GAKUTO International Series
**Mathematical Sciences and Applications, Vol.32(2010)**
*Current Advances in Nonlinear Analysis and Related Topics, pp.239–254*

GAKKOTOSHO
TOKYO JAPAN

# ON SOME DOUBLY NONLINEAR PARABOLIC EQUATIONS

Dedicated to Professor Mitsuharu Ôtani on the occasion of his 60th birthday

Goro Akagi
Shibaura Institute of Technology, College of Systems Engineering and Science,
307 Fukasaku, Minuma-ku, Saitama-shi, Saitama 337-8570, JAPAN.
E-mail: g-akagi@sic.shibaura-it.ac.jp

**Abstract.** This paper exhibits characteristic behaviors of solutions $u = u(x,t)$ for some class of doubly nonlinear parabolic equations such as

$$|u_t|^{p-2}u_t - \Delta u = \lambda u \quad \text{for} \;\; x \in \Omega \subset \mathbb{R}^N, \; t > 0$$

and

$$|u_t|^{p-2}u_t - \Delta u + |u|^{p-2}u = \lambda u \quad \text{for} \;\; x \in \Omega \subset \mathbb{R}^N, \; t > 0,$$

where $p > 2$ and $\lambda \in \mathbb{R}$. Our method of analysis is based on classical separation of variable method as well as variational method.

AMS Subject Classification: 35K55, 35J60

# 1 Introduction

In the recent development of nonlinear analysis, various classes of doubly nonlinear problems have been intensively studied. Gurtin [7] proposed a generalized form of Ginzburg-Landau equation,

$$\rho(u_t)u_t - \Delta u + W'(u) = 0 \quad \text{for } x \in \Omega \subset \mathbb{R}^N, \ t > 0,$$

where $u = u(x,t)$ is an order parameter, $\rho \geq 0$ is a constitutive modulus, and $W$ stands for a double-well potential, and its mathematical analysis was recently started. One of characteristic features of such doubly nonlinear problems is the lack of uniqueness of solutions for the initial-boundary value problems. Segatti [9] treated the case that $\rho$ is bounded and proved the existence of global (in time) solutions. He also pointed out the non-uniqueness of solutions and investigated the large-time behavior of solutions by employing the notion of generalized semiflow developed by J.M. Ball [4] for multi-valued dynamical systems. Recently, the author extended his results to more general cases, in which $\rho$ might have a polynomial growth order (see [1, 2]).

The main purpose of the present paper is to exhibit characteristic behaviors of solutions for such doubly nonlinear problems compared to those for usual nonlinear parabolic equations. To this end, we particularly treat the initial-boundary value problems for a doubly nonlinear parabolic equation,

$$|u_t|^{p-2}u_t - \Delta u = \lambda u \quad \text{in } \Omega \times (0,\infty),$$

as well as that with nonlinear absorption,

$$|u_t|^{p-2}u_t - \Delta u + |u|^{p-2}u = \lambda u \quad \text{in } \Omega \times (0,\infty),$$

where $\Omega$ is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $p > 2$ and $\lambda \in \mathbb{R}$. The second equation can be regarded as a special form of Gurtin's generalized Ginzburg-Landau equations with $\rho(u_t) = |u_t|^{p-2}$ and $W'(u) = |u|^{p-2}u - \lambda u$. Moreover, in case $|u_t|^{p-2}u_t$ is replaced by a linear form $u_t$, there are a number of contributions, and it is well known that the initial-boundary value problem admits a *unique* global (in time) solution and the solution converges to some equilibrium at $t = \infty$.

Our analysis relies upon very classical separation of variable method, and then, each equation will be divided into a nonlinear ODE in time and a nonlinear elliptic equation in space. We explicitly solve the ODE in time, and moreover, we employ variational method to solve the elliptic equation.

Finally, we also investigate the asymptotic behavior of general energy solutions for the doubly nonlinear parabolic equation with nonlinear absorption by using a standard energy method.

**Notation.** We denote by $\|\cdot\|_p$ the standard norm of $L^p$-space, i.e.,

$$\|w\|_p = \left( \int_\Omega |w(x)|^p dx \right)^{1/p} \quad \text{for } w \in L^p(\Omega).$$

# 2 Equations without nonlinear absorption

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ and assume that $p > 2$. In this section, we are concerned with the following initial-boundary value problem for a doubly nonlinear parabolic equation (without nonlinear absorption):

$$|u_t|^{p-2}u_t - \Delta u = \lambda u \quad \text{in} \quad \Omega \times (0,\infty), \tag{2.1}$$
$$u = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty), \tag{2.2}$$
$$u = u_0 \quad \text{on} \quad \Omega \times \{0\} \tag{2.3}$$

with a given function $u_0 : \Omega \to \mathbb{R}$ and a constant $\lambda \in \mathbb{R}$. We particularly address ourselves into explicitly constructing solutions by employing separation of variable method.

Substituting $u(x,t) = \rho(t)\psi(x)$ with unknown functions $\rho : [0,\infty) \to [0,\infty)$ and $\psi : \Omega \to \mathbb{R}$ into (2.1), we obtain

$$|\dot\rho|^{p-2}\dot\rho|\psi|^{p-2}\psi = \rho\left(\Delta\psi + \lambda\psi\right),$$

where $\dot\rho$ denotes the time-derivative of $\rho$. Then equation (2.1) with (2.2) is equivalently rewritten into the following system:

$$|\dot\rho|^{p-2}\dot\rho = \sigma\rho, \;\; \rho \geq 0 \;\; \text{in} \;\; (0,\infty), \tag{2.4}$$
$$-\Delta\psi + \sigma|\psi|^{p-2}\psi = \lambda\psi \;\; \text{in} \;\; \Omega, \quad \psi = 0 \;\; \text{on} \;\; \partial\Omega \tag{2.5}$$

for an arbitrary constant $\sigma \in \mathbb{R}$.

## 2.1 Multiple unbounded solutions

In this subsection, we explicitly construct non-trivial solutions of (2.1)–(2.3) for $u_0 = 0$ (hence $u \equiv 0$ solves the problem). Here we also note that each solution is unbounded as $t \to \infty$.

Here and thereafter, we denote by $\lambda_1 = \lambda_1(\Omega) > 0$ and $\psi_1$ the principal eigen-value and a corresponding eigen-function, respectively, for the eigen-value problem:

$$-\Delta\psi = \lambda\psi \;\; \text{in} \;\; \Omega, \quad \psi = 0 \;\; \text{on} \;\; \partial\Omega. \tag{2.6}$$

Our first result reads,

**Theorem 2.1.** *Let $\Omega$ be a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and assume*

$$2 < p, \quad \lambda_1 < \lambda \quad \textit{and} \quad u_0 = 0. \tag{2.7}$$

*Then (2.1)–(2.3) admits at least two non-trivial solutions $u^+(x,t)$ and $u^-(x,t)$ of separable form such that $u^\pm(x,t)$ are unbounded in $H^1_0(\Omega) \cap L^p(\Omega)$ at the rate $t^{1/(2-p')}$ as $t \to \infty$, where $p'$ denotes the Hölder conjugate of $p$, i.e., $1/p + 1/p' = 1$.*

*Proof.* Let us solve (2.4) and (2.5) with $\sigma = 1$. The initial condition $u(x,0) = \rho(0)\psi(x) \equiv 0$ yields $\rho(0) = 0$. Equation (2.4) with $\sigma = 1$ is transformed into

$$\dot\rho = |\rho|^{p'-2}\rho, \;\; \rho \geq 0 \;\; \text{in} \; (0,\infty), \quad \rho(0) = 0. \tag{2.8}$$

Then (2.4) can be explicitly solved by

$$\rho(t) = c_p t^{1/(2-p')} \quad \text{for} \ \ t \geq 0$$

with $c_p := (2-p')^{1/(2-p')} > 0$.

In order to solve the elliptic equation (2.5) with $\sigma = 1$, i.e.,

$$-\Delta\psi + |\psi|^{p-2}\psi = \lambda\psi \ \text{ in } \ \Omega, \quad \psi = 0 \ \text{ on } \ \partial\Omega, \tag{2.9}$$

we exploit a standard variational approach. Let us define a functional $I^+$ on $X := H_0^1(\Omega) \cap L^p(\Omega)$ by

$$I^+(\psi) := \frac{1}{2}\|\nabla\psi\|_2^2 + \frac{1}{p}\|\psi\|_p^p - \frac{\lambda}{2}\|\psi\|_2^2 \quad \text{for} \ \ \psi \in X. \tag{2.10}$$

Since $p > 2$, we can easily observe that $I^+$ is bounded from below in $X$. Moreover, by using a standard technique (see, e.g., [8]), one can check the so-called Palais-Smale condition for $I^+$ in $X$, i.e., for any sequence $(\psi_n)$ in $X$, if $I^+(\psi_n)$ is bounded and $d_X I^+(\psi_n)$ $(= -\Delta\psi_n + |\psi_n|^{p-2}\psi_n - \lambda\psi_n) \to 0$ strongly in $X^*$, then there exists a strong convergent subsequence of $(\psi_n)$ in $X$. For the convenience of the reader, let us give a proof.

Since $I^+$ is bounded from below in $X$ and $I^+(\psi_n)$ is bounded, it follows that $\|\nabla\psi_n\|_2 + \|\psi_n\|_p \leq C$. Hence since $X$ is reflexive, up to a subsequence, we have

$$u_n \to u \quad \text{weakly in } X,$$

and moreover, since $X$ is compactly embedded into $L^2(\Omega)$, we also obtain

$$u_n \to u \quad \text{strongly in } L^2(\Omega).$$

We find by assumption that

$$-\Delta\psi_n + |\psi_n|^{p-2}\psi_n = d_X I^+(\psi_n) + \lambda\psi_n \to \lambda\psi \quad \text{strongly in } X^*.$$

Hence by virtue of the demiclosedness of the maximal monotone operator $u \mapsto -\Delta u + |u|^{p-2}u$ in $X \times X^*$, we deduce that

$$-\Delta\psi + |\psi|^{p-2}\psi = \lambda\psi.$$

Thus $\psi$ solves (2.9). Moreover, multiplying (2.9) by $\psi$ and integrating this over $\Omega$, we get

$$\|\nabla\psi\|_2^2 + \|\psi\|_p^p = \lambda\|\psi\|_2^2.$$

Consequently, it follows that

$$\begin{aligned} \limsup_{n\to\infty} \|\nabla\psi_n\|_2^2 &= \lim_{n\to\infty} \langle d_X I^+(\psi_n), \psi_n\rangle_X - \liminf_{n\to\infty} \|\psi_n\|_p^p + \lambda \lim_{n\to\infty} \|\psi_n\|_2^2 \\ &\leq -\|\psi\|_p^p + \lambda\|\psi\|_2^2 = \|\nabla\psi\|_2^2. \end{aligned}$$

Here we used the weak lower-semicontinuity of $\|\cdot\|_p$ on $L^p(\Omega)$. Since $X$ is uniformly convex, we have

$$u_n \to u \quad \text{strongly in } X.$$

Thus $I^+$ satisfies the Palais-Smale condition.

Thanks to a standard minimizing method, $I^+$ admits a global minimizer $\hat{\psi} \in X$ (hence $\hat{\psi}$ solves (2.9)). Moreover, we note that

$$\begin{aligned} I^+(s\psi_1) &= \frac{1}{2}\|\nabla\psi_1\|_2^2 s^2 + \frac{1}{p}\|\psi_1\|_p^p s^p - \frac{\lambda}{2}\|\psi_1\|_2^2 s^2 \\ &= \frac{\lambda_1 - \lambda}{2}\|\psi_1\|_2^2 s^2 + \frac{1}{p}\|\psi_1\|_p^p s^p \quad \text{for} \quad s > 0, \end{aligned}$$

where $\psi_1 \neq 0$ denotes an eigen-function corresponding to the principal eigen-value for (2.6). Here we also used the fact that $\|\nabla\psi_1\|_2^2 = \lambda_1\|\psi_1\|_2^2$. Since $\lambda > \lambda_1$, it holds that $I^+(s\psi_1) < 0$ for $s > 0$ small enough. Thus we deduce that

$$I^+(\hat{\psi}) = \min_{\psi \in X} I^+(\psi) < 0,$$

which implies $\hat{\psi} \neq 0$. Moreover, from the symmetry, $I^+(-\psi) = I^+(\psi)$, we find that $-\hat{\psi}$ also minimizes $I^+$, and therefore it also solves (2.9). Consequently, (2.9) possesses two non-trivial solutions $\pm\hat{\psi}$.

Therefore the functions $u^+(x,t) = \rho(t)\hat{\psi}(x)$ and $u^-(x,t) = -\rho(t)\hat{\psi}(x)$ solve (2.1)–(2.3), and moreover, $u^\pm$ are unbounded at the rate $t^{1/(2-p')}$ as $t \to \infty$. Thus we complete a proof. □

**Remark 2.2.** 1. In case $p > 2$ (equivalently, $1 < p' < 2$), the function $\rho \mapsto |\rho|^{p'-2}\rho$ is not locally Lipschitz continuous at $\rho = 0$. However, in case $1 < p < 2$ (equivalently, $p' > 2$), one can ensure that the function $\rho \mapsto |\rho|^{p'-2}\rho$ is locally Lipschitz continuous on $\mathbb{R}$, and therefore, the solution of (2.8) is unique. Then since $\rho \equiv 0$ obviously solve (3.22), there is no non-trivial solution.

2. In case $\lambda \leq \lambda_1$, we can derive the strict convexity of $I^+$ in $X$. Then $I^+$ uniquely possesses a global minimizer $\psi = 0$.

If $\lambda$ coincides with the $i$-th eigen-value for (2.6) with some $i = 2, 3, \ldots$, by virtue of Theorem 2.1, we can construct multiple solutions $u = u(x,t)$ starting from some stationary solution $\psi$ (equivalently, eigen-function corresponding to $\lambda$) besides the stationary solution $u(x,t) \equiv \psi(x)$.

**Corollary 2.3.** *Let $\Omega$ be a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and assume $p > 2$. Let $\lambda$ be the $i$-th eigen-value for* (2.6) *with some $i = 2, 3, \ldots$ and let $\psi$ be an eigen-function corresponding to $\lambda$. Then* (2.1)–(2.3) *with $u_0 = \psi$ admits at least two unbounded solutions of separable form.*

*Proof.* Put $v = u - \psi$. Then $v$ solves (2.1)–(2.3) with $u_0 = 0$, since $\Delta v + \lambda v = \Delta u + \lambda u$. By Theorem 2.1 there exist at least two unbounded solutions $v^\pm$ of (2.1)–(2.3) with $u_0 = 0$. Hence the functions

$$u^\pm(x,t) = \psi(x) + v^\pm(x,t) \quad \text{for} \quad (x,t) \in \Omega \times (0,\infty)$$

solve (2.1)–(2.3) with $u_0 = \psi$ and are unbounded as $t \to \infty$. □

## 2.2 Solutions vanishing in finite time

We next construct solutions of (2.1)–(2.3) which vanish in finite time. Here we can also treat negative $\lambda$, however, some restriction will be imposed on the exponent $p$. To state more details, we use the notation,

$$2^* = \frac{2N}{N-2} \quad \text{for} \quad N > 3 \qquad \text{and} \qquad 2^* = \infty \quad \text{for} \quad N = 1, 2,$$

which is called Sobolev's critical exponent associated with the continuous (respectively, compact) embedding

$$H_0^1(\Omega) \hookrightarrow L^p(\Omega),$$

provided that $p \leq 2^*$ (respectively, $p < 2^*$). Now, our result is stated as follows.

**Theorem 2.4.** *Let $\Omega$ be a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Assume that*

$$2 < p < 2^* \qquad \textit{and} \qquad \lambda \in \mathbb{R}. \tag{2.11}$$

*Then there exist infinitely many solutions of separable form for* (2.1), (2.2) *such that all these solutions vanish at a finite time.*

*Proof.* Put $\sigma = -1$ in (2.4) and (2.5). Then we have

$$|\dot\rho|^{p-2}\dot\rho = -\rho, \ \rho \geq 0 \ \text{ in } (0,\infty), \qquad \rho(0) = r, \tag{2.12}$$

$$-\Delta\psi = \lambda\psi + |\psi|^{p-2}\psi \ \text{ in } \Omega, \qquad \psi = 0 \ \text{ on } \partial\Omega \tag{2.13}$$

with an arbitrary constant $r > 0$. Equation (2.12) can be explicitly solved by

$$\rho(t) = \Big[r^{2-p'} - (2-p')t\Big]_+^{1/(2-p')} \qquad \text{for} \quad t > 0,$$

where $[s]_+$ denotes the positive part of $s \in \mathbb{R}$. Here we observe that $\rho$ vanishes at a finite time

$$T := \frac{r^{2-p'}}{2-p'} > 0.$$

We proceed to treat (2.13). Equation (2.13) has already been studied well (see, e.g., [8]). Define a functional $I^-$ on $X := H_0^1(\Omega)$ by

$$I^-(\psi) := \frac{1}{2}\|\nabla\psi\|_2^2 - \frac{1}{p}\|\psi\|_p^p - \frac{\lambda}{2}\|\psi\|_2^2 \quad \text{for} \quad \psi \in X.$$

The existence of infinitely many critical points of $I^-$ has been proved by using the so-called fountain theorem (see Theorem 2.5 of [5], Theorem 3.6 of [11]). For the reader's convenience, let us show an outline of proof. The fountain theorem on separable Hilbert spaces for the antipodal action is described as follows.

**Theorem 2.5** (Fountain theorem [5])**.** *Let $\{e_j\}_{j\in\mathbb{N}}$ be a complete orthogonal system of a separable Hilbert space $X$. Set*

$$Y_k := \operatorname{span}\{e_j\}_{j\leq k}, \quad Z_k := \overline{\operatorname{span}\{e_j\}_{j\geq k}}^X \qquad \textit{for} \quad k \in \mathbb{N}.$$

*Assume that $I \in C^1(X;\mathbb{R})$ satisfies $I(-\psi) = I(\psi)$ and the Palais-Smale condition in $X$. If for all $k \in \mathbb{N}$ there exist $\rho_k > r_k > 0$ such that*

(i) $a_k := \inf\{I(\psi);\ \psi \in Z_k, \|\nabla\psi\|_2 = r_k\} \to \infty$ *as* $k \to \infty$,

(ii) $b_k := \max\{I(\psi);\ \psi \in Y_k, \|\nabla\psi\|_2 = \rho_k\} \le 0$,

*then* $I$ *possesses a sequence of critical points* $(\hat{\psi}_n)$ *such that* $I(\pm\hat{\psi}_n) \to \infty$ *as* $n \to \infty$.

To apply this theorem to our problem, let $\{e_j\}_{j\in\mathbb{N}}$ be a complete orthogonal system of $X = H_0^1(\Omega)$.

*Claim* 1. *For each* $k \in \mathbb{N}$, *there exist* $\rho_k > r_k > 0$ *such that* (i) *and* (ii) *are satisfied with* $I$ *replaced by* $I^-$.

We see, by $p > 2$,

$$I^-(\psi) \ge \frac{1}{2}\|\nabla\psi\|_2^2 - C(\|\psi\|_p^p + 1) \quad \text{for} \ \ \psi \in X$$

with some constant $C > 0$. Set $\beta_k := \sup\{\|\psi\|_p;\ \psi \in Z_k,\ \|\nabla\psi\|_2 = 1\} > 0$. Then it follows that

$$I^-(\psi) \ge \frac{1}{2}\|\nabla\psi\|_2^2 - C\,(\beta_k^p\|\nabla\psi\|_2^p + 1) =: J(\psi) \quad \text{for} \ \ \psi \in Z_k.$$

Put $r_k = (Cp\beta_k^p)^{1/(2-p)}$. Then we obtain

$$\begin{aligned} J(r_k\psi) &= \max_{r>0} J(r\psi) \\ &= \max_{r>0}\left(\frac{r^2}{2} - C(\beta_k^p r^p + 1)\right) \\ &= \frac{p-2}{2p}(Cp)^{-2/(p-2)}\beta_k^{-(2p)/(p-2)} - C \quad \text{for} \ \ \psi \in Z_k \ \text{ and } \ \|\nabla\psi\|_2 = 1. \end{aligned}$$

Hence assertion (i) can be checked if $\beta_k \to 0$ as $k \to \infty$. Since $\beta_k$ is non-increasing and bounded from below, $\beta_k \to \beta$ as $k \to \infty$. On the other hand, for every $k \in \mathbb{N}$, there exists $\psi_k \in Z_k$ such that $\|\nabla\psi_k\|_2 = 1$ and $\|\psi_k\|_p > \beta_k/2$. By definition of $\psi_k$, up to a subsequence, we have $\psi_k \to 0$ weakly in $X$, and moreover, strongly in $L^p(\Omega)$. Thus $\beta = 0$.

We next show (ii). Since all norms of $Y_k$ are equivalent to each other (because $\dim Y_k < \infty$), we see

$$I^-(\psi) \le \frac{1}{2}\|\nabla\psi\|_2^2 - C\|\nabla\psi\|_2^p \quad \text{for} \ \ \psi \in Y_k$$

with a constant $C > 0$. Hence $I^-(\psi) \le 0$ for all $\psi \in Y_k$ satisfying $\|\nabla\psi\|_2 = \rho_k$ with some constant $\rho_k > r_k$ large enough. Thus (ii) follows.

*Claim* 2. *The Palais-Smale condition holds for* $I^-$ *in* $X$.

Indeed, let $(\psi_n)$ be a sequence in $X$ such that $I^-(\psi_n)$ is bounded and $d_X I^-(\psi_n) \to 0$ strongly in $X^*$. It follows that

$$\langle d_X I^-(\psi_n), \psi_n\rangle_X = \|\nabla\psi_n\|_2^2 - \|\psi_n\|_p^p - \lambda\|\psi_n\|_2^2.$$

Hence choosing $\varepsilon \in (1/p, 1/2)$ and using Hölder's and Young's inequalities, we see

$$\begin{aligned}
&I^-(\psi_n) - \varepsilon \langle d_X I^-(\psi_n), \psi_n \rangle_X \\
&= \left(\frac{1}{2} - \varepsilon\right) \|\nabla \psi_n\|_2^2 + \left(\varepsilon - \frac{1}{p}\right) \|\psi_n\|_p^p - \left(\frac{1}{2} - \varepsilon\right) \lambda \|\psi_n\|_2^2 \\
&\geq \left(\frac{1}{2} - \varepsilon\right) \|\nabla \psi_n\|_2^2 + \frac{1}{2}\left(\varepsilon - \frac{1}{p}\right) \|\psi_n\|_p^p - C \\
&\geq \delta \|\nabla \psi_n\|_2^2 - C
\end{aligned}$$

with $\delta := 1/2 - \varepsilon > 0$ and some constant $C \geq 0$. On the other hand, by assumption, the left-hand-side can be estimated from above as follows:

$$I^-(\psi_n) - \varepsilon \langle d_X I^-(\psi_n), \psi_n \rangle_X \leq C_1 + \varepsilon \|d_X I^-(\psi_n)\|_{X^*} \|\nabla \psi_n\|_2 \leq C_2 + \frac{\delta}{2} \|\nabla \psi_n\|_2^2$$

with some constants $C_1, C_2 \geq 0$. Therefore comparing both sides, we deduce that $\|\nabla \psi_n\|_2$ is bounded as $n \to \infty$.

Since $X$ is reflexive, up to a subsequence, we infer that

$$\psi_n \to \psi \quad \text{weakly in } X,$$

and moreover, since $X = H_0^1(\Omega)$ is compactly embedded in $L^p(\Omega)$ by assumption, it follows that

$$\psi_n \to \psi \quad \text{strongly in } L^p(\Omega).$$

Hence we observe that

$$-\Delta \psi_n = d_X I^-(\psi_n) + |\psi_n|^{p-2}\psi_n + \lambda \psi_n \to |\psi|^{p-2}\psi + \lambda \psi \quad \text{strongly in } X^*.$$

By the (weak) closedness of $-\Delta$, the limit $\psi$ solves (2.13). Moreover, we also have

$$\|\nabla \psi\|_2^2 = \|\psi\|_p^p + \lambda \|\psi\|_2^2.$$

Note that

$$\begin{aligned}
\|\nabla \psi_n\|_2^2 &= \langle d_X I^-(\psi_n), \psi_n \rangle_X + \|\psi_n\|_p^p + \lambda \|\psi_n\|_2^2 \\
&\to \|\psi\|_p^p + \lambda \|\psi\|_2^2 = \|\nabla \psi\|_2^2.
\end{aligned}$$

Since $X$ is uniformly convex, we conclude that

$$\psi_n \to \psi \quad \text{strongly in } X.$$

Hence $I^-$ satisfies the Palais-Smale condition in $X$.

Obviously, $I^-(-\psi) = I^-(\psi)$. Combining these claims and using the fountain theorem (see Theorem 2.5 of [5], Theorem 3.6 of [11]), we assure that $I^-$ admits infinitely many critical points $\hat{\psi}_n \in X$ (hence $\hat{\psi}_n$ solves (2.9) for $n \in \mathbb{N}$) such that $I^-(\hat{\psi}_n) \to \infty$ as $n \to \infty$. Moreover, $-\hat{\psi}_n$ also solves (2.9).

Set

$$u_n^\pm(x,t) := \pm \rho(t) \hat{\psi}_n(x) \quad \text{for} \quad (x,t) \in \Omega \times (0, \infty).$$

Then $u_n^\pm$ solve (2.1), (2.2) and vanish at $t = T$. □

**Remark 2.6.** As for the Sobolev critical case, i.e., $p = 2^*$, Brézis-Nirenberg [6] proved the existence of positive solutions $\hat{\psi}_1$ for (2.9) by assuming

$$\lambda \in \begin{cases} (0, \lambda_1) & \text{if } N \geq 4, \\ (\lambda_*, \lambda_1) & \text{if } N = 3 \end{cases} \tag{2.14}$$

with some $\lambda_* \in [0, \lambda_1)$, where $\lambda_1$ denotes the principal eigenvalue for (2.6). Hence under these assumptions, we can construct sign-definite non-trivial solutions $u^\pm(x,t) = \pm\rho(t)\hat{\psi}_1(x)$ of (2.1), (2.2) such that $u^\pm$ vanish at a finite time $T > 0$.

As in Corollary 2.3, we also obtain the following corollary.

**Corollary 2.7.** *Let $\Omega$ be a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and assume that $2 < p < 2^*$. Let $\lambda$ be an eigen-value for* (2.6) *and let $\psi$ be a corresponding eigen-function. Then* (2.1), (2.2) *admits infinitely many solutions of separable form such that all these solutions coincide with $\psi$ at a finite time.*

**Remark 2.8.** Assume that

$$2 < p < 2^* \quad \text{and} \quad \lambda_1 < \lambda,$$

where $\lambda_1$ stands for the principal eigen-value of (2.6). Then we can construct a solution of (2.1)–(2.3) whose orbit splits into three branches at a finite time. Indeed, by Theorem 2.4, there exists a non-trivial solution $v$ of (2.1)–(2.3) such that $v$ vanishes at a finite time $T > 0$. By Theorem 2.1, we also have two unbounded non-trivial solution $w^\pm$ of (2.1)–(2.3) with $u_0 = 0$. Now, put

$$\begin{aligned} u^\pm(x,t) &:= \begin{cases} v(x,t) & \text{if } t \in [0,T], \\ w^\pm(x, t-T) & \text{if } t \in (T, \infty), \end{cases} \\ u^0(x,t) &:= \begin{cases} v(x,t) & \text{if } t \in [0,T], \\ 0 & \text{if } t \in (T, \infty). \end{cases} \end{aligned}$$

Then $u^\pm$ and $u^0$ coincide with each other for all $t \in [0,T]$ and solve (2.1)–(2.3). Moreover, the common orbit bifurcates into three branches at $t = T$.

# 3 Equations with nonlinear absorption

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. This section is devoted to the following initial-boundary value problem for a doubly nonlinear parabolic equation with nonlinear absorption:

$$|u_t|^{p-2}u_t - \Delta u + |u|^{p-2}u = \lambda u \quad \text{in} \quad \Omega \times (0, \infty), \tag{3.15}$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty), \tag{3.16}$$

$$u = u_0 \quad \text{on} \quad \Omega \times \{0\} \tag{3.17}$$

with an initial data $u_0 : \Omega \to \mathbb{R}$ and $\lambda \in \mathbb{R}$. Our analysis here will also rely on the method of separation of variable. Substitute $u(x,t) = \theta(t)\psi(x)$ into (3.15)–(3.17). Then we obtain

$$|\dot{\theta}|^{p-2}\dot{\theta} + |\theta|^{p-2}\theta = \sigma\theta, \ \theta \geq 0 \ \text{ in } \ (0,\infty), \tag{3.18}$$

$$-\Delta\psi + \sigma|\psi|^{p-2}\psi = \lambda\psi \ \text{ in } \ \Omega, \quad \psi = 0 \ \text{ on } \ \partial\Omega \tag{3.19}$$

for any $\sigma \in \mathbb{R}$.

## 3.1 Heteroclinic orbits

We first construct heteroclinic orbits for (3.15)–(3.17) by solving (3.18), (3.19) with $\sigma = 1$. The stationary problem for (3.15)–(3.17) can be written as

$$-\Delta\psi + |\psi|^{p-2}\psi = \lambda\psi \ \text{ in } \ \Omega, \quad \psi = 0 \ \text{ on } \ \partial\Omega, \tag{3.20}$$

which is (3.19) with $\sigma = 1$. In the following theorem, we construct solutions which start from the trivial solution for (3.20) and arrive at some non-trivial one in finite time.

**Theorem 3.1.** *Suppose that*

$$2 < p, \quad \lambda_1 < \lambda \quad \text{and} \quad u_0 = 0, \tag{3.21}$$

*where* $\lambda_1$ *is the principal eigenvalue of* (2.6). *Then* (3.15)–(3.17) *admits at least two non-trivial solutions* $u^+(x,t)$ *and* $u^-(x,t)$ *of separable form. Moreover, there exists a non-trivial stationary solution* $\hat{\psi}$ *such that* $u^\pm(\cdot,t) = \pm\hat{\psi}$ *for all* $t \geq T$ *with*

$$T = \frac{1}{p-2}\mathrm{B}\left(\frac{2-p'}{p-2}, 2-p'\right),$$

*where* $\mathrm{B}(\cdot,\cdot)$ *denotes the Euler beta function.*

To prove this theorem, we first explicitly solve the ode problem,

$$|\dot{\theta}|^{p-2}\dot{\theta} + |\theta|^{p-2}\theta = \theta, \ \theta \geq 0 \ \text{ in } \ (0,\infty), \quad \theta(0) = 0, \tag{3.22}$$

in the following lemma.

**Lemma 3.2.** *Set*

$$\Phi(\theta) := \int_0^\theta s^{1-p'}\left(1 - s^{p-2}\right)^{1-p'} ds \quad \text{for } \ \theta \in [0,1].$$

*Then the following* (i)–(iv) *are satisfied.*

(i) *The function* $\Phi$ *is well defined, continuous and strictly increasing on* $[0,1]$. *In particular,* $\Phi(0) = 0$ *and* $\Phi(1) = T$.

(ii) *The function* $\theta := \Phi^{-1}$ *is well defined, continuous and strictly increasing on* $[0,T]$. *In particular,* $\theta(0) = 0$ *and* $\theta(T) = 1$.

(iii) *The function* $\theta$ *belongs to* $C^1([0,T]) \cap C^2(0,T)$. *Moreover,* $\dot{\theta}(0) = \dot{\theta}(T) = 0$ *and* $\theta$ *is convex (respectively, concave) on the interval in which* $\theta(t)^{p-2} < 1/(p-1)$ *(respectively,* $\theta(t)^{p-2} > 1/(p-1)$*).*

(iv) *Extend* $\theta$ *by constant into* $\mathbb{R}$ *as follows:*

$$\theta(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ \Phi^{-1}(t) & \text{if } 0 < t < T, \\ 1 & \text{if } T \leq t. \end{cases}$$

*Then,* $\theta$ *belongs to* $C^1(\mathbb{R})$ *and solves* (3.22) *in* $\mathbb{R}$.

*Proof.* Put $\xi := s^{p-2}$. By the definition of $\Phi$,

$$\Phi(\theta) = \frac{1}{p-2} \int_0^{\theta^{p-2}} \xi^{P-1}(1-\xi)^{Q-1} d\xi \tag{3.23}$$

with $P := (2-p')/(p-2) > 0$ and $Q := 2-p' > 0$. We note that the integrand of (3.23) is continuous in $(0,1)$. In case $\theta < 1$, since $Q - 1 < 0$, it follows that

$$\int_\varepsilon^{\theta^{p-2}} \xi^{P-1}(1-\xi)^{Q-1} d\xi \leq \frac{1}{P}\left(\theta^{2-p'} - \varepsilon^P\right)(1-\theta^{p-2})^{Q-1}$$

for any $\varepsilon \in (0, \theta^{p-2})$. Hence $\Phi(\theta)$ is well defined by passing to the limit as $\varepsilon \to 0_+$, and moreover,

$$\Phi(\theta) = \frac{1}{p-2} \lim_{\varepsilon \to 0_+} \int_\varepsilon^{\theta^{p-2}} \xi^{P-1}(1-\xi)^{Q-1} d\xi \leq \frac{1}{Q}\theta^{2-p'}(1-\theta^{p-2})^{Q-1},$$

which also implies $\Phi(0) = 0$ by letting $\theta \to 0_+$. In case $\theta = 1$, we find that

$$\Phi(1) = \frac{1}{p-2} \int_0^1 \xi^{P-1}(1-\xi)^{Q-1} d\xi = \frac{1}{p-2} \mathrm{B}(P,Q) =: T,$$

where $\mathrm{B}(\cdot,\cdot)$ denotes the Euler beta function. Since $\Phi'(\theta) = \theta^{1-p'}(1-\theta^{p-2})^{1-p'}$ is positive and continuous in $(0,1)$, it follows that $\Phi(\theta)$ is of class $C^1$ and strictly increasing. Thus (i) has been proved.

Let us prove (ii). Since $\Phi$ is strictly increasing and continuous on $[0,1]$, we can take the inverse function $\theta := \Phi^{-1}$ from $[0,T]$ into $[0,1]$ and observe that $\theta$ is strictly increasing and continuous on $[0,T]$. Moreover, by definition, it is obvious that $\theta(0) = 0$ and $\theta(T) = 1$.

Concerning (iii), since $\Phi \in C^1(0,1)$ and $\Phi' > 0$, we have $\theta \in C^1(0,T)$ and

$$\dot{\theta}(t) = \frac{1}{\Phi'(\theta)} = \theta(t)^{p'-1}\left(1-\theta(t)^{p-2}\right)^{p'-1} \quad \text{for } t \in (0,T). \tag{3.24}$$

Hence we find by (ii) that $\dot{\theta}(0) = \dot{\theta}(T) = 0$ and $\theta \in C^1([0,T])$. Furthermore,

$$\begin{aligned} \ddot{\theta}(t) &= (p'-1)\theta(t)^{p'-2}\left(1-\theta(t)^{p-2}\right)^{p'-2}\left[1-(p-1)\theta(t)^{p-2}\right]\dot{\theta}(t) \\ &= (p'-1)\theta(t)^{2p'-3}\left(1-\theta(t)^{p-2}\right)^{2p'-3}\left[1-(p-1)\theta(t)^{p-2}\right] \end{aligned} \tag{3.25}$$

for all $t \in (0,T)$. It implies $\theta \in C^2(0,T)$. Moreover, $\theta(t)$ is convex if $\theta(t)^{p-2} \le 1/(p-1)$, and concave otherwise.

As for (iv), it follows immediately from (3.24) that

$$\dot\theta(t)^{p-1} = \theta(t)\left(1-\theta(t)^{p-2}\right) \quad \text{for all } \ t \in (0,T).$$

Thus $\theta$ solves (3.22) on $[0,T]$. Moreover, let us extend $\theta$ by constant into $\mathbb{R}$. Then $\theta \in C^1(\mathbb{R})$ and it solves (3.15) in $\mathbb{R}$. □

**Remark 3.3.** The function $\theta$ given in Lemma 3.2 is not of class $C^2$ at $t = 0, T$, provided that $p > 3$ (equivalently, $2p' - 3 < 0$). Indeed, by (3.25), $\ddot\theta(t)$ diverges as $t \to 0_+$ or $t \to T_-$.

We are ready to prove Theorem 3.1.

*Proof of Theorem 3.1.* The elliptic problem,

$$-\Delta\psi + |\psi|^{p-2}\psi = \lambda\psi \ \text{ in } \ \Omega, \quad \psi = 0 \ \text{ on } \ \partial\Omega, \tag{3.26}$$

which is the stationary problem for (3.15)–(3.17), has already been solved in the last section (see Theorem 2.1), and (3.26) admits at least two non-trivial solutions $\hat\psi, -\hat\psi$. Therefore combining this with Lemma 3.2 and putting $u^\pm(x,t) = \pm\theta(t)\hat\psi(x)$, we can obtain the conclusion of Theorem 3.1. □

**Remark 3.4.** Due to Lemma 3.2, the function $u^+(x,t) = \theta(t)\hat\psi(x)$ solves (3.15)–(3.17) in $\mathbb{R}$ such that $u^+(\cdot,t) = \hat\psi$ for $t \ge T$ and $u^+(\cdot,t) = 0$ for $t \le 0$. Hence the complete orbit $\gamma(u^+) := \{u^+(\cdot,t);\ t \in \mathbb{R}\}$ of $u^+$ becomes a heteroclinic orbit connecting two equilibria $\hat\psi$, 0 of the dynamical system generated by (3.15)–(3.17).

## 3.2 Solutions vanishing in finite time

We next solve (3.18), (3.19) with $\sigma = -1$ and construct a solution vanishing at a finite time for (3.15)–(3.17). Our result reads,

**Theorem 3.5.** *Let $\Omega$ be a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Suppose that*

$$2 < p < 2^* \quad \textit{and} \quad \lambda \in \mathbb{R}. \tag{3.27}$$

*Then* (3.15), (3.16) *admits infinitely many non-trivial solutions of separable form such that all these solutions vanish at a finite time.*

Equation (3.19) with $\sigma = -1$ is written as

$$-\Delta\psi = \lambda\psi + |\psi|^{p-2}\psi \ \text{ in } \ \Omega, \quad \psi = 0 \ \text{ on } \ \partial\Omega. \tag{3.28}$$

In Theorem 2.4, we have proved that (3.28) has infinitely many non-trivial solutions $\pm\hat\psi_n$ under (3.27). Next, let us solve the Cauchy problem for the ode,

$$|\dot\theta|^{p-2}\dot\theta + |\theta|^{p-2}\theta = -\theta,\ \theta \ge 0 \ \text{ in } \ (0,\infty), \quad \theta(0) = \theta_0 > 0. \tag{3.29}$$

**Lemma 3.6.** *For $\theta_0 > 0$, define*

$$\Psi(\theta) := \int_\theta^{\theta_0} s^{1-p'} \left(1 + s^{p-2}\right)^{1-p'} ds \quad \text{for } \theta \in [0, \theta_0].$$

*Then the following* (i)–(iv) *are satisfied.*

(i) *The function $\Psi$ is well defined, continuous and strictly decreasing on $[0, \theta_0]$. In particular, $\Psi(\theta_0) = 0$ and $\Psi(0) = T$, where $T$ is given by*

$$T = \int_0^{\theta_0} s^{1-p'}(1 + s^{p-2})^{1-p'} ds \in (0, \infty).$$

(ii) *The function $\theta := \Psi^{-1}$ is well defined, continuous and strictly decreasing on $[0, T]$. In particular, $\theta(0) = \theta_0$ and $\theta(T) = 0$.*

(iii) *The function $\theta$ belongs to $C^1([0, T]) \cap C^2([0, T))$. Moreover,*

$$\dot\theta(0) = -\theta_0^{p'-1}(1 + \theta_0^{p-2})^{p'-1} < 0 \quad \text{and} \quad \dot\theta(T) = 0$$

*and $\theta$ is convex.*

(iv) *Extend $\theta$ by zero onto $[0, \infty)$. Then $\theta$ solves* (3.29) *on $[0, \infty)$.*

*Proof.* The function $\Psi$ is obviously well defined on $(0, \theta_0]$. We note that

$$\int_\varepsilon^{\theta_0} s^{1-p'}(1 + s^{p-2})^{1-p'} ds \le \frac{C_{\theta_0}}{2 - p'} \left(\theta_0^{2-p'} - \varepsilon^{2-p'}\right) \to \frac{C_{\theta_0}}{2 - p'} \theta_0^{2-p'} \quad \text{as } \varepsilon \to 0_+$$

with a constant $C_{\theta_0} > 0$ depending on $\theta_0$. Hence $\Psi$ is well defined on $[0, \theta_0]$, in particular, $T := \Psi(0)$ is finite. The rest of (i) can be easily seen. Moreover, (ii) follows from (i) as in Lemma 3.6.

As for (iii), we find that

$$\dot\theta(t) = -\theta(t)^{p'-1} \left(1 + \theta(t)^{p-2}\right)^{p'-1} \quad \text{for } t \in (0, T).$$

Hence $\theta$ belongs to $C^1([0, T])$ and solves (3.29) on $[0, T]$. Moreover,

$$\ddot\theta(t) = (p' - 1)\theta(t)^{2p'-3} \left(1 + \theta(t)^{p-2}\right)^{2p'-3} \left[1 + (p - 1)\theta(t)^{p-2}\right]$$

for $t \in [0, T)$, which yields the rest of (iii). Extend $\theta$ onto $[0, \infty)$ by

$$\theta(t) = \begin{cases} \Psi^{-1}(t) & \text{if } 0 \le t \le T, \\ 0 & \text{if } T < t. \end{cases}$$

Since $\dot\theta(T) = 0$, we observe that $\theta \in C^1([0, \infty))$, and moreover, $\theta$ solves (3.29) on $[0, \infty)$. □

*Proof of Theorem 3.5.* Put $u_n^\pm(x, t) := \pm\theta(t)\hat\psi_n(x, t)$. Then $u_n^\pm$ solve (3.15), (3.16) and vanish at $t = T$. Thus Theorem 3.5 has been proved. □

# 4 Asymptotic behavior of general solutions

In the final section, we are concerned with asymptotic behavior of global solutions for (3.15)–(3.17) with general initial data $u_0$. Let us state our main result here.

**Theorem 4.1.** *Let $\Omega$ be a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and suppose that*

$$2 < p \quad \text{and} \quad \lambda \in \mathbb{R}.$$

*Let $(t_n)$ be a sequence on $[0,\infty)$ such that $t_n \to \infty$. Let $u$ be a global solution of (3.15)–(3.17) with an initial data $u_0 \in H_0^1(\Omega) \cap L^p(\Omega)$. Then there exist a subsequence $(n')$ of $(n)$ and a stationary solution $\hat{\psi}$ such that*

$$u(t_{n'}) \to \hat{\psi} \quad \text{strongly in } H_0^1(\Omega) \cap L^p(\Omega).$$

*Proof.* Multiply (3.15) by $u_t(x,t)$ and integrate this over $\Omega$ to get

$$\|u_t(t)\|_p^p + \frac{d}{dt} J(u(t)) = 0 \quad \text{for a.e.} \quad t > 0 \tag{4.30}$$

with a functional defined on $H_0^1(\Omega) \cap L^p(\Omega)$ by

$$J(w) := \frac{1}{2}\|\nabla w\|_2^2 + \frac{1}{p}\|w\|_p^p - \frac{\lambda}{2}\|w\|_2^2 \quad \text{for} \quad w \in H_0^1(\Omega) \cap L^p(\Omega).$$

Thus we obtain

$$\int_0^\infty \|u_t(t)\|_p^p dt \quad < \quad \infty, \tag{4.31}$$

$$\sup_{t \geq 0} \left( \|\nabla u(t)\|_2^2 + \|u(t)\|_p^p \right) \quad < \quad \infty. \tag{4.32}$$

By (4.31) we can take $\theta_n \in [t_n, t_n + 1]$ such that

$$u_t(\theta_n) \to 0 \quad \text{strongly in } L^p(\Omega),$$

which also yields that $|u_t|^{p-2}u_t(\theta_n) \to 0$ strongly in $L^{p'}(\Omega)$. In the rest of this proof, we write $u_n = u(\theta_n)$ if no confusion can arise. By (4.32), there exist a subsequence $(n')$ of $(n)$ and $\hat{\psi} \in H_0^1(\Omega) \cap L^p(\Omega)$ such that

$$\begin{aligned} u_{n'} \to \hat{\psi} \qquad & \text{weakly in } H_0^1(\Omega) \cap L^p(\Omega), \\ & \text{strongly in } L^q(\Omega) \ \text{ for any } \ q < \max\{2^*, p\}. \end{aligned}$$

Furthermore, we observe

$$|u_{n'}|^{p-2}u_{n'} \to |\hat{\psi}|^{p-2}\hat{\psi} \quad \text{weakly in } L^{p'}(\Omega),$$

since $u_n(x) \to \hat{\psi}(x)$ for a.e. $x \in \Omega$. By (3.15),

$$-\Delta\hat{\psi} + |\hat{\psi}|^{p-2}\hat{\psi} = \lambda\hat{\psi} \ \text{ in } \Omega, \quad \hat{\psi} = 0 \ \text{ on } \ \partial\Omega.$$

Hence $\hat{\psi}$ becomes a stationary solution of (3.15)–(3.17).

Multiplying (3.15) by $u_{n'}(x)$ and integrating this over $\Omega$, we have

$$\|\nabla u_{n'}\|_2^2 = \lambda \|u_{n'}\|_2^2 - \|u_{n'}\|_p^p - \left\langle |u_t|^{p-2} u_t(\theta_{n'}), u_{n'} \right\rangle_{L^p}. \tag{4.33}$$

By using the convergences obtained above and the lower semicontinuity of $\|\cdot\|_p$ in the weak topology of $L^p(\Omega)$,

$$\limsup_{n'\to\infty} \|\nabla u_{n'}\|_2^2 = \lambda \|\hat{\psi}\|_2^2 - \liminf_{n'\to\infty} \|u_{n'}\|_p^p \leq \lambda \|\hat{\psi}\|_2^2 - \|\hat{\psi}\|_p^p = \|\nabla \hat{\psi}\|_2^2.$$

By the lower semicontinuity of $\|\nabla \cdot\|_2$ in the weak topology of $H_0^1(\Omega)$, we have

$$\|\nabla u_{n'}\|_2 \to \|\nabla \hat{\psi}\|_2.$$

Using (4.33) again, we have

$$\|u_{n'}\|_p \to \|\hat{\psi}\|_p.$$

Therefore since $H_0^1(\Omega)$ and $L^p(\Omega)$ are uniformly convex, we deduce that

$$u(\theta_{n'}) \to \hat{\psi} \quad \text{strongly in } H_0^1(\Omega) \cap L^p(\Omega).$$

Let us finally prove the convergence of $u(t)$ along $t_{n'} \to \infty$. By (4.31), we have

$$\begin{aligned} \|u(\theta_{n'}) - u(t_{n'})\|_p &= \left\| \int_{t_{n'}}^{\theta_{n'}} u_t(t)dt \right\|_p \\ &\leq \left( \int_{t_{n'}}^{\theta_{n'}} \|u_t(t)\|_p^p dt \right)^{1/p} (\theta_{n'} - t_{n'})^{1/p'} \\ &\leq \left( \int_{t_{n'}}^{\infty} \|u_t(t)\|_p^p dt \right)^{1/p} \to 0, \end{aligned}$$

since $\theta_{n'} \in [t_{n'}, t_{n'}+1]$. Hence

$$u(t_{n'}) \to \hat{\psi} \quad \text{strongly in } L^p(\Omega).$$

Moreover, since $J(u(t))$ is non-increasing for $t > 0$ and $J(u(\theta_{n'})) \to J(\hat{\psi})$, we assure that

$$J(u(t)) \to J(\hat{\psi}) \quad \text{as } t \to \infty.$$

It follows that

$$\begin{aligned} \frac{1}{2}\|\nabla u(t_{n'})\|_2^2 &= J(u(t_{n'})) - \frac{1}{p}\|u(t_{n'})\|_p^p + \frac{\lambda}{2}\|u(t_{n'})\|_2^2 \\ &\to J(\hat{\psi}) - \frac{1}{p}\|\hat{\psi}\|_p^p + \frac{\lambda}{2}\|\hat{\psi}\|_2^2 = \frac{1}{2}\|\nabla \hat{\psi}\|_2^2. \end{aligned}$$

Therefore $u(t_{n'})$ converges to $\hat{\psi}$ strongly in $H_0^1(\Omega) \cap L^p(\Omega)$ as $n \to \infty$ by the uniform convexity of $H_0^1(\Omega)$. Thus our proof is completed. □

# References


[1] G. Akagi, Doubly nonlinear evolution equations with non-monotone perturbations in reflexive Banach spaces, submitted.

[2] G. Akagi, Global attractors for doubly nonlinear evolution equations with non-monotone perturbations, submitted.

[3] G. Akagi and R. Kajikiya, Asymptotic behavior of solutions for $p$-Laplace parabolic equations, submitted.

[4] J.M. Ball, Continuity properties and global attractors of generalized semiflows and the Navier-Stokes equations, J. Nonlinear Science, **7**(1997), 475–502.

[5] T. Bartsch, Infinitely many solutions of a symmetric Dirichlet problem, Nonlinear Analysis, **20**(1993), 1205–1216.

[6] H. Brézis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math., **36**(1983), 437–477.

[7] M.E. Gurtin, Generalized Ginzburg-Landau and Cahn-Hilliard equations based on a microforce balance, Physica D, **92**(1996), 178–192.

[8] P.H., Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics, 65, Published for the Conference Board of the Mathematical Sciences, Washington, DC, the American Mathematical Society, Providence, RI, 1986.

[9] A. Segatti, Global attractor for a class of doubly nonlinear abstract evolution equations, Discrete Contin. Dyn. Syst., **14**(2006), 801–820.

[10] M. Struwe, *Variational methods, Applications to nonlinear partial differential equations and Hamiltonian systems. Third edition*, Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics, 34, Springer-Verlag, Berlin, 2000.

[11] M. Willem, *Minimax theorems*, Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston, 1996.