DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
SUPPLEMENT 2007

Website: www.aimSciences.org

pp. **18–27**

ON A CERTAIN DEGENERATE PARABOLIC EQUATION ASSOCIATED WITH THE INFINITY-LAPLACIAN

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Abstract. The comparison, uniqueness and existence of viscosity solutions to the Cauchy-Dirichlet problem are proved for a degenerate parabolic equation of the form $u_t = \Delta_\infty u$, where Δ_∞ denotes the so-called infinity-Laplacian given by $\Delta_\infty u = \sum_{i,j=1}^N u_{x_i} u_{x_j} u_{x_i x_j}$. Our proof relies on a coercive regularization of the equation, barrier function arguments and the stability of viscosity solutions.

1. Introduction.

Aronsson [2] introduces the so-called infinity-Laplacian Δ_∞ given by

$$\Delta_\infty \phi(x) = \sum_{i,j=1}^N \frac{\partial \phi}{\partial x_i}(x) \frac{\partial \phi}{\partial x_j}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) \tag{1}$$

to investigate the existence of absolutely minimizing Lipschitz extensions (AMLE's for short) of functions g defined only on the boundary $\partial\Omega$ of a domain Ω in $\mathbb{R}^N$ into Ω. Here the AMLE of g into Ω means a function $u \in W^{1,\infty}(\Omega)$ satisfying that $u = g$ on $\partial\Omega$ and that for every open subset U of Ω and $\phi \in W^{1,\infty}(U)$, if $u - \phi \in W_0^{1,\infty}(U)$, then

$$|Du|_{L^\infty(U)} \leq |D\phi|_{L^\infty(U)}.$$

More precisely, the following elliptic problem is proposed in [2] as an Euler equation of the above variational problem for smooth AMLE's:

$$\Delta_\infty u = 0 \ \text{ in } \Omega, \quad u = g \ \text{ on } \partial\Omega. \tag{2}$$

Aronsson [3] also reveals various properties of classical solutions of (2) in $N = 2$; particularly, it is somewhat important that if u is a non-constant classical solution, then $|\nabla u| > 0$ in Ω, which also implies that in general (2) does not admit classical solutions (this fact is clearly described in [15, p. 55]).

Jensen [15] employs the notion of viscosity solutions as a weak solution of (2) and proves the existence and uniqueness of AMLE's under somewhat general assumptions, and moreover, it is also shown that u is a viscosity solution of (2) if and only if u is the AMLE of g.

2000 *Mathematics Subject Classification.* Primary: 35K55, 35K65; Secondary: 35D05.
Key words and phrases. Degenerate parabolic equation, infinity-Laplacian, viscosity solution.
G. Akagi is supported by Shibaura Institute of Technology grant for Project Research, #211459.

Furthermore, various problems related to elliptic equations associated with the infinity Laplacian, e.g., the regularity of solutions, Harnack's inequality, limiting problems associated with p-Laplacian as $p \to +\infty$, eigenvalue problem, L^∞-inequality of the Poincaré type, have been vigorously studied by many authors (see, e.g., [4], [6], [5], [7], [10], [11], [12], [14], [17], [22]). On the other hand, to the best of the authors' knowledge, parabolic problems associated with the infinity-Laplacian have not been studied yet except in [9], [21] and [16].

This paper is concerned with the following parabolic problem:

$$u_t = \Delta_\infty u \qquad \text{in} \quad Q := \Omega \times (0,T), \tag{3}$$

$$u = \varphi \qquad \text{on} \quad \mathcal{P}Q, \tag{4}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$, $\mathcal{P}Q$ denotes the parabolic boundary of $Q = \Omega \times (0,T)$ and u_t denotes the time-derivative of $u = u(x,t)$ (see the notation in the end of this section). The main purpose of this paper is to investigate the comparison, uniqueness and existence of viscosity solutions $u = u(x,t)$ of the Cauchy-Dirichlet problem (3), (4).

Another type of parabolic equation associated with the infinity-Laplacian is also studied by Juutinen and Kawohl in [16], where they treat the following:

$$u_t = \frac{\Delta_\infty u}{|Du|^2} \quad \text{in } Q. \tag{5}$$

They investigate the existence and uniqueness of solutions of the Cauchy-Dirichlet problem for (5) with initial-boundary data φ, and moreover, they deal with the Cauchy problem for the case $\Omega = \mathbb{R}^N$ as well. To prove the existence, they introduce approximate problems of the form $(u_{\varepsilon,\delta})_t = \varepsilon \Delta u_{\varepsilon,\delta} + \Delta_\infty u_{\varepsilon,\delta}/(|Du_{\varepsilon,\delta}|^2 + \delta)$ with $\varepsilon, \delta > 0$, and establish boundary Hölder estimates of their solutions by constructing barrier functions.

To prove the existence for (3), (4), we introduce the following approximate problems with $\varepsilon > 0$:

$$(u_\varepsilon)_t = \varepsilon \left(|Du_\varepsilon|^2 + \varepsilon\right) \Delta u_\varepsilon + \Delta_\infty u_\varepsilon \quad \text{in } Q \tag{6}$$

and prove the existence of classical solutions u_ε for the Cauchy-Dirichlet problems for (6) with initial-boundary data φ. Moreover, as in [16], we employ barrier function arguments to establish a priori estimates for the solutions u_ε. Our proof of establishing a priori estimates is inspired by [16].

In the next section, we state our main results on the comparison, uniqueness and existence of viscosity solutions of the Cauchy-Dirichlet problem (3), (4). Section 3 is devoted to our proof of the existence result.

Notation: Throughout this paper, we use the following notation: $Q = \Omega \times (0,T)$, $\mathcal{S}Q = \partial\Omega \times (0,T)$, $\mathcal{B}Q = \Omega \times \{0\}$, $\mathcal{C}Q = \partial\Omega \times \{0\}$, $\mathcal{P}Q = \mathcal{S}Q \cup \mathcal{B}Q \cup \mathcal{C}Q$,

$$\phi_t = \frac{\partial \phi}{\partial t}, \quad D_i = \frac{\partial}{\partial x_i}, \quad D = (D_1, D_2, \ldots, D_N), \quad D^2_{ij} = \frac{\partial^2}{\partial x_i \partial x_j},$$

and D^2 denotes the $N \times N$ matrix whose (i,j)-th element is D^2_{ij}. Furthermore, we also use the Einstein summation convention, where we sum over repeated Greek indices. As for the definitions of function spaces such as $C^{2,1}$, H^α and $H^{\ell,\ell/2}$ and (semi-)norms, we refer the reader to [19, pp. 7-8]. Moreover, we denote by $Lip(Q)$ the class of Lipschitz continuous functions in Q, and we simply denote by $|\cdot|_\infty$ the sup-norm in the corresponding space if no confusion arises.

2. **Main Results.** Before stating our main results, we give a couple of notation and definitions to be used. Set

$$P(s,p,X) := p_i p_j X_{ij} - s, \quad (s,p,X) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N,$$

where $\mathbb{S}^N$ denotes the set of all symmetric $N \times N$ matrices. We are then concerned with viscosity solutions of (3) given in the following.

Definition 1. Let Ω be a domain in $\mathbb{R}^N$ and let $Q = \Omega \times (0,T)$. A function $u \in USC(Q) := \{$upper semicontinuous functions $u : Q \to \mathbb{R}\}$ is said to be a *viscosity subsolution* in Q of (3) if

$$P(\phi_t(\hat{x},\hat{t}), D\phi(\hat{x},\hat{t}), D^2\phi(\hat{x},\hat{t})) \geq 0$$

for all $(\hat{x},\hat{t}) \in Q$ and $\phi \in C^{2,1}(Q)$ satisfying $u - \phi$ attains its local maximum at $(\hat{x},\hat{t})$.

A function $u \in LSC(Q) := \{$lower semicontinuous functions $u : Q \to \mathbb{R}\}$ is said to be a *viscosity supersolution* in Q of (3) if

$$P(\phi_t(\hat{x},\hat{t}), D\phi(\hat{x},\hat{t}), D^2\phi(\hat{x},\hat{t})) \leq 0$$

for all $(\hat{x},\hat{t}) \in Q$ and $\phi \in C^{2,1}(Q)$ satisfying $u - \phi$ attains its local minimum at $(\hat{x},\hat{t})$.

Moreover, $u \in C(Q)$ is said to be a *viscosity solution* in Q of (3) if it is both a viscosity subsolution and a viscosity supersolution in Q of (3).

Furthermore, viscosity solutions of the Cauchy-Dirichlet problem (3), (4) are defined as follows:

Definition 2. A function $u \in USC(\overline{Q})$ (resp., $LSC(\overline{Q})$) is said to be a viscosity subsolution (resp., supersolution) in Q of (3), (4) if u is a viscosity subsolution (resp., supersolution) in Q of (3), $u \leq \varphi$ (resp., $u \geq \varphi$) on $\mathcal{P}Q$. Furthermore, $u \in C(\overline{Q})$ is a viscosity solution in Q of (3), (4) if it is both a viscosity subsolution and a viscosity supersolution in Q of (3), (4).

Applying Theorem 8.2 and related remarks of [8], the comparison principle for (3), (4) is immediately derived, and moreover, it also implies the continuous dependence on initial-boundary data φ and the uniqueness of solutions.

Theorem 1 (Comparison and uniqueness)**.** *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ and let $u \in USC(\overline{Q})$ and $v \in LSC(\overline{Q})$ be a viscosity subsolution and a viscosity supersolution in $Q = \Omega \times (0,T)$ of* (3)*, respectively, such that $u \leq v$ on $\mathcal{P}Q$. Then $u \leq v$ in Q.*

In particular, let $\varphi_1, \varphi_2 \in C(\overline{Q})$ and let u_1 and u_2 be viscosity solutions in Q of (3), (4) *with the initial-boundary data φ_1 and φ_2, respectively. Then it follows that*

$$\sup_{(x,t)\in Q} |u_1(x,t) - u_2(x,t)| \leq \sup_{(x,t)\in \mathcal{P}Q} |\varphi_1(x,t) - \varphi_2(x,t)|, \tag{7}$$

which also implies the uniqueness of solutions.

Proof of Theorem 1. Due to Theorem 8.2 of [8], the comparison part follows immediately. Now, let u_1 and u_2 be viscosity solutions of (3), (4) with the initial-boundary data φ_1 and φ_2, respectively, and put $w^{\pm}(x,t) := u_2(x,t) \pm \sup_{(x,t)\in\mathcal{P}Q} |\varphi_1(x,t) - \varphi_2(x,t)|$. Then the functions w^- and w^+ become a viscosity subsolution and a viscosity supersolution of (3), (4) with φ replaced by φ_1 respectively. Thus we have

$$w^- \leq u_1 \leq w^+ \ \text{ in } Q,$$

which implies (7). In particular, if $\varphi_1 = \varphi_2$ on $\mathcal{P}Q$, then the uniqueness of solutions follows. □

As for the existence of solution, we first introduce the following assumption.

$$\left.\begin{array}{l}\text{For all } x_0 \in \partial\Omega, \text{ there exists } y_0 \in \mathbb{R}^N \text{ such that } |x_0 - y_0| = R \\ \text{and } \{x \in \mathbb{R}^N; |x - y_0| < R\} \cap \Omega = \emptyset \text{ for some positive constant } R \\ \text{independent of } x_0.\end{array}\right\} \quad (8)$$

This assumption is employed only for the construction of approximate solutions in classical sense (see Theorem 4.4 of [19, Chap. VI, p. 560]). Now, our result reads:

Theorem 2 (Existence)**.** *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ and let $Q = \Omega \times (0, T)$. Suppose that* (8) *is satisfied. Then, for every $\varphi \in C(\overline{Q})$, the Cauchy-Dirichlet problem* (3), (4) *admits a viscosity solution $u \in C(\overline{Q})$ in Q such that*

$$\sup_{(x,t)\in Q} |u(x,t)| \le \sup_{(x,t)\in \mathcal{P}Q} |\varphi(x,t)|. \quad (9)$$

3. **Proof of Theorem 2.** In this section, we give a proof of Theorem 2, which is concerned with the existence of viscosity solutions of the Cauchy-Dirichlet problem (3), (4). Firstly we deal with the case $\varphi \in H^{2+\alpha,1+\alpha/2}(\overline{Q})$ for some $\alpha \in (0,1)$. We then introduce the following approximation of (3), (4) for each $\varepsilon \in (0,1)$.

$$(u_\varepsilon)_t = \varepsilon \left(|Du_\varepsilon|^2 + \varepsilon\right) \Delta u_\varepsilon + \Delta_\infty u_\varepsilon \quad \text{in} \quad Q, \quad (10)$$

$$u_\varepsilon = \varphi \quad \text{on} \quad \mathcal{P}Q. \quad (11)$$

Define $a_{ij}^\varepsilon \in C^\infty(\mathbb{R}^N)$ and $P_\varepsilon \in C(\mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N)$ by

$$a_{ij}^\varepsilon(p) := \varepsilon(|p|^2 + \varepsilon)\delta_{ij} + p_i p_j, \quad i, j = 1, 2, \ldots, N, \quad p \in \mathbb{R}^N$$

and

$$P_\varepsilon(s, p, X) := a_{ij}^\varepsilon(p) X_{ij} - s, \quad (s, p, X) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N.$$

Then (10) is rewritten into

$$P_\varepsilon((u_\varepsilon)_t(x,t), Du_\varepsilon(x,t), D^2 u_\varepsilon(x,t)) = 0, \quad (x,t) \in Q.$$

Moreover, we observe that

$$\varepsilon(|p|^2 + \varepsilon)|\xi|^2 \le a_{ij}^\varepsilon(p)\xi_i\xi_j \le \left\{\varepsilon(|p|^2 + \varepsilon) + |p|^2\right\} |\xi|^2$$

for all $\xi \in \mathbb{R}^N$, and furthermore

$$\left|\frac{\partial a_{ij}^\varepsilon}{\partial p_k}\right| (1 + |p|)^3 \le C(1 + |p|)^4, \quad i, j, k = 1, 2, \ldots, N.$$

Thus, since Ω satisfies (8), Theorem 4.4 of [19, Chap. VI, p. 560] ensures that the Cauchy-Dirichlet problem (10), (11) admits a classical solution $u_\varepsilon \in C(\overline{Q}) \cap H^{2+\alpha,1+\alpha/2}(Q)$ for each $\varepsilon \in (0,1)$.

We now proceed to establish a priori estimates for classical solutions u_ε of the Cauchy-Dirichlet problems (10), (11) for each $\varepsilon \in (0,1)$. To derive the convergence of u_ε as $\varepsilon \to +0$, thanks to the stability of viscosity solutions, it suffices to obtain a Hölder estimate for u_ε on $\overline{Q}$, which implies the precompactness of u_ε in $C(\overline{Q})$. The following lemma provides an L^∞-estimate for u_ε.

Lemma 1 (L^∞-estimate)**.** *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ and let $u \in C(\overline{Q}) \cap C^{2,1}(Q)$ be a classical solution in $Q = \Omega \times (0,T)$ of the Cauchy-Dirichlet problem* (10)*,* (11) *with $\varphi \in C(\overline{Q})$. Then we have*

$$|u|_\infty \leq |\varphi|_\infty.$$

Proof of Lemma 1. The function $w^+(x,t) \equiv |\varphi|_\infty$ (resp., $w^-(x,t) \equiv -|\varphi|_\infty$) becomes a classical supersolution (resp., subsolution) in Q of (10), (11), so the classical comparison principle (see, e.g., Theorem 9.1 of [20, p. 213]) implies that $|u|_\infty \leq |\varphi|_\infty$. □

We have several steps to establish a Hölder estimate for u_ε in Q. The first step is concerned with a Lipschitz estimate for $u_\varepsilon(x,\cdot)$ at $t = 0$ (see Lemma 2), and the second step yields a Lipschitz estimate at any $t \in (0,T)$ (see Lemma 3). In the third step, we estimate a Hölder constant of $u_\varepsilon(\cdot,t)$ on $\partial\Omega$ (see Lemma 4). Hence these three steps imply a boundary Hölder estimate on $\mathcal{P}Q$ (see Lemma 5). Finally, we derive a global Hölder estimate for u_ε in Q from the boundary Hölder estimate (see Lemma 6). Our derivations of these estimates are due to the similar barrier function argument as in [16], and we also employ the translation invariance of the equation (10) to extend Lipschitz and Hölder estimates established only on the boundary, e.g., $t = 0$, $\partial\Omega$, $\mathcal{P}Q$, as in [18] (a similar argument using the translation invariance of an equation is also found in [13, Corollary 2.11]).

Lemma 2 (Lipschitz estimate for $u_\varepsilon(x,\cdot)$ at $t = 0$)**.** *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ and let $u \in C(\overline{Q}) \cap C^{2,1}(Q)$ be a classical solution in $Q = \Omega \times (0,T)$ of the Cauchy-Dirichlet problem* (10)*,* (11) *with $\varphi \in C^{2,1}(\overline{Q})$. Then it follows that*

$$|u(x,t) - \varphi(x,0)| \leq M_1 t \quad \text{for all } t \in (0,T) \text{ and } x \in \Omega, \tag{12}$$

where $M_1 := 2(|D\varphi|_\infty^2 + 1)|D^2\varphi|_\infty + |\varphi_t|_\infty$.

Proof of Lemma 2. Put $w^\pm(x,t) = \varphi(x,0) \pm M_1 t$ and observe that

$$\begin{aligned}
&P(w_t^+(x,t), Dw^+(x,t), D^2w^+(x,t)) \\
&= -M_1 + a_{ij}^\varepsilon(D\varphi(x,0))D_{ij}^2\varphi(x,0) \\
&\leq -M_1 + \varepsilon(|D\varphi|_\infty^2 + \varepsilon)|D^2\varphi|_\infty + |D\varphi|_\infty^2|D^2\varphi|_\infty \leq 0
\end{aligned}$$

for all $(x,t) \in Q$. Moreover, if $(x,t) \in \mathcal{P}Q$, then

$$\begin{aligned}
w^+(x,t) &= \varphi(x,0) + M_1 t \\
&= \varphi(x,t) - \varphi(x,t) + \varphi(x,0) + M_1 t \\
&\geq \varphi(x,t) - |\varphi_t|_\infty t + M_1 t \geq \varphi(x,t).
\end{aligned}$$

We can also deduce that $P(w_t^-(x,t), Dw^-(x,t), D^2w^-(x,t)) \geq 0$ for all $(x,t) \in Q$ and $w^- \leq \varphi$ on $\mathcal{P}Q$. Therefore the classical comparison principle ensures that $w^- \leq u \leq w^+$ in Q. Hence we obtain (12). □

By using the translation invariance of the equations (10) and the above lemma, we can obtain a Lipschitz estimate for $u_\varepsilon(x,\cdot)$ in $(0,T)$.

Lemma 3 (Lipschitz estimate for $u_\varepsilon(x,\cdot)$ in $(0,T)$)**.** *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ and let $u \in C(\overline{Q}) \cap C^{2,1}(Q)$ be a classical solution in*

$Q = \Omega \times (0,T)$ *of the Cauchy-Dirichlet problem* (10), (11) *with* $\varphi \in C^{2,1}(\overline{Q})$. *Then it follows that*

$$|u(x,t) - u(x,s)| \le M_1|t-s| \quad \textit{for all } t,s \in (0,T) \textit{ and } x \in \Omega, \tag{13}$$

where $M_1 = 2(|D\varphi|_\infty^2 + 1)|D^2\varphi|_\infty + |\varphi_t|_\infty$.

Proof of Lemma 3. Let $h \in (-T,T)$ be fixed and set $Q_h = \Omega \times (h, T+h)$. Putting $v(x,t) = u(x,t-h)$, we see that v remains to be a classical solution in Q_h of (10), (11) with φ replaced by $\varphi(\cdot, \cdot - h)$. Hence, by Lemma 2, we infer that

$$|v(x,t) - u(x,t)| \le M_1|h| \quad \text{for all } (x,t) \in \mathcal{B}(Q \cap Q_h).$$

Here we used the fact that $t = \max\{0,h\}$ if $(x,t) \in \mathcal{B}(Q \cap Q_h)$. Thus we can derive $u \le v + M_1|h|$ on $\mathcal{B}(Q \cap Q_h)$. Moreover, if $(x,t) \in \mathcal{S}(Q \cap Q_h)$, then we see that $(x,t) \in \mathcal{S}Q$, which implies that

$$\begin{aligned} v(x,t) + M_1|h| &= u(x,t-h) + M_1|h| \\ &= \varphi(x,t-h) + M_1|h| \\ &\ge \varphi(x,t) = u(x,t). \end{aligned}$$

Therefore, since $u(x,t) \le v(x,t) + M_1|h|$ for all $(x,t) \in \mathcal{P}(Q \cap Q_h)$ and $v + M_1|h|$ also becomes a classical supersolution in $Q \cap Q_h$ of (10), it follows that $u \le v + M_1|h|$ in $Q \cap Q_h$. Repeating the above argument with $v + M_1|h|$ replaced by $v - M_1|h|$, we can deduce that $v - M_1|h| \le u \le v + M_1|h|$ in $Q \cap Q_h$, which also gives $|u(x,t) - u(x,t-h)| \le M_1|h|$ for all $(x,t) \in Q \cap Q_h$. Furthermore, from the arbitrariness of h, we can verify (13). □

We next establish a Hölder estimate for $u_\varepsilon(\cdot,t)$ on $\partial\Omega$.

Lemma 4 (Hölder estimate for $u_\varepsilon(\cdot,t)$ on $\partial\Omega$)**.** *Let* Ω *be a bounded domain in* $\mathbb{R}^N$ *with boundary* $\partial\Omega$ *and let* $\alpha \in (0,1)$ *and* $R > 0$ *be fixed. Let* $u \in C(\overline{Q}) \cap C^{2,1}(Q)$ *be a classical solution in* $Q = \Omega \times (0,T)$ *of the Cauchy-Dirichlet problem* (10), (11) *with* $\varphi \in C(\overline{Q})$ *satisfying*

$$|\varphi_t|_\infty < \infty$$
$$\textit{and} \quad \langle \varphi \rangle_{x,Q}^\alpha := \sup\left\{ \frac{|\varphi(x,t) - \varphi(y,t)|}{|x-y|^\alpha}; x,y \in \Omega, x \ne y, t \in [0,T] \right\} < \infty.$$

Then there exist constants $\varepsilon_0 = \varepsilon_0(N,\alpha,R) > 0$ *and* $M_2 = M_2(|\varphi|_\infty, |\varphi_t|_\infty, \langle\varphi\rangle_{x,Q}^\alpha, N, \alpha, R) \ge 0$ *such that if* $\varepsilon < \varepsilon_0$ *then*

$$|u(x,t) - \varphi(x_0,t_0)| \le M_2(|x-x_0|^\alpha + t_0 - t)$$
$$\textit{for all } (x_0,t_0) \in \mathcal{S}Q,\ x \in \Omega \cap B_R(x_0) \textit{ and } \ t \in (\max\{0, t_0-1\}, t_0),$$
$$\textit{where } B_R(x_0) := \{x \in \mathbb{R}^N; |x-x_0| < R\}.$$

In particular, the same conclusion also follows with $\Omega \cap B_R(x_0)$ *replaced by* Ω *by choosing* $R > 0$ *enough large.*

Proof of Lemma 4. Let $(x_0,t_0) \in \mathcal{S}Q$ and $\alpha \in (0,1)$ be fixed and define

$$w^+(x,t) = \varphi(x_0,t_0) + \kappa|x-x_0|^\alpha + \rho(t_0-t)$$

for all $x \in B_R(x_0) := \{x \in \mathbb{R}^N; |x-x_0| < R\}$ and all $t < t_0$ with positive constants κ and ρ which will be determined later. Observing that

$$w_t^+(x,t) = -\rho, \quad D_i w^+(x,t) = \kappa\alpha|x-x_0|^{\alpha-2}(x-x_0)_i,$$
$$D_{ij}^2 w^+(x,t) = \kappa\alpha(\alpha-2)|x-x_0|^{\alpha-4}(x-x_0)_i(x-x_0)_j + \kappa\alpha|x-x_0|^{\alpha-2}\delta_{ij},$$

we then see that

$$\Delta_\infty w^+(x,t) = x_0|^4(\kappa\alpha)^3(\alpha-1)|x-x_0|^{3\alpha-4}.$$

Thus it follows that

$$\begin{aligned}&-w_t^+(x,t) + a_{ij}^\varepsilon(Dw^+(x,t))D_{ij}^2 w^+(x,t)\\ =&\rho + (\kappa\alpha)^3\left\{\varepsilon(\alpha-2+N)+\alpha-1\right\}|x-x_0|^{3\alpha-4}\\ &+\varepsilon^2\kappa\alpha(\alpha-2+N)|x-x_0|^{\alpha-2}.\end{aligned}$$

Here taking $\varepsilon > 0$ enough small such that

$$\varepsilon(\alpha-2+N)+\alpha-1 < \frac{1}{2}(\alpha-1),$$

we have

$$\begin{aligned}&-w_t^+(x,t) + a_{ij}^\varepsilon(Dw^+(x,t))D_{ij}^2 w^+(x,t)\\ <&\rho + \frac{(\kappa\alpha)^3}{2}(\alpha-1)|x-x_0|^{3\alpha-4} + \varepsilon^2\kappa\alpha(\alpha-2+N)|x-x_0|^{\alpha-2}\\ =&\rho+\kappa\alpha|x-x_0|^{\alpha-2}\left\{\frac{(\kappa\alpha)^2}{2}(\alpha-1)|x-x_0|^{2\alpha-2}+\varepsilon^2(\alpha-2+N)\right\}\\ \le&\rho+\kappa\alpha|x-x_0|^{\alpha-2}\left\{\frac{(\kappa\alpha)^2}{2}(\alpha-1)R^{2\alpha-2}+\varepsilon^2(\alpha-2+N)\right\},\end{aligned}$$

where we used the fact that $|x-x_0| < R$. Note that

$$\frac{(\kappa\alpha)^2}{2}(\alpha-1)R^{2\alpha-2}+\varepsilon^2(\alpha-2+N) \le \frac{(\kappa\alpha)^2}{4}(\alpha-1)R^{2\alpha-2},$$

provided that $\kappa \ge 1$ and ε is enough small so that

$$\frac{\alpha^2}{4}(\alpha-1)R^{2\alpha-2}+\varepsilon^2(\alpha-2+N) \le 0.$$

Thus

$$\begin{aligned}&-w_t^+(x,t) + a_{ij}^\varepsilon(Dw^+(x,t))D_{ij}^2 w^+(x,t)\\ \le&\rho+\frac{(\kappa\alpha)^3}{4}(\alpha-1)R^{2\alpha-2}|x-x_0|^{\alpha-2}\\ \le&\rho+\frac{(\kappa\alpha)^3}{4}(\alpha-1)R^{3\alpha-4}.\end{aligned}$$

Therefore taking κ enough large such that $4\rho \le (\kappa\alpha)^3(1-\alpha)R^{3\alpha-4}$, we conclude that

$$-w_t^+(x,t) + a_{ij}^\varepsilon(Dw^+(x,t))D_{ij}^2 w^+(x,t) \le 0$$

for all $x \in B_R(x_0)\cap\Omega$ and all $t < t_0$.

We next prove that $w^+ \ge u$ on $\mathcal{P}((B_R(x_0)\cap\Omega)\times(t_0-1,t_0))$ for the case that $t_0 > 1$. To do so, we divide our proof to the following three cases:

(i): Let $x \in (\partial B_R(x_0))\cap\Omega$ and $t < t_0$ be fixed. From the fact that $|x-x_0| = R$, we then see that

$$w^+(x,t) = \varphi(x_0,t_0)+\kappa R^\alpha+\rho(t_0-t) \ge \varphi(x_0,t_0)+\kappa R^\alpha \ge |\varphi|_\infty \ge u(x,t),$$

provided that $\kappa \ge 2|\varphi|_\infty/R^\alpha$.

(ii): Let $x \in B_R(x_0) \cap \partial\Omega$ and $t < t_0$ be fixed. Since $\varphi(x,t) = u(x,t)$, it follows that

$$w^+(x,t) = \varphi(x,t) - \varphi(x,t) + \varphi(x_0,t_0) + \kappa|x - x_0|^\alpha + \rho(t_0 - t) \ge u(x,t),$$

provided that $\kappa \ge \langle\varphi\rangle^\alpha_{x,Q}$ and $\rho \ge |\varphi_t|_\infty$.

(iii): Let $x \in B_R(x_0) \cap \Omega$ and let $t = t_0 - 1$ be fixed. Then

$$w^+(x,t) = \varphi(x_0,t_0) + \kappa|x - x_0|^\alpha + \rho \ge \varphi(x_0,t_0) + \rho \ge |\varphi|_\infty \ge u(x,t),$$

provided that $\rho \ge 2|\varphi|_\infty$.

Now as for the case where $t_0 < 1$, we use $(B_R(x_0) \cap \Omega) \times (0, t_0)$ instead of the cylinder used in the last case. Then it is easily seen that, for $x \in B_R(x_0) \cap \Omega$ and $t = 0$,

$$w^+(x,0) = \varphi(x_0,t_0) + \kappa|x - x_0|^\alpha + \rho t_0 \ge \varphi(x,0) = u(x,0),$$

provided that $\kappa \ge \langle\varphi\rangle^\alpha_{x,Q}$ and $\rho \ge |\varphi_t|_\infty$.

Therefore the comparison principle ensures that

$$u \le w^+ \quad \text{on} \quad \overline{B_R(x_0) \cap \Omega} \times [\max\{0, t_0 - 1\}, t_0].$$

Repeating the same argument with the function $w^-(x,t) := \varphi(x_0,t_0) - \kappa|x-x_0|^\alpha - \rho(t_0 - t)$, we can also obtain $w^- \le u$ on $\overline{B_R(x_0) \cap \Omega} \times [\max\{0, t_0 - 1\}, t_0]$. Consequently, we can deduce that

$$|u(x,t) - \varphi(x_0,t_0)| \le \kappa|x - x_0|^\alpha + \rho(t_0 - t)$$

for all $(x_0,t_0) \in \mathcal{S}Q$ and $x \in B_R(x_0) \cap \Omega$ and $t \in [\max\{0, t_0 - 1\}, t_0]$. □

Thus Lemmas 2 and 4 imply the following:

Lemma 5 (Hölder estimate on $\mathcal{P}Q$)**.** *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ and let $\alpha \in (0,1)$. Suppose that* (8) *is satisfied. Let $u \in C(\overline{Q}) \cap C^{2,1}(Q)$ be a classical solution in $Q = \Omega \times (0,T)$ of the Cauchy-Dirichlet problem* (10), (11) *with $\varepsilon \in (0, \varepsilon_0)$ and $\varphi \in C^{2,1}(\overline{Q})$. Then it follows that*

$$|u(x,t) - \varphi(x_0,t_0)| \le M_3 \left(|x - x_0|^\alpha + |t - t_0|\right) \tag{14}$$
$$\text{for all } (x_0,t_0) \in \mathcal{P}Q \text{ and } (x,t) \in Q,$$

where $M_3 = M_1 + M_2 + \langle\varphi\rangle^{(\alpha)}_{x,Q}$.

Proof of Lemma 5. For the case: $(x_0,t_0) \in \mathcal{S}Q$, by virtue of Lemmas 3 and 4,

$$\begin{aligned} |u(x,t) - \varphi(x_0,t_0)| &\le |u(x,t) - u(x,t_0)| + |u(x,t_0) - \varphi(x_0,t_0)| \\ &\le M_1|t_0 - t| + M_2|x_0 - x|^\alpha. \end{aligned}$$

For the case: $(x_0,t_0) \in \mathcal{B}Q$, that is, $t_0 = 0$, by Lemma 2, we also have

$$\begin{aligned} |u(x,t) - \varphi(x_0,t_0)| &\le |u(x,t) - \varphi(x,0)| + |\varphi(x,0) - \varphi(x_0,0)| \\ &\le M_1 t + \langle\varphi\rangle^{(\alpha)}_{x,Q}|x_0 - x|^\alpha. \end{aligned}$$

Hence (14) follows. □

Now, we extend the above Hölder estimate on the parabolic boundary $\mathcal{P}Q$ into the parabolic domain Q in the following lemma, which is derived from Theorem 6 of [18], but for the completeness we give a proof.

Lemma 6 (Global Hölder estimate)**.** *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ and let $\alpha \in (0,1)$. Suppose that* (8) *is satisfied. Let $u \in C(\overline{Q}) \cap C^{2,1}(Q)$ be a classical solution in $Q = \Omega \times (0,T)$ of the Cauchy-Dirichlet problem* (10)*,* (11) *with $\varepsilon \in (0, \varepsilon_0)$ and $\varphi \in C^{2,1}(\overline{Q})$. Then it follows that*

$$|u(x,t) - u(y,s)| \le M_3 \left(|x-y|^\alpha + |t-s|\right) \text{ for all } (x,t),(y,s) \in Q, \quad (15)$$

where $M_3 = M_1 + M_2 + \langle\varphi\rangle_{x,Q}^{(\alpha)}$.

Proof of Lemma 6. Let $h := (h_x, h_t) \in \mathbb{R}^N \times \mathbb{R}$ be fixed and let $Q + h := \{(x,t) \in \mathbb{R}^{N+1}; (x - h_x, t - h_t) \in Q\}$. Moreover, put $v(x,t) = u(x - h_x, t - h_t)$. We then find that v still remains to be a classical solution in $Q+h$ of (10), (11) with φ replaced by $\varphi(\cdot - h_x, \cdot - h_t)$. Then, by Lemma 5, we can assure that, for $(x,t) \in \mathcal{P}\{Q \cap (Q+h)\}$, $|v(x,t) - u(x,t)| \le M_3 |h|_{\alpha,1}$, where $|h|_{\alpha,1} := |h_x|^\alpha + |h_t|$; hence, $v - M_3|h|_{\alpha,1} \le u \le v + M_3|h|_{\alpha,1}$ on $\mathcal{P}\{Q \cap (Q+h)\}$. Furthermore, since $v \pm M_3|h|_{\alpha,1}$ also become classical solutions in $Q \cap (Q+h)$ of (10), the classical comparison theorem ensures that $v - M_3|h|_{\alpha,1} \le u \le v + M_3|h|_{\alpha,1}$ in $Q \cap (Q+h)$. From the arbitrariness of h, we can verify (15). □

By virtue of the global Hölder estimate for u_ε in Lemma 6 and Ascoli-Arzela's compactness theorem, taking a sequence $\varepsilon_n \to +0$, we can deduce that

$$u_{\varepsilon_n} \to u \quad \text{uniformly on } \overline{Q} \quad (16)$$

as $\varepsilon_n \to +0$. We also note that

$$P_\varepsilon(s,p,X) \to P(s,p,X) \text{ as } \varepsilon \to +0, \text{ for all } (s,p,X) \in \mathbb{R} \times \mathbb{R}^N \times S^N.$$

Therefore the stability of viscosity solutions (see, e.g., Section 6 of [8]) ensures that the limit u becomes a viscosity solution of (3), (4).

Secondly we proceed to the case $\varphi \in C(\overline{Q})$. By virtue of Weierstrass's approximation theorem (see, e.g., 1.29 Corollary of [1, p. 10]), we can take an approximate sequence $\varphi_n \in H^{2+\alpha, 1+\alpha/2}(\overline{Q})$ such that $\varphi_n \to \varphi$ uniformly on $\overline{Q}$. Hence, due to the last case, there exists a viscosity solution u_n of (3), (4) with φ replaced by φ_n. Moreover, by Theorem 1,

$$\sup_{(x,t)\in Q} |u_n(x,t) - u_m(x,t)| \le \sup_{(x,t)\in \mathcal{P}Q} |\varphi_n(x,t) - \varphi_m(x,t)| \to 0$$

as $n, m \to +\infty$. Thus (u_n) forms a Cauchy sequence in $C(\overline{Q})$, so $u_n \to u$ uniformly on $\overline{Q}$. Therefore, from the stability of viscosity solution, u also becomes a viscosity solution of (3), (4) with the initial data $\varphi \in C(\overline{Q})$. Furthermore, as in Lemma 1, (9) follows immediately. This completes our proof of Theorem 2.

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Received September 2006; revised January 2007.

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