WEIGHTED ENERGY-DISSIPATION APPROACH TO DOUBLY-NONLINEAR PROBLEMS ON THE HALF LINE

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Abstract. We discuss a variational approach to abstract doubly-nonlinear evolution systems defined on the time half line $t > 0$. This relies on the minimization of Weighted Energy-Dissipation (WED) functionals, namely a family of ε-dependent functionals defined over entire trajectories. We prove WED functionals admit minimizers and that the corresponding Euler-Lagrange system, which is indeed an elliptic-in-time regularization of the original problem, is strongly solvable. Such WED minimizers converge, up to subsequences, to a solution of the doubly-nonlinear system as $\varepsilon \to 0$. The analysis relies on a specific estimate on WED minimizers, which is specifically tailored to the unbounded time interval case. In particular, previous results on the bounded time interval are extended and generalized. Applications of the theory to classes of nonlinear PDEs are also presented.

1. Introduction

We are concerned with the analysis of the *Weighted Energy-Dissipation* (WED) variational approach to the abstract doubly-nonlinear Cauchy problem on the time half line $t > 0$ defined as

$$\xi(t) + \eta(t) = 0 \quad \text{in } V^* \text{ for a.e. } t > 0, \tag{1.1}$$

$$\xi(t) = \mathrm{d}_V \psi(u'(t)) \quad \text{in } V^* \text{ for a.e. } t > 0, \tag{1.2}$$

$$\eta(t) \in \partial\phi(u(t)) \quad \text{in } V^* \text{ for a.e. } t > 0, \tag{1.3}$$

$$u(0) = u_0. \tag{1.4}$$

Here $\psi, \phi : V \to (-\infty, \infty]$ are convex functionals defined on a Banach space V with dual V^*, ψ has p-growth for some $p > 1$, d_V denotes the Gâteaux derivative, and ∂ is the subdifferential in the sense of convex analysis from V to V^*.

The doubly-nonlinear relations (1.1)-(1.3) stand as an abstract balance systems between conservative and dissipative actions. The former is modeled by the subgradient $\partial\phi$ of the *energy* ϕ whereas the latter correspond to the derivative $\mathrm{d}_V\psi$ of the *dissipation potential* ψ. As such, system (1.1)-(1.3) (possibly combined with nonzero forcing in the right-hand side of (1.1), here neglected for simplicity) appears ubiquitously in applications. In particular, the choice of a p-homogeneous dissipation ψ corresponds to a *gradient flow* for $p = 2$, a rate-independent flow for $p = 1$, and a general *doubly-nonlinear flow* in all other cases. Note that system (1.1)-(1.3) has already been considered under a variety of different settings, see [5, 8, 10, 11, 30] for some reference result.

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The aim of this paper is to propose a variational formalism for the treatment of the Cauchy problem (1.1)-(1.4) of the time half line $t > 0$. This consists in introducing a parameter-dependent family of functionals $I_\varepsilon : L^p(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; V) \to (-\infty, \infty]$ defined over entire trajectories and given by

$$I_\varepsilon(u) = \begin{cases} \displaystyle\int_0^\infty \mathrm{e}^{-t/\varepsilon}\big(\varepsilon\psi(u'(t)) + \phi(u(t))\big)\mathrm{d}t & \text{if } u \in K(u_0), \\ \infty & \text{else,} \end{cases}$$

$$K(u_0) = \{u \in W^{1,p}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; V) : u(0) = u_0 \text{ and } \phi(u) \in L^1(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t)\}$$

and in verifying that its minimizers u_ε converge as $\varepsilon \to 0$ (up to subsequences) to solutions of the problem (1.1)-(1.4). The interest in such a variational approach lies in the possibility of reformulating the differential system (1.1)-(1.4) in terms of a convex minimization problem, combined in a limiting procedure. This reformulation allows to apply the techniques of the modern calculus of variations to the differential problem, in particular the Direct Method, Γ-convergence, and relaxation.

The role of the exponential weight in I_ε is revealed by computing the corresponding Euler-Lagrange equation. In the current setting these read

$$-\varepsilon\xi_\varepsilon'(t) + \xi_\varepsilon(t) + \eta_\varepsilon(t) = 0 \quad \text{in } X^*, \text{ for a.e. } t > 0, \tag{1.5}$$

$$\xi_\varepsilon(t) = \mathrm{d}_V\psi(u_\varepsilon'(t)) \quad \text{in } V^*, \text{ for a.e. } t > 0, \tag{1.6}$$

$$\eta_\varepsilon(t) \in \partial_X\phi_X(u_\varepsilon(t)) \quad \text{in } X^*, \text{ for a.e. } t > 0, \tag{1.7}$$

$$u_\varepsilon(0) = u_0. \tag{1.8}$$

Here, X denotes a Banach space compactly embedded into V and $\partial_X\phi_X$ is the subdifferential from X to X^* of the restriction ϕ_X of ϕ to X. In particular, the minimizers of the WED functionals u_ε solve an *elliptic-in-time regularization* of the target problem (1.1)-(1.4).

Elliptic-regularization techniques have to be traced back at least to Lions and Oleinik [18, 26], see also Lions & Magenes [19] for a linear theory and Barbu [8] for solvability of nonlinear differential equations. Its variational version via WED functionals is already mentioned in the classical textbook by Evans [14, Problem 3, p. 487] and has been used by Ilmanen [16] in the context of Brakke mean-curvature flow of varifolds, and by Hirano [15] in connection with periodic solutions of gradient flows.

The WED formalism has then been considered in the context of abstract rate-independent systems Mielke & Ortiz [25], see also the subsequent [23], and then applied for modeling crack-front propagation in brittle materials in Larsen, Ortiz, & Richardson [17].

The already mentioned [25] presents a first discussion of the WED approach in the linear gradient flow case. Relaxation is then discussed in Conti & Ortiz [12]. The full extent of the classical theory for convex potentials is the recovered in [24] and applied to mean curvature evolution of Cartesian surfaces in [34]. Bögelein, Duzaar, & Marcellini [9] exploit the WED approach in order to find a variational solution to $u_t - \nabla\cdot f(x, u, \nabla u) + \partial_u f(x, u, \nabla u) = 0$ where $u : \Omega\times(0, T) \to \mathbb{R}^d$ and the field f is convex in $(u, \nabla u)$. The gradient-flow theory has then been extended to nonconvex potentials [4] and nonpotential perturbations [22]. A generalization for curves of maximal slope in metric spaces is also available [28, 29].

A celebrated conjecture by De Giorgi [13] pertains the hyperbolic version of the WED technique to recover solutions of the semilinear wave equation. Results on this conjecture in the positive are in [35] (for the bounded time interval case) and in Serra & Tilli [31] (for the original, unbounded time interval case). Extensions to mixed hyperbolic-parabolic equations and to some different classes of nonlinear energies have also been presented [20, 21, 32].

The doubly-nonlinear system (1.1)-(1.3) is considered [1, 2, 3] in the bounded time interval case $t \in (0,T)$. There, the Euler-Lagrange equation corresponding to I_ε (with T instead of ∞) features the final Neumann condition $\xi_\varepsilon(T) = 0$, which turns out to be crucial for obtaining a priori estimates.

The focus of the paper is on the time half line case instead. We prove that the WED functionals admit minimizers u_ε for all $\varepsilon > 0$, that these strongly solve the Euler-Lagrange problem (1.5)-(1.8), and that $u_\varepsilon \to u$ up to subsequences as $\varepsilon \to 0$, where u solves (1.1)-(1.4). This is indeed the assertion of our main result, Theorem 1.

Differently from [1, 2, 3], the final Neumann condition $\xi_\varepsilon(T) = 0$ is obviously here unavailable and one can rely exclusively on weaker integrability conditions for $t \to \infty$. Correspondingly, we are forced here to adapt the technique from [31] and to obtain a priori estimates from minimality by comparing with time-rescaled minimizers. On the other hand, in contrast to [1, 2, 3] or [31], the estimate here delivers the pointwise boundedness of the energy. This in turn allows for a generalization of the abstract theory. In particular, we present in Section 5 a novel existence results for an integropartial differential equation of Kirchhoff type.

Note that the target problem (1.1)-(1.4) may admit multiple solutions. On the other hand, minimizers of the WED functional are unique whenever ψ or ϕ is strictly convex. A by-product of our theory is the proof that WED minimizers u_ε strongly solve the Euler-Lagrange problem (1.5)-(1.8) in the whole half line $t > 0$. Such a strong solvability is, to the best of our knowledge, new.

The paper is organized as follows. We list the assumptions and state our main result in Section 2 we enlist the assumptions and state our main result. Section 3 is devoted to a proof of the existence for strong solutions to the Euler-Lagrange problem (1.5)-(1.8). We deal with the limit $\varepsilon \to 0$ in Section 4. Finally, applications of the abstract theory to nonlinear PDE problems are presented in Section 5.

2. Main result

We devote this section to the statement of our main result. Let $(V, |\cdot|_V)$ be a uniformly convex Banach space. We indicate by V^* its dual and by $\langle\cdot,\cdot\rangle_V$ the corresponding duality pairing. Moreover, let $(X, |\cdot|_X)$ be a reflexive Banach space, densely and compactly embedded in V, with dual space X^*, and duality pairing $\langle\cdot,\cdot\rangle_X$. Assume $\psi : V \to [0,\infty)$ to be a Gâteaux differentiable and convex and $\phi : V \to [0,\infty)$ to be proper, lower semicontinuous, and convex. Let $p \in (1,\infty)$ be fixed and assume that there exist a strictly positive constant C and a nondecreasing function $\ell : \mathbb{R}_+ \to \mathbb{R}_+$ such that the following conditions hold:

$$|u|_V^p \le C(\psi(u) + 1) \quad \forall u \in V,\ \psi(0) = 0; \tag{2.1}$$

$$|\mathrm{d}_V \psi(u)|_{V^*}^{p'} \le C(|u|_V^p + 1) \quad \forall u \in V,\ p' = p/(p-1); \tag{2.2}$$

$$D(\phi) \subset X\,;\ \text{for each } c \text{ the set } \{u \in X : \phi(u) \le c\} \text{ is bounded in } X; \tag{2.3}$$

$$|\eta|_{X^*} \le \ell(|u|_V + \phi(u)) \quad \forall \eta \in \partial_X \phi_X(u). \tag{2.4}$$

As a consequence, there exists a constant C' such that

$$|u|_V^p \leq C'(\langle \mathrm{d}_V\psi(u), u\rangle_V + 1) \quad \forall u \in V, \tag{2.5}$$

$$\psi(u) \leq \psi(0) + \langle \mathrm{d}_V\psi(u), u\rangle_V \leq C'\,(|u|_V^p + 1) \quad \forall u \in V. \tag{2.6}$$

In particular, we have that $D(I_\varepsilon) = \{u \in W^{1,p}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; V) \ : \ \phi(u) \in L^1(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t), u(0) = u_0\}$.

For the sake of later reference, we shall introduce also the coercivity assumption

$$|u|_X \leq C(\phi(u) + 1) \quad \forall u \in D(\phi) \tag{2.7}$$

for some $C > 0$, which is stronger than (2.3). Under the stronger (2.7) we readily check that $D(I_\varepsilon) \subset W^{1,p}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; V) \cap L^1(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; X)$.

Our main result reads as follows.

Theorem 1 (WED variational approach)**.** *Assume* (2.1)-(2.4)*. Then:*

i) *The* WED *functional I_ε admits global minimizers u_ε in $K(u_0)$. Additionally, if either ϕ or ψ is strictly convex, the minimizer is unique.*
ii) *For every minimizer u_ε of I_ε on $K(u_0)$ there exists $\eta_\varepsilon \in \partial_X\phi_X(u_\varepsilon)$ such that, by letting $\xi_\varepsilon = \mathrm{d}_V\psi(u_\varepsilon')$, the triple*

$$(u_\varepsilon, \eta_\varepsilon, \xi_\varepsilon) \in W^{1,p}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; V) \times L^\infty(\mathbb{R}_+; X^*) \times L^{p'}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; V^*),$$

is a strong solution of the Euler-Lagrange problem (1.5)-(1.8)*. Under assumption* (2.7)*, minimizers of I_ε in $K(u_0)$ and strong solutions of the Euler-Lagrange system coincide.*
iii) *There exists a subsequence $\varepsilon_k \to 0$ such that $(u_{\varepsilon_k}, \eta_{\varepsilon_k}, \xi_{\varepsilon_k}) \to (u, \eta, \xi)$ weakly in*

$$W^{1,p}(0,T;V) \cap L^m(0,T;X) \times L^m(0,T;X^*) \times L^{p'}(0,T;V^*)$$

for all $T > 0$, $m > 1$ where (u, η, ξ) is a strong solution to the doubly-nonlinear problem (1.1)-(1.4)*. In case X is separable, we additionally have convergences $\eta_{\varepsilon_k} \to \eta$ weakly-star in $L^\infty(\mathbb{R}_+; X^*)$ and $u_{\varepsilon_k} \to u$ weakly-star in $L^\infty(\mathbb{R}_+; X)$.*

Theorem 1.i-ii is proved in Section 3 by means of a regularization argument whereas the $\varepsilon \to 0$ limit in Theorem 1.iii is ascertained in Section 4 instead.

Before moving on, let us comment that the assumptions (2.1)-(2.4) are weaker than the one in [2], which deals with the bounded time interval case. Indeed, we ask here a weaker growth condition on ϕ, see (2.3)-(2.4), which allows linearly growing energies. We present in Section 5 an example of an integropartial differential equation whose variational formulation fulfills (2.1)-(2.4) but cannot be treated under the frame of [2].

3. Existence of solutions to the Euler-Lagrange problem

In this section we check Theorem 1.i-ii, by proving the existence of a solution u_ε to the Euler-Lagrange problem (1.5)-(1.8), which also minimizes the WED functional I_ε. To this end, we first consider an approximate functional $I_{\varepsilon\lambda}$, obtained from I_ε via Moreau-Yosida regularization (here $\lambda > 0$ is small), and prove that it admits a minimizer $u_{\varepsilon\lambda}$. We then prove uniform estimates for $u_{\varepsilon\lambda}$, which allow us to identify the limit $u_\varepsilon = \lim_{\lambda\to 0} u_{\varepsilon\lambda}$ as a solution to the Euler-Lagrange system (1.5)-(1.8). This solution u_ε turns out to minimize the WED functional I_ε as well.

3.1. **Weighted Lebesgue-Bochner spaces.** Let us start by recalling some basic facts about weighted Lebesgue-Bochner spaces. Let $q \in [1,\infty)$ and $\varepsilon > 0$ be given, and B be a Banach space. We recall that the space $L^q(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; B)$ is defined as follows

$$L^q(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; B) = \left\{ u \in \mathcal{M}(\mathbb{R}_+; B) \colon t \mapsto |u(t)|_B^q \mathrm{e}^{-t/\varepsilon} \in L^1(\mathbb{R}_+) \right\}$$

Here, $\mathcal{M}(\mathbb{R}_+; B)$ stands for the space of strongly measurable functions with values in B. One can check that

$$L^{q_2}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; B) \hookrightarrow L^{q_1}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; B) \quad \text{if } 1 \le q_1 \le q_2 < \infty.$$

Moreover, it also holds true that

$$L^\infty(\mathbb{R}_+; B) \hookrightarrow L^q(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; B) \quad \text{if } 1 \le q < \infty.$$

Moreover, Sobolev spaces $W^{1,p}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; B)$ are also defined analogously.

We now prove a Poincaré-type inequality, which will be of use in the analysis.

Lemma 2 (Poincaré inequality)**.** *Let $\varepsilon > 0$, $1 \le p < \infty$, and B be a reflexive Banach space. Let $u \in W^{1,p}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; B)$. Then, there exists a constant $C = C(\varepsilon, p)$ such that*

$$\|u\|^p_{L^p(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; B)} \le C\left(|u(0)|_B^p + \|u'\|^p_{L^p(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; B)} \right) \quad \forall u \in W^{1,p}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; B). \tag{3.1}$$

Proof. For the sake of notational simplicity, we focus on the case $\varepsilon = 1$, and the general case can be proved analogously. Let $u \in L^1(\mathbb{R}_+, \mathrm{e}^{-t}\mathrm{d}t) := L^1(\mathbb{R}_+, \mathrm{e}^{-t}\mathrm{d}t; \mathbb{R})$ and assume $u \ge 0$ a.e. in $\mathbb{R}_+$. Then, one readily checks that

$$\int_0^t u(s)\mathrm{e}^{-s}\mathrm{d}s \le \|u\|_{L^1(\mathbb{R}_+, \mathrm{e}^{-t}\mathrm{d}t)} \quad \forall t > 0. \tag{3.2}$$

Define now $v(t) := \int_0^t u(s)\mathrm{d}s$ for $t > 0$. By integrating by parts in (3.2) and using $v(0) = 0$, we have

$$v(t)\mathrm{e}^{-t} + \int_0^t v(s)\mathrm{e}^{-s}\mathrm{d}s \le \|u\|_{L^1(\mathbb{R}_+, \mathrm{e}^{-t}\mathrm{d}t)} \quad \forall t > 0.$$

As u (and hence v) is non-negative, taking the limit as $t \to \infty$ we get

$$\|u\|_{L^1(\mathbb{R}_+, \mathrm{e}^{-t}\mathrm{d}t)} \ge \int_0^\infty v(t)\mathrm{e}^{-t}\mathrm{d}t = \int_0^\infty \mathrm{e}^{-t}\left(\int_0^t u(s)\mathrm{d}s \right)\mathrm{d}t. \tag{3.3}$$

Let now u be in $W^{1,p}(\mathbb{R}_+, \mathrm{e}^{-t}\mathrm{d}t; B)$ and assume first that $p = 1$. We estimate

$$\int_0^\infty |u(t) - u(0)|_B \mathrm{e}^{-t}\mathrm{d}t \le \int_0^\infty \mathrm{e}^{-t}\left(\int_0^t |u'(s)|_B \mathrm{d}s \right)\mathrm{d}t.$$

Applying (3.3), we get

$$\int_0^\infty |u(t) - u(0)|_B \mathrm{e}^{-t}\mathrm{d}t \le \| |u'|_B \|_{L^1(\mathbb{R}_+, \mathrm{e}^{-t}\mathrm{d}t)} = \|u'\|_{L^1(\mathbb{R}_+, \mathrm{e}^{-t}\mathrm{d}t; B)}. \tag{3.4}$$

As $\int_0^\infty |u(0)|_B \mathrm{e}^{-t}\mathrm{d}t = |u(0)|_B \int_0^\infty \mathrm{e}^{-t}\mathrm{d}t = |u(0)|_B$, the bound (3.4) implies the assertion with $C = 1$ for $p = 1$.

Let now $p > 1$. For all $\delta > 0$ there exists some constant $C_\delta \ge 0$ such that

$$|u(t)|_B^p - |u(0)|_B^p \le \int_0^t \left(\delta |u(s)|_B^p + C_\delta |u'(s)|_B^p \right) \mathrm{d}s \quad \forall t > 0. \tag{3.5}$$

Indeed, by denoting the duality mapping between B and B^* by $F_B : B \to B^*$, and by using the Young inequality, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}|u(t)|_B^p = p\left\langle |u(t)|_B^{p-2}F_Bu(t), u'(t)\right\rangle_B \le \delta|u(t)|_B^p + C_\delta|u'(t)|_B^p,$$

which yields (3.5) upon integrating both sides over $(0,t)$. By choosing $\delta = 1/2$ in (3.5) and exploiting (3.3), we conclude that

$$\begin{aligned}\int_0^\infty |u(t)|_B^p \mathrm{e}^{-t}\mathrm{d}t &= \int_0^\infty \mathrm{e}^{-t}\left(\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}|u(s)|_B^p \mathrm{d}s\right)\mathrm{d}t + \int_0^\infty \mathrm{e}^{-t}\,|u(0)|_B^p\,\mathrm{d}t \\ &\le \int_0^\infty \mathrm{e}^{-t}\left(\int_0^t \frac{1}{2}|u(s)|_B^p + C_{1/2}|u'(s)|_B^p \mathrm{d}s\right)\mathrm{d}t + \int_0^\infty \mathrm{e}^{-t}\,|u(0)|_B^p\,\mathrm{d}t \\ &\le \frac{1}{2}\int_0^\infty |u(t)|_B^p\mathrm{e}^{-t}\mathrm{d}t + C_{1/2}\int_0^\infty \mathrm{e}^{-t}|u'(t)|_B^p\mathrm{d}t + \int_0^\infty \mathrm{e}^{-t}\,|u(0)|_B^p\,\mathrm{d}t,\end{aligned}$$

whence the inequality (3.1) follows. □

Remark 3. Note that the integrability of the time derivative u' does not imply higher integrability on u on $\mathbb{R}_+$. In particular, the inclusion $W^{1,p}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; B) \subset L^q(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; B)$ does not hold if $p < q$. Indeed, let $p < q$, $B = \mathbb{R}$, and α be such that $q\varepsilon > \alpha > p\varepsilon$. Then,

$$\frac{p}{\alpha} < \frac{1}{\varepsilon} < \frac{q}{\alpha}.$$

Thus, the function $\gamma : t \to \mathrm{e}^{t/\alpha}$ belongs to $W^{1,p}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; B)$ as $\gamma' = (1/\alpha)\gamma$ and

$$\int_0^\infty \gamma^p(t)\mathrm{e}^{-t/\varepsilon}\mathrm{d}t = \int_0^\infty \mathrm{e}^{tp/\alpha - t/\varepsilon}\mathrm{d}t = \int_0^\infty \mathrm{e}^{t(p/\alpha - 1/\varepsilon)}\mathrm{d}t < \infty.$$

On the other hand, we have that $\gamma \notin L^q(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t)$. Indeed,

$$\int_0^\infty \gamma^q(t)\mathrm{e}^{-t/\varepsilon}\mathrm{d}t = \int_0^\infty \mathrm{e}^{t(q/\alpha - 1/\varepsilon)}\mathrm{d}t = \infty.$$

We have hence checked that the exponent in the Poincaré inequality (3.1) is sharp as

$$W^{1,p}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; B) \not\subset L^q(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; B) \quad \forall q > p.$$

3.2. **Approximating functional $I_{\varepsilon\lambda}$.** Let $I_{\varepsilon\lambda} : L^p(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; V) \to (-\infty, \infty]$ be

$$I_{\varepsilon\lambda}(u) := \begin{cases} \displaystyle\int_0^\infty \mathrm{e}^{-t/\varepsilon}\big(\varepsilon\psi(u'(t)) + \phi_\lambda(u(t))\big)\mathrm{d}t & \text{if } u \in K_\lambda(u_0), \\ \infty & \text{else,} \end{cases}$$

$$K_\lambda(u_0) := \{u \in W^{1,p}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; V) : u(0) = u_0 \text{ and } \phi_\lambda(u) \in L^1(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t)\}$$

where $\lambda > 0$ and ϕ_λ denotes the Moreau-Yosida regularization of ϕ (see, e.g., [7]), namely

$$\phi_\lambda(u) := \inf_{v\in V}\left(\frac{1}{2\lambda}|u - v|_V^2 + \phi(v)\right) = \frac{1}{2\lambda}|u - J_\lambda u|_V^2 + \phi(J_\lambda u), \tag{3.6}$$

where J_λ is the resolvent of $\partial_V\phi$ at level λ, namely the solution operator $J_\lambda : u \mapsto J_\lambda u$ to

$$F_V(J_\lambda u - u) + \lambda\partial_V\phi(J_\lambda u) = 0 \quad \forall u \in V, \tag{3.7}$$

where $F_V : V \to V^*$ denotes the duality mapping between V and V^*. Note that

$$D(I_{\varepsilon\lambda}) = K_\lambda(u_0) = \{u \in W^{1,p}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; V) : u(0) = u_0, \ \ \phi_\lambda(u) \in L^1(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t)\}$$

and that $I_{\varepsilon\lambda}$ can be decomposed as

$$I_{\varepsilon\lambda} = I^1_\varepsilon + I^2_{\varepsilon\lambda},$$

where the functionals $I^1_\varepsilon, I^2_{\varepsilon\lambda} : L^p(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; V) \to (-\infty, \infty]$ are defined by

$$I^1_\varepsilon(u) = \begin{cases} \displaystyle\int_0^\infty \mathrm{e}^{-t/\varepsilon}\varepsilon\psi(u'(t))\mathrm{d}t & \text{if } u \in W^{1,p}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; V),\ u(0) = u_0, \\ \infty & \text{else,} \end{cases}$$

and

$$I^2_\varepsilon(u) = \begin{cases} \displaystyle\int_0^\infty \mathrm{e}^{-t/\varepsilon}\phi_\lambda(u(t))\mathrm{d}t & \text{if } \phi_\lambda(u) \in L^1(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t), \\ \infty & \text{else,} \end{cases}$$

with domains

$$D(I^1_\varepsilon) = \{u \in W^{1,p}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; V) : u(0) = u_0\},$$
$$D(I^2_{\varepsilon\lambda}) = \{u \in L^p(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; V) \colon \phi_\lambda(u) \in L^1(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t)\}.$$

The functional $I_{\varepsilon\lambda}$ is proper, lower semicontinuous, and convex. Moreover, thanks to inequality (3.1) it is coercive on $L^p(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; V)$. The Direct Method ensures that there exists a minimizer $u_{\varepsilon\lambda}$ in the closed, convex set $K_\lambda(u_0)$.

3.3. **A priori estimates.** We now derive a priori estimates for $u_{\varepsilon\lambda}$ in order to pass to the limit for $\lambda \to 0$ (and next for $\varepsilon \to 0$). As mentioned in Introduction, we adapt the variational technique introduced by Serra & Tilli [31], see also [20, 21] and [9]. In what follows, the symbols C will denote a strictly positive constant independent on ε, λ, and T, possibly varying from line to line.

We begin with rescaling orbits as follows: For each $u : \mathbb{R}_+ \to V$ set

$$\widetilde{u}(s) := u(t) \quad \text{with} \quad t = \varepsilon s.$$

Then, one has

$$u'(t) = \widetilde{u}'(s)\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\widetilde{u}'(s)}{\varepsilon} \quad \text{and} \quad \mathrm{d}t = \varepsilon\mathrm{d}s$$

and we can rewrite

$$I_{\varepsilon\lambda}(u) = \varepsilon\int_0^\infty \mathrm{e}^{-s}\big(\varepsilon\psi(\widetilde{u}'(s)/\varepsilon) + \phi_\lambda(\widetilde{u}(s))\big)\mathrm{d}s =: \varepsilon J_{\varepsilon\lambda}(\widetilde{u}).$$

Let now $u_{\varepsilon\lambda}$ minimize $I_{\varepsilon\lambda}$ on $K_\lambda(u_0)$. Correspondingly $\widetilde{u}_{\varepsilon\lambda}$ minimizes $J_{\varepsilon\lambda}$ on $\widetilde{K}_\lambda(u_0) := \{\widetilde{u} : u \in K_\lambda\,(u_0)\}$. As $\phi_\lambda \le \phi$, $t \mapsto u_0 \in \widetilde{K}_\lambda(u_0)$, $\phi(u_0) < \infty$, and $\psi(0) = 0$, we get

$$J_{\varepsilon\lambda}(\widetilde{u}_{\varepsilon\lambda}) = \min_{v\in\widetilde{K}_\lambda(u_0)} J_{\varepsilon\lambda}(v) \le J_{\varepsilon\lambda}(u_0) = \int_0^\infty \mathrm{e}^{-s}\phi_\lambda(u_0)\mathrm{d}s$$
$$\le \int_0^\infty \mathrm{e}^{-s}\phi(u_0)\mathrm{d}s = \phi(u_0) = J_\varepsilon(u_0),$$

where J_ε is defined via

$$\varepsilon J_\varepsilon(\widetilde{u}) := \varepsilon\int_0^\infty \mathrm{e}^{-s}\big(\varepsilon\psi(\widetilde{u}(s)/\varepsilon) + \phi(\widetilde{u}(s))\big)\mathrm{d}s = I_\varepsilon(u).$$

In the remainder of this subsection, we simply write $\widetilde{u}$ instead of $\widetilde{u}_{\varepsilon\lambda}$ for the sake of notational simplicity. Let $\eta \in C_{\rm c}^\infty([0,\infty))$ be given, $g' = \eta$ with $g(0) = 0$, and, for small $\delta > 0$, define

$$\varphi_\delta(r) := r - \delta g(r) \quad \forall r \geq 0.$$

Note that $\varphi_\delta : \mathbb{R}_+ \to \mathbb{R}_+$ is a diffeomorphism for every $\delta > 0$ sufficiently small. Set

$$\widetilde{u}_\delta(s) := \widetilde{u}(\varphi_\delta(s))$$

and let π_δ be the inverse function of φ_δ, that is,

$$\sigma = \varphi_\delta(s) \quad \text{iff} \quad s = \pi_\delta(\sigma) = \varphi_\delta^{-1}(\sigma).$$

By using $\widetilde{u}_\delta(s) = \widetilde{u}\left(\varphi_\delta(s)\right) = \widetilde{u}(\sigma)$, and

$$\frac{\mathrm{d}\widetilde{u}_\delta(s)}{\mathrm{d}s} = \widetilde{u}'\left(\varphi_\delta(s)\right)\varphi_\delta'(s) = \widetilde{u}'(\sigma)\varphi_\delta'\left(\pi_\delta(\sigma)\right)$$

we rewrite

$$\begin{aligned} J_{\varepsilon\lambda}(\widetilde{u}_\delta) &= \int_0^\infty \mathrm{e}^{-s}\big(\varepsilon\psi(\widetilde{u}_\delta'(s)/\varepsilon) + \phi_\lambda(\widetilde{u}_\delta(s))\big)\mathrm{d}s \\ &= \int_0^\infty \mathrm{e}^{-\pi_\delta(\sigma)}\left(\varepsilon\psi\left(\frac{\widetilde{u}'(\sigma)}{\varepsilon}\varphi_\delta'\left(\pi_\delta(\sigma)\right)\right) + \phi_\lambda(\widetilde{u}(\sigma))\right)\pi_\delta'(\sigma)\mathrm{d}\sigma. \end{aligned}$$

We now compute the derivative

$$\begin{aligned} \frac{\mathrm{d}J_{\varepsilon\lambda}(\widetilde{u}_\delta)}{\mathrm{d}\delta} &= \int_0^\infty \mathrm{e}^{-\pi_\delta(\sigma)}\left(-\frac{\mathrm{d}}{\mathrm{d}\delta}\pi_\delta(\sigma)\right)\left(\varepsilon\psi\left(\frac{\widetilde{u}'(\sigma)}{\varepsilon}\varphi_\delta'\left(\pi_\delta(\sigma)\right)\right) + \phi_\lambda(\widetilde{u}(\sigma))\right)\pi_\delta'(\sigma)\mathrm{d}\sigma \\ &+ \int_0^\infty \mathrm{e}^{-\pi_\delta(\sigma)}\varepsilon\left\langle \mathrm{d}_V\psi\left(\frac{\widetilde{u}'(\sigma)}{\varepsilon}\varphi_\delta'\left(\pi_\delta(\sigma)\right)\right), \frac{\widetilde{u}'(\sigma)}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\delta}\left(\varphi_\delta'\left(\pi_\delta(\sigma)\right)\right)\right\rangle \pi_\delta'(\sigma)\mathrm{d}\sigma \\ &+ \int_0^\infty \mathrm{e}^{-\pi_\delta(\sigma)}\left(\varepsilon\psi\left(\frac{\widetilde{u}'(\sigma)}{\varepsilon}\varphi_\delta'\left(\pi_\delta(\sigma)\right)\right) + \phi_\lambda(\widetilde{u}(\sigma))\right)\left(\frac{\mathrm{d}}{\mathrm{d}\delta}\pi_\delta'(\sigma)\right)\mathrm{d}\sigma. \end{aligned}$$

From condition

$$\left.\frac{\mathrm{d}J_{\varepsilon\lambda}(\widetilde{u}_\delta)}{\mathrm{d}\delta}\right|_{\delta=0} = 0$$

and by using the fact that

$$\begin{aligned} \varphi_\delta(s)|_{\delta=0} &= s, \qquad \pi_\delta(\sigma)|_{\delta=0} = \sigma, \\ \pi_\delta'(\sigma) &= \frac{1}{\varphi_\delta'(s)} = \frac{1}{1-\delta g'(s)}, \qquad \pi_\delta'(\sigma)|_{\delta=0} = 1, \\ \varphi_\delta'(s) &= 1 - \delta g'(s), \qquad \varphi_\delta'(s)|_{\delta=0} = 1, \\ \frac{\mathrm{d}}{\mathrm{d}\delta}\varphi_\delta'\left(\pi_\delta(\sigma)\right) &= \frac{\mathrm{d}}{\mathrm{d}\delta}\big(1 - \delta g'\left(\pi_\delta(\sigma)\right)\big) = -g'\left(\pi_\delta(\sigma)\right) - \delta g''\left(\pi_\delta(\sigma)\right)\frac{\mathrm{d}}{\mathrm{d}\delta}\pi_\delta(\sigma), \\ \left.\frac{\mathrm{d}}{\mathrm{d}\delta}\varphi_\delta'\left(\pi_\delta(\sigma)\right)\right|_{\delta=0} &= -g'(\sigma), \\ \frac{\mathrm{d}}{\mathrm{d}\delta}\pi_\delta'(\sigma) &= \frac{g'(s)}{(1-\delta g'(s))^2}, \qquad \left.\frac{\mathrm{d}}{\mathrm{d}\delta}\pi_\delta'(\sigma)\right|_{\delta=0} = g'(\sigma), \\ \left.\frac{\mathrm{d}}{\mathrm{d}\delta}\pi_\delta(\sigma)\right|_{\delta=0} &= \lim_{\delta\to 0}\frac{\pi_\delta(\sigma)-\pi_0(\sigma)}{\delta} = \lim_{\delta\to 0}\frac{s-\sigma}{\delta} \\ &= \lim_{\delta\to 0}\frac{s-\varphi_\delta(s)}{\delta} = \lim_{\delta\to 0}\frac{s-s+\delta g(s)}{\delta} = g(s), \end{aligned}$$

we infer that

$$0=\int_0^\infty \mathrm{e}^{-\sigma}\left(-g(\sigma)+g'(\sigma)\right)L(\sigma)\mathrm{d}\sigma-\varepsilon\int_0^\infty \mathrm{e}^{-\sigma}\left\langle \mathrm{d}_V\psi\left(\widetilde{u}'(\sigma)/\varepsilon\right),\widetilde{u}'(\sigma)/\varepsilon\right\rangle g'(\sigma)\mathrm{d}\sigma \tag{3.8}$$

where we set

$$L(\sigma):=\varepsilon\psi\left(\widetilde{u}'(\sigma)/\varepsilon\right)+\phi_\lambda(\widetilde{u}(\sigma)). \tag{3.9}$$

Define now $H:[0,\infty)\to\mathbb{R}$ as

$$H(\sigma):=\int_\sigma^\infty \mathrm{e}^{-\tau}L(\tau)\mathrm{d}\tau\geq 0. \tag{3.10}$$

Then, $H(\sigma)\to 0$ as $\sigma\to\infty$ since $J_{\varepsilon\lambda}(\widetilde{u})<\infty$. We also note that

$$H'(\sigma)=-\mathrm{e}^{-\sigma}L(\sigma). \tag{3.11}$$

Therefore, by using $g(0)=0$, one obtains

$$-\int_0^\infty \mathrm{e}^{-\sigma}g(\sigma)L(\sigma)\mathrm{d}\sigma=\int_0^\infty g(\sigma)H'(\sigma)\mathrm{d}\sigma=-\int_0^\infty g'(\sigma)H(\sigma)\mathrm{d}\sigma.$$

The latter allows us to deduce from (3.8) that

$$\begin{aligned}0&=\int_0^\infty\left(-H(\sigma)+\mathrm{e}^{-\sigma}L(\sigma)-\varepsilon\mathrm{e}^{-\sigma}\left\langle \mathrm{d}_V\psi\left(\widetilde{u}'(\sigma)/\varepsilon\right),\widetilde{u}'(\sigma)/\varepsilon\right\rangle\right)g'(\sigma)\mathrm{d}\sigma\\&=\int_0^\infty\left(H(\sigma)-\mathrm{e}^{-\sigma}L(\sigma)+\varepsilon\mathrm{e}^{-\sigma}\left\langle \mathrm{d}_V\psi\left(\widetilde{u}'(\sigma)/\varepsilon\right),\widetilde{u}'(\sigma)/\varepsilon\right\rangle\right)\eta(\sigma)\mathrm{d}\sigma.\end{aligned}$$

From the arbitrariness of η, we conclude that

$$\mathrm{e}^\sigma H(\sigma)-L(\sigma)+\varepsilon\left\langle \mathrm{d}_V\psi\left(\widetilde{u}'(\sigma)/\varepsilon\right),\widetilde{u}'(\sigma)/\varepsilon\right\rangle=0\quad\text{for a.e. }\sigma>0. \tag{3.12}$$

Set now $E:[0,\infty)\to\mathbb{R}$ as

$$E(\sigma):=\int_0^\sigma\varepsilon\left\langle \mathrm{d}_V\psi(\widetilde{u}'(\tau)/\varepsilon),\widetilde{u}'(\tau)/\varepsilon\right\rangle\mathrm{d}\tau+\mathrm{e}^\sigma H(\sigma). \tag{3.13}$$

Then, by (3.12) and (3.11), one has

$$E'(\sigma)=\varepsilon\left\langle \mathrm{d}_V\psi(\widetilde{u}'(\sigma)/\varepsilon),\widetilde{u}'(\sigma)/\varepsilon\right\rangle+\mathrm{e}^\sigma H(\sigma)+\mathrm{e}^\sigma H'(\sigma)=L(\sigma)-L(\sigma)=0.$$

Therefore, $E(\sigma)\equiv E_0=H(0)$ is constant. As H is nonnegative, we deduce from (3.13) that

$$\int_0^\infty\varepsilon\left\langle \mathrm{d}\psi(\widetilde{u}'(\tau)/\varepsilon),\widetilde{u}'(\tau)/\varepsilon\right\rangle\mathrm{d}\tau\leq E_0.$$

Moreover, since $\widetilde{u}$ is a minimizer of $J_{\varepsilon\lambda}$, one has that

$$E_0=H(0)=\int_0^\infty\mathrm{e}^{-s}L(s)\mathrm{d}s=J_{\varepsilon\lambda}(\widetilde{u})\leq J_{\varepsilon\lambda}(u_0)=\phi_\lambda(u_0)\leq\phi(u_0). \tag{3.14}$$

Consequently, we obtain that

$$\int_0^\infty\varepsilon\left\langle \mathrm{d}_V\psi(\widetilde{u}'(s)/\varepsilon),\widetilde{u}'(s)/\varepsilon\right\rangle\mathrm{d}s\leq\phi(u_0),$$

which can be rewritten as

$$\int_0^\infty\left\langle \mathrm{d}_V\psi(u'(t)),u'(t)\right\rangle\mathrm{d}t\leq\phi(u_0). \tag{3.15}$$

By means of assumption (2.5), for every $T>0$ we get

$$\|u'\|^p_{L^p(\mathbb{R}_+,\mathrm{e}^{-t/\varepsilon}\mathrm{d}t;V)} \le C\int_0^\infty (\langle \mathrm{d}_V\psi(u'(t)),u'(t)\rangle + 1)\,\mathrm{e}^{-t/\varepsilon}\mathrm{d}t \le C, \tag{3.16}$$

$$\|u'\|^p_{L^p(0,T;V)} \le \int_0^T (\langle \mathrm{d}_V\psi(u'(t)),u'(t)\rangle + C)\,\mathrm{d}t \le C + CT, \tag{3.17}$$

as well as

$$\sup_{t\in[0,T]} |u(t)|_V \le C(T). \tag{3.18}$$

Assumption (2.2) yields

$$\|\mathrm{d}_V\psi(u')\|_{L^{p'}(\mathbb{R}_+,\mathrm{e}^{-t/\varepsilon}\mathrm{d}t;V^*)} \le C,$$
$$\|\mathrm{d}_V\psi(u')\|_{L^{p'}(0,T;V^*)} \le C(T).$$

Fix now $\tau>0$. By letting $s=t/\varepsilon$ again, we can write

$$\int_{\varepsilon\tau}^{\varepsilon\tau+\varepsilon} \phi_\lambda(u(t))\mathrm{d}t = \varepsilon\int_\tau^{\tau+1}\phi_\lambda(\widetilde{u}(s))\mathrm{d}s.$$

Recalling definitions (3.9), (3.10), (3.13), formula (3.11), the fact that $E(\cdot)$ is constant, and estimate (3.14), we have

$$\begin{aligned}\int_{\varepsilon\tau}^{\varepsilon\tau+\varepsilon} \phi_\lambda(u(t))\mathrm{d}t &= \varepsilon\int_\tau^{\tau+1}\phi_\lambda(\widetilde{u}(s))\mathrm{d}s \le \varepsilon\mathrm{e}^{\tau+1}\int_\tau^{\tau+1}\mathrm{e}^{-s}\phi_\lambda(\widetilde{u}(s))\mathrm{d}s\\ &\le \varepsilon\mathrm{e}^{\tau+1}\int_\tau^{\tau+1}\mathrm{e}^{-s}L(s)\mathrm{d}s \le \varepsilon\mathrm{e}^{\tau+1}\int_\tau^{\infty}\mathrm{e}^{-s}L(s)\mathrm{d}s\\ &= \varepsilon\mathrm{e}^{\tau+1}\int_\tau^\infty(-H'(s))\,\mathrm{d}s = \varepsilon\mathrm{e}^{\tau+1}H(\tau) \le \varepsilon\mathrm{e}E(\tau) = \varepsilon\mathrm{e}E_0.\end{aligned}$$

Here, we used the inequality $e^\tau H(\tau)\le E(\tau)$, which directly follows from (3.13), and the convexity of ψ. By setting $T=\varepsilon\tau$, the latter implies

$$\frac{1}{\varepsilon}\int_T^{T+\varepsilon}\phi_\lambda(u(t))\mathrm{d}t \le C \quad \forall T>0,\, \varepsilon>0.$$

Thus, $\phi_\lambda(u(\cdot))$ is bounded in the Morrey space $L^{1,1}([0,\infty)) = L^\infty([0,\infty))$, and hence,

$$\sup_{t\in[0,\infty)}\phi_\lambda(u(t)) \le C. \tag{3.19}$$

In particular,

$$\int_0^T \phi_\lambda(u(t))\mathrm{d}t \le CT \quad \forall T>0. \tag{3.20}$$

Let $\eta = \partial_V\phi_\lambda(u)$. Since $\phi(J_\lambda u)\le\phi_\lambda(u)$ and $\eta\in\partial_V\phi(J_\lambda u)\subset\partial_X\phi_X(J_\lambda u)$, it follows from assumptions (2.3), (2.4) that

$$\sup_{t\in[0,\infty)}|\eta(t)|_{X^*} + \sup_{t\in[0,\infty)}|J_\lambda u(t)|_X \le C. \tag{3.21}$$

3.4. **Representation of subdifferentials.** In order to derive the Euler-Lagrange equation for $I_{\varepsilon\lambda}$, we prepare here some representation result. Denote by $I_{\varepsilon\lambda,\mathcal{V}}$, $I^1_{\varepsilon,\mathcal{V}}$, and $I^2_{\varepsilon\lambda,\mathcal{V}}$ the restrictions to

$$\mathcal{V} = L^{\max\{p,2\}}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; V)$$

of $I_{\varepsilon\lambda}$, I^1_ε, and $I^2_{\varepsilon\lambda}$, respectively. We have the following.

Lemma 4 (Identification of $\partial_\mathcal{V} I^1_{\varepsilon,\mathcal{V}}$). *We have*

$$D(\partial_\mathcal{V} I^1_{\varepsilon,\mathcal{V}}) = \{u \in D(I^1_\varepsilon) : \mathrm{d}_V\psi(u') \in W^{1,p'}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}; V^*)\} \cap \mathcal{V},$$

$$\partial_\mathcal{V} I^1_{\varepsilon,\mathcal{V}}(u)(t) = -\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon \mathrm{e}^{-t/\varepsilon}\mathrm{d}_V\psi(u'(t))) \quad \forall u \in D(\partial_\mathcal{V} I^1_\varepsilon), \textit{ for a.e. } t > 0.$$

Proof. Define $\mathcal{A} : \mathcal{V} \to \mathcal{V}^*$ as

$$D(\mathcal{A}) = \{u \in D(I^1_\varepsilon) : \mathrm{d}_V\psi(u') \in W^{1,p'}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}; V^*)\} \cap \mathcal{V},$$

$$\mathcal{A}(u)(t) = -\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon \mathrm{e}^{-t/\varepsilon}\mathrm{d}_V\psi(u'(t))) \quad \forall u \in D(\mathcal{A}), \text{ for a.e. } t > 0,$$

so that the assertion corresponds to

$$\partial_\mathcal{V} I^1_{\varepsilon,\mathcal{V}} = \mathcal{A} \text{ and } D(\partial_\mathcal{V} I^1_{\varepsilon,\mathcal{V}}) = D(\mathcal{A}).$$

We start with proving inclusion $\partial_\mathcal{V} I^1_{\varepsilon,\mathcal{V}} \subset \mathcal{A}$. Set $\mathcal{W} := W^{1,p}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; V) \cap \mathcal{V}$. Define the functionals $J_1, J_2 : \mathcal{W} \to [0,\infty)$ by

$$J_1(u) := \int_0^\infty \mathrm{e}^{-t/\varepsilon}\psi(u'(t))\mathrm{d}t, \quad J_2(u) := \begin{cases} 0 & \text{if } u(0) = u_0, \\ \infty & \text{else,} \end{cases}$$

and denote by $I^1_{\varepsilon,\mathcal{W}}$ the restriction of I^1_ε to $\mathcal{W}$. Thus, $I^1_{\varepsilon,\mathcal{W}} = J_1 + J_2$. One can easily check that J_1 is Gâteaux differentiable in $\mathcal{W}$ and

$$\langle \mathrm{d}_\mathcal{W} J_1(u), e\rangle_\mathcal{W} = \int_0^\infty \mathrm{e}^{-t/\varepsilon} \langle \mathrm{d}_V\psi(u'(t)), e'(t)\rangle_V \,\mathrm{d}t \quad \forall e \in \mathcal{W}.$$

Moreover, J_2 is proper, lower semicontinuous, and convex in $\mathcal{W}$, and we have

$$\langle f, e\rangle_\mathcal{W} = 0 \text{ for all } f \in \partial_\mathcal{W} J_2(u) \text{ and } e \in \mathcal{W} \text{ with } e(0) = 0.$$

Since $D(J_1) = \mathcal{W}$, we deduce that

$$\partial_\mathcal{W} I^1_{\varepsilon,\mathcal{W}} = \mathrm{d}_\mathcal{W} J_1 + \partial_\mathcal{W} J_2$$

with domain

$$D(\partial_\mathcal{W} I^1_{\varepsilon,\mathcal{W}}) = \{u \in \mathcal{W} : u(0) = u_0\}.$$

As we have $D\left(I^1_\varepsilon\right) \cap \mathcal{V} \subset \mathcal{W} \subset \mathcal{V}$, it follows that $\partial_\mathcal{V} I^1_{\varepsilon,\mathcal{V}} \subset \partial_\mathcal{W} I^1_{\varepsilon,\mathcal{W}}$. Let now $f \in \partial_\mathcal{V} I^1_{\varepsilon,\mathcal{V}}(u)$ (hence $u(0) = u_0$). Then,

$$\int_0^\infty \mathrm{e}^{-t/\varepsilon} \langle \mathrm{d}_V\psi(u'(t)), e'(t)\rangle_V \,\mathrm{d}t = \int_0^\infty \langle f(t), e(t)\rangle_V \,\mathrm{d}t$$

for all $e \in \mathcal{W}$ with $e(0) = 0$. Hence, $\mathrm{d}_V\psi(u'(\cdot)) \in W^{1,p'}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; V^*)$ and

$$f(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-t/\varepsilon}\mathrm{d}_V\psi(u'(t))\right) \text{ for a.e. } t > 0.$$

Thus, $u \in D(\mathcal{A})$ and $f = \mathcal{A}(u)$.

We now prove the converse inclusion $\partial_\mathcal{V} I^1_{\varepsilon,\mathcal{V}} \supset \mathcal{A}$. To this aim, let $u \in D(\mathcal{A})$, $v \in D(I_\varepsilon) \cap \mathcal{V}$, $f \in \mathcal{A}(u)$, $T > 0$, and $\varphi \in C^\infty([0,\infty))$ be a nonincreasing cut-off function with $\varphi(t) = 1$ for

$t \le T$ and $\varphi(t) = 0$ for $t \ge T+1$. By recalling the definition of $\mathcal{A}$, integrating by parts, and by using the fact that $u(0) = u_0 = v(0)$ and the definition of subdifferential, we have

$$\langle \varphi f, v-u\rangle_{\mathcal{V}} = \int_0^{T+1} \langle \varphi(t) f(t), v(t)-u(t)\rangle_V \,\mathrm{d}t$$

$$= \int_0^{T+1} \left\langle -\varphi(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon \mathrm{e}^{-t/\varepsilon}\mathrm{d}_V\psi(u'(t))\right), v(t)-u(t)\right\rangle_V \mathrm{d}t$$

$$= \int_0^{T+1} \varphi(t)\varepsilon \mathrm{e}^{-t/\varepsilon}\langle \mathrm{d}_V\psi(u'(t)), v'(t)-u'(t)\rangle_V \,\mathrm{d}t$$

$$+ \int_T^{T+1} \varphi'(t)\varepsilon \mathrm{e}^{-t/\varepsilon}\langle \mathrm{d}_V\psi(u'(t)), v(t)-u(t)\rangle_V \,\mathrm{d}t$$

$$\le \int_0^{T+1} \varphi(t)\varepsilon \mathrm{e}^{-t/\varepsilon}\left(\psi(v'(t))-\psi(u'(t))\right)\mathrm{d}t$$

$$+ \varepsilon|\varphi'|_{L^\infty(T,T+1)}\left(\int_T^{T+1} \mathrm{e}^{-t/\varepsilon}\frac{1}{p'}|\mathrm{d}_V\psi(u'(t))|_{V^*}^{p'}\mathrm{d}t + \int_T^{T+1}\mathrm{e}^{-t/\varepsilon}\frac{1}{p}|v(t)-u(t)|_V^p\mathrm{d}t\right).$$

Note that, as $\mathrm{d}_V\psi(u') \in L^{p'}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; V^*)$ and $u, v \in \mathcal{V} \subset L^p(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; V)$,

$$\int_T^{T+1} \mathrm{e}^{-t/\varepsilon}\frac{1}{p'}|\mathrm{d}_V\psi(u'(t))|_{V^*}^{p'}\mathrm{d}t + \int_T^{T+1}\mathrm{e}^{-t/\varepsilon}\frac{1}{p}|v(t)-u(t)|_V^p\mathrm{d}t \to 0 \quad \text{for} \quad T\to\infty.$$

Thus, as T and φ are arbitrary, we have

$$\langle f, v-u\rangle_{\mathcal{V}} \le \int_0^\infty \varepsilon \mathrm{e}^{-t/\varepsilon}\left(\psi(v'(t))-\psi(u'(t))\right)\mathrm{d}t = I_\varepsilon^1(v) - I_\varepsilon^1(u),$$

which implies $f \in \partial_{\mathcal{V}} I^1_{\varepsilon,\mathcal{V}}(u)$. □

Note that $\partial_V\phi_\lambda : V \to \mathbb{R}$ is demicontinuous (i.e. strong-weak continuous) and single-valued. As a consequence, by (3.6), the functional $I^2_{\varepsilon\lambda,\mathcal{V}} : \mathcal{V} \to (-\infty,\infty]$ is such that $D(I^2_{\varepsilon\lambda,\mathcal{V}}) = D(\partial_{\mathcal{V}} I^2_{\varepsilon\lambda,\mathcal{V}}) = \mathcal{V} = L^{\max\{2,p\}}(\mathbb{R}_+, \mathrm{e}^{-\varepsilon/t}\mathrm{d}t; V)$. As $\partial_{\mathcal{V}} I^1_{\varepsilon,\mathcal{V}} + \partial_{\mathcal{V}} I^2_{\varepsilon\lambda,\mathcal{V}}$ is maximal monotone in $\mathcal{V}\times\mathcal{V}^*$, we have that

$$\partial_{\mathcal{V}} I_{\varepsilon\lambda,\mathcal{V}} = \partial_{\mathcal{V}} I^1_{\varepsilon,\mathcal{V}} + \partial_{\mathcal{V}} I^2_{\varepsilon\lambda,\mathcal{V}}. \tag{3.22}$$

3.5. **Euler-Lagrange equation for $I_{\varepsilon\lambda}$.** Let us first observe that every minimizer $u_{\varepsilon\lambda}$ of $I_{\varepsilon\lambda}$ belongs to $\mathcal{V} = L^{\max\{2,p\}}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; V)$. Indeed, from the definition of ϕ_λ, we have

$$\phi_\lambda(u_{\varepsilon\lambda}(t)) \ge \frac{1}{2\lambda}|u_{\varepsilon\lambda}(t) - J_\lambda u_{\varepsilon\lambda}(t)|_V^2 \quad \text{for a.e. } t>0.$$

By multiplying by $\mathrm{e}^{-t/\varepsilon}$, integrating over $\mathbb{R}_+$, and using (3.19), we obtain

$$\frac{1}{2\lambda}\int_0^\infty |u_{\varepsilon\lambda}(t) - J_\lambda u_{\varepsilon\lambda}(t)|_V^2 \mathrm{e}^{-t/\varepsilon}\mathrm{d}t \le \int_0^\infty \phi_\lambda(u_{\varepsilon\lambda}(t))\mathrm{e}^{-t/\varepsilon}\mathrm{d}t \le C\varepsilon.$$

Thanks to (3.21), we hence have $u_{\varepsilon\lambda} \in \mathcal{V}$. In particular

$$\inf\{I_{\varepsilon\lambda}(v) \colon v \in L^p(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; V)\} = I_{\varepsilon\lambda}(u_{\varepsilon\lambda}) = I_{\varepsilon\lambda,\mathcal{V}}(u_{\varepsilon\lambda}) = \inf_{v\in\mathcal{V}} I_{\varepsilon\lambda}(v),$$

which also implies

$$\partial_{\mathcal{V}} I_{\varepsilon\lambda,\mathcal{V}}(u_{\varepsilon\lambda}) \ni 0.$$

Thanks to the decomposition (3.22) and the fact that $u_{\varepsilon\lambda} \in \mathcal{V}$, we have that

$$0 \in \partial_{\mathcal{V}} I^1_{\varepsilon}(u_{\varepsilon\lambda}) + \partial_{\mathcal{V}} I^2_{\varepsilon\lambda}(u_{\varepsilon\lambda}).$$

Recalling Lemma 4 and noting that

$$f \in \partial_{\mathcal{V}} I^2_{\varepsilon\lambda,\mathcal{V}}(u) \text{ if and only if } f(t) = \mathrm{e}^{-t/\varepsilon}\partial_V \phi_\lambda(u(t)) \text{ for a.e. } t > 0,$$

we deduce that $u_{\varepsilon\lambda}$ fulfills

$$-\varepsilon\xi'_{\varepsilon\lambda}(t) + \xi_{\varepsilon\lambda}(t) + \eta_{\varepsilon\lambda}(t) = 0 \text{ in } V^* \text{ for a.e. } t > 0, \tag{3.23}$$

$$\xi_{\varepsilon\lambda}(t) = \mathrm{d}_V\psi(u'_{\varepsilon\lambda}(t)) \text{ in } V^* \text{ for a.e. } t > 0, \tag{3.24}$$

$$\eta_{\varepsilon\lambda}(t) = \partial_V\phi_\lambda(u_{\varepsilon\lambda}(t)) \text{ in } V^* \text{ for a.e. } t > 0, \tag{3.25}$$

$$u_{\varepsilon\lambda}(0) = u_0. \tag{3.26}$$

Finally, we close this subsection with deriving the rest of a priori estimates: a comparison in equation (3.23) yields

$$\|\varepsilon\,(\mathrm{d}_V\psi(u'))'\,\|_{L^\infty(\mathbb{R}_+;X^*)+L^{p'}(\mathbb{R}_+,\mathrm{e}^{-t/\varepsilon}\mathrm{d}t;V^*)} \le C,$$

$$\|\varepsilon\,(\mathrm{d}_V\psi(u'))'\,\|_{L^\infty(0,T;X^*)+L^{p'}(0,T;V^*)} \le C(T),$$

which imply

$$\sup_{t\in[0,T]} |\varepsilon \mathrm{d}_V\psi(u')|_{X^*} \le C(T).$$

3.6. **Passage to the limit $\lambda \to 0$.** Let $u_{\varepsilon\lambda}$ be a minimizer of $I_{\varepsilon\lambda}$ and $\eta_{\varepsilon\lambda} = \partial_V\phi_\lambda(u_{\varepsilon\lambda})$ and $\xi_{\varepsilon\lambda} = \mathrm{d}_V\psi(u'_{\varepsilon\lambda})$. We have proved that $(u_{\varepsilon\lambda}, \eta_{\varepsilon\lambda}, \xi_{\varepsilon\lambda})$ solves (3.23)-(3.26). Thanks to the uniform estimates obtained above, for every $T > 0$ fixed, we deduce the following convergences for some not relabeled subsequence $\lambda \to 0$

$$u_{\varepsilon\lambda} \to u_\varepsilon \text{ weakly in } W^{1,p}(0,T;V), \tag{3.27}$$

$$J_\lambda u_{\varepsilon\lambda} \to v_\varepsilon \text{ weakly in } L^m(0,T;X) \ \ \forall m > 1, \tag{3.28}$$

$$\xi_{\varepsilon\lambda} \to \xi_\varepsilon \text{ weakly in } L^{p'}(0,T;V^*), \tag{3.29}$$

$$\eta_{\varepsilon\lambda} \to \eta_\varepsilon \text{ weakly in } L^m(0,T;X^*) \ \ \forall m > 1, \tag{3.30}$$

$$\xi'_{\varepsilon\lambda} \to \xi'_\varepsilon \text{ weakly in } L^m(0,T;X^*) + L^{p'}(0,T;V^*) \ \ \forall m > 1, \tag{3.31}$$

for some limits

$$v_\varepsilon \in L^\infty(\mathbb{R}_+;X),\ u_\varepsilon \in W^{1,p}(\mathbb{R}_+,\mathrm{e}^{-t/\varepsilon}\mathrm{d}t;V),$$

$$\eta_\varepsilon \in L^\infty(\mathbb{R}_+;X^*),\ \xi_\varepsilon \in L^{p'}(\mathbb{R}_+,\mathrm{e}^{-t/\varepsilon}\mathrm{d}t;V^*).$$

In case X is separable, we additionally have convergences

$$J_\lambda u_{\varepsilon\lambda} \to v_\varepsilon \text{ weakly star in } L^\infty(\mathbb{R}_+;X),$$

$$\eta_{\varepsilon\lambda} \to \eta_\varepsilon \text{ weakly star in } L^\infty(\mathbb{R}_+;X^*).$$

Moving from convergences (3.27)-(3.31) we can pass to the limit in equation (3.23) and get

$$-\varepsilon\xi'_\varepsilon + \xi_\varepsilon + \eta_\varepsilon = 0 \text{ in } X^*, \text{ a.e. in } \mathbb{R}_+.$$

We now prove the convergence

$$J_\lambda u_{\varepsilon\lambda} \to v_\varepsilon \text{ strongly in } C([0,T];V). \tag{3.32}$$

To this aim, let us define $w_{\varepsilon\lambda} := J_\lambda u_{\varepsilon\lambda} - u_{\varepsilon\lambda}$. By (3.7) and the convexity of ϕ, we have

$$\langle F_V(w_{\varepsilon\lambda}(t+h)) - F_V(w_{\varepsilon\lambda}(t)), J_\lambda u_{\varepsilon\lambda}(t+h) - J_\lambda u_{\varepsilon\lambda}(t)\rangle_V \le 0,$$

which, together with estimate (3.18) and the fact that $|J_\lambda a|_V \le C(|a|_V + 1)$ for all $a \in V$, yields

$$\langle F_V(w_{\varepsilon\lambda}(t+h)) - F_V(w_{\varepsilon\lambda}(t)), w_{\varepsilon\lambda}(t+h) - w_{\varepsilon\lambda}(t)\rangle_V \le C(T)|u_{\varepsilon\lambda}(t+h) - u_{\varepsilon\lambda}(t)|_V. \quad (3.33)$$

Estimate (3.17) implies that $u_{\varepsilon\lambda} : [0,T] \to V$ is equicontinuous, for every $T > 0$ fixed, and that the right-hand side of (3.33) goes to 0 as $h \to 0$, uniformly for $t \in [0,T]$ and $\lambda > 0$. Since V is uniformly convex, thanks to [27], for each $R > 0$ there exists a strictly increasing function m_R on $[0,\infty)$ such that $m_R(0) = 0$ and

$$m_R(|u-v|_V) \le \langle F_V(u) - F_V(v), u - v\rangle_V \ \text{ for } u, v \in B_R := \{w \in V : |w|_V \le R\}.$$

In particular, $w_{\varepsilon\lambda} : [0,T] \to V$ are equicontinuous for all $T > 0$ and so are $J_\lambda u_{\varepsilon\lambda}$. As a consequence of the compact embedding $X \hookrightarrow\hookrightarrow V$ and of [33, Theorem 3], we deduce the strong convergence (3.32). Furthermore, by using estimate (3.20) and (3.6), we conclude that

$$\int_0^T |J_\lambda u_{\varepsilon\lambda}(t) - u_{\varepsilon\lambda}(t)|_V^2 \mathrm{d}t \le 2\lambda \int_0^T \phi_\lambda(u_{\varepsilon\lambda}(t))\mathrm{d}t \le \lambda C(T) \to 0.$$

In particular, we get $v_\varepsilon = u_\varepsilon$ and

$$u_{\varepsilon\lambda}(t) \to u_\varepsilon(t) \ \text{ strongly in } L^2(0,T;V), \ \text{ for all } T > 0,$$

which yields

$$u_{\varepsilon\lambda}(t) \to u_\varepsilon(t) \text{ strongly in } V, \ \text{ for a.e. } t > 0.$$

By following the argument of [2, Section 3.3], one can prove that $\xi_\varepsilon(t) = \mathrm{d}_V\psi(u'_\varepsilon(t))$ and $\eta_\varepsilon(t) \in \partial_X\phi_X(u_\varepsilon(t))$ for almost every $t \in (0,T)$ and we get the identifications (3.24)-(3.25) as T is arbitrary. In particular, we have proved that the Euler-Lagrange problem (1.5)-(1.8) admits a strong solution on the half line $t > 0$.

3.7. **Minimization of the WED functional I_ε.** Our next aim is to prove that the above-determined limit u_ε indeed minimizes I_ε on $K(u_0)$. Note that $K(u_0) \subset K_\lambda(u_0)$ as $\phi_\lambda \le \phi$. By passing to the limit as $\lambda \to 0$ and using the dominated convergence theorem, we have

$$I_{\varepsilon\lambda}(v) \to I_\varepsilon(v) \quad \forall v \in K(u_0).$$

As $u_{\varepsilon\lambda}$ is a global minimizer of $I_{\varepsilon\lambda}$, we have

$$I_{\varepsilon\lambda}(v) \ge I_{\varepsilon\lambda}(u_{\varepsilon\lambda}) \quad \forall v \in K(u_0).$$

Furthermore, convergences (3.27)-(3.28) and the lower semicontinuity of $u \mapsto \int_0^T \mathrm{e}^{-t/\varepsilon}\varepsilon\psi(u'(t))\mathrm{d}t$ and of $u \to \int_0^T \mathrm{e}^{-t/\varepsilon}\phi(u(t))\,\mathrm{d}t$ in $L^p(0,T;V)$ for every $T > 0$, give us

$$\begin{aligned}\liminf_{\lambda\to0} I_{\varepsilon\lambda}(u_{\varepsilon\lambda}) &\ge \liminf_{\lambda\to0} \int_0^T \mathrm{e}^{-t/\varepsilon}\left(\varepsilon\psi(u'_{\varepsilon\lambda}(t)) + \phi_\lambda(u_{\varepsilon\lambda}(t))\right)\mathrm{d}t \\ &= \liminf_{\lambda\to0} \int_0^T \mathrm{e}^{-t/\varepsilon}\left(\varepsilon\psi(u'_{\varepsilon\lambda}(t)) + \phi(J_\lambda u_{\varepsilon\lambda}(t))\right)\mathrm{d}t \\ &\ge \int_0^T \mathrm{e}^{-t/\varepsilon}\big(\varepsilon\psi(u'_\varepsilon(t)) + \phi\big(u_\varepsilon(t)\big)\big)\,\mathrm{d}t.\end{aligned}$$

Taking the supremum for $T > 0$, we deduce that

$$I_\varepsilon(v) \ge I_\varepsilon(u_\varepsilon) \quad \forall v \in K(u_0).$$

Namely, u_ε minimizes I_ε on $K(u_0)$. As a consequence, we deduce that $u_\varepsilon \in D(I_\varepsilon)$.

In case either ψ or ϕ is strictly convex, the functional I_ε turns out to be strictly convex in $L^p(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; V)$. In particular, the minimizer is unique.

If the WED functional is not strictly convex, we proceed by penalization. Let $\widehat{u}_\varepsilon$ be a minimizer of I_ε over $K(u_0)$. We define the penalized functionals $\widehat{I}_\varepsilon$, $\widehat{I}_{\varepsilon\lambda}$ by

$$\begin{aligned} \widehat{I}_\varepsilon(u) &:= \int_0^\infty \mathrm{e}^{-t/\varepsilon}\big(\varepsilon\psi(u'(t)) + \widehat{\phi}\,(u(t))\,\big)\mathrm{d}t, \\ \widehat{\phi}(u) &:= \phi\,(u) + \frac{c}{p}|u - \widehat{u}_\varepsilon|_V^p \\ \widehat{I}_{\varepsilon\lambda}(u) &:= \int_0^\infty \mathrm{e}^{-t/\varepsilon}\big(\varepsilon\psi(u'(t)) + \widehat{\phi}_\lambda(u(t))\big)\mathrm{d}t, \\ \widehat{\phi}_\lambda\,(u(t)) &:= \phi_\lambda\,(u) + \frac{c}{p}|u - \widehat{u}_\varepsilon|_V^p, \end{aligned}$$

where ϕ_λ is the Moreau-Yosida regularization of ϕ and c is a strictly positive constant. Note that $\widehat{u}_\varepsilon$ is the unique global minimizer of the strictly convex functional $\widehat{I}_\varepsilon$ and

$$\widehat{I}_\varepsilon(\widehat{u}_\varepsilon) = I_\varepsilon(\widehat{u}_\varepsilon) = \min_{v\in K(u_0)} I_\varepsilon(v).$$

Arguing as above we can show that $\widehat{I}_{\varepsilon\lambda}$ admits a minimizer $\widetilde{u}_{\varepsilon\lambda}$. Moreover, it fulfills for almost every $t > 0$

$$\begin{aligned} -\varepsilon\widetilde{\xi}'_{\varepsilon\lambda}(t) + \widetilde{\xi}_{\varepsilon\lambda}(t) + \widetilde{\eta}_{\varepsilon\lambda}(t) &= -c|\widetilde{u}_{\varepsilon\lambda}(t) - \widehat{u}_\varepsilon(t)|_V^{p-2} F_V(\widetilde{u}_{\varepsilon\lambda}(t) - \widehat{u}_\varepsilon(t)) \text{ in } V^*, \\ \widetilde{\xi}_{\varepsilon\lambda}(t) &= \mathrm{d}_V\psi(\widetilde{u}'_{\varepsilon\lambda}(t)) \text{ in } V^*, \\ \widetilde{\eta}_{\varepsilon\lambda}(t) &= \partial_V\phi_\lambda(\widetilde{u}_{\varepsilon\lambda}(t)) \text{ in } V^*, \\ \widetilde{u}_{\varepsilon\lambda}(0) &= u_0. \end{aligned}$$

Arguing as in Section 3.6, we can deduce uniform estimates for $\widetilde{u}_{\varepsilon\lambda}$ and prove the following convergences as $\lambda \to 0$, for all $T > 0$,

$$\begin{aligned} &\widetilde{u}_{\varepsilon\lambda} \to \widetilde{u}_\varepsilon \text{ weakly in } W^{1,p}(0,T;V), \\ &\widetilde{\xi}_{\varepsilon\lambda} \to \widetilde{\xi}_\varepsilon \text{ weakly in } L^{p'}(0,T;V^*), \\ &\widetilde{\eta}_{\varepsilon\lambda} \to \widetilde{\eta}_\varepsilon \text{ weakly in } L^m(0,T;X^*) \quad \forall m > 1, \\ &\widetilde{\xi}'_{\varepsilon\lambda} \to \widetilde{\xi}'_\varepsilon \text{ weakly in } L^m(0,T;X^*) + L^{p'}(0,T;V^*) \quad \forall m > 1, \end{aligned}$$

where $(\widetilde{u}_\varepsilon, \widetilde{\eta}_\varepsilon, \widetilde{\xi}_\varepsilon)$ solves for almost every $t > 0$

$$-\varepsilon\widetilde{\xi}'_\varepsilon(t) + \widetilde{\xi}_\varepsilon(t) + \widetilde{\eta}_\varepsilon(t) = -c|\widetilde{u}_\varepsilon(t) - \widehat{u}_\varepsilon(t)|_V^{p-2} F_V(\widetilde{u}_\varepsilon(t) - \widehat{u}_\varepsilon(t)) \text{ in } X^*, \tag{3.34}$$

$$\widetilde{\xi}_\varepsilon(t) = \mathrm{d}_V\psi(\widetilde{u}'_\varepsilon(t)) \text{ in } V^*, \tag{3.35}$$

$$\widetilde{\eta}_\varepsilon(t) \in \partial_X\phi_X(\widetilde{u}_\varepsilon(t)) \text{ in } X^*, \tag{3.36}$$

$$\widetilde{u}_\varepsilon(0) = u_0. \tag{3.37}$$

Moreover, $\widetilde{u}_\varepsilon$ minimizes $\widehat{I}_\varepsilon$, which is strictly convex. This implies that $\widetilde{u}_\varepsilon = \widehat{u}_\varepsilon$. Finally, by directly substituting $\tilde{u}_\varepsilon = \widehat{u}_\varepsilon$ into (3.34)-(3.37), we check that $\widehat{u}_\varepsilon$ solves the Euler-Lagrange problem (1.5)-(1.8).

Assume now the stronger coercivity condition (2.7) and let $(u_\varepsilon, \eta_\varepsilon, \xi_\varepsilon)$ solve the Euler-Lagrange problem (1.5)-(1.8). Given any $v \in W^{1,p}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; V) \cap L^1(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; X)$ with $v(0) = 0$

one can check that

$$\int_0^\infty \mathrm{e}^{-t/\varepsilon}\big(\langle\varepsilon\xi_\varepsilon'(t), v'(t)\rangle_V + \langle\eta_\varepsilon(t), v(t)\rangle_X\big)\mathrm{d}t = 0. \tag{3.38}$$

Indeed, it suffices to test (1.5) with φv, where $\varphi \in C^\infty([0,\infty))$ is a nonincreasing cut-off function with $\varphi(t) = 1$ for $t \le T$ and $\varphi(t) = 0$ for $t \ge T+1$, and take the limit as $T \to \infty$. Let now $w \in K(u_0)$ be given and set $v = w - u_\varepsilon$. By (2.7), we find that $v \in L^1(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; X) \cap W^{1,p}(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\mathrm{d}t; V)$. By using the convexity of ϕ and ψ we get by (3.38) that

$$I_\varepsilon(w) - I_\varepsilon(u_\varepsilon) \ge \int_0^\infty \mathrm{e}^{-t/\varepsilon}\big(\langle\varepsilon\xi_\varepsilon'(t), v'(t)\rangle_V + \langle\eta_\varepsilon(t), v(t)\rangle_X\big)\mathrm{d}t = 0$$

so that u_ε minimizes I_ε on $K(u_0)$. This concludes the proof of Theorem 1.i-ii.

4. The causal limit

Let us now proceed to the proof of Theorem 1.iii by checking that, up to subsequences, u_ε converges to a strong solution of (1.1)-(1.4). This limit is usually referred to as *causal limit* as it connects the *noncausal*, elliptic-in-time Euler-Lagrange system to the *causal* target problem (1.1)-(1.4).

Starting from the uniform estimates derived in Section 3.3 and using the lower semicontinuity of norms and of ϕ, we deduce the following bounds, for all $T > 0$,

$$\|u_\varepsilon'\|_{L^p(0,T;V)} + \|\xi_\varepsilon\|_{L^{p'}(0,T;V^*)} + \|\varepsilon\xi_\varepsilon'\|_{L^\infty(0,T;X^*)+L^{p'}(0,T;V^*)} \le C(T),$$
$$\|u_\varepsilon\|_{L^\infty(\mathbb{R}_+;X)} + \sup_{t\ge 0}\phi(u_\varepsilon(t)) + \|\eta_\varepsilon\|_{L^\infty(\mathbb{R}_+;X^*)} \le C,$$

which, up to not relabeled subsequences, imply the following convergences for all $T > 0$ as $\varepsilon \to 0$

$$u_\varepsilon \to u \text{ weakly in } W^{1,p}(0,T;V) \text{ and strongly in } C([0,T];V), \tag{4.1}$$
$$u_\varepsilon \to u \text{ weakly in } L^m(0,T;X) \quad \forall m > 1, \tag{4.2}$$
$$\xi_\varepsilon \to \xi \text{ weakly in } L^{p'}(0,T;V^*), \tag{4.3}$$
$$\varepsilon\xi_\varepsilon' \to 0 \text{ weakly in } L^m(0,T;X^*) + L^{p'}(0,T;V^*) \quad \forall m > 1, \tag{4.4}$$
$$\eta_\varepsilon \to \eta \text{ weakly in } L^m(0,T;X^*) \quad \forall m > 1, \tag{4.5}$$

for some limits

$$u \in W^{1,p}(0,T;V) \cap L^\infty(\mathbb{R}_+;X), \ \ \xi \in L^{p'}(0,T;V^*), \ \ \eta \in L^\infty(\mathbb{R}_+;X^*) \quad \forall T > 0.$$

Note that, in case X is separable, we additionally have the convergences

$$u_\varepsilon \to u \text{ weakly star in } L^\infty(\mathbb{R}_+;X),$$
$$\eta_\varepsilon \to \eta \text{ weakly star in } L^\infty(\mathbb{R}_+;X^*).$$

Convergences (4.3)-(4.5) are sufficient in order to pass to the limit in equation (1.5) and obtain

$$\xi + \eta = 0 \text{ in } X^* \text{ a.e. in } \mathbb{R}_+. \tag{4.6}$$

We now check that $\eta \in \partial_X\phi_X(u)$. Let us start observing that, for every $v \in X$, $\varphi \in C_c^\infty([0,\infty))$, and $t \ge 0$, we have

$$\langle\varepsilon\xi_\varepsilon(t)\varphi(t), v\rangle_X = \left\langle\int_\infty^t \varepsilon(\xi_\varepsilon\varphi)', v\right\rangle_X = \int_\infty^t \langle\varepsilon\xi_\varepsilon', v\rangle_X\,\varphi + \int_\infty^t \langle\varepsilon\xi_\varepsilon, v\rangle_X\,\varphi' \to 0.$$

Let us now compute that

$$\begin{aligned}\int_0^\infty \langle \eta_\varepsilon, u_\varepsilon\varphi\rangle_X &= \int_0^\infty \langle \varepsilon\xi_\varepsilon', u_\varepsilon\varphi\rangle_X - \int_0^\infty \langle \xi_\varepsilon, u_\varepsilon\varphi\rangle_V \\ &= -\langle \varepsilon\xi_\varepsilon(0), u_\varepsilon(0)\varphi(0)\rangle_X - \int_0^\infty \left(\langle \varepsilon\xi_\varepsilon, u_\varepsilon'\varphi\rangle_V + \langle \varepsilon\xi_\varepsilon, u_\varepsilon\varphi'\rangle_V \right) - \int_0^\infty \langle \xi_\varepsilon, u_\varepsilon\varphi\rangle_V \\ &\to -\int_0^\infty \langle \xi, u\varphi\rangle_V .\end{aligned}$$

Therefore, we have

$$\lim_{\varepsilon\to 0}\int_0^\infty \langle \eta_\varepsilon, u_\varepsilon\varphi\rangle_X = -\int_0^\infty \langle \xi, u\varphi\rangle_V = \int_0^\infty \langle \eta, u\varphi\rangle_X .$$

As φ is arbitrary, we conclude that

$$\lim_{\varepsilon\to 0}\int_0^T \langle \eta_\varepsilon, u_\varepsilon\rangle_X = \int_0^T \langle \eta, u\rangle_X \qquad \forall T>0.$$

Thus, by using the demiclosedness of the maximal monotone operator $\partial_X\phi_X$ in $L^m(0,T;X)\times L^{m'}(0,T;X^*)$, we deduce that $\eta(t)\in\partial_X\phi_X(u(t))$ for almost every $t>0$, namely relation (1.3). Moreover, we find by (4.6) that $\eta(t)\in V^*$ for almost every $t>0$. Thus, thanks to [2, Prop. 2.1], $\eta(t)\in\partial_V\phi(u(t))$ for almost every $t>0$ and equation (4.6) actually holds in V^* for a.e. $t>0$. This proves (1.1) and (1.3).

In order to identify the limit of ξ_ε as a subgradient of ψ at u_ε', we use the following lemma.

Lemma 5. *Let $\varphi\in C^1([0,\infty))$ be nonincreasing with $\varphi(t)=1$ for $t\le T$ and $\varphi(t)=0$ for $t\ge T+1$. Then, there exists a constant $C>0$ such that*

$$\int_0^\infty \langle \xi_\varepsilon, u_\varepsilon'\varphi\rangle_V \le \varepsilon C\phi(u_0)+\phi(u_0)+\int_T^{T+1}\phi(u_\varepsilon)\varphi'.$$

Proof. We test equation (3.23) with $u_{\varepsilon\lambda}'\varphi$ to get

$$\int_0^\infty \langle \xi_{\varepsilon\lambda}, u_{\varepsilon\lambda}'\varphi\rangle_V = \varepsilon\int_0^\infty \langle \xi_{\varepsilon\lambda}', u_{\varepsilon\lambda}'\varphi\rangle_V - \int_0^\infty \langle \eta_{\varepsilon\lambda}, u_{\varepsilon\lambda}'\varphi\rangle_V . \tag{4.7}$$

Denote by ψ^* the Fenchel conjugate of ψ. By observing that $u_{\varepsilon\lambda}'\in\partial_{V^*}\psi^*(\xi_{\varepsilon\lambda})$, we estimate

$$\begin{aligned}\varepsilon\int_0^\infty \langle \xi_{\varepsilon\lambda}', u_{\varepsilon\lambda}'\rangle_V\,\varphi &= \varepsilon\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\psi^*(\xi_{\varepsilon\lambda})\varphi = -\varepsilon\psi^*(\xi_{\varepsilon\lambda}(0))\varphi(0) - \varepsilon\int_0^\infty \psi^*(\xi_{\varepsilon\lambda})\varphi' \\ &\le \varepsilon C\int_T^{T+1}\psi^*(\xi_{\varepsilon\lambda}) \le \varepsilon C\int_0^\infty \langle \xi_{\varepsilon\lambda}, u_{\varepsilon\lambda}'\rangle_V \overset{(3.15)}{\le} \varepsilon C\phi(u_0).\end{aligned}$$

Here we used nonnegativity of ψ^* (from $\psi^*\ge-\psi(0)=0$) and φ, the definition of subdifferential, and estimate (3.15). Note that, by chain-rule, we also have

$$\langle \eta_{\varepsilon\lambda}, u_{\varepsilon\lambda}'\rangle_V = \frac{\mathrm{d}}{\mathrm{d}t}\phi_\lambda(u_{\varepsilon\lambda}) \ \text{ a.e. in } \ \mathbb{R}_+.$$

Therefore, by integrating by parts, we can derive

$$\begin{aligned}\int_0^\infty \langle \eta_{\varepsilon\lambda}, u_{\varepsilon\lambda}'\rangle_V\,\varphi &= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\phi_\lambda(u_{\varepsilon\lambda})\varphi = -\phi_\lambda(u_{\varepsilon\lambda}(0))\varphi(0) - \int_T^{T+1}\phi_\lambda(u_{\varepsilon\lambda})\varphi' \\ &\ge -\phi_\lambda(u_0) - \int_T^{T+1}\phi(J_\lambda u_{\varepsilon\lambda})\varphi'.\end{aligned}$$

Hence, by substituting it into (4.7) and by using the lower semicontinuity of $u \longmapsto -\int_T^{T+1} \phi(u)\varphi'$ in $L^p(T, T+1; V)$, convergences (3.27), (3.29), (3.32), and the pointwise convergence $\phi_\lambda(v) \to \phi(v)$ for all $v \in D(\phi)$, we get

$$\begin{aligned}
\int_0^\infty \langle \xi_\varepsilon, u_\varepsilon' \rangle_V \varphi &= \lim_{\lambda \to 0} \int_0^\infty \langle \xi_{\varepsilon\lambda}, u_{\varepsilon\lambda}' \rangle_V \varphi \\
&\le \limsup_{\lambda \to 0} \varepsilon \int_0^\infty \langle \xi_{\varepsilon\lambda}', u_{\varepsilon\lambda}' \rangle_V \varphi + \limsup_{\lambda \to 0} \left(- \int_0^\infty \langle \eta_{\varepsilon\lambda}, u_{\varepsilon\lambda}' \rangle_V \varphi \right) \\
&\le \varepsilon C \phi(u_0) + \lim_{\lambda \to 0} \phi_\lambda(u_0) - \liminf_{\lambda \to 0} \int_T^{T+1} \phi(J_\lambda u_{\varepsilon\lambda}) \, (-\varphi') \\
&\le \varepsilon C \phi(u_0) + \phi(u_0) + \int_T^{T+1} \phi(u_\varepsilon) \varphi'.
\end{aligned}$$

This completes the proof. □

Let φ be given as in Lemma 5. Then, by using the lower semicontinuity of ϕ and convergence (4.1), we obtain

$$\begin{aligned}
\limsup_{\varepsilon \to 0} \int_0^\infty \langle \xi_\varepsilon, u_\varepsilon' \rangle_V \varphi &\le \lim_{\varepsilon \to 0} \{ \varepsilon C \phi(u_0) \} + \phi(u_0) + \limsup_{\varepsilon \to 0} \int_T^{T+1} \phi(u_\varepsilon) \varphi' \\
&\le \phi(u_0) - \liminf_{\varepsilon \to 0} \int_T^{T+1} \phi(u_\varepsilon) \, (-\varphi') \le \phi(u_0) + \int_T^{T+1} \phi(u) \varphi' \\
&= - \int_0^\infty \langle \eta, u' \rangle_V \varphi = \int_0^\infty \langle \xi, u' \rangle_V \varphi.
\end{aligned}$$

From the arbitrariness of φ, by using the demiclosedness of maximal monotone operators, we deduce that $\xi(t) = \mathrm{d}_V \psi(u'(t))$ for almost every $t > 0$, namely (1.2). This concludes the proof of Theorem 1.

5. Applications

The abstract theory developed in the present paper provides direct extension of the results in [2] to unbounded time intervals. In particular, all doubly-nonlinear PDE systems from [2] can be investigated on the time half line by means of Theorem 1. For instance, one can consider the problem

$$\alpha(\partial_t u(x,t)) - \nabla \cdot \big(a(x) |\nabla u(x,t)|^{m-2} \nabla u(x,t) \big) = 0 \quad \text{for } (x,t) \in \Omega \times \mathbb{R}_+, \tag{5.1}$$

$$u(x,t) = 0 \quad \text{for } (x,t) \in \partial\Omega \times \mathbb{R}_+ \tag{5.2}$$

$$u(x,0) = u_0(x) \quad \text{for} x \in \Omega, \tag{5.3}$$

where Ω is a nonempty, open, and bounded subset of $\mathbb{R}^d$ with smooth boundary $\partial\Omega$. Here, $\alpha : \mathbb{R} \to \mathbb{R}$ is a maximal monotone operator such that there exists $p > 1$ and a positive constant C such that

$$C|s|^p - \frac{1}{C} \le A(s) := \int_0^s \alpha(r) \mathrm{d}r \quad \text{and} \quad |\alpha(s)|^{p'} \le C(|s|^p + 1) \quad \forall s \in \mathbb{R}. \tag{5.4}$$

The coefficient $a : \Omega \to \mathbb{R}_+$ is measurable, bounded, and uniformly positive, almost everywhere in Ω, and $u_0 \in W^{1,m}(\Omega)$, $a \in L^\infty(\Omega)$. Theorem 1 implies that the minimizers u_ε of the WED functional

$$I_\varepsilon(u) = \int_0^\infty \mathrm{e}^{-t/e} \left(\varepsilon \int_\Omega A(\partial_t u(x,t))\mathrm{d}x + \frac{1}{m} \int_\Omega a(x)|\nabla u(x,t)|^m \mathrm{d}x \right) \mathrm{d}t$$

converge, up to subsequences, to solutions of system (5.1)-(5.3) as $\varepsilon \to 0$. We refer to Section 7 of [2] for additional details.

As already mentioned, the assumptions on the energy functional (2.3)-(2.4) are more general than the analogous ones in [2]. There ϕ is assumed to show a polynomial behavior $\phi(u) \sim |u|_X^m$ for $m > 1$. Here, linearly growing energy potentials ϕ can be considered instead (i.e., $m = 1$). The possibility of this extension originates from the different estimation technique.

We give now an explicit example of a doubly-nonlinear problem which is treatable within this abstract frame but is not included into [2]. Let us consider the following nonlocal, Kirchhoff-type integropartial differential equation

$$\left(\int_\Omega |\nabla u|^2 \right)^{1/2} \alpha(\partial_t u) - \Delta u = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \tag{5.5}$$

$$u = 0 \quad \text{on } \partial\Omega \times \mathbb{R}_+, \tag{5.6}$$

$$u(0) = u_0 \quad \text{in } \Omega, \tag{5.7}$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is nondecreasing, continuous, and satisfies (5.4) for some $p \in (1, 2d/(d-2)^+)$. In order to apply our abstract theory, we set $V = L^p(\Omega)$, $X = H_0^1(\Omega)$,

$$\psi(u) = \int_\Omega A(u), \quad \phi(u) = \begin{cases} \left(\int_\Omega |\nabla u|^2 \right)^{1/2} = |u|_{H_0^1(\Omega)} = |u|_X & \text{if } u \in X, \\ \infty & \text{else.} \end{cases}$$

Note that ϕ is lower semicontinuous in $X = D(\phi) = D(\partial\phi)$, which implies that ϕ is lower semicontinuous in V as well. Moreover, ϕ is convex and has bounded sublevels in X. Namely, it satisfies (2.3). Moreover, ϕ is Gâteaux differentiable in $X \setminus \{0\}$. Indeed, for every $u \in X \setminus \{0\}$, $x \in X$, and $h \in \mathbb{R}$, we have

$$\begin{aligned} \phi(u+hx) &= \left(\int_\Omega |\nabla u + h\nabla x|^2 \right)^{1/2} = \left(\int_\Omega |\nabla u|^2 + 2h\nabla u \nabla x + \mathrm{o}(h) \right)^{1/2} \\ &= \left(\int_\Omega |\nabla u|^2 \right)^{1/2} + h \int_\Omega \nabla u \cdot \nabla x \left(\int_\Omega |\nabla u|^2 \right)^{-1/2} + \mathrm{o}(h) \\ &= \phi(u) + h \langle -\Delta u, x \rangle_X \left(\int_\Omega |\nabla u|^2 \right)^{-1/2} + \mathrm{o}(h). \end{aligned}$$

Thus, the Gâteaux differential $\mathrm{d}_X \phi_X$ of ϕ is given by

$$\mathrm{d}_X \phi_X : u \longmapsto \left(\int_\Omega |\nabla u|^2 \right)^{-1/2} \langle -\Delta u, \cdot \rangle_X \quad \forall u \in X \setminus \{0\}.$$

We now prove that

$$\partial_X \phi_X(0) = \{ \eta \in X^* : |\eta|_{X^*} \leq 1 \} =: B^1_{X^*}.$$

Indeed, fix $\eta \in \partial_X \phi_X(0)$. Then, by the definition of subdifferential

$$\langle \eta, u \rangle_X \leq \phi(u) - \phi(0) = \phi(u) = |u|_X \text{ for all } u \in X.$$

Thus, taking the supremum over the set $\{u \in X : |u|_X \leq 1\}$, we get $|\eta|_{X^*} \leq 1$, which yields $\partial_X \phi_X(0) \subset B^1_{X^*}$. Conversely, fix $\eta \in B^1_{X^*}$. For all $u \in X$ we have

$$\langle \eta, u \rangle_X \leq |\eta|_{X^*} |u|_X \leq |u|_X = \phi(u).$$

Thus, $\eta \in \partial_X \phi_X(0)$. The arbitrariness of η ensures that $B^1_{X^*} \subset \partial_X \phi_X(0)$. Furthermore, note that, for all $u \in X \setminus \{0\}$ and $\eta = \mathrm{d}_X \phi_X(u)$, we have

$$\begin{aligned} |\eta|_{X^*} &= \sup_{x \in X,\ |x|_X = 1} \langle \eta, x \rangle_X = \left(\int_\Omega |\nabla u|^2 \right)^{-1/2} \sup_{x \in H^1_0(\Omega),\ |x|_{H^1_0(\Omega)} = 1} \int_\Omega \nabla u \cdot \nabla x \\ &\leq \sup_{x \in H^1_0(\Omega),\ |x|_{H^1_0(\Omega)} = 1} \left(\int_\Omega |\nabla u|^2 \right)^{-1/2} \int_\Omega |\nabla u|^2 \int_\Omega |\nabla x|^2 \\ &= \left(\int_\Omega |\nabla u|^2 \right)^{1/2} = \phi(u). \end{aligned}$$

In particular, ϕ satisfies assumption (2.4). We refer to [2, Section 6] to check that assumptions (2.1) and (2.2) are satisfied.

The direct application of Theorem 1 entails the following.

Theorem 6. *Let the above assumptions be satisfied. Then, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ system*

$$\begin{aligned} \left(\int_\Omega |\nabla u|^2 \right)^{1/2} \left(-\varepsilon \partial_t \left(\alpha(\partial_t u) \right) + \alpha(\partial_t u) \right) - \Delta u = 0 \quad &\text{in } \Omega \times \mathbb{R}_+, \\ u = 0 \quad &\text{on } \partial\Omega \times \mathbb{R}_+, \\ u(0) = u_0 \quad &\text{in } \Omega, \end{aligned}$$

admits a solution $u_\varepsilon \in W^{1,p}(0, T; L^p(\Omega)) \cap L^\infty(\mathbb{R}_+; H^1_0(\Omega))$ with $\alpha(\partial_t u) \in W^{1,p'}(0, T, L^{p'}(\Omega))$ for all $T > 0$. Moreover, there exists a non-relabeled subsequence u_ε such that $u_\varepsilon \to u$ strongly in $C([0, T]; L^p(\Omega))$ for all $T > 0$, where $u \in W^{1,p}(0, T; L^p(\Omega)) \cap L^\infty(\mathbb{R}_+; H^1_0(\Omega))$ for all $T > 0$ is a solution of (5.5)-(5.7).

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