GAKUTO International Series
Mathematical Sciences and Applications, Vol.20(2004)
Nonlinear Partial Differential Equations and Their Applications, pp.418–429

Asymptotic behavior of solutions for parabolic equations associated with p-Laplacian as p tends to infinity

Goro Akagi
Department of Applied Physics, School of Science and Engineering,
Waseda University
4-1 Okubo 3-chōme, Shinjuku-ku, Tokyo, 169-8555 Japan
E-mail: goro@toki.waseda.jp

Abstract. In this paper, the asymptotic behavior of solutions for some quasilinear parabolic equation associated with p-Laplacian as $p \to +\infty$ will be discussed by investigating the convergence of the functional corresponding to p-Laplacian. Moreover some abstract theory of Mosco convergence of functionals as well as evolution equations governed by subdifferentials is also employed.

The author is supported by Waseda University Grant for Special Research Projects, #2003A-123.
AMS Subject Classification 34G20, 40A30, 47J35

1 Introduction

Several authors have already studied the convergence of the solutions for the following initial-boundary value problem as well as generalized ones as $p \to +\infty$.

$$(\mathrm{P})_p \quad \begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \Delta_p u(x,t) = f(x,t), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where Δ_p denotes the so-called p-Laplacian given by $\Delta_p u := \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ and Ω denotes a domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$. Their works were motivated by a couple of physical topics, e.g., sandpile growth [3, 11], Bean's critical-state model for type-II superconductivity [6, 12, 13], river networks [11] and so on.

In order to investigate the asymptotic behavior of the solutions for $(\mathrm{P})_p$ as $p \to +\infty$, we point out the variational structure of p-Laplacian in $L^2(\Omega)$, i.e. define $\varphi_p : L^2(\Omega) \to [0, +\infty]$ as follows:

$$\varphi_p(u) \quad = \quad \begin{cases} \dfrac{1}{p} \displaystyle\int_\Omega |\nabla u(x)|^p dx & \text{if } \ u \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise;} \end{cases}$$

then the subdifferential $\partial_{L^2(\Omega)} \varphi_p(u)$ of φ_p at u in $L^2(\Omega)$ coincides with $-\Delta_p u$ equipped with the homogeneous Dirichlet boundary condition $u|_{\partial\Omega} = 0$ in the sense of distribution (the definition of subdifferentials will be given in Section 3).

Moreover it is well known that $(\mathrm{P})_p$ is reduced to the following Cauchy problem.

$$(1) \qquad \begin{cases} \dfrac{du}{dt}(t) + \partial_{L^2(\Omega)} \varphi_p(u(t)) = f(t) \ \text{ in } L^2(\Omega), \quad 0 < t < T, \\ u(0) = u_0. \end{cases}$$

According to [3] and [6], the strong solutions u_p of (1) converge to u_∞ as $p \to +\infty$ and the limit u_∞ becomes the unique strong solution of the following Cauchy problem.

$$\begin{cases} \dfrac{du}{dt}(t) + \partial_{L^2(\Omega)} \varphi_\infty(u(t)) \ni f(t) \ \text{ in } L^2(\Omega), \quad 0 < t < T, \\ u(0) = u_0, \end{cases}$$

where $\varphi_\infty : L^2(\Omega) \to [0, +\infty]$ is given by

$$\varphi_\infty(u) \quad = \quad \begin{cases} 0 & \text{if } \ u \in K, \\ +\infty & \text{otherwise} \end{cases}$$

with

$$K \ := \ \left\{ u \in H_0^1(\Omega); |\nabla u(x)| \le 1 \quad \text{for a.e. } x \in \Omega \right\}.$$

Then one may conjecture that φ_p converges to φ_∞ in a sense as $p \to +\infty$; however rigorous proof of this conjecture has not been provided yet.

In this paper, we prove that φ_p converges to φ_∞ in the sense of Mosco as $p \to +\infty$. Moreover by employing the abstract theory of Mosco convergence of functionals as well as evolution equations governed by subdifferentials developed by Attouch (see e.g. [5]), we also discuss the convergence of the solutions for (1) as $p \to +\infty$. Furthermore we deal with a couple of other types of quasilinear parabolic equations as well.

2 Mosco Convergence of φ_p as $p \to +\infty$

From now on, we denote by $\Psi(X)$ the set of all proper lower semi-continuous convex functionals ϕ from a Hilbert space X into $(-\infty, +\infty]$, where *"proper"* means that $\phi \not\equiv +\infty$. Now Mosco convergence is defined in the following

Definition 2.1. *Let (φ_n) be a sequence in $\Psi(X)$ and let $\varphi \in \Psi(X)$. Then $\varphi_n \to \varphi$ on X in the sense of Mosco as $n \to +\infty$ if the following* (i) *and* (ii) *are all satisfied*:

(i) *For all $u \in D(\varphi)$, there exists a sequence (u_n) in X such that $u_n \to u$ strongly in X and $\varphi_n(u_n) \to \varphi(u)$.*

(ii) *Let (u_k) be a sequence in X such that $u_k \to u$ weakly in X as $k \to +\infty$ and let (n_k) be a subsequence of (n). Then $\liminf_{k\to+\infty} \varphi_{n_k}(u_k) \geq \varphi(u)$.*

Our main result is then stated as follows.

Theorem 2.2. *Suppose that Ω is bounded and let (p_n) be a sequence in $(1, +\infty)$ such that $p_n \to +\infty$ as $n \to +\infty$. Then it follows that*

$$\varphi_{p_n} \to \varphi_\infty \quad \text{on } L^2(\Omega) \text{ in the sense of Mosco as } p_n \to +\infty.$$

Proof We first prove that

$$(2) \qquad \begin{cases} \forall u \in D(\varphi_\infty),\ \exists (u_n) \subset L^2(\Omega); \\ u_n \to u \text{ strongly in } L^2(\Omega) \text{ and } \varphi_{p_n}(u_n) \to \varphi_\infty(u) \text{ as } n \to +\infty. \end{cases}$$

Let $u \in D(\varphi_\infty) = K$ and set $u_n := u$ for all $n \in \mathbf{N}$. Then since $K \subset W_0^{1,p_n}(\Omega)$ for all $n \in \mathbf{N}$, it follows immediately that

$$\begin{aligned} 0 \leq \varphi_{p_n}(u_n) &= \frac{1}{p_n} \int_\Omega |\nabla u(x)|^{p_n} dx \\ &\leq \frac{1}{p_n} |\Omega| \to 0 = \varphi_\infty(u) \quad \text{as } p_n \to +\infty. \end{aligned}$$

Hence (2) holds.

We next show that

$$(3)\qquad \begin{cases} \forall (u_k) \subset L^2(\Omega) \text{ satisfying } u_k \to u \text{ weakly in } L^2(\Omega) \text{ as } k \to +\infty, \\ \forall (n_k) \subset (n), \ \liminf_{k\to+\infty} \varphi_{p_{n_k}}(u_k) \geq \varphi_\infty(u). \end{cases}$$

For the case where $u \in D(\varphi_\infty) = K$, it is easily seen that

$$\liminf_{k\to+\infty} \varphi_{p_{n_k}}(u_k) \ \geq \ 0 = \varphi_\infty(u).$$

For the case where $u \notin K$, we give a proof by contradiction. To do this, suppose that

$$\begin{aligned} &\exists (u_k) \subset L^2(\Omega), \ \exists (n_k) \subset (n); \ u_k \to u \text{ weakly in } L^2(\Omega) \text{ as } k \to +\infty, \\ &\liminf_{k\to+\infty} \varphi_{p_{n_k}}(u_k) < \varphi_\infty(u) = +\infty. \end{aligned}$$

Then by taking a subsequence (k') of (k), we can deduce

$$\varphi_{p_{n_{k'}}}(u_{k'}) \ \leq \ C \quad \forall k' \in \mathbf{N},$$

which implies

$$\begin{aligned} \left(\int_\Omega |\nabla u_{k'}(x)|^{p_{n_{k'}}} dx \right)^{1/p_{n_{k'}}} &\leq \left\{ p_{n_{k'}} \varphi_{p_{n_{k'}}}(u_{k'}) \right\}^{1/p_{n_{k'}}} \\ &\leq (p_{n_{k'}} C)^{1/p_{n_{k'}}} \to 1 \quad \text{as } k' \to +\infty. \end{aligned}$$

For simplicity of notation, we write p and u_p for $p_{n_{k'}}$ and $u_{k'}$ respectively. Moreover it follows that

$$\begin{aligned} \left(\int_\Omega |\nabla u_p(x)|^q dx \right)^{1/q} &\leq \left(\int_\Omega |\nabla u_p(x)|^p dx \right)^{1/p} |\Omega|^{(p-q)/(pq)} \\ &\to |\Omega|^{1/q} \quad \text{as } p \to +\infty, \end{aligned}$$

which implies that (∇u_p) is bounded in $(L^q(\Omega))^N$. Thus for each $q \in (1, +\infty)$, we can take a subsequence (p_q) of (p) such that

$$\nabla u_{p_q} \to \nabla u \qquad \text{weakly in } (L^q(\Omega))^N.$$

Moreover we can derive $u \in H_0^1(\Omega)$ from the case where $q = 2$. In the rest of this proof, we drop q in p_q. Furthermore it follows that

$$\begin{aligned} \left(\int_\Omega |\nabla u(x)|^q dx \right)^{1/q} &\leq \liminf_{p\to+\infty} \left(\int_\Omega |\nabla u_p(x)|^q dx \right)^{1/q} \\ &\leq \liminf_{p\to+\infty} \left(\int_\Omega |\nabla u_p(x)|^p dx \right)^{1/p} |\Omega|^{(p-q)/(pq)} \\ &\leq \lim_{p\to+\infty} (pC)^{1/p} |\Omega|^{(p-q)/(pq)} = |\Omega|^{1/q}. \end{aligned}$$

Hence letting $q \to +\infty$, we conclude that

$$|\nabla u(x)| \ \leq \ 1 \quad \text{for a.e. } x \in \Omega,$$

which contradicts the fact that $u \notin K$; therefore (3) holds true. [Q.E.D.]

Remark 2.3. Theorem 2.2 is still valid if φ_p and φ_∞ are replaced by the following ψ_p and ψ_∞ respectively:

$$\psi_p(u) \;=\; \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega |\nabla u(x)|^p dx + \int_{\partial\Omega} j(u(x))d\Gamma & \text{if } \; u \in W^{1,p}(\Omega), \; j(u(\cdot)) \in L^1(\partial\Omega), \\ +\infty & \text{otherwise} \end{cases}$$

and

$$\psi_\infty(u) \;=\; \begin{cases} \displaystyle\int_{\partial\Omega} j(u(x))d\Gamma & \text{if } \; u \in \tilde{K}, \; j(u(\cdot)) \in L^1(\partial\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where $j \in \Psi(\mathbf{R})$ and $\tilde{K}$ is given by

$$\tilde{K} \;:=\; \left\{u \in L^2(\Omega); |\nabla u(x)| \leq 1 \quad \text{for a.e. } x \in \Omega\right\}.$$

We here note that ψ_p and ψ_∞ belong to $\Psi(L^2(\Omega))$ and $\partial_{L^2(\Omega)}\psi_p(u)$ coincides with $-\Delta_p u$ equipped with the following boundary condition:

$$-|\nabla u|^{p-2}\frac{\partial u}{\partial n}(x) \in \partial_{\mathbf{R}} j(u(x)), \quad x \in \partial\Omega \tag{4}$$

in the distribution sense.

3 Asymptotic Behavior of Solutions as $p \to +\infty$

In order to investigate the convergence of the solutions for $(\mathrm{P})_p$ as $p \to +\infty$, we first deal with the following abstract Cauchy problem denoted by $\mathrm{CP}_H(\varphi, f, u_0)$ in a Hilbert space H.

$$\mathrm{CP}_H(\varphi, f, u_0) \quad \begin{cases} \dfrac{du}{dt}(t) + \partial_H\varphi(u(t)) \ni f(t) \;\text{ in } H, \quad 0 < t < T, \\ u(0) = u_0, \end{cases}$$

where $\partial_H\varphi$ denotes the subdifferential of $\varphi \in \Psi(H)$, $f \in L^1(0,T;H)$ and $u_0 \in H$.

We here review the definition of subdifferentials. Let X be a Hilbert space and let $\phi \in \Psi(X)$. The subdifferential $\partial_X\phi(u)$ of ϕ at u in X is then defined as follows:

$$\partial_X\phi(u) \;:=\; \{\xi \in X; \phi(v) - \phi(u) \geq (\xi, v-u)_X \quad \forall v \in D(\phi)\},$$

where $(\cdot,\cdot)_X$ denotes the inner product of X and $D(\phi)$ is the *effective domain* of ϕ given by

$$D(\phi) \;:=\; \{u \in X; \phi(u) < +\infty\}.$$

Moreover the domain $D(\partial_X\phi)$ of $\partial_X\phi$ is defined by

$$D(\partial_X\phi) \;:=\; \{u \in D(\phi); \partial_X\phi(u) \neq \emptyset\}.$$

Now solutions of $\mathrm{CP}_H(\varphi, f, u_0)$ are defined as follows.

Definition 3.1. *A function $u \in C([0,T];H)$ is said to be a strong solution of $\mathrm{CP}_H(\varphi, f, u_0)$, if the following* (i)-(iii) *are all satisfied*:

(i) *u is an H-valued absolutely continuous function on $[0,T]$.*

(ii) *$u(t) \in D(\partial_H\varphi)$ for a.e. $t \in (0,T)$ and there exists a section $g(t) \in \partial_H\varphi(u(t))$ such that*

$$\frac{du}{dt}(t) + g(t) = f(t) \text{ in } H \qquad \text{for a.e. } t \in (0,T).$$

(iii) $u(0) = u_0$.

Moreover a function $u \in C([0,T];H)$ is said to be a weak solution of $\mathrm{CP}_H(\varphi, f, u_0)$ if there exist sequences $(f_n) \subset L^1(0,T;H)$, $(u_{0,n}) \subset H$ and $(u_n) \subset C([0,T];H)$ such that u_n is the strong solution of $\mathrm{CP}_H(\varphi, f_n, u_{0,n})$, $f_n \to f$ strongly in $L^1(0,T;H)$ and $u_n \to u$ strongly in $C([0,T];H)$.

It is well known that $\mathrm{CP}_H(\varphi, f, u_0)$ has a unique strong (resp. unique weak) solution if $u_0 \in D(\varphi)$ and $f \in L^2(0,T;H)$ (resp. $u_0 \in \overline{D(\varphi)}^H$ and $f \in L^1(0,T;H)$) (see e.g. Brézis [7], [8], Kenmochi [10]).

We next discuss the convergence of the solutions u_n for $\mathrm{CP}_H(\varphi_n, f_n, u_{0,n})$ when $\varphi_n \to \varphi$ on H in the sense of Mosco, $f_n \to f$ strongly in $L^2(0,T;H)$ and $u_{0,n} \to u_0$ strongly in H as $n \to +\infty$. To this end, we employ the following theorem, whose proof can be found in [4] and [5].

Theorem 3.2. *Let $\varphi_n, \varphi \in \Psi(H)$ be such that*

$$\varphi_n \to \varphi \text{ on } H \text{ in the sense of Mosco as } n \to +\infty.$$

Moreover let $f_n, f \in L^2(0,T;H)$ be such that

$$f_n \to f \qquad \text{strongly in } L^2(0,T;H)$$

and let $u_{0,n} \in \overline{D(\varphi_n)}^H$ and $u_0 \in \overline{D(\varphi)}^H$ be such that

$$u_{0,n} \to u_0 \qquad \text{strongly in } H.$$

Then the weak solutions u_n of $\mathrm{CP}_H(\varphi_n, f_n, u_{0,n})$ converge to u as $n \to +\infty$ in the following sense:

$$u_n \to u \qquad \text{strongly in } C([0,T];H),$$
$$\sqrt{t}\frac{du_n}{dt} \to \sqrt{t}\frac{du}{dt} \qquad \text{strongly in } L^2(0,T;H).$$

Moreover the limit u is the unique weak solution of $\mathrm{CP}_H(\varphi, f, u_0)$.

In particular, if $\varphi_n(u_{0,n}) \to \varphi(u_0) < +\infty$ as $n \to +\infty$, then the limit u becomes the strong solution of $\mathrm{CP}_H(\varphi, f, u_0)$ *and*

$$\frac{du_n}{dt} \to \frac{du}{dt} \qquad \text{strongly in } L^2(0,T;H).$$

We are now concerned with solutions of $(\mathrm{P})_p$ defined as follows.

Definition 3.3. *A function $u \in C([0,T];L^2(\Omega))$ is said to be a strong solution of* $(\mathrm{P})_p$, *if the following* (i)-(iii) *are all satisfied*:

(i) *u is an $L^2(\Omega)$-valued absolutely continuous function on* $[0,T]$.

(ii) *$u(t) \in W_0^{1,p}(\Omega)$ for a.e. $t \in (0,T)$ and the following equality holds*:
$$\int_\Omega \frac{\partial u}{\partial t}(x,t)v(x)dx + \int_\Omega |\nabla u|^{p-2}\nabla u(x,t)\cdot\nabla v(x)dx = \int_\Omega f(x,t)v(x)dx$$
for all $v \in C_0^\infty(\Omega)$ and a.e. $t \in (0,T)$.

(iii) $u(0) = u_0$.

Moreover a function $u \in C([0,T];L^2(\Omega))$ is said to be a weak solution of $(\mathrm{P})_p$ if there exist sequences $(f_n) \subset L^1(0,T;L^2(\Omega))$, $(u_{0,n}) \subset L^2(\Omega)$ and $(u_n) \subset C([0,T];L^2(\Omega))$ such that u_n is the strong solution of $(\mathrm{P})_p$ with u_0 and f replaced by $u_{0,n}$ and f_n respectively, $f_n \to f$ strongly in $L^1(0,T;L^2(\Omega))$ and $u_n \to u$ strongly in $C([0,T];L^2(\Omega))$.

Then $(\mathrm{P})_p$ is reduced to $\mathrm{CP}_{L^2(\Omega)}(\varphi_p, f, u_0)$. Hence we deal with $\mathrm{CP}_{L^2(\Omega)}(\varphi_p, f, u_0)$ instead of $(\mathrm{P})_p$ in the rest of this section. We now discuss the convergence of the solutions u_p for $\mathrm{CP}_{L^2(\Omega)}(\varphi_p, f_p, u_{0,p})$ when $f_p \to f$ strongly in $L^2(0,T;L^2(\Omega))$ and $u_{0,p} \to u_0$ strongly in $L^2(\Omega)$ as $p \to +\infty$. On account of Theorems 2.2 and 3.2, we have the following

Theorem 3.4. *Suppose that Ω is bounded and let (p_n) be a sequence in $[2,+\infty)$ such that $p_n \to +\infty$ as $n \to +\infty$. Moreover let $f_n, f \in L^2(0,T;L^2(\Omega))$, $u_{0,n} \in L^2(\Omega)$ and $u_0 \in K$ be such that*

$$\begin{aligned} f_n &\to f && \text{strongly in } L^2(0,T;L^2(\Omega)), \\ u_{0,n} &\to u_0 && \text{strongly in } L^2(\Omega). \end{aligned}$$

Then the weak solutions u_n of $\mathrm{CP}_{L^2(\Omega)}(\varphi_{p_n}, f_n, u_{0,n})$ converge to u as $n \to +\infty$ in the following sense:

$$\begin{aligned} u_n &\to u && \text{strongly in } C([0,T];L^2(\Omega)), \\ \sqrt{t}\frac{du_n}{dt} &\to \sqrt{t}\frac{du}{dt} && \text{strongly in } L^2(0,T;L^2(\Omega)). \end{aligned}$$

Moreover the limit u is the unique weak solution of $\mathrm{CP}_{L^2(\Omega)}(\varphi_\infty, f, u_0)$.

In particular, if $(1/p_n)\int_\Omega |\nabla u_{0,n}(x)|^{p_n}dx \to 0$ as $n \to +\infty$, then the limit u becomes the strong solution of $\mathrm{CP}_{L^2(\Omega)}(\varphi_\infty, f, u_0)$ *and*

$$\frac{du_n}{dt} \to \frac{du}{dt} \qquad \text{strongly in } L^2(0,T;L^2(\Omega)).$$

Proof By Theorem 2.2, we have already known that $\varphi_{p_n} \to \varphi_\infty$ on $L^2(\Omega)$ in the sense of Mosco as $p_n \to +\infty$. Hence Theorem 3.2 completes the proof. [Q.E.D.]

Remark 3.5. (1) In [6], they prove the strong convergence of strong solutions u_p for $\mathrm{CP}_{L^2(\Omega)}(\varphi_p, f, u_0)$ in $C([0,T]; L^2(\Omega))$ when $f \in L^2(0,T; L^2(\Omega))$ and $u_0 \in K$. On the other hand, noting that

$$\frac{1}{p}\int_\Omega |\nabla u_0(x)|^p dx \quad \leq \quad \frac{1}{p}|\Omega| \to 0 \qquad \text{as } p \to +\infty,$$

we find that $\varphi_p(u_0) \to \varphi_\infty(u_0)$ as $p \to +\infty$; hence we can derive the strong convergence of u_p in $W^{1,2}(0,T; L^2(\Omega))$ from Theorem 2.2. From this observation, our approach would have the advantage over previous studies.

(2) On account of Remark 2.3, the same conclusion as in Theorem 3.4 can be drawn for $(\mathrm{P})_p$ with the boundary condition (4), which is transcribed as $\mathrm{CP}_{L^2(\Omega)}(\psi_p, f, u_0)$. For this case, $\mathrm{CP}_{L^2(\Omega)}(\psi_\infty, f, u_0)$ corresponds to the limiting problem of $\mathrm{CP}_{L^2(\Omega)}(\psi_p, f, u_0)$ as $p \to +\infty$.

4 Applications to Other Equations

Our argument in the proofs of Theorems 2.2 and 3.4 is also valid for other types of quasilinear parabolic equations; in this section, we give a couple of examples. First we deal with the following parabolic system.

$$(\mathbf{P})_p \quad \begin{cases} \dfrac{\partial \mathbf{u}}{\partial t}(x,t) + \nabla \times \left\{ |\nabla \times \mathbf{u}|^{p-2} \nabla \times \mathbf{u}(x,t) \right\} = \mathbf{f}(x,t), & (x,t) \in \Omega \times (0,T), \\ \nabla \cdot \mathbf{u}(x,t) = 0, & (x,t) \in \Omega \times (0,T), \\ \mathbf{u}(x,t) = \mathbf{0}, & (x,t) \in \partial\Omega \times (0,T), \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), & x \in \Omega, \end{cases}$$

where $\mathbf{u} : \Omega \times (0,T) \to \mathbf{R}^3$ and Ω denotes a simply connected bounded domain in $\mathbf{R}^3$ with smooth boundary $\partial\Omega$. In [13], H-M. Yin et al have already studied the asymptotic behavior of solutions for $(\mathbf{P})_p$ with another type of boundary condition as $p \to +\infty$. Their work is motivated by Bean's critical-state model for type-II superconductivity and its approximation.

To reformulate $(\mathbf{P})_p$, we introduce

$$\begin{aligned} \mathbf{L}^p(\Omega) &:= (L^p(\Omega))^3, \quad 1 < p < +\infty, \quad \mathbf{L}^2_\sigma(\Omega) := \overline{\mathbf{C}^\infty_{0,\sigma}(\Omega)}^{\mathbf{L}^2(\Omega)}, \\ \mathbf{H}^1(\Omega) &:= (H^1(\Omega))^3, \quad \mathbf{H}^1_{0,\sigma}(\Omega) := \overline{\mathbf{C}^\infty_{0,\sigma}(\Omega)}^{\mathbf{H}^1(\Omega)}, \end{aligned}$$

where $\mathbf{C}^\infty_{0,\sigma}(\Omega) := \{\mathbf{u} \in (C^\infty_0(\Omega))^3; \nabla \cdot \mathbf{u} = 0\}$, with norms

$$|\mathbf{u}|_{\mathbf{L}^p(\Omega)} \;:=\; \left(\int_\Omega |\mathbf{u}(x)|^p dx \right)^{1/p}, \quad |\mathbf{u}|_{\mathbf{L}^2_\sigma(\Omega)} := |\mathbf{u}|_{\mathbf{L}^2(\Omega)},$$

$$|\mathbf{u}|_{\mathbf{H}^1_{0,\sigma}(\Omega)} \quad := \quad |\mathbf{u}|_{\mathbf{H}^1(\Omega)} := \left(|\mathbf{u}|^2_{\mathbf{L}^2(\Omega)} + \sum_{i,j=1,2,3} \left| \frac{\partial u_j}{\partial x_i} \right|^2_{L^2(\Omega)} \right)^{1/2},$$

where $\mathbf{u} := (u_1, u_2, u_3)$. Solutions of $(\mathbf{P})_p$ are defined in the following

Definition 4.1. *A function $\mathbf{u} \in C([0,T]; \mathbf{L}^2_\sigma(\Omega))$ is said to be a strong solution of $(\mathbf{P})_p$, if the following* (i)-(iii) *are all satisfied*:

(i) $\mathbf{u}$ *is an $\mathbf{L}^2_\sigma(\Omega)$-valued absolutely continuous function on* $[0,T]$.

(ii) $\mathbf{u}(t) \in \mathbf{H}^1_{0,\sigma}(\Omega)$, $\nabla \times \mathbf{u}(t) \in \mathbf{L}^p(\Omega)$ *for a.e.* $t \in (0,T)$ *and the following equality holds*:

$$\int_\Omega \frac{\partial \mathbf{u}}{\partial t}(x,t) \cdot \mathbf{v}(x) dx + \int_\Omega |\nabla \times \mathbf{u}|^{p-2} \nabla \times \mathbf{u}(x,t) \cdot \nabla \times \mathbf{v}(x) dx$$
$$= \int_\Omega \mathbf{f}(x,t) \cdot \mathbf{v}(x) dx \quad \text{for all } \mathbf{v} \in \mathbf{C}^\infty_{0,\sigma}(\Omega) \text{ and a.e. } t \in (0,T).$$

(iii) $\mathbf{u}(0) = \mathbf{u}_0$.

Moreover a function $\mathbf{u} \in C([0,T]; \mathbf{L}^2_\sigma(\Omega))$ is said to be a weak solution of $(\mathbf{P})_p$ if there exist sequences $(\mathbf{f}_n) \subset L^1(0,T; \mathbf{L}^2_\sigma(\Omega))$, $(\mathbf{u}_{0,n}) \subset \mathbf{L}^2_\sigma(\Omega)$ and $(\mathbf{u}_n) \subset C([0,T]; \mathbf{L}^2_\sigma(\Omega))$ such that $\mathbf{u}_n$ is the strong solution of $(\mathbf{P})_p$ with $\mathbf{f}$ and $\mathbf{u}_0$ replaced by $\mathbf{f_n}$ and $\mathbf{u}_{0,n}$ respectively, $\mathbf{f}_n \to \mathbf{f}$ strongly in $L^1(0,T; \mathbf{L}^2_\sigma(\Omega))$ and $\mathbf{u}_n \to \mathbf{u}$ strongly in $C([0,T]; \mathbf{L}^2_\sigma(\Omega))$.

Now define the functional Φ_p on $\mathbf{L}^2_\sigma(\Omega)$ as follows.

$$\Phi_p(\mathbf{u}) \quad = \quad \begin{cases} \dfrac{1}{p} \displaystyle\int_\Omega |\nabla \times \mathbf{u}(x)|^p dx & \text{if } \mathbf{u} \in \mathbf{H}^1_{0,\sigma}(\Omega),\ \nabla \times \mathbf{u} \in \mathbf{L}^p(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Then by Theorem 6.1 of [9, Chap.7], we find

$$|\mathbf{u}|_{\mathbf{H}^1(\Omega)} \quad \leq \quad C |\nabla \times \mathbf{u}|_{\mathbf{L}^2(\Omega)} \quad \forall \mathbf{u} \in \mathbf{H}^1_{0,\sigma}(\Omega). \tag{5}$$

Hence it is easily seen that $\Phi_p \in \Psi(\mathbf{L}^2_\sigma(\Omega))$ and $(\mathbf{P})_p$ is equivalent to $\mathrm{CP}_{\mathbf{L}^2_\sigma(\Omega)}(\Phi_p, \mathbf{f}, \mathbf{u}_0)$. Furthermore define

$$\mathbf{K} \quad := \quad \left\{ \mathbf{u} \in \mathbf{H}^1_{0,\sigma}(\Omega); |\nabla \times \mathbf{u}(x)| \leq 1 \ \text{ for a.e. } x \in \Omega \right\}$$

and

$$\Phi_\infty(\mathbf{u}) \quad = \quad \begin{cases} 0 & \text{if } \mathbf{u} \in \mathbf{K}, \\ +\infty & \text{if } \mathbf{u} \in \mathbf{L}^2_\sigma(\Omega) \setminus \mathbf{K}. \end{cases}$$

Then we also find that $\Phi_\infty \in \Psi(\mathbf{L}^2_\sigma(\Omega))$. Now repeating the same argument as in the proof of Theorem 2.2, we can derive the following

Theorem 4.2. *Let (p_n) be a sequence in $[2, +\infty)$ such that $p_n \to +\infty$ as $n \to +\infty$. Then we have*

$$\Phi_{p_n} \to \Phi_\infty \qquad \text{on } \mathbf{L}^2_\sigma(\Omega) \text{ in the sense of Mosco as } p_n \to +\infty.$$

Proof Just as in the proof of Theorem 2.2, we can immediately verify

$$(6)\qquad \begin{cases} \forall \mathbf{u} \in D(\Phi_\infty),\ \exists(\mathbf{u}_n) \subset \mathbf{L}^2_\sigma(\Omega); \\ \mathbf{u}_n \to \mathbf{u} \text{ strongly in } \mathbf{L}^2_\sigma(\Omega) \text{ and } \Phi_{p_n}(\mathbf{u}_n) \to \Phi_\infty(\mathbf{u}) \text{ as } n \to +\infty. \end{cases}$$

Hence to complete the proof, it suffices to show that

$$(7)\qquad \begin{cases} \forall(\mathbf{u}_k) \subset \mathbf{L}^2_\sigma(\Omega) \text{ satisfying } \mathbf{u}_k \to \mathbf{u} \text{ weakly in } \mathbf{L}^2_\sigma(\Omega) \text{ as } k \to +\infty, \\ \forall(n_k) \subset (n),\ \liminf_{k\to+\infty} \Phi_{p_{n_k}}(\mathbf{u}_k) \geq \Phi_\infty(\mathbf{u}). \end{cases}$$

For the case where $\mathbf{u} \in D(\Phi_\infty) = \mathbf{K}$, it is obvious that $\liminf_{k\to+\infty} \Phi_{p_{n_k}}(\mathbf{u}_k) \geq 0 = \Phi_\infty(\mathbf{u})$; for the case where $\mathbf{u} \notin \mathbf{K}$, conversely suppose that

$$\begin{aligned} &\exists(\mathbf{u}_k) \subset \mathbf{L}^2_\sigma(\Omega),\ \exists(n_k) \subset (n);\ \mathbf{u}_k \to \mathbf{u} \text{ weakly in } \mathbf{L}^2_\sigma(\Omega) \text{ as } k \to +\infty, \\ &\liminf_{k\to+\infty} \Phi_{p_{n_k}}(\mathbf{u}_k) < \Phi_\infty(\mathbf{u}) = +\infty. \end{aligned}$$

Then we can extract a subsequence (k') of (k) such that

$$\Phi_{p_{n_{k'}}}(\mathbf{u}_{k'}) \ \leq\ C \quad \forall k' \in \mathbf{N}.$$

For simplicity of notation, we write p and $\mathbf{u}_p$ for $p_{n_{k'}}$ and $\mathbf{u}_{k'}$ respectively. Hence we have

$$\begin{aligned} \left(\int_\Omega |\nabla \times \mathbf{u}_p(x)|^p dx\right)^{1/p} &\leq \{p\Phi_p(\mathbf{u}_p)\}^{1/p} \\ &\leq (pC)^{1/p} \to 1 \quad \text{as } p \to +\infty, \end{aligned}$$

which implies

$$\begin{aligned} \left(\int_\Omega |\nabla \times \mathbf{u}_p(x)|^q dx\right)^{1/q} &\leq \left(\int_\Omega |\nabla \times \mathbf{u}_p(x)|^p dx\right)^{1/p} |\Omega|^{(p-q)/(pq)} \\ &\to |\Omega|^{1/2} \quad \text{as } p \to +\infty. \end{aligned}$$

Therefore for each $q \in [2, +\infty)$, we can extract a subsequence (p_q) of (p) such that

$$\nabla \times \mathbf{u}_{p_q} \to \nabla \times \mathbf{u} \qquad \text{weakly in } \mathbf{L}^q(\Omega).$$

Moreover, by (5), we can also obtain $\mathbf{u} \in \mathbf{H}^1_{0,\sigma}(\Omega)$. From now on, we drop q in p_q. Furthermore it follows that

$$\begin{aligned} \left(\int_\Omega |\nabla \times \mathbf{u}(x)|^q dx\right)^{1/q} &\leq \liminf_{p\to+\infty} \left(\int_\Omega |\nabla \times \mathbf{u}_p(x)|^q dx\right)^{1/q} \\ &\leq \liminf_{p\to+\infty} \left(\int_\Omega |\nabla \times \mathbf{u}_p(x)|^p dx\right)^{1/p} |\Omega|^{(p-q)/(pq)} \\ &\leq \lim_{p\to+\infty} (pC)^{1/p} |\Omega|^{(p-q)/(pq)} = |\Omega|^{1/q}. \end{aligned}$$

Hence passing to the limit $q \to +\infty$, we deduce that

$$|\nabla \times \mathbf{u}(x)| \ \leq\ 1 \quad \text{for a.e. } x \in \Omega,$$

which implies $\mathbf{u} \in \mathbf{K}$. However this contradicts our assumption that $\mathbf{u} \notin \mathbf{K}$. Hence (7) holds true. [Q.E.D.]

Therefore by Theorems 3.2 and 4.2, we obtain the following

Theorem 4.3. *Let (p_n) be a sequence in $[2, +\infty)$ such that $p_n \to +\infty$ as $n \to +\infty$. Moreover let $\mathbf{f}_n, \mathbf{f} \in L^2(0, T; \mathbf{L}^2_\sigma(\Omega))$, $\mathbf{u}_{0,n} \in \mathbf{L}^2_\sigma(\Omega)$ and $\mathbf{u}_0 \in \mathbf{K}$ be such that*

$$\begin{aligned} \mathbf{f}_n \to \mathbf{f} \qquad & \textit{strongly in } L^2(0, T; \mathbf{L}^2_\sigma(\Omega)), \\ \mathbf{u}_{0,n} \to \mathbf{u}_0 \qquad & \textit{strongly in } \mathbf{L}^2_\sigma(\Omega). \end{aligned}$$

Then the weak solutions $\mathbf{u}_n$ of $\mathrm{CP}_{\mathbf{L}^2_\sigma(\Omega)}(\Phi_{p_n}, \mathbf{f}_n, \mathbf{u}_{0,n})$ converge to $\mathbf{u}$ as $n \to +\infty$ in the following sense:

$$\begin{aligned} \mathbf{u}_n \to \mathbf{u} \qquad & \textit{strongly in } C([0, T]; \mathbf{L}^2_\sigma(\Omega)), \\ \sqrt{t}\frac{d\mathbf{u_n}}{dt} \to \sqrt{t}\frac{d\mathbf{u}}{dt} \qquad & \textit{strongly in } L^2(0, T; \mathbf{L}^2_\sigma(\Omega)). \end{aligned}$$

Moreover the limit $\mathbf{u}$ is the unique weak solution of $\mathrm{CP}_{\mathbf{L}^2_\sigma(\Omega)}(\Phi_\infty, \mathbf{f}, \mathbf{u}_0)$.

In particular, if $(1/p_n) \int_\Omega |\nabla \times \mathbf{u}_{0,n}(x)|^{p_n} dx \to 0$ as $n \to +\infty$, then the limit $\mathbf{u}$ becomes a strong solution of $\mathrm{CP}_{\mathbf{L}^2_\sigma(\Omega)}(\Phi_\infty, \mathbf{f}, \mathbf{u}_0)$ and

$$\frac{d\mathbf{u}_n}{dt} \to \frac{d\mathbf{u}}{dt} \qquad \textit{strongly in } L^2(0, T; \mathbf{L}^2_\sigma(\Omega)).$$

Furthermore we can also investigate the convergence of the solutions for the following parabolic equations:

$$(\mathrm{P})^t_p \quad \frac{\partial u}{\partial t}(x, t) - \Delta^\gamma_p u(x, t) = f(x, t), \quad (x, t) \in \Omega \times (0, T),$$

where Δ^γ_p is defined by

$$\Delta^\gamma_p u(x) \ := \ \nabla \cdot \left\{ \left(\frac{1}{\gamma(x, t)} \right)^p |\nabla u(x)|^{p-2} \nabla u(x) \right\}$$

for some function $\gamma : \Omega \times (0, T) \to \mathbf{R}$. This generalization is motivated by some macroscopic model for type-II superconductivity (see [1] and [2] for more details).

Moreover the porous medium equation $(\mathrm{PM})_m$ also falls within the scope of our approach.

$$(\mathrm{PM})_m \quad \frac{\partial u}{\partial t}(x, t) - \Delta |u|^{m-2} u(x, t) = f(x, t), \quad (x, t) \in \Omega \times (0, T).$$

In [1], the asymptotic behavior of the solutions for $(\mathrm{PM})_m$ as $m \to +\infty$ is discussed.

References

[1] Akagi, G. and Ôtani, M., Convergence of functionals and its applications to parabolic equations, preprint.

[2] Akagi, G. and Ôtani, M., Time-dependent constraint problems arising from macroscopic critical-state models for type-II superconductivity and their approximations, preprint.

[3] Aronsson, G., Evans, L. C. and Wu, Y., Fast/slow diffusion and growing sandpile, J. Diff. Eq., **131**(1996), 304-335.

[4] Attouch, H., Convergence de fonctionnelles convexes, Journées d'Analyse Non Linéaire (Proc. Conf., Besanéon, 1977), Lecture Notes in Math., 665, Springer, Berlin, 1978, pp.1-40.

[5] Attouch, H., *Variational Convergence for Functions and Operators*, Applicable Mathematics Series. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[6] Barrett, J. W. and Prigozhin, L., Bean's critical-state model as the $p \to \infty$ limit of an evolutionary p-Laplacian equation, Nonlinear analysis, **42**(2000), 977-993.

[7] Brézis, H., *Operateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*, Math Studies, Vol.5 North-Holland, Amsterdam/New York, (1973).

[8] Brézis, H., *Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations, Contributions to Nonlinear Functional Analysis, ed. Zarantonello, E.*, Academic Press, New York-London (1971), pp.101-156.

[9] Duvaut, G. and Lions, J. L., Les inéquations en mécanique et en physique. Travaux et Recherches Mathématiques, No. 21. Dunod, Paris, 1972.

[10] Kenmochi, N., Solvability of nonlinear evolution equations with time-dependent constraints and applications, Bull. Fac. Education, Chiba Univ., **30**(1981), 1-87.

[11] Prigozhin, L., Sandpiles and river networks: extended systems with nonlocal interactions. Phys. Rev. E (3) **49**(1994), 1161-1167.

[12] Yin, H. M., On a p-Laplacian type of evolution system and applications to the Bean model in the type-II superconductivity theory, Quart. Appl. Math., **59**(2001), 47-66.

[13] Yin, H. M., Li, B. Q. and Zou, J., A degenerate evolution system modeling Bean's critical-state type-II superconductors, Discrete and Continuous Dynmical Systems, **8**(2002), 781-794.