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On some macroscopic models for type-II superconductivity

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Abstract

This paper is concerned with a variational inequality with a time-dependent constraint, which arises from some macroscopic models of type-II superconductivity, as well as its approximate problems associated with generalized p-Laplace operators. We prove the existence and uniqueness of solutions for each problem by establishing an abstract theory for doubly nonlinear evolution equations governed by two time-dependent subdifferential operators in reflexive Banach spaces.

Keywords: Doubly nonlinear evolution equation; Subdifferential operator; Bean's critical-state model

1. Introduction

Barrett and Prigozhin [2] studied the dynamics of internal magnetic fields in type-II superconductors under non-stationary external magnetic fields from macroscopic points of view for two different geometric conditions; infinitely long cylindrical superconductor with parallel fields, and thin film superconductor with perpendicular fields. They employed Bean's critical-state model, which describes the correspondence between electric field **E**

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and current density $\mathbf{j}$ given by

$$
\text{(B)} \quad \mathbf{E} \parallel \mathbf{j}, \quad |\mathbf{E}| \in \begin{cases} 0 & \text{if } |\mathbf{j}| < j_c, \\ [0, +\infty) & \text{if } |\mathbf{j}| = j_c, \\ \emptyset & \text{if } |\mathbf{j}| > j_c \end{cases}
$$

with the critical current density j_c, and derived the following variational inequality (P) from the Maxwell system and Bean's critical-state model (B).

$$
\text{(P)} \quad \begin{cases} b\left(\dfrac{\partial h}{\partial t}(\cdot, t),\ v - h(\cdot, t)\right) \geqslant -\left\langle \dfrac{\partial h_e}{\partial t}(\cdot, t), v - h(\cdot, t)\right\rangle_{H_0^1} \\ \quad \forall v \in K^t,\ 0 < t < T, \\ h(\cdot, t) \in K^t, \quad 0 < t < T, \quad h(\cdot, 0) = h_0, \end{cases}
$$

where Ω is a bounded domain in $\mathbf{R}^2$ with smooth boundary $\partial\Omega$, h_e denotes the x_3-component of external magnetic field, h denotes the x_3-component of internal magnetic field (resp., stream function of sheet current density) for long cylinder case (resp., for thin film case), and h_0 is initial data of h; moreover, the convex set K^t is defined by

$$
K^t := \{u \in H_0^1(\Omega);\ |\nabla u(x)| \leqslant j_c(x, t) \text{ for a.e. } x \in \Omega\}
$$

and $b(\cdot, \cdot)$ is defined as follows:

$$
b(u, v) := \begin{cases} \displaystyle\int_\Omega u(x) v(x)\,\mathrm{d}x & \text{for long cylinder case,} \\ \displaystyle\int_\Omega \int_\Omega \frac{\nabla_x u(x) \cdot \nabla_{x'} v(x')}{4\pi |x - x'|}\,\mathrm{d}x\,\mathrm{d}x' & \text{for thin film case.} \end{cases}
$$

In [2], the critical current density j_c is assumed to be constant; however, from the view point of experimental physics, it would be more reasonable to suppose that j_c is inhomogeneous in space and time. In this paper, we assume that the critical current density j_c depends on x and t, i.e., $j_c = j_c(x, t)$; then, the variational inequality (P) is regarded as a time-dependent constraint problem, since the convex set K^t depends on t.

On the other hand, (B) has strong nonlinearity, so the following approximation of (B) is often employed instead of Bean's model (B):

$$
\text{(B)}_p \quad \mathbf{E} := \left|\frac{\mathbf{j}}{j_c}\right|^{p-2} \frac{\mathbf{j}}{j_c}
$$

for a large enough number p. Moreover, the following equation is also derived from the Maxwell system and the power approximation (B)_p:

$$
\text{(P)}_p \quad \begin{cases} b\left(\dfrac{\partial h_p}{\partial t}(\cdot, t),\ v\right) + \displaystyle\int_\Omega \left(\frac{1}{j_c(x, t)}\right)^p |\nabla h_p(x, t)|^{p-2} \nabla h_p(x, t) \cdot \nabla v(x)\,\mathrm{d}x \\ \quad = -\left\langle \dfrac{\partial h_e}{\partial t}(\cdot, t), v\right\rangle_{W_0^{1,p}} \quad \forall v \in W_0^{1,p}(\Omega), \quad 0 < t < T, \\ h_p(\cdot, 0) = h_0. \end{cases}
$$

In the next section, we develop an abstract theory of the following evolution equation governed by two time-dependent subdifferential operators $\partial\psi^t$, $\partial\varphi^t$ in the reflexive Banach space V^*:

$$\text{(E)}\quad \partial\psi^t\left(\frac{\mathrm{d}u}{\mathrm{d}t}(t)\right) + \partial\varphi^t(u(t)) \ni f(t) \quad \text{in } V^*, \quad 0<t<T,$$

since both problems (P) and $(\mathrm{P})_p$ can be reduced into such evolution equations. As for the case where both ψ^t and φ^t are independent of t, (E) has already been studied by several authors (see e.g. [4] and its references). Moreover, in Section 3, we discuss the solvability of (P) and $(\mathrm{P})_p$ by using the abstract theory developed in Section 2.

2. Abstract theory

First we reduce (P) and $(\mathrm{P})_p$ to evolution equations governed by time-dependent subdifferential operators. To do this, we recall the definition of subdifferential operators in the following:

Definition 2.1. Let $\Phi(X)$ be the set of all convex lower-semicontinuous functionals φ from a topological linear space X into $(-\infty, +\infty]$ satisfying $\varphi \not\equiv +\infty$. Then the subdifferential operator $\partial_{X,X^*}\varphi : X \to 2^{X^*}$ of $\varphi \in \Phi(X)$ is defined by

$$\partial_{X,X^*}\varphi(u) := \{\xi \in X^*;\ \varphi(v) - \varphi(u) \geqslant \langle \xi, v-u\rangle_X \ \forall v \in D(\varphi)\},$$

$$D(\varphi) := \{u \in X;\ \varphi(u) < +\infty\}$$

with the domain $D(\partial_{X,X^*}\varphi) := \{u \in D(\varphi);\ \partial_{X,X^*}\varphi(u) \neq \emptyset\}$, where $\langle\cdot,\cdot\rangle_X$ denotes the duality pairing between X and its dual space X^*. For simplicity of notation, we shall write $\partial\varphi$ and $\langle\cdot,\cdot\rangle$ instead of $\partial_{X,X^*}\varphi$ and $\langle\cdot,\cdot\rangle_X$, respectively, if no confusion arises.

Now we set

$$V_0 := \begin{cases} L^2(\Omega) & \text{(long cylinder case)},\\ H^{1/2}_{00}(\Omega) & \text{(thin film case)}, \end{cases}$$

where

$$H^{1/2}_{00}(\Omega) := \left\{ \chi \in H^{1/2}(\Omega);\quad \tilde{\chi} := \begin{cases} \chi & \text{in } \Omega,\\ 0 & \text{in } \mathbf{R}^2\backslash\Omega \end{cases} \in H^{1/2}(\mathbf{R}^2) \right\},$$

equipped with the norm

$$|u|_{V_0} := \begin{cases} |u|_{L^2(\Omega)} = \left(\displaystyle\int_\Omega |u(x)|^2\,\mathrm{d}x\right)^{1/2} & \text{(long cylinder case)},\\ |u|_{H^{1/2}_{00}(\Omega)} = \left(|u|^2_{H^{1/2}(\Omega)} + \displaystyle\int_\Omega \frac{|u(x)|^2}{\mathrm{dist}(x,\partial\Omega)}\,\mathrm{d}x\right)^{1/2} & \text{(thin film case)}. \end{cases}$$

Then we have $V_0 \subset L^2(\Omega) \subset V_0^*$ with continuous densely defined natural injections. Moreover, from the definition of $b(\cdot, \cdot)$, we can verify

$$b(u, v) = b(v, u) \quad \text{for all } u, v \in V_0, \tag{1}$$

$$C_1 |u|_{V_0}^2 \leqslant b(u, u) \quad \text{for all } u \in V_0, \tag{2}$$

$$|b(u, v)| \leqslant C_2 |u|_{V_0} |v|_{V_0} \quad \text{for all } u, v \in V_0, \tag{3}$$

where $C_1, C_2 > 0$ (see [2] for their proofs). Define the functional $\psi : V_0 \to [0, +\infty)$ by

$$\psi(u) := \tfrac{1}{2} b(u, u) \quad \forall u \in V_0.$$

Then we can easily see

$$\psi \in \Phi(V_0) \quad \text{and} \quad \langle \partial\psi(u), v \rangle = b(u, v) \quad \forall u, v \in V_0.$$

Furthermore, we define $\varphi_\infty^t : V_0 \to [0, +\infty]$ by

$$\varphi_\infty^t(u) := \begin{cases} 0 & \text{if } u \in K^t, \\ +\infty & \text{otherwise.} \end{cases}$$

It then follows that $\varphi_\infty^t \in \Phi(V_0)$, $D(\partial\varphi_\infty^t) = D(\varphi_\infty^t) = K^t$, and

$$f \in \partial\varphi_\infty^t(u) \quad \text{if and only if} \quad u \in K^t \quad \text{and} \quad \langle f, v - u \rangle \leqslant 0 \quad \forall v \in K^t$$

under the following condition on j_c:

$$0 < \delta_0 \leqslant j_c(x, t) \quad \text{for a.e. } x \in \Omega \quad \text{and} \quad j_c(\cdot, t) \in L^\infty(\Omega) \quad \forall t \in [0, T]. \tag{4}$$

Hence (P) is reduced to the following evolution equation:

$$(\mathrm{CP})_\infty \quad \partial\psi\left(\frac{\mathrm{d}h}{\mathrm{d}t}(t)\right) + \partial\varphi_\infty^t(h(t)) \ni -\frac{\mathrm{d}h_e}{\mathrm{d}t}(t) \quad \text{in } V_0^*,\ 0 < t < T, \quad h(0) = h_0.$$

On the other hand, the functional $\varphi_p^t : V_0 \to [0, +\infty]$ defined by

$$\varphi_p^t(u) := \begin{cases} \dfrac{1}{p} \int_\Omega \left(\dfrac{|\nabla u(x)|}{j_c(x, t)} \right)^p \mathrm{d}x & \text{if } u \in H_0^1(\Omega) \quad \text{and} \quad |\nabla u| / j_c(\cdot, t) \in L^p(\Omega), \\ +\infty & \text{otherwise} \end{cases}$$

satisfies $\varphi_p^t \in \Phi(V_0)$, $D(\varphi_p^t) = W_0^{1,p}(\Omega)$ and

$$\langle \partial\varphi_p^t(u), v \rangle = \int_\Omega \left(\frac{1}{j_c(x, t)} \right)^p |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x) \, \mathrm{d}x \quad \forall v \in W_0^{1,p}(\Omega)$$

under the assumption (4). Thus $(\mathrm{P})_p$ is rewritten as

$$(\mathrm{CP})_p \quad \partial\psi\left(\frac{\mathrm{d}h_p}{\mathrm{d}t}(t)\right) + \partial\varphi_p^t(h_p(t)) = -\frac{\mathrm{d}h_e}{\mathrm{d}t}(t) \quad \text{in } V_0^*,\ 0 < t < T, \quad h_p(0) = h_0.$$

Therefore we will deal with the following abstract Cauchy problem (CP) with two time-dependent subdifferential operators $\partial\psi^t, \partial\varphi^t : V \to 2^{V^*}$ in the dual space V^* of a real reflexive Banach space V to investigate the solvability of (P) and $(\mathrm{P})_p$; moreover, the scope of our abstract theory to be developed here can cover further generalized physical settings as well as (P) and $(\mathrm{P})_p$.

$$\text{(CP)} \quad \partial\psi^t\left(\frac{\mathrm{d}u}{\mathrm{d}t}(t)\right) + \partial\varphi^t(u(t)) \ni f(t) \quad \text{in } V^*,\ 0<t<T, \quad u(0)=u_0,$$

where φ^t and ψ^t belong to $\Phi(V)$ for every $t \in [0, T]$ and f is a given function from $[0, T]$ into V^*. Moreover, we suppose that there exists a Hilbert space H whose dual space is identified with itself such that V is densely and continuously embedded in H, which implies the continuous embedding $H^* \subset V^*$. The notion of strong solution of (CP) is defined as follows.

Definition 2.2. A function $u \in C([0, T]; V)$ is said to be a strong solution of (CP), if the following conditions are all satisfied:

(a) u is a V-valued absolutely continuous function on $[0, T]$, and $u(+0) = u_0$.
(b) $u(t) \in D(\partial\varphi^t)$, $\mathrm{d}u(t)/\mathrm{d}t \in D(\partial\psi^t)$ for a.e. $t \in (0, T)$
and there exist sections $g(t) \in \partial\varphi^t(u(t))$ and $\eta(t) \in \partial\psi^t(\mathrm{d}u(t)/\mathrm{d}t)$ such that

$$\eta(t) + g(t) = f(t) \quad \text{in } V^* \quad \text{for a.e. } t \in (0, T). \tag{5}$$

(c) The function $t \mapsto \varphi^t(u(t))$ is differentiable for a.e. $t \in (0, T)$ and the function $t \mapsto \psi^t(\mathrm{d}u(t)/\mathrm{d}t)$ is integrable on $(0, T)$.

To state our results on the solvability of (CP), we introduce the following assumptions for some number $q \in (1, +\infty)$.

(A1) There exist constants $\alpha > 0$ and $C_3 \geqslant 0$ such that
$\alpha|u|_V^q \leqslant \psi^t(u) + C_3$ for all $u \in D(\psi^t)$ and $t \in [0, T]$.
(A2) There exist a constant $C_4 \geqslant 0$ and a function $a \in L^1(0, T)$ such that
$|\eta|_{V^*}^{q'} \leqslant C_4\psi^t(u) + a(t)$ for all $[u, \eta] \in \partial\psi^t$ and $t \in [0, T]$.

Define the mapping $\mathscr{B} : L^q(0, T; V) \to L^{q'}(0, T; V^*)$ as follows:

$$\mathscr{B}u \ni f \quad \text{if and only if} \quad \partial\psi^t(u(t)) \ni f(t) \quad \text{for a.e. } t \in (0, T)$$

for all $[u, f] \in L^q(0, T; V) \times L^{q'}(0, T; V^*)$. Then $\mathscr{B}$ is said to be a weakly closed mapping from $L^q(0, T; V)$ into $L^{q'}(0, T; V^*)$ if $\eta \in \mathscr{B}(u)$ whenever $u_n \to u$ weakly in $L^q(0, T; V)$, $\eta_n \in \mathscr{B}(u_n)$ and $\eta_n \to \eta$ weakly in $L^{q'}(0, T; V^*)$.

(A3) $\mathscr{B}$ is a weakly closed mapping from $L^q(0, T; V)$ into $L^{q'}(0, T; V^*)$.

The following (A3)′ will be used to verify the uniqueness of solutions for (CP).

(A3)′ For every $t \in [0, T]$, $D(\partial\varphi^t) \subset D(\partial\psi^t)$, the graph of $\partial\psi^t$ is linear,
i.e., $\alpha\xi + \beta\eta \in \partial\psi^t(\alpha u + \beta v)$ $\forall[u, \xi], [v, \eta] \in \partial\psi^t$, $\forall\alpha, \beta \in \mathbf{R}$,
and $\partial\psi^t$ is symmetric, where "symmetric" means $\langle\xi, v\rangle = \langle\eta, u\rangle$
$\forall[u, \xi], [v, \eta] \in D(\partial\psi^t)$.

Remark 2.3. On account of the demiclosedness of maximal monotone operators and Mazur's lemma, condition (A3) is derived from (A3)$'$.

The next condition is concerned with the compactness of the level set of φ^t in V.

(A4) There exist a Banach space X and a non-decreasing function ℓ defined on $[0, +\infty)$ such that X is compactly embedded in V and $|u|_X \leqslant \ell(|\varphi^t(u)| + |u|_H)$ for all $u \in D(\partial\varphi^t)$ and $t \in [0, T]$.

From now on, we write $\{\varphi^t\}_{t\in[0,T]} \in \Phi(V, [0, T]; \alpha, \beta, C_0, r)$ for some functions $\alpha, \beta : [0, T] \to \mathbf{R}$ and numbers $C_0 \in \mathbf{R}$, $r \in (1, +\infty)$ if the following (i) and (ii) hold true.

(i) $\varphi^t \in \Phi(V)$ for all $t \in [0, T]$.
(ii) $\exists\delta > 0,\ \forall t_0 \in [0, T],\ \forall u_0 \in D(\varphi^{t_0}),\ \exists u : I_\delta(t_0) := [t_0 - \delta, t_0 + \delta] \cap [0, T] \to V;$

$$|u(t) - u_0|_V \leqslant |\alpha(t) - \alpha(t_0)|\{|\varphi^{t_0}(u_0)| + C_0\}^{1/r},$$

$$\varphi^t(u(t)) \leqslant \varphi^{t_0}(u_0) + |\beta(t) - \beta(t_0)|\{|\varphi^{t_0}(u_0)| + C_0\} \quad \forall t \in I_\delta(t_0).$$

We now employ the following condition for the t-smoothness of functionals:

(A5) There exist functions $\alpha_1 \in W^{1,\rho}(0, T)$ with $\rho = \max\{q, 2\}$, $\alpha_2 \in W^{1,r}(0, T),\ \beta_i \in W^{1,1}(0, T)\ (i = 1, 2)$ and constants $C_0 \in \mathbf{R}$, $r \in (1, +\infty)$ such that $\{\varphi^t\}_{t\in[0,T]} \in \Phi(V, [0, T]; \alpha_1, \beta_1, C_0, \rho)$ and $\{\psi^t\}_{t\in[0,T]} \in \Phi(V, [0, T]; \alpha_2, \beta_2, C_0, r)$.

The following (A5)$'$ is also required to prove the uniqueness of solutions for (CP):

(A5)$'$ There exist functions $\alpha_2, \beta_2 \in W^{1,1}(0, T)$ such that $\{\psi^t\}_{t\in[0,T]} \in \Phi(V, [0, T]; \alpha_2, \beta_2, 0, q)$.

Now our main result is stated as follows.

Theorem 2.4. *Let* $q \in (1, +\infty)$ *and suppose that* (A1)–(A5) *are all satisfied. Then for all* $f \in L^{q'}(0, T; V^*)$ *and* $u_0 \in D(\varphi^0)$, (CP) *has at least one strong solution* u *on* $[0, T]$ *satisfying* $u \in W^{1,q}(0, T; V)$ *and* $g, \eta \in L^{q'}(0, T; V^*)$, *where* $g(t)$ *and* $\eta(t)$ *denote the sections of* $\partial\varphi^t(u(t))$ *and* $\partial\psi^t(\mathrm{d}u(t)/\mathrm{d}t)$ *in* (5), *respectively*.

In particular, suppose that (A1) *with* $C_3 = 0$, (A2) *with* $a \equiv 0$, (A3)$'$ *and* (A5)$'$ *are satisfied and* $\psi^0(0) = 0$. *Then the solution is unique*.

Proof. The proof consists of 4 steps. For simplicity, we assume that $\psi^t, \varphi^t \geqslant 0$; however, the following argument is valid also for the general case.

Approximation: We introduce approximate problems for (CP) in the Hilbert space H. To this end, we define the extensions of ψ^t and φ^t on H as follows:

$$(u) := \begin{cases} \phi(u) & \text{if } u \in V, \\ +\infty & \text{otherwise}, \end{cases} \qquad \phi = \psi^t,\ \varphi^t.$$

Then we can consider the following Cauchy problems:

$$(\mathrm{CP})_\lambda \quad \begin{cases} \lambda \dfrac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t) + \eta_\lambda(t) + \partial_H \varphi^t_{H,\lambda}(u_\lambda(t)) = f_\lambda(t) \quad \text{in } H, \quad 0<t<T, \\ \eta_\lambda(t) \in \partial_H \psi^t_H\left(\dfrac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t)\right), \quad u_\lambda(0) = u_0, \end{cases}$$

where $\partial_H \varphi^t_{H,\lambda}$ denotes the Yosida approximation of $\partial_H \varphi^t_H$, and (f_λ) denotes a sequence in $C([0,T];H)$ such that $f_\lambda \to f$ strongly in $L^{q'}(0,T;V^*)$ as $\lambda \to +0$. Since $(\lambda I + \partial_H \psi^t_H)^{-1}$ and $\partial_H \varphi^t_{H,\lambda}$ become Lipschitz continuous in H, where I denotes the identity in H, the mapping $u \mapsto A(t,u) := \left(\lambda I + \partial_H \psi^t_H\right)^{-1}\{f_\lambda(t) - \partial_H \varphi^t_{H,\lambda}(u)\}$ also becomes Lipschitz continuous with respect to u for each $t \in [0,T]$. Moreover, (A5) ensures the continuity of the mapping $t \mapsto A(t,u)$ for each $u \in H$. Consequently, since $(\mathrm{CP})_\lambda$ is equivalent to the following

$$\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t) = A(t, u_\lambda(t)) \quad \text{in } H, \quad 0<t<T, \quad u_\lambda(0) = u_0.$$

Hence, Theorem 1.4 of [3] assures the existence of a unique strong solution u_λ for $(\mathrm{CP})_\lambda$ satisfying $u_\lambda \in C^1([0,T];H)$, $\mathrm{d}u_\lambda(t)/\mathrm{d}t \in D(\partial_H \psi^t_H)$ for all $t \in [0,T]$ and $\mathrm{d}u_\lambda/\mathrm{d}t \in C_w([0,T];V)$, where $C_w([0,T];V)$ denotes the set of all weakly continuous functions from $[0,T]$ into V (see [1] for more details).

A priori estimates: We first establish the following a priori estimates for $u_\lambda(t)$.

Lemma 2.5. *There exists a constant M_1 such that*

$$\sup_{t\in[0,T]} \varphi^t_{H,\lambda}(u_\lambda(t)) \leqslant M_1, \tag{6}$$

$$\lambda \int_0^T \left|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t)\right|_H^2 \mathrm{d}t + \int_0^T \left|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t)\right|_V^q \mathrm{d}t + \int_0^T \psi^t\left(\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t)\right) \mathrm{d}t \leqslant M_1, \tag{7}$$

$$\sup_{t\in[0,T]} |u_\lambda(t)|_V \leqslant M_1 \tag{8}$$

for all $\lambda \in (0,1]$.

Proof. We note that (A5) yields $\psi^t(u) \leqslant \langle \eta, u\rangle + C(|\eta|_{V^*} + 1)$ for all $[u,\eta] \in \partial\psi^t$ and $t \in [0,T]$ (see [1] for more details). Multiplying $(\mathrm{CP})_\lambda$ by $\mathrm{d}u_\lambda(t)/\mathrm{d}t$, we get

$$\begin{aligned} &\lambda\left|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t)\right|_H^2 + \psi^t\left(\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t)\right) + \left\langle g_\lambda(t), \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t)\right\rangle \\ &\quad \leqslant \left\langle f_\lambda(t), \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t)\right\rangle + C(|\eta_\lambda(t)|_{V^*} + 1) \end{aligned}$$

for a.e. $t \in (0, T)$, where $g_\lambda(t) := \partial_H \varphi^t_{H,\lambda}(u_\lambda(t))$. Now by (A5), it follows that

$$\begin{aligned}
&\left| \frac{\mathrm{d}}{\mathrm{d}t} \varphi^t_{H,\lambda}(u_\lambda(t)) - \left\langle g_\lambda(t), \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t) \right\rangle \right| \\
&\quad \leqslant |\dot{\alpha}_1(t)| |g_\lambda(t)|_{V^*} \{\varphi^t_{H,\lambda}(J^t_{H,\lambda} u_\lambda(t)) + C_0\}^{1/\rho} \\
&\qquad + |\dot{\beta}_1(t)| \{\varphi^t_{H,\lambda}(J^t_{H,\lambda} u_\lambda(t)) + C_0\},
\end{aligned} \tag{9}$$

where $\rho = \max\{q, 2\}$ and $J^t_{H,\lambda}$ denotes the resolvent of $\partial_H \varphi^t_H$, i.e., $J^t_{H,\lambda} u = (I + \lambda \partial_H \varphi^t_H)^{-1} u$ (see [1] for its proof). Hence by (A1) and $(\mathrm{CP})_\lambda$, we have

$$\begin{aligned}
&\lambda \left| \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t) \right|_H^2 + \frac{\alpha}{2} \left| \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t) \right|_V^q + \frac{1}{2} \psi^t \left(\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t) \right) + \frac{\mathrm{d}}{\mathrm{d}t} \varphi^t_{H,\lambda}(u_\lambda(t)) \\
&\quad \leqslant |\dot{\alpha}_1(t)| \left\{ |f_\lambda(t)|_{V^*} + \lambda \left| \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t) \right|_{V^*} + |\eta_\lambda(t)|_{V^*} \right\} \{\varphi^t_{H,\lambda}(u_\lambda(t)) + C_0\}^{1/\rho} \\
&\qquad + |\dot{\beta}_1(t)| \{\varphi^t_{H,\lambda}(u_\lambda(t)) + C_0\} + |f_\lambda(t)|_{V^*} \left| \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t) \right|_V \\
&\qquad + C(|\eta_\lambda(t)|_{V^*} + 1) + \frac{C_3}{2}.
\end{aligned} \tag{10}$$

Therefore since $\rho = \max\{q, 2\}$, it follows from (A1) and (A2) that

$$\begin{aligned}
&\frac{\lambda}{2} \left| \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t) \right|_H^2 + \frac{\alpha}{2} \left| \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t) \right|_V^q + \frac{1}{4} \psi^t \left(\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t) \right) + \frac{\mathrm{d}}{\mathrm{d}t} \varphi^t_{H,\lambda}(u_\lambda(t)) \\
&\quad \leqslant C\{|\dot{\alpha}_1(t)|^\rho + |\dot{\beta}_1(t)| + |f_\lambda(t)|_{V^*}^{q'} + |a(t)| + 1\} \\
&\qquad + C\{|\dot{\alpha}_1(t)|^\rho + |\dot{\beta}_1(t)|\} \varphi^t_{H,\lambda}(u_\lambda(t)).
\end{aligned} \tag{11}$$

Moreover, integrating (11) over $(0, t)$, we get

$$\begin{aligned}
&\frac{\lambda}{2} \int_0^t \left| \frac{\mathrm{d}u_\lambda}{\mathrm{d}\tau}(\tau) \right|_H^2 \mathrm{d}\tau + \frac{\alpha}{2} \int_0^t \left| \frac{\mathrm{d}u_\lambda}{\mathrm{d}\tau}(\tau) \right|_V^q \mathrm{d}\tau + \frac{1}{4} \int_0^t \psi^\tau \left(\frac{\mathrm{d}u_\lambda}{\mathrm{d}\tau}(\tau) \right) \mathrm{d}\tau + \varphi^t_{H,\lambda}(u_\lambda(t)) \\
&\quad \leqslant \varphi^0(u_0) + C \left\{ \int_0^T |\dot{\alpha}_1(\tau)|^\rho \, \mathrm{d}\tau + \int_0^T |\dot{\beta}_1(\tau)| \, \mathrm{d}\tau + \int_0^T |f_\lambda(\tau)|_{V^*}^{q'} \, \mathrm{d}\tau \right. \\
&\qquad \left. + \int_0^T |a(\tau)| \, \mathrm{d}\tau + 1 \right\} + C \int_0^t \{|\dot{\alpha}_1(\tau)|^\rho + |\dot{\beta}_1(\tau)|\} \varphi^\tau_{H,\lambda}(u_\lambda(\tau)) \, \mathrm{d}\tau
\end{aligned} \tag{12}$$

for all $t \in [0, T]$. Thus Gronwall's inequality implies (6). Moreover, (7) follows from (6) and (12). Furthermore, since $u_\lambda(0) = u_0$, we can derive (8) from (7). $\square$

Lemma 2.6. *There exists a constant M_2 such that*

$$\int_0^T |\eta_\lambda(t)|_{V^*}^{q'}\,\mathrm{d}t \leqslant M_2, \tag{13}$$

$$\int_0^T |g_\lambda(t)|_{V^*}^{\sigma}\,\mathrm{d}t \leqslant M_2 \tag{14}$$

for all $\lambda \in (0, 1]$, where $\sigma = \min\{q', 2\}$.

Proof. By (A2), we get

$$\int_0^T |\eta_\lambda(t)|_{V^*}^{q'}\,\mathrm{d}t \leqslant C_4 \int_0^T \psi^t\left(\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t)\right)\mathrm{d}t + \int_0^T a(t)\,\mathrm{d}t,$$

which together with (7) implies (13). Since (f_λ) is bounded in $L^{q'}(0, T; V^*)$ and $g_\lambda(t) = f_\lambda(t) - \lambda(\mathrm{d}u_\lambda(t)/\mathrm{d}t) - \eta_\lambda(t)$, (7) and (13) yield (14). □

Lemma 2.7. *There exists a constant M_3 such that*

$$\sup_{t\in[0,T]} \varphi^t(J_{H,\lambda}^t u_\lambda(t)) \leqslant M_3, \tag{15}$$

$$\sup_{t\in[0,T]} |J_{H,\lambda}^t u_\lambda(t)|_X \leqslant M_3 \tag{16}$$

for all $\lambda \in (0, 1]$.

Proof. By (A5), we can verify $|J_{H,\lambda}^t u|_H \leqslant C(|u|_H + 1)$ for all $u \in H$ and $t \in [0, T]$. Moreover, we get by (6),

$$\sup_{t\in[0,T]} \varphi_H^t(J_{H,\lambda}^t u_\lambda(t)) \leqslant \sup_{t\in[0,T]} \varphi_{H,\lambda}^t(u_\lambda(t)) \leqslant C. \tag{17}$$

Hence (16) follows immediately from (A4), (8) and (15). □

Proof of Theorem 2.4 (*continued*).

Convergence: From a priori estimates established above, we can take a sequence (λ_n) in $(0, 1]$ such that $\lambda_n \to +0$ as $n \to +\infty$ and the following Lemmas 2.8 and 2.9 hold.

Lemma 2.8. *There exists $u \in W^{1,q}(0, T; V)$ such that*

$$u_{\lambda_n} \to u \quad \text{weakly in } W^{1,q}(0, T; V), \tag{18}$$

$$\lambda_n \frac{\mathrm{d}u_{\lambda_n}}{\mathrm{d}t} \to 0 \quad \text{strongly in } L^2(0, T; H), \tag{19}$$

$$J_{H,\lambda_n}^t u_{\lambda_n} \to u \quad \text{strongly in } C([0, T]; V). \tag{20}$$

Proof. We can derive (18) and (19) immediately from (7) and (8). Moreover, since V is continuously embedded in H, it follows that

$$\begin{aligned}
&|J^t_{H,\lambda}u_\lambda(t) - J^s_{H,\lambda}u_\lambda(s)|_H \\
&\quad \leqslant |J^t_{H,\lambda}u_\lambda(t) - J^s_{H,\lambda}u_\lambda(t)|_H + |u_\lambda(t) - u_\lambda(s)|_H \\
&\quad \leqslant |J^t_{H,\lambda}u_\lambda(t) - J^s_{H,\lambda}u_\lambda(t)|_H + C\left(\int_0^T \left|\frac{\mathrm{d}u_\lambda}{\mathrm{d}\tau}(\tau)\right|_V^q \mathrm{d}\tau\right)^{1/q} (t-s)^{1/q'}.
\end{aligned}$$

Hence, since (A5) assures that for any bounded set B in H, the function $t \mapsto J^t_{H,\lambda}u$ is equi-continuous on $[0,T] \times B$ (see [1] for its proof), the function $t \mapsto J^t_{H,\lambda}u_\lambda(t)$ becomes equi-continuous in $C([0,T];H)$ for all $\lambda \in (0,1]$. Hence since X is compactly embedded in V, by Theorem 5 of [6], there exists $v \in C([0,T];V)$ such that

$$J^t_{H,\lambda_n}u_{\lambda_n} \to v \quad \text{strongly in } C([0,T];V). \tag{21}$$

On the other hand, we have by (14),

$$\int_0^T |u_{\lambda_n}(t) - J^t_{H,\lambda_n}u_{\lambda_n}(t)|^\sigma_{V^*}\,\mathrm{d}t = \lambda_n^\sigma \int_0^T |g_{\lambda_n}(t)|^\sigma_{V^*}\,\mathrm{d}t \leqslant \lambda_n^\sigma M_2 \to 0 \quad \text{as } \lambda_n \to 0.$$

Therefore it follows from (18) and (21) that $v = u$, which implies (20). □

Lemma 2.9. *There exist $g, \eta \in L^{q'}(0,T;V^*)$ such that*

$$\eta_{\lambda_n} \to \eta \quad \textit{weakly in } L^{q'}(0,T;V^*), \tag{22}$$

$$g_{\lambda_n} \to g \quad \textit{weakly in } L^{\sigma}(0,T;V^*). \tag{23}$$

Moreover, we have

$$\eta(t) \in \partial\psi^t\left(\frac{\mathrm{d}u}{\mathrm{d}t}(t)\right), \quad g(t) \in \partial\varphi^t(u(t)), \quad \eta(t)+g(t)=f(t) \quad \textit{for a.e. } t \in (0,T). \tag{24}$$

Proof. It is easily seen that (13) and (14) imply (22) and (23), respectively. Moreover, by Proposition 1.1 of [5], we can derive $\eta(t) \in \partial\psi^t(\mathrm{d}u(t)/\mathrm{d}t)$ from (18), (22) and (A3). Furthermore, by the demiclosedness of subdifferential operators, it follows from (20) and (23) that $g(t) \in \partial\varphi^t(u(t))$. Finally, since $f_{\lambda_n} \to f$ strongly in $L^{q'}(0,T;V^*)$, by $(\mathrm{CP})_\lambda$, (19), (22) and (23), we can deduce that $g = f - \eta \in L^{q'}(0,T;V^*)$. □

Since $u \in W^{1,q}(0,T;V)$ and $g \in L^{q'}(0,T;V^*)$, the function $t \mapsto \varphi^t(u(t))$ is differentiable for a.e. $t \in (0,T)$. Moreover, by (7) and (18), the function $t \mapsto \psi^t(\mathrm{d}u(t)/\mathrm{d}t)$ is integrable on $(0,T)$. Consequently, u becomes a strong solution of (CP).

Proof of Theorem 2.4 (*continued*).

Uniqueness: Let u_1 and u_2 be strong solutions of (CP). Then there exist $g_i(t) \in \partial\varphi^t(u_i(t))$ and $\eta_i(t) \in \partial\psi^t(\mathrm{d}u_i(t)/\mathrm{d}t)$ $(i = 1, 2)$ such that

$$\eta_1(t) - \eta_2(t) + g_1(t) - g_2(t) = 0 \quad \text{for a.e. } t \in (0, T). \tag{25}$$

Multiplying (25) by $w(t) := u_1(t) - u_2(t)$ and using the monotonicity of $\partial\varphi^t$, we get $\langle \xi(t), w(t)\rangle \leqslant 0$ for a.e. $t \in (0, T)$, where $\xi(t) := \eta_1(t) - \eta_2(t)$. Now since the graph of $\partial\psi^t$ is linear, we have $\xi(t) \in \partial\psi^t(\mathrm{d}w(t)/\mathrm{d}t)$. Hence by (A3)$'$, we also deduce $\langle \zeta(t), \mathrm{d}w(t)/\mathrm{d}t\rangle \leqslant 0$ for a.e. $t \in (0, T)$, where $\zeta(t) \in \partial\psi^t(w(t))$. Therefore (A2) with $a \equiv 0$ and (A5)$'$ yield

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\psi^t(w(t)) &\leqslant |\dot{\alpha}_2(t)||\zeta(t)|_{V^*}\psi^t(w(t))^{1/q} + |\dot{\beta}_2(t)|\psi^t(w(t)) \\ &\leqslant \{C_4^{1/q'}|\dot{\alpha}_2(t)| + |\dot{\beta}_2(t)|\}\psi^t(w(t)). \end{aligned}$$

Integrating this over $(0, t)$, we get

$$\psi^t(w(t)) \leqslant \psi^0(w(0)) + \int_0^t \{C_4^{1/q'}|\dot{\alpha}_2(\tau)| + |\dot{\beta}_2(\tau)|\}\psi^\tau(w(\tau))\,\mathrm{d}\tau.$$

Thus Gronwall's inequality implies $\psi^t(w(t)) = 0$ for all $t \in [0, T]$, since $\psi^0(w(0)) = 0$. Hence (A1) with $C_3 = 0$ implies $w(t) = 0$ for all $t \in [0, T]$, which completes the proof. □

3. Solvability of (P) and (P)$_p$

We now prove the existence and uniqueness of solutions for (P) as well as for (P)$_p$ by applying the preceding abstract theory. We are here concerned with solutions of (P) or (P)$_p$ defined in the following.

Definition 3.1. A function $u \in W^{1,2}(0, T; V_0)$ is said to be a solution of (P)$_p$ (resp., (P)), if $u(t)$ satisfies (P)$_p$ (resp., (P)) for a.e. $t \in (0, T)$, and $u(+0) = h_0$.

We can derive (A1) and (A2) with ψ^t replaced by ψ and $q = 2$ immediately from (2) and (3), respectively. Moreover, since $\partial\psi$ is linear and independent of t, (1) implies (A3)$'$ with ψ^t replaced by ψ.

We now suppose that (4) and the following condition are satisfied.

$$\begin{aligned} &\mathrm{d}h_\mathrm{e}/\mathrm{d}t \in L^2(0, T; V_0^*), \quad j_\mathrm{c}(x, t) = \pi(x)\phi(t), \\ &\quad \pi \in L^\infty(\Omega), \quad \phi \in W^{1,2}(0, T). \end{aligned} \tag{26}$$

It then follows that

$$|u|_{H_0^1} \leqslant C \sup_{\tau\in[0,T]} |j_\mathrm{c}(\cdot, \tau)|_{L^\infty} \quad \forall u \in D(\varphi^t_\infty) = K^t,$$

since $|\nabla u|_{L^\infty} \leqslant j_\mathrm{c}(x, t) \leqslant \sup_{\tau\in[0,T]} |j_\mathrm{c}(\cdot, \tau)|_{L^\infty}$ for all $u \in K^t$. Hence since $H_0^1(\Omega)$ is compactly embedded in V_0, (A4) is satisfied with X and φ^t replaced by $H_0^1(\Omega)$ and φ^t_∞, respectively. Moreover, let $t_0 \in [0, T]$ and $u_0 \in D(\varphi_\infty^{t_0})$ be fixed and put $u(t) = \phi(t)u_0/\phi(t_0)$ for all $t \in [0, T]$. Then we have

$$|u(t) - u_0|_{V_0} \leqslant \left| \frac{\phi(t_0) - \phi(t)}{\phi(t_0)} \right| C |\nabla u_0|_{L^\infty}$$
$$\leqslant \frac{C}{\delta_0} |\pi|_{L^\infty} \sup_{\tau \in [0,T]} |j_c(\cdot, \tau)|_{L^\infty} |\phi(t_0) - \phi(t)|,$$

$$\varphi_\infty^t(u(t)) = \varphi_\infty^{t_0}(u_0).$$

Hence (A5) is satisfied with ψ^t, φ^t replaced by ψ, φ_∞^t, respectively and $q = 2$, since $\phi \in W^{1,2}(0, T)$. Therefore setting $V = V_0$ and $H = L^2(\Omega)$ and applying Theorem 2.4 to (P), we can deduce:

Theorem 3.2. *Suppose that* (4) *and* (26) *are satisfied. Then for any* $h_0 \in K^0$, (P) *has a unique solution.*

As for $(\mathrm{P})_p$, we can also verify (A4) and (A5) with ψ^t, φ^t and X replaced by ψ, φ_p^t and $H_0^1(\Omega)$, respectively and $q = 2$ (see [1] for more details). Hence by Theorem 2.4, we obtain

Theorem 3.3. *Let* $p \in [2, +\infty)$ *and suppose that* (4) *and* (26) *are satisfied. Then for any* $h_0 \in W_0^{1,p}(\Omega)$, $(\mathrm{P})_p$ *has a unique solution.*

Remark 3.4. Barrett and Prigozhin [2] proved that h_p converges to the unique solution h of (P) strongly in $C([0, T]; V_0)$ as $p \to +\infty$ for the case where j_c is constant. Akagi and Ôtani [1] also obtained similar results for the case where j_c depends on space and time; moreover, they introduced a new way of approximation of Bean's critical-state model which makes it possible to estimate the difference between the solution of (P) and solutions for approximate equations.

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