

WELL-POSEDNESS AND LARGE-TIME BEHAVIORS OF SOLUTIONS FOR A PARABOLIC EQUATION INVOLVING $p(x)$ -LAPLACIAN

GORO AKAGI

Department of Machinery and Control Systems,
College of Systems Engineering and Science, Shibaura Institute of Technology
307 Fukasaku, Minuma-ku, Saitama-shi, Saitama 337-8570 Japan

KEI MATSUURA

Research Institute for Science and Engineering,
Waseda University
Okubo 3-4-1, Shinjuku-ku, Tokyo 169-8555

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ABSTRACT. This paper is concerned with the initial-boundary value problem for a nonlinear parabolic equation involving the so-called $p(x)$ -Laplacian. A subdifferential approach is employed to obtain a well-posedness result as well as to investigate large-time behaviors of solutions.

1. Introduction. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. The so-called $p(x)$ -Laplacian is given by

$$\Delta_{p(x)}\phi(x) := \nabla \cdot \left(|\nabla\phi(x)|^{p(x)-2} \nabla\phi(x) \right)$$

with a function $p = p(x)$ from $\overline{\Omega}$ into $(1, \infty)$. In the present paper we study the well-posedness and asymptotic behaviors of solutions $u = u(x, t)$ as $t \rightarrow \infty$ for the following initial-boundary value problem:

$$\partial_t u = \Delta_{p(x)} u + f \quad \text{in } \Omega \times (0, \infty), \quad (1)$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, \infty), \quad (2)$$

$$u = u_0 \quad \text{in } \Omega, \quad (3)$$

where $\partial_t u = \partial u / \partial t$ and $f : \Omega \times (0, \infty) \rightarrow \mathbb{R}$ and $u_0 : \Omega \rightarrow \mathbb{R}$ are given functions.

The $p(x)$ -Laplacian with a variable exponent $p(x)$ is deeply related to generalized Lebesgue and Sobolev spaces, $L^{p(x)}$ and $W^{1,p(x)}$. There have been many contributions to nonlinear elliptic problems associated with the $p(x)$ -Laplacian (see, e.g., [23] for a thorough overview of the recent advantages) from various view points. Moreover, parabolic equations involving the $p(x)$ -Laplacian have been proposed in the

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study of image restoration (see [13]) as well as in some model of electrorheological fluids (see [14], [15], [26]). A mathematical analysis was also done for such problems by Acerbi and Mingione [1, 2] and by Acerbi, Mingione and Seregin [3]. In [22], some nonlinear parabolic problem proposed by [13] was studied in a weak formulation and an existence result for weak solutions was established. Antontsev and Shmarev studied parabolic equations with anisotropic $p(x, t)$ -Laplace operators and proved existence, uniqueness, extinction in finite time, decay and blow-up of solutions in [5, 6, 7, 8].

Equation (1) was very recently studied by Bendahmane, Wittbold and Zimmermann in [10], where the well-posedness of a renormalized solution is proved for L^1 -data. Moreover, the existence and uniqueness of entropy solutions and the equivalence between two notions of solutions are also discussed by Zhang and Zhou in [28]. In this paper we treat L^2 -solutions for (1)–(3), and we prove the well-posedness and reveal large-time behaviors of solutions by using subdifferential calculus.

This paper is composed of four sections. In Section 2, we recall the definition of variable exponent Lebesgue spaces, $L^{p(x)}(\Omega)$, as well as Sobolev spaces, $W^{1,p(x)}(\Omega)$. Moreover, some properties of these spaces will be also exhibited to be used later. In Section 3, we prove the well-posedness of the Cauchy-Dirichlet problem (1)–(3) by using a theory of evolution equations governed by subdifferential operators. Moreover, we also treat the periodic problem for (1). In Section 4, we discuss asymptotic behaviors of solutions $u = u(x, t)$ for (1)–(3) as $t \rightarrow \infty$.

Further results on qualitative properties of solutions for (1)–(3) (e.g., extinction/decay rates of solutions and limit problems as $p(x) \rightarrow \infty$) will be reported in our forthcoming paper [4].

Notation. We write $(s)_+ := \max\{s, 0\}$ for $s \in \mathbb{R}$. Let $\|\cdot\|_q$ denote the usual norm of $L^q(\Omega)$ -spaces for $1 \leq q \leq \infty$. Moreover, $(\cdot, \cdot)_{L^2}$ denotes the usual inner product of the Hilbert space $L^2(\Omega)$, i.e., $(u, v)_{L^2} = \int_{\Omega} u(x)v(x)dx$.

2. Generalized Lebesgue and Sobolev spaces. This section is devoted to some preliminary facts on Lebesgue and Sobolev spaces with variable exponents (see [24], [16, 17], [20] for an introduction to this field). Let Ω be a bounded domain in \mathbb{R}^N . Let p be a measurable function from Ω to $[1, \infty)$. We write

$$p^+ := \operatorname{ess\,sup}_{x \in \Omega} p(x), \quad p^- := \operatorname{ess\,inf}_{x \in \Omega} p(x).$$

Define a Lebesgue space with a variable exponent $p(x)$, which is a special sort of Musielak-Orlicz spaces (see [25]), by

$$L^{p(x)}(\Omega) := \left\{ u : \Omega \rightarrow \mathbb{R}; \text{ measurable in } \Omega \text{ and } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\}$$

with a Luxemburg-type norm

$$\|u\|_{p(x)} := \inf \left\{ \lambda > 0; \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

The following proposition plays an important role to establish energy estimates (see Theorem 1.3 of [20] for a proof).

Proposition 1. *It holds that*

$$\sigma^-(\|w\|_{p(x)}) \leq \int_{\Omega} |w(x)|^{p(x)} dx \leq \sigma^+(\|w\|_{p(x)}) \quad \text{for all } w \in L^{p(x)}(\Omega)$$

with the strictly increasing functions,

$$\sigma^-(s) := \min\{s^{p^-}, s^{p^+}\}, \quad \sigma^+(s) := \max\{s^{p^-}, s^{p^+}\} \quad \text{for } s \geq 0.$$

We next define variable exponent Sobolev spaces $W^{1,p(x)}(\Omega)$ as follows:

$$W^{1,p(x)}(\Omega) := \left\{ u \in L^{p(x)}(\Omega); \frac{\partial u}{\partial x_i} \in L^{p(x)}(\Omega) \quad \text{for all } i = 1, 2, \dots, N \right\}$$

with the norm $\|u\|_{W^{1,p(x)}(\Omega)} := (\|u\|_{p(x)}^2 + \|\nabla u\|_{p(x)}^2)^{1/2}$, where $\|\nabla u\|_{p(x)}$ denotes the $L^{p(x)}(\Omega)$ -norm of $|\nabla u|$. Furthermore, let $W_0^{1,p(x)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

The following proposition is concerned with the uniform convexity of $L^{p(x)}$ and $W^{1,p(x)}$ (see [25] for its proof).

Proposition 2. *If $p \in C(\overline{\Omega})$, $1 < p^-$ and $p^+ < \infty$, then $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are uniformly convex. Hence they are reflexive.*

Let us exhibit Poincaré and Sobolev inequalities (see [18], [21], [27] and references therein for more details). To do so, we introduce the Zhikov-Fan condition:

$$|p(x) - p(x')| \leq \frac{A}{\log(1/|x - x'|)} \quad \text{for all } x, x' \in \Omega \text{ with } |x - x'| \leq \delta \quad (4)$$

with some $A, \delta > 0$. This condition follows from a Hölder continuity of p over $\overline{\Omega}$.

Proposition 3. *Let Ω be a bounded domain in \mathbb{R}^N with smooth boundary $\partial\Omega$.*

(i) *If (4) holds, then*

$$\|w\|_{p(x)} \leq C \|\nabla w\|_{p(x)} \quad \text{for all } w \in W_0^{1,p(x)}(\Omega).$$

In particular, the space $W_0^{1,p(x)}(\Omega)$ has a norm $\|\cdot\|_{1,p(x)}$ given by

$$\|w\|_{1,p(x)} := \|\nabla w\|_{p(x)} \quad \text{for } w \in W_0^{1,p(x)}(\Omega),$$

which is equivalent to $\|\cdot\|_{W^{1,p(x)}(\Omega)}$.

(ii) *If $p \in C(\overline{\Omega})$, $q : \Omega \rightarrow [1, \infty)$ is measurable, and $\inf_{x \in \Omega} (p^*(x) - q(x)) > 0$ with $p^*(x) := Np(x)/(N - p(x))_+$, then $W^{1,p(x)}(\Omega)$ is continuously and compactly embedded in $L^{q(x)}(\Omega)$. Hence it follows that*

$$\|w\|_{q(x)} \leq C \|w\|_{W^{1,p(x)}(\Omega)} \quad \text{for all } w \in W^{1,p(x)}(\Omega). \quad (5)$$

3. Well-posedness. In this section, we prove the well-posedness of (1)–(3) by using a theory of evolution equations governed by subdifferential operators. Let us begin with the definition of solutions for (1)–(3).

Definition 3.1. A function $u \in C([0, \infty); L^2(\Omega))$ is said to be a solution of (1)–(3), if the following conditions are all satisfied:

- $u \in W_{loc}^{1,2}((0, \infty); L^2(\Omega)) \cap C_w((0, \infty); W_0^{1,p(x)}(\Omega))$, where C_w denotes the class of weakly continuous functions, and $\Delta_{p(x)} u \in L_{loc}^2((0, \infty); L^2(\Omega))$,
- $u(0) = u_0$,
- u satisfies (1) for a.e. $x \in \Omega$ and $t > 0$.

We reduce the initial-boundary value problem (1)–(3) into the Cauchy problem for an abstract evolution equation. Let $H := L^2(\Omega)$ and define $\varphi : H \rightarrow [0, \infty]$ by

$$\varphi(w) = \begin{cases} \int_{\Omega} \frac{1}{p(x)} |\nabla w(x)|^{p(x)} dx & \text{if } w \in W_0^{1,p(x)}(\Omega), \\ \infty & \text{otherwise} \end{cases} \quad (6)$$

with $D(\varphi) := \{w \in H; \varphi(w) < \infty\}$. In order to prove the well-posedness for (1)–(3), the most crucial point lies in checking the lower semicontinuity of the functional φ in $H = L^2(\Omega)$.

Lemma 3.2. *In addition to (4), suppose that $1 < p^-$ and $p^+ < \infty$. The function φ is proper, convex and lower semicontinuous in H .*

PROOF OF LEMMA 3.2 It is obvious that φ is proper and convex in H . Let $\mu \in \mathbb{R}$ be fixed and set

$$[\varphi \leq \mu] := \{u \in H; \varphi(u) \leq \mu\}.$$

Let (u_n) be a sequence on $[\varphi \leq \mu]$ such that $u_n \rightarrow u$ strongly in H . By Proposition 1, it follows that

$$\frac{1}{p^+} \sigma^- (\|u_n\|_{1,p(x)}) \leq \frac{1}{p^+} \int_{\Omega} |\nabla u_n(x)|^{p(x)} dx \leq \varphi(u_n) \leq \mu.$$

Hence $\|u_n\|_{1,p(x)} \leq \lambda$ with a constant λ independent of n . Since $W_0^{1,p(x)}(\Omega)$ is reflexive by $1 < p^-$ and $p^+ < \infty$, we can take a subsequence of (n) denoted again by the same letter such that $u_n \rightarrow u$ weakly in $W_0^{1,p(x)}(\Omega)$.

Let $\hat{\varphi}$ be the restriction of φ to $W_0^{1,p(x)}(\Omega)$. Then $\hat{\varphi}$ is of class C^1 in $W_0^{1,p(x)}(\Omega) \cap L^2(\Omega)$ (see Proposition 4 below), and moreover, $\hat{\varphi}$ is convex. Hence $\hat{\varphi}$ becomes weakly lower semicontinuous in $W_0^{1,p(x)}(\Omega)$. Therefore we have $\liminf_{n \rightarrow \infty} \hat{\varphi}(u_n) \geq \hat{\varphi}(u) = \varphi(u)$, which together with the fact that $\hat{\varphi}(u_n) = \varphi(u_n) \leq \mu$ implies that $u \in [\varphi \leq \mu]$. Thus we conclude that $[\varphi \leq \mu]$ is closed in H , and therefore, φ is lower semicontinuous in H . \square

One can verify the following proposition (see [19]).

Proposition 4. *In addition to (4), suppose that $1 < p^-$ and $p^+ < \infty$. The restriction $\hat{\varphi}$ of φ to $W_0^{1,p(x)}(\Omega) \cap L^2(\Omega)$ is of class C^1 . Moreover, the Fréchet derivative $d\hat{\varphi}(u)$ of $\hat{\varphi}$ at u coincides with $-\Delta_{p(x)}u$ furnished with $u|_{\partial\Omega} = 0$ in the sense of distribution.*

The subdifferential operator $\partial\varphi : H \rightarrow H$ of φ is given by

$$\partial\varphi(u) := \{\xi \in H; \varphi(v) - \varphi(u) \geq (\xi, v - u)_H \text{ for all } v \in D(\varphi)\} \quad \text{for } u \in D(\varphi)$$

with $(\cdot, \cdot)_H = (\cdot, \cdot)_{L^2}$ and the domain $D(\partial\varphi) = \{u \in D(\varphi); \partial\varphi(u) \neq \emptyset\}$. Since $\partial\varphi(u) \subset d\hat{\varphi}(u)$ for all $u \in D(\partial\varphi)$, we have $\partial\varphi(u) = -\Delta_{p(x)}u$ with $u|_{\partial\Omega} = 0$ in $H = L^2(\Omega)$. Thus the initial-boundary value problem (1)–(3) is reduced into the following Cauchy problem:

$$\frac{du}{dt}(t) + \partial\varphi(u(t)) = f(t) \text{ in } H \text{ for } t > 0, \quad (7)$$

$$u(0) = u_0. \quad (8)$$

Such an abstract evolution equation was well studied in 1970s and fundamental results have already been established for Cauchy problem and for periodic problem mainly by H. Brézis (see Chap. III of [11]). Hence one can immediately assure that

Theorem 3.3 (Cauchy problem). *In addition to (4), suppose that $1 < p^-$ and $p^+ < \infty$. Then for all $f \in L^2_{loc}([0, \infty); L^2(\Omega))$ and $u_0 \in L^2(\Omega)$, there exists a unique solution $u = u(x, t)$ of the initial-boundary value problem (1)–(3) such that the function $t \mapsto \varphi(u(t))$ is absolutely continuous in $(0, \infty)$.*

In particular, if u_0 belongs to $W_0^{1,p(x)}(\Omega)$, then

$$u \in W_{loc}^{1,2}([0, \infty); L^2(\Omega)) \cap C_w([0, \infty); W_0^{1,p(x)}(\Omega))$$

and the function $t \mapsto \varphi(u(t))$ is absolutely continuous on $[0, \infty)$.

Furthermore, the solution u of (1)–(3) continuously depends on initial data u_0 and f in the following sense: Let u_i be the unique solution of (1)–(3) with $u_0 = u_{0,i} \in L^2(\Omega)$ and $f = f_i \in L^2_{loc}([0, \infty); L^2(\Omega))$ for $i = 1, 2$. Then it follows that

$$\|u_1(t) - u_2(t)\|_2 \leq \|u_{0,1} - u_{0,2}\|_2 + \int_0^t \|f_1(\tau) - f_2(\tau)\|_2 d\tau \quad \text{for all } t \geq 0.$$

Remark 1. The assertion for the case that $u_0 \in W_0^{1,p(x)}(\Omega)$ also follows from Theorem 6.1 of [6], where the $p(x, t)$ -Laplacian is treated.

In case $p^- \geq 2$, one can also assure that u is strongly continuous from $(0, \infty)$ (resp., $[0, \infty)$) into $W_0^{1,p(x)}(\Omega)$ in Theorem 3.3 for $u_0 \in L^2(\Omega)$ (resp., $u_0 \in W_0^{1,p(x)}(\Omega)$). Since $\varphi(u(\cdot))$ is continuous, this fact immediately follows from the following proposition:

Proposition 5 (Modular convergence and strong convergence). *Assume that $p^- \geq 2$. Let (w_n) be a sequence in $W_0^{1,p(x)}(\Omega)$ such that*

$$w_n \rightarrow w \quad \text{weakly in } L^2(\Omega) \quad \text{and} \quad \varphi(w_n) \rightarrow \varphi(w)$$

with some $w \in W_0^{1,p(x)}(\Omega)$. Then w_n strongly converges to w in $W_0^{1,p(x)}(\Omega)$.

Proof. Since $p(x) \geq 2$ for a.e. $x \in \Omega$, by the following fundamental inequality (see, e.g., [12]):

$$\left| \frac{\mathbf{a} + \mathbf{b}}{2} \right|^q + \left| \frac{\mathbf{a} - \mathbf{b}}{2} \right|^q \leq \frac{1}{2} (|\mathbf{a}|^q + |\mathbf{b}|^q) \quad \text{for all } \mathbf{a}, \mathbf{b} \in \mathbb{R}^N \text{ when } q \geq 2,$$

which is used in a proof of Clarkson's first inequality, we derive

$$\begin{aligned} & \left| \frac{\nabla w_n(x) + \nabla w(x)}{2} \right|^{p(x)} + \left| \frac{\nabla w_n(x) - \nabla w(x)}{2} \right|^{p(x)} \\ & \leq \frac{1}{2} \left(|\nabla w_n(x)|^{p(x)} + |\nabla w(x)|^{p(x)} \right) \quad \text{for a.e. } x \in \Omega, \end{aligned}$$

which gives

$$\varphi\left(\frac{w_n + w}{2}\right) + \varphi\left(\frac{w_n - w}{2}\right) \leq \frac{1}{2} (\varphi(w_n) + \varphi(w)).$$

On the other hand, since $w_n \rightarrow w$ weakly in $L^2(\Omega)$ and φ is weakly lower-semicontinuous in $L^2(\Omega)$, we observe that

$$\varphi(w) \leq \liminf_{n \rightarrow \infty} \varphi\left(\frac{w_n + w}{2}\right).$$

Combining these facts with the assumption that $\varphi(w_n) \rightarrow \varphi(w)$, we deduce that

$$\liminf_{n \rightarrow \infty} \varphi\left(\frac{w_n - w}{2}\right) = 0,$$

which together with Proposition 1 implies $w_n \rightarrow w$ strongly in $W_0^{1,p(x)}(\Omega)$. It completes our proof. \square

Let us next discuss the existence and the uniqueness of periodic solutions.

Corollary 1 (Periodic problem). *In addition to (4), assume that*

$$\max\{1, 2N/(N+2)\} < p^- \quad \text{and} \quad p^+ < \infty. \quad (9)$$

Then for any $T > 0$ and $f \in L^2(0, T; L^2(\Omega))$, there exists a unique solution u for (1), (2) such that $u(\cdot, 0) = u(\cdot, T)$.

Proof. To prove the existence of periodic solutions, it suffices to check the coercivity of φ in $L^2(\Omega)$. Since $2N/(N+2) < p^-$ (equivalently, $2 < \inf_{x \in \Omega} p^*(x)$), by Propositions 1 and 3, we observe that

$$\begin{aligned} \varphi(v) &\geq \frac{1}{p^+} \int_{\Omega} |\nabla v(x)|^{p(x)} dx \\ &\geq \frac{1}{p^+} \sigma^-(\|v\|_{1,p(x)}) \geq \frac{C}{p^+} \min \left\{ \|v\|_2^{p^+}, \|v\|_2^{p^-} \right\} \end{aligned}$$

with a constant $C \geq 0$. Since $p^- > 1$, the functional φ is coercive, that is,

$$\lim_{\|v\|_2 \rightarrow \infty} \frac{\varphi(v)}{\|v\|_2} = \infty.$$

By Corollary 3.4 of [11], for any $T > 0$ and $f \in L^2(0, T; H)$, there exists a solution of (7) such that $u(0) = u(T)$. Moreover, since φ is strictly convex, the periodic solution is unique. \square

Remark 2. In the proof described above, the Sobolev inequality (5) (i.e., the continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^2(\Omega)$) is used, but the compactness of the embedding is not employed. Hence the continuous embedding is sufficient for this proof, so one can replace (4) and (9) by the following:

$$1 < p^-, \frac{2N}{N+2} \leq p^-, p^+ < \infty \quad \text{and} \quad p \text{ is Lipschitz continuous on } \bar{\Omega}. \quad (10)$$

We note that the Lipschitz continuity of p is more restrictive than (4) (see [16], [18]).

4. Large-time behaviors of solutions. This section is concerned with large-time behaviors of solutions.

Theorem 4.1 (Large-time behavior of solutions). *Assume (4) and (9). Let $f_{\infty} \in L^2(\Omega)$ and $f \in L_{loc}^2([0, \infty); L^2(\Omega))$ be such that*

$$f(\cdot) - f_{\infty} \in L^2(0, \infty; L^2(\Omega)), \quad (11)$$

$$f(t) \rightarrow f_{\infty} \quad \text{weakly in } L^2(\Omega) \quad \text{as } t \rightarrow \infty. \quad (12)$$

Let $u = u(x, t)$ be the unique solution for (1)–(3) with some $u_0 \in W_0^{1,p(x)}(\Omega)$. Then there exists $\phi \in W_0^{1,p(x)}(\Omega)$ such that

$$u(t) \rightarrow \phi \quad \text{strongly in } L^2(\Omega) \quad \text{as } t \rightarrow \infty, \quad (13)$$

$$\varphi(u(t)) \rightarrow \varphi(\phi) \quad \text{as } t \rightarrow \infty, \quad (14)$$

where φ is given by (6). Moreover, ϕ solves

$$-\Delta_{p(x)} \phi = f_{\infty} \quad \text{in } \Omega, \quad \phi = 0 \quad \text{on } \partial\Omega. \quad (15)$$

Proof. We recall that (1) is reduced into (7). Multiplying (7) by $u'(t)$ in $H = L^2(\Omega)$ and using the chain-rule for subdifferentials, we have

$$\begin{aligned} \|u'(t)\|_2^2 + \frac{d}{dt}\varphi(u(t)) &= (f(t) - f_\infty, u'(t))_{L^2} + (f_\infty, u'(t))_{L^2} \\ &\leq \frac{1}{2}\|f(t) - f_\infty\|_2^2 + \frac{1}{2}\|u'(t)\|_2^2 + \frac{d}{dt}(f_\infty, u(t))_{L^2}, \end{aligned}$$

which yields

$$\frac{1}{2}\|u'(t)\|_2^2 + \frac{d}{dt}\{\varphi(u(t)) - (f_\infty, u(t))_{L^2}\} \leq \frac{1}{2}\|f(t) - f_\infty\|_2^2 \quad \text{for a.e. } t > 0.$$

Here set a function $E : [0, \infty) \rightarrow \mathbb{R}$ by

$$E(t) := \varphi(u(t)) - (f_\infty, u(t))_{L^2} - \frac{1}{2} \int_0^t \|f(\tau) - f_\infty\|_2^2 d\tau \quad \text{for } t > 0.$$

Then

$$\frac{1}{2}\|u'(t)\|_2^2 + \frac{d}{dt}E(t) \leq 0 \quad \text{for a.e. } t > 0.$$

Hence the function $t \mapsto E(t)$ is non-increasing for $t > 0$. Moreover, since $f - f_\infty \in L^2(0, \infty; L^2(\Omega))$ and φ is coercive in $L^2(\Omega)$ by (9), there exists a constant $M > 0$ such that $E(t) \geq -M$ for all $t \geq 0$. Therefore we find that

$$\int_0^\infty \|u'(t)\|_2^2 dt < \infty. \quad (16)$$

Now, let (t_n) be an arbitrary sequence on $[0, \infty)$ such that $t_n \rightarrow \infty$. Then by (16) one can take $\theta_n \in [t_n, t_n + 1]$ such that $u'(\theta_n) \rightarrow 0$ strongly in $L^2(\Omega)$. Hence it follows from (12) that

$$\partial\varphi(u(\theta_n)) = f(\theta_n) - u'(\theta_n) \rightarrow f_\infty \quad \text{weakly in } L^2(\Omega). \quad (17)$$

On the other hand, observing that

$$E(t) \leq E(0) = \varphi(u_0) - (f_\infty, u_0)_{L^2}$$

and exploiting the facts that $f - f_\infty \in L^2(0, \infty; L^2(\Omega))$ and the coercivity of φ in $L^2(\Omega)$ again, we deduce that $\sup_{t \geq 0} \varphi(u(t)) < \infty$, which together with Proposition 1 implies

$$\sup_{t \geq 0} \|u(t)\|_{1,p(x)} < \infty.$$

Hence there exists a subsequence (n') of (n) such that

$$u(\theta_{n'}) \rightarrow \phi \quad \text{weakly in } W_0^{1,p(x)}(\Omega), \quad (18)$$

which together with (ii) of Proposition 3 by $p^- > 2N/(N+2)$ yields

$$u(\theta_{n'}) \rightarrow \phi \quad \text{strongly in } L^2(\Omega). \quad (19)$$

Combining these facts with (17), one deduce from the demiclosedness of $\partial\varphi$ that

$$\partial\varphi(\phi) = f_\infty, \quad (20)$$

which is equivalent to an L^2 -formulation of (15).

Furthermore, from the definition of subdifferentials, we see

$$\varphi(u(\theta_{n'})) \leq \varphi(\phi) + (\partial\varphi(u(\theta_{n'})), u(\theta_{n'}) - \phi)_{L^2} \rightarrow \varphi(\phi).$$

Hence since the weak lower-semicontinuity of φ also gives

$$\liminf_{n' \rightarrow \infty} \varphi(u(\theta_{n'})) \geq \varphi(\phi),$$

we obtain

$$\varphi(u(\theta_{n'})) \rightarrow \varphi(\phi).$$

Since the solution of (20) is unique by the strict convexity of φ (equivalently, the strict monotonicity of $\partial\varphi$), we can also verify the convergence of $u(\theta_n)$ and $\varphi(u(\theta_n))$ without taking a subsequence of (n) .

We next derive the convergence of $u(t)$ along the prescribed sequence $t_n \rightarrow \infty$. Recall (16) to obtain

$$\begin{aligned} \|u(t_n) - u(\theta_n)\|_2 &\leq \left(\int_{t_n}^{\theta_n} \|u'(\tau)\|_2^2 d\tau \right)^{1/2} \sqrt{\theta_n - t_n} \\ &\leq \left(\int_{t_n}^{\infty} \|u'(\tau)\|_2^2 d\tau \right)^{1/2} \rightarrow 0. \end{aligned}$$

Thus we have

$$u(t_n) \rightarrow \phi \quad \text{strongly in } L^2(\Omega).$$

On the other hand, note by (19) that

$$E(\theta_n) \rightarrow \varphi(\phi) - (f_\infty, \phi)_{L^2} - \frac{1}{2} \int_0^\infty \|f(\tau) - f_\infty\|_2^2 d\tau.$$

Since $E(\cdot)$ is monotone, $E(t_n)$ also converges to the same limit as $t_n \rightarrow \infty$. Consequently, we conclude that

$$\varphi(u(t_n)) = E(t_n) + (f_\infty, u(t_n))_{L^2} + \frac{1}{2} \int_0^{t_n} \|f(\tau) - f_\infty\|_2^2 d\tau \rightarrow \varphi(\phi).$$

It completes our proof. \square

By Proposition 5 we can immediately obtain the following corollary for the case that $p^- \geq 2$.

Corollary 2. *Assume $p^- \geq 2$. Under the same assumptions as in Theorem 4.1,*

$$u(t) \rightarrow \phi \quad \text{strongly in } W_0^{1,p(x)}(\Omega) \quad \text{as } t \rightarrow \infty. \quad (21)$$

Let us further discuss the case that $f(t) \equiv f_\infty$ with $p^- \geq 2$.

Theorem 4.2. *Assume (4) and (9). Suppose that $f(t) \equiv f_\infty \in L^2(\Omega)$ and $p^- \geq 2$.*

(i) *If $p^+ > 2$ then there exist constants $c_1 > 0$ and $t_1 \geq 0$ such that*

$$\|u(t) - \phi\|_2 \leq \|u_0 - \phi\|_2 (c_1 t + 1)^{-1/(p^+ - 2)} \quad \text{for all } t \geq t_1. \quad (22)$$

(ii) *If $p^+ = 2$ then there exists a constant $c_2 > 0$ such that*

$$\|u(t) - \phi\|_2 \leq \|u_0 - \phi\|_2 e^{-c_2 t} \quad \text{for all } t \geq 0. \quad (23)$$

Proof. Assertion (ii) follows immediately, since $p(x) \equiv 2$ by assumption. Hence we prove only (i). Let ϕ be a solution of (15). Then

$$\partial_t u - \Delta_{p(x)} u = f_\infty = -\Delta_{p(x)} \phi.$$

Multiply this by $u(t) - \phi$ in $L^2(\Omega)$ to get

$$\frac{1}{2} \frac{d}{dt} \|u(t) - \phi\|_2^2 + \int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u - |\nabla \phi|^{p(x)-2} \nabla \phi \right) \cdot (\nabla u - \nabla \phi) dx = 0.$$

Since $p(x) \geq 2$ for a.e. $x \in \Omega$, by Tartar's inequality, one can take $\omega > 0$ such that

$$\omega |\mathbf{a} - \mathbf{b}|^{p(x)} \leq \left(|\mathbf{a}|^{p(x)-2} \mathbf{a} - |\mathbf{b}|^{p(x)-2} \mathbf{b} \right) \cdot (\mathbf{a} - \mathbf{b})$$

for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$ and a.e. $x \in \Omega$. Thus we obtain

$$\frac{1}{2} \frac{d}{dt} \|u(t) - \phi\|_2^2 + \omega \int_{\Omega} |\nabla u - \nabla \phi|^{p(x)} dx \leq 0.$$

By Propositions 1, it follows that

$$\frac{1}{2} \frac{d}{dt} \|u(t) - \phi\|_2^2 + \omega \sigma^- (\|\nabla u - \nabla \phi\|_{p(x)}) \leq 0.$$

Here by assumption that $2N/(N+2) < p^-$ and Proposition 3, we see

$$\|w\|_2 \leq \mathcal{C} \|w\|_{1,p(x)} \quad \text{for all } w \in W_0^{1,p(x)}(\Omega)$$

with some constant $\mathcal{C} > 0$. Hence setting $\rho(t) := \|u(t) - \phi\|_2^2$, one obtain

$$\rho'(t) + \alpha \sigma^- (\rho(t)^{1/2}) \leq 0 \quad \text{for a.e. } t > 0$$

with a constant $\alpha > 0$ depending only on $\sigma^-(\cdot)$, \mathcal{C} and ω .

In case $\rho(0) = \|u_0 - \phi\|_2^2 > 1$, one can write

$$\rho'(t) + \alpha \rho(t)^{p^-/2} \leq 0 \quad \text{for a.e. } t \in (0, t_1)$$

with $t_1 := \sup\{\tau > 0; \rho(t) > 1 \text{ for all } t \in [0, \tau]\} > 0$. Hence

$$\rho(t) \leq \begin{cases} \|u_0 - \phi\|_2^2 \left(1 + \frac{p^- - 2}{2} \|u_0 - \phi\|_2^{p^- - 2} \alpha t\right)^{-2/(p^- - 2)} & \text{if } p^- > 2, \\ \|u_0 - \phi\|_2^2 e^{-\alpha t} & \text{if } p^- = 2 \end{cases}$$

for all $t \in [0, t_1]$. Hence $\rho(t)$ attains 1 at a finite time. In case $\rho(0) \leq 1$, we have

$$\rho'(t) + \alpha \rho(t)^{p^+/2} \leq 0 \quad \text{for a.e. } t > 0,$$

which yields

$$\rho(t) \leq \|u_0 - \phi\|_2^2 \left(1 + \frac{p^+ - 2}{2} \|u_0 - \phi\|_2^{p^+ - 2} \alpha t\right)^{-2/(p^+ - 2)} \quad \text{for all } t \geq 0.$$

Consequently, we obtain (22) with some constant $c_1 > 0$. \square

Remark 3. (i) By Theorem 3.11 of [11], one can derive (13) and prove that ϕ solves (15) under $u_0 \in L^2(\Omega)$ and $f \in L_{loc}^2([0, \infty); L^2(\Omega))$ satisfying $f(\cdot) - f_{\infty} \in L^1(0, \infty; L^2(\Omega))$. However, the convergence of $\varphi(u(t))$ (equivalently, the convergence of $u(t)$ in $W_0^{1,p(x)}(\Omega)$ when $p^- \geq 2$) does not directly follow.

(ii) From the smoothing effect for (1)–(3) (see Theorem 3.3), one can also assure the same conclusion as in Theorem 4.1 for the wider class of initial data: $u_0 \in L^2(\Omega)$. Indeed, even if $u_0 \in L^2(\Omega)$, then $u(\tau)$ belongs to $W_0^{1,p(x)}(\Omega)$ for any $\tau > 0$. Hence our proof runs as before by replacing u_0 with $u(\tau)$.

(iii) In Theorem 4.2, assumptions (4) and (9) can be replaced by (10), since the compactness of the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^2(\Omega)$ is not required in a proof.

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E-mail address: g-akagi@shibaura-it.ac.jp

E-mail address: kei@kurenai.waseda.jp