsilon|\xi|_{V^*}^\sigma + C_\varepsilon \ell_2(\varphi(u) + |u|_V)$, where $\sigma := \min\{2, p'\}$,
for a.e. $t \in (0, T)$ and all $u \in D(\partial_V \varphi)$, $g \in B(t, u)$ and $\xi \in \partial_V \varphi(u)$.

(A5) Let $S \in (0, T]$ and let $\{u_n\}$ and $\{\xi_n\}$ be sequences in $C([0, S]; V)$ and $L^\sigma(0, S; V^*)$, respectively, with $\sigma := \min\{2, p'\}$, such that $u_n \to u$ strongly in $C([0, S]; V)$, $[u_n(t), \xi_n(t)] \in \partial_V \varphi$ for a.e. $t \in (0, S)$, and $\sup_{t \in [0,S]} \varphi(u_n(t)) + \int_0^S |u_n'(t)|_H^p dt + \int_0^S |\xi_n(t)|_{V^*}^\sigma dt$ is bounded for all $n \in \mathbb{N}$, and let $\{g_n\}$ be a sequence in $L^{p'}(0, S; V^*)$ such that $g_n(t) \in B(t, u_n(t))$ for a.e. $t \in (0, S)$, $g_n \to g$ weakly in $L^{p'}(0, S; V^*)$. Then, $\{g_n\}$ is precompact in $L^{p'}(0, S; V^*)$ and $g(t) \in B(t, u(t))$ for a.e. $t \in (0, S)$.

(A6) Let $S \in (0, T]$ and let $u \in W^{1,p}(0, S; V)$ be such that $\sup_{t \in [0,S]} \varphi(u(t)) < +\infty$ and suppose that there exists $\xi \in L^{p'}(0, S; V^*)$ such that $\xi(t) \in \partial_V \varphi(u(t))$ for a.e. $t \in (0, S)$. Then, there exists a V^*-valued strongly measurable function g such that $g(t) \in B(t, u(t))$ for a.e. $t \in (0, S)$. Moreover, the set $B(t, u)$ is convex for all $t \in (0, T)$ and $u \in D(B(t, \cdot))$.

Now, our result on local (in time) existence is stated as follows:

Theorem 2.1 (Akagi [1]**)** *Let $p \in (1, \infty)$ and $T > 0$ be given. Suppose that* (A1)-(A6) *are all satisfied. Then, for all $f \in L^{p'}(0, T; V^*)$ and $u_0 \in D(\varphi)$, there exists $T_* = T_*(\varphi(u_0) + |u_0|_H + \|f\|_{L^{p'}(0,T;V^*)}) \in (0, T]$ such that* (CP) *admits at least one strong solution $u \in W^{1,p}(0, T_*; V)$ on $[0, T_*]$.*

3 Application to (IBVP)

Applying Theorem 2.1 to (IBVP), we have the following existence result.

Theorem 3.1 (Akagi [1]**)** *Let $T > 0$ and suppose that*

$$2 \leq p < m^* := \begin{cases} \frac{mN}{N-m} & \text{if } m < N, \\ +\infty & \text{if } m \geq N \end{cases} \quad \text{and} \quad 1 < q < \frac{m^*}{p'} + 1.$$

Then, for all $f \in L^{p'}(0, T; L^{p'}(\Omega))$ and $u_0 \in W_0^{1,m}(\Omega)$, there exists $T_* = T_*(\varphi(u_0) + \|f\|_{L^{p'}(0,T;L^{p'}(\Omega))}) > 0$ such that (IBVP) *admits at least one solution $u \in W^{1,p}(0, T_*; L^p(\Omega))$ on $[0, T_*]$.*

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