ON SOME DOUBLY-NONLINEAR PARABOLIC EQUATIONS POSED IN $\mathbb{R}^d$

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Dedicated to Professor Pierluigi Colli on the occasion of his 65th birthday

Abstract. In this manuscript, existence of strong solutions to the Cauchy problem for a doubly-nonlinear parabolic equation posed in $\mathbb{R}^d$ is proved based on Colli's result [16], which extends the celebrated Colli-Visintin theory to Banach space settings, as well as the *localized Minty's trick*, which can also cover a wide class of PDEs in unbounded domains and which may enable us to overcome difficulties in identification of weak limits arising from the lack of compact embeddings due to the unboundedness of domains.

1. Introduction

In this paper, we shall consider the following Cauchy problem for a doubly-nonlinear parabolic equation posed in $\mathbb{R}^d$:

$$|\partial_t u|^{p-2}\partial_t u - \Delta_m u = f \qquad \text{in } \ \mathbb{R}^d \times (0,\infty), \tag{1.1}$$

$$u(\cdot,0) = u_0 \qquad \text{in } \ \mathbb{R}^d, \tag{1.2}$$

where $1 < m, p < \infty$, Δ_m stands for the so-called m-Laplacian given by

$$\Delta_m u := \operatorname{div}\left(|\nabla u|^{m-2}\nabla u\right)$$

and $f = f(x,t)$ and $u_0 = u_0(x)$ are given data. To the author's best knowledge, existence of (strong) solutions to the initial-boundary value problem for (1.1) posed in *bounded domains* was first proved by Pierluigi Colli [16] for nondifferentiable f (cf. see [12, 11] for differentiable f). In [16], an abstract theory is established for a doubly-nonlinear evolution equation of the form,

$$A\left(\frac{\mathrm{d}u}{\mathrm{d}t}(t)\right) + B(u(t)) \ni f(t) \ \text{ in } W, \quad 0 < t < T, \tag{1.3}$$

where $A : W \to W^*$ and $B : D(B) \subset W \to W^*$ are (possibly multi-valued) maximal monotone operators from a real Banach space W, reflexive and

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strictly convex, to its dual space W^*, and moreover, it is an extension of the celebrated Colli-Visintin theory [17] in the Hilbert space setting. The doubly-nonlinear evolution equation (1.3) has been vigorously studied by many authors from various points of views (see, e.g., [27, 31, 4, 22, 24, 26, 25, 29, 2, 7, 3, 23]). The doubly-nonlinear parabolic equation (1.1) was also studied by several authors, but there are fewer results than those on (1.3).

According to [30, §3.4.2], Equation (1.1) with $m = 2$ is called a *dual filtration equation* associated with the nonlinear diffusion equation,

$$\partial_t \rho = \Delta \left(|\rho|^{p'-2} \rho \right) \quad \text{in } \mathbb{R}^d \times (0, \infty), \tag{1.4}$$

where p' denotes the Hölder conjugate of p, that is, $p' = p/(p-1)$. Indeed, the solution u of (1.1) corresponds to the Newton potential of the density ρ, that is, $u = (-\Delta)^{-1}\rho$, when $d \geq 3$. Such a dual equation appears in the study of uniqueness of distributional solutions to (1.4).

As for the bounded domain case, Hynd and Lindgren [18] proved that every nontrivial weak solution to the Cauchy-Dirichlet problem for (1.1) with $m = p$ posed in an arbitrary bounded domain Ω decreases the p-Rayleigh quotient,

$$R(u(t)) = \frac{\|\nabla u(t)\|^p_{L^p(\Omega)}}{\|u(t)\|^p_{L^p(\Omega)}},$$

along its evolution, and moreover, an appropriately rescaled solution converges to a limit. In addition, if the limit is nontrivial, it is a ground state of the eigenvalue problem for the Dirichlet p-Laplacian, equivalently, an optimizer of the Poincaré inequality in $W^{1,p}_0(\Omega)$. They also applied such an observation to the study of the infinity-Laplace operator (see also [19]).

On the other hand, existence of solutions for the Cauchy problem (1.1), (1.2) in $\mathbb{R}^d$ may still be open to question. Indeed, in the abstract theory established in [16] (as well as in [17]), a compact embedding $D(B) \hookrightarrow W$ is assumed and plays a crucial role, and therefore, it cannot be directly applied to the Cauchy problem (1.1), (1.2) posed in $\mathbb{R}^d$; indeed, as we shall see, in the present setting, the above-mentioned (abstract) embedding corresponds to the following:

$$D^{1,m}_p(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d),$$

which is still continuous but no longer compact due to the unboundedness of the domain. Here $D^{1,m}_p(\mathbb{R}^d)$ denotes the function space defined by

$$D^{1,m}_p(\mathbb{R}^d) := \overline{C^\infty_c(\mathbb{R}^d)}^{\|\cdot\|_{D^{1,m}_p}}$$

equipped with the norm $\|\cdot\|_{D^{1,m}_p}$ given by

$$\|w\|_{D^{1,m}_p} := \|\nabla w\|_{L^m(\mathbb{R}^d)} + \|w\|_{L^p(\mathbb{R}^d)} \quad \text{for } \ w \in C^\infty_c(\mathbb{R}^d).$$

The main purpose of this paper is to prove existence of (strong) solutions to the Cauchy problem (1.1), (1.2) by developing a localized Minty's trick (see [9, §2]) for the doubly-nonlinear parabolic equation (1.1). Throughout this paper, we are concerned with strong solutions to the Cauchy problem (1.1), (1.2) in the following sense:

Definition 1.1 (Strong solution). Let $T > 0$, $u_0 \in D_p^{1,m}(\mathbb{R}^d)$ and $f \in L^{p'}(0,T;L^{p'}(\mathbb{R}^d))$. A function $u \in C([0,T];L^p(\mathbb{R}^d))$ is called a *strong solution* on $[0,T]$ to the Cauchy problem (1.1), (1.2), if the following (i)–(iii) hold true:

(i) u belongs to $W^{1,p}(0,T;L^p(\mathbb{R}^d)) \cap C([0,T];D_p^{1,m}(\mathbb{R}^d))$ and $\Delta_m u$ lies on $L^{p'}(0,T;L^{p'}(\mathbb{R}^d))$,

(ii) it holds that

$$|\partial_t u|^{p-2}\partial_t u - \Delta_m u = f \quad \text{a.e. in } \ \mathbb{R}^d \times (0,T),$$

(iii) it further holds that

$$u(\cdot,t) \to u_0 \quad \text{ strongly in } D_p^{1,m}(\mathbb{R}^d) \ \text{ as } \ t \to 0_+.$$

The main result of this paper reads,

Theorem 1.2. *Let $T > 0$ and let $1 < m, p < \infty$ be such that $p < m^* := dm/(d-m)_+$. For any $f \in L^{p'}(0,T;L^{p'}(\mathbb{R}^d))$ and $u_0 \in D_p^{1,m}(\mathbb{R}^d)$, the Cauchy problem* (1.1), (1.2) *admits a strong solution $u = u(x,t)$ on $[0,T]$ in the sense of Definition* 1.1 *such that the following maximal regularity estimate holds*:

$$\int_0^T \left\| |\partial_t u(\cdot,t)|^{p-2}\partial_t u(\cdot,t)\right\|_{L^{p'}(\mathbb{R}^d)}^{p'} \,\mathrm{d}t + \int_0^T \|\Delta_m u(\cdot,t)\|_{L^{p'}(\mathbb{R}^d)}^{p'} \,\mathrm{d}t$$
$$\leq C\left(\|\nabla u_0\|_{L^m(\mathbb{R}^d)}^m + \int_0^T \|f(\cdot,t)\|_{L^{p'}(\mathbb{R}^d)}^{p'} \,\mathrm{d}t \right) \tag{1.5}$$

for some constant $C \geq 0$ depending only on m, p.

This paper consists of four sections. The next section is devoted to recalling some preliminary facts which will be used to prove the main result of the present paper. In Section 3, we give a proof of Theorem 1.2. In Section 4, we shall provide concluding remarks. Moreover, in Appendix §A, we also present a proof of a chain-rule formula for subdifferentials in reflexive Banach spaces (see Proposition 2.2 below) based only on a classical subdifferential calculus for the convenience of the reader.

2. Preliminaries

Let us first briefly review an abstract theory established in [16] concerning the Cauchy problem for the doubly-nonlinear evolution equation (1.3). Let W and W^* be a reflexive and strictly convex Banach space and its dual space, respectively. Let $A : W \to W^*$ and $B : D(B) \subset W \to W^*$ be (possibly multi-valued) maximal monotone operators. Moreover, we introduce the following assumptions for $1 < p < +\infty$:

(A1) There exist positive constants C_1, C_2, C_3 such that

$$C_1 \|w\|_W^p \le \langle z, w \rangle_W + C_2 \quad \text{for} \quad [w,z] \in G(A),$$
$$\|z\|_{W^*}^{p'} \le C_3 (\|w\|_W^p + 1) \quad \text{for} \quad [w,z] \in G(A),$$

where $\langle \cdot, \cdot \rangle_W$ denotes the duality pairing between W and W^* and $G(A) \subset W \times W^*$ stands for the graph of A.

(A2) $B = \partial\psi$ is the subdifferential operator of a proper lower-semicontinuous convex functional $\psi : W \to (-\infty, +\infty]$.

(A3) There exists a reflexive Banach space V densely and compactly embedded in W such that $D(\psi) \subset V$ and

$$\|w\|_W^p + \psi(w) \to \infty,$$

whenever $w \in D(\psi)$ and $\|w\|_V \to +\infty$.

Here we also recall the *effective domain* $D(\psi) := \{w \in W : \psi(w) < +\infty\}$ as well as the *subdifferential operator* $\partial\psi : W \to 2^{W^*}$ defined by

$$\partial\psi(w) := \{\xi \in W^* : \psi(v) - \psi(w) \ge \langle \xi, v - w \rangle_W \quad \text{for } v \in D(\psi)\}$$

for $w \in D(\psi)$ with domain $D(\partial\psi) := \{w \in D(\psi) : \partial\psi(w) \ne \emptyset\}$. Then we recall

Theorem 2.1 ([16, Theorem 1]). *Under the assumptions* (A1)–(A3) *with some* $1 < p < +\infty$, *for every* $f \in L^{p'}(0,T;W^*)$ *and* $u_0 \in D(\psi)$ *there exists a triplet*

$$u \in W^{1,p}(0,T;W) \cap L^\infty(0,T;V), \quad v, w \in L^{p'}(0,T;W^*)$$

such that

$$w(t) + v(t) = f(t), \ w(t) \in A(u'(t)), \ v(t) \in B(u(t)) \quad \textit{for a.e. } t \in (0,T),$$
$$u(0) = u_0,$$

where $u' := (\mathrm{d}/\mathrm{d}t)u$.

The following chain-rule formula is used in [16] for proving the theorem above and also plays a crucial role in the present paper to derive a priori estimates as well as to identify weak limits of nonlinear terms.

Proposition 2.2 (Chain-rule formula for subdifferentials [16]). *Let B be a reflexive Banach space and let B^* be its dual space. Let $\phi : B \to (-\infty, +\infty]$ be a proper lower semicontinuous convex functional and denote by $\partial\phi : B \to 2^{B^*}$ the subdifferential operator of ϕ. Let $1 < p < +\infty$, let I be an open interval, and let $u \in W^{1,p}(I;B)$ be such that $u(t) \in D(\partial\phi)$ for a.e. $t \in I$. Suppose that there exists $g \in L^{p'}(I;B^*)$ such that $g(t) \in \partial\phi(u(t))$ for a.e. $t \in I$. Then the function $t \mapsto \phi(u(t))$ is absolutely continuous on $\bar{I}$, and moreover, it holds that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(u(t)) = \left\langle \xi, \frac{\mathrm{d}u}{\mathrm{d}t}(t) \right\rangle_B \quad \textit{for any } \ \xi \in \partial\phi(u(t)) \ \ \textit{and a.e. } t \in I. \tag{2.1}$$

For the convenience of the reader, we shall give a proof of this proposition in Appendix §A. We close this section with recalling the so-called Minty's trick for maximal monotone operators (see, e.g., [15, Lemma 1.3]).

PROPOSITION 2.3 (Minty's trick). *Let A be a (possibly multi-valued) maximal monotone operator from a Banach space B into its dual space B^*. Let $u_n \in D(A)$ and $\xi_n \in A(u_n)$ be such that $u_n \to u$ weakly in B, $\xi_n \to \xi$ weakly star in B^* and*

$$\limsup_{n\to\infty} \langle \xi_n, u_n \rangle_B \le \langle \xi, u \rangle_B$$

for some $u \in B$ and $\xi \in B^*$. Then $u \in D(A)$ and $\xi \in A(u)$. Moreover,

$$\lim_{n\to\infty} \langle \xi_n, u_n \rangle_B = \langle \xi, u \rangle_B.$$

3. PROOF OF THEOREM 1.2

We divide a proof of Theorem 1.2 into four steps, each of which corresponds to the following subsections. In what follows, we denote by B_R the open ball centered at the origin of radius $R > 0$ and fix $T > 0$ arbitrarily.

3.1. **Approximation.** For each $n \in \mathbb{N}$, we consider the following Cauchy-Dirichlet problem as an approximation of the Cauchy problem (1.1), (1.2):

$$|\partial_t u_n|^{p-2}\partial_t u_n - \Delta_m u_n = f \qquad \text{in } B_n \times (0,T), \tag{3.1}$$

$$u_n = 0 \qquad \text{on } \partial B_n \times (0,T), \tag{3.2}$$

$$u_n(\cdot,0) = u_{0,n} \qquad \text{in } B_n, \tag{3.3}$$

where $u_{0,n} \in C_c^\infty(\mathbb{R}^d)$ is a smooth approximation of u_0 such that

$$\operatorname{supp} u_{0,n} \subset B_n, \quad u_{0,n} \to u_0 \text{ in } D_p^{1,m}(\mathbb{R}^d),$$

that is, $u_{0,n} \to u_0$ in $L^p(\mathbb{R}^d)$ and $\nabla u_{0,n} \to \nabla u_0$ in $L^m(\mathbb{R}^d;\mathbb{R}^d)$, as $n \to \infty$. Then thanks to Colli's abstract theory (see Theorem 2.1), the Cauchy-Dirichlet problem (3.1)–(3.3) admits a strong solution on $[0,T]$,

$$u_n \in W^{1,p}(0,T;L^p(B_n)) \cap L^\infty(0,T;W_0^{1,m}(B_n)).$$

Indeed, we set

$$W = L^p(B_n), \quad V = W_0^{1,m}(B_n),$$

$$A(w) = |w|^{p-2}w \quad \text{for } w \in L^p(B_n),$$

$$\psi(w) = \begin{cases} \frac{1}{m}\int_{B_n} |\nabla w(x)|^m \,\mathrm{d}x & \text{if } w \in W_0^{1,m}(B_n), \\ +\infty & \text{otherwise.} \end{cases}$$

Then $B(w) = \partial\psi(w)$ coincides with $-\Delta_m w$ for $w \in D(\partial\psi) = \{w \in W_0^{1,m}(B_n) \colon \Delta_m w \in L^{p'}(B_n)\}$. Moreover, one can easily check all the assumptions (A1)–(A3) of Theorem 2.1, provided that $p < m^*$, which is used to check the compact embedding $V \hookrightarrow W$ (i.e., $W_0^{1,m}(B_n) \hookrightarrow L^p(B_n)$).

3.2. **A priori estimates.** In this subsection, we derive a priori estimates for the approximate solutions (u_n) uniformly for $n \to +\infty$ in a simple energy method with the use of the chain-rule formula in Proposition 2.2. Here and henceforth, we shall denote by u_n again the zero extension of u_n obtained above onto the whole $\mathbb{R}^d$ when no confusion can arise.

Testing (3.1) by $\partial_t u_n$, we have

$$\begin{aligned}\|\partial_t u_n(t)\|^p_{L^p(B_n)} &+ \langle -\Delta_m u_n(t), \partial_t u_n(t)\rangle_{L^p(B_n)} \\ &= \langle f(t), \partial_t u_n(t)\rangle_{L^p(B_n)} \\ &\leq \|f(t)\|_{L^{p'}(B_n)} \|\partial_t u_n(t)\|_{L^p(B_n)} \end{aligned} \tag{3.4}$$

for a.e. $t \in (0,T)$. Here we employ Proposition 2.2 to observe that the function $t \mapsto \frac{1}{m}\|\nabla u_n(t)\|^m_{L^m(B_n)}$ is absolutely continuous on $[0,T]$ and

$$\begin{aligned}\langle -\Delta_m u_n(t), \partial_t u_n(t)\rangle_{L^p(B_n)} &= \langle \partial\psi(u_n(t)), \partial_t u_n(t)\rangle_{L^p(B_n)} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\psi(u_n(t)) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{m}\|\nabla u_n(t)\|^m_{L^m(B_n)}\right)\end{aligned}$$

for a.e. $t \in (0,T)$. Here we emphasize that the above facts are not trivial, since the functional ψ is not Fréchet differentiable in $L^p(B_n)$ (although the restriction of ψ onto $W^{1,m}_0(B_n)$ is Fréchet differentiable in $W^{1,m}_0(B_n)$) and u_n is differentiable in time in the strong topology of $L^p(B_n)$ (but not in $W^{1,m}_0(B_n)$), and therefore, the chain-rule for subdifferentials in reflexive Banach spaces (see Proposition 2.2) plays an essential role.

Therefore integrating both sides of (3.4) over $(0,t)$ (and using Young's inequality as well), we infer that

$$\begin{aligned}\frac{1}{p'}\int_0^t \|\partial_t u_n(s)\|^p_{L^p(B_n)}\,\mathrm{d}s &+ \frac{1}{m}\|\nabla u_n(t)\|^m_{L^m(B_n)} \\ &\leq \frac{1}{p'}\int_0^t \|f(s)\|^{p'}_{L^{p'}(B_n)}\,\mathrm{d}s + \frac{1}{m}\|\nabla u_{0,n}\|^m_{L^m(B_n)}\end{aligned} \tag{3.5}$$

for any $t \in [0,T]$. Thus we obtain the boundedness of the approximate solutions (u_n) in $W^{1,p}(0,T;L^p(\mathbb{R}^d))$ as well as that of (∇u_n) in $L^\infty(0,T;L^m(\mathbb{R}^d;\mathbb{R}^d))$ for $n \in \mathbb{N}$. Indeed, we find that

$$u_n(t) = u_{0,n} + \int_0^t \partial_t u_n(s)\,\mathrm{d}s \quad \text{in } L^p(\mathbb{R}^d) \qquad \text{for} \quad t \geq 0,$$

whence it follows that

$$\|u_n(t)\|_{L^p(\mathbb{R}^d)} \leq \|u_{0,n}\|_{L^p(\mathbb{R}^d)} + \int_0^t \|\partial_t u_n(s)\|_{L^p(\mathbb{R}^d)}\,\mathrm{d}s \qquad \text{for} \quad t \geq 0.$$

Hence we infer from the boundedness of $(\partial_t u_n)$ in $L^p(0,T;L^p(\mathbb{R}^d))$ (see (3.5)) that

$$\sup_{t\in[0,T]} \|u_n(t)\|_{L^p(\mathbb{R}^d)} \le C.$$

Moreover, we have

$$\int_0^T \left\||\partial_t u_n(s)|^{p-2}\partial_t u_n(s)\right\|_{L^{p'}(\mathbb{R}^d)}^{p'} \, \mathrm{d}s = \int_0^T \|\partial_t u_n(s)\|_{L^p(\mathbb{R}^d)}^p \, \mathrm{d}s \le C,$$

which along with (3.1) implies

$$\int_0^T \left\|\overline{-\Delta_m u_n}(s)\right\|_{L^{p'}(\mathbb{R}^d)}^{p'} \, \mathrm{d}s \le C.$$

Here $\overline{-\Delta_m u_n}$ is given by

$$\overline{-\Delta_m u_n} = \begin{cases} -\Delta_m u_n & \text{in } \ B_n \times (0,T), \\ f & \text{in } \ (\mathbb{R}^d \setminus B_n) \times (0,T). \end{cases}$$

Then we see that

$$|\partial_t u_n|^{p-2}\partial_t u_n + \left(\overline{-\Delta_m u_n}\right) = f \quad \text{a.e. in } \mathbb{R}^d \times (0,T). \tag{3.6}$$

3.3. **Convergence.** From the a priori estimates obtained so far, we can immediately derive, up to a (not relabeled) subsequence, that

$$u_n \to u \qquad \text{weakly in } W^{1,p}(0,T;L^p(\mathbb{R}^d)), \tag{3.7}$$

$$\nabla u_n \to \nabla u \qquad \text{weakly star in } L^\infty(0,T;L^m(\mathbb{R}^d;\mathbb{R}^d)), \tag{3.8}$$

$$|\partial_t u_n|^{p-2}\partial_t u_n \to \chi \qquad \text{weakly in } L^{p'}(0,T;L^{p'}(\mathbb{R}^d)), \tag{3.9}$$

$$\overline{-\Delta_m u_n} \to \xi \qquad \text{weakly in } L^{p'}(0,T;L^{p'}(\mathbb{R}^d)) \tag{3.10}$$

for some $u \in W^{1,p}(0,T;L^p(\mathbb{R}^d)) \cap L^\infty(0,T;D_p^{1,m}(\mathbb{R}^d))$ as well as $\chi, \xi \in L^{p'}(0,T;L^{p'}(\mathbb{R}^d))$. Moreover, we derive from (3.6) that

$$\chi + \xi = f \quad \text{in } L^{p'}(0,T;L^{p'}(\mathbb{R}^d)). \tag{3.11}$$

It still remains to identify the weak limits of nonlinear terms, i.e., χ and ξ, as well as to check the initial condition along with the regularity $u \in C([0,T];D_p^{1,m}(\mathbb{R}^d))$. There arises a significant difference from the bounded domain case (say, in $\Omega \subset \mathbb{R}^d$ as in [16]), where the strong compactness of (u_n) in $C([0,T];L^p(\Omega))$ can be proved from the compact embedding $W_0^{1,m}(\Omega) \hookrightarrow L^p(\Omega)$ with the aid of the Aubin-Lions-Simon lemma, and therefore, the weak limits can be identified via standard Minty's trick. On the other hand, in the whole domain case (i.e., $\Omega = \mathbb{R}^d$), due to the lack of compactness of the embedding $D_p^{1,m}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$, we cannot derive the strong compactness of (u_n) in $C([0,T];L^p(\mathbb{R}^d))$. Instead, we can still verify that, for any $R > 0$, up to a (not relabeled) subsequence,

$$u_n \to u \quad \text{strongly in } C([0,T];L^p(B_R) \cap L^m(B_R)) \tag{3.12}$$

by using the Aubin-Lions-Simon lemma (see [28, Theorem 3]), since $W^{1,m}(B_R)$ is compactly embedded in $L^q(B_R)$ for $1 \leq q < m^*$ and $0 < R < \infty$. It also leads us to obtain, up to a (not relabeled) subsequence of (n),

$$u_n \to u \quad \text{a.e. in } \mathbb{R}^d \times (0,T) \tag{3.13}$$

thanks to a diagonal argument. However, these facts are still insufficient to identify the weak limits immediately. Indeed, both χ and ξ cannot be identified only from the pointwise convergence (3.13) as well as the weak convergences. To be more precise, χ is the weak limit of the power nonlinearity of the *time-derivative* $\partial_t u_n$, and moreover, roughly speaking, ξ is the weak limit of the m-Laplacian, which includes the *gradient* ∇u_n. Here we stress that the pointwise convergence has been proved for u_n itself, but not for their derivatives, $\partial_t u_n$ and ∇u_n.

Before closing this subsection, let us check the initial condition from the facts obtained so far. Recalling (3.3) and using (3.12), for each $R > 0$ we find in particular that $u_n(0) \to u(0)$ strongly in $L^p(B_R)$ by taking a (not relabeled) subsequence of (n). Moreover, using the fact that $u_n(0) = u_{0,n} \to u_0$ strongly in $L^p(\mathbb{R}^d)$, we infer via a diagonal argument that

$$u(0) = u_0 \ \text{ a.e. in } \mathbb{R}^d. \tag{3.14}$$

Moreover, we may prove that

$$u_n(t) \to u(t) \quad \text{weakly in } L^p(\mathbb{R}^d) \ \text{ for } \ t \geq 0. \tag{3.15}$$

Indeed, we see from (3.7) and (3.14) that

$$\begin{aligned} u_n(t) - u(t) &= u_n(t) - u_{0,n} - (u(t) - u_0) + u_{0,n} - u_0 \\ &= \int_0^t (\partial_t u_n(s) - \partial_t u(s))\, \mathrm{d}s + u_{0,n} - u_0 \\ &\to 0 \quad \text{weakly in } L^p(\mathbb{R}^d) \ \text{ for } \ t \geq 0. \end{aligned}$$

3.4. **Identification of weak limits via localized Minty's trick.** To overcome the difficulty mentioned above, we shall employ an idea of the *localized Minty's trick* developed in [9] for doubly-nonlinear diffusion equations. To do so, we fix $R > 0$ and localize the equation onto the ball B_R by multiplying both sides of (3.1) for $n > R$ by a smooth cut-off (in space) function $\rho \geq 0$ whose support is the closure of B_R. Then it follows that

$$\rho|\partial_t u_n|^{p-2}\partial_t u_n - \rho\, \Delta_m u_n = \rho f \ \text{ in } B_R \times (0,T).$$

Here and henceforth, we also denote by u_n again the restriction of u_n onto B_R when no confusion can arise. We next test it by $\varphi \in X := W^{1,m}(B_R)$. We here note that neither u_n nor φ may vanish on the boundary ∂B_R. However, since ρ vanishes on ∂B_R (hence so does $\rho\varphi$), we can observe that

$$\int_{B_R} \rho|\partial_t u_n|^{p-2}(\partial_t u_n)\, \varphi \, \mathrm{d}x$$

$$
\begin{aligned}
&= \int_{B_R} \rho\,(\Delta_m u_n)\,\varphi\,\mathrm{d}x + \int_{B_R} \rho f \varphi\,\mathrm{d}x \\
&= -\int_{B_R} |\nabla u_n|^{m-2}\nabla u_n \cdot \nabla(\varphi\rho)\,\mathrm{d}x + \int_{B_R} \rho f\varphi\,\mathrm{d}x \\
&= -\int_{B_R} |\nabla u_n|^{m-2}\nabla u_n \cdot (\nabla\varphi)\rho\,\mathrm{d}x \\
&\quad - \int_{B_R} |\nabla u_n|^{m-2}\nabla u_n \cdot (\nabla\rho)\varphi\,\mathrm{d}x + \int_{B_R} \rho f\varphi\,\mathrm{d}x \\
&=: -\langle A(u_n),\varphi\rangle_X + \langle F(u_n),\varphi\rangle_{L^m(B_R)} + \int_{B_R} \rho f\varphi\,\mathrm{d}x,
\end{aligned}
$$

where $A : X \to X^*$ and $F : X \to L^{m'}(B_R) \subset X^*$ are defined by

$$
\begin{aligned}
\langle A(w), v\rangle_X &= \int_{B_R} |\nabla w|^{m-2}\nabla w \cdot (\nabla v)\rho\,\mathrm{d}x \quad \text{ for } \ v \in X, \\
\langle F(w), v\rangle_{L^m(B_R)} &= -\int_{B_R} |\nabla w|^{m-2}\nabla w \cdot (\nabla\rho) v\,\mathrm{d}x \quad \text{ for } \ v \in L^m(B_R)
\end{aligned}
$$

for $w \in X$. Namely, u_n solves the following auxiliary evolution equation,

$$
\rho|\partial_t u_n|^{p-2}\partial_t u_n + A(u_n) = F(u_n) + \rho f \quad \text{in } X^*, \quad 0 < t < T. \tag{3.16}
$$

Here we also note that $A : X \to X^*$ is maximal monotone, since it is obviously monotone and continuous (see [13, Chap. II, Theorem 1.3]). We shall achieve the identification of the weak limit ξ of $\overline{-\Delta_m u_n}$ (see (3.10)) by identifying weak limits of $A(u_n)$ and $F(u_n)$ above.

To this end, we first find that $A(u_n)$ is bounded in $L^\infty(0,T;X^*)$, and therefore, there exists $\zeta \in L^\infty(0,T;X^*)$ such that, up to a (not relabeled) subsequence,

$$
A(u_n) \to \zeta \quad \text{ weakly star in } L^\infty(0,T;X^*),
$$

which in particular implies

$$
\int_0^T \langle A(u_n),\varphi\rangle_X\,\mathrm{d}t \to \int_0^T \langle\zeta,\varphi\rangle_X\,\mathrm{d}t
$$

for $\varphi \in W^{1,m}(B_R)$. On the other hand, by virtue of the boundedness of (∇u_n) in $L^\infty(0,T;L^m(\mathbb{R}^d;\mathbb{R}^d))$, there exists $\eta \in L^\infty(0,T;L^{m'}(\mathbb{R}^d;\mathbb{R}^d))$ such that

$$
|\nabla u_n|^{m-2}\nabla u_n \to \eta \quad \text{ weakly star in } L^\infty(0,T;L^{m'}(\mathbb{R}^d;\mathbb{R}^d)),
$$

up to a (not relabeled) subsequence. Hence we can also derive from (3.9) and (3.16) that

$$
\begin{aligned}
\int_0^T \langle A(u_n),\varphi\rangle_X\,\mathrm{d}t &= \int_0^T \langle \rho f - \rho|\partial_t u_n|^{p-2}\partial_t u_n + F(u_n),\varphi\rangle_X\,\mathrm{d}t \\
&\to \int_0^T \int_{B_R} \rho f\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_{B_R} \rho\chi\varphi\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

$$- \int_0^T \int_{B_R} \eta \cdot (\nabla \rho) \varphi \, \mathrm{d}x \, \mathrm{d}t$$

for $\varphi \in W^{1,m}(B_R)$. Thus we have

$$\int_0^T \langle \zeta, \varphi \rangle_X \, \mathrm{d}t = \int_0^T \int_{B_R} \rho f \varphi \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_{B_R} \rho \chi \varphi \, \mathrm{d}x \, \mathrm{d}t \\ - \int_0^T \int_{B_R} \eta \cdot (\nabla \rho) \varphi \, \mathrm{d}x \, \mathrm{d}t \tag{3.17}$$

for $\varphi \in W^{1,m}(B_R)$.

Now, in order to apply Proposition 2.3 to identify the weak limit ζ of $A(u_n)$, we calculate

$$\int_0^T \langle A(u_n), u_n \rangle_X \, \mathrm{d}t \\ \stackrel{(3.16)}{=} \int_0^T \int_{B_R} \rho f u_n \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_{B_R} \rho |\partial_t u_n|^{p-2} (\partial_t u_n) u_n \, \mathrm{d}x \, \mathrm{d}t \\ + \int_0^T \langle F(u_n), u_n \rangle_{L^m(B_R)} \, \mathrm{d}t.$$

Hence (3.12) yields

$$\int_0^T \langle F(u_n), u_n \rangle_{L^m(B_R)} \, \mathrm{d}t = - \int_0^T \int_{B_R} |\nabla u_n|^{m-2} \nabla u_n \cdot (\nabla \rho) u_n \, \mathrm{d}x \, \mathrm{d}t \\ \to - \int_0^T \int_{B_R} \eta \cdot (\nabla \rho) u \, \mathrm{d}x \, \mathrm{d}t$$

and

$$\int_0^T \int_{B_R} \rho |\partial_t u_n|^{p-2} (\partial_t u_n) u_n \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_{B_R} \rho \chi u \, \mathrm{d}x \, \mathrm{d}t.$$

Therefore we see that

$$\int_0^T \langle A(u_n), u_n \rangle_X \, \mathrm{d}t \\ \to \int_0^T \int_{B_R} \rho f u \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_{B_R} \rho \chi u \, \mathrm{d}x \, \mathrm{d}t \\ - \int_0^T \int_{B_R} \eta \cdot (\nabla \rho) u \, \mathrm{d}x \, \mathrm{d}t \\ \stackrel{(3.17)}{=} \int_0^T \langle \zeta, u \rangle_X \, \mathrm{d}t.$$

Thus applying Minty's trick to the maximal monotone operator $A : X \to X^*$ (see Proposition 2.3), we conclude that

$$u \in D(A), \quad \zeta = A(u),$$

that is,

$$\int_0^T \langle \zeta, \varphi \rangle_X \,\mathrm{d}t = \int_0^T \langle A(u), \varphi \rangle_X \,\mathrm{d}t = \int_0^T \int_{B_R} |\nabla u|^{p-2} \nabla u \cdot (\nabla \varphi) \rho \,\mathrm{d}x \,\mathrm{d}t$$

for $\varphi \in W^{1,m}(B_R)$ (see also [20, Proposition 1.1]). Moreover, we also obtain

$$\int_0^T \langle A(u_n), u_n \rangle_X \,\mathrm{d}t \to \int_0^T \langle A(u), u \rangle_X \,\mathrm{d}t, \tag{3.18}$$

whence it follows that

$$\int_0^T \int_{B_R} |\nabla u_n - \nabla u|^m \rho \,\mathrm{d}x \,\mathrm{d}t \to 0$$

from the uniform convexity of the weighted L^m norm with $1 < m < \infty$. Moreover, this also helps us to obtain

$$\eta = |\nabla u|^{m-2} \nabla u;$$

thus the weak limit of $F(u_n)$ has been identified as well.

Now, let $\phi \in C_c^\infty(\mathbb{R}^d \times (0,T))$ be fixed and take $R > 0$ large enough that $\operatorname{supp} \phi(\cdot, t) \subset B_{R/2}$ for all $t \in (0,T)$. Moreover, (re)take $\rho \in C_c^\infty(\mathbb{R}^d)$ satisfying

$$\rho \geq 0, \quad \rho \equiv 1 \text{ on } \overline{B}_{R/2}, \quad \operatorname{supp} \rho = \overline{B}_R.$$

Then noting that $\rho\phi = \phi$ in $\operatorname{supp} \phi$, we infer from (3.10) that

$$\begin{aligned}\lim_{n\to\infty} \int_0^T \langle -\Delta_m u_n, \rho\phi \rangle_{W_0^{1,m}(B_n)} \,\mathrm{d}t &= \lim_{n\to\infty} \int_0^T \langle -\Delta_m u_n, \phi \rangle_{L^p(B_{R/2})} \,\mathrm{d}t \\ &= \lim_{n\to\infty} \int_0^T \langle \overline{-\Delta_m u_n}, \phi \rangle_{L^p(\mathbb{R}^d)} \,\mathrm{d}t \\ &= \int_0^T \langle \xi, \phi \rangle_{L^p(\mathbb{R}^d)} \,\mathrm{d}t\end{aligned}$$

and

$$\begin{aligned}&\lim_{n\to\infty} \int_0^T \langle -\Delta_m u_n, \rho\phi \rangle_{W_0^{1,m}(B_n)} \,\mathrm{d}t \\ &\quad = \lim_{n\to\infty} \int_0^T \langle A(u_n), \phi \rangle_X \,\mathrm{d}t - \lim_{n\to\infty} \int_0^T \langle F(u_n), \phi \rangle_{L^m(B_R)} \,\mathrm{d}t \\ &\quad = \int_0^T \langle A(u), \phi \rangle_X \,\mathrm{d}t - \int_0^T \langle F(u), \phi \rangle_{L^m(B_R)} \,\mathrm{d}t \\ &\quad = \int_0^T \int_{B_R} |\nabla u|^{m-2} \nabla u \cdot (\nabla \phi) \rho \,\mathrm{d}x \,\mathrm{d}t \\ &\qquad + \int_0^T \int_{B_R} |\nabla u|^{m-2} \nabla u \cdot (\nabla \rho) \phi \,\mathrm{d}x \,\mathrm{d}t\end{aligned}$$

$$= \int_0^T \int_{\mathbb{R}^d} |\nabla u|^{m-2} \nabla u \cdot \nabla \phi \, \mathrm{d}x \, \mathrm{d}t.$$

Here we also used the fact that $\nabla \rho \equiv 0$ on $\operatorname{supp} \phi$. Thus from the arbitrariness of $\phi \in C_c^\infty(\mathbb{R}^d \times (0,T))$, the weak limit ξ turns out to fulfill

$$\xi(t) = -\Delta_m u(t) \quad \text{in } L^{p'}(\mathbb{R}^d) \quad \text{for a.e. } t \in (0,T). \tag{3.19}$$

We next check $u \in C([0,T]; D_p^{1,m}(\mathbb{R}^d))$. Since u belongs to $C([0,T]; L^p(\mathbb{R}^d))$ as well as $L^\infty(0,T; D_p^{1,m}(\mathbb{R}^d))$, we find that $t \mapsto u(t)$ is weakly continuous on $[0,T]$ with values in $D_p^{1,m}(\mathbb{R}^d)$ (see [21]). Furthermore, we set

$$B = L^p(\mathbb{R}^d), \quad \phi(w) = \begin{cases} \frac{1}{m} \int_{\mathbb{R}^d} |\nabla w|^m \, \mathrm{d}x & \text{if } w \in D_p^{1,m}(\mathbb{R}^d), \\ \infty & \text{otherwise.} \end{cases} \tag{3.20}$$

Then $\partial \phi(w)$ coincides with $-\Delta_m w$ in $L^{p'}(\mathbb{R}^d)$ for

$$w \in D(\partial \phi) = \{ w \in D_p^{1,m}(\mathbb{R}^d) \colon \Delta_m w \in L^{p'}(\mathbb{R}^d) \},$$

and moreover, all the assumptions for Proposition 2.2 can be checked easily. Thus recalling that $-\Delta_m u = \xi \in L^{p'}(0,T; L^{p'}(\mathbb{R}^d))$ and $u \in W^{1,p}(0,T; L^p(\mathbb{R}^d))$ and exploiting Proposition 2.2, we can deduce that the function $t \mapsto \|\nabla u(t)\|_{L^m(\mathbb{R}^d)}$ is (absolutely) continuous on $[0,T]$, and therefore, from the uniform convexity of $\| \cdot \|_{D_p^{1,m}(\mathbb{R}^d)}$, we conclude that

$$u \in C([0,T]; D_p^{1,m}(\mathbb{R}^d)),$$

which in particular yields

$$u(t) \to u_0 \quad \text{strongly in } D_p^{1,m}(\mathbb{R}^d) \quad \text{as } t \to 0_+.$$

We finally identify the weak limit χ of $|\partial_t u_n|^{p-2} \partial_t u_n$. To this end, testing (3.1) by $\partial_t u_n$, by a simple calculation, we have

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} |\partial_t u_n|^{p-2} (\partial_t u_n) \partial_t u_n \, \mathrm{d}x \, \mathrm{d}t \\
&\quad = \int_0^T \langle \Delta_m u_n, \partial_t u_n \rangle_{L^p(B_n)} \, \mathrm{d}t + \int_0^T \langle f, \partial_t u_n \rangle_{L^p(B_n)} \, \mathrm{d}t \\
&\quad = -\int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{m} \int_{B_n} |\nabla u_n|^m \, \mathrm{d}x \right) \mathrm{d}t + \int_0^T \langle f, \partial_t u_n \rangle_{L^p(B_n)} \, \mathrm{d}t \\
&\quad = -\frac{1}{m} \int_{\mathbb{R}^d} |\nabla u_n(T)|^m \, \mathrm{d}x + \frac{1}{m} \int_{\mathbb{R}^d} |\nabla u_{0,n}|^m \, \mathrm{d}x \\
&\qquad + \int_0^T \langle f, \partial_t u_n \rangle_{L^p(\mathbb{R}^d)} \, \mathrm{d}t,
\end{aligned}$$

where u_n has been extended by zero onto $\mathbb{R}^d \setminus B_n$ with the same notation in the last line. It further yields

$$\limsup_{n \to \infty} \int_0^T \int_{\mathbb{R}^d} |\partial_t u_n|^{p-2} (\partial_t u_n) \partial_t u_n \, \mathrm{d}x \, \mathrm{d}t$$

$$\le -\frac{1}{m}\int_{\mathbb{R}^d} |\nabla u(T)|^m \, \mathrm{d}x + \frac{1}{m}\int_{\mathbb{R}^d} |\nabla u_0|^m \, \mathrm{d}x + \int_0^T \int_{\mathbb{R}^d} f \partial_t u \, \mathrm{d}x \, \mathrm{d}t.$$

Here we used (3.15) as well as the weak lower-semicontinuity of ϕ defined by (3.20) in $L^p(\mathbb{R}^d)$. Employing Proposition 2.2 again, we note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{m}\int_{\mathbb{R}^d} |\nabla u(t)|^m \, \mathrm{d}x\right) = \langle -\Delta_m u(t), \partial_t u(t)\rangle_{L^p(\mathbb{R}^d)} \quad \text{for a.e. } t \in (0,T).$$

Integrating both sides over $(0,T)$ and recalling (3.14), we infer that

$$\begin{aligned} &\frac{1}{m}\int_{\mathbb{R}^d} |\nabla u(T)|^m \, \mathrm{d}x - \frac{1}{m}\int_{\mathbb{R}^d} |\nabla u_0|^m \, \mathrm{d}x \\ &\quad = \int_0^T \langle -\Delta_m u(t), \partial_t u(t)\rangle_{L^p(\mathbb{R}^d)} \, \mathrm{d}t. \end{aligned} \tag{3.21}$$

Therefore one has

$$\begin{aligned} &\limsup_{n\to\infty} \int_0^T \int_{\mathbb{R}^d} |\partial_t u_n|^p \, \mathrm{d}x \, \mathrm{d}t \\ &\quad \le \int_0^T \langle \Delta_m u + f, \partial_t u\rangle_{L^p(\mathbb{R}^d)} \, \mathrm{d}t = \int_0^T \int_{\mathbb{R}^d} \chi \partial_t u \, \mathrm{d}x \, \mathrm{d}t. \end{aligned}$$

Hence thanks to Proposition 2.3 along with the maximal monotonicity of the operator $w \mapsto |w|^{p-2}w$ in $L^p(\mathbb{R}^d \times (0,T)) \times L^{p'}(\mathbb{R}^d \times (0,T))$, we can conclude that

$$\chi = |\partial_t u|^{p-2}\partial_t u \quad \text{a.e. in } \mathbb{R}^d \times (0,T),$$

and as a by-product,

$$\lim_{n\to\infty} \int_0^T \int_{\mathbb{R}^d} |\partial_t u_n|^p \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_{\mathbb{R}^d} |\partial_t u|^p \, \mathrm{d}x \, \mathrm{d}t,$$

which along with the uniform convexity of the weighted L^p norm gives

$$\partial_t u_n \to \partial_t u \quad \text{strongly in } L^p(0,T; L^p(\mathbb{R}^d)).$$

Therefore we also obtain

$$|\partial_t u_n|^{p-2}\partial_t u_n \to |\partial_t u|^{p-2}\partial_t u \quad \text{strongly in } L^{p'}(0,T; L^{p'}(\mathbb{R}^d)),$$

which may be of independent interest.

Thus u turns out to be a strong solution on $[0,T]$ of the Cauchy problem (1.1), (1.2) in the sense of Definition 1.1. The maximal regularity estimate (1.5) follows immediately by testing (1.1) with $\partial_t u$ and using the chain-rule formula in Proposition 2.2 again. This completes the proof. □

4. Concluding remarks

We close this paper with the following concluding remarks on possible extensions as well as open questions: Theorem 1.2 can be extended to more general settings, for instance, one may consider the inclusion,

$$\beta(\partial_t u) - \Delta_m u \ni f \quad \text{in } \mathbb{R}^d \times (0,T)$$

instead of (1.1). Here the power nonlinearity of the time-derivative in (1.1) was replaced by a maximal monotone graph β in $\mathbb{R}\times\mathbb{R}$ with $D(\beta)=\mathbb{R}$ under a p-growth condition, that is, there exist positive constants c_1, c_2 such that

$$c_1|s|^p \leq bs \quad \text{and} \quad |b|^{p'} \leq c_2|s|^p \quad \text{for} \ \ s\in\mathbb{R} \ \ \text{and} \ \ b\in\beta(s).$$

As we saw in §3.4, the *localized Minty's trick* enables us to overcome difficulties arising from the lack of compact embeddings due to the unboundedness of domains. It can also be applied to the periodic problem (see [8]), for which we may add a lower order term to the equation, a variable exponent setting (see [5]) and a Musielak-Orlicz setting (see [6]). It is further applicable to other PDEs with nonlinear (possibly degenerate) elliptic operators in divergence form. On the other hand, another noncompact setting where $p=m^*$ is a different story and still remains open.

Finally, we exhibit several open questions. It may also be interesting to discuss *smoothing effect* of solutions to the doubly-nonlinear parabolic equation (1.1). To be more precise, our question is whether a strong solution can be constructed for more general initial data. Furthermore, it is also challenging to prove existence of solutions to the Cauchy problem (1.1), (1.2) for growing initial data; indeed, the (linear) diffusion equation admits a solution for initial data growing (at most like $\mathrm{e}^{|x|^2}$) at infinity, and moreover, nonlinear diffusion equations such as the porous medium and fast diffusion equations (even doubly-nonlinear diffusion equations) do so (see [9] and references therein). On the other hand, it is rather delicate to establish local energy estimates for the doubly-nonlinear parabolic equation (1.1) concerned in the present paper. The uniqueness of solutions is also widely open; however, it is noteworthy that doubly-nonlinear parabolic equations such as (1.1) may violate the uniqueness of solutions, depending on boundary conditions as well as on reaction terms (if exist). See [16, pp.186,187] for a celebrated counter example on the Cauchy-Neumann problem, which admits homogeneous (in space) nontrivial solutions for the zero initial and Neumann data. The uniqueness of solutions for the Cauchy problem (1.1), (1.2) as well as for the Cauchy-Dirichlet problem posed in bounded domains may still be open, unless either p or m is equal to 2 (i.e., "single" nonlinear cases). We further refer the reader to [1], which presents a counter example of the uniqueness of solutions to the Cauchy-Dirichlet problem for (1.1) with reaction terms posed in bounded domains.

Appendix A. Proof of Chain-rule formula

In this appendix, for the convenience of the reader, we exhibit a proof of Proposition 2.2, which is concerned with a chain-rule formula for subdifferentials in a reflexive Banach space B, based on the classical subdifferential calculus (see [14, 13]) only (cf. see also [10], where the proposition is proved in the frame of the metric gradient flow theory). In what follows, I denotes a non-empty open interval of $\mathbb{R}$, e.g., $I=(0,T)$. Of course, the case where $B=H$ is a Hilbert space is well understood (see, e.g., Brézis [14, Lemma

3.3]). On the other hand, concerning non-Hilbertian settings, one has to compensate the lack of Fréchet differentiability of the Moreau-Yosida regularization ϕ_λ of a convex functional $\phi : B \to (-\infty, +\infty]$ in order to prove the absolute continuity of the function $t \mapsto \phi(u(t))$ for $u \in W^{1,p}(0,T;B)$ with $1 < p < \infty$. Indeed, in general, even Gâteaux differentiable functions can violate chain-rule formulae (even in the $\mathbb{R}^d$ case).

Proof of Proposition 2.2. We prove the absolute continuity of the function $t \mapsto \phi(u(t))$ on $\bar{I}$. As in the Hilbertian case, let $\phi_\lambda : B \to \mathbb{R}$ be the Moreau-Yosida regularization of ϕ (see, e.g., [13] in reflexive Banach spaces); however, ϕ_λ is no longer Fréchet differentiable in B, and therefore, it is still unclear whether the chain-rule formula holds for ϕ_λ or not. On the other hand, in the Hilbert space setting, the proof of the assertion (see [14, Lemma 3.3]) starts with the chain-rule formula for the Moreau-Yosida regularization.

Let us begin with the case where $u \in C^1(\bar{I}; B)$. Since ϕ_λ is Gâteaux differentiable in B (see [13]), we may observe that

$$
\begin{aligned}
&\frac{\phi_\lambda(u(t+h)) - \phi_\lambda(u(t))}{h} \\
&\quad = \frac{\phi_\lambda(u(t) + u'(t)h) - \phi_\lambda(u(t))}{h} + \frac{\phi_\lambda(u(t+h)) - \phi_\lambda(u(t) + u'(t)h)}{h} \\
&\quad \to \left\langle \mathrm{d}\phi_\lambda(u(t)), u'(t) \right\rangle_B
\end{aligned}
$$

for all $t \in I$. Here u' stands for the derivative $(\mathrm{d}/\mathrm{d}t)u$ of u and $\mathrm{d}\phi_\lambda$ denotes the Gâteaux derivative of ϕ_λ. Indeed, we note that

$$
\begin{aligned}
&\phi_\lambda(u(t+h)) - \phi_\lambda(u(t) + u'(t)h) \\
&\qquad \le \langle \mathrm{d}\phi_\lambda(u(t+h)), u(t+h) - u(t) - u'(t)h \rangle \\
&\qquad \le \|\mathrm{d}\phi_\lambda(u(t+h))\|_{B^*} \|u(t+h) - u(t) - u'(t)h\|_B.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\phi_\lambda(u(t+h)) - \phi_\lambda(u(t) + u'(t)h) \\
&\qquad \ge -\|\mathrm{d}\phi_\lambda(u(t) + u'(t)h)\|_{B^*} \|u(t+h) - u(t) - u'(t)h\|_B.
\end{aligned}
$$

Hence dividing both sides by $h \neq 0$ and taking a limit $h \to 0$ and using the boundedness of the operator $\mathrm{d}\phi_\lambda = (\partial\phi)_\lambda$, which is the Yosida approximation of $\partial\phi$, we deduce that

$$
\lim_{h \to 0} \left\| \frac{\phi_\lambda(u(t+h)) - \phi_\lambda(u(t) + u'(t)h)}{h} \right\|_B = 0 \quad \text{for all} \quad t \in I.
$$

Here we used the fact that

$$
\lim_{h \to 0} \left\| \frac{u(t+h) - u(t)}{h} - u'(t) \right\|_B = 0 \quad \text{for all} \quad t \in I
$$

from the assumption $u \in C^1(\bar{I}; B)$. Thus we have proved that the function $t \mapsto \phi_\lambda(u(t))$ is differentiable everywhere in $(0,T)$ and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_\lambda(u(t)) = \left\langle \mathrm{d}\phi_\lambda(u(t)), u'(t) \right\rangle_B \quad \text{for all } \ t \in I.$$

Hence integrating both sides over (s,t), we obtain

$$\phi_\lambda(u(t)) - \phi_\lambda(u(s)) = \int_s^t \langle \mathrm{d}\phi_\lambda(u(\sigma)), u'(\sigma)\rangle_B \,\mathrm{d}\sigma \tag{A.1}$$

for any $s,t \in \bar{I}$. We next consider the case where $u \in W^{1,p}(I;B)$. Choosing a sequence (u_n) in $C^1(\bar{I};B)$ such that $u_n \to u$ strongly in $W^{1,p}(I;B)$, we observe that

$$\phi_\lambda(u_n(t)) - \phi_\lambda(u_n(s)) = \int_s^t \langle \mathrm{d}\phi_\lambda(u_n(\sigma)), u_n'(\sigma)\rangle_B \,\mathrm{d}\sigma$$

for any $s,t \in \bar{I}$. Since $u_n \to u$ in $C(\bar{I};B)$ as well, ϕ_λ is continuous in B and $\mathrm{d}\phi_\lambda : B \to B^*$ is demicontinuous (see [15, Lemma 1.3]), we can verify that (A.1) holds true again for any $s,t \in \bar{I}$ with the aid of Vitali's convergence theorem (indeed, we note that $\mathrm{d}\phi_\lambda : B \to B^*$ is bounded; see [15, Lemma 1.3]).

Next, pass to the limit as $\lambda \to 0_+$. Noting that $\phi_\lambda(u(t)) \to \phi(u(t))$ and using the fact that

$$\|\mathrm{d}\phi_\lambda(u(\sigma))\|_{B^*} = \|(\partial\phi)_\lambda(u(\sigma))\|_{B^*} \le \|g(\sigma)\|_{B^*} \in L^{p'}(I),$$

we can deduce that

$$|\phi(u(t)) - \phi(u(s))| \le \int_s^t \rho(\sigma)\,\mathrm{d}\sigma$$

for any $s,t \in \bar{I}$ satisfying $s < t$. Here $\rho(\sigma) = \|g(\sigma)\|_{B^*}\|u'(\sigma)\|_B \in L^1(I)$. Therefore, $t \mapsto \phi(t)$ turns out to be absolutely continuous on $\bar{I}$.

The rest of proof (for deriving the identity (2.1)) runs as in the Hilbertian case (see [14, Lemma 3.3]); however, we give a proof for completeness. Let $t \in I$ be such that u and $\phi(u(\cdot))$ are differentiable at t and $u(t) \in D(\partial\phi)$ and let $\xi \in \partial\phi(u(t))$. Indeed, the set of such $t \in I$ has full measure in I, since $u \in W^{1,p}(I;B)$ and B is reflexive. For $h > 0$, we see that

$$\langle \xi, u(t+h) - u(t)\rangle_B \le \phi(u(t+h)) - \phi(u(t))$$

from the definition of subdifferential. Dividing both sides by h and passing to the limit as $h \to 0_+$, we infer that

$$\langle \xi, u'(t)\rangle_B \le \frac{\mathrm{d}}{\mathrm{d}t}\phi(u(t)).$$

Repeating the same argument as above with $h < 0$, we can also derive the inverse inequality. Thus (2.1) follows. This completes the proof. □

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