DOUBLY NONLINEAR EVOLUTION EQUATIONS
AND BEAN’S CRITICAL-STATE MODEL
FOR TYPE-II SUPERCONDUCTIVITY

Goro Akagi

Abstract. This paper is intended as an investigation of the solvability of Cauchy
problem for doubly nonlinear evolution equation of the form
\[ \frac{dv(t)}{dt} + \partial V \phi_t(u(t)) \ni f(t), \quad v(t) \in \partial H \psi(u(t)), \quad 0 < t < T, \]
where \( \partial V \phi_t \) and \( \partial H \psi \) are subdifferential operators, and
\( \partial V \phi_t \) depends on \( t \) explicitly. Our method of proof relies on chain rules for \( t \)-dependent
subdifferentials and an appropriate boundedness condition on \( \partial V \phi_t \); however, it does
not require either a strong monotonicity condition or a boundedness condition on
\( \partial H \psi \). Moreover, an initial-boundary value problem for a nonlinear parabolic equation
arising from an approximation of Bean’s critical-state model for type-II supercon-
ductivity is also treated as an application of our abstract theory.

1. Introduction. Various types of doubly nonlinear evolution equations have been
studied by many authors (see, e.g., \([5, 10, 3, 14, 16, 15, 11]\)), and their results were
applied to quasilinear parabolic equations arising from physics, biology, mechanics
and so on. This paper is concerned with doubly nonlinear evolution equations
governed by time-dependent subdifferential operators in reflexive Banach spaces.

Let \( V \) and \( V^* \) be a real reflexive Banach space and its dual space, respectively,
and let \( H \) be a Hilbert space whose dual space \( H^* \) is identified with itself \( H \) such
that \( V \) is continuously and densely embedded in \( H \). Then, we consider
\[ \frac{dv(t)}{dt} + \partial V \phi_t(u(t)) \ni f(t), \quad v(t) \in \partial H \psi(u(t)), \quad 0 < t < T, \quad (1) \]
where \( \partial V \phi_t \) and \( \partial H \psi \) denote subdifferential operators of proper lower semi-continuous
convex functionals \( \phi_t \) and \( \psi \) defined on \( V \) and \( H \), respectively, and \( f \) is a given func-
tion from \( (0, T) \) into \( V^* \). We here emphasize that \( \phi_t \) depends on \( t \) explicitly, and
this is one of main features of our problem.

In this paper, we aim at constructing a solution of Cauchy problem for (1) without
imposing either a strong monotonicity condition (cf. \([14]\)) or a boundedness
condition (cf. \([10, 15]\)) on \( \partial H \psi \). To this end, we employ the chain rules for subdif-
fferentials of \( t \)-dependent functionals developed in \([2]\) and make use of an appropriate
boundedness condition on \( \partial V \phi_t \). In Section 2, our main result on (1) is stated and
proved.

Key words and phrases. Doubly nonlinear evolution equation, time-dependent subdifferential,
reflexive Banach space, Bean model.

The author is supported by Waseda University Grant for Special Research Projects, \#2004A-366.
In Section 3, as an application of our abstract theory, we deal with the following initial-boundary value problem arising from some macroscopic model for type-II superconductivity:

\[
\begin{aligned}
\text{(IBVP) } \quad \left\{ 
\begin{array}{ll}
\frac{\partial}{\partial t}|u|^{\sigma-2}u(x,t) - \Delta u(x,t) = f(x,t), & (x,t) \in \Omega \times (0,T), \\
\frac{\partial u}{\partial n}(x,t) = -g(x,t)|u|^{\sigma-2}u(x,t), & (x,t) \in \partial\Omega \times (0,T), \\
|u|^{\sigma-2}u(x,0) = j_0(x), & x \in \Omega,
\end{array}
\right.
\end{aligned}
\]

where \(\sigma > 1\), \(f : \Omega \times (0,T) \to \mathbb{R}\) and \(g : \partial\Omega \times (0,T) \to \mathbb{R}\) are given.

2. Doubly Nonlinear Evolution Equation. Let \(V\) and \(V^*\) be a real reflexive Banach space and its dual space, respectively, and let \(H\) be a Hilbert space whose dual space \(H^*\) is identified with itself \(H\) such that

\[V \subset H \equiv H^* \subset V^*\]

with densely defined and continuous canonical injections.

In this section, we discuss the existence of solutions for the following abstract Cauchy problem:

\[
\text{(CP) } \quad \left\{ 
\begin{array}{ll}
\frac{dv}{dt} + \partial_V \varphi^s(u(t)) \ni f(t), & v(t) \in \partial_H \psi(u(t)), \\
v(0) = v_0,
\end{array}
\right.
\]

where \(\partial_V \varphi^s\) and \(\partial_H \psi\) denote subdifferential operators of proper lower semi-continuous convex functionals \(\varphi^s : V \to [0, +\infty]\) and \(\psi : H \to [0, +\infty]\), respectively, for every \(t \in [0,T]\).

We here recall the definition of subdifferential operator. Let \(\Phi(X)\) be the set of all proper lower-semicontinuous convex functionals \(\varphi\) from a reflexive Banach space \(X\) into \((-\infty, +\infty]\), where “proper” means \(\varphi \neq +\infty\). Then, the subdifferential \(\partial_{X,X^*} \varphi(u)\) of \(\varphi \in \Phi(X)\) at \(u\) is given by

\[
\partial_{X,X^*} \varphi(u) := \{ \xi \in X^* : \langle \varphi(v) - \varphi(u), \xi \rangle \geq \langle \xi, v - u \rangle_X \quad \forall v \in D(\varphi) \},
\]

where \(\langle \cdot , \cdot \rangle_X\) denotes the duality pairing between \(X\) and \(X^*\) and \(D(\varphi) := \{ u \in X ; \varphi(u) < +\infty \}\). Hence we can define the subdifferential operator \(\partial_X X^* \cdot \varphi : X \to 2^{X^*}\) \(u \mapsto \partial_X X^* \cdot \varphi(u)\) with the domain \(D(\partial_X X^* \cdot \varphi) := \{ u \in D(\varphi) ; \partial_X X^* \cdot \varphi(u) \neq \emptyset \}\).

For simplicity of notation, we shall write \(\partial_X \varphi\) and \(\langle \cdot , \cdot \rangle_X\) instead of \(\partial_X X^* \cdot \varphi\) and \(\langle \cdot , \cdot \rangle_X\), respectively, if no confusion can arise. It is well known that the graph of every subdifferential operator \(\partial_X \varphi\) becomes maximal monotone in \(X \times X^*\).

In particular, if \(X\) is a Hilbert space \(H\) whose dual space is identified with itself, i.e., \(H \equiv H^*\), then the subdifferential \(\partial_H \varphi(u)\) of \(\varphi \in \Phi(H)\) at \(u\) can be written by

\[
\partial_H \varphi(u) = \{ \xi \in H ; \langle \varphi(v) - \varphi(u), \xi \rangle \geq \langle \xi, v - u \rangle_H \quad \forall v \in D(\varphi) \},
\]

since \(\langle \cdot , \cdot \rangle_H\) coincides with the inner product \(\langle \cdot , \cdot \rangle_H\) of \(H\); moreover, we can always find a unique element of least norm in \(\partial_H \varphi(u)\), which is called minimal section of \(\partial_H \varphi(u)\) and denoted by \((\partial_H \varphi)^\circ (u)\), for every \(u \in D(\partial_H \varphi)\).

We are concerned with strong solutions of (CP) defined below.
Theorem 1. A pair of functions \((u, v) : [0, T] \rightarrow V \times V^*\) is said to be a strong solution of (CP) on \([0, T]\) if the following (i)-(iv) hold true:

(i) \(v\) is a \(V^*\)-valued absolutely continuous function on \([0, T]\);
(ii) \(u(t) \in D(\partial_H \psi) \cap D(\partial_V \varphi^f)\) for a.e. \(t \in (0, T)\);
(iii) There exist sections \(v(t) \in \partial_H \psi(u(t))\) and \(g(t) \in \partial_V \varphi^f(u(t))\) such that
\[
\frac{dv}{dt}(t) + g(t) = 0 \quad \text{in } V^*, \text{ for a.e. } t \in (0, T);
\]
(iv) \(v(t) \rightharpoonup v_0\) strongly in \(V^*\) and weakly in \(H\) as \(t \to +0\).

Our basic assumptions are the following. Let \(p, q \in (1, +\infty)\) be fixed.

(A\(\varphi^f\)) There exist functions \(\alpha \in W^{1,q}(0, T), \beta \in W^{1,1}(0, T)\) and a constant \(\delta > 0\) such that for every \(t_0 \in [0, T]\) and \(x_0 \in D(\varphi^f)\), we can take a function \(x : I_{\delta}(t_0) := [t_0 - \delta, t_0 + \delta] \cap [0, T] \to V\) satisfying:
\[
\begin{align*}
|\alpha(t) - \alpha(t_0)| \leq |\varphi^f(x_0)| + 1, \\
|\varphi^f(x(t))| \leq \varphi^f(x_0) + |\beta(t) - \beta(t_0)| \leq 0 \quad \forall t \in I_{\delta}(t_0).
\end{align*}
\]

(A1) There exists a constant \(C_1\) such that
\[
|u_v^p| \leq C_1 \{\varphi^f(u) + 1\} \quad \forall u \in D(\varphi^f), \forall t \in [0, T].
\]

(A2) There exists a constant \(C_2\) such that
\[
|\xi|^q_v \leq C_2 \{\varphi^f(u) + 1\} \quad \forall [u, \xi] \in \partial_V \varphi^f, \forall t \in [0, T].
\]

(A3) There exists a non-decreasing function \(\ell_1 : \mathbb{R} \to \mathbb{R}\) such that
\[
\varphi^f(J_{\lambda} u) \leq \varphi^f(u) + \lambda \ell_1(\varphi^f(u))\{\partial_H \psi \lambda(u)\}^f_H + 1, \quad \forall \lambda > 0, \forall u \in D(\varphi^f),
\]
\[
\forall t \in [0, T], \text{ where } J_\lambda \text{ denotes the resolvent of } \partial_H \psi, \text{ i.e., } J_\lambda = (I + \lambda \partial_H \psi)^{-1}
\]
with the identity \(I\) in \(H\), and \(\partial_H \psi \lambda\) is the Yosida approximation of \(\partial_H \psi\).

(A4) \(V\) is compactly embedded in \(H\).

Our main result is now stated as follows:

Theorem 1. Suppose that (A\(\varphi^f\)), (A1)-(A4) are all satisfied with \(p, q \in (1, +\infty)\). Then, for every \(f \in L^2(0, T; H) \cap W^{1,p}(0, T; V^*)\) and \(v_0 \in \{\partial_H \psi^c(u_0); u_0 \in D(\varphi^f) \cap D(\partial_H \psi)\}\), (CP) admits at least one strong solution \((u, v)\) such that
\[
u \in L^\infty(0, T; V^*), \quad v \in C_w([0, T]; H) \cap W^{1,\infty}(0, T; V^*), \quad g \in L^\infty(0, T; V^*),
\]
where \(g\) denotes the section of \(\partial_V \varphi^f(u(t))\) given in (3) and \(C_w([0, T]; H)\) denotes the set of all continuous functions from \([0, T]\) into \(H\) equipped with the weak topology \(\sigma(H, H)\).

Before describing the proof of Theorem 1, we recall a couple of useful properties of the Legendre-Fenchel transform \(\varphi^*\) of \(\varphi \in \Phi(X)\) defined by
\[
\varphi^*(u) := \sup_{v \in X} \{\langle u, v \rangle - \varphi(v)\} \quad \forall u \in X^*.
\]

It is well known that \(\varphi^*\) belongs to \(\Phi(X^*)\), and the following identity holds:
\[
\varphi^*(f) = \langle f, u \rangle - \varphi(u) \quad \forall [u, f] \in \partial_X \phi,
\]
which also implies \(u \in \partial_X \varphi^*(f)\), i.e., \(\partial_X \phi^* = (\partial_X \phi)^{-1}\).

Moreover, we give a remark on the assumption (A3).
Remark 1. By virtue of (A3), we have the following:
\[ \langle g, \partial_H \psi_\lambda(u) \rangle_H \geq -\ell_1(\psi'(u)) \left\{ |\partial_H \psi_\lambda(u)|_H^2 + 1 \right\} \quad \forall u \in D(\partial_V \psi'), \quad \forall g \in \partial_V \psi'(u) \cap H. \tag{4} \]

Indeed, from the definition of the Yosida approximation \( \partial_H \psi_\lambda \), we see
\[ \langle g, \partial_H \psi_\lambda(u) \rangle_H = \frac{1}{\lambda} \langle g, u - J_\lambda u \rangle_H \geq \frac{1}{\lambda} \left\{ \varphi'(u) - \varphi'(J_\lambda u) \right\}. \]
Thus (A3) implies (4).

We now proceed to the proof of theorem 1.

Proof of Theorem 1. We introduce the following approximate problem for \((CP)\).
\[
(CP)_\lambda \quad \left\{ \begin{array}{l}
\frac{d u_\lambda}{dt}(t) + \frac{d}{dt} \partial_H \psi_\lambda(u_\lambda(t)) + \partial_H \varphi'_H(u_\lambda(t)) \ni f_\lambda(t), \quad 0 < t < T, \\
u_\lambda(0) = u_0,
\end{array} \right.
\]
where \(u_0 \in D(\psi^\lambda) \cap D(\partial_H \psi)\) satisfies \(v_0 = (\partial_H \psi)^\lambda(u_0)\); \(\psi_\lambda\) denotes the Moreau-Yosida regularization of \(\psi\); \(f_\lambda \in C^1([0, T]; H)\) and \(f_\lambda \to f\) strongly in \(L^2(0, T; V^*)\) and weakly in \(L^2(0, T; H) \cap W^{1,\varphi}(0, T; V^*)\); moreover, \(\varphi'_H\) is the extension of \(\varphi'\) on \(H\) defined by
\[
\varphi'_H(u) := \begin{cases} 
\varphi'(u) & \text{if } u \in V, \\
+\infty & \text{otherwise}.
\end{cases}
\]
We here notice that \(\varphi'_H \in \Phi(H)\) and \(D(\varphi') = D(\varphi'_H), \partial_H \varphi'_H(u) \subset \partial_V \varphi'(u)\) for all \(u \in V\) and \(t \in [0, T]\). Moreover, note that the mapping \(u \mapsto \lambda u + \partial_H \psi_\lambda(u)\) is Lipschitz continuous in \(H\) and satisfies
\[
\lambda|u - v|_H^2 \leq \langle Au - Av, u - v \rangle_H \quad \forall u, v \in H.
\]
Hence in much the same way as in the proof of Theorem 2.8.1 of [13], we can verify the existence of solutions \(u_\lambda \in W^{1,2}(0, T; H)\) for \((CP)_\lambda\).

For simplicity, we write \(u_\lambda(t)\) and \(v_0,\lambda\) instead of \(\partial_H \psi_\lambda(u_\lambda(t))\) and \(\partial_H \psi_\lambda(u_0)\), respectively; moreover, put \(g_\lambda(t) := \frac{d}{dt}(f_\lambda(t) - \lambda du_\lambda(t)/dt - dv_0,\lambda(t)/dt) \in \partial_H \varphi'_H(u_\lambda(t))\).

Now we have the following a priori estimates.

Lemma 1. There exists a constant \(C\) such that
\[
\mathcal{L} \int_0^T \left| \frac{du_\lambda}{dt}(t) \right|_H^2 dt \leq C, \tag{5}
\]
\[
\sup_{t \in [0, T]} \varphi'(u_\lambda(t)) \leq C. \tag{6}
\]

Proof of Lemma 1. Multiply \((CP)_\lambda\) by \(du_\lambda(t)/dt\) to get
\[
\lambda \left| \frac{du_\lambda}{dt}(t) \right|_H^2 + \langle \frac{d}{dt}(f_\lambda(t)), \frac{du_\lambda}{dt}(t) \rangle_H + \left( g_\lambda(t), \frac{du_\lambda}{dt}(t) \right)_H = \left( f_\lambda(t), \frac{du_\lambda}{dt}(t) \right)_H
\]
for a.e. \(t \in (0, T)\). We here notice that
\[
\left( f_\lambda(t), \frac{du_\lambda}{dt}(t) \right)_H = \frac{d}{dt}(f_\lambda(t), u_\lambda(t))_H - \left( \frac{df_\lambda}{dt}(t), u_\lambda(t) \right)_H
\]
and
\[
0 \leq \left( \frac{dv_0,\lambda}{dt}(t), \frac{du_\lambda}{dt}(t) \right)_H.
\]
since $v_\lambda(t) \in \partial_H \psi_\lambda(u_\lambda(t))$ and $\partial_H \psi_\lambda$ is monotone in $H$. Moreover, by $(A_\varphi')$, Lemma 2.12 of [2] implies

$$
\left| \frac{d}{dt} \varphi_\lambda'(u_\lambda(t)) - \left( g_\lambda(t), \frac{du_\lambda}{dt}(t) \right)_H \right|
\leq |\dot{\alpha}(t)||g_\lambda(t)||\varphi_\lambda'(u_\lambda(t)) + 1|^{1/\gamma} + |\dot{\beta}(t)||\varphi_\lambda'(u_\lambda(t)) + 1|
$$

for a.e. $t \in (0, T)$. Hence integrating (8) over $(0, T)$, we obtain

$$
\lambda \int_0^t \left| \frac{du_\lambda}{dt}(\tau) \right|^2_H d\tau + \varphi_\lambda'(u_\lambda(t))
\leq \varphi_\lambda'(u_0) + C \left\{ \int_0^T |\dot{\alpha}(\tau)| d\tau + \int_0^T |\dot{\beta}(\tau)| d\tau + \sup_{\tau \in [0, T]} |f_\lambda(\tau)|_{V'} \right\}^2
\leq |f_\lambda(t)||v_\lambda(t)|_H + (g_\lambda(t), v_\lambda(t))_H \leq |f_\lambda(t)||v_\lambda(t)|_H
$$

for a.e. $t \in (0, T)$. Moreover, integrating both sides over $(0, t)$ and using (A1) and (A2), we have

$$
\lambda \int_0^t \left| \frac{du_\lambda}{dt}(\tau) \right|^2_H d\tau + \varphi_\lambda'(u_\lambda(t))
\leq \varphi_\lambda'(u_0) + C \left\{ \int_0^T |\dot{\alpha}(\tau)| d\tau + \int_0^T |\dot{\beta}(\tau)| d\tau + \sup_{\tau \in [0, T]} |f_\lambda(\tau)|_{V'} \right\}^2
\leq |f_\lambda(t)||v_\lambda(t)|_H + (g_\lambda(t), v_\lambda(t))_H
$$

Thus Gronwall’s inequality yields (5) and (6).

Lemma 2. There exists a constant $C$ such that

$$
\sup_{t \in [0, T]} |v_\lambda(t)|_H \leq C. \tag{7}
$$

Proof of Lemma 2. Multiplying $(CP)_\lambda$ by $v_\lambda(t)$, we have

$$
\lambda \frac{d}{dt} \psi_\lambda(u_\lambda(t)) + \frac{1}{2} \frac{d}{dt} |v_\lambda(t)|_H^2 + (g_\lambda(t), v_\lambda(t))_H \leq |f_\lambda(t)||v_\lambda(t)|_H \tag{8}
$$

for a.e. $t \in (0, T)$. Moreover, by virtue of Remark 1 and (6), it follows that

$$
(g_\lambda(t), v_\lambda(t))_H \geq -\ell_1(C) \left\{ |v_\lambda(t)|_H^2 + 1 \right\}. \tag{9}
$$

Hence integrating (8) over $(0, t)$ and applying Gronwall’s inequality, we get (7).

Lemma 3. There exists a constant $C$ such that

$$
\sup_{t \in [0, T]} |u_\lambda(t)|_V \leq C, \tag{10}
$$

$$
\sup_{t \in [0, T]} |g_\lambda(t)|_{V'} \leq C, \tag{11}
$$

$$
\int_0^T \left| \frac{d\lambda}{dt}(t) \right|^2_{V'} dt \leq C, \tag{12}
$$

$$
\sup_{t \in [0, T]} |J_\lambda u_\lambda(t)|_V \leq C, \tag{13}
$$

where $J_\lambda$ denotes the resolvent of $\partial_H \psi$, that is, $J_\lambda := (I + \lambda \partial_H \psi)^{-1}$. 

\hfill \Box
Proof of Lemma 3. A priori estimates (10) and (11) follow immediately from (A1), (A2) and (6). Moreover, by using (CP)$_\lambda$ and the a priori estimates (5) and (11), we can verify (12), since $f_\lambda$ is bounded in $L^2(0,T;V^*)$. Finally, by virtue of (A1) and (A3), we can deduce (13) from (6) and (7).

From these a priori estimates, we can take a sequence $\lambda_n$ in $(0,1]$ such that $\lambda_n \to +0$ and the following lemmas hold.

**Lemma 4.** There exist $u \in L^\infty(0,T;V)$ and $v \in C_w([0,T];H) \cap W^{1,2}(0,T;V^*)$ such that
\[
\lambda_n \frac{du_{\lambda_n}}{dt} \to 0 \quad \text{strongly in } L^2(0,T;H),
\]
\[
u_{\lambda_{\lambda_n}} \to u \quad \text{weakly in } L^2(0,T;V),
\]
\[
J_{\lambda_{n}}u_{\lambda_{n}} \to u \quad \text{weakly in } L^2(0,T;V),
\]
\[
v_{\lambda_{n}} \to v \quad \text{weakly in } L^2(0,T;H) \cap W^{1,2}(0,T;V^*),
\]
\[
v_{\lambda_{n}}(T) \to v(T) \quad \text{weakly in } H.
\]
Moreover, we have $v(t) \in \partial_H \psi(u(t))$ for a.e. $t \in (0,T)$. Furthermore, $v(t) \to v_0$ strongly in $V^*$ and weakly in $H$ as $t \to +0$.

**Proof of Lemma 4.** First, (14) and (15) are derived immediately from (5) and (10), respectively. Moreover, we can deduce $v \in L^\infty(0,T;V)$ from (10) (see the proof of Lemma 4 of [1] for more details). Now (13) yields $J_{\lambda_{n}}u_{\lambda_{n}} \to w$ weakly in $L^2(0,T;V)$ for some $w \in L^2(0,T;V)$. We then claim $w = u$. Indeed, by the definition of $\partial_H \psi$, it follows from (7) that
\[
sup_{t \in [0,T]} |u(t) - J_{\lambda_{n}}u_{\lambda_{n}}(t)|_H = \lambda \sup_{t \in [0,T]} |v_{\lambda_{n}}(t)|_H \leq C \to 0
\]
as $\lambda \to 0$. Thus we obtain $w = u$. Now (17) follows from (7) and (12). Moreover, since $v_{\lambda_{n}}$ is bounded in $L^\infty(0,T;H) \cap W^{1,2}(0,T;V^*)$, by (A4), Ascoli’s compactness lemma ensures (18). Moreover, by (7), we can also verify $v \in L^\infty(0,T;H)$; hence, since $L^\infty(0,T;H) \cap C([0,T];V^*) \subset C_w([0,T];H)$, we have $v \in C_w([0,T];H)$. Furthermore, (7) and (18) also yield (19).

Now note that (16) and (18) imply
\[
\int_0^T (v_{\lambda_{n}}(t), J_{\lambda_{n}}u_{\lambda_{n}}(t))_H dt = \int_0^T \langle v_{\lambda_{n}}(t), J_{\lambda_{n}}u_{\lambda_{n}}(t) \rangle_H dt \\
\to \int_0^T \langle v(t), u(t) \rangle_H dt = \int_0^T (v(t), u(t))_H dt
\]
as $\lambda_n \to 0$, and (16) also yields
\[
J_{\lambda_{n}}u_{\lambda_{n}} \to u \quad \text{weakly in } L^2(0,T;H).
\]
Hence, by Lemma 1.3 of [4] and Proposition 1.1 of [12], it follows from (17), (20) and (21) that $v(t) \in \partial_H \psi(u(t))$ for a.e. $t \in (0,T)$.

Finally, we check the initial condition for $v$. By (12), it follows that
\[
|v_{\lambda_{n}}(t) - v_{0,\lambda_{n}}|_{V^*} \leq \int_0^T \left| \frac{dv_{\lambda_{n}}(r)}{dr} \right|_{V^*} dr \leq C^{1/2} \sqrt{T},
\]
which together with (18) and the fact that $v_{0,\lambda_{n}} \to v_0$ strongly in $H$ implies
\[
v(t) \to v_0 \quad \text{strongly in } V^* \text{ as } t \to +0.
\]
Moreover, since \( v \in C_w([0,T]; H) \), we also deduce that \( v(t) \to v_0 \) weakly in \( H \).

Finally, we prove the convergence of \( g_{\lambda_n} \) in the following:

**Lemma 5.** There exists a function \( g \in L^\infty(0,T; V^*) \) such that
\[
g_{\lambda_n} \to g \quad \text{weakly in } L^2(0,T; V^*). \tag{22}
\]
Moreover, \( g(t) = f(t) - dv(t)/dt \in \partial_V \varphi'(u(t)) \) for a.e. \( t \in (0,T) \), and \( dv/dt \in L^\infty(0,T; V^*) \).

**Proof of Lemma 5.** By (11) and Lemma 4, there exists a function \( g \in L^2(0,T; V^*) \) such that (22) holds true and \( g = f - dv/dt \); moreover, we can also verify \( g \in L^\infty(0,T; V^*) \), which implies \( dv/dt \in L^\infty(0,T; V^*) \). So it remains to show that \( g(t) \in \partial_V \varphi'(u(t)) \) for a.e. \( t \in (0,T) \). Multiply \( g_{\lambda_n}(t) \) by \( u_{\lambda_n}(t) \) and integrate this over \((0,T)\). We then see
\[
\int_0^T \langle g_{\lambda_n}(t), u_{\lambda_n}(t) \rangle \, dt = \int_0^T (f_{\lambda_n}(t), u_{\lambda_n}(t))_H dt - \lambda_n \int_0^T \left( \frac{dv_{\lambda_n}}{dt}(t), u_{\lambda_n}(t) \right)_H dt
\]
\[
- \int_0^T \left( \frac{dv_{\lambda_n}}{dt}(t), J_{\lambda_n} u_{\lambda_n}(t) \right)_H dt - \int_0^T \left( \frac{dv_{\lambda_n}}{dt}(t), u_{\lambda_n}(t) - J_{\lambda_n} u_{\lambda_n}(t) \right)_H dt
\]
\[
= \int_0^T (f_{\lambda_n}(t), u_{\lambda_n}(t))_H dt - \frac{\lambda_n}{2} |u_{\lambda_n}(T)|_H^2 + \frac{\lambda_n}{2} |u_0|_H^2 - \psi^*(v_{\lambda_n}(T)) + \psi^*(v_{0,\lambda_n}) - \frac{\lambda_n}{2} |v_{\lambda_n}(T)|_H^2 + \frac{\lambda_n}{2} |v_{0,\lambda_n}|_H^2,
\]
since the fact that \( v_{\lambda_n}(t) \in \partial_H \psi(J_{\lambda_n} u_{\lambda_n}(t)) \) is equivalent to that \( J_{\lambda_n} u_{\lambda_n}(t) \in \partial_H \psi^*(v_{\lambda_n}(t)) \), where \( \psi^* \) denotes the Legendre-Fenchel transform of \( \psi \in \Phi(H) \), i.e.,
\[
\psi^*(u) := \sup_{w \in H} \{ \langle u, w \rangle_H - \psi(w) \}. \quad \text{Here since } v_{0,\lambda_n} \in \partial_H \psi(J_{\lambda_n} u_0), v_0, \lambda_n \to 0 \text{ and } J_{\lambda_n} u_0 \to u_0 \text{ strongly in } H, \text{ we get}
\]
\[
\limsup_{\lambda_n \to 0} \psi^*(v_{0,\lambda_n}) \leq \lim_{\lambda_n \to 0} (v_{0,\lambda_n}, J_{\lambda_n} u_0)_H - \liminf_{\lambda_n \to 0} \psi(J_{\lambda_n} u_0) \leq \psi^*(v_0),
\]
which together with the lower semi-continuity of \( \psi^* \) implies \( \psi^*(v_{0,\lambda_n}) \to \psi^*(v_0) \). Hence, by virtue of Lemma 4, we can deduce
\[
\limsup_{\lambda_n \to 0} \int_0^T \langle g_{\lambda_n}(t), u_{\lambda_n}(t) \rangle \, dt \leq \int_0^T \langle f(t), u(t) \rangle \, dt - \psi^*(v(T)) + \psi^*(v_0).
\]
We here notice that
\[
\psi^*(v(T)) \geq \sup_{w \in V} \{ \langle v(T), w \rangle - \psi(w) \} =: (\psi|_V)^*(v(T)),
\]
\[
v_0 \in \partial_H \psi(u_0) \subset \partial_V \psi^*(v_0),
\]
\[
\psi^*(v_0) = \langle v_0, u_0 \rangle_H - \psi(u_0) = \langle v_0, u_0 \rangle - \psi(u_0) = (\psi|_V)^*(v_0),
\]
where \( \psi|_V \) denotes the restriction of \( \psi \) on \( V \); moreover, we also observe \( u(t) \in \partial_V \psi^*(v(T)) \), since \( v(t) \in \partial_H \psi(u(t)) \subset \partial_V \psi^*(v(u(t))) \). Therefore, by Lemmas 2.10 and 2.11 of [2], we find that
\[
-\psi^*(v(T)) + \psi^*(v_0) \leq - (\psi|_V)^*(v(T)) + (\psi|_V)^*(v_0)
\]
\[
= - \int_0^T \left( \frac{dv}{dt}(t), u(t) \right) \, dt.
\]
Thus Lemma 1.3 of [4] and Proposition 1.1 of [12] imply $g(t) = f(t) - dv(t)/dt \in \partial V \phi^d(t)$ for a.e. $t \in (0, T)$. \hfill \Box

3. Macroscopic Model for Type-II Superconductivity. In 1964, C. P. Bean [6] proposed a macroscopic critical-state model for type-II superconductivity characterized by the following correspondence between the electric field $e$ and the current density $j$:

\[(B) \quad e \parallel j, \quad |j| \in \begin{cases} 1 & \text{if } |e| > 0, \\ [0, 1] & \text{if } |e| = 0. \end{cases} \]

Moreover, the following power approximation $(B)_\sigma$ is often used in place of $(B)$:

\[(B)_\sigma \quad j = |e|^{\sigma - 2} e \]

with $\sigma \in (1, +\infty)$ enough close to 1, because of the strong nonlinearity of $(B)$.

On the other hand, just as in [9], the dynamics of the electric field $e(x, t) = (0, 0, u(x, t))$ and the current density $j(x, t) = (0, 0, j(x, t))$ in an infinitely long cylindrical type-II superconductor $\Omega \times (-\infty, +\infty)$ can be described by

\[
\frac{\partial}{\partial t} j(x, t) - \Delta u(x, t) = f(x, t), \quad (x, t) \in \Omega \times (0, T),
\]

where $\Omega$ denotes a bounded domain in $\mathbb{R}^2$. Furthermore, the current-voltage law $(B)_\sigma$ is reduced into the following 2 dimensional form:

\[
j(x, t) = |u|^{\sigma - 2} u(x, t), \quad (x, t) \in \Omega \times (0, T). \tag{24}\]

In this section, we apply the abstract theory developed in Section 2 to the initial-boundary value problem (IBVP), which is obtained by imposing the following nonlinear boundary condition and an initial condition on (23) with (24):

\[
\frac{\partial u}{\partial n}(x, t) = -g(x, t) j(x, t), \quad (x, t) \in \partial \Omega \times (0, T), \tag{25}\]

under the following assumptions:

\[
g \in W^{1,1}(0, T; L^\infty(\partial \Omega)), \quad g(x, t) \geq g_0 > 0, \quad (x, t) \in \partial \Omega \times (0, T). \tag{26}\]

To this end, we reduce (IBVP) into an abstract Cauchy problem in the form of (CP). Put $V := H^1(\Omega)$ and $H := L^2(\Omega)$ equipped with the norms: $| \cdot |_V := |\nabla \cdot |_{L^2(\Omega)} + | \cdot |_{L^2(\partial \Omega)}$ and $| \cdot |_H = | \cdot |_{L^2(\Omega)}$, respectively. Then, since $\Omega$ is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial \Omega$, we observe that $V$ is densely and continuously embedded in $H$. Moreover, $V$ is also continuously embedded in $L^r(\partial \Omega)$ for any $r \in [1, +\infty)$, and $| \cdot |_V$ is equivalent to the norm $\| \cdot \|_V := |\nabla \cdot |_{L^2(\Omega)} + | \cdot |_{L^r(\partial \Omega)}$. Now define $\phi^\delta_\sigma : V \to [0, +\infty)$ and $\psi_\sigma : H \to [0, +\infty]$ as follows:

\[
\phi^\delta_\sigma(u) := \frac{1}{2} \int_\Omega |\nabla u(x)|^2 dx + \frac{1}{\sigma} \int_{\partial \Omega} g(x, t)|u(x)|^\sigma d\Gamma \quad \forall u \in V,
\]

\[
\psi_\sigma(u) := \begin{cases} \frac{1}{\sigma} \int_\Omega |u(x)|^\sigma dx & \text{if } u \in L^\sigma(\Omega), \\ +\infty & \text{otherwise}. \end{cases}
\]

\[
\leqno (\sigma)\]

DOUBLY NONLINEAR EVOLUTION EQUATIONS AND BEAN’S MODEL 37
Then, we can easily see that $\phi'_\rho \in \Phi(V)$, $\psi_\rho \in \Phi(H)$ and $\partial_\nu \phi'_\rho(u)$ coincides with $-\Delta u$ equipped with the boundary condition (25) in the distribution sense. Moreover, $\partial_H \psi_\rho(u)$ is equivalent to $|u(\cdot)|^\rho u(\cdot)$ in $H$. Hence (IBVP) is rewritten as

\[
(CP)_{\sigma} \quad \begin{cases}
\frac{dv}{dt}(t) + \partial_\nu \phi'_\rho(u(t)) = f(t), \\
v(t) = \partial_H \psi_\rho(u(t)), \\
v(0) = v_0.
\end{cases}
\]

In order to apply Theorem 1 to (CP)$_\sigma$, we prepare the following lemma.

**Lemma 6.** Suppose that (26) is satisfied. Then, (A1)-(A4) and (A$\varphi'$) are satisfied with $\varphi'$ and $\psi$ replaced by $\phi'_\rho$ and $\psi_\rho$, respectively.

**Proof of Lemma 6.** For all $u \in V$, we have, by (26),

\[
\phi'_\rho(u) \geq \frac{1}{2} \int_\Omega |\nabla u(x)|^2 dx + \frac{\rho_0}{\sigma} \int_{\partial \Omega} |u(x)|^\rho d\Gamma \geq C\{ |u|_V^{\rho_0} - 1 \},
\]

where $\rho := \min\{2, \sigma\} > 1$. Hence (A1) holds with $\varphi'$ and $p$ replaced by $\phi'_\rho$ and $\rho$, respectively. Moreover, for every $[u, \xi] \in \partial_\nu \phi'_\rho$ and $w \in V$, we observe

\[
(\xi, w) = \int_\Omega \nabla u(x) \cdot \nabla w(x) dx + \int_{\partial \Omega} g(x, t)|u|^\rho u(x)w(x) d\Gamma \leq |\nabla u|_{L^2(\Omega)} |\nabla w|_{L^2(\Omega)} + |g|_{L^{\infty}(\partial \Omega \times (0, T))} |u|_{L^\rho(\partial \Omega)} |w|_{L^\sigma(\partial \Omega)},
\]

which implies

\[
|\xi|_{V'} \leq C\{ |\nabla u|_{L^2(\Omega)} + |u|_{L^\rho(\partial \Omega)}^{\sigma-1} \} \quad \forall u \in V, \forall t \in [0, T].
\]

Thus (A2) follows with $\varphi' = \phi'_\rho$ and $q' = \min\{2, \sigma'\} > 1$.

Now let $J_\lambda$ and $j_\lambda$ be the resolvents of $\partial_H \psi_\rho$ and $\partial_\nu (\sigma^{-1} |\cdot|)$, respectively. Then, $(J_\lambda u)(x)$ coincides with $j_\lambda(u(x))$ for a.e. $x \in \Omega$, so $|(J_\lambda u)(x)| \leq |u(x)|$ and $|(J_\lambda u)(x + h) - (J_\lambda u)(x)| \leq |u(x + h) - u(x)|$ for a.e. $x \in \Omega$ (see the proof of Corollary 16 of [8]). Hence it follows that

\[
\phi'_\rho(J_\lambda u) = \frac{1}{2} \int_\Omega |\nabla (J_\lambda u)(x)|^2 dx + \frac{1}{\sigma} \int_{\partial \Omega} g(x, t)|(J_\lambda u)(x)|^\rho d\Gamma \leq \frac{1}{2} \int_\Omega |\nabla u(x)|^2 dx + \frac{1}{\sigma} \int_{\partial \Omega} g(x, t)|u(x)|^\rho d\Gamma.
\]

Therefore (A3) is satisfied with $\varphi'$ and $\psi$ replaced by $\phi'_\rho$ and $\psi_\rho$, respectively. Moreover, since $\Omega$ is bounded in $\mathbb{R}^2$ and $\partial \Omega$ is smooth, (A4) also holds true.

Let $t_0 \in [0, T]$ and $u_0 \in D(\phi''_\rho)$ be fixed and define $u(t) \equiv u_0$. Then we see

\[
\phi'(u(t)) = \phi'(u_0) + \frac{1}{\sigma} \int_{\partial \Omega} \{ g(x, t) - g(x, t_0) \} |u_0(x)|^\rho d\Gamma \leq \phi''_\rho(u_0) \equiv 0.
\]

and $|u(t) - u(t)|_V \equiv 0$. Hence $(\Lambda \varphi')$ is satisfied with $\varphi' = \phi'_\rho$, $\alpha \equiv 0$, $\beta = g^{-1}_0 \int_\Omega |\partial_\nu g(\cdot, \tau)|_{L^\infty(\partial \Omega)} d\tau$ and an arbitrary number $q \in [0, +\infty)$.

Consequently, we have:
Theorem 2. Suppose that (26) is satisfied and \( \sigma \in (1, +\infty) \). Then, for every \( j_0 \in \left\{ v \in L^2(\Omega); \exists u_0 \in H^1(\Omega); |u_0|^{\sigma-2}u_0 = v \right\} \) and \( f \in L^2(\Omega \times (0, T)) \cap W^{1,\rho}(0, T; (H^1(\Omega))^*) \) with \( \rho := \min\{2, \sigma\} \), there exists \( u \in \Omega \times (0, T) \to \mathbb{R} \) such that
\[
u \in L^\infty(0, T; H^1(\Omega)), \quad |u|^{\sigma-2}u \in C_w(0, T; L^2(\Omega)) \cap W^{1,\infty}(0, T; (H^1(\Omega))^*) \cap L^\infty(0, T; L^r(\partial\Omega)) \]
for any \( r \in [1, +\infty) \),
\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial t}|u|^{\sigma-2}u(t, \cdot) + \int_\Omega \nabla u(x, t) \cdot \nabla w(x) dx \\
+ \int_{\partial\Omega} g(x, t)|u|^{\sigma-2}u(t, x)w(x) d\Gamma = \int_\Omega f(x, t)w(x) dx,
\end{array} \right.
\]
for a.e. \( t \in (0, T) \),
\[
|u|^{\sigma-2}u(t, \cdot) \to j_0 \text{ strongly in } (H^1(\Omega))^* \text{ and weakly in } L^2(\Omega) \text{ as } t \to +0.
\]

REFERENCES


E-mail address: goro@toki.waseda.jp