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A MINIMIZATION APPROACH TO GRADIENT FLOWS OF NONCONVEX ENERGIES

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Abstract. This note is concerned with a minimization approach to gradient flows in Hilbert spaces H driven by nonconvex energies which can be decomposed into a sum of a convex part and a smooth part. More precisely, we introduce a *Weighted Energy-Dissipation* (*WED* for short) *functional* $\mathcal{W}_\varepsilon$, which consists of a dissipation part and an energy part along with an exponential weight in time with a parameter $\varepsilon > 0$ and which is defined for each trajectory $u : [0, T] \to H$ satisfying an initial condition. Eventually, gradient flows are obtained as limits of minimizers u_ε of WED functionals $\mathcal{W}_\varepsilon$ as the parameter ε goes to zero.

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1 Variational formulation of gradient flows

Gradient flows arise in the description of various sorts of phenomena with energy-dissipation (e.g., phase transition, diffusion), and have attracted a large number of contributions from numerous points of view. In this study, we discuss a (natural) variational principle for gradient flows.

We start with an abstract gradient flow in a Hilbert space H for an energy functional $E : H \to \mathbb{R}$. Gradient flows $u : [0, T] \to H$ of E are generated by the evolution equation

$$u'(t) = -\mathrm{d}E(u(t)) \;\text{ in } H, \quad t \in (0, T), \tag{1}$$

where $u' = \mathrm{d}u/\mathrm{d}t$ and $\mathrm{d}E$ denotes some suitable functional derivative of E. Such a gradient flow often appears in the study of PDEs. For instance, we give

Example 1.1 (Sublinear heat equation)**.** Set $H = L^2(\Omega)$ with a smooth bounded domain $\Omega \subset \mathbb{R}^N$ and define

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{1}{q}\int_\Omega |u(x)|^q \mathrm{d}x \quad \text{ for } u \in D(E) \subset H$$

with $1 < q < 2$ and effective domain $D(E) = H^1_0(\Omega)$. Then, the abstract evolution equation (1) corresponds to a suitable variational formulation of the *sublinear heat equation*

$$\partial_t u - \Delta u - |u|^{q-2}u = 0 \;\text{ in } \Omega \times (0, T)$$

along with the homogeneous Dirichlet boundary condition. □

Concerning variational formulations for gradient flow, let us recall the approaches based on

(i) time-discretization (e.g., discrete Morse flow method),

(ii) the Brézis-Ekeland variational principle (see [8, 9]),

(iii) *Weighted Energy-Dissipation* (*WED* for short) *functionals.*

Example 1.2 (Time-discretization of gradient flow)**.** The time-discretization is a classical variational method to approximate gradient flows. One can incrementally obtain u_n from the previous u_{n-1} by solving the semi-discretized problem for (1),

$$\frac{u_n - u_{n-1}}{h} = -\mathrm{d}E(u_n),$$

where $h > 0$ is a time step. The latter corresponds to the Euler-Lagrange equation of the functional,

$$I_n(w) := \frac{1}{2}|w|^2_H + hE(w) - (u_{n-1}, w)_H \quad \text{for } \; w \in H.$$

Under appropriate assumptions, u_n can be obtained by minimizing I_n. These minimizers $\{u_n\}$ form an approximation for the gradient flow generated by (1). □

Example 1.3 (The Brézis-Ekeland variational principle [8, 9])**.** Let φ be a proper, lower semicontinuous, convex functional on a Hilbert space H. Brézis and Ekeland observed that

$$u'(t) + \partial\varphi(u(t)) \ni 0, \quad u(0) = u_0 \quad \text{iff} \quad J(u) = \inf_{D(J)} J = 0,$$

where J is a functional on $L^2(0,T;H)$ given by

$$J(u) := \int_0^T \Big(\varphi(u(t)) + \varphi^*(-u'(t))\Big)\mathrm{d}t + \frac{1}{2}|u(T)|_H^2 - \frac{1}{2}|u_0|_H^2$$

with the domain $D(J) := \{u \in W^{1,2}(0,T;H) \colon u(0) = u_0,\ \varphi(u(\cdot)), \varphi^*(-u'(\cdot)) \in L^1(0,T)\}$. Here, φ^* is the convex conjugate of φ defined by

$$\varphi^*(v) := \sup_{u\in H} \{(v,u)_H - \varphi(u)\} \quad \text{for } \ v \in H. \quad \square$$

In this note, we address a third approach based on the Weighted Energy-Dissipation (WED) functional. Let us briefly outline this approach. Let $E : H \to \mathbb{R}$ be a *convex* energy and consider the Cauchy problem,

$$\begin{cases} u'(t) = -\mathrm{d}E(u(t)) \ \text{ in } H, \quad 0 < t < T, \\ u(0) = u_0. \end{cases} \tag{2}$$

The WED functional for (2) is defined by

$$\mathcal{W}_\varepsilon(u) := \int_0^T e^{-t/\varepsilon} \left(\frac{\varepsilon}{2}|u'(t)|_H^2 + E(u(t))\right) \mathrm{d}t$$

for $u \in \mathcal{H} := L^2(0,T;H)$ satisfying $u(0) = u_0$. The minimization approach using WED functionals is formulated as follows: Let u_ε be the *unique* (by strict convexity) minimizer of $\mathcal{W}_\varepsilon(u)$ subject to $u_\varepsilon(0) = u_0$. Then, the minimizer u_ε approximates the gradient flow u of E for $\varepsilon > 0$ sufficiently small (more precisely, $u_\varepsilon \to u$ strongly in $C([0,T];H)$ as $\varepsilon \to 0$).

Example 1.4 (WED approach to the heat equation)**.** As a simple example, let us treat the heat equation,

$$\text{(Heat)} \begin{cases} \partial_t u - \Delta u = 0 \ \text{ in } Q, \\ u = 0 \ \text{ on } \partial\Omega \times (0,T), \quad u(\cdot,0) = u_0 \ \text{ in } \Omega, \end{cases}$$

where $Q := \Omega \times (0,T)$. For each $\varepsilon > 0$, let us define the WED functional,

$$\mathcal{W}_\varepsilon(u) := \iint_Q e^{-t/\varepsilon} \left(\frac{\varepsilon}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla u|^2\right) \mathrm{d}x\mathrm{d}t$$

for $u \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ with $u(\cdot,0) = u_0$. Then, the Euler-Lagrange equation $\mathrm{d}\mathcal{W}_\varepsilon(u) = 0$ provides the elliptic-in-time regularizations of (Heat):

$$\text{(Heat)}_\varepsilon \begin{cases} -\varepsilon\partial_t^2 u + \partial_t u - \Delta u = 0 \ \text{ in } Q, \\ u|_{\partial\Omega} = 0, \quad u(\cdot,0) = u_0, \quad \partial_t u(\cdot,T) = 0. \end{cases}$$

Indeed, $\mathcal{W}_\varepsilon$ resembles an $(N+1)$-dimensional Dirichlet integral over Q, and then, for smooth test functions ϕ satisfying $\phi(0) = 0$ (from initial constraint), we observe that

$$\begin{aligned}
0 = (\mathrm{d}\mathcal{W}_\varepsilon(u), \phi)_{L^2(Q)} &= \iint_Q e^{-t/\varepsilon} \left(\varepsilon \partial_t u \partial_t \phi + \nabla u \cdot \nabla \phi\right) \mathrm{d}x\mathrm{d}t \\
&= \int_\Omega \varepsilon e^{-t/\varepsilon} \partial_t u \phi \ \mathrm{d}x \Big|_{t=0}^{t=T} - \iint_Q \varepsilon \partial_t \left(e^{-t/\varepsilon} \partial_t u\right) \phi \ \mathrm{d}x\mathrm{d}t \\
&\quad - \iint_Q e^{-t/\varepsilon} \Delta u \phi \ \mathrm{d}x\mathrm{d}t \\
&= \int_\Omega \varepsilon e^{-T/\varepsilon} \partial_t u(T) \phi(T) \ \mathrm{d}x - \iint_Q \varepsilon \left(e^{-t/\varepsilon} \partial_t^2 u - \frac{1}{\varepsilon} e^{-t/\varepsilon} \partial_t u\right) \phi \ \mathrm{d}x\mathrm{d}t \\
&\quad - \iint_Q e^{-t/\varepsilon} \Delta u \phi \ \mathrm{d}x\mathrm{d}t \\
&= \int_\Omega \varepsilon e^{-T/\varepsilon} \partial_t u(T) \phi(T) \ \mathrm{d}x + \iint_Q e^{-t/\varepsilon} \left(-\varepsilon \partial_t^2 u + \partial_t u - \Delta u\right) \phi \ \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

From the arbitrariness of $\phi(T)$ and ϕ, we obtain $(\mathrm{Heat})_\varepsilon$. Then $u_\varepsilon \to u$ strongly in $C([0,T]; L^2(\Omega))$ as $\varepsilon \to 0$ and u solves (Heat). This procedure is known as elliptic-in-time regularization technique (see, e.g., [16]). □

Let us briefly review previous studies on this topic. Ilmanen [12] introduced *translative functional*, which is based on a similar idea to WED functionals, to prove the partial regularity of Brakke mean curvature flow of varifolds. See also [6] and [11], where similar variational methods are employed to construct approximate solutions for some nonlinear evolution equations. The term "WED functional" was first introduced by Mielke-Ortiz and Conti-Ortiz in [17] and [10], where rate-independent systems are studied for some application to Mechanics. Moreover, a discrete version of WED functional for rate-independent process is also studied in detail by Mielke-Stefanelli [19].

Thereafter, variational formulations using WED functionals have been done for gradient flows in various settings. Mielke-Stefanelli [18] provided an WED approach to gradient flows generated by

$$u' + \partial\varphi(u) \ni f \quad \text{in a Hilbert space } H \tag{3}$$

for lower semicontinuous convex energies $\varphi : H \to (-\infty, \infty]$ and proved the convergence of minimizers as $\varepsilon \to 0$. Furthermore, Akagi-Stefanelli [4, 3] also presented an WED formulation for generalized gradient flows as

$$\mathrm{d}\psi(u') + \partial\varphi(u) \ni 0 \quad \text{in a dual space } V^* \text{ of a reflexive Banach space } V$$

with convex *dissipation functional* ψ and convex energy φ defined on V, by introducing the corresponding WED functional given by

$$\mathcal{W}_\varepsilon(u) = \int_0^T e^{-t/\varepsilon} \left(\varepsilon\psi(u') + \varphi(u)\right) \mathrm{d}t$$

for all trajectories $u : (0,T) \to V$ satisfying the initial condition.

Rossi et al [22, 21] extended the WED approach to metric gradient flows, which are formulated as curves of maximal slope for φ, that is, absolutely continuous curves u with values in a Polish space (X,d) satisfying the relation,

$$\varphi(u(t)) + \frac{1}{2}\int_0^t |u'|^2(\tau)\mathrm{d}\tau + \frac{1}{2}\int_0^t |\partial\varphi|^2(u(\tau))\mathrm{d}\tau = \varphi(u_0) \quad \text{for all } t \geq 0, \tag{4}$$

where $|u'|(t)$ denotes the metric derivative of u at t and $|\partial\varphi|$ stands for the local slope of φ (see [5] for more details). The relation above is a metric counterpart of the following well-known energy identity for the gradient flow (3) in a Hilbert space:

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(u(t)) = (\partial\varphi(u(t)), u'(t))_H = -\frac{1}{2}|\partial\varphi(u(t))|_H^2 - \frac{1}{2}|u'(t)|_H^2,$$

which is derived from a chain-rule for subdifferential as well as (3). The WED functional for (4) is

$$\mathcal{W}_\varepsilon(u) = \int_0^\infty e^{-t/\varepsilon}\left(\frac{\varepsilon}{2}|u'|^2(t) + \varphi(u(t))\right)\mathrm{d}t$$

for $u : (0,\infty) \to X$ satisfying $u(0) = u_0$.

Concerning nonlinear diffusion such as porous medium and fast diffusion equations and the Stefan problem, Akagi and Stefanelli [2] presented a WED formulation by *dualization* for another type of generalized gradient flow,

$$\frac{\mathrm{d}}{\mathrm{d}t}\partial\psi(u) + \partial\varphi(u) \ni g \ \text{ in } V^*,$$

which corresponds to the above-mentioned nonlinear diffusion models.

The WED approach is also applicable to other types of problems, e.g., wave equations and Lagrangian systems without dissipation. In contrast to dissipative systems such as gradient flows, these issues are originally formulated in a variational fashion, and then, solutions are obtained as critical points (mostly, saddle points) of the corresponding action functionals. On the other hand, applying the WED approach to these issues, one can reformulate solutions by *minimization*, and moreover, one can provide a variational formulation for wave equations and Lagrangian systems, possibly including some dissipation such as friction (see [14, 15]).

Notably, for the semilinear wave equation

$$\partial_t^2 u - \Delta u + |u|^{p-2}u = 0 \ \text{ in } \mathbb{R}^N \times (0,\infty),$$

the WED functional is given by

$$\mathcal{W}_\varepsilon(u) = \int_0^T \int_{\mathbb{R}^N} e^{-t/\varepsilon}\left(\frac{\varepsilon^2}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{p}|u|^p\right)\mathrm{d}x\mathrm{d}t$$

for all trajectories satisfying appropriate initial constraints. The convergence of minimizers of $\mathcal{W}_\varepsilon$ to a solution of the semilinear wave equation corresponds to a conjecture by De Giorgi (see [25], [23]).

Here we emphasize that all these works are done for convex (or λ-convex) energies. The aim of this note is to extend the WED framework for gradient flows of *non-convex energies*. We particularly address the case of an energy functional E which is decomposed into the sum of a convex (but possibly nonsmooth) part φ and a smooth (but possibly nonconvex) part ϕ, i.e., $E = \varphi + \phi$.

2 WED formulation for nonconvex energies

Let H be a Hilbert space and let $E : H \to (-\infty, \infty]$ be a *non-convex* energy of the form

$$E(u) := \varphi(u) + \phi(u) \quad \text{for } \ u \in H,$$

where $\varphi : H \to [0, \infty]$ is a proper, lower semicontinuous, convex functional with the *effective domain* $D(\varphi) := \{u \in H : \varphi(u) < \infty\}$ and $\phi : H \to \mathbb{R}$ is of class C^1, that is, the gradient operator ϕ' (in the sense of Fréchet differential) is continuous on H (the notions of functional derivative are briefly reviewed in Appendix §A). Then, the effective domain $D(E)$ of E coincides with $D(\varphi)$.

Here we remark that if ϕ is of class $C^{1,1}$ over H (i.e., ϕ' is Lipschitz continuous on H), then E becomes λ-convex, and therefore, such a case falls within the scope of [18].

Let us consider a gradient flow $u : [0, T] \to H$ of the energy functional E starting from $u_0 \in H$. Namely, u is generated by the Cauchy problem,

$$\text{(GF)} \begin{cases} u'(t) + \partial_F E(u(t)) \ni 0 \ \text{ in } H, \quad 0 < t < T, \\ u(0) = u_0, \end{cases}$$

where $\partial_F E$ stands for the *Fréchet subdifferential* of E given by

$$\xi \in \partial_F E(v) \quad \Leftrightarrow \quad \liminf_{w \to v \text{ in } H} \frac{E(w) - E(v) - (\xi, w - v)_H}{|w - v|_H} \geq 0$$

with the domain $D(\partial_F E) := \{u \in D(E) : \partial_F E(u) \neq \emptyset\}$. Then, we remark that

$$\partial_F E = \partial\varphi + \phi' \quad \text{and} \quad D(\partial_F E) = D(\partial\varphi)$$

(see, e.g., [5, Corollary 1.4.5]).

Before defining an WED functional for (GF), let us set up assumptions.

(A1) The convex part φ is proper, lower semicontinuous, and convex in H. Moreover, the smooth part ϕ is of class C^1 over H. It also holds that $u_0 \in D(\partial_F E) = D(\partial\varphi)$.

(A2) The energy functional E is bounded from below, that is,

$$E(u) \geq -C_1 \quad \text{for all } \ u \in H \tag{5}$$

for some constant $C_1 \geq 0$. There exist a Banach space X compactly embedded in H and a non-decreasing function ℓ_1 on $[0, \infty)$ such that

$$|u|_X \leq \ell_1(|u|_H)\,(|E(u)| + 1) \quad \forall u \in D(E). \tag{6}$$

(A3) There exists a constant $C_2 \geq 0$ such that

$$|\phi'(u)|_H^2 \leq C_2\left(|u|_H^2 + 1\right) \quad \forall u \in H,$$

where ϕ' denotes the Fréchet derivative of ϕ.

Remark 2.1. By (A3), we find that

$$|\phi(u)| \le C(|u|_H^2 + 1) \quad \text{for all} \ \ u \in H.$$

Indeed, by the mean value theorem, for any $u \in H$, one can take $\lambda \in (0,1)$ such that

$$\phi(u) - \phi(0) = (\phi'(\lambda u), u)_H,$$

which together with (A3) implies

$$|\phi(u)| \le |\phi(0)| + |\phi'(\lambda u)|_H |u|_H \le C\left(|u|_H^2 + 1\right),$$

where C is a constant depending only on $\phi(0)$ and C_2. □

We define the WED functional $\mathcal{W}_\varepsilon$ for (GF) by

$$\mathcal{W}_\varepsilon(u) := \begin{cases} \displaystyle\int_0^T e^{-t/\varepsilon}\left(\frac{\varepsilon}{2}|u'(t)|_H^2 + E(u(t))\right)\mathrm{d}t \\ \qquad\qquad \text{if} \quad u \in W^{1,2}(0,T;H),\ u(0) = u_0, \\ \qquad\qquad\qquad u(t) \in D(E) \text{ for a.e. } t \in (0,T),\ E(u(\cdot)) \in L^1(0,T), \\ \infty \qquad\quad \text{else} \end{cases}$$

for $u \in \mathcal{H} := L^2(0,T;H)$ with norm $\|u\|_{\mathcal{H}} := (\int_0^T |u(t)|_H^2 \mathrm{d}t)^{1/2}$. The minimization approach presented here for (GF) consists of the following steps:

- Firstly, we find minimizers u_ε of $\mathcal{W}_\varepsilon$ for $\varepsilon > 0$.
- We then take a limit of minimizers u_ε as $\varepsilon \to \infty$.
- Eventually, we prove that the limit u of u_ε is a solution of (GF).

Our main result reads as follows.

Theorem 2.2 (Minimization approach to (GF)). *Suppose that* (A1)–(A3) *hold. Then, it holds that*:

(i) *For each $\varepsilon > 0$, $\mathcal{W}_\varepsilon$ admits a global minimizer over $\mathcal{H}$.*

Let u_ε be a global or local minimizer of $\mathcal{W}_\varepsilon$.

(ii) *Then u_ε belongs to $W^{2,2}(0,T;H)$ and solves*

$$\text{(EL)} \begin{cases} -\varepsilon u_\varepsilon''(t) + u_\varepsilon'(t) + \partial_F E(u_\varepsilon(t)) \ni 0 \ \ \textit{in } H, \quad 0 < t < T, \\ u_\varepsilon(0) = u_0, \quad u_\varepsilon'(T) = 0. \end{cases}$$

(iii) *There exists $u \in W^{1,2}(0,T;H)$ such that, up to a subsequence,*

$$u_\varepsilon \to u \quad \textit{strongly in } C([0,T];H), \qquad \textit{weakly in } W^{1,2}(0,T;H) \quad \textit{as} \ \ \varepsilon \to 0.$$

Moreover, the limit u solves (GF).

3 Outline of proof

In this section, we give an outline of proof for Theorem 2.2.

Step 1. For $\varepsilon > 0$, the WED functional $\mathcal{W}_\varepsilon$ admits a global minimizer.

We prove the existence of global minimizer of $\mathcal{W}_\varepsilon$ by employing the *Direct Method* of the calculus of variations. By (5), one observes that

$$\mathcal{W}_\varepsilon(u) \geq -C_1 \int_0^T e^{-t/\varepsilon} \mathrm{d}t \quad \text{for all} \quad u \in \mathcal{H}.$$

Then $\mathcal{W}_\varepsilon$ admits a minimizing sequence $u_n \in D(\mathcal{W}_\varepsilon)$, i.e., $\mathcal{W}_\varepsilon(u_n)$ converges to the infimum of $\mathcal{W}_\varepsilon$ over $\mathcal{H}$. In particular, $\mathcal{W}_\varepsilon(u_n)$ is bounded, so we find by (5) that

$$\int_0^T |u_n'(t)|_H^2 \mathrm{d}t \leq C,$$

which along with the initial constraint $u_n(0) = u_0$ also implies that

$$\sup_{t \in [0,T]} |u_n(t)|_H \leq C.$$

Using (A2), we also deduce that

$$\int_0^T |u_n(t)|_X \mathrm{d}t \leq C.$$

Therefore one can verify, up to a subsequence, that u_n converges to u weakly in $W^{1,2}(0,T;H)$. Moreover, since X is compactly embedded in H, by Theorem 3 of [24], we deduce, up to a subsequence, that u_n converges to u strongly in $C([0,T];H)$. Hence, u satisfies the initial constraint, $u(0) = u_0$. From the fact that ϕ is continuous on H and Remark 2.1, it follows that

$$\int_0^T e^{-t/\varepsilon} \phi(u_n(t)) \mathrm{d}t \to \int_0^T e^{-t/\varepsilon} \phi(u(t)) \mathrm{d}t.$$

Due to the weakly lower semicontinuity of the convex part of $\mathcal{W}_\varepsilon$, one has

$$\liminf_{n \to \infty} \int_0^T e^{-t/\varepsilon} \left(\frac{\varepsilon}{2} |u_n(t)|_H^2 + \varphi(u_n(t)) \right) \mathrm{d}t \geq \int_0^T e^{-t/\varepsilon} \left(\frac{\varepsilon}{2} |u(t)|_H^2 + \varphi(u(t)) \right) \mathrm{d}t.$$

Thus we obtain

$$\liminf_{n \to \infty} \mathcal{W}_\varepsilon(u_n) \geq \mathcal{W}_\varepsilon(u).$$

However, since the left-hand-side converges to the infimum of $\mathcal{W}_\varepsilon$, the limit u minimizes $\mathcal{W}_\varepsilon$ over $\mathcal{H}$.

Step 2. Every minimizer u_ε of $\mathcal{W}_\varepsilon$ over $\mathcal{H}$ solves (EL).

Let u_ε be a global or local minimizer of $\mathcal{W}_\varepsilon$ over $\mathcal{H}$. Then one can take $r > 0$ such that

$$\mathcal{W}_\varepsilon(u_\varepsilon) \le \mathcal{W}_\varepsilon(v) \quad \text{for all} \quad v \in B(u_\varepsilon; r) \tag{7}$$

with the neighborhood $B(u_\varepsilon; r) := \{v \in \mathcal{H} \colon \|u_\varepsilon - v\|_{\mathcal{H}} < r\}$ of u_ε in $\mathcal{H}$.

In this step, we shall prove that every minimizer u_ε solves the Euler-Lagrange equation (EL). Here it is noteworthy that the WED functional $\mathcal{W}_\varepsilon$ involves the (possibly) nonsmooth part φ and the initial constraint, $u(0) = u_0$. In particular, the solution of (EL) is more delicate than in usual variational problems for smooth functionals without constraint. We divide $\mathcal{W}_\varepsilon$ into a sum of two parts, $\mathcal{W}_\varepsilon = \mathcal{C}_\varepsilon + \mathcal{S}_\varepsilon$, where $\mathcal{C}_\varepsilon$ and $\mathcal{S}_\varepsilon$ are defined by

$$\mathcal{C}_\varepsilon(u) := \begin{cases} \displaystyle\int_0^T e^{-t/\varepsilon} \left(\frac{\varepsilon}{2} |u'(t)|_H^2 + \varphi(u(t)) \right) \mathrm{d}t \\ \qquad \text{if} \quad u \in W^{1,2}(0,T;H), \ u(0) = u_0, \\ \qquad\qquad u(t) \in D(\varphi) \text{ for a.e. } t \in (0,T), \ \varphi(u(\cdot)) \in L^1(0,T), \\ +\infty \quad \text{else} \end{cases}$$

and

$$\mathcal{S}_\varepsilon(u) := \int_0^T e^{-t/\varepsilon} \phi(u(t)) \mathrm{d}t$$

for $u \in \mathcal{H}$. Then $D(\mathcal{C}_\varepsilon) = D(\mathcal{W}_\varepsilon)$ and $D(\mathcal{S}_\varepsilon) = \mathcal{H}$ by Remark 2.1.

Let $g \in \partial \mathcal{C}_\varepsilon(u)$. Then, u minimizes the convex functional

$$v \mapsto \mathcal{C}_\varepsilon(v) - \int_0^T (g(t), v(t))_H \mathrm{d}t = \mathcal{C}_\varepsilon(v) - \int_0^T e^{-t/\varepsilon} \left(e^{t/\varepsilon} g(t), v(t) \right)_H \mathrm{d}t$$

defined on $\mathcal{H}$. It has already been proved in [18, Theorem 3.1] that u solves

$$-\varepsilon u'' + u' + \partial\varphi(u) \ni e^{t/\varepsilon} g \ \text{ in } H, \quad 0 < t < T,$$
$$u(0) = u_0, \quad u'(T) = 0, \quad u \in W^{2,2}(0,T;H),$$

and that u satisfies the following *maximal regularity estimate*:

$$\varepsilon \|u''\|_{\mathcal{H}} + \|u'\|_{\mathcal{H}} + \|\xi\|_{\mathcal{H}} \le \|e^{t/\varepsilon} g\|_{\mathcal{H}} + \varepsilon |(\partial\varphi)^\circ(u_0)|_H^2 + 2\varphi(u_0), \tag{8}$$

where $\xi := e^{\varepsilon/t} g + \varepsilon u'' - u'$ is a section of $\partial\varphi(u)$ and $(\partial\varphi)^\circ(u_0)$ stands for the minimal section of $\partial\varphi(u_0)$ (see [7]).

As for the smooth part, we first prove that $\mathcal{S}_\varepsilon$ is Gâteaux differentiable over $\mathcal{H}$ and derive a representation of the derivative,

$$\mathcal{S}'_\varepsilon(u) = e^{-t/\varepsilon} \phi'(u(\cdot)) \quad \text{for} \quad u \in \mathcal{H}. \tag{9}$$

Let $u, e \in \mathcal{H}$ and $h \in \mathbb{R}$. Since ϕ is of class C^1 over H, one notes that

$$\frac{\phi(u(t) + he(t)) - \phi(u(t))}{h} \to (\phi'(u(t)), e(t))_H \quad \text{strongly in } H \ \text{ as } \ h \to 0$$

for a.e. $t \in (0,T)$. Then by virtue of the mean value theorem, it follows that

$$\left| e^{-t/\varepsilon} \frac{\phi(u(t)+he(t))-\phi(u(t))}{h} \right| \le e^{-t/\varepsilon} |\phi'(u(t)+h\theta e(t))|_H |e(t)|_H$$

with $\theta \in (0,1)$ depending on t. Here we note by (A3) that

$$|\phi'(u(t)+h\theta e(t))|_H \le C\left(|u(t)|_H + |e(t)|_H + 1\right).$$

Therefore we deduce that

$$\left| e^{-t/\varepsilon} \frac{\phi(u(t)+he(t))-\phi(u(t))}{h} \right| \le e^{-t/\varepsilon} \left(|u(t)|_H^2 + |e(t)|_H^2 + 1\right) \in L^1(0,T).$$

Thus, due to Lebesgue's dominated convergence theorem, we deduce that

$$\int_0^T e^{-t/\varepsilon} \frac{\phi(u(t)+he(t))-\phi(u(t))}{h} \mathrm{d}t \to \int_0^T e^{-t/\varepsilon} \left(\phi'(u(t)), e(t)\right)_H \mathrm{d}t$$

as $h \to 0$. This concludes that $\mathcal{W}_\varepsilon$ is Gâteaux differentiable in $\mathcal{H}$ and (9) follows.

We further claim that $\mathcal{S}_\varepsilon'$ is continuous in $\mathcal{H}$. Indeed, let $u_n \to u$ strongly in $\mathcal{H}$. Then one can take $g \in L^1(0,T)$ and a non-relabeled subsequence of (n) such that $u_n(t) \to u(t)$ strongly in H for a.e. $t \in (0,T)$ and $|u_n(t)|_H^2 \le g(t)$. By assumption $\phi \in C^1(H;\mathbb{R})$, it follows that

$$e^{-t/\varepsilon}\phi'(u_n(t)) \to e^{-t/\varepsilon}\phi'(u(t)) \quad \text{strongly in } H \quad \text{for a.e. } t \in (0,T).$$

Moreover, by (A3), we find that

$$|e^{-t/\varepsilon}\phi'(u_n(t))|_H^2 \le Ce^{-2t/\varepsilon}(|u_n(t)|_H^2 + 1) \le Ce^{-2t/\varepsilon}(g(t)+1) \in L^1(0,T).$$

Thus, by Lebesgue's dominated convergence theorem we deduce that $S_\varepsilon'(u_n) \to S_\varepsilon'(u)$ strongly in $\mathcal{H}$. Therefore, $\mathcal{S}_\varepsilon'$ is continuous on $\mathcal{H}$. Since the Gâteaux derivative is continuous in $\mathcal{H}$, one can check that $\mathcal{S}_\varepsilon$ is also Fréchet differentiable over $\mathcal{H}$.

Recall (7) and let $w \in D(\mathcal{W}_\varepsilon) = D(\mathcal{C}_\varepsilon)$ and $\theta \in (0,1)$. If θ is small enough, $u_\varepsilon + \theta(w - u_\varepsilon)$ belongs to $D(\mathcal{C}_\varepsilon) \cap B(u_\varepsilon; r)$. Substitute $u_\varepsilon + \theta(w-u_\varepsilon) = (1-\theta)u_\varepsilon + \theta w$ for v. Then we have, by the convexity of $\mathcal{C}_\varepsilon$,

$$\begin{aligned} \mathcal{C}_\varepsilon(u_\varepsilon) + \mathcal{S}_\varepsilon(u_\varepsilon) &\le \mathcal{C}_\varepsilon((1-\theta)u_\varepsilon + \theta w) + \mathcal{S}_\varepsilon(u_\varepsilon + \theta(w-u_\varepsilon)) \\ &\le (1-\theta)\mathcal{C}_\varepsilon(u_\varepsilon) + \theta\mathcal{C}_\varepsilon(w) + \mathcal{S}_\varepsilon(u_\varepsilon + \theta(w-u_\varepsilon)), \end{aligned}$$

which yields that

$$\mathcal{S}_\varepsilon(u_\varepsilon) - \mathcal{S}_\varepsilon(u_\varepsilon + \theta(w-u_\varepsilon)) \le \theta\left(\mathcal{C}_\varepsilon(w) - \mathcal{C}_\varepsilon(u_\varepsilon)\right).$$

Dividing both sides by $\theta > 0$ and passing to the limit as $\theta \to 0_+$, we get

$$(-\mathcal{S}_\varepsilon'(u_\varepsilon), w - u_\varepsilon)_H \le \mathcal{C}_\varepsilon(w) - \mathcal{C}_\varepsilon(u_\varepsilon).$$

From the arbitrariness of $w \in D(\mathcal{C}_\varepsilon)$, one obtains

$$-\mathcal{S}_\varepsilon'(u_\varepsilon) \in \partial\mathcal{C}_\varepsilon(u_\varepsilon).$$

Thus we conclude that

$$\begin{aligned} -\varepsilon u_\varepsilon'' + u_\varepsilon' + \partial\varphi(u_\varepsilon) \ni -e^{t/\varepsilon}\mathcal{S}_\varepsilon'(u_\varepsilon) = -\phi'(u_\varepsilon) \ \text{ in } H, \quad 0 < t < T,\\ u(0) = u_0, \quad u'(T) = 0, \quad u \in W^{2,2}(0,T;H). \end{aligned}$$

Hence, u_ε solves (ER). Furthermore, one can derive from (8) that

$$\varepsilon\|u_\varepsilon''\|_{\mathcal{H}} + \|u_\varepsilon'\|_{\mathcal{H}} + \|\xi_\varepsilon\|_{\mathcal{H}} \le \|\phi'(u_\varepsilon)\|_{\mathcal{H}} + \varepsilon|(\partial\varphi)^\circ(u_0)|_H^2 + 2\varphi(u_0) \tag{10}$$

with $\xi_\varepsilon := -\phi'(u_\varepsilon(\cdot)) + \varepsilon u_\varepsilon'' - u_\varepsilon' \in \partial\varphi(u_\varepsilon(\cdot))$.

Step 3. Convergence of minimizers u_ε as $\varepsilon \to 0$

In this step, we shall prove the convergence of minimizers u_ε of $\mathcal{W}_\varepsilon$ along a subsequence of $\varepsilon \to 0$. A crucial task is to derive uniform estimates for the minimizers. To this end, we employ the fact that all minimizers solve (EL) and establish energy estimates of solutions for (EL).

Here we only exhibit a formal argument to obtain uniform estimates for u_ε. For simplicity, we omit the subscript ε. Multiply (EL) by u' to get

$$-\varepsilon(u''(t), u'(t))_H + |u'(t)|_H^2 + \frac{\mathrm{d}}{\mathrm{d}t}E(u(t)) = 0.$$

Noting $(u''(t), u'(t))_H = (1/2)(\mathrm{d}/\mathrm{d}t)|u'(t)|_H^2$ and integrating it over $(0,T)$, we deduce that

$$\frac{\varepsilon}{2}|u'(0)|_H^2 + \int_0^T |u'(t)|_H^2\mathrm{d}t + E(u(T)) = \frac{\varepsilon}{2}|u'(T)|_H^2 + E(u_0).$$

Here we use that $E(u(T)) \ge -C_1$ by (A2) and $u'(T) = 0$ in order to get

$$\varepsilon|u'(0)|_H^2 + \int_0^T |u'(t)|_H^2\mathrm{d}t \le C.$$

Moreover, integrating (EL)$\times u'(t)$ over $(0,t)$, we have

$$\frac{\varepsilon}{2}|u'(0)|_H^2 + \int_0^t |u'(\tau)|_H^2\mathrm{d}\tau + E(u(t)) \le \frac{\varepsilon}{2}|u'(t)|_H^2 + E(u_0).$$

Integrate both sides over $(0,T)$ again. We obtain

$$\int_0^T E(u(t))\mathrm{d}t \le C + \frac{\varepsilon}{2}\int_0^T |u'(t)|_H^2\mathrm{d}t \le C.$$

By (A3), we get

$$\int_0^T |\phi'(u(t))|_H^2 \mathrm{d}t \leq C_2 \left(\int_0^T |u(t)|_H^2 \mathrm{d}t + 1 \right) \leq C.$$

Recalling the maximal regularity estimate (10), one obtains

$$\int_0^T |\varepsilon u''(t)|_H^2 \mathrm{d}t + \int_0^T |\xi(t)|_H^2 \mathrm{d}t \leq C.$$

Combining all these estimates, one can derive the convergence of minimizers u_ε as $\varepsilon \to 0$, and moreover, employing the maximal monotonicity of $\partial\varphi$ and the continuity of ϕ' in H, one can identify the limit of ξ_ε.

4 Application to sublinear heat equations

Let us consider the following sublinear heat equation:

$$(\mathrm{P}) \begin{cases} \partial_t u - \Delta u - |u|^{q-2}u = 0 & \text{in } Q := \Omega \times (0,T), \\ u|_{\partial\Omega} = 0, \quad u(\cdot, 0) = u_0, \end{cases}$$

where $1 < q < 2$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$.

Set $H = L^2(\Omega)$ and an energy functional,

$$E(u) := \varphi(u) + \phi(u) \quad \text{for } \ u \in H$$

with a convex part $\varphi : H \to [0, \infty]$ and a smooth part $\phi : H \to \mathbb{R}$ given by

$$\varphi(u) := \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega |\nabla u|^2 \mathrm{d}x & \text{if } \ u \in H_0^1(\Omega), \\ \infty & \text{else,} \end{cases}$$

$$\phi(u) := \begin{cases} -\dfrac{1}{q} \displaystyle\int_\Omega |u|^q \mathrm{d}x & \text{if } \ u \in L^q(\Omega), \\ \infty & \text{else.} \end{cases}$$

Here we find that φ is proper, lower semicontinuous and convex in H and its effective domain $D(\varphi)$ coincides with $H_0^1(\Omega)$. Moreover, $\partial\varphi$ coincides with the Dirichlet Laplacian $-\Delta$. On the other hand, one can prove the following in a standard way.

Proposition 4.1. *The smooth part ϕ is of class C^1 over H. Moreover, $\phi'(u) = -|u|^{q-2}u$.*

Here we emphasize that ϕ is not of class $C^{1,1}$. Indeed, $\phi'(u) = -|u|^{q-2}u$ is not Lipschitz continuous at $u = 0$ due to the fact that $1 < q < 2$.

Thus (P) is reduced into the Cauchy problem,

$$u'(t) + \partial_F E(u(t)) = 0 \ \text{ in } H, \quad 0 < t < T, \quad u(0) = u_0.$$

Let us briefly check (A1)–(A3) for this setting. (A1) has already been checked. As for (A2), since $q < 2$, there exists $C_1 \geq 0$ such that

$$\frac{1}{q}\int_\Omega |u|^q \mathrm{d}x \leq \frac{1}{4}\int_\Omega |\nabla u|^2 \mathrm{d}x + C_1 = \frac{1}{2}\varphi^1(u) + C_1 \quad \text{for all } u \in H^1_0(\Omega),$$

which implies $E(u) \geq -C_1$ for all $u \in L^2(\Omega)$. Concerning (A3), one can take $C_2 \geq 0$ such that

$$|\phi'(u)|_H^2 = \int_\Omega |u|^{2(q-1)} \mathrm{d}x \leq C_2 \left(|u|_H^2 + 1\right).$$

Therefore Theorem 2.2 is applicable to (P).

5 Final remarks

We close this note with giving a couple of remarks.

(i) The WED formalism enables us to apply variational tools such as Direct Method of the calculus of variations, critical point theories, Γ-convergence, and relaxation to analyze gradient flows.

(ii) The WED formalism also provides numerical schemes for gradient flows. Actually, one may obtain approximate solutions by directly minimizing WED functionals. This perspective would be particularly helpful to treat moving boundary problems and free boundary problems.

(iii) As for further generalizations, one may consider gradient flows with nonconvex energies which are unbounded from below, some relaxation of assumptions (A1)–(A3), how to describe smoothing effect in this formulation, and an extension of the formulation to Banach space settings.

(iv) Moreover, one can also apply Theorem 2.2 to other types of PDEs such as the Allen-Cahn equation, $\partial_t u - \Delta u + u^3 - u = 0$, as well as quasilinear (e.g., p-Laplace) heat equations with sublinear nonlinearity.

In the forthcoming paper [1], we shall further discuss the case that the energy functional E is the difference of two *possibly nonsmooth* convex functionals and also relax the assumption (A3) on the boundedness of the nonmonotone part. This case can cover Allen-Cahn equations with more general potentials. In this case, due to the nonsmoothness of the nonconvex part, one needs a couple of additional steps to derive the equivalence between minimizers of the WED functional and solutions of the Euler-Lagrange equation as well as to prove the existence of minimizers.

A Functional derivative

In this appendix, let us recall the notions of Fréchet derivative, Gâteaux derivative and subdifferential. A functional $\psi : H \to \mathbb{R}$ is said to be *Fréchet differentiable* in H if for each $u \in H$ there exists $\xi \in H$ such that

$$\frac{\psi(u+v) - \psi(u) - (\xi, v)_H}{|v|_H} = 0, \qquad \text{whenever } \ v \to 0 \ \text{ strongly in } H.$$

Moreover, ψ is said to be *Gâteaux differentiable* in H if for each $u \in H$ there exists $\xi \in H$ such that

$$\lim_{h \to 0} \frac{\psi(u+hv) - \psi(u)}{h} = (\xi, v)_H \quad \text{ for any } \ v \in H.$$

If ψ is Gâteaux differentiable, then it is also Fréchet differentiable and derivatives in both senses coincide.

In both notions above, ξ is called a Fréchet or Gâteaux derivative of ψ at u and denoted by $\psi'(u)$. Furthermore, one can define a *gradient operator* $\psi' : u \mapsto \psi'(u)$ from H into itself.

Now, let ψ be a proper (i.e., $\psi \not\equiv \infty$), lower semicontinuous, convex functional. Then, $\partial\psi : H \to H$ is called a *subdifferential operator* of ψ defined by

$$\partial\psi(u) := \{\xi \in H : \psi(v) - \psi(u) \geq (\xi, v-u)_H \quad \forall v \in D(\psi)\} \quad \text{ for } \ u \in D(\psi)$$

with the domain $D(\partial\psi) := \{u \in D(\psi) : \partial\psi(u) \neq \emptyset\}$. If ψ is Gâteaux differentiable, then $\partial\psi(u) = \{\psi'(u)\}$.

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