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Subdifferential approach to degenerate parabolic equations

Dedicated to Professor Nobuyuki Kenmochi on the Occasion of His 60th Birthday

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Abstract. A new framework is proposed to deal with degenerate parabolic equations such as $u_t(x,t) - \Delta_p u(x,t) - |u|^{q-2}u(x,t) = f(x,t)$, $x \in \Omega$, $t > 0$, where $1 < p, q < +\infty$ and Δ_p is the so-called p-Laplacian given by $\Delta_p u := \nabla \cdot (|\nabla u|^{p-2}\nabla u)$. Such a degenerate parabolic equation can be reduced to an abstract evolution equation governed by subdifferential operators in an appropriate reflexive Banach space. However, the most of studies on evolution equations governed by subdifferential operators have been done so far only in Hilbert space settings.

Let V and V^* be a reflexive Banach space and its dual space, respectively, and suppose that there exists a Hilbert space H such that $V \subset H \equiv H^* \subset V^*$ continuously and densely. In this paper, sufficient conditions for the existence of local or global (in time) solutions of Cauchy problems for evolution equations of the form: $du(t)/dt + \partial_V \varphi^1(u(t)) - \partial_V \varphi^2(u(t)) \ni f(t)$ in V^*, $0 < t < T$, where $\partial_V \varphi^i$ $(i = 1, 2)$ are subdifferential operators of proper lower semicontinuous convex functionals $\varphi^i : V \to (-\infty, +\infty]$, are provided for the case $u_0 \in D(\varphi^1)$ (resp. $u_0 \in \overline{D(\varphi^1)}^H$) by using the theory of subdifferential operators. Moreover, these results are also applied to the initial-boundary value problem for the degenerate parabolic equation described above, and in particular, if $p \leq q$ and $u_0 \in W_0^{1,p}(\Omega)$ (resp. $u_0 \in L^2(\Omega)$), then the initial-boundary value problem admits a time-local solution under $q < p^*$ (resp. $q < (N+2)p/2$), where p^* denotes the so-called Sobolev's critical exponent.

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1 Introduction

Subdifferential operator theory is often utilized for constructing a solution of degenerate parabolic equations, because it enables us to take account of energy structures of equations as well as to employ useful properties of maximal monotone operators. In particular, energy structures play an important role in studies on degenerate parabolic equations.

We introduce a new framework to deal with initial-boundary value problems of degenerate parabolic equations such as

$$\text{(NHE)} \quad \begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \Delta_p u(x,t) - |u|^{q-2}u(x,t) = f(x,t), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where Ω denotes a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$, $1 < q < +\infty$ and Δ_p is the so-called p-Laplacian given by

$$\Delta_p \phi(x) \ := \ \nabla \cdot (|\nabla \phi(x)|^{p-2} \nabla \phi(x)), \quad 1 < p < +\infty.$$

More precisely, we reduce (NHE) to Cauchy problem of an abstract evolution equation in the dual space V^* of a reflexive Banach space V of the form:

$$\text{(1. 1)} \qquad \frac{du}{dt}(t) + \partial_V \varphi^1(u(t)) - \partial_V \varphi^2(u(t)) \ni f(t) \ \text{ in } V^*, \quad 0 < t < T,$$

where $\partial_V \varphi^i : V \to 2^{V^*}$ $(i = 1,2)$ denote subdifferential operators of functionals $\varphi^i : V \to (-\infty, +\infty]$ and $f : (0,T) \to V^*$ is given, in Section 3; moreover, we provide sufficient conditions for the existence of solutions of Cauchy problem for (1. 1) in Section 4. To do this, we also recall various properties of subdifferential operators (see Sect. 2), which will be employed in major 3 steps, i.e., approximation of equations, establishing a priori estimates and convergence of approximate solutions, to construct a solution of (1. 1).

The existence of solutions for (NHE) has already been studied by several authors. Tsutsumi [12] provided sufficient conditions for the existence of local or global (in time) solutions for (NHE) by using Galerkin's method and energy method. On the other hand, Ôtani [9, 10] and Ishii [6] developed abstract theories of (1. 1) in the Hilbert space setting, where V must be a Hilbert space whose dual space is identified with V, and applied their abstract theories to (NHE). However, it has been an open problem for a long time whether there exists a time-local solution of (NHE) with $u_0 \in W_0^{1,p}(\Omega)$ under $q < p^*$, where p^* denotes the so-called Sobolev's critical exponent given by $p^* := Np/(N-p)$ if $p < N$; $p^* := +\infty$ if $p \geq N$, because of the restriction on the choice of base spaces in [9, 10] and [6] (see [2] for more details).

In Section 5, we apply our abstract theory developed in Section 4 to (NHE) for both cases: $u_0 \in W_0^{1,p}(\Omega)$ and $u_0 \in L^2(\Omega)$, and derive sufficient conditions for the existence of local or global (in time) solutions of (NHE). Particularly, if $p \leq q$, then we can assure that (NHE) with $u_0 \in W_0^{1,p}(\Omega)$ (resp. $u_0 \in L^2(\Omega)$) admits a local solution under $q < p^*$ (resp. $q < (N+2)p/N$).

2 Subdifferential Operators in Reflexive Banach Spaces

The theory of subdifferential operators has been developed by many mathematicians (see e.g. [4], [5], [3], [8]), and in this section, some of their results will be reviewed to be used later.

Let X be a reflexive Banach space and let $\Phi(X)$ denote the set of all lower semicontinuous convex functions ϕ from X into $(-\infty, +\infty]$ satisfying $\phi \not\equiv +\infty$. Then the subdifferential $\partial_X\phi(u)$ of $\phi \in \Phi(X)$ at u is defined by

$$\partial_X\phi(u) \;:=\; \{\xi \in X^*; \phi(v) - \phi(u) \geq \langle \xi, v-u\rangle_X \quad \forall v \in D(\phi)\},$$

where $\langle \cdot, \cdot\rangle_X$ denotes the natural duality between X and X^* and the effective domain $D(\phi)$ of ϕ is given by

$$D(\phi) \;:=\; \{u \in X; \phi(u) < +\infty\}.$$

Then the subdifferential operator $\partial_X\phi$ of ϕ can be defined by

$$\partial_X\phi : X \to 2^{X^*};\ u \mapsto \partial_X\phi(u)$$

with the domain $D(\partial_X\phi) := \{u \in D(\phi); \partial_X\phi(u) \neq \emptyset\}$. It is well known that every subdifferential operator forms a maximal monotone graph in $X \times X^*$.

In particular, if X is a Hilbert space H whose dual space H^* is identified with H, then the subdifferential $\partial_H\phi(u)$ of $\phi \in \Phi(H)$ at u can be written by

$$\partial_H\phi(u) \;=\; \{\xi \in H; \phi(v) - \phi(u) \geq (\xi, v-u)_H \quad \forall v \in D(\phi)\},$$

where $(\cdot, \cdot)_H$ denotes the inner product in H, and furthermore the subdifferential $\partial_H\phi$ also becomes a maximal monotone operator from H into itself. Hence we can define the resolvent J^ϕ_λ and the Yosida approximation $(\partial_H\phi)_\lambda$ of $\partial_H\phi$, which become Lipschitz continuous in H with Lipschitz constants 1 and $2/\lambda$, respectively. Moreover, the Moreau-Yosida regularization ϕ_λ of ϕ is defined by

$$\phi_\lambda(u) \;:=\; \inf_{v\in H}\left\{\frac{1}{2\lambda}|u-v|_H^2 + \phi(v)\right\} \quad \forall u \in H,$$

and the following proposition holds true:

Proposition 2.1 *Let $\phi \in \Phi(H)$. Then ϕ_λ is a Fréchet differentiable convex function from H into* **R**. *Moreover, it follows that*

$$\phi_\lambda(u) = \frac{1}{2\lambda}|u - J^\phi_\lambda u|_H^2 + \phi(J^\phi_\lambda u) = \frac{\lambda}{2}|(\partial_H\phi)_\lambda(u)|_H^2 + \phi(J^\phi_\lambda u).$$

Furthermore, the following (1)-(3) *hold.*

(1) $\partial_H(\phi_\lambda) = (\partial_H\phi)_\lambda$, *where $\partial_H(\phi_\lambda)$ is the subdifferential* (*Fréchet derivative*) *of ϕ_λ.*

(2) $\phi(J^\phi_\lambda u) \leq \phi_\lambda(u) \leq \phi(u)$ *for all $u \in H$ and $\lambda > 0$.*

(3) $\phi_\lambda(u) \to \phi(u)$ *as* $\lambda \to +0$ *for all* $u \in H$.

As for evolution problems generated by subdifferential operators, we often use the following chain rule for subdifferential operators.

Proposition 2.2 *Let* $\phi \in \Phi(X)$ *and let* $u \in W^{1,p}(0,T;X)$ *with* $p \in (1,+\infty)$. *Suppose that there exists* $g \in L^{p'}(0,T;X^*)$ *such that* $g(t) \in \partial_X \phi(u(t))$ *for a.e.* $t \in (0,T)$. *Then the function* $t \mapsto \phi(u(t))$ *is differentiable for a.e.* $t \in (0,T)$ *and the following holds true.*

$$\frac{d}{dt}\phi(u(t)) = \left\langle h(t), \frac{du}{dt}(t) \right\rangle_X \quad \forall h(t) \in \partial_X \phi(u(t)), \quad \textit{for a.e. } t \in (0,T).$$

Furthermore, for all $\phi \in \Phi(X)$, we can define the functional Ψ on $\mathcal{X} := L^p(0,T;X)$ with $p \in (1,+\infty)$:

$$\Psi(u) \;:=\; \begin{cases} \displaystyle\int_0^T \phi(u(t))dt & \text{if } \phi(u(\cdot)) \in L^1(0,T), \\ +\infty & \text{otherwise.} \end{cases}$$

Then we note that for all $u \in \mathcal{X}$ and $f \in \mathcal{X}^*$, it follows that $f \in \partial_{\mathcal{X}} \Psi(u)$ if and only if $f(x) \in \partial_X \phi(u(x))$ for a.e. $x \in \Omega$.

Finally, we recall the closedness of graphs of maximal monotone operators. Throughout this paper, we use the same letter A for the graph of A.

Proposition 2.3 *Let* A *be a maximal monotone operator from* X *into* X^* *and let* $[u_n, \xi_n] \in A$. *Moreover, suppose that*

$$u_n \to u \quad \textit{weakly in } X, \quad \xi_n \to \xi \quad \textit{weakly in } X^*, \quad \limsup_{n\to+\infty} \langle \xi_n, u_n \rangle_X \le \langle \xi, u \rangle_X.$$

Then $[u,\xi] \in A$ *and* $\langle \xi_n, u_n \rangle_X \to \langle \xi, u \rangle_X$.

3 Reduction of (NHE) to an Evolution Equation

In order to employ nice properties of subdifferential operators described in the last section and construct a solution of (NHE), we reduce (NHE) to an evolution equation governed by the difference of two subdifferential operators in an appropriate reflexive Banach space.

Now suppose that $2N/(N+2) \le p$ and $q \le p^*$. Then it is easily seen that $W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset W^{-1,p'}(\Omega)$, $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ with continuous and densely defined canonical injections.

Moreover, we define the functionals φ_p, $\psi_q : W_0^{1,p}(\Omega) \to [0,+\infty)$ in the following:

$$\varphi_p(u) := \frac{1}{p}\int_\Omega |\nabla u(x)|^p dx, \quad \psi_q(u) := \frac{1}{q}\int_\Omega |u(x)|^q dx \quad \forall u \in W_0^{1,p}(\Omega).$$

Then φ_p and ψ_q belong to $\Phi(W_0^{1,p}(\Omega))$, and furthermore $\partial_{W_0^{1,p}}\varphi_p(u)$ and $\partial_{W_0^{1,p}}\psi_q(u)$ coincide with $-\Delta_p u$ equipped with homogeneous Dirichlet boundary condition $u|_{\partial\Omega} = 0$ and $|u|^{q-2}u$, respectively, in the sense of distribution. Thus putting $u(t) := u(\cdot,t) \in W_0^{1,p}(\Omega)$, we can reduce (NHE) to the following Cauchy problem:

$$(\mathrm{CP})_{p,q} \quad \begin{cases} \dfrac{du}{dt}(t) + \partial_{W_0^{1,p}}\varphi_p(u(t)) - \partial_{W_0^{1,p}}\psi_q(u(t)) = f(t) \ \text{ in } W^{-1,p'}(\Omega), \quad 0 < t < T, \\ u(0) = u_0. \end{cases}$$

4 Abstract Theory

In this section, we establish an abstract theory on evolution equations governed by the difference of two subdifferential operators in reflexive Banach spaces to verify the existence of solutions for $(\mathrm{CP})_{p,q}$.

Let V and let V^* be a reflexive Banach space and its dual space, respectively, and suppose that there exists a Hilbert space H whose dual space H^* is identified with H such that $V \subset H \equiv H^* \subset V^*$ with continuous and densely defined canonical injections.

Now we consider the following Cauchy problem:

$$(\mathrm{CP}) \quad \begin{cases} \dfrac{du}{dt}(t) + \partial_V \varphi^1(u(t)) - \partial_V \varphi^2(u(t)) \ni f(t) \ \text{ in } V^*, \quad 0 < t < T, \\ u(0) = u_0, \end{cases}$$

where $\partial_V \varphi^i$ $(i = 1, 2)$ denote the subdifferential operators of $\varphi^i \in \Phi(V)$ and $f : (0, T) \to V^*$. Solutions of (CP) are defined in the following:

Definition 4.1 *A function $u \in C([0, S]; H)$ is said to be a strong solution of* (CP) *on $[0, S]$, if the following conditions are satisfied:*

(i) *$u(t)$ is a V^*-valued absolutely continuous function on $[0, S]$.*

(ii) *$u(t) \to u_0$ strongly in H as $t \to +0$.*

(iii) *$u(t) \in D(\partial_V \varphi^1) \cap D(\partial_V \varphi^2)$ for a.e. $t \in (0, S)$ and there exist sections $g^i(t) \in \partial_V \varphi^i(u(t))$ $(i = 1, 2)$ such that*

$$\frac{du}{dt}(t) + g^1(t) - g^2(t) = f(t) \text{ in } V^* \text{ for a.e. } t \in (0, S). \tag{4. 1}$$

Furthermore, a function $u \in C([0, S); H)$ is said to be a strong solution of (CP) *on $[0, S)$, if u is a strong solution of* (CP) *on $[0, \tau]$ for any $\tau < S$.*

Throughout the present paper, we denote by C a non-negative constant, which may vary from line to line, and $\mathcal{L}$ denotes the set of all non-decreasing functions from $[0, +\infty)$ into itself.

First, we treat the case where $u_0 \in D(\varphi^1)$. To state results on the existence of solutions for (CP), we introduce the following assumptions: Let $p \in (1, +\infty)$ be fixed.

(A1) $|u|_V^p - C_1|u|_H^2 - C_2 \le C_3\varphi^1(u)$ $\forall u \in D(\varphi^1)$ for some $C_1, C_2, C_3 \ge 0$.

(A2) $D(\varphi^1) \subset D(\partial_V \varphi^2)$. Furthermore, if $\{u_n\}$ is a sequence such that $\int_0^T |\varphi^1(u_n(t))|dt + \sup_{t\in[0,T]} |u_n(t)|_H + \int_0^T |du_n(t)/dt|_{V^*}dt$ is bounded, then for every $g_n(\cdot) \in \partial_V \varphi^2(u_n(\cdot))$, $\{g_n\}$ becomes a precompact subset in $L^{p'}(0, T; V^*)$.

(A3) There exists an extension $\tilde{\varphi}^2 \in \Phi(H)$ of φ^2, i.e., $\tilde{\varphi}^2(u) = \varphi^2(u)$ $\forall u \in V$, such that $\varphi^1(J_\lambda u) \le \ell_1(\varphi^1(u) + \ell_2(|u|_H))$ $\forall \lambda \in (0, 1]$, $\forall u \in D(\varphi^1)$, where $\ell_i \in \mathcal{L}$ $(i = 1, 2)$ and J_λ denotes the resolvent of $\partial_H \tilde{\varphi}^2$.

(A4) $\varphi^2(u) \le k\varphi^1(u) + C_4|u|_H^2 + C_5$ $\quad \forall u \in D(\varphi^1)$ for some $k \in [0, 1)$, $C_4, C_5 \ge 0$.

Theorem 4.2 (Akagi-Ôtani [2]) *Assume that* (A1), (A2), (A3) *and* (A4) *hold. Then for all* $u_0 \in D(\varphi^1)$ *and* $f \in W^{1,p'}(0,T;V^*)$, (CP) *has a strong solution* u *on* $[0,T]$ *satisfying*:

$$(4.\ 2) \qquad \begin{cases} u \in C_w([0,T];V) \cap W^{1,2}(0,T;H), \\ u(t) \in D(\partial_V \varphi^1) \cap D(\partial_V \varphi^2) \quad \text{for a.e. } t \in (0,T), \\ g^1 \in L^2(0,T;V^*), \quad g^2 \in C([0,T];V^*), \\ \sup_{t\in[0,T]} \varphi^1(u(t)) < +\infty, \quad \varphi^2(u(\cdot)) \in C([0,T]), \end{cases}$$

where g^i $(i = 1,2)$ *are the sections of* $\partial_V \varphi^i(u(\cdot))$ *satisfying* (4. 1) *and* $C_w([0,T];V)$ *denotes the set of all* V*-valued weakly continuous functions on* $[0,T]$.

In order to prove Theorem 4.2, we introduce the following approximate problems:

$$(\mathrm{CP})_\lambda \quad \begin{cases} \dfrac{du_\lambda}{dt}(t) + \partial_H \varphi^1_H(u_\lambda(t)) - \partial_H \tilde{\varphi}^2_\lambda(u_\lambda(t)) \ni f_\lambda(t) \ \text{ in } H, \quad 0 < t < T, \\ u_\lambda(0) = u_0, \end{cases}$$

where φ^1_H is the extension of φ^1 given by

$$\varphi^1(u) \ := \ \begin{cases} \varphi^1(u) & \text{if } \ u \in V, \\ +\infty & \text{if } \ u \in H \setminus V, \end{cases}$$

and $\tilde{\varphi}^2_\lambda$ denotes the Moreau-Yosida regularization of $\tilde{\varphi}^2$, and $f_\lambda \in C^1([0,T];H)$ satisfies $f_\lambda \to f$ strongly in $W^{1,p'}(0,T;V^*)$. By Proposition 2.1, $\partial_H \tilde{\varphi}^2_\lambda$ is Lipschitz continuous in H, so $(\mathrm{CP})_\lambda$ admits a unique strong solution u_λ on $[0,T]$. Moreover, multiplying $(\mathrm{CP})_\lambda$ by $du_\lambda(t)/dt$ and integrating this over $(0,t)$, by virtue of Proposition 2.2, we can deduce from (A4) that $\int_0^T |du_\lambda(t)/dt|^2_H \, dt + \sup_{t\in[0,T]} \varphi^1(u_\lambda(t)) \le C$, which together with (A1) and (A3) implies that u_λ and $J_\lambda u_\lambda$ are bounded in $L^\infty(0,T;V)$. Furthermore, we can derive the convergence of u_λ as $\lambda \to +0$; moreover, Proposition 2.3 ensures that the limit becomes a strong solution of (CP) on $[0,T]$, where we also used the fact that $\partial_H \tilde{\varphi}^2_\lambda(u_\lambda(t)) \in \partial_H \tilde{\varphi}^2(J_\lambda u_\lambda(t))$ (see [2] for more details).

Moreover, we can verify the existence of local (in time) solutions of (CP) without assuming (A4), which seems to be somewhat restrictive from the aspect of applications to (NHE).

Theorem 4.3 (Akagi-Ôtani [2]) *Assume that* (A1), (A2) *and* (A3) *hold. Then for all* $u_0 \in D(\varphi^1)$ *and* $f \in W^{1,p'}(0,T;V^*)$, *there exists a number* $T_0 \in (0,T]$ *such that* (CP) *has a strong solution* u *on* $[0,T_0]$ *satisfying* (4. 2) *with* T *replaced by* T_0.

As for the global (in time) existence, we introduce the following.

$$(\mathrm{A5}) \qquad \alpha\varphi^1(u) \le \langle \xi - \eta, u \rangle + \ell_3(\varphi^2(u)) \cdot \varphi^1(u) \quad \forall [u,\xi] \in \partial_V \varphi^1, \ \forall [u,\eta] \in \partial_V \varphi^2,$$

where $\alpha > 0$ and ℓ_3 denotes a non-decreasing continuous function from $[0,+\infty)$ to $\mathbf{R}$ satisfying $\ell_3(0) = 0$. Then we have:

Theorem 4.4 (Akagi-Ôtani [2]) *In addition to all the assumptions in Theorem* 4.3, *assume that* $C_1 = C_2 = 0$ *in* (A1), $\varphi^2 \geq 0$ *and* (A5) *is satisfied. Let* δ_0 *be a positive number such that* $\ell_3(\delta_0) < \alpha$. *Then, for all* $R > 0$, *there exists a positive number* δ_R *such that for all* $T > 0$ *and* (u_0, f) *belonging to*

$$\begin{aligned} X^T_{\delta_R,R} \;:=\; \Bigg\{ & (u_0, f) \in D(\varphi^1) \times W^{1,p'}(0,T;V^*); \\ & \varphi^1(u_0) + \int_0^T |f(\tau)|^{p'}_{V^*} d\tau + \int_0^T \left| \frac{df}{d\tau}(\tau) \right|^{p'}_{V^*} d\tau \leq R, \quad \varphi^2(u_0) < \delta_0, \\ & |u_0|_H + \left\{ \max\left(1, \frac{1}{T}\right) \left\| |f(\cdot)|^{p'}_{V^*} \right\|_{1,T} \right\}^{1/p} < \delta_R \Bigg\}, \end{aligned}$$

where

$$\left\| |f(\cdot)|^{p'}_{V^*} \right\|_{1,T} \;:=\; \begin{cases} \displaystyle\int_0^T |f(\tau)|^{p'}_{V^*} d\tau & \text{if } T < 1, \\ \displaystyle\sup_{t \in [1,T]} \int_{t-1}^t |f(\tau)|^{p'}_{V^*} d\tau & \text{if } T \geq 1, \end{cases}$$

(CP) *has a strong solution* u *on* $[0,T]$ *satisfying* (4. 2).

Secondly, we also deal with the case where $u_0 \in \overline{D(\varphi^1)}^H$. To this end, let us introduce the following.

(A6) There exists $\ell_4 \in \mathcal{L}$ such that
$$|\xi|^{p'}_{V^*} \leq \ell_4(|u|_H) \left\{ \varphi^1(u) + 1 \right\} \quad \forall [u, \eta] \in \partial_V \varphi^1.$$

(A7) For all $\varepsilon > 0$, there exists a constant $C_\varepsilon \geq 0$ such that
$$|\eta|^{p'}_{V^*} \leq \varepsilon \varphi^1(u) + C_\varepsilon \ell_5(|u|_H) \quad \forall [u, \eta] \in \partial_V \varphi^2, \text{ where } \ell_5 \in \mathcal{L}.$$

As for time-local existence, we have:

Theorem 4.5 *Suppose that* (A1), (A2), (A3), (A6) *and* (A7) *are satisfied. Moreover, assume that* $\partial_H \tilde{\varphi}^2(0) \ni 0$, *where* $\tilde{\varphi}^2$ *is given by* (A3). *Then for all* $u_0 \in \overline{D(\varphi^1)}^H$ *and* $f \in L^\rho(0,T;V^*)$ *with* $\rho > p'$, *there exists a number* $T_0 = T_0(|u_0|_H, \|f\|_{L^\rho(0,T;V^*)}) \in (0,T]$ *such that* (CP) *admits at least one strong solution* u *on* $[0, T_0]$ *satisfying*

$$\text{(4. 3)} \qquad \begin{cases} u \in L^p(0,T_0;V) \cap C([0,T_0];H) \cap W^{1,p'}(0,T_0;V^*), \\ g^1, g^2 \in L^{p'}(0,T_0;V^*), \quad \varphi^1(u(\cdot)), \varphi^2(u(\cdot)) \in L^1(0,T_0), \end{cases}$$

where g^i $(i = 1, 2)$ *are the sections of* $\partial_V \varphi^i(u(\cdot))$ *satisfying* (4. 1).

Proof of Theorem 4.5 Let us introduce the following Cauchy problems:

$$(\mathrm{CP})_{r,n} \quad \begin{cases} \dfrac{du_n}{dt}(t) + \partial_V \varphi^1_r(u_n(t)) - \partial_V \varphi^2(u_n(t)) \ni f_n(t) \text{ in } V^*, \quad 0 < t < T, \\ u_n(0) = u_{0,n}, \end{cases}$$

where $\varphi_r^1 : V \to (-\infty, +\infty]$ is defined as follows

$$\varphi_r^1(u) \quad := \quad \begin{cases} \varphi^1(u) & \text{if } |u|_H \le r, \\ +\infty & \text{otherwise} \end{cases}$$

for an enough large number $r \in \mathbf{R}$ satisfying $|u_0|_H < r$ and $D(\varphi_r^1) \ne \emptyset$, and $f_n \in W^{1,p'}(0,T;V^*)$ and $u_{0,n} \in D(\varphi^1)$ satisfy

$$(4.\ 4) \qquad f_n \to f \qquad \text{strongly in } L^{p'}(0,T;V^*) \text{ and weakly in } L^{\rho}(0,T;V^*),$$

$$(4.\ 5) \qquad u_{0,n} \to u_0 \qquad \text{strongly in } H.$$

In the rest of this proof, we denote by C_r a constant depending on r but independent of n, which may vary from line to line.

It is easily seen that (A1) and (A2) hold with φ^1 replaced by φ_r^1. Since $\partial_H \tilde{\varphi}^2(0) \ni 0$, we note that $J_\lambda 0 = 0$, where J_λ denotes the resolvent of $\partial_H \tilde{\varphi}^2$ (see (A3)); hence it follows that $|J_\lambda u|_H \le |u|_H$ for all $u \in H$. Therefore, by (A3), it follows that $\varphi_r^1(J_\lambda u) = \varphi^1(J_\lambda u) \le \ell_1(\varphi^1(u) + \ell_2(|u|_H)) = \ell_1(\varphi_r^1(u) + \ell_2(|u|_H))$ for all $u \in D(\varphi_r^1)$. Hence (A3) is satisfied with φ^1 replaced by φ_r^1. Moreover, we can deduce from (A7) that

$$\varphi^2(u) \le \varphi^2(0) + \langle \eta, u \rangle \le \frac{1}{2}\varphi^1(u) + C_r = \frac{1}{2}\varphi_r^1(u) + C_r \quad \forall u \in D(\varphi_r^1),$$

where $\eta \in \partial_V \varphi^2(u)$. Thus Theorem 4.2 ensures the existence of strong solutions u_n for $(\mathrm{CP})_{r,n}$ on $[0,T]$. Since $u_n(t) \in D(\varphi_r^1)$ for all $t \in [0,T]$, it follows immediately that

$$(4.\ 6) \qquad \sup_{t \in [0,T]} |u_n(t)|_H \quad \le \quad r.$$

Now multiplying $(\mathrm{CP})_{r,n}$ by $u_n(t) - v_0$ for some $v_0 \in D(\varphi_r^1)$, we get by (A1) and (A7),

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|u_n(t) - v_0|_H^2 + \varphi_r^1(u_n(t)) \\ &\quad \le \quad \varphi_r^1(v_0) + |g_n^2(t)|_{V^*}|u_n(t) - v_0|_V + |f_n(t)|_{V^*}|u_n(t) - v_0|_V \\ &\quad \le \quad \varphi_r^1(v_0) + \frac{1}{2}\varphi^1(u_n(t)) + C_r + C\left(|f_n(t)|_{V^*}^{p'} + |v_0|_V^p\right), \end{aligned}$$

where $g_n^2(t)$ denotes the section of $\partial_V \varphi^2(u_n(t))$ as in (4. 1). Hence

$$\frac{1}{2}\frac{d}{dt}|u_n(t) - v_0|_H^2 + \frac{1}{2}\varphi^1(u_n(t)) \quad \le \quad \varphi_r^1(v_0) + C_r + C\left(|f_n(t)|_{V^*}^{p'} + |v_0|_V^p\right)$$

for a.e. $t \in (0,T)$. Moreover, integrating this over $(0,t)$, we obtain

$$\begin{aligned} (4.\ 7) \qquad &\frac{1}{2}|u_n(t) - v_0|_H^2 + \frac{1}{2}\int_0^t \varphi^1(u_n(\tau))d\tau \\ &\quad \le \quad \frac{1}{2}|u_{0,n} - v_0|_H^2 + \left\{\varphi_r^1(v_0) + C|v_0|_V^p + C_r\right\}t + C\int_0^t |f_n(\tau)|_{V^*}^{p'} d\tau. \end{aligned}$$

Since $\{u_{0,n}\}$ and $\{f_n\}$ are bounded in H and $L^{p'}(0,T;V^*)$, respectively, by Proposition 2.1 of [3, Chap. II], we have

$$(4.\ 8) \qquad \int_0^T |\varphi^1(u_n(t))|dt \quad \le \quad C_r,$$

which together with (A1) yields

$$\int_0^T |u_n(t)|_V^p dt \ \leq \ C_r. \tag{4. 9}$$

Moreover, by (A1), it follows from (4. 7) that

$$\begin{aligned}
&\frac{1}{2}|u_n(t) - v_0|_H^2 + \frac{1}{2C_3}\int_0^t |u_n(\tau)|_V^2 d\tau \\
&\leq \ \frac{1}{2}|u_{0,n} - v_0|_H^2 + \left\{\varphi_r^1(v_0) + C|v_0|_V^p + \frac{C_1}{2C_3}r^2 + \frac{C_2}{2C_3} + C_r\right\} t \\
&\quad + C\left(\int_0^T |f_n(\tau)|_{V^*}^\rho d\tau\right)^{p'/\rho} t^{(\rho-p')/\rho}.
\end{aligned}$$

Now determine r such that

$$\frac{1}{2}|u_0 - v_0|_H^2 \ < \ \frac{1}{4}\left(r - |v_0|_H\right)^2 \tag{4. 10}$$

and take a positive number $T_0 \in (0, T]$ depending on r and $\|f\|_{L^\rho(0,T;V^*)}$ such that

$$\begin{aligned}
&\left\{\varphi_r^1(v_0) + C|v_0|_V^p + \frac{C_1}{2C_3}r^2 + \frac{C_2}{2C_3} + C_r\right\} T_0 \\
&\quad + C\left(\int_0^T |f(\tau)|_{V^*}^\rho d\tau + 1\right)^{p'/\rho} T_0^{(\rho-p')/\rho} \ < \ \frac{1}{4}\left(r - |v_0|_H\right)^2.
\end{aligned}$$

Hence there exists $N_0 \in \mathbf{N}$ such that $\sup_{t\in[0,T_0]} |u_n(t)|_H < r$ for all $n \geq N_0$.

Noting that $u \in D(\partial_V\varphi^1)$ and $\partial_V\varphi_r^1(u) = \partial_V\varphi^1(u)$ if $u \in D(\partial_V\varphi_r^1)$ and $|u|_H < r$ (see [5]), we deduce that $\partial_V\varphi_r^1(u_n(t)) = \partial_V\varphi^1(u_n(t))$ for all $t \in [0, T_0]$ and $n \geq N_0$.

Furthermore, by (A6), we have

$$\int_0^{T_0} |g_n^1(t)|_{V^*}^{p'} dt \ \leq \ C_r, \tag{4. 11}$$

where $g_n^1(t) := f_n(t) - du_n(t)/dt + g_n^2(t)$. By (A7), it follows from (4. 6) and (4. 8) that

$$\int_0^T |g_n^2(t)|_{V^*}^{p'} dt \ \leq \ C_r. \tag{4. 12}$$

From the fact that $du_n(t)/dt := f_n(t) - g_n^1(t) + g_n^2(t)$, we also find that

$$\int_0^{T_0} \left|\frac{du_n}{dt}(t)\right|_{V^*}^{p'} dt \ \leq \ C_r. \tag{4. 13}$$

By grace of these a priori estimates, we can take a subsequence $\{n'\}$ of $\{n\}$ such that

$$u_{n'} \ \to \ u \quad \text{weakly in } L^p(0,T_0;V) \cap W^{1,p'}(0,T_0;V^*), \tag{4. 14}$$

$$g_{n'}^1 \ \to \ g^1 \quad \text{weakly in } L^{p'}(0,T_0;V^*), \tag{4. 15}$$

$$g_{n'}^2 \ \to \ g^2 \quad \text{weakly in } L^{p'}(0,T_0;V^*). \tag{4. 16}$$

Hence $u \in C([0,T_0];H)$. Moreover, we claim that

$$(4.\ 17) \qquad u_{n'}(t) \ \rightarrow \ u(t) \quad \text{weakly in } V^* \text{ for all } t \in [0,T_0].$$

Indeed, for any $q \in (1,+\infty)$, we can take a subsequence $\{n'_q\}$ of $\{n'\}$ such that $u_{n'_q} \rightarrow u$ weakly in $L^q(0,t;V^*)$ for all $t \in [0,T_0]$; therefore we see

$$\begin{aligned} \|u-u_0\|_{L^q(0,t;V^*)} &\leq \liminf_{n'_q \to +\infty} \|u_{n'_q} - u_{0,n'_q}\|_{L^q(0,t;V^*)} \\ &= \liminf_{n'_q \to +\infty} \left\{ \int_0^t \left| \int_0^\tau \frac{du_{n'_q}}{ds}(s)ds \right|_{V^*}^q d\tau \right\}^{1/q} \leq C_r^{1/p'} \left(\frac{p}{p+q} \right)^{1/q} t^{1/p+1/q}. \end{aligned}$$

Thus letting $q \rightarrow +\infty$, we obtain $\sup_{\tau\in[0,t]} |u(\tau)-u_0|_{V^*} \leq C_r^{1/p'} t^{1/p}$, which implies that $u(t) \rightarrow u_0$ strongly in V^* as $t \rightarrow +0$. Moreover, we find that

$$\begin{aligned} \langle u_{n'}(t) - u(t), \phi \rangle &= \int_0^t \left\langle \frac{du_{n'}}{d\tau}(\tau) - \frac{du}{d\tau}(\tau), \phi \right\rangle d\tau + \langle u_{0,n'} - u_0, \phi \rangle \\ &\rightarrow 0 \quad \text{as } n' \rightarrow +\infty \quad \text{for all } \phi \in V, \end{aligned}$$

which implies (4. 17). Furthermore, by (4. 6), for every $t \in [0,T_0]$, we can extract a subsequence $\{n'_t\}$ of $\{n'\}$ such that

$$(4.\ 18) \qquad u_{n'_t}(t) \ \rightarrow \ u(t) \quad \text{weakly in } H.$$

Now, by (A2), it follows that

$$(4.\ 19) \qquad g^2_{n'} \ \rightarrow \ g^2 \quad \text{strongly in } L^{p'}(0,T_0;V^*).$$

Therefore, by Proposition 2.3, it follows from (4. 14) and (4. 19) that $g^2(t) \in \partial_V \varphi^2(u(t))$ for a.e. $t \in (0,T_0)$.

We next claim that $g^1(t) \in \partial_V \varphi^1(u(t))$ for a.e. $t \in (0,T_0)$. Indeed, calculating

$$\begin{aligned} &\int_0^{T_0} \langle g^1_{n'}(t), u_{n'}(t) \rangle dt \\ &\quad = \int_0^{T_0} \langle f_{n'}(t), u_{n'}(t) \rangle dt - \int_0^{T_0} \left\langle \frac{du_{n'}}{dt}(t), u_{n'}(t) \right\rangle dt + \int_0^{T_0} \langle g^2_{n'}(t), u_{n'}(t) \rangle dt \\ &\quad = \int_0^{T_0} \langle f_{n'}(t), u_{n'}(t) \rangle dt - \frac{1}{2}|u_{n'}(T_0)|_H^2 + \frac{1}{2}|u_{0,n'}|_H^2 + \int_0^{T_0} \langle g^2_{n'}(t), u_{n'}(t) \rangle dt, \end{aligned}$$

we infer

$$\begin{aligned} &\limsup_{n' \to +\infty} \int_0^{T_0} \langle g^1_{n'}(t), u_{n'}(t) \rangle dt \\ &\quad \leq \int_0^{T_0} \langle f(t), u(t) \rangle dt - \frac{1}{2}|u(T_0)|_H^2 + \frac{1}{2}|u_0|_H^2 + \int_0^{T_0} \langle g^2(t), u(t) \rangle dt \\ &\quad = \int_0^{T_0} \left\langle f(t) - \frac{du}{dt}(t) + g^2(t), u(t) \right\rangle dt. \end{aligned}$$

Hence, by Proposition 2.3, it follows from (4. 14) and (4. 15) that

$$g^1(t) \;=\; f(t) - \frac{du}{dt}(t) + g^2(t) \in \partial_V \varphi^1(u(t)) \;\text{ for a.e. } t \in (0, T_0).$$

Finally, we prove that the limit u satisfies the initial condition, i.e., $u(t) \to u_0$ strongly in H as $t \to +0$. To this end, we employ the following auxiliary problem:

$$(\mathrm{CP})_0 \quad \frac{dv}{dt}(t) + \partial_V \varphi^1(v(t)) \ni f(t) \text{ in } V^*, \quad 0 < t < T, \quad v(0) = u_0.$$

By (A1) and (A6), the existence of a unique strong solution v is ensured by Theorem 3.2 of [1]; moreover, v belongs to $v \in L^p(0,T;V) \cap C([0,T];H) \cap W^{1,p'}(0,T;V^*)$.

Now multiplying $(\mathrm{CP})_{r,n} - (\mathrm{CP})_0$ by $w_n(t) := u_n(t) - v(t)$ and noting that $\partial_V \varphi_r^1(u_n(t)) = \partial_V \varphi^1(u_n(t))$ for all $t \in [0, T_0]$ and $n \ge N_0$, we find that

$$\frac{1}{2}\frac{d}{dt}|w_n(t)|_H^2 \;\le\; |g_n^2(t)|_{V^*}|w_n(t)|_V + |f_n(t) - f(t)|_{V^*}|w_n(t)|_V$$

for a.e. $t \in (0, T_0)$. Therefore, by (A1) and (A7), for any $\varepsilon > 0$, there exists a constant C_ε depending on ε such that

$$\frac{1}{2}\frac{d}{dt}|w_n(t)|_H^2 \;\le\; \varepsilon\left\{\varphi^1(u_n(t)) + |u_n(t)|_V^p + |v(t)|_V^p\right\} + C_\varepsilon\left\{|f_n(t) - f(t)|_{V^*}^{p'} + 1\right\}.$$

Hence integrating this over $(0,t)$, we get

$$\begin{aligned}\frac{1}{2}|w_n(t)|_H^2 \;\le\;& \frac{1}{2}|u_{0,n} - u_0|_H^2 + \varepsilon\left\{\int_0^T |\varphi^1(u_n(\tau))| d\tau + \int_0^T |u_n(\tau)|_V^p d\tau + \int_0^T |v(\tau)|_V^p d\tau\right\} \\ &+ C_\varepsilon\left\{\int_0^T |f_n(\tau) - f(\tau)|_{V^*}^{p'} d\tau + t\right\}.\end{aligned}$$

Thus (4. 8) and (4. 9) yield

$$|w_n(t)|_H^2 \;\le\; |u_{0,n} - u_0|_H^2 + \varepsilon C + 2C_\varepsilon\left\{\int_0^T |f_n(t) - f(t)|_{V^*}^{p'} dt + t\right\}.$$

Since $w_{n_t'}(t) \to u(t) - v(t)$ weakly in H, it follows from (4. 4) and (4. 5) that

$$|u(t) - v(t)|_H^2 \;\le\; \liminf_{n_t' \to +\infty} |w_{n_t'}(t)|_H^2 \le \varepsilon C + 2C_\varepsilon t.$$

Hence $u(t) \to u_0$ strongly in H as $t \to +0$. Consequently, u becomes a strong solution of (CP) on $[0, T_0]$. ∎

Before describing the results on the global (in time) existence, we prepare the following lemma concerned with the maximal existence time of solutions for (CP) defined by

$$T_{max} \;:=\; \sup\left\{T_0 \in (0,T];\ (\mathrm{CP}) \text{ has a strong solution on } [0,T_0]\right\}.$$

By virtue of Theorem 4.5, we can verify the following lemma.

Lemma 4.6 *Let T_{max} be the maximal existence time of solutions for* (CP) *and suppose that $T_{max} < T$. Then it follows that $\lim_{t\to T_{max}-0} |u(t)|_H = +\infty$.*

As for the global existence, our result is stated as follows.

Theorem 4.7 *In addition to the same assumptions as in Theorem* 4.5, *suppose that*

(A8) *There exist constants $\alpha > 0$ and $C_6 \geq 0$ such that*
$$\alpha\varphi^1(u) \leq \langle \xi - \eta, u\rangle + C_6\left(|u|_H^2 + 1\right) \quad \forall[u,\xi] \in \partial_V\varphi^1,\ \forall\eta \in \partial_V\varphi^2(u).$$

Then (CP) *has a strong solution u on $[0,T]$ satisfying* (4. 3) *with $T_0 = T$.*

Proof of Theorem 4.7 Let T_{max} be the maximal existence time of solutions for (CP) and suppose that $T_{max} < T$. Moreover, let u be a strong solution of (CP) on $[0, T_{max})$. Then multiplying (CP) by $u(t)$, we get by (A8),

$$\frac{1}{2}\frac{d}{dt}|u(t)|_H^2 + \alpha\varphi^1(u(t)) \;\; \leq \;\; |f(t)|_{V^*}|u(t)|_V + C_6\left(|u(t)|_H^2 + 1\right)$$

for a.e. $t \in (0, T_{max})$. Moreover, (A1) implies

$$\frac{1}{2}\frac{d}{dt}|u(t)|_H^2 + \frac{\alpha}{2}\varphi^1(u(t)) \;\; \leq \;\; C\left(|f(t)|_{V^*}^{p'} + |u(t)|_H^2 + 1\right).$$

Hence integrating this over $(0,t)$, by Proposition 2.1 of [3, Chap. II] and Gronwall's inequality, we have $\sup_{t\in[0,T_{max})} |u(t)|_H \leq C$, which contradicts Lemma 4.6; therefore, (CP) admits a strong solution on $[0,T]$. ∎

Furthermore, we can also derive the global existence by assuming an additional growth condition on $\partial_V\varphi^2$ and the smallness of u_0 and f in an appropriate sense.

Theorem 4.8 *In addition to the same assumptions as in Theorem* 4.5, *suppose that $\varphi^1(0) = 0$, $\ell_5(x) = o(x^p)$ as $x \to 0$ in* (A7), *and $C_1 = C_2 = 0$ in* (A1). *Then there exists a constant $\delta > 0$ independent of T such that for any $f \in L^{p'}(0,T;V^*)$ and $u_0 \in \overline{D(\varphi^1)}^H$ satisfying $|u_0|_H + |||f(\cdot)|_{V^*}^{p'}||_{1,T}^{1/p} < \delta$, where $|||f(\cdot)|_{V^*}^{p'}||_{1,T}$ is given in Theorem* 4.4, (CP) *admits a strong solution u on $[0,T]$ satisfying* (4. 3) *with $T_0 = T$.*

Proof of Theorem 4.8 Let T_{max} be the maximal existence time of solutions for (CP) and let u be a strong solution of (CP) on $[0, T_{max})$. Now suppose that $T_{max} < T$. By Lemma 4.6, it then follows that $|u(t)|_H \to +\infty$ as $t \to T_{max} - 0$.

Multiply (CP) by $u(t)$. By (A1) with $C_1 = C_2 = 0$ and (A6), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u(t)|_H^2 + \varphi^1(u(t)) \;\; &\leq \;\; \varphi^1(0) + |f(t)|_{V^*}|u(t)|_V + |g^2(t)|_{V^*}|u(t)|_V \\ &\leq \;\; \frac{1}{2}\varphi^1(u(t)) + C\left\{|f(t)|_{V^*}^{p'} + \ell_5(|u(t)|_H)\right\}.\end{aligned}$$

Since V is continuously embedded in H, by (A1) with $C_1 = C_2 = 0$, it follows that

$$\frac{1}{2}\frac{d}{dt}|u(t)|_H^2 + \gamma|u(t)|_H^p \;\; \leq \;\; C\left\{|f(t)|_{V^*}^{p'} + \ell_5(|u(t)|_H)\right\} \tag{4. 20}$$

for some positive constant γ independent of T.

From the fact that $\ell_5(x) = o(x^p)$ as $x \to 0$, there exists a constant $\delta_0 > 0$ such that

$$(4.\ 21) \qquad \ell_5(x) \ \leq \ \frac{\gamma}{2C} x^p \quad \forall x \in [0, \delta_0].$$

Now, if $|u_0|_H < \delta_0/2$, then there exists $T_* \in (0, T_{max})$ such that $|u(t)|_H < \delta_0$ for all $t \in [0, T_*)$ and $|u(T_*)|_H = \delta_0$. Hence combining (4. 20) and (4. 21), we obtain

$$\frac{1}{2}\frac{d}{dt}|u(t)|_H^2 + \frac{\gamma}{2}|u(t)|_H^p \ \leq \ C|f(t)|_{V^*}^{p'}$$

for a.e. $t \in (0, T_*)$, where γ and C are independent of T_*. Hence, by Lemma 4.4 of [1], there exists a positive constant $\delta < \delta_0/2$ independent of T_* such that if

$$|u_0|_H + \left\| |f(\cdot)|_{V^*}^{p'} \right\|_{1,T}^{1/p} \ \leq \ \delta,$$

then $\sup_{t \in [0, T_*)} |u(t)|_H \leq 3\delta_0/4$, which contradicts the definition of T_*. Thus $T_{max} = T$. ∎

5 Applications to (NHE)

Now we apply the preceding abstract theory to (NHE) and derive sufficient conditions for the existence of local or global (in time) solutions of (NHE).

5.1 The Case: $u_0 \in W_0^{1,p}(\Omega)$

By grace of Theorems 4.2-4.4, we obtain the followings (see [2] for their proofs).

Theorem 5.1 (Akagi-Ôtani [2]) *Assume that* $p \leq q$ *and*

$$(5.\ 1) \qquad 2N/(N+2) \leq p, \quad q < p^*.$$

Then, for all $u_0 \in W_0^{1,p}(\Omega)$ *and* $f \in W^{1,p'}(0, T; W^{-1,p'}(\Omega))$, *there exists a number* $T_0 \in (0, T]$ *such that* (NHE) *has a weak solution* u *on* $[0, T_0]$ *satisfying*:

$$(5.\ 2) \qquad u \in C_w([0, T_0]; W_0^{1,p}(\Omega)) \cap C([0, T_0]; L^q(\Omega)) \cap W^{1,2}(0, T_0; L^2(\Omega)).$$

Theorem 5.2 (Akagi-Ôtani [2]) *Assume* (5. 1) *holds and* $p < q$. *Let* R *be an arbitrary positive number, and let* δ *be a positive number such that* $\delta < C(p,q)^{-p/(q-p)}$, *where* $C(p,q)$ *denotes the best possible constant for the Sobolev-Poincaré-type inequality*: $|u|_{L^q(\Omega)} \leq C(p,q)|u|_V$. *Then there exists a positive number* δ_R *independent of* T *such that if* u_0 *and* f *satisfy*

$$\frac{1}{p}|u_0|_V^p + \int_0^T |f(\tau)|_{V^*}^{p'} d\tau + \int_0^T \left| \frac{df}{d\tau}(\tau) \right|_{V^*}^{p'} d\tau \leq R,$$

$$|u_0|_{L^q(\Omega)} < \delta, \ |u_0|_{L^2(\Omega)} + \left\{ \max\left(1, \frac{1}{T}\right) \left\| |f(\cdot)|_{V^*}^{p'} \right\|_{1,T} \right\}^{1/p} < \delta_R,$$

then (NHE) *has a weak solution* u *on* $[0, T]$ *satisfying* (5. 2) *with* T_0 *replaced by* T.

Theorem 5.3 (Akagi-Ôtani [2]) *Assume* (5. 1) *holds and* $p > q$. *Then, for all* $u_0 \in W_0^{1,p}(\Omega)$ *and* $f \in W^{1,p'}(0, T; W^{-1,p'}(\Omega))$, (NHE) *has a weak solution* u *on* $[0, T]$ *satisfying* (5. 2) *with* T_0 *replaced by* T.

5.2 The Case: $u_0 \in L^2(\Omega)$

As for the case where $u_0 \in L^2(\Omega)$, we employ Theorems 4.5, 4.7 and 4.8.

Lemma 5.4 *Assume that*

$$(5.\ 3) \qquad 2N/(N+2) \le p < N, \quad q < (N+2)p/N.$$

Then (A1), (A2), (A3), (A6) *and* (A7) *hold true with* $\varphi^1 = \varphi_p$, $\varphi^2 = \psi_q$ *and* $C_1 = C_2 = 0$.

Proof of Lemma 5.4. Since $\varphi_p(u) = |u|_V^p/p$, (A1) with $C_1 = C_2 = 0$ and $C_3 = p$ follows at once. Moreover, just as in [2], we can also verify (A3) with φ^1 and φ^2 replaced by φ_p and ψ_q, respectively.

Let $[u, \eta] \in \partial_V \psi_q$ be fixed. Then since $r(q-1) < r(p^*-1) = p^*$, where r denotes the Hölder conjugate of the Sobolev critical exponent p^*, i.e., $r := (p^*)'$, we find that

$$\langle \eta, \phi \rangle \ = \ \int_\Omega |u|^{q-2}u(x)\phi(x)dx \le |u|^{q-1}_{L^{r(q-1)}}|\phi|_{L^{p^*}} \quad \forall \phi \in V.$$

Now, if $r(q-1) > 2$, then

$$(5.\ 4) \qquad |u|_{L^{r(q-1)}} \ \le \ C|u|_H^\theta |u|_V^{(1-\theta)}, \qquad \theta := \left(\frac{1}{r(q-1)} - \frac{1}{p^*}\right) / \left(\frac{1}{2} - \frac{1}{p^*}\right),$$

which implies $|\eta|_{V^*}^{p'} \le C|u|_H^{\theta(q-1)p'}\varphi_p(u)^{(1-\theta)(q-1)p'/p}$. Since $q < (N+2)p/N$, it is easily seen that

$$(5.\ 5) \qquad (1-\theta)(q-1) \ < \ p-1 = p/p'.$$

Hence, for any $\varepsilon > 0$, there exists a constant C_ε such that $|\eta|_{V^*}^{p'} \le \varepsilon\varphi_p(u) + C_\varepsilon \ell_6(|u|_H)$, where $\ell_6(\cdot)$ is given by

$$(5.\ 6) \qquad \ell_6(x) \ = \ x^{\sigma\theta(q-1)p'}, \qquad 1/\sigma + (1-\theta)(q-1)p'/p = 1.$$

Hence (A7) is satisfied with ℓ_5, φ^1 and φ^2 replaced by ℓ_6, φ_p and ψ_q, respectively. On the other hand, if $r(q-1) \le 2$, then $|\eta|_{V^*}^{p'} \le C|u|_H^{p'(q-1)}$.

Let $\mu \in \mathbf{R}$ and let u_n be such that

$$\int_0^T \varphi_p(u_n(t))dt + \sup_{t\in[0,T]} |u_n(t)|_H + \int_0^T \left|\frac{du_n}{dt}(t)\right|_{V^*} dt \ \le \ \mu.$$

Then since V is compactly embedded in $L^q(\Omega)$, Theorem 5 of [11] ensures that $\{u_n\}$ becomes precompact in $L^p(0,T;L^q(\Omega))$.

For the case where $q > 2$, note that

$$\left||s|^{q-2}s - |s'|^{q-2}s'\right| \ \le \ (q-1)\left(|s|^{q-2} + |s'|^{q-2}\right)|s-s'|$$

for all $s, s' \in \mathbf{R}$. Hence, if $r(q-1) > 2$, then recalling (5. 4), we obtain

$$\begin{aligned}
&\left\||u_n|^{q-2}u_n - |u|^{q-2}u\right\|_{L^{p'}(0,T;V^*)} \\
&\le \ C_{\mu,\alpha}\left(\|u_n\|^{(1-\theta)(q-2)}_{L^p(0,T;V)} + \|u\|^{(1-\theta)(q-2)}_{L^p(0,T;V)}\right)\|u_n - u\|^\alpha_{L^p(0,T;H)}\|u_n - u\|^{(1-\theta)}_{L^p(0,T;V)},
\end{aligned}$$

where $C_{\mu,\alpha}$ denotes a constant depending on μ, q, θ, α and $\sup_{t\in[0,T]} |u(t)|_H$ but not on n and t, and $\alpha \in (0, \theta]$ is given by $\alpha + (1-\theta)(q-2) + (1-\theta) \leq p/p'$. On the other hand, if $r(q-1) \leq 2$, then we have

$$\begin{aligned} &\left\| |u_n|^{q-2}u_n - |u|^{q-2}u \right\|_{L^{p'}(0,T;V^*)} \\ &\leq \; C\left(\|u_n\|^{q-2}_{L^\infty(0,T;H)} + \|u\|^{q-2}_{L^\infty(0,T;H)}\right) \|u_n - u\|^{1/p}_{L^\infty(0,T;H)} \|u_n - u\|^{1/p'}_{L^1(0,T;H)}. \end{aligned}$$

Now since $q > 2$ and $\{u_n\}$ is precompact in $L^p(0,T;H)$; therefore we can verify that $\{|u_n|^{q-2}u_n\}$ becomes precompact in $L^{p'}(0,T;V^*)$.

For the case where $q \leq 2$, we note that u_n belongs to $C([0,T];L^q(\Omega))$; hence we can easily derive the precompactness of $\{|u_n|^{q-2}u_n\}$ in $L^{p'}(0,T;V^*)$. Consequently, (A2) is satisfied with φ^1 and φ^2 replaced by φ_p and ψ_q, respectively.

Moreover, it is easily seen that (A6) is satisfied with φ^1 replaced by φ_p. ∎

Theorem 5.5 *Suppose that* (5. 3) *is satisfied. Then for all* $u_0 \in L^2(\Omega)$ *and* $f \in L^\rho(0,T;W^{-1,p'}(\Omega))$ *with* $\rho > p'$, *there exists a number* $T_0 = T_0(|u_0|_{L^2}, \|f\|_{L^\rho(0,T;W^{-1,p'})}) \in (0,T]$ *such that* (NHE) *has a weak solution* u *on* $[0,T_0]$ *satisfying*:

$$u \in L^p(0,T_0;W^{1,p}_0(\Omega)) \cap C([0,T_0];L^2(\Omega)) \cap W^{1,p'}(0,T_0;W^{-1,p'}(\Omega)). \tag{5. 7}$$

Proof of Theorem 5.5 Lemma 5.4 and Theorem 4.5 ensure the existence of weak solutions for (NHE) on $[0,T_0]$ for some $T_0 > 0$. ∎

Theorem 5.6 *Suppose that* (5. 3) *is satisfied and* $p < q$. *Then there exists a positive number* δ *independent of* T *such that if* $u_0 \in L^2(\Omega)$ *and* $f \in L^\rho(0,T;W^{-1,p'}(\Omega))$ *satisfy*

$$|u_0|_{L^2(\Omega)} + \left\| |f(\cdot)|^{p'}_{V^*} \right\|^{1/p}_{1,T} < \delta, \tag{5. 8}$$

then (NHE) *has a weak solution* u *on* $[0,T]$ *satisfying* (5. 7) *with* T_0 *replaced by* T.

Proof of Theorem 5.6 We claim that (5. 6) implies $\ell_6(x) = o(x^p)$ as $x \to 0$; indeed, since $p < q$, it follows that $\sigma\theta(q-1)p' > p$, where θ is given in (5. 4). Moreover, we also note that $p'(q-1) > \sigma p'\theta(q-1)$. Furthermore, it is obvious that $\varphi_p(0) = 0$. Hence, by Theorem 4.8, (NHE) admits a weak solution on $[0,T]$ for u_0 and f satisfying (5. 8). ∎

Theorem 5.7 *Assume* (5. 1) *and* $q < p$. *Then for all* $u_0 \in L^2(\Omega)$ *and* $f \in L^\rho(0,T;W^{-1,p'}(\Omega))$ *with* $\rho > p'$, (NHE) *has a weak solution* u *on* $[0,T]$ *satisfying* (5. 7) *with* T_0 *replaced by* T.

Proof of Theorem 5.7 We here employ Theorem 4.7 to derive the global existence of weak solutions on $[0,T]$ for (NHE). To this end, it suffices to show that (A8) holds with φ^1 and φ^2 replaced by φ_p and ψ_q, respectively. Let $[u,\xi] \in \partial_V\varphi_p$ and $\eta = \partial_V\psi_q(u)$ be fixed. Since $q < p$, it then follows that

$$\langle \xi - \eta, u\rangle = \int_\Omega |\nabla u(x)|^p dx - \int_\Omega |u(x)|^q dx \geq \frac{1}{2}\int_\Omega |\nabla u(x)|^p dx - C,$$

which implies the desired conclusion. ∎

Remark 5.8 We can also discuss the case where $p \geq N$ by establishing further a priori estimates for $t(du_n/dt)$ and $t\varphi^1(u_n(t))$ and using Gagliardo-Nirenberg's inequality in the proof of Theorem 4.5.

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