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DOUBLY NONLINEAR EVOLUTION EQUATIONS AND BEAN'S CRITICAL-STATE MODEL FOR TYPE-II SUPERCONDUCTIVITY

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Abstract. This paper is intended as an investigation of the solvability of Cauchy problem for doubly nonlinear evolution equation of the form $dv(t)/dt + \partial\varphi^t(u(t)) \ni f(t)$, $v(t) \in \partial\psi(u(t))$, $0 < t < T$, where $\partial\varphi^t$ and $\partial\psi$ are subdifferential operators, and $\partial\varphi^t$ depends on t explicitly. Our method of proof relies on chain rules for t-dependent subdifferentials and an appropriate boundedness condition on $\partial\varphi^t$; however, it does not require either a strong monotonicity condition or a boundedness condition on $\partial\psi$. Moreover, an initial-boundary value problem for a nonlinear parabolic equation arising from an approximation of Bean's critical-state model for type-II superconductivity is also treated as an application of our abstract theory.

1. **Introduction.** Various types of doubly nonlinear evolution equations have been studied by many authors (see, e.g., [5, 10, 3, 14, 16, 15, 11]), and their results were applied to quasilinear parabolic equations arising from physics, biology, mechanics and so on. This paper is concerned with doubly nonlinear evolution equations governed by time-dependent subdifferential operators in reflexive Banach spaces.

Let V and V^* be a real reflexive Banach space and its dual space, respectively, and let H be a Hilbert space whose dual space H^* is identified with itself H such that V is continuously and densely embedded in H. Then, we consider

$$\frac{dv}{dt}(t) + \partial_V \varphi^t(u(t)) \ni f(t), \quad v(t) \in \partial_H \psi(u(t)), \quad 0 < t < T, \tag{1}$$

where $\partial_V \varphi^t$ and $\partial_H \psi$ denote subdifferential operators of proper lower semi-continuous convex functionals φ^t and ψ defined on V and H, respectively, and f is a given function from $(0, T)$ into V^*. We here emphasize that φ^t depends on t explicitly, and this is one of main features of our problem.

In this paper, we aim at constructing a solution of Cauchy problem for (1) without imposing either a strong monotonicity condition (cf. [14]) or a boundedness condition (cf. [10, 15]) on $\partial_H \psi$. To this end, we employ the chain rules for subdifferentials of t-dependent functionals developed in [2] and make use of an appropriate boundedness condition on $\partial_V \varphi^t$. In Section 2, our main result on (1) is stated and proved.

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In Section 3, as an application of our abstract theory, we deal with the following initial-boundary value problem arising from some macroscopic model for type-II superconductivity:

$$\text{(IBVP)}\quad \begin{cases} \dfrac{\partial}{\partial t}|u|^{\sigma-2}u(x,t) - \Delta u(x,t) = f(x,t), & (x,t) \in \Omega \times (0,T), \\ \dfrac{\partial u}{\partial n}(x,t) = -g(x,t)|u|^{\sigma-2}u(x,t), & (x,t) \in \partial\Omega \times (0,T), \\ |u|^{\sigma-2}u(x,0) = j_0(x), & x \in \Omega, \end{cases}$$

where $\sigma > 1$, $f : \Omega \times (0,T) \to \mathbb{R}$ and $g : \partial\Omega \times (0,T) \to \mathbb{R}$ are given.

2. **Doubly Nonlinear Evolution Equation.** Let V and V^* be a real reflexive Banach space and its dual space, respectively, and let H be a Hilbert space whose dual space H^* is identified with itself H such that

$$V \subset H \equiv H^* \subset V^* \tag{2}$$

with densely defined and continuous canonical injections.

In this section, we discuss the existence of solutions for the following abstract Cauchy problem:

$$\text{(CP)}\quad \begin{cases} \dfrac{dv}{dt}(t) + \partial_V \varphi^t(u(t)) \ni f(t), \quad v(t) \in \partial_H \psi(u(t)), \quad 0 < t < T, \\ v(0) = v_0, \end{cases}$$

where $\partial_V \varphi^t$ and $\partial_H \psi$ denote subdifferential operators of proper lower semi-continuous convex functionals $\varphi^t : V \to [0,+\infty]$ and $\psi : H \to [0,+\infty]$, respectively, for every $t \in [0,T]$.

We here recall the definition of subdifferential operator. Let $\Phi(X)$ be the set of all proper lower-semicontinuous convex functionals ϕ from a reflexive Banach space X into $(-\infty,+\infty]$, where "proper" means $\phi \not\equiv +\infty$. Then, the subdifferential $\partial_{X,X^*}\phi(u)$ of $\phi \in \Phi(X)$ at u is given by

$$\partial_{X,X^*}\phi(u) \quad := \quad \{\xi \in X^*; \phi(v) - \phi(u) \geq \langle \xi, v - u \rangle_X \quad \forall v \in D(\phi)\},$$

where $\langle \cdot,\cdot \rangle_X$ denotes the duality pairing between X and X^* and $D(\phi) := \{u \in X; \phi(u) < +\infty\}$. Hence we can define the subdifferential operator $\partial_{X,X^*}\phi : X \to 2^{X^*}; u \mapsto \partial_{X,X^*}\phi(u)$ with the domain $D(\partial_{X,X^*}\phi) := \{u \in D(\phi); \partial_{X,X^*}\phi(u) \neq \emptyset\}$. For simplicity of notation, we shall write $\partial_X \phi$ and $\langle \cdot,\cdot \rangle$ instead of $\partial_{X,X^*}\phi$ and $\langle \cdot,\cdot \rangle_X$, respectively, if no confusion can arise. It is well known that the graph of every subdifferential operator $\partial_X \phi$ becomes maximal monotone in $X \times X^*$.

In particular, if X is a Hilbert space H whose dual space is identified with itself, i.e., $H \equiv H^*$, then the subdifferential $\partial_H \phi(u)$ of $\phi \in \Phi(H)$ at u can be written by

$$\partial_H \phi(u) \quad = \quad \{\xi \in H; \phi(v) - \phi(u) \geq (\xi, v - u)_H \quad \forall v \in D(\phi)\},$$

since $\langle \cdot,\cdot \rangle_H$ coincides with the inner product $(\cdot,\cdot)_H$ of H; moreover, we can always find a unique element of least norm in $\partial_H \phi(u)$, which is called minimal section of $\partial_H \phi(u)$ and denoted by $(\partial_H \phi)^\circ(u)$, for every $u \in D(\partial_H \phi)$.

We are concerned with strong solutions of (CP) defined below.

Definition 1. A pair of functions $(u, v) : [0, T] \to V \times V^*$ is said to be a strong solution of (CP) on $[0, T]$ if the following (i)-(iv) hold true:

(i) v is a V^*-valued absolutely continuous function on $[0, T]$;

(ii) $u(t) \in D(\partial_H \psi) \cap D(\partial_V \varphi^t)$ for a.e. $t \in (0, T)$;

(iii) There exist sections $v(t) \in \partial_H \psi(u(t))$ and $g(t) \in \partial_V \varphi^t(u(t))$ such that

$$\frac{dv}{dt}(t) + g(t) = 0 \text{ in } V^*, \text{ for a.e. } t \in (0, T); \tag{3}$$

(iv) $v(t) \to v_0$ strongly in V^* and weakly in H as $t \to +0$.

Our basic assumptions are the following. Let $p, q \in (1, +\infty)$ be fixed.

(Aφ^t) There exist functions $\alpha \in W^{1,q}(0, T)$, $\beta \in W^{1,1}(0, T)$ and a constant $\delta > 0$ such that for every $t_0 \in [0, T]$ and $x_0 \in D(\varphi^{t_0})$, we can take a function $x : I_\delta(t_0) := [t_0 - \delta, t_0 + \delta] \cap [0, T] \to V$ satisfying:

$$\begin{cases} |x(t) - x_0|_V & \le |\alpha(t) - \alpha(t_0)| \left\{\varphi^{t_0}(x_0) + 1\right\}^{1/q}, \\ \varphi^t(x(t)) & \le \varphi^{t_0}(x_0) + |\beta(t) - \beta(t_0)| \left\{\varphi^{t_0}(x_0) + 1\right\} \quad \forall t \in I_\delta(t_0). \end{cases}$$

(A1) There exists a constant C_1 such that
$|u|_V^p \le C_1\{\varphi^t(u) + 1\} \quad \forall u \in D(\varphi^t), \ \forall t \in [0, T]$.

(A2) There exists a constant C_2 such that
$|\xi|_{V^*}^{q'} \le C_2\{\varphi^t(u) + 1\} \quad \forall [u, \xi] \in \partial_V \varphi^t, \ \forall t \in [0, T]$.

(A3) There exists a non-decreasing function $\ell_1 : \mathbb{R} \to \mathbb{R}$ such that
$\varphi^t(J_\lambda u) \le \varphi^t(u) + \lambda \ell_1(\varphi^t(u))\{|\partial_H \psi_\lambda(u)|_H^2 + 1\}, \ \forall \lambda > 0, \ \forall u \in D(\varphi^t)$,
$\forall t \in [0, T]$, where J_λ denotes the resolvent of $\partial_H \psi$, i.e., $J_\lambda = (I + \lambda \partial_H \psi)^{-1}$ with the identity I in H, and $\partial_H \psi_\lambda$ is the Yosida approximation of $\partial_H \psi$.

(A4) V is compactly embedded in H.

Our main result is now stated as follows:

Theorem 1. *Suppose that* (Aφ^t), (A1)-(A4) *are all satisfied with* $p, q \in (1, +\infty)$. *Then, for every* $f \in L^2(0, T; H) \cap W^{1,p'}(0, T; V^*)$ *and* $v_0 \in \{(\partial_H \psi)^\circ(u_0); u_0 \in D(\varphi^0) \cap D(\partial_H \psi)\}$, (CP) *admits at least one strong solution* (u, v) *such that*

$$u \in L^\infty(0, T; V). \quad v \in C_w([0, T]; H) \cap W^{1,\infty}(0, T; V^*), \quad g \in L^\infty(0, T; V^*),$$

where g denotes the section of $\partial_V \varphi^t(u(t))$ given in (3) *and $C_w([0, T]; H)$ denotes the set of all continuous functions from $[0, T]$ into H equipped with the weak topology $\sigma(H, H)$.*

Before describing the proof of Theorem 1, we recall a couple of useful properties of the Legendre-Fenchel transform ϕ^* of $\phi \in \Phi(X)$ defined by

$$\phi^*(u) \quad := \quad \sup_{v \in X} \{\langle u, v\rangle - \phi(v)\} \quad \forall u \in X^*.$$

It is well known that ϕ^* belongs to $\Phi(X^*)$, and the following identity holds:

$$\phi^*(f) \quad = \quad \langle f, u\rangle - \phi(u) \quad \forall [u, f] \in \partial_X \phi,$$

which also implies $u \in \partial_{X^*} \phi^*(f)$, i.e., $\partial_{X^*} \phi^* = (\partial_X \phi)^{-1}$.

Moreover, we give a remark on the assumption (A3).

Remark 1. By virtue of (A3), we have the following:

$$(g, \partial_H \psi_\lambda(u))_H \geq -\ell_1(\varphi^t(u)) \left\{ |\partial_H \psi_\lambda(u)|_H^2 + 1 \right\} \quad \forall u \in D(\partial_V \varphi^t), \ \forall g \in \partial_V \varphi^t(u) \cap H. \tag{4}$$

Indeed, from the definition of the Yosida approximation $\partial_H \psi_\lambda$, we see

$$(g, \partial_H \psi_\lambda(u))_H = \frac{1}{\lambda}(g, u - J_\lambda u)_H \geq \frac{1}{\lambda} \left\{ \varphi^t(u) - \varphi^t(J_\lambda u) \right\}.$$

Thus (A3) implies (4).

We now proceed to the proof of theorem 1.

Proof of Theorem 1. We introduce the following approximate problem for (CP).

$$(\mathrm{CP})_\lambda \quad \begin{cases} \lambda \dfrac{du_\lambda}{dt}(t) + \dfrac{d}{dt}\partial_H \psi_\lambda(u_\lambda(t)) + \partial_H \varphi_H^t(u_\lambda(t)) \ni f_\lambda(t), & 0 < t < T, \\ u_\lambda(0) = u_0, \end{cases}$$

where $u_0 \in D(\varphi^0) \cap D(\partial_H \psi)$ satisfies $v_0 = (\partial_H \psi)^\circ(u_0)$; ψ_λ denotes the Moreau-Yosida regularization of ψ; $f_\lambda \in C^1([0,T];H)$ and $f_\lambda \to f$ strongly in $L^2(0,T;V^*)$ and weakly in $L^2(0,T;H) \cap W^{1,p'}(0,T;V^*)$; moreover, φ_H^t is the extension of φ^t on H defined by

$$\varphi_H^t(u) := \begin{cases} \varphi^t(u) & \text{if } u \in V, \\ +\infty & \text{otherwise.} \end{cases}$$

We here notice that $\varphi_H^t \in \Phi(H)$ and $D(\varphi^t) = D(\varphi_H^t)$, $\partial_H \varphi_H^t(u) \subset \partial_V \varphi^t(u)$ for all $u \in V$ and $t \in [0,T]$. Moreover, note that the mapping $u \mapsto Au := \lambda u + \partial_H \psi_\lambda(u)$ is Lipschitz continuous in H and satisfies

$$\lambda |u - v|_H^2 \leq (Au - Av, u - v)_H \quad \forall u, v \in H.$$

Hence in much the same way as in the proof of Theorem 2.8.1 of [13], we can verify the existence of solutions $u_\lambda \in W^{1,2}(0,T;H)$ for $(\mathrm{CP})_\lambda$.

For simplicity, we write $v_\lambda(t)$ and $v_{0,\lambda}$ instead of $\partial_H \psi_\lambda(u_\lambda(t))$ and $\partial_H \psi_\lambda(u_0)$, respectively; moreover, put $g_\lambda(t) := f_\lambda(t) - \lambda du_\lambda(t)/dt - dv_\lambda(t)/dt \in \partial_H \varphi_H^t(u_\lambda(t))$. Now we have the following a priori estimates.

Lemma 1. *There exists a constant C such that*

$$\lambda \int_0^T \left| \frac{du_\lambda}{dt}(t) \right|_H^2 dt \leq C, \tag{5}$$

$$\sup_{t \in [0,T]} \varphi^t(u_\lambda(t)) \leq C. \tag{6}$$

Proof of Lemma 1. Multiply $(\mathrm{CP})_\lambda$ by $du_\lambda(t)/dt$ to get

$$\lambda \left| \frac{du_\lambda}{dt}(t) \right|_H^2 + \left(\frac{dv_\lambda}{dt}(t), \frac{du_\lambda}{dt}(t) \right)_H + \left(g_\lambda(t), \frac{du_\lambda}{dt}(t) \right)_H = \left(f_\lambda(t), \frac{du_\lambda}{dt}(t) \right)_H$$

for a.e. $t \in (0,T)$. We here notice that

$$\left(f_\lambda(t), \frac{du_\lambda}{dt}(t) \right)_H = \frac{d}{dt}(f_\lambda(t), u_\lambda(t))_H - \left(\frac{df_\lambda}{dt}(t), u_\lambda(t) \right)_H$$

and

$$0 \leq \left(\frac{dv_\lambda}{dt}(t), \frac{du_\lambda}{dt}(t) \right)_H,$$

since $v_\lambda(t) \in \partial_H \psi_\lambda(u_\lambda(t))$ and $\partial_H \psi_\lambda$ is monotone in H. Moreover, by (Aφ^t), Lemma 2.12 of [2] implies

$$\begin{aligned} &\left| \frac{d}{dt}\varphi_H^t(u_\lambda(t)) - \left(g_\lambda(t), \frac{du_\lambda}{dt}(t) \right)_H \right| \\ &\quad \leq \ |\dot\alpha(t)||g_\lambda(t)|_{V^*}\{\varphi_H^t(u_\lambda(t)) + 1\}^{1/q} + |\dot\beta(t)|\{\varphi_H^t(u_\lambda(t)) + 1\} \end{aligned}$$

for a.e. $t \in (0, T)$. Hence, by using (A2), we obtain

$$\begin{aligned} &\lambda \left| \frac{du_\lambda}{dt}(t) \right|_H^2 + \frac{d}{dt}\varphi_H^t(u_\lambda(t)) \\ &\quad \leq \ C\left\{ |\dot\alpha(t)| + |\dot\beta(t)| \right\} \left\{ \varphi_H^t(u(t)) + 1 \right\} + \frac{d}{dt}(f_\lambda(t), u_\lambda(t))_H - \left(\frac{df_\lambda}{dt}(t), u_\lambda(t) \right)_H \end{aligned}$$

for a.e. $t \in (0, T)$. Moreover, integrating both sides over $(0, t)$ and using (A1) and (A2), we have

$$\begin{aligned} &\lambda \int_0^t \left| \frac{du_\lambda}{d\tau}(\tau) \right|_H^2 d\tau + \varphi_H^t(u_\lambda(t)) \\ &\quad \leq \ \varphi_H^0(u_0) + C\Bigg\{ \int_0^T |\dot\alpha(\tau)| d\tau + \int_0^T |\dot\beta(\tau)| d\tau + \sup_{\tau\in[0,T]} |f_\lambda(\tau)|_{V^*}^{p'} \\ &\qquad + |u_0|_V^p + \int_0^T \left| \frac{df_\lambda}{d\tau}(\tau) \right|_{V^*}^{p'} d\tau + 1 \Bigg\} + \frac{1}{2}\varphi_H^t(u_\lambda(t)) \\ &\qquad + C\int_0^t \left\{ |\dot\alpha(\tau)| + |\dot\beta(\tau)| + 1 \right\} \varphi_H^\tau(u_\lambda(\tau)) d\tau. \end{aligned}$$

Thus Gronwall's inequality yields (5) and (6). □

Lemma 2. *There exists a constant C such that*

$$\sup_{t\in[0,T]} |v_\lambda(t)|_H \leq C. \tag{7}$$

Proof of Lemma 2. Multiplying $(\mathrm{CP})_\lambda$ by $v_\lambda(t)$, we have

$$\lambda \frac{d}{dt}\psi_\lambda(u_\lambda(t)) + \frac{1}{2}\frac{d}{dt}|v_\lambda(t)|_H^2 + (g_\lambda(t), v_\lambda(t))_H \leq |f_\lambda(t)|_H |v_\lambda(t)|_H \tag{8}$$

for a.e. $t \in (0.T)$. Moreover, by virtue of Remark 1 and (6), it follows that

$$(g_\lambda(t), v_\lambda(t))_H \geq -\ell_1(C)\left\{ |v_\lambda(t)|_H^2 + 1 \right\}. \tag{9}$$

Hence integrating (8) over $(0, t)$ and applying Gronwall's inequality, we get (7). □

Lemma 3. *There exists a constant C such that*

$$\sup_{t\in[0,T]} |u_\lambda(t)|_V \leq C, \tag{10}$$

$$\sup_{t\in[0,T]} |g_\lambda(t)|_{V^*} \leq C, \tag{11}$$

$$\int_0^T \left| \frac{dv_\lambda}{dt}(t) \right|_{V^*}^2 dt \leq C, \tag{12}$$

$$\sup_{t\in[0,T]} |J_\lambda u_\lambda(t)|_V \leq C, \tag{13}$$

where J_λ denotes the resolvent of $\partial_H \psi$, that is, $J_\lambda := (I + \lambda\partial_H \psi)^{-1}$.

Proof of Lemma 3. A priori estimates (10) and (11) follow immediately from (A1), (A2) and (6). Moreover, by using $(\mathrm{CP})_\lambda$ and the a priori estimates (5) and (11), we can verify (12), since f_λ is bounded in $L^2(0,T;V^*)$. Finally, by virtue of (A1) and (A3), we can deduce (13) from (6) and (7). □

From these a priori estimates, we can take a sequence λ_n in $(0,1]$ such that $\lambda_n \to +0$ and the following lemmas hold.

Lemma 4. *There exist* $u \in L^\infty(0,T;V)$ *and* $v \in C_w([0,T];H) \cap W^{1,2}(0,T;V^*)$ *such that*

$$\lambda_n \frac{du_{\lambda_n}}{dt} \to 0 \qquad \text{strongly in } L^2(0,T;H), \tag{14}$$

$$u_{\lambda_n} \to u \qquad \text{weakly in } L^2(0,T;V), \tag{15}$$

$$J_{\lambda_n} u_{\lambda_n} \to u \qquad \text{weakly in } L^2(0,T;V), \tag{16}$$

$$v_{\lambda_n} \to v \qquad \text{weakly in } L^2(0,T;H) \cap W^{1,2}(0,T;V^*), \tag{17}$$

$$\text{strongly in } C([0,T];V^*), \tag{18}$$

$$v_{\lambda_n}(T) \to v(T) \qquad \text{weakly in } H. \tag{19}$$

Moreover, we have $v(t) \in \partial_H \psi(u(t))$ *for a.e.* $t \in (0,T)$. *Furthermore,* $v(t) \to v_0$ *strongly in* V^* *and weakly in* H *as* $t \to +0$.

Proof of Lemma 4. First, (14) and (15) are derived immediately from (5) and (10), respectively. Moreover, we can deduce $v \in L^\infty(0,T;V)$ from (10) (see the proof of Lemma 4 of [1] for more details). Now (13) yields $J_{\lambda_n} u_{\lambda_n} \to w$ weakly in $L^2(0,T;V)$ for some $w \in L^2(0,T;V)$. We then claim $w = u$. Indeed, by the definition of $\partial_H \psi_\lambda$, it follows from (7) that

$$\sup_{t\in[0,T]} |u_\lambda(t) - J_\lambda u_\lambda(t)|_H = \lambda \sup_{t\in[0,T]} |v_\lambda(t)|_H \le \lambda C \to 0$$

as $\lambda \to 0$. Thus we obtain $w = u$. Now (17) follows from (7) and (12). Moreover, since v_λ is bounded in $L^\infty(0,T;H) \cap W^{1,2}(0,T;V^*)$, by (A4), Ascoli's compactness lemma ensures (18). Moreover, by (7), we can also verify $v \in L^\infty(0,T;H)$; hence, since $L^\infty(0,T;H) \cap C([0,T];V^*) \subset C_w([0,T];H)$, we have $v \in C_w([0,T];H)$. Furthermore, (7) and (18) also yield (19).

Now note that (16) and (18) imply

$$\begin{aligned}\int_0^T (v_{\lambda_n}(t), J_{\lambda_n} u_{\lambda_n}(t))_H dt &= \int_0^T \langle v_{\lambda_n}(t), J_{\lambda_n} u_{\lambda_n}(t)\rangle dt \\ &\to \int_0^T \langle v(t), u(t)\rangle dt = \int_0^T (v(t), u(t))_H dt \end{aligned} \tag{20}$$

as $\lambda_n \to 0$, and (16) also yields

$$J_{\lambda_n} u_{\lambda_n} \to u \qquad \text{weakly in } L^2(0,T;H). \tag{21}$$

Hence, by Lemma 1.3 of [4] and Proposition 1.1 of [12], it follows from (17), (20) and (21) that $v(t) \in \partial_H \psi(u(t))$ for a.e. $t \in (0,T)$.

Finally, we check the initial condition for v. By (12), it follows that

$$|v_{\lambda_n}(t) - v_{0,\lambda_n}|_{V^*} \le \int_0^t \left| \frac{dv_{\lambda_n}}{d\tau}(\tau) \right|_{V^*} d\tau \le C^{1/2}\sqrt{t},$$

which together with (18) and the fact that $v_{0,\lambda_n} \to v_0$ strongly in H implies

$$v(t) \to v_0 \qquad \text{strongly in } V^* \text{ as } t \to +0.$$

Moreover, since $v \in C_w([0,T];H)$, we also deduce that $v(t) \to v_0$ weakly in H. $\square$

Finally, we prove the convergence of g_{λ_n} in the following:

Lemma 5. *There exists a function* $g \in L^\infty(0,T;V^*)$ *such that*

$$g_{\lambda_n} \to g \qquad \text{weakly in } L^2(0,T;V^*). \tag{22}$$

Moreover, $g(t) = f(t) - dv(t)/dt \in \partial_V \varphi^t(u(t))$ *for a.e.* $t \in (0,T)$, *and* $dv/dt \in L^\infty(0,T;V^*)$.

Proof of Lemma 5. By (11) and Lemma 4, there exists a function $g \in L^2(0,T;V^*)$ such that (22) holds true and $g = f - dv/dt$; moreover, we can also verify $g \in L^\infty(0,T;V^*)$, which implies $dv/dt \in L^\infty(0,T;V^*)$. So it remains to show that $g(t) \in \partial_V \varphi^t(u(t))$ for a.e. $t \in (0,T)$. Multiply $g_{\lambda_n}(t)$ by $u_{\lambda_n}(t)$ and integrate this over $(0,T)$. We then see

$$\begin{aligned}
&\int_0^T \langle g_{\lambda_n}(t), u_{\lambda_n}(t)\rangle \, dt \\
&= \int_0^T (f_{\lambda_n}(t), u_{\lambda_n}(t))_H dt - \lambda_n \int_0^T \left(\frac{du_{\lambda_n}}{dt}(t), u_{\lambda_n}(t)\right)_H dt \\
&\quad - \int_0^T \left\langle \frac{dv_{\lambda_n}}{dt}(t), J_{\lambda_n} u_{\lambda_n}(t)\right\rangle dt - \int_0^T \left\langle \frac{dv_{\lambda_n}}{dt}(t), u_{\lambda_n}(t) - J_{\lambda_n} u_{\lambda_n}(t)\right\rangle dt \\
&= \int_0^T (f_{\lambda_n}(t), u_{\lambda_n}(t))_H dt - \frac{\lambda_n}{2}|u_{\lambda_n}(T)|_H^2 + \frac{\lambda_n}{2}|u_0|_H^2 \\
&\quad -\psi^*(v_{\lambda_n}(T)) + \psi^*(v_{0,\lambda_n}) - \frac{\lambda_n}{2}|v_{\lambda_n}(T)|_H^2 + \frac{\lambda_n}{2}|v_{0,\lambda_n}|_H^2,
\end{aligned}$$

since the fact that $v_{\lambda_n}(t) \in \partial_H \psi(J_{\lambda_n} u_{\lambda_n}(t))$ is equivalent to that $J_{\lambda_n} u_{\lambda_n}(t) \in \partial_H \psi^*(v_{\lambda_n}(t))$, where ψ^* denotes the Legendre-Fenchel transform of $\psi \in \Phi(H)$, i.e., $\psi^*(u) := \sup_{w \in H}\{(u,w)_H - \psi(w)\}$. Here since $v_{0,\lambda_n} \in \partial_H \psi(J_{\lambda_n} u_0)$, $v_{0,\lambda_n} \to v_0$ and $J_{\lambda_n} u_0 \to u_0$ strongly in H, we get

$$\limsup_{\lambda_n \to 0} \psi^*(v_{0,\lambda_n}) = \lim_{\lambda_n \to 0} (v_{0,\lambda_n}, J_{\lambda_n} u_0)_H - \liminf_{\lambda_n \to 0} \psi(J_{\lambda_n} u_0) \le \psi^*(v_0),$$

which together with the lower semi-continuity of ψ^* implies $\psi^*(v_{0,\lambda_n}) \to \psi^*(v_0)$. Hence, by virtue of Lemma 4, we can deduce

$$\limsup_{\lambda_n \to 0} \int_0^T \langle g_{\lambda_n}(t), u_{\lambda_n}(t)\rangle \, dt \quad \le \quad \int_0^T \langle f(t), u(t)\rangle dt - \psi^*(v(T)) + \psi^*(v_0).$$

We here notice that

$$\begin{aligned}
&\psi^*(v(T)) \ge \sup_{w \in V}\{\langle v(T), w\rangle - \psi(w)\} =: (\psi|_V)^*(v(T)), \\
&v_0 \in \partial_H \psi(u_0) \subset \partial_V(\psi|_V)(u_0), \\
&\psi^*(v_0) = (v_0, u_0)_H - \psi(u_0) = \langle v_0, u_0\rangle - \psi(u_0) = (\psi|_V)^*(v_0),
\end{aligned}$$

where $\psi|_V$ denotes the restriction of ψ on V; moreover, we also observe $u(t) \in \partial_{V^*}(\psi|_V)^*(v(t))$, since $v(t) \in \partial_H \psi(u(t)) \subset \partial_V(\psi|_V)(u(t))$. Therefore, by Lemmas 2.10 and 2.11 of [2], we find that

$$\begin{aligned}
-\psi^*(v(T)) + \psi^*(v_0) \quad &\le \quad -(\psi|_V)^*(v(T)) + (\psi|_V)^*(v_0) \\
&= \quad -\int_0^T \left\langle \frac{dv}{dt}(t), u(t)\right\rangle dt.
\end{aligned}$$

Thus Lemma 1.3 of [4] and Proposition 1.1 of [12] imply $g(t) = f(t) - dv(t)/dt \in \partial_V \varphi^t(t)$ for a.e. $t \in (0,T)$. □

3. **Macroscopic Model for Type-II Superconductivity.** In 1964, C. P. Bean [6] proposed a macroscopic critical-state model for type-II superconductivity characterized by the following correspondence between the electric field $\mathbf{e}$ and the current density $\mathbf{j}$:

$$\text{(B)} \qquad \mathbf{e} \parallel \mathbf{j}, \qquad |\mathbf{j}| \in \begin{cases} 1 & \text{if } |\mathbf{e}| > 0, \\ [0,1] & \text{if } |\mathbf{e}| = 0. \end{cases}$$

Moreover, the following power approximation (B)_σ is often used in place of (B):

$$\text{(B)}_\sigma \qquad \mathbf{j} = |\mathbf{e}|^{\sigma-2}\mathbf{e}$$

with $\sigma \in (1, +\infty)$ enough close to 1, because of the strong nonlinearity of (B).

On the other hand, just as in [9], the dynamics of the electric field $\mathbf{e}(x,t) = (0,0,u(x,t))$ and the current density $\mathbf{j}(x,t) = (0,0,j(x,t))$ in an infinitely long cylindrical type-II superconductor $\Omega \times (-\infty, +\infty)$ can be described by

$$\frac{\partial}{\partial t} j(x,t) - \Delta u(x,t) = f(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{23}$$

where Ω denotes a bounded domain in $\mathbb{R}^2$. Furthermore, the current-voltage law (B)_σ is reduced into the following 2 dimensional form:

$$j(x,t) \quad = \quad |u|^{\sigma-2}u(x,t), \quad (x,t) \in \Omega \times (0,T). \tag{24}$$

In this section, we apply the abstract theory developed in Section 2 to the initial-boundary value problem (IBVP), which is obtained by imposing the following nonlinear boundary condition and an initial condition on (23) with (24):

$$\frac{\partial u}{\partial n}(x,t) = -g(x,t) j(x,t), \quad (x,t) \in \partial\Omega \times (0,T), \tag{25}$$

under the following assumptions:

$$g \in W^{1,1}(0,T; L^\infty(\partial\Omega)), \quad g(x,t) \ge g_0 > 0, \quad (x,t) \in \partial\Omega \times (0,T). \tag{26}$$

To this end, we reduce (IBVP) into an abstract Cauchy problem in the form of (CP). Put $V := H^1(\Omega)$ and $H := L^2(\Omega)$ equipped with the norms: $|\cdot|_V := |\nabla \cdot|_{L^2(\Omega)} + |\cdot|_{L^2(\partial\Omega)}$ and $|\cdot|_H = |\cdot|_{L^2(\Omega)}$, respectively. Then, since Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, we observe that V is densely and continuously embedded in H. Moreover, V is also continuously embedded in $L^r(\partial\Omega)$ for any $r \in [1, +\infty)$, and $|\cdot|_V$ is equivalent to the norm $\|\cdot\|_V := |\nabla \cdot|_{L^2(\Omega)} + |\cdot|_{L^r(\partial\Omega)}$. Now define $\phi^t_\sigma : V \to [0, +\infty)$ and $\psi_\sigma : H \to [0, +\infty]$ as follows:

$$\begin{aligned} \phi^t_\sigma(u) \quad &:= \quad \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx + \frac{1}{\sigma}\int_{\partial\Omega} g(x,t)|u(x)|^\sigma d\Gamma \quad \forall u \in V, \\ \psi_\sigma(u) \quad &:= \quad \begin{cases} \dfrac{1}{\sigma}\displaystyle\int_\Omega |u(x)|^\sigma dx & \text{if } u \in L^\sigma(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

Then, we can easily see that $\phi_\sigma^t \in \Phi(V)$, $\psi_\sigma \in \Phi(H)$ and $\partial_V \phi_\sigma^t(u)$ coincides with $-\Delta u$ equipped with the boundary condition (25) in the distribution sense. Moreover, $\partial_H \psi_\sigma(u)$ is equivalent to $|u(\cdot)|^\sigma u(\cdot)$ in H. Hence (IBVP) is rewritten as

$$
(\mathrm{CP})_\sigma \quad \begin{cases} \dfrac{dv}{dt}(t) + \partial_V \phi_\sigma^t(u(t)) = f(t), \quad v(t) = \partial_H \psi_\sigma(u(t)), \quad 0 < t < T, \\ v(0) = v_0. \end{cases}
$$

In order to apply Theorem 1 to $(\mathrm{CP})_\sigma$, we prepare the following lemma.

Lemma 6. *Suppose that* (26) *is satisfied. Then,* (A1)-(A4) *and* (Aφ^t) *are satisfied with φ^t and ψ replaced by ϕ_σ^t and ψ_σ, respectively.*

Proof of Lemma 6. For all $u \in V$, we have, by (26),

$$
\phi_\sigma^t(u) \geq \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx + \frac{g_0}{\sigma}\int_{\partial\Omega} |u(x)|^\sigma d\Gamma \geq C\{|u|_V^\rho - 1\},
$$

where $\rho := \min\{2, \sigma\} > 1$. Hence (A1) holds with φ^t and p replaced by ϕ_σ^t and ρ, respectively. Moreover, for every $[u, \xi] \in \partial_V \phi_\sigma^t$ and $w \in V$, we observe

$$
\begin{aligned}
\langle \xi, w \rangle &= \int_\Omega \nabla u(x) \cdot \nabla w(x) dx + \int_{\partial\Omega} g(x,t)|u|^{\sigma-2} u(x) w(x) d\Gamma \\
&\leq |\nabla u|_{L^2(\Omega)} |\nabla w|_{L^2(\Omega)} + |g|_{L^\infty(\partial\Omega\times(0,T))} |u|_{L^\sigma(\partial\Omega)}^{\sigma-1} |w|_{L^\sigma(\partial\Omega)},
\end{aligned}
$$

which implies

$$
|\xi|_{V^*} \leq C\{|\nabla u|_{L^2(\Omega)} + |u|_{L^\sigma(\partial\Omega)}^{\sigma-1}\} \quad \forall u \in V, \ \forall t \in [0,T].
$$

Thus (A2) follows with $\varphi^t = \phi_\sigma^t$ and $q' = \min\{2, \sigma'\} > 1$.

Now let J_λ and j_λ be the resolvents of $\partial_H \psi_\sigma$ and $\partial_{\mathbb{R}}(\sigma^{-1}|\cdot|^\sigma)$, respectively. Then, $(J_\lambda u)(x)$ coincides with $j_\lambda(u(x))$ for a.e. $x \in \Omega$, so $|(J_\lambda u)(x)| \leq |u(x)|$ and $|(J_\lambda u)(x+h) - (J_\lambda u)(x)| \leq |u(x+h) - u(x)|$ for a.e. $x \in \Omega$ (see the proof of Corollary 16 of [8]). Hence it follows that

$$
\begin{aligned}
\phi_\sigma^t(J_\lambda u) &= \frac{1}{2}\int_\Omega |\nabla (J_\lambda u)(x)|^2 dx + \frac{1}{\sigma}\int_{\partial\Omega} g(x,t)|(J_\lambda u)(x)|^\sigma d\Gamma \\
&\leq \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx + \frac{1}{\sigma}\int_{\partial\Omega} g(x,t)|u(x)|^\sigma d\Gamma.
\end{aligned}
$$

Therefore (A3) is satisfied with φ^t and ψ replaced by ϕ_σ^t and ψ_σ, respectively. Moreover, since Ω is bounded in $\mathbb{R}^2$ and, $\partial\Omega$ is smooth, (A4) also holds true.

Let $t_0 \in [0,T]$ and $u_0 \in D(\phi_\sigma^{t_0}) = V$ be fixed and define $u(t) \equiv u_0$. We then see

$$
\begin{aligned}
\phi^t(u(t)) &= \phi^{t_0}(u_0) + \frac{1}{\sigma}\int_{\partial\Omega} \{g(x,t) - g(x,t_0)\}|u_0(x)|^\sigma d\Gamma \\
&\leq \phi_\sigma^{t_0}(u_0) + |g(\cdot,t) - g(\cdot,t_0)|_{L^\infty(\partial\Omega)} \sigma^{-1} |u_0|_{L^\sigma(\partial\Omega)}^\sigma \\
&\leq \phi_\sigma^{t_0}(u_0) + g_0^{-1}|g(\cdot,t) - g(\cdot,t_0)|_{L^\infty(\partial\Omega)} \left\{\phi_\sigma^{t_0}(u_0) + 1\right\}
\end{aligned}
$$

and $|u(t) - u_0|_V \equiv 0$. Hence (Aφ^t) is satisfied with $\varphi^t = \phi_\sigma^t$, $\alpha \equiv 0$, $\beta = g_0^{-1}\int_0^t |\partial_\tau g(\cdot,\tau)|_{L^\infty(\partial\Omega)} d\tau$ and an arbitrary number $q \in [0, +\infty)$. □

Consequently, we have:

Theorem 2. *Suppose that* (26) *is satisfied and* $\sigma \in (1, +\infty)$. *Then, for every* $j_0 \in \{v \in L^2(\Omega); \exists u_0 \in H^1(\Omega); |u_0|^{\sigma-2}u_0 = v\}$ *and* $f \in L^2(\Omega \times (0,T)) \cap W^{1,\rho'}(0,T;(H^1(\Omega))^*)$ *with* $\rho := \min\{2,\sigma\}$, *there exists* $u : \Omega \times (0,T) \to \mathbb{R}$ *such that*

$$u \in L^\infty(0,T;H^1(\Omega)),$$

$$|u|^{\sigma-2}u \in C_w([0,T];L^2(\Omega)) \cap W^{1,\infty}(0,T;(H^1(\Omega))^*) \cap L^\infty(0,T;L^r(\partial\Omega))$$

for any $r \in [1,+\infty)$,

$$\begin{cases} \left\langle \frac{\partial}{\partial t}|u|^{\sigma-2}u(\cdot,t), w \right\rangle + \int_\Omega \nabla u(x,t) \cdot \nabla w(x) dx \\ \quad + \int_{\partial\Omega} g(x,t)|u|^{\sigma-2}u(x,t)w(x)d\Gamma = \int_\Omega f(x,t)w(x)dx, \\ \quad \forall w \in H^1(\Omega), \quad \text{for a.e. } t \in (0,T), \end{cases}$$

$|u|^{\sigma-2}u(\cdot,t) \to j_0$ *strongly in* $(H^1(\Omega))^*$ *and weakly in* $L^2(\Omega)$ *as* $t \to +0$.

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