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ENERGY SOLUTIONS OF THE CAUCHY-NEUMANN PROBLEM FOR POROUS MEDIUM EQUATIONS

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Abstract. The existence of energy solutions to the Cauchy-Neumann problem for the porous medium equation of the form $v_t - \Delta(|v|^{m-2}v) = \alpha v$ with $m \geq 2$ and $\alpha \in \mathbb{R}$ is proved, by reducing the equation to an evolution equation involving two subdifferential operators and exploiting subdifferential calculus recently developed by the author.

1. **Introduction.** Let us consider the existence of solutions $u = u(x,t)$ to the Cauchy-Neumann problem (CNP) for the porous medium equation,

$$\begin{aligned} &\frac{\partial v}{\partial t} - \Delta u = \alpha v, \quad u = |v|^{m-2}v, && (x,t) \in \Omega \times (0,T), \\ &\frac{\partial u}{\partial n} = 0, && (x,t) \in \partial\Omega \times (0,T), \\ &v(\cdot,0) = v_0, && x \in \Omega, \end{aligned}$$

where $\alpha \in \mathbb{R}$, $m \geq 2$, $T > 0$, Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $N \in \mathbb{N}$.

As for Cauchy-Dirichlet problems, two major nonlinear semigroup approaches are widely used to treat porous medium type equations such as

$$\frac{\partial v}{\partial t} - \Delta u = 0, \quad u \in \beta(v), \quad (x,t) \in \Omega \times (0,T) \tag{1}$$

with a maximal monotone graph β in $\mathbb{R} \times \mathbb{R}$. One is an "$L^1$-framework" based on the m-accretive operator theory. Brézis and Strauss [6] proved the m-accretivity in $X := L^1(\Omega)$ of the operator $A : X \to X$ given by $Au := -\Delta\beta(u)$ equipped with the homogeneous Dirichlet boundary condition. Hence due to an abstract theory of Crandall and Liggett, the operator A generates a continuous contraction semigroup $S(t)$ in X, and moreover, $S(t)v_0$ is the unique generalized solution of the Cauchy-Dirichlet problem for (1).

The other is an ''H^{-1}-framework" based on the subdifferential operator theory. Let $X := H^{-1}(\Omega)$ be a Hilbert space with the inner product $(u,v)_{H^{-1}(\Omega)} :=$

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$\langle u, (-\Delta)^{-1}v\rangle_{H^1_0(\Omega)}$ and define a function $\phi : X \to (-\infty, \infty]$ by

$$\phi(v) := \begin{cases} \int_\Omega j(v(x))dx & \text{if } \ v \in L^1(\Omega) \ \text{ and } \ j(v(\cdot)) \in L^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where j is a primitive function of β, i.e., $\beta = \partial_{\mathbb{R}} j$. Then the subdifferential operator $\partial_X \phi : X \to X$ satisfies a representation formula, $f \in \partial_X \phi(u)$ if and only if $f(\cdot) \in -\Delta\beta(u(\cdot))$. Hence the Cauchy-Dirichlet problem for (1) is reduced to the Cauchy problem for an evolution equation governed by the subdifferential operator $\partial_X \phi$ in the Hilbert space X, and one can prove the well-posedness for the Cauchy problem by using a general theory due to Brézis (see [5]).

These two approaches have been developed to cover the initial-boundary value problems for (1) with other boundary conditions. Alikakos and Rostamian [3] found that the operator A still generates a continuous contraction semigroup in $L^1(\Omega)$ even for the Neumann boundary condition, and Bénilan et al [4] also dealt with a wider class of nonlinear boundary conditions. As for the subdifferential approach, Damlamian [7] and Damlamian-Kenmochi [8] developed an "$(H^1)^*$-framework" for the Robin boundary condition with $n_0 > 0$,

$$\frac{\partial u}{\partial n} + n_0 u = 0 \quad \text{on } \ \partial\Omega \times (0, T), \tag{2}$$

and their results enable us to treat (1) with (2) for any maximal monotone graph β (see also [11]). The $(H^1)^*$-framework is also applied to the Neumann boundary condition by Damlamian [7] (in case β is Lipschitz continuous) and by Kubo-Lu [10] (in case $\beta(v) = -1/v$), however, some restrictions are always imposed on β, and there seems to be no contribution of subdifferential approach which can cover the Cauchy-Neumann problem for the porous medium equation (i.e., $\beta(u) = |u|^{m-2}u$).

Table 1. Comparison of frameworks for the PM-type equation (1)

Framework	Base space X	Boundary condition	Nonlinearity of β
L^1	$L^1(\Omega)$	D, N, R	any for "D"
H^{-1}	$H^{-1}(\Omega)$	D	any
$(H^1)^*$	$(H^1(\Omega))^*$	R	any
		N	restricted

(D = Dirichlet, N = Neumann, R = Robin)

In studies of the asymptotic behavior of solutions for (CNP), energy identities (or inequalities) play crucial roles. However, the semigroup approaches described above have not been designed so well that one can directly obtain energy inequalities sufficient for the analysis.

The purpose of this paper is to prove the existence of weak solutions for (CNP) and derive energy inequalities even for $\alpha \neq 0$ by performing subdifferential calculus. We particularly exploit techniques recently developed by the author [1, 2] for nonlinear evolution equations involving two subdifferential operators in reflexive Banach spaces.

2. **Reduction to an evolution equation.**

Set $V = H^1(\Omega)$ with the norm

$$|\cdot|_V := \left(|\cdot|^2_{L^2} + |\nabla \cdot|^2_{L^2}\right)^{1/2},$$

and put $H = L^2(\Omega)$ with the norm $|\cdot|_H$ and the inner product $(\cdot,\cdot)_H$. Then

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*$$

with compact densely defined canonical injections. Here, H is identified with its dual space. Define a mapping A from V into V^* by

$$\langle Av, w\rangle_V = \int_\Omega \nabla v(x)\cdot\nabla w(x)dx \quad \text{ for } v, w \in V.$$

Let us introduce the following Cauchy problem for an evolution equation as a weak form of (CNP),

$$\frac{dv}{dt}(t) + A(|v|^{m-2}v(t)) = \alpha v(t) \;\text{ in } V^*, \quad 0 < t < T, \tag{3}$$

$$v(0) = v_0. \tag{4}$$

Furthermore, define functionals $\varphi : V \to [0,\infty)$ and $\psi : H \to [0,\infty)$ by

$$\varphi(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2\, dx \quad \text{ for } \; u \in V,$$

$$\psi(u) = \frac{1}{m'}\int_\Omega |u(x)|^{m'}\, dx \quad \text{for } \; u \in H,$$

where $m' := m/(m-1)$, and $\partial_V\varphi : V \to V^*$ and $\partial_H\psi : H \to H$ are given by

$$\partial_V\varphi(u) \;:=\; \{g \in V^*;\; \varphi(v) - \varphi(u) \ge \langle g, v-u\rangle_V \;\text{ for all } v \in V\},$$

$$\partial_H\psi(u) \;:=\; \{g \in H;\; \psi(v) - \psi(u) \ge (g, v-u)_H \;\text{ for all } v \in H\}.$$

Then $\partial_V\varphi(u)$ and $\partial_H\psi(u)$ coincide with Au and $|u|^{m'-2}u(\cdot)$ respectively (here and henceforth, we write $|r|^q r = |r|^{q+1}\mathrm{sgn}(r)$ for $r \in \mathbb{R}$ and $q > -1$). Therefore by setting $u := |v|^{m-2}v$ (hence, $v = |u|^{m'-2}u$), we can equivalently rewrite the Cauchy problem (3), (4) to

$$\frac{dv}{dt}(t) + \partial_V\varphi(u(t)) = \alpha v(t) \;\text{ in } V^*, \qquad 0 < t < T, \tag{5}$$

$$v(t) = \partial_H\psi(u(t)), \qquad 0 < t < T, \tag{6}$$

$$v(0) = v_0. \tag{7}$$

We shall treat (5)–(7) to discuss the existence of solutions for (CNP).

3. **Main result.** Define a function $J : H^1(\Omega) \to \mathbb{R}$ by

$$J(u) := \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx - \frac{\alpha}{m'}\int_\Omega |u(x)|^{m'} dx \quad \text{ for } u \in H^1(\Omega).$$

We denote by $C_w([0,T]; H^1(\Omega))$ the set of all $H^1(\Omega)$-valued weakly continuous functions on $[0,T]$. We are concerned with solutions of (CNP) given as follows.

Definition 3.1. A pair of functions $(u, v) : [0,T] \to H^1(\Omega) \times L^m(\Omega)$ is said to be a *weak solution* of (CNP) on $[0,T]$ if the following (i)-(iii) hold true:

(i) $u \in C([0,T]; L^{m'}(\Omega)) \cap C_w([0,T]; H^1(\Omega))$ and
$v \in C([0,T]; L^m(\Omega)) \cap W^{1,\infty}(0,T; (H^1(\Omega))^*)$;

(ii) (5) and (6) hold for a.e. $t \in (0,T)$;

(iii) $v(0) = v_0$.

A weak solution (u,v) on $[0,T]$ is particularly said to be an *energy solution* of (CNP) on $[0,T]$ if the following (iv), (v) are satisfied:

(iv) $|u|^{(m'-2)/2}u = |v|^{(m-2)/2}v \in L^2(0,T; H^1(\Omega)) \cap W^{1,2}(0,T; L^2(\Omega))$;

(v) $J(u(\cdot))$ and $|v(\cdot)|^2_{L^2(\Omega)}$ are differentiable for a.e. $t \in (0,T)$, and the following two energy inequalities hold for a.e. $t \in (0,T)$:

$$c_m \left| \frac{d}{d\tau} |v|^{(m-2)/2} v(t) \right|^2_{L^2(\Omega)} + \frac{d}{dt} J(u(t)) \quad \leq \quad 0, \tag{8}$$

$$\frac{1}{2} \frac{d}{dt} |v(t)|^2_{L^2(\Omega)} + c_m \left| \nabla \left(|u|^{(m'-2)/2} u \right)(t) \right|^2_{L^2(\Omega)} \quad \leq \quad \alpha |v(t)|^2_{L^2(\Omega)} \tag{9}$$

with $c_m := 4/(mm') > 0$.

Our main result reads,

Theorem 3.2. *Let $m \in [2,\infty)$ and $T > 0$ be fixed. Then for every $v_0 \in L^m(\Omega)$ satisfying $u_0 := |v_0|^{m-2} v_0 \in H^1(\Omega)$, the Cauchy-Neumann problem* (CNP) *admits a unique energy solution (u,v) on $[0,T]$.*

To prove this theorem, we first recall the notion of the Legendre-Fenchel transform of convex functions. The Legendre-Fenchel transform $\psi^* : H \to [0,\infty]$ of ψ is defined as follows:

$$\psi^*(v) := \sup_{u \in H} \{(v,u)_H - \psi(u)\} \qquad \text{for} \quad v \in H.$$

We note that $\partial_H \psi^* = (\partial_H \psi)^{-1}$, i.e., $\partial_H \psi^*(v) = |v|^{m-2} v$, and for any $[u,v] \in \partial_H \psi$,

$$\psi^*(v) = \frac{1}{m} |u|^{m'}_{L^{m'}(\Omega)} = \frac{m'}{m} \psi(u), \quad |v|^{(m-2)/2} v = |u|^{(m'-2)/2} u. \tag{10}$$

We next prepare the following proposition.

Proposition 1. *There exists a constant $C \geq 0$ such that*

$$|u|^2_V \quad \leq \quad C \left(\varphi(u) + \psi(u)^{2/m'} \right) \quad \textit{for all} \quad u \in V, \tag{11}$$

$$|\partial_V \varphi(u)|^2_{V^*} \quad \leq \quad C \varphi(u) \quad \textit{for all} \quad u \in V. \tag{12}$$

Proof. By Gagliardo-Nirenberg's inequality,

$$|u|_H \leq |u|^\theta_V |u|^{1-\theta}_{L^{m'}(\Omega)} \qquad \text{for} \quad u \in V$$

with some $\theta \in (0,1)$. Hence

$$|u|^2_V \leq \frac{1}{2} |u|^2_V + C |u|^2_{L^{m'}(\Omega)} + |\nabla u|^2_H \quad \text{for} \quad u \in V.$$

Thus (11) follows from the definitions of φ and ψ. Moreover, (12) is well known as the boundedness of $-\Delta : H^1(\Omega) \to (H^1(\Omega))^*$ equipped with $\partial u / \partial n = 0$ on $\partial\Omega$. $\square$

4. Proof of Theorem 3.2.

4.1. **Uniqueness.** Let (u,v) and $(\hat{u},\hat{v})$ be weak solutions of (CNP) on $[0,T]$ with an initial data $v_0 \in V^*$. Subtract (3) with v from that with $\hat{v}$. Then

$$\frac{d}{dt} (v(t) - \hat{v}(t)) + A \left(|v|^{p-2} v(t) - |\hat{v}|^{p-2} \hat{v}(t) \right) = \alpha (v(t) - \hat{v}(t)).$$

Since $\langle v(t), 1 \rangle_V = \langle \hat{v}(t), 1 \rangle_V = \langle v_0, 1 \rangle_V e^{\alpha t}$ for all $t \in [0,T]$, it follows that $v(t) - \hat{v}(t) \in V_0^* := \{w \in V^*;\ \langle w, 1 \rangle_V = 0\}$. Hence multiplying this by $F(v(t) - \hat{v}(t))$ with $F := A^{-1}$ defined on V_0^*, we have

$$\frac{1}{2} \frac{d}{dt} |\nabla F(v(t) - \hat{v}(t))|^2_H + \left\langle A \left(|v|^{p-2} v(t) - |\hat{v}|^{p-2} \hat{v}(t) \right), F\left(v(t) - \hat{v}(t)\right) \right\rangle_V$$
$$= \alpha |\nabla F(v(t) - \hat{v}(t))|^2_H.$$

Here we used the fact that

$$\begin{aligned}&\left\langle \frac{d}{dt}\left(v(t)-\hat{v}(t)\right), F(v(t)-\hat{v}(t))\right\rangle_V \\ &= \left\langle A\circ F(v(t)-\hat{v}(t)), \frac{d}{dt}F(v(t)-\hat{v}(t))\right\rangle_V = \frac{1}{2}\frac{d}{dt}|\nabla F(v(t)-\hat{v}(t))|_H^2.\end{aligned}$$

Moreover, we notice that

$$\begin{aligned}&\left\langle A\left(|v|^{p-2}v(t)-|\hat{v}|^{p-2}\hat{v}(t)\right), F\left(v(t)-\hat{v}(t)\right)\right\rangle_V \\ &= \left(|v|^{p-2}v(t)-|\hat{v}|^{p-2}\hat{v}(t), v(t)-\hat{v}(t)\right)_H \geq 0.\end{aligned}$$

Thus

$$\frac{1}{2}\frac{d}{dt}|\nabla F(v(t)-\hat{v}(t))|_H^2 \leq \alpha|\nabla F(v(t)-\hat{v}(t))|_H^2.$$

Hence integrating both sides over $(0,t)$ and applying Gronwall's inequality, we have

$$|\nabla F(v(t)-\hat{v}(t))|_H^2 \leq e^{2|\alpha|t}|\nabla F(v(0)-\hat{v}(0))|_H^2 = 0 \quad \text{for all} \quad t\in[0,T],$$

which implies $v=\hat{v}$. Thus every weak solution of (CNP) is unique.

4.2. **Approximation.** To prove the existence part, let us introduce the following approximate problems of (5)–(7) for $\varepsilon,\lambda\in(0,1]$:

$$\frac{d}{dt}\left[\varepsilon u_\lambda^\varepsilon(t)+v_\lambda^\varepsilon(t)\right]+\partial_H\tilde{\varphi}_\lambda(u_\lambda^\varepsilon(t)) = \alpha v_\lambda^\varepsilon(t) \ \text{ in } H, \qquad 0<t<T, \tag{13}$$

$$v_\lambda^\varepsilon(t) = \partial_H\psi(u_\lambda^\varepsilon(t)), \qquad 0<t<T, \tag{14}$$

$$\varepsilon u_\lambda^\varepsilon(0)+v_\lambda^\varepsilon(0) = \varepsilon u_0+v_0, \tag{15}$$

where $\tilde{\varphi}$ denotes the extension by infinity of φ onto H given by

$$\tilde{\varphi}(u) = \begin{cases}\varphi(u) & \text{for} \quad u\in V,\\ \infty & \text{for} \quad u\in H\setminus V\end{cases}$$

and $\partial_H\tilde{\varphi}_\lambda$ denotes the Yosida approximation of $\partial_H\tilde{\varphi}$. For abbreviation, we write u_λ and v_λ instead of u_λ^ε and v_λ^ε, respectively, if no confusion arises. Put

$$x_\lambda(t) := \varepsilon u_\lambda(t)+v_\lambda(t) \quad \text{for} \quad t\in[0,T].$$

Then (13) is written of the form

$$\frac{dx_\lambda}{dt}(t)+\partial_H\tilde{\varphi}_\lambda(u_\lambda(t)) = \alpha v_\lambda(t) \quad \text{in } H, \quad 0<t<T. \tag{16}$$

Here we note that $u_\lambda(t)=(\varepsilon I+\partial_H\psi)^{-1}x_\lambda(t)$ and $v_\lambda(t)=x_\lambda(t)-\varepsilon u_\lambda(t)$. Hence since $\partial_H\tilde{\varphi}_\lambda$ and $(\varepsilon I+\partial_H\psi)^{-1}$ are Lipschitz continuous in H, the Cauchy problem for (16) admits a unique strong solution $x_\lambda\in C^1([0,T];H)$ on $[0,T]$. Furthermore, $u_\lambda, v_\lambda\in W^{1,\infty}(0,T;H)$.

4.3. **Estimates.** Here and henceforth, we denote by C_T (respectively, $C_{\varepsilon,T}$) a constant independent of ε, λ (respectively, λ) and it may vary from line to line.

Let us recall the following inequality:

$$(a-b)\left(|a|^{p-2}a-|b|^{p-2}b\right) \geq c_p\left||a|^{(p-2)/2}a-|b|^{(p-2)/2}b\right|^2 \quad \text{for} \quad a,b\in\mathbb{R} \tag{17}$$

with $c_p := 4/(pp') > 0$ for $p \in (1, \infty)$. Let J_λ be the resolvent of $\partial_H \tilde{\varphi}$, i.e., $J_\lambda := (I + \lambda \partial_H \tilde{\varphi})^{-1}$. Then it follows that

$$\begin{aligned}(\partial_H \psi(u), \partial_H \tilde{\varphi}_\lambda(u))_H &\geq (\partial_H \psi(J_\lambda u), \partial_H \tilde{\varphi}_\lambda(u))_H \\ &\geq c_m \left| \nabla \left(|J_\lambda u|^{(m'-2)/2} J_\lambda u(\cdot) \right) \right|_H^2 \quad \text{for } u \in V \quad (18)\end{aligned}$$

with $c_m := 4/(mm') > 0$ (see also Appendix of [1]). Hence multiplying (13) by $v_\lambda(t)$ and integrating this over $(0, t)$, we get

$$\begin{aligned}&\varepsilon \psi(u_\lambda(t)) + \frac{1}{2} |v_\lambda(t)|_H^2 + c_m \int_0^t |\nabla \Phi_\lambda^\varepsilon(\tau)|_H^2 d\tau \\ &\leq \varepsilon \psi(u_0) + \frac{1}{2} |v_0|_H^2 + \alpha \int_0^t |v_\lambda(\tau)|_H^2 \, d\tau, \qquad (19)\end{aligned}$$

where $\Phi_\lambda^\varepsilon(\tau) := |J_\lambda u_\lambda^\varepsilon|^{(m'-2)/2} J_\lambda u_\lambda^\varepsilon(\cdot, \tau)$. By Gronwall's inequality,

$$\begin{aligned}&\frac{1}{2} |v_\lambda^\varepsilon(t)|_H^2 + c_m \int_0^t |\nabla \Phi_\lambda^\varepsilon(\tau)|_H^2 d\tau \leq \left(\varepsilon \psi(u_0) + \frac{1}{2} |v_0|_H^2 \right) e^{2|\alpha| t} \\ &\qquad \text{for all } t \in [0, T] \text{ and } \varepsilon, \lambda \in (0, 1]. \qquad (20)\end{aligned}$$

Multiplying (13) by $u_\lambda(t)$, we derive

$$\frac{\varepsilon}{2} \frac{d}{dt} |u_\lambda(t)|_H^2 + \frac{d}{dt} \psi^*(v_\lambda(t)) + \tilde{\varphi}_\lambda(u_\lambda(t)) \leq \alpha (v_\lambda(t), u_\lambda(t))_H = \alpha m \psi^*(v_\lambda(t))$$

for a.e. $t \in (0, T)$. Integrate both sides over $(0, t)$ and apply Gronwall's inequality. It then follows that

$$\begin{aligned}&\frac{1}{m} |v_\lambda(t)|_{L^m}^m = \psi^*(v_\lambda(t)) \leq \left(\frac{\varepsilon}{2} |u_0|_H^2 + \psi^*(v_0) \right) e^{|\alpha| m t} \\ &\qquad \text{for all } \ t \in [0, T] \ \text{ and } \ \varepsilon, \lambda \in (0, 1], \qquad (21)\end{aligned}$$

which also implies

$$\sup_{t \in [0, T]} \varepsilon |u_\lambda(t)|_H^2 + \int_0^T \tilde{\varphi}_\lambda(u_\lambda(t)) \, dt \leq C_T. \qquad (22)$$

Since $u_\lambda(\cdot, t) = |v_\lambda|^{m-2} v_\lambda(\cdot, t)$, it follows from (17) that

$$\begin{aligned}&(v_\lambda(t+h) - v_\lambda(t), u_\lambda(t+h) - u_\lambda(t))_H \\ &\geq c_m \left| |v_\lambda|^{(m-2)/2} v_\lambda(t+h) - |v_\lambda|^{(m-2)/2} v_\lambda(t) \right|_H^2\end{aligned}$$

for a.e. $t \in (0, T)$ and $h \geq 0$. Thus $|v_\lambda|^{(m-2)/2} v_\lambda \in W^{1,2}(0, T; H)$ and

$$c_m \left| \frac{d}{dt} |v_\lambda|^{(m-2)/2} v_\lambda(t) \right|_H^2 \leq \left(\frac{dv_\lambda}{dt}(t), \frac{du_\lambda}{dt}(t) \right)_H \qquad \text{for a.e.} \ \ t \in (0, T).$$

Therefore multiply (13) by $du_\lambda(t)/dt$ to obtain

$$\begin{aligned}&\varepsilon \left| \frac{du_\lambda}{dt}(t) \right|_H^2 + c_m \left| \frac{d}{dt} |v_\lambda|^{(m-2)/2} v_\lambda(t) \right|_H^2 + \frac{d}{dt} \tilde{\varphi}_\lambda(u_\lambda(t)) \\ &\leq \alpha \left(v_\lambda(t), \frac{du_\lambda}{dt}(t) \right)_H \\ &\leq C_{\varepsilon, T} |v_\lambda(t)|_H^2 + \frac{\varepsilon}{2} \left| \frac{du_\lambda}{dt}(t) \right|_H^2. \qquad (23)\end{aligned}$$

Moreover, by integrating both sides over $(0,t)$ and using (20), we get

$$\frac{\varepsilon}{2}\int_0^t \left|\frac{du_\lambda}{d\tau}(\tau)\right|_H^2 d\tau + c_m \int_0^t \left|\frac{d}{d\tau}|v_\lambda|^{(m-2)/2}v_\lambda(\tau)\right|_H^2 d\tau + \tilde{\varphi}_\lambda(u_\lambda(t) \leq C_{\varepsilon,T} \tag{24}$$

for all $t \in [0,T]$.

Since (18) implies $\psi(J_\lambda u) \leq \psi(u) = (m/m')\psi^*(v)$ with $v = \partial_H \psi(u)$ for all $\lambda \in (0,1]$ and $u \in V$ (see also [2]), it follows from (11) and (12) that

$$\sup_{t\in[0,1]} |J_\lambda u_\lambda(t)|_V + \sup_{t\in[0,T]} |\partial_H \tilde{\varphi}_\lambda(u_\lambda(t))|_{V^*} \leq C_{\varepsilon,T}. \tag{25}$$

Moreover, by (16)

$$\sup_{t\in[0,T]} |x_\lambda(t)|_H \leq C_T, \quad \sup_{t\in[0,T]} \left|\frac{dx_\lambda}{dt}(t)\right|_{V^*} \leq C_{\varepsilon,T}. \tag{26}$$

4.4. **Convergence as $\lambda \to +0$.** From these a priori estimates, we can derive the following convergences by taking a sequence $\lambda_n \to +0$.

$$v_{\lambda_n} \to v \quad \text{weakly star in } L^\infty(0,T;L^m(\Omega)), \tag{27}$$

$$u_{\lambda_n} \to u \quad \text{weakly star in } L^\infty(0,T;H), \tag{28}$$

$$J_{\lambda_n} u_{\lambda_n}(\cdot) \to \hat{u} \quad \text{weakly star in } L^\infty(0,T;V), \tag{29}$$

$$\partial_H \tilde{\varphi}_{\lambda_n}(u_{\lambda_n}(\cdot)) \to g \quad \text{weakly star in } L^\infty(0,T;V^*), \tag{30}$$

$$x_{\lambda_n} \to x \quad \text{weakly star in } W^{1,\infty}(0,T;V^*), \tag{31}$$

which also gives $x = \varepsilon u + v$. We note that $\partial_H \tilde{\varphi}_{\lambda_n}(u_{\lambda_n}(t)) \in \partial_V \varphi(J_{\lambda_n} u_{\lambda_n}(t))$ and $\partial_V \varphi : V \to V^*$ is bounded and linear. Hence $g(t) = \partial_V \varphi(u(t))$ for a.e. $t \in (0,T)$.

Since H is compactly embedded in V^*, by Ascoli's compactness theorem (see, e.g., [12]), we can deduce from (26) that

$$x_{\lambda_n} \to x \quad \text{strongly in } C([0,T];V^*), \tag{32}$$

which also implies $x(0) = \varepsilon u_0 + v_0$. Since the resolvent J_λ is non-expansive in H, it follows from (24) that

$$\int_0^T \left|\frac{d}{dt}J_\lambda u_\lambda(t)\right|_H^2 dt \leq \int_0^T \left|\frac{du_\lambda}{dt}(t)\right|_H^2 dt \leq C_{\varepsilon,T}.$$

Hence, by (25), it also holds that

$$J_{\lambda_n} u_{\lambda_n}(\cdot) \to \hat{u} \quad \text{strongly in } C([0,T];H). \tag{33}$$

Moreover, noting that

$$|u_\lambda(t) - J_\lambda u_\lambda(t)|_H^2 \leq 2\lambda \tilde{\varphi}_\lambda(u_\lambda(t)) \leq 2\lambda C_{\varepsilon,T} \quad \text{for all } t \in [0,T],$$

we also obtain $u = \hat{u}$ and

$$u_{\lambda_n} \to u \quad \text{strongly in } C([0,T];H). \tag{34}$$

Thus the demiclosedness of $\partial_H \psi$ in $H \times H$ ensures that $v(t) = \partial_H \psi(u(t))$ for all $t \in [0,T]$. Since $\partial_H \psi$ is continuous from $L^{m'}(\Omega)$ into $L^m(\Omega)$, we particularly observe

$$v_{\lambda_n} \to v \quad \text{strongly in } L^2(0,T;H), \tag{35}$$

which will be used to derive an energy inequality. Therefore u solves

$$\frac{d}{dt}\left[\varepsilon u^\varepsilon(t) + v^\varepsilon(t)\right] + \partial_V \varphi(u^\varepsilon(t)) = \alpha v^\varepsilon(t) \ \text{ in } V^*, \quad 0 < t < T, \tag{36}$$

$$v^\varepsilon(t) = \partial_H \psi(u^\varepsilon(t)), \qquad 0 < t < T, \tag{37}$$

$$\varepsilon u^\varepsilon(0) + v^\varepsilon(0) = \varepsilon u_0 + v_0. \tag{38}$$

Let us derive energy inequalities for $(u^\varepsilon, v^\varepsilon)$. From (23) and the fact that

$$\left(v_\lambda^\varepsilon(t), \frac{du_\lambda^\varepsilon}{dt}(t)\right)_H = \frac{d}{dt}\psi(u_\lambda^\varepsilon(t)) \quad \text{ for a.e. } \ t \in (0,T), \tag{39}$$

we have

$$\varepsilon \int_0^t \left|\frac{du_\lambda^\varepsilon}{d\tau}(\tau)\right|_H^2 d\tau + c_m \int_0^t \left|\frac{d}{d\tau}|v_\lambda^\varepsilon|^{(m-2)/2} v_\lambda^\varepsilon(\tau)\right|_H^2 d\tau + \tilde{\varphi}_\lambda(u_\lambda^\varepsilon(t)) - \alpha\psi(u_\lambda^\varepsilon(t)) \\ \leq \tilde{\varphi}_\lambda(u_0) - \alpha\psi(u_0) \tag{40}$$

for all $t \in (0,T)$. Here, by the definition of subdifferentials, (21) and (34) yield

$$\limsup_{n\to\infty} \psi(u_{\lambda_n}^\varepsilon(t)) \leq \psi(u^\varepsilon(t)) + \lim_{n\to\infty} |v_{\lambda_n}^\varepsilon(t)|_H |u_{\lambda_n}^\varepsilon(t) - u^\varepsilon(t)|_H = \psi(u^\varepsilon(t))$$

for all $t \in [0,T]$. Thus since ψ is lower semicontinuous in H, we have

$$\psi(u_{\lambda_n}^\varepsilon(t)) \to \psi(u^\varepsilon(t)) \quad \text{ for all } \ t \in [0,T]. \tag{41}$$

Moreover, (24) yields

$$|v_{\lambda_n}^\varepsilon|^{(m-2)/2} v_{\lambda_n}^\varepsilon \to \chi^\varepsilon \quad \text{ weakly in } W^{1,2}(0,T;H) \tag{42}$$

with some $\chi^\varepsilon \in W^{1,2}(0,T;H)$. From the demiclosedness of maximal monotone operators and (10), we can verify that $\chi^\varepsilon(t) = |u^\varepsilon|^{(m'-2)/2} u^\varepsilon(t) = |v^\varepsilon|^{(m-2)/2} v^\varepsilon(t)$ for a.e. $t \in (0,T)$. Thus passing to the limit in (40) as $\lambda \to +0$, we infer that

$$\varepsilon \int_0^t \left|\frac{du^\varepsilon}{d\tau}(\tau)\right|_H^2 d\tau + c_m \int_0^t \left|\frac{d}{d\tau}|v^\varepsilon|^{(m-2)/2} v^\varepsilon(\tau)\right|_H^2 d\tau \\ + \varphi(u^\varepsilon(t)) - \alpha\psi(u^\varepsilon(t)) \leq \varphi(u_0) - \alpha\psi(u_0) \tag{43}$$

for all $t \in [0,T]$. Here we used (41) and the weak lower semicontinuity of φ and norms. Furthermore, by (19) and (35),

$$\varepsilon\psi(u^\varepsilon(t)) + \frac{1}{2}|v^\varepsilon(t)|_H^2 + c_m \int_0^t \left|\nabla\left(|u^\varepsilon|^{(m'-2)/2} u^\varepsilon\right)(\tau)\right|_H^2 d\tau \\ \leq \quad \varepsilon\psi(u_0) + \frac{1}{2}|v_0|_H^2 + \alpha\int_0^t |v^\varepsilon(\tau)|_H^2 \, d\tau \quad \text{ for all } \ t \in [0,T]. \tag{44}$$

4.5. **Convergence as $\varepsilon \to +0$.** We next derive convergences of $(u^\varepsilon, v^\varepsilon)$ by passing to the limit as $\varepsilon \to +0$. To do so, we first derive from (21) and (22) that

$$\sup_{t\in[0,T]} \left\{|v^\varepsilon(t)|_{L^m}^m + \psi(u^\varepsilon(t)) + \varepsilon|u^\varepsilon(t)|_H^2\right\} \leq C_T. \tag{45}$$

Moreover, combining (43) with this fact, we obtain

$$\varepsilon \int_0^t \left|\frac{du^\varepsilon}{d\tau}(\tau)\right|_H^2 d\tau + c_m \int_0^t \left|\frac{d}{d\tau}|v^\varepsilon|^{(m-2)/2} v^\varepsilon(\tau)\right|_H^2 d\tau + \varphi(u^\varepsilon(t)) \leq C_T \tag{46}$$

for all $t \in [0,T]$ and $\varepsilon \in (0,1]$. Therefore by Proposition 1 and (36),

$$\sup_{t\in[0,T]} \left\{ |u^\varepsilon(t)|_V + |\partial_V \varphi(u^\varepsilon(t))|_{V^*} + \left| \frac{dx^\varepsilon}{dt}(t) \right|_{V^*} + |x^\varepsilon(t)|_H \right\} \leq C_T. \tag{47}$$

Then we can take a sequence $\varepsilon_n \to +0$ such that

$$\varepsilon_n \frac{du^{\varepsilon_n}}{dt} \to 0 \quad \text{strongly in } L^2(0,T;H), \tag{48}$$
$$v^{\varepsilon_n} \to v \quad \text{weakly star in } L^\infty(0,T;L^m(\Omega)), \tag{49}$$
$$u^{\varepsilon_n} \to u \quad \text{weakly star in } L^\infty(0,T;V), \tag{50}$$
$$\varepsilon_n u^{\varepsilon_n} \to 0 \quad \text{strongly in } C([0,T];H), \tag{51}$$
$$\partial_V \varphi(u^{\varepsilon_n}(\cdot)) \to g \quad \text{weakly star in } L^\infty(0,T;V^*), \tag{52}$$
$$x^{\varepsilon_n} \to v \quad \text{weakly star in } W^{1,\infty}(0,T;V^*), \tag{53}$$
$$\text{strongly in } C([0,T];V^*). \tag{54}$$

Hence we also have

$$v^{\varepsilon_n} \to v \quad \text{strongly in } C([0,T];V^*), \tag{55}$$

which gives $v(t) = \partial_H \psi(u(t))$ for a.e. $t \in (0,T)$ (see Proposition 1.1 of [9]). Moreover, we can also verify that $g(t) = \partial_V \varphi(u(t))$ for a.e. $t \in (0,T)$. Therefore u becomes a strong solution of (5)–(7) on $[0,T]$.

Finally, we establish energy inequalities. Since $v \in W^{1,2}(0,T;V^*)$ and $u \in L^2(0,T;V)$, the function $t \mapsto \psi^*(v(t))$ is absolutely continuous on $[0,T]$. Note that

$$\psi^*(v(t)) = \frac{1}{m}|v(t)|^m_{L^m(\Omega)} = \frac{1}{m}|u(t)|^{m'}_{L^{m'}(\Omega)} \quad \text{for all } t \in [0,T]$$

and recall that $L^{m'}(\Omega)$ and $L^m(\Omega)$ are uniformly convex. Hence we deduce that $u \in C([0,T];L^{m'}(\Omega))$ and $v \in C([0,T];L^m(\Omega))$. On the other hand, it follows from (50) and (55) that

$$\int_0^T \psi(u^{\varepsilon_n}(t))\, dt \leq \int_0^T \psi(u(t))dt + \int_0^T \langle v^{\varepsilon_n}(t), u^{\varepsilon_n}(t) - u(t)\rangle_V\, dt$$
$$\to \int_0^T \psi(u(t))dt.$$

From the uniform convexity of $L^{m'}(0,T;L^{m'}(\Omega))$ and $L^m(0,T;L^m(\Omega))$, we obtain

$$u^{\varepsilon_n} \to u \quad \text{strongly in } L^{m'}(0,T;L^{m'}(\Omega)), \tag{56}$$
$$v^{\varepsilon_n} \to v \quad \text{strongly in } L^m(0,T;L^m(\Omega)). \tag{57}$$

Then since $u \in C([0,T];L^{m'}(\Omega))$, it follows that $u^{\varepsilon_n}(t) \to u(t)$ strongly in $L^{m'}(\Omega)$ for all $[0,T]$, which gives

$$\psi(u^{\varepsilon_n}(t)) \to \psi(u(t)) \quad \text{for each } t \in [0,T]. \tag{58}$$

Moreover, since φ is lower semicontinuous in the topology of $L^{m'}(\Omega)$, we deduce

$$\liminf_{n\to\infty} \varphi(u^{\varepsilon_n}(t)) \geq \varphi(u(t)) \quad \text{for each } t \in [0,T]. \tag{59}$$

Furthermore, we observe that

$$\sup_{t\in[0,T]} \left| |v^{\varepsilon_n}|^{(m-2)/2} v^{\varepsilon_n}(t) \right|_H^2 = \sup_{t\in[0,T]} |v^{\varepsilon_n}(t)|^m_{L^m(\Omega)} \leq C_T.$$

Hence since H is compactly embedded in V^*, we get, by (46)

$$|v^{\varepsilon_n}|^{(m-2)/2}v^{\varepsilon_n} \to \chi \quad \text{weakly in } W^{1,2}(0,T;H), \tag{60}$$

$$|v^{\varepsilon_n}|^{(m-2)/2}v^{\varepsilon_n} \to \chi \quad \text{strongly in } C([0,T];V^*). \tag{61}$$

Recalling (10) and (50), we deduce that $\chi(t) = |v|^{(m-2)/2}v(t)$ for a.e. $t \in (0,T)$. Passing to the limit in (43) and (44) as $\varepsilon \to 0$ and recalling the uniqueness of weak solution for (CNP), we can verify

$$c_m \int_s^t \left| \frac{d}{d\tau}|v|^{(m-2)/2}v(\tau) \right|_H^2 d\tau + \varphi(u(t)) - \alpha\psi(u(t)) \le \varphi(u(s)) - \alpha\psi(u(s)),$$

$$\frac{1}{2}|v(t)|_H^2 + c_m \int_s^t \left| \nabla \left(|u|^{(m'-2)/2}u \right)(\tau) \right|_H^2 d\tau \le \frac{1}{2}|v(s)|_H^2 + \alpha \int_s^t |v(\tau)|_H^2 \, d\tau$$

for all $t, s \in [0,T]$ with $s \le t$. Dividing both sides of each inequality by $t-s$ and letting $s \to t-0$, we obtain (8) and (9) for a.e. $t \in (0,T)$.

Remark 1. (i) We can prove the same conclusion as in Theorem 3.2 with $H^1(\Omega)$ replaced by $H^1_0(\Omega)$ also for the Cauchy-Dirichlet problem in a similar way.

(ii) Our approach presented here can be applied to proving the existence of weak solutions for doubly nonlinear parabolic equation of the form

$$\frac{\partial v}{\partial t} - \Delta_p u = \alpha v, \quad u = |v|^{m-2}v, \quad (x,t) \in \Omega \times (0,T),$$

and moreover, we can also deal with the case that $v_0 \in L^m(\Omega)$ and the fast diffusion case, $m \in (1,2)$. These attempts will appear in a forthcoming paper.

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