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DOUBLY NONLINEAR EVOLUTION EQUATIONS IN REFLEXIVE BANACH SPACES

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Abstract. Let V and V^* be a reflexive Banach space and its dual space, respectively, and let H be a Hilbert space whose dual space H^* is identified with itself H such that

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*$$

with continuous and densely defined canonical injections. This paper is concerned with Cauchy problems for doubly nonlinear evolution equations governed by subdifferential operators with non-monotone perturbations of the form:

$$\text{(CP)} \quad \begin{cases} \partial_V \psi(u'(t)) + \partial_V \varphi(u(t)) + B(u(t)) \ni f \quad \text{in } V^*, \quad 0 < t < T, \\ u(0) = u_0, \end{cases}$$

where $\partial_V \psi, \partial_V \varphi : V \to 2^{V^*}$ denote subdifferential operators of proper, lower semicontinuous and convex functions $\psi, \varphi : V \to (-\infty, +\infty]$, respectively, and $f \in V^*$ and $u_0 \in V$ are given data. Moreover, let B be a (possibly) multi-valued operator from V into V^* such that B may be non-monotone in $V \times V^*$.

In this paper, after reviewing author's recent results on sufficient conditions for the local (in time) existence of strong solutions to (CP) as well as those for the global existence, the long-time behavior of global strong solutions for (CP) is discussed. It is emphasized that our abstract framework is established in a reflexive Banach space setting and can cover evolution problems without any gradient structures. Furthermore, solutions of (CP) may not be unique, so the usual semigroup approach to dynamical systems is not effective for our setting. In this paper, the theory of generalized semiflow due to J.M. Ball is exploited to treat dynamical systems generated by (CP).

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1 Introduction

The theory of evolution equations governed by subdifferential operators is well known as a powerful tool to analyze the existence and uniqueness of solutions as well as their asymptotic behaviors for nonlinear parabolic PDEs now. Developments of the theory started with the study of a simple form:

$$u'(t) + \partial_H \varphi(u(t)) \ni 0, \quad 0 < t < T \tag{1. 1}$$

in a Hilbert space H (see Brézis [15]) as a special case of the nonlinear semigroup theory due to Kōmura [26], and Brézis's abstract theory for (1. 1) was applied to degenerate parabolic equations associated with the p-Laplace operator and porous medium equations. His theory has been generalized in various directions by many mathematicians, and some of such generalizations succeeded triumphantly in applications to nonlinear parabolic equations. Indeed, many free boundary problems, e.g., Stefan problem (see, e.g., §3.5 of [23]) were solved through a generalization with time-dependent subdifferentials due to Kenmochi [22, 23], Attouch et al, Yamada [35] and so on. Non-monotone perturbation theories for (1. 1) also extended the applicability of subdifferential approaches to degenerate parabolic equations with blow-up terms, Navier-Stokes equation (see Ôtani-Yamada [30], Ôtani [28, 29]), Allen-Cahn equation, Cahn-Hilliard equation (see Kenmochi et al [24]), and so on. Moreover, doubly nonlinear problems naturally arise in the description of phase transition phenomena, so abstract theories are also established for doubly nonlinear evolution equations of the form

$$\partial_H \psi(u'(t)) + \partial_H \varphi(u(t)) \ni f \quad \text{in } H, \quad 0 < t < T \tag{1. 2}$$

with two subdifferential operators $\partial_H \psi$ and $\partial_H \varphi$ by Barbu [12], Arai [8], Colli-Visintin [19] (see also [18], [6], [31, Sect. 11], [32] and [9]). Another type of doubly nonlinear problems are also treated by Barbu [14] and Kenmochi-Pawlow [25] (see also [20], [34], [27], [36], [31, Sect. 11], [3]). On the other hand, evolution equations governed by subdifferential operators were originally studied only in Hilbert space settings. However, several authors (e.g., Brézis [15], Kenmochi [22], Barbu [14] and Colli [18]) made attempts to establish abstract theories which enable us to treat them in V-V^* frameworks with reflexive Banach spaces V and their dual spaces V^* (see also Akagi-Ôtani [4, 5, 6], Akagi [3], Aso et al [10]). These contributions provide us more useful frameworks to handle PDEs with severe nonlinearities.

The author recently made an attempt to develop a new framework which can unify these branches in [1], where doubly nonlinear problems governed by (time-dependent) subdifferential operators with non-monotone perturbations in reflexive Banach spaces are treated. In this paper, we deal with an autonomous version of the problems treated in [1]. More precisely, let V and V^* be a reflexive Banach space and its dual space, respectively, and let H be a Hilbert space whose dual space H^* is identified with itself such that

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*$$

with continuous and densely defined canonical injections. Let $\partial_V \psi, \partial_V \varphi : V \to 2^{V^*}$ stand for the subdifferential operators of proper, lower semicontinuous and convex functions ψ

and φ, respectively, from V into $(-\infty, +\infty]$. Moreover, let B be a (possibly) multi-valued mapping from V into V^* such that B may be non-monotone in $V \times V^*$. Then one can consider the following Cauchy problem:

$$\text{(CP)} \quad \begin{cases} \partial_V \psi(u'(t)) + \partial_V \varphi(u(t)) + B(u(t)) \ni f \quad \text{in } V^*, \quad 0 < t < T, \\ u(0) = u_0, \end{cases}$$

where $u'(t) = du(t)/dt$, and $f \in V^*$ and $u_0 \in V$ are given. In Sections 3 and 4, we briefly review the results of [1] on the existence of solutions for (CP).

The main purpose of this paper is to reveal the long-time behaviors of the global solutions of (CP), in particular, the existence of global attractors; however, since solutions of (CP) may not be unique, the usual semi-group approach to dynamical systems could be no longer valid. Therefore we employ the notion of generalized semiflow proposed by J.M. Ball [11] to investigate the asymptotic behavior of solutions for (CP).

Segatti in [32] studies a doubly nonlinear gradient system in the Hilbert space setting, i.e., $V = V^* = H$ and the perturbation term in (CP) has a special form, $B(u) = -\lambda u$ with $\lambda > 0$, when growth conditions of linear order are imposed on $\partial_H \psi$. He constructed global (in time) solutions and proved the existence of global attractors by using the notion of generalized semiflow and establishing a dissipative estimate for a Lyapunov functional $J(u) := \varphi(u) - \frac{\lambda}{2}|u|_H^2$. However, in our setting, the perturbation term has no explicit form, in particular, it might have no gradient structure. Moreover, since we work in a Banach space setting, there also arises some technical difficulties. These difficulties will be solved and the existence of a global attractor for a generalized semiflow generated by (CP) will be demonstrated in §5.

Notation. We denote by C a non-negative constant, which does not depend on the elements of the corresponding space or set and may vary from line to line. Let I be a section in $\mathbb{R}$ and let E be a set. Then $AC(I; E)$ (respectively, $AC(I)$) stands for the set of all E-valued (respectively, real-valued) absolutely continuous functions defined on I. Moreover, the set of all proper (i.e., $\phi \not\equiv +\infty$), lower semicontinuous and convex functions ϕ from E into $(-\infty, +\infty]$ is denoted by $\Phi(E)$.

2 Abstract setting and basic assumptions

Let V and V^* be a real reflexive Banach space and its dual space, respectively, and let H be a real Hilbert space whose dual space H^* is identified with itself such that

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^* \tag{2. 1}$$

with continuous and densely defined canonical injections. Let $\psi, \varphi \in \Phi(V)$ and let $\partial_V \psi$ and $\partial_V \varphi$ be the subdifferential operators of ψ and φ respectively. Moreover, let B be a (possibly non-monotone) mapping from V into 2^{V^*}. We consider the following Cauchy problem.

$$\text{(CP)} \quad \begin{cases} \partial_V \psi(u'(t)) + \partial_V \varphi(u(t)) + B(u(t)) \ni f \quad \text{in } V^*, \quad 0 < t < +\infty, \\ u(0) = u_0, \end{cases}$$

where $f \in V^*$ and $u_0 \in V$ are given data. Here and henceforth, we are concerned with strong solutions of (CP) defined as follows:

Definition 2.1. *For $T \in (0, \infty)$, a function $u \in AC([0,T];V)$ is said to be a strong solution of* (CP) *on $[0,T]$, if the following conditions are satisfied*:

(i) $u(0) = u_0$;

(ii) *there exists a negligible set $N \subset (0,T)$, i.e., the Lebesgue measure of N is zero, such that $u(t) \in D(\partial_V \varphi)$ and $u'(t) \in D(\partial_V \psi)$ for all $t \in [0,T] \setminus N$, and moreover, there exist sections $\eta(t) \in \partial_V \psi(u'(t))$, $\xi(t) \in \partial_V \varphi(u(t))$ and $g(t) \in B(u(t))$ such that*

$$\eta(t) + \xi(t) + \lambda g(t) = f \quad \text{in } V^* \text{ for all } t \in [0,T] \setminus N, \tag{2. 2}$$

(iii) *$u(t) \in D(\varphi)$ for all $t \in [0,T]$, and the function $t \mapsto \varphi(u(t))$ is absolutely continuous on $[0,T]$.*

For $T \in (0, \infty]$, a function $u \in AC([0,T);V)$ is said to be a strong solution of (CP) *on $[0,T)$, if u is a strong solution of* (CP) *on $[0,S]$ for every $S \in (0,T)$.*

Let us introduce basic assumptions on ψ, φ and B with parameters $p \in (1, +\infty)$ and $T > 0$.

(A1) There exist constants $C_1 > 0$ and $C_2 \geq 0$ such that

$$C_1 |u|_V^p \leq \psi(u) + C_2 \quad \text{for all } u \in D(\psi).$$

(A2) There exist constants $C_3, C_4 \geq 0$ such that

$$|\eta|_{V^*}^{p'} \leq C_3 \psi(u) + C_4 \quad \text{for all } [u, \eta] \in \partial_V \psi.$$

Here we give the following proposition for later use (see §5).

Proposition 2.2. *Suppose that* (A2) *is satisfied. Then there exist constants $C_5 > 0$, $C_6 \geq 0$ such that*

$$C_5 \psi(u) \leq \langle \eta, u \rangle + C_6 \quad \textit{for all } \ [u, \eta] \in \partial_V \psi.$$

As for φ, we employ the following compactness condition.

(Φ1) There exist a reflexive Banach space X and a non-decreasing function ℓ_1 on $[0, +\infty)$ such that X is compactly embedded in V and

$$|u|_X \leq \ell_1([\varphi(u)]_+ + |u|_H) \quad \text{for all } u \in D(\partial_V \varphi),$$

where $[s]_+ := \max\{s, 0\} \geq 0$ for $s \in \mathbb{R}$.

Concerning the non-monotone mapping B, we impose the following assumptions.

(B1) $D(\partial_V \varphi) \subset D(B)$. There exists a non-decreasing function ℓ_2 on $[0, +\infty)$ satisfying the following: for all $\varepsilon > 0$, there exists a constant $C_\varepsilon \geq 0$ such that

$$|g|_{V^*}^{p'} \leq \varepsilon |\xi|_{V^*}^{\sigma} + C_\varepsilon \ell_2(|\varphi(u)| + |u|_V), \quad \sigma := \min\{2, p'\}$$

for all $u \in D(\partial_V \varphi)$, $g \in B(u)$ and $\xi \in \partial_V \varphi(u)$.

(B2) For $S \in (0, T]$, let $\{u_n\}$ and $\{\xi_n\}$ be sequences in $C([0, S]; V)$ and $L^\sigma(0, S; V^*)$ with $\sigma := \min\{2, p'\}$, respectively, such that

$$u_n \to u \text{ strongly in } C([0, S]; V), \quad [u_n(t), \xi_n(t)] \in \partial_V \varphi \text{ for a.e. } t \in (0, S),$$
$$\sup_{t \in [0,S]} |\varphi(u_n(t))| + \int_0^S |u_n'(t)|_H^p dt + \int_0^S |\xi_n(t)|_{V^*}^\sigma dt \text{ is bounded for all } n \in \mathbb{N}.$$

Moreover, let $\{g_n\}$ be a sequence in $L^{p'}(0, S; V^*)$ such that

$$g_n(t) \in B(u_n(t)) \text{ for a.e. } t \in (0, S), \quad g_n \to g \text{ weakly in } L^{p'}(0, S; V^*).$$

Then $\{g_n\}$ is precompact in $L^{p'}(0, S; V^*)$ and $g(t) \in B(u(t))$ for a.e. $t \in (0, S)$.

(B3) For $S \in (0, T]$, let $u \in C([0, S]; V) \cap W^{1,p}(0, S; H)$ be such that

$$\sup_{t \in [0,S]} |\varphi(u(t))| < +\infty.$$

Suppose that there exists $\xi \in L^{p'}(0, S; V^*)$ such that $\xi(t) \in \partial_V \varphi(u(t))$ for a.e. $t \in (0, S)$. Then there exists a V^*-valued strongly measurable function g such that $g(t) \in B(u(t))$ for a.e. $t \in (0, S)$. Moreover, the set $B(u)$ is convex for all $u \in D(B)$.

Remark 2.3. We can assume that $\psi \geq 0$ and $\varphi \geq 0$ without any loss of generality. Indeed, putting $\hat{\psi} := \psi + C_2$ and using (A1), we find that $\hat{\psi} \geq 0$, $D(\hat{\psi}) = D(\psi)$, $D(\partial_V \hat{\psi}) = D(\partial_V \psi)$ and $\partial_V \hat{\psi} = \partial_V \psi$. As for φ, from the fact that $\varphi \in \Phi(V)$ and (Φ1), the extension $\tilde{\varphi}$ of φ onto H defined by

$$\tilde{\varphi}(u) := \begin{cases} \varphi(u) & \text{if } u \in V, \\ +\infty & \text{if } u \in H \setminus V \end{cases}$$

belongs to $\Phi(H)$. Hence there exist $u^* \in H$ and $\mu \in \mathbb{R}$ such that $\tilde{\varphi}(u) \geq (u^*, u)_H + \mu$ for all $u \in H$ (see, e.g., Proposition 2.1 of [13, p. 51]). Thus we have $\hat{\varphi}(u) := \varphi(u) - (u^*, u)_H - \mu \geq 0$ for all $u \in V$, and moreover, it holds that $D(\hat{\varphi}) = D(\varphi)$, $D(\partial_V \hat{\varphi}) = D(\partial_V \varphi)$ and $\partial_V \hat{\varphi} = \partial_V \varphi - u^*$. Therefore the evolution equation in (CP) is equivalent to the following:

$$\partial_V \hat{\psi}(u'(t)) + \partial_V \hat{\varphi}(u(t)) + B(u(t)) \ni \hat{f} := f - u^*.$$

Moreover, (A1), (A2), (Φ1) and (B1)-(B3) are all satisfied with ψ and φ replaced by $\hat{\psi}$ and $\hat{\varphi}$ respectively.

3 Local existence of solutions

The existence of local (in time) solutions for (CP) has already been proved under the preceding assumptions by the author in [1], where (CP) is treated in a more general setting; more precisely, ψ, B and f may explicitly depend on t.

Theorem 3.1 (Local existence, [1]). *Let $p \in (1, +\infty)$ and $T > 0$ be given. Suppose that* (A1), (A2), (Φ1), (B1)–(B3) *are all satisfied. Then, for all $f \in V^*$ and $u_0 \in D(\varphi)$, there exists $T_* = T_*(\varphi(u_0) + |u_0|_H + |f|_{V^*}) \in (0, T]$ such that* (CP) *admits at least one strong solution $u \in W^{1,p}(0, T_*; V)$ on $[0, T_*]$ satisfying*

$$\eta, \xi, g \in L^{p'}(0, T_*; V^*), \quad \varphi(u(\cdot)) \in W^{1,1}(0, T_*),$$

where $\eta(t)$, $\xi(t)$ and $g(t)$ denote the sections of $\partial_V \psi(u'(t))$, $\partial_V \varphi(u(t))$ and $B(u(t))$, respectively, as in (2. 2) *for a.e. $t \in (0, T_*)$.*

Let us show the outline of proof for this theorem (see [1] for more detail).

Phase 1. We introduce the following approximate problems of (CP) for all $\lambda \in [0, 1]$:

$$(\mathrm{CP})_\lambda \quad \begin{cases} \lambda u'(t) + \partial_V \psi(u'(t)) + \partial_H \tilde{\varphi}_\lambda(u(t)) + B(J_\lambda u(t)) \ni f & \text{in } V^*, \\ u(0) = u_0, \end{cases}$$

where $\tilde{\varphi}$ is the extension of φ onto H and J_λ and $\partial_H \tilde{\varphi}_\lambda$ denote the resolvent and the Yosida approximation of $\partial_H \tilde{\varphi}$ respectively. To construct strong solutions for $(\mathrm{CP})_\lambda$ on $[0, T_*]$ with some $T_* \in (0, T]$ independent of $\lambda \in (0, 1]$, we prepare the following two steps.

Phase 1, Step 1. We first prove the existence and uniqueness of solutions for the following unperturbed problems for a given function $g \in L^{p'}(0, T; V^*)$ (see also Remark 3.2).

$$(\mathrm{CP})_{\lambda,g} \quad \begin{cases} \lambda u'(t) + \partial_V \psi(u'(t)) + \partial_H \tilde{\varphi}_\lambda(u(t)) + g(t) \ni f & \text{in } V^*, \\ u(0) = u_0. \end{cases}$$

Phase 1, Step 2. We next define the mapping $\mathcal{F}_S$ on $L^{p'}(0, S; V^*)$ by

$$\mathcal{F}_S : g \mapsto B(J_\lambda u(\cdot)), \qquad \text{where } u \text{ is a solution of } (\mathrm{CP})_{\lambda,g} \text{ on } [0, S].$$

Then applying Kakutani-Ky Fan's fixed point theorem for multi-valued mappings (see Corollary 2 to Theorem (6. 3) of [17, p. 75]), we find a fixed point $g_\lambda \in L^{p'}(0, T_*; V^*)$ of $\mathcal{F}_{T_*}$ with some T_* independent of $\lambda \in (0, 1]$. Let u_λ be a strong solution of $(\mathrm{CP})_{\lambda,g}$ with $g = g_\lambda$ on $[0, T_*]$. Then u_λ solves $(\mathrm{CP})_\lambda$ on $[0, T_*]$.

Phase 2. Establishing a priori estimates and deriving the convergences for u_λ as $\lambda \to +0$, we can obtain local (in time) solutions for (CP).

Remark 3.2. For the case where $V = V^* = H$ is a Hilbert space, one can easily prove the uniqueness of strong solutions for $(\mathrm{CP})_{\lambda,g}$. Indeed, $(\mathrm{CP})_{\lambda,g}$ can be rewritten into

$$u'(t) = (\lambda I + \partial_H \psi)^{-1} (f - g(t) - \partial_H \tilde{\varphi}_\lambda(u(t))) \quad \text{in } H, \quad 0 < t < T,$$

and we observe that the mapping $u \mapsto (\lambda I + \partial_H \psi)^{-1} (f - g(t) - \partial_H \tilde{\varphi}_\lambda(u))$ becomes Lipschitz continuous in H for every $t \in [0,T]$. Hence the uniqueness of strong solutions follows immediately. However, for the case where V is not a Hilbert space, the mapping $(\lambda I + \partial_V \psi)^{-1} : V^* \to V$ is no longer Lipschitz continuous. In [1], the uniqueness of solutions for $(\mathrm{CP})_{\lambda,g}$ is proved by using the Lipschitz continuity of $\partial_H \tilde{\varphi}_\lambda$ only and Gronwall's inequality.

4 Global existence of solutions

As for the global (in time) existence, [1] also presents the following two theorems.

Theorem 4.1 (Global existence, [1]). *Let $p \in (1,+\infty)$ and $T > 0$ be fixed. Suppose that* (A1), (A2), (Φ1), (B2), (B3) *and the following* (B4) *are satisfied.*

(B4) *$D(\partial_V \varphi) \subset D(B)$. For all $\varepsilon > 0$, there exists a constant $C_\varepsilon \geq 0$ such that*

$$|g|_{V^*}^{p'} \leq \varepsilon |\xi|_{V^*}^{\sigma} + C_\varepsilon \left\{ |\varphi(u)| + |u|_V^p + 1 \right\}, \quad \sigma := \min\{2, p'\}$$

for all $u \in D(\partial_V \varphi)$, $g \in B(u)$ and $\xi \in \partial_V \varphi(u)$.

Then, for all $f \in V^*$ and $u_0 \in D(\varphi)$, there exists a strong solution $u \in W^{1,p}(0,T;V)$ of (CP) *on $[0,T]$ such that*

$$\eta, \xi, g \in L^{p'}(0,T;V^*), \quad \varphi(u(\cdot)) \in W^{1,1}(0,T), \tag{4. 1}$$

where $\eta(t)$, $\xi(t)$ and $g(t)$ denote the sections of $\partial_V \psi(u'(t))$, $\partial_V \varphi(u(t))$ and $B(u(t))$, respectively, as in (2. 2) *for a.e. $t \in (0,T)$.*

As in §6.1 of [1], we can prove the following proposition, which will be used in §5.

Proposition 4.2. *Let $p \in (1,+\infty)$ and $T > 0$ be given. Assume that* (B4) *holds. Let u be a strong solution of* (CP) *on $[0,T]$. Then for any $\gamma > 0$, there exist a constant $C_\gamma \geq 0$ independent of u, T, u_0 and f such that*

$$|g(t)|_{V^*}^{p'} \leq C_\gamma \left(|f|_{V^*}^{p'} + 1 \right) + C_\gamma \left\{ |\varphi(u(t))| + |u(t)|_V^p \right\} + \gamma \psi\left(u'(t)\right),$$

where $g(t)$ denotes the section of $B(u(t))$ as in (2. 2)*, for a.e. $t \in (0,T)$.*

Furthermore, in the next theorem, the global existence is assured for small data u_0 and f in a proper sense by imposing the following (B5) instead of (B4).

(B5) There exist a positive constant C_7 and non-decreasing functions ℓ_i ($i = 3,4,5$) on $[0,+\infty)$ such that $\lim_{s \to +0} \ell_i(s) = 0$ and

$$C_7 \varphi(u) \leq \langle \xi + g, u \rangle + \ell_3(\varphi(u))\varphi(u), \tag{4. 2}$$

$$|u|_V^p \leq \ell_4(\varphi(u))\varphi(u), \tag{4. 3}$$

for all $u \in D(\partial_V \varphi)$, $\xi \in \partial_V \varphi(u)$, $g \in B(u)$, and moreover, for all $\varepsilon > 0$, there exists a constant $C_\varepsilon \geq 0$ such that

$$|g|_{V^*}^{p'} \leq \varepsilon |\xi|_{V^*}^{p'} + C_\varepsilon \ell_5(\varphi(u))\varphi(u) \tag{4. 4}$$

for all $u \in D(\partial_V \varphi)$, $\xi \in \partial_V \varphi(u)$, $g \in B(u)$.

Theorem 4.3 (Global existence for small data)**.** *Let $p \in (1, +\infty)$ and $T > 0$ be fixed. Suppose that* (A1), (A2), (Φ1), (B1)–(B3) *and* (B5) *are all satisfied with $C_2 = C_4 = 0$ and $\psi(0) \equiv 0$. Then there exists $\delta > 0$ independent of T such that if $|f|_{V^*} + \varphi(u_0) < \delta$, then* (CP) *admits a strong solution $u \in W^{1,p}(0, T; V^*)$ on $[0, T]$ such that* (4. 1) *holds true.*

5 Long-time behavior of solutions

As we mentioned in §1, solutions of (CP) may not be unique. Hence the usual approach to dynamical systems based on the semigroup theory is not valid for our problem. Hence we exploit the theory of generalized semiflow proposed by J.M. Ball [11], which can be adapted to dynamical systems generated by equations whose solutions are not unique, instead of the usual one. Our strategy of proof is based on that of [32] and techniques developed in [1] for (CP).

5.1 Theory of generalized semiflow

The notion of generalized semiflow is first introduced by J.M. Ball [11]. He also extend the notion of global attractor to generalized semiflows and provide a criterion of the existence of global attractors. We first recall the definition of generalized semiflow.

Definition 5.1. *Let X be a metric space with metric $d_X = d_X(\cdot, \cdot)$. A family $\mathcal{G}$ of maps $\varphi : [0, +\infty) \to X$ is said to be a generalized semiflow in X, if the following four conditions are all satisfied*:

(H1) (Existence) *For each $x \in X$ there exists $\varphi \in \mathcal{G}$ such that $\varphi(0) = x$*;

(H2) (Translation invariance) *If $\varphi \in \mathcal{G}$ and $\tau \geq 0$, then the map φ^τ also belongs to $\mathcal{G}$, where $\varphi^\tau(t) := \varphi(t + \tau)$ for $t \in [0, +\infty)$*;

(H3) (Concatenation invariance) *If $\varphi_1, \varphi_2 \in \mathcal{G}$ and $\varphi_2(0) = \varphi_1(\tau)$ for some $\tau \geq 0$, then the map ψ, the concatenation of φ_1 and φ_2 at τ, defined by*

$$\psi(t) := \begin{cases} \varphi_1(t) & \text{if } t \in [0, \tau], \\ \varphi_2(t - \tau) & \text{if } t \in (\tau, +\infty) \end{cases}$$

also belongs to $\mathcal{G}$;

(H4) (Upper semicontinuity) *If $\varphi_n \in \mathcal{G}$, $x \in X$ and $\varphi_n(0) \to x$ in X, then there exist a subsequence $\{n'\}$ of $\{n\}$ and $\varphi \in \mathcal{G}$ such that $\varphi_{n'}(t) \to \varphi(t)$ for each $t \in [0, +\infty)$.*

Let $\mathcal{G}$ be a generalized semiflow in a metric space X. We define a mapping $T(t) : 2^X \to 2^X$ by

$$T(t)E := \{\varphi(t);\ \varphi \in \mathcal{G} \text{ and } \varphi(0) \in E\} \quad \text{for } E \subset X \tag{5. 1}$$

for each $t \geq 0$. One can check from (H1)–(H3) that $\{T(t)\}_{t \geq 0}$ satisfies the semi-group properties, that is, (i) $T(0)$ is the identity mapping in 2^X; (ii) $T(t)T(s) = T(t + s)$ for all $t, s \geq 0$.

Moreover, global attractors for generalized semiflows are defined as follows.

Definition 5.2. *Let $\mathcal{G}$ be a generalized semiflow in a metric space X and let $\{T(t)\}_{t\geq 0}$ be the family of mappings defined as in* (5. 1)*. A set $\mathcal{A} \subset X$ is said to be a global attractor for the generalized semiflow $\mathcal{G}$ if the following* (i)–(iii) *hold.*

(i) *$\mathcal{A}$ is compact in X;*

(ii) *$\mathcal{A}$ is invariant under $T(t)$, i.e., $T(t)\mathcal{A} = \mathcal{A}$, for all $t \geq 0$;*

(iii) *$\mathcal{A}$ attracts any bounded subsets B of X by $\{T(t)\}_{t\geq 0}$, i.e.,*

$$\lim_{t\to+\infty} \operatorname{dist}(T(t)B, \mathcal{A}) = 0,$$

where $\operatorname{dist}(\cdot,\cdot)$ *is defined by*

$$\operatorname{dist}(A, B) := \sup_{a\in A} \inf_{b\in B} d_X(a, b) \quad \text{for } A, B \subset X.$$

As in the standard theory of dynamical systems for (single-valued) semi-group operators, we can also introduce the notion of ω-limit set.

Definition 5.3. *Let $\mathcal{G}$ be a generalized semiflow in a metric space X. For $E \subset X$, the ω-limit set of E for $\mathcal{G}$ is given as follows.*

$$\omega(E) := \Big\{x \in X;\ \text{there exist sequences } \{\varphi_n\} \text{ in } \mathcal{G} \text{ and } \{t_n\} \text{ on } [0, +\infty) \\ \text{such that } \varphi_n(0) \text{ is bounded and belongs to } E \\ \text{for all } n \in \mathbb{N}, \quad t_n \to +\infty \text{ and } \varphi_n(t_n) \to x\Big\}.$$

In order to prove the existence of global attractors for generalized semiflows, we employ the following theorem due to J.M. Ball [11].

Theorem 5.4 (J.M. Ball [11])**.** *A generalized semiflow $\mathcal{G}$ in a metric space X has a global attractor $\mathcal{A}$ if and only if the following two conditions are satisfied.*

(i) *$\mathcal{G}$ is point dissipative, that is,*

$$\exists a \text{ bounded set } B \subset X,\ \forall\varphi \in \mathcal{G},\ \exists\tau = \tau(\varphi) \geq 0,\ \forall t \geq \tau, \quad \varphi(t) \in B;$$

(ii) *$\mathcal{G}$ is asymptotically compact, that is, for any sequences $\{\varphi_n\}$ in $\mathcal{G}$ and $\{t_n\}$ on $[0, +\infty)$, if $\{\varphi_n(0)\}$ is bounded in X and $t_n \to +\infty$, then $\{\varphi_n(t_n)\}$ is precompact in X.*

Moreover, $\mathcal{A}$ is a unique global attractor for $\mathcal{G}$ and given by

$$\mathcal{A} = \bigcup \{\omega(B);\ B \text{ is a bounded set in } X\} = \omega(X).$$

Furthermore, $\mathcal{A}$ is the maximal compact invariant subset of X under the family of mappings $\{T(t)\}_{t\geq 0}$.

The following proposition gives a sufficient condition for the asymptotic compactness of generalized semiflows.

Proposition 5.5 (J.M. Ball [11]). *Let $\mathcal{G}$ be a generalized semiflow in a metric space X. If $\mathcal{G}$ satisfies the following conditions*:

(i) *$\mathcal{G}$ is eventually bounded, that is, for any bounded set $D \subset X$, there exist $\tau = \tau(D) \geq 0$ and a bounded set $B = B(D) \subset X$ such that*

$$\bigcup_{t \geq \tau} T(t)D \subset B,$$

(ii) *$\mathcal{G}$ is compact, that is, for any sequence $\{u_n\}$ in $\mathcal{G}$, if $\{u_n(0)\}$ is bounded in X, then there exists a subsequence $\{n'\}$ of $\{n\}$ such that $\{u_{n'}(t)\}$ is convergent in X for each $t > 0$,*

then $\mathcal{G}$ is asymptotically compact.

5.2 Formation of a generalized semiflow

Our analysis of the large-time behavior of solutions for (CP) is based on the theory of generalized semiflow briefly reviewed in §5.1. In this subsection, we first define the set $\mathcal{G}$ and next prove that it forms a generalized semiflow in a metric space.

Let $X := D(\varphi)$ be a metric space equipped with the distance $d_X(\cdot, \cdot)$ defined by

$$d_X(u, v) := |u - v|_V + |\varphi(u) - \varphi(v)| \quad \text{for} \ \ u, v \in X$$

and define

$$\mathcal{G} := \{u \in AC([0, +\infty); X);\ u \text{ is a strong solution of (CP) on } [0, +\infty) \text{ with some } u_0 \in X\}.$$

Then we have:

Theorem 5.6. *Let $p \in (1, +\infty)$ be fixed. Assume that* (A1), (A2), (Φ1), (B2)–(B4) *are satisfied for any $T > 0$. Then $\mathcal{G}$ is a generalized semiflow in X.*

Proof. It suffices to check the four conditions (H1)–(H4) (see Definition 5.1). The assertion (H1) follows from Theorem 4.1. Moreover, it is easily seen that (H2) and (H3) are satisfied. Hence it remains to check (H4).

Let $u_n \in \mathcal{G}$ and $v \in X$ be such that $u_n(0) \to v$ in X. Then multiplying (CP) with $u = u_n$ by $u_n'(t)$, we have

$$\langle \eta_n(t), u_n'(t) \rangle + \langle \xi_n(t), u_n'(t) \rangle + \langle g_n(t), u_n'(t) \rangle = \langle f, u_n'(t) \rangle$$

with sections $\eta_n(t) \in \partial_V \psi(u_n'(t))$, $\xi_n(t) \in \partial_V \varphi(u_n(t))$, $g_n(t) \in B(u_n(t))$ for a.e. $t \in (0, +\infty)$. We then derive from (A2)$'$ and the chain rule for subdifferentials that

$$C_5 \psi\left(u_n'(t)\right) - C_6 + \frac{d}{dt} \varphi(u_n(t)) \leq \left(|f|_{V^*} + |g_n(t)|_{V^*}\right) |u_n'(t)|_V.$$

Furthermore, by Young's inequality and (A1),

$$\frac{3}{4}C_5\psi\left(u_n'(t)\right)+\frac{d}{dt}\varphi(u_n(t))\le C\left(|f|_{V^*}^{p'}+|g_n(t)|_{V^*}^{p'}+1\right). \tag{5. 2}$$

By Proposition 4.2 and (A1),

$$\begin{aligned}\frac{C_5}{4}\psi(u_n'(t))+\frac{C_5}{4}\left(C_1|u_n'(t)|_V^p-C_2\right)+\frac{d}{dt}\varphi(u_n(t))&\\ \le\ C\left(|f|_{V^*}^{p'}+1\right)+C\left\{\varphi(u_n(t))+|u_n(t)|_V^p\right\}.&\end{aligned} \tag{5. 3}$$

Thus

$$\begin{aligned}&\frac{C_5}{4}\psi(u_n'(t))+\frac{d}{dt}\left\{\varphi(u_n(t))+|u_n(t)|_V^p\right\}\\ &\quad\le\ C\left(|f|_{V^*}^{p'}+1\right)+C\left\{\varphi(u_n(t))+|u_n(t)|_V^p\right\}\end{aligned} \tag{5. 4}$$

for a.e. $t\in(0,+\infty)$. Hence integrating both sides over $(0,t)$ and applying Gronwall's inequality, we deduce that

$$\sup_{t\in[0,T]}\left\{\varphi(u_n(t))+|u_n(t)|_V^p\right\}\le\left\{\varphi(u_0)+|u_0|_V^p+CT\left(|f|_{V^*}^{p'}+1\right)\right\}e^{CT} \tag{5. 5}$$

with an arbitrary positive number $T>0$. Hence (Φ1) implies that $\{u_n(t)\}_{n\in\mathbb{N}}$ is precompact in V for each $t>0$. Furthermore, we can also deduce that

$$\int_0^T\psi(u_n'(t))dt\le C_T, \tag{5. 6}$$

$$\int_0^T|u_n'(t)|_V^p dt\le C_T \tag{5. 7}$$

with a constant $C_T\ge 0$. Here and henceforth, C_T denotes a non-negative constant independent of n and t but possibly depending on T and may vary from line to line. Performing a diagonal process, we can verify

$$u_{n_k}(t)\to u(t)\quad\text{strongly in } V\ \text{ for each }\ t\in[0,+\infty) \tag{5. 8}$$

with some $u\in C([0,+\infty);V)$ and a subsequence $\{n_k\}$ of $\{n\}$ (see [2] for more details).

We next prove that $u\in\mathcal{G}$, that is, u is a strong solution of (CP) on $[0,+\infty)$. Recall (5. 5)–(5. 7) with an arbitrary $T>0$ and use (A2), Proposition 4.2 and (2. 2) to get

$$\int_0^T|\eta_n(t)|_{V^*}^{p'}dt\le C_T, \tag{5. 9}$$

$$\int_0^T|g_n(t)|_{V^*}^{p'}dt\le C_T, \tag{5. 10}$$

$$\int_0^T|\xi_n(t)|_{V^*}^{p'}dt\le C_T, \tag{5. 11}$$

respectively, for all $n \in \mathbb{N}$. Hence we can take a subsequence of $\{n_k\}$, which is denoted by $\{n_k\}$ again, such that

$$
\begin{aligned}
&(5.\ 12) \qquad & u_{n_k} \to u \qquad & \text{weakly in } W^{1,p}(0,T;V),\\
&(5.\ 13) \qquad & & \text{strongly in } C([0,T];V),\\
&(5.\ 14) \qquad & \eta_{n_k} \to \eta \qquad & \text{weakly in } L^{p'}(0,T;V^*),\\
&(5.\ 15) \qquad & g_{n_k} \to g \qquad & \text{weakly in } L^{p'}(0,T;V^*),\\
&(5.\ 16) \qquad & \xi_{n_k} \to \xi \qquad & \text{weakly in } L^{p'}(0,T;V^*)
\end{aligned}
$$

with some $\eta, g, \xi \in L^{p'}(0,T;V^*)$. Therefore from the demiclosedness of subdifferential operators we can derive that $\xi(t) \in \partial_V \varphi(u(t))$ for a.e. $t \in (0,T)$, and moreover, (B2) implies that $g(t) \in B(u(t))$ for a.e. $t \in (0,T)$ and

$$(5.\ 17) \qquad g_{n_k} \to g \qquad \text{strongly in } L^{p'}(0,T;V^*).$$

Furthermore, multiply $\eta_{n_k}(t)$ by $u'_{n_k}(t)$ and integrate this over $(0,T)$. It then follows from (5. 12) and (5. 17) that

$$(5.\ 18) \qquad \limsup_{n_k \to +\infty} \int_0^T \left\langle \eta_{n_k}(t), u'_{n_k}(t) \right\rangle dt \le -\varphi(u(T)) + \varphi(v) + \int_0^T \langle f - g(t), u'(t) \rangle \, dt.$$

Here we also used the fact that $u_n(0) \to v$ in X, in particular, $\varphi(u_n(0)) \to \varphi(v)$. Thus by Proposition 2.1 of [1], we obtain $\eta(t) \in \partial_V \psi(u'(t))$ for a.e. $t \in (0,T)$, which implies that u is a strong solution of (CP) with $u_0 = v$ on $[0,T]$. From the arbitrariness of T, we can also verify that u becomes a strong solution of (CP) on $[0,+\infty)$. Hence u belongs to $\mathcal{G}$.

Finally, we show that $\varphi(u_{n_k}(t)) \to \varphi(u(t))$ for each $t \in [0,+\infty)$, by taking a subsequence of $\{n_k\}$ if necessary. To do so, we first derive a convergence of $\varphi(u_{n_k}(t))$ for a.e. $t \in (0,+\infty)$.

Lemma 5.7. *It follows that*

$$\liminf_{n_k \to +\infty} \varphi(u_{n_k}(t)) = \varphi(u(t)) \qquad \textit{for a.e.} \quad t \in (0,+\infty).$$

Proof. Let $T > 0$ be arbitrarily given. Then by (5. 11), Fatou's lemma ensures that

$$p(\cdot) := \liminf_{n \to +\infty} |\xi_n(\cdot)|_{V^*}^{p'} \in L^1(0,T).$$

Hence $p(t) < +\infty$ for a.e. $t \in (0,T)$. Thus we get, by (5. 8),

$$
\begin{aligned}
\liminf_{n_k \to +\infty} \varphi(u_{n_k}(t)) &\le \varphi(u(t)) + p(t) \left(\lim_{n_k \to +\infty} |u_{n_k}(t) - u(t)|_V \right)\\
&= \varphi(u(t)) \qquad \text{for a.e. } t \in (0,T).
\end{aligned}
$$

Combining this fact with the lower semicontinuity of φ, we can prove this lemma. □

Continuation of proof of (H4). We next exhibit the convergence of $\varphi(u_{n_k}(t))$ at every $t \in [0, +\infty)$. Recalling (5. 3), we find

$$\frac{d}{dt}\zeta_n(t) \leq 0 \quad \text{for a.e. } t \in (0, +\infty),$$

where ζ_n is an absolutely continuous function from $[0, +\infty)$ into $\mathbb{R}$ given by

$$\zeta_n(t) := \varphi(u_n(t)) - Ct\left(|f|_{V^*}^{p'} + 1\right) - M_2 \int_0^t \left\{\varphi(u_n(\tau)) + |u_n(\tau)|_V^p\right\} d\tau$$

for $t \in [0, +\infty)$. Hence ζ_n is non-increasing on $[0, +\infty)$. Applying Helly's lemma (see Lemma 3.3.3 of [7]) and a diagonal process to our situation (see [2] for more details), we have

$$\lim_{n_k \to +\infty} \zeta_{n_k}(t) = \phi(t) \quad \text{for all } \ t \in [0, +\infty). \tag{5. 19}$$

It remains only to reveal the representation of ϕ.

Lemma 5.8. *For each $t \in [0, +\infty)$, it follows that*

$$\phi(t) = \varphi(u(t)) - Ct\left(|f|_{V^*}^{p'} + 1\right) - M_2 \int_0^t \left\{\varphi(u(\tau)) + |u(\tau)|_V^p\right\} d\tau.$$

Proof. Let $T > 0$ be fixed. We can then derive that

$$\int_0^t \varphi(u_{n_k}(\tau))d\tau \to \int_0^t \varphi(u(\tau))d\tau \quad \text{for all } \ t \in [0, T]$$

from the definition of subdifferential together with (5. 13) and (5. 16). Hence the definition of ζ_n and Lemma 5.7 yield

$$\liminf_{n_k \to +\infty} \zeta_{n_k}(t) = \zeta(t) \quad \text{for a.e. } \ t \in (0, T),$$

where $\zeta \in AC([0, +\infty))$ is given by

$$\zeta(t) := \varphi(u(t)) - Ct\left(|f|_{V^*}^{p'} + 1\right) - M_2 \int_0^t \left\{\varphi(u(\tau)) + |u(\tau)|_V^p\right\} d\tau.$$

Therefore we can obtain $\phi(t) = \zeta(t)$ for a.e. $t \in (0, +\infty)$ from the arbitrariness of $T > 0$. Thus since ζ is continuous and ϕ is non-increasing on $[0, \infty)$, we can verify

$$\phi(t) = \zeta(t) \quad \text{for all } \ t \in [0, +\infty)$$

by (5. 19) (see [2] for more details). □

Continuation of Proof of (H4). From Lemma 5.8, we conclude that $u_{n_k}(t) \to u(t)$ in X for each $t \in [0, +\infty)$. Thus (H4) follows. □

5.3 Existence of global attractors

This subsection is devoted to proving the existence of global attractors for the generalized semiflow $\mathcal{G}$. To this end, we first introduce the following structure condition.

(S1) There exist constants $\alpha > 0$ and $C_8 \geq 0$ such that

$$\alpha \{\varphi(u) + |u|_V^p\} \leq \langle \xi + g, u \rangle + C_8$$

for all $u \in D(\partial_V \varphi) \cap D(B)$, $\xi \in \partial_V \varphi(u)$ and $g \in B(u)$.

Furthermore, we also introduce the following assumption on the growth order of B, which is more restrictive than (B4).

(B6) $D(\partial_V \varphi) \subset D(B)$. There exists a number $\gamma \in (0,1)$ satisfying that: for all $\varepsilon > 0$, there exists a constant $C_\varepsilon \geq 0$ such that

$$|g|_{V^*}^{p'} \leq \varepsilon |\xi|_{V^*}^{\sigma} + C_\varepsilon \{|\varphi(u)| + |u|_V^p + 1\}^\gamma, \quad \sigma := \min\{2, p'\}$$

for all $u \in D(\partial_V \varphi)$, $g \in B(u)$ and $\xi \in \partial_V \varphi(u)$.

Theorem 5.9. *Let $p \in (1, +\infty)$ be fixed and assume that* (A1), (A2), (Φ1), (B2), (B3), (B6) *and* (S1) *are satisfied for any $T > 0$. Then the generalized semiflow $\mathcal{G}$ has a unique global attractor $\mathcal{A}$, which is given by*

$$\mathcal{A} := \bigcup \{\omega(B); B \textit{ is a bounded subset of } X\} = \omega(X).$$

Furthermore, $\mathcal{A}$ is the maximal compact invariant subset of X.

In order to prove Theorem 5.9, we prepare a couple of lemmas.

Lemma 5.10. *Under the same assumptions as in Theorem* 5.9, *there exist a constant $R \geq 0$ and an increasing function $T_0(\cdot)$ on $[0, +\infty)$ such that*

$$\varphi(u(t)) + |u(t)|_V^p \leq R \qquad \textit{for all } u_0 \in X,\ u \in \mathcal{G} \textit{ satisfying } u(0) = u_0 \textit{ and } t \geq T_0(\varphi(u_0) + |u_0|_V^p). \tag{5. 20}$$

Proof. Let $u_0 \in X$ and let $u \in \mathcal{G}$ be such that $u(0) = u_0$. By (S1), we find that

$$\alpha \{\varphi(u(t)) + |u(t)|_V^p\} \leq C \left(|f|_{V^*}^{p'} + |\eta(t)|_{V^*}^{p'} + 1 \right) + \frac{\alpha}{2} |u(t)|_V^p$$

with sections $\xi(t) \in \partial_V \varphi(u(t))$, $g(t) \in B(u(t))$, $\eta(t) \in \partial_V \psi(u'(t))$ for a.e. $t \in (0, +\infty)$. Thus by (A2) we can take a constant $c_0 \geq 0$ such that

$$\frac{\alpha}{2} \{\varphi(u(t)) + |u(t)|_V^p\} \leq C \left(|f|_{V^*}^{p'} + 1 \right) + c_0 \psi(u'(t)) \tag{5. 21}$$

for a.e. $t \in (0, +\infty)$. As in (5. 4), taking an enough small number $\sigma > 0$, we can deduce from (B6) that

$$\begin{aligned} &\frac{C_5}{4} \psi(u'(t)) + \frac{d}{dt} \{\varphi(u(t)) + \sigma |u(t)|_V^p\} \\ &\quad \leq \ C \left(|f|_{V^*}^{p'} + 1 \right) + C \{\varphi(u(t)) + |u(t)|_V^p\}^\gamma \end{aligned} \tag{5. 22}$$

for a.e. $t \in (0, +\infty)$ with the constant $\gamma \in (0, 1)$ of (B6).

Now, set $\phi(t) := \varphi(u(t)) + \sigma|u(t)|_V^p$. Multiplying (5. 21) by an enough small number and adding this to (5. 22), we can derive

$$\frac{d\phi}{dt}(t) + \beta\phi(t) \leq F := C\left(|f|_{V^*}^{p'} + 1\right) \quad \text{for a.e.} \quad t \in (0, +\infty)$$

with a positive number β (see [2] for more details). By standard techniques for differential inequalities, we have

$$\phi(t) \leq \frac{F}{\beta} + \phi(0)e^{-\beta t} \quad \text{for all} \quad t \in [0, +\infty),$$

in particular,

$$\phi(t) \leq \frac{F}{\beta} + 1 \quad \text{for all} \quad t \geq \log(\phi(0) + 1)/\beta.$$

Thus, by putting $R := F/(\sigma\beta) + 1/\sigma$ and $T_0(\cdot) := \log(\cdot + 1)/\beta$, we obtain (5. 20). □

Hence we obtain the following lemma.

Lemma 5.11. *Under the same assumptions as in Theorem* 5.9*, the following* (i) *and* (ii) *are satisfied.*

(i) *$\mathcal{G}$ is point dissipative.*

(ii) *$\mathcal{G}$ is eventually bounded.*

Proof. Let $R \geq 0$ and $T_0(\cdot)$ be the constant and the increasing function given by Lemma 5.10 respectively. Moreover, we write

$$B_r := \{v \in X;\ \varphi(v) + |v|_V^p \leq r\} \quad \text{for} \quad r > 0$$

in the following.
Proof of (i). Put $B := B_R$. Let $u \in \mathcal{G}$ and set $\tau := T_0(\varphi(u(0)) + |u(0)|_V^p)$. Then by Lemma 5.10, we can deduce that $u(t) \in B$ for all $t \geq \tau$.
Proof of (ii). Let D be a bounded set in X. Then we can take $R_1 \in (0, +\infty)$ such that $D \subset B_{R_1}$. Moreover, put $\tau := T_0(R_1)$ and $B := B_R$. Then by Lemma 5.10, for any $u \in \mathcal{G}$ with $u(0) \in D \subset B_{R_1}$, it follows that $u(t) \in B$ for all $t \geq \tau$. □

Concerning the compactness of $\mathcal{G}$, we have:

Lemma 5.12. *Under the same assumptions as in Theorem* 5.9*, $\mathcal{G}$ is compact.*

Proof. Since $u_n(0)$ is bounded in X, i.e., $|u_n(0)|_V + \varphi(u_n(0)) \leq C$ with some constant C independent of n, the estimates (5. 5)–(5. 7), (5. 9)–(5. 11) and the convergences (5. 8), (5. 12)–(5. 17) are established with an arbitrary $T > 0$ as in the proof of (H4) (see the proof of Theorem 5.6). Moreover, we can also verify that there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\varphi(u_{n_k}(t)) \to \varphi(u(t)) \quad \text{for all} \quad t \in (0, +\infty). \tag{5. 23}$$

Thus $u_{n_k}(t) \to u(t)$ in X for each positive t. This completes our proof. □

Finally, these lemmas prove Theorem 5.9 immediately.

Proof of Theorem 5.9. Thanks to Theorem 5.4 and Lemmas 5.11 and 5.12, we obtain our desired conclusion. □

We close this paper with the following final remark.

Remark 5.13. (1) One can also discuss the existence of global attractors for

$$\partial_V \psi(u'(t)) + \partial_V \varphi(u(t)) + \lambda B(u(t)) \ni f \ \text{ in } V^*, \quad 0 < t < T$$

with an enough small $\lambda \geq 0$, even if (B4) is satisfied instead of (B6).

(2) We can apply the preceding abstract theory to the initial-boundary value problem for doubly nonlinear parabolic equations such as

$$\alpha(u_t) - \Delta_m u + g(u) = f \quad \text{and} \quad \alpha(u_t) - \Delta u + h(u, \nabla u) = f,$$

where α is a maximal monotone operator in $\mathbb{R}$ satisfying a growth condition of order p, and g (respectively, h) is a continuous function from $\mathbb{R}$ (respectively, $\mathbb{R} \times \mathbb{R}^N$) into $\mathbb{R}$ satisfying appropriate growth conditions, to investigate the existence of solutions and their asymptotic behaviors, in particular, the existence of global attractors. Generalized forms of the Allen-Cahn equation particularly fall within our framework. Moreover, we emphasize that our framework can cover the case where h depends on the gradient of u.

These generalization and application of our work reported in the current paper have been discussed in [2] with more details of the arguments in §5.

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