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EVOLUTION EQUATIONS AND SUBDIFFERENTIALS IN BANACH SPACES

Goro Akagi and Mitsuharu Ôtani

Department of Applied Physics, School of Science and Technology,
Waseda University,
4-1, Okubo 3-chome, Shinjuku-ku,
Tokyo, 169-8555, Japan

Abstract. Sufficient conditions for the existence of strong solutions to the Cauchy problem are given for the evolution equation $du(t)/dt + \partial\varphi^1(u(t)) - \partial\varphi^2(u(t)) \ni f(t)$ in V^*, where $\partial\varphi^i$ is the so-called subdifferential operator from a Banach space V into its dual space V^* $(i = 1, 2)$.

Studies for this equation in the Hilbert space framework has been done by several authors. However the study in the V-V^* setting is not pursued yet.

Our method of proof relies on some approximation arguments in a Hilbert space. To carry out this procedure, it is assumed that there exists a Hilbert space H satisfying $V \subset H \equiv H^* \subset V^*$ with densely defined continuous injections.

As an application of our abstract theory, the initial-boundary value problem is discussed for the nonlinear heat equation: $u_t(x,t) - \Delta_p u(x,t) - |u|^{q-2}u(x,t) = f(x,t)$, $x \in \Omega$, $u|_{\partial\Omega} = 0$, $t \geq 0$, where Ω is a bounded domain in $\mathbb{R}^N$. In particular, the local existence of solutions is assured under the so-called subcritical condition, i.e., $q < p^*$, where p^* denotes Sobolev's critical exponent, provided that the initial data u_0 belongs to $W_0^{1,p}(\Omega)$.

1. **Introduction.**

Let V be a real reflexive Banach space and let H be a real Hilbert space whose dual space H^* is identified with H such that

$$V \subset H \equiv H^* \subset V^*, \tag{1}$$

where V^* denotes the dual space of V and each injection is densely defined and continuous.

In this paper, we investigate sufficient conditions for the existence of strong solutions of the following Cauchy problem:

$$\text{(CP)} \quad \frac{du}{dt}(t) + \partial\varphi^1(u(t)) - \partial\varphi^2(u(t)) \ni f(t) \ \text{ in } V^*, \quad u(0) = u_0,$$

where $\partial\varphi^1$ (resp. $\partial\varphi^2$) is the subdifferential of a proper lower semicontinuous convex function φ^1 (resp. φ^2) from V into $(-\infty, +\infty]$.

The theory of evolution equations governed by subdifferentials was first studied by Brézis (see [4], [5]) and its generalization in various directions has been vigorously studied by many people. Among them, Ôtani [11] (see also [6], [8] and [12]) studied

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(CP) in the Hilbert space framework (i.e., $V^* = H$) and applied his abstract results to the following initial-boundary value problem:

$$\text{(NHE)} \begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \Delta_p u(x,t) - |u|^{q-2}u(x,t) = f(x,t), \quad (x,t) \in \Omega \times [0,T], \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases}$$

where Δ_p is the so-called p-Laplacian given by $\Delta_p u(x) := \mathrm{div}(|\nabla u(x)|^{p-2}\nabla u(x))$ and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$.

The existence of weak solutions of (NHE) is also studied by using Faedo-Galerkin's method in [9] and [13]. The theory of perturbations in the Hilbert space framework (such as in [6], [8], [11], [12]) has an advantage over Faedo-Galerkin's method in that it can assure a better regularity of solutions. At the same time, however, it needs a stronger restriction on the growth order q of the perturbed term $|u|^{q-2}u$ when $p \neq 2$, which is caused by the loss of the elliptic estimate for the p-Laplacian with $p \neq 2$.

It is readily suggested from the study of nonlinear elliptic equations that the perturbation theory for subdifferentials in the V-V^* setting should remedy the deficiency mentioned above. However the study in this direction is not fully pursued even for the non-perturbed case where $\partial\varphi^2 \equiv 0$, whose study can be founded in [2] and [7]. The main purpose of this paper is to present an abstract setting which can cover the deficiency of the Hilbert space setting. In fact, as an application of our abstract results, we can assure the local existence of solutions of (NHE) under the so-called subcritical growth conditions, $q < p^*$, (i.e., $W_0^{1,p}$ is compactly embedded in L^q), for all $u_0 \in W_0^{1,p}(\Omega)$. This fact has been conjectured but left as an open problem for a long time.

2. **Main results.** Let $\Phi(X)$ be the set of all *proper lower semicontinuous convex functions* φ from X into $(-\infty, +\infty]$, where "*proper*" means that the *effective domain* $D(\varphi)$ of φ defined by $D(\varphi) := \{u \in X; \varphi(u) < +\infty\}$ is not empty. Define the subdifferential $\partial_X\varphi$ of φ by

$$\partial_X\varphi(u) := \{f \in X^*; \varphi(v) - \varphi(u) \geq {}_{X^*}\langle f, v-u\rangle_X \text{ for all } v \in D(\varphi)\}$$

with domain $D(\partial_X\varphi) = \{u \in X; \partial_X\varphi(u) \neq \emptyset\}$, where ${}_{X^*}\langle\cdot,\cdot\rangle_X$ denotes the natural duality between X and X^*.

Let $\varphi^1,\ \varphi^2 \in \Phi(V)$ be such that $D(\varphi^1) \cap D(\varphi^2) \neq \emptyset$. For simplicity of notation, we write $\partial\varphi$ and $\langle\cdot,\cdot\rangle$ instead of $\partial_V\varphi$ and ${}_{V^*}\langle\cdot,\cdot\rangle_V$ respectively if no confusion arises.

Here and henceforth, we are concerned with *strong solutions* of (CP) in the following sense.

Definition 1. A function $u \in C([0,T]; V^*)$ is said to be a strong solution of (CP) on $[0,T]$, if the following conditions are satisfied:

(**1**) $u(t)$ is a V^*-valued absolutely continuous function on $(0,T]$,

(**2**) $u(+0) = u_0$,

(**3**) $u(t) \in D(\partial\varphi^1) \cap D(\partial\varphi^2)$ for a.e. $t \in (0,T)$
and there exist sections $g^1(t) \in \partial\varphi^1(u(t))$ and $g^2(t) \in \partial\varphi^2(u(t))$ satisfying:

$$\frac{du}{dt}(t) + g^1(t) - g^2(t) = f(t) \text{ in } V^*, \text{ a.e. on } (0,T). \tag{2}$$

Throughout the present paper, we denote by C, C_i $(i \in \mathbb{N})$ positive constants which do not depend on the elements of the corresponding space or set. Moreover let us denote by $\mathcal{L}$ the set of all monotone non-decreasing functions from $[0,+\infty)$ into itself. For $p \in (1,+\infty)$, p' designates the Hölder conjugate of p, i.e., $p' = p/(p-1)$.

Our basic assumptions are the following.

(A1) $|u|_V^p - C_1|u|_H^2 - C_2 \le C_3\varphi^1(u) \quad \forall u \in D(\varphi^1)$,

(A2) $D(\varphi^1) \subset D(\partial\varphi^2)$. Furthermore, if $\sup_{t\in[0,T]}\{\varphi^1(u_n(t)) + |u_n(t)|_H\} + \int_0^T |du_n(t)/dt|_H^2 dt$ is bounded, then for every $g_n(\cdot) \in \partial\varphi^2(u_n(\cdot))$, $\{g_n\}$ forms a precompact subset in $C([0,T];V^*)$,

(A3) There exists an extension $\tilde{\varphi}^2 \in \Phi(H)$ of φ^2, i.e., $\tilde{\varphi}^2(u) = \varphi^2(u) \quad \forall u \in V$, such that $\varphi^1(J_\lambda u) \le l_1\left(\varphi^1(u) + l_2(|u|_H)\right) \ \forall\lambda > 0,\ \forall u \in D(\varphi^1)$, where $l_i \in \mathcal{L}$ $(i = 1,2)$, J_λ denotes the resolvent of $\partial_H\tilde{\varphi}^2$, that is, $J_\lambda = (I + \lambda\, \partial_H\tilde{\varphi}^2)^{-1}$,

(A4) $\varphi^2(u) \le k\varphi^1(u) + C_4|u|_H^2 + C_5 \quad \forall u \in D(\varphi^1),\ 0 \le k < 1$.

We note that the continuity of φ^2 can be derived from (A2).

Proposition 1. *Assume that* (A2) *is satisfied. Let $u_n \in D(\varphi^1)$ be such that $u_n \to u$ weakly in V and $\varphi^1(u_n)$ is bounded. Then there exists a subsequence $u_{n'}$ of u_n such that $\varphi^2(u_{n'}) \to \varphi^2(u)$.*

Proof of Proposition 1. Let $u_n \in D(\varphi^1)$ be such that $u_n \to u$ weakly in V as $n \to +\infty$ and $\varphi^1(u_n)$ is bounded. Then from the fact that $\varphi^2 \in \Phi(V)$, it follows that

$$\varphi^2(u) \quad \le \quad \liminf_{n\to+\infty} \varphi^2(u_n). \tag{3}$$

On the other hand, let $g_n \in \partial\varphi^2(u_n)$ and set $v_n(t) = u_n$ and $h_n(t) = g_n$ for all $t \in [0,T]$. Then we see that $\sup_{t\in[0,T]}\{\varphi^1(v_n(t)) + |v_n(t)|_H\} = \varphi^1(u_n) + |u_n|_H$ is bounded, $dv_n/dt \equiv 0$ and $h_n(\cdot) \in \partial\varphi^2(v_n(\cdot))$. By (A2), we can extract a subsequence n' of n such that $h_{n'} \to h$ strongly in $C([0,T];V^*)$, which implies $g_{n'}$ becomes a strongly convergent sequence in V^*.

Hence since $\varphi^2(u_{n'}) \le \varphi^2(u) + \langle g_{n'}, u_{n'} - u\rangle$, we get

$$\limsup_{n'\to+\infty} \varphi^2(u_{n'}) \quad \le \quad \varphi^2(u) + \lim_{n'\to+\infty}\langle g_{n'}, u_{n'} - u\rangle = \varphi^2(u). \tag{4}$$

Therefore it follows from (3) and (4) that $\lim_{n'\to+\infty}\varphi^2(u_{n'}) = \varphi^2(u)$. □

Our main results are stated as follows.

Theorem 1. *Assume that* (A1), (A2), (A3) *and* (A4) *hold. Then for all $u_0 \in D(\varphi^1)$ and $f \in W^{1,p'}(0,T;V^*)$,* (CP) *has a strong solution u on $[0,T]$ satisfying:*

$$\left.\begin{array}{c} u \in C_w([0,T];V) \cap W^{1,2}(0,T;H), \\ u(t) \in D(\partial\varphi^1) \cap D(\partial\varphi^2) \quad \text{for a.e. } t \in (0,T), \\ g^1 \in L^2(0,T;V^*), \quad g^2 \in C([0,T];V^*), \\ \sup_{t\in[0,T]} \varphi^1(u(t)) < +\infty, \quad \varphi^2(u(\cdot)) \in C([0,T]), \end{array}\right\} \tag{5}$$

where g^i $(i = 1,2)$ are the sections of $\partial\varphi^i$ satisfying (2) *and $C_w([0,T];V)$ denotes the set of all V-valued weakly continuous functions on $[0,T]$.*

Theorem 2. *Assume that* (A1), (A2) *and* (A3) *hold. Then for all $u_0 \in D(\varphi^1)$ and $f \in W^{1,p'}(0,T;V^*)$, there exists a number $T_0 \in (0,T]$ such that* (CP) *has a strong solution u on $[0,T_0]$ satisfying* (5) *with T replaced by T_0.*

Remark 1. We can give another type of sufficient conditions for the global existence when (A4) is not satisfied. Moreover the asymptotic behavior of solutions and the smoothing effect for the solution of (CP) with the initial data $u_0 \in \overline{D(\varphi^1)}^H$ falls within the scope of our V-V^* setting. These topics will be discussed in our forthcoming papers.

3. **Proof of Main results.**

3.1. **Proof of Theorem 1.**

One of key steps of our proof is to introduce approximation problems for (CP) in the Hilbert space H. To this end, we first define the extension of φ^1 on H by

$$\varphi_H^1(u) = \begin{cases} \varphi^1(u) & \text{if } u \in V, \\ +\infty & \text{if } u \in H/V. \end{cases}$$

Then, by virtue of (A1), we can verify that $\varphi_H^1 \in \Phi(H)$ (see [2]).

Now our approximation problems for (CP) are given by

$$(\mathrm{CP})_\lambda \quad \frac{du_\lambda}{dt}(t) + \partial_H \varphi_H^1(u_\lambda(t)) - \partial_H \tilde{\varphi}_\lambda^2(u_\lambda(t)) \ni f_\lambda(t) \ \text{ in } H, \quad u_\lambda(0) = u_0,$$

where f_λ belongs to $C^1([0,T];H)$ such that $f_\lambda \to f$ strongly in $W^{1,p'}(0,T;V^*)$ as $\lambda \to 0$, $\tilde{\varphi}^2$ is the extension of φ^2 on H given in (A3) and $\partial_H \tilde{\varphi}_\lambda^2$ denotes the Yosida approximation of $\partial_H \tilde{\varphi}^2$. We note by Proposition 2.11 of [4] that $\partial_H \tilde{\varphi}_\lambda^2 = \partial_H (\tilde{\varphi}_\lambda^2)$.

Since $\partial_H \tilde{\varphi}_\lambda^2$ is Lipschitz continuous in H, Proposition 3.12 of [4] assures the existence of a unique strong solution u_λ of $(\mathrm{CP})_\lambda$ on $[0,T]$ satisfying:

$$u_\lambda \in W^{1,2}(0,T;H), \quad u_\lambda(t) \in D(\partial_H \varphi_H^1) \ \text{ a.e. on } (0,T),$$
$$t \mapsto \varphi_H^1(u_\lambda(t)), \ \tilde{\varphi}_\lambda^2(u_\lambda(t)) \text{ are absolutely continuous on } [0,T].$$

Here and henceforth, we can assume that $\varphi^1 \geq 0$ without any loss of generality. We are going to establish a priori estimates in the following Lemmas 1-3.

Lemma 1. *There exists a constant M_1 such that*

$$\sup_{t \in [0,T]} |u_\lambda(t)|_H \leq M_1, \tag{6}$$

$$\sup_{t \in [0,T]} \varphi^1(u_\lambda(t)) \leq M_1, \tag{7}$$

$$\int_0^T \left| \frac{du_\lambda}{dt}(t) \right|_H^2 dt \leq M_1, \tag{8}$$

$$\sup_{t \in [0,T]} |u_\lambda(t)|_V \leq M_1. \tag{9}$$

Proof of Lemma 1. Multiply $(\mathrm{CP})_\lambda$ by $du_\lambda(t)/dt$. Then, by Lemma 3.3 of [4], we obtain

$$\left| \frac{du_\lambda}{dt}(t) \right|_H^2 + \frac{d}{dt} \varphi_H^1(u_\lambda(t)) - \frac{d}{dt} \tilde{\varphi}_\lambda^2(u_\lambda(t)) = \left(f_\lambda(t), \frac{du_\lambda}{dt}(t) \right)_H, \tag{10}$$

where $(\cdot,\cdot)_H$ denotes the inner product of H. Hence, integrating (10) over $(0,t)$, we have

$$\begin{aligned}\int_0^t \left|\frac{du_\lambda}{d\tau}(\tau)\right|_H^2 d\tau + \varphi_H^1(u_\lambda(t)) + \tilde{\varphi}_\lambda^2(u_0) \qquad (11)\\ = \varphi_H^1(u_0) + \tilde{\varphi}_\lambda^2(u_\lambda(t)) + \langle f_\lambda(t), u_\lambda(t)\rangle - \langle f_\lambda(0), u_0\rangle\\ - \int_0^t \left\langle \frac{df_\lambda}{d\tau}(\tau), u_\lambda(\tau)\right\rangle d\tau.\end{aligned}$$

By (A1) and (A4), it follows that

$$\begin{aligned}&\int_0^t \left|\frac{du_\lambda}{d\tau}(\tau)\right|_H^2 d\tau + (1-k)\varphi^1(u_\lambda(t)) \qquad (12)\\ &\leq \varphi^1(u_0) - \tilde{\varphi}_\lambda^2(u_0) + C_4|u_\lambda(t)|_H^2 + C_5\\ &\quad + \{C_3\varphi^1(u_\lambda(t)) + C_1|u_\lambda(t)|_H^2 + C_2\}^{1/p}|f_\lambda(t)|_{V^*} + |u_0|_V|f_\lambda(0)|_{V^*}\\ &\quad + \int_0^t \left\{C_3\varphi^1(u_\lambda(\tau)) + C_1|u_\lambda(\tau)|_H^2 + C_2\right\}^{1/p}\left|\frac{df_\lambda}{d\tau}(\tau)\right|_{V^*} d\tau.\end{aligned}$$

From the fact that $\frac{d}{dt}|u_\lambda(t)|_H \leq |\frac{du_\lambda}{dt}(t)|_H$, Young's inequality and Gronwall's inequality imply

$$\begin{aligned}|u_\lambda(t)|_H^2 + \varphi^1(u_\lambda(t)) \leq C\Bigg\{&|u_0|_H^2 + \varphi^1(u_0) + |\tilde{\varphi}_\lambda^2(u_0)| + |u_0|_V^p + C_2 + C_5\\ &+ \sup_{\tau\in[0,T]}|f_\lambda(\tau)|_{V^*}^{p'} + \int_0^T \left|\frac{df_\lambda}{d\tau}(\tau)\right|_{V^*}^{p'} d\tau\Bigg\},\end{aligned}$$

where C depends on $C_1,\ C_3,\ C_4,\ k$ and T. Since f_λ is bounded in $W^{1,p'}(0,T;V^*)$ and $\tilde{\varphi}_\lambda^2(u_0)$ is bounded, it follows that (6) and (7) hold. Moreover, (6), (7) and (12) imply (8). Furthermore, by (A1), we get

$$|u_\lambda(t)|_V^p \leq C_1|u_\lambda(t)|_H^2 + C_2 + C_3\varphi^1(u_\lambda(t)).$$

Hence, (6) and (7) imply (9). □

Lemma 2. *There exists a constant M_2 such that*

$$\sup_{t\in[0,T]}|J_\lambda u_\lambda(t)|_H \leq M_2, \qquad (13)$$

$$\sup_{t\in[0,T]}\varphi^1(J_\lambda u_\lambda(t)) \leq M_2, \qquad (14)$$

$$\sup_{t\in[0,T]}|J_\lambda u_\lambda(t)|_V \leq M_2, \qquad (15)$$

$$\int_0^T \left|\frac{d}{dt}J_\lambda u_\lambda(t)\right|_H^2 dt \leq M_2. \qquad (16)$$

Proof of Lemma 2. Since J_λ is non-expansive in H (see [5, P102]), we can derive (13) from (6). By (A3), (6) and (7) yield (14), which together with (A1) and (13) implies (15). Moreover since $|J_\lambda u_\lambda(t+h) - J_\lambda u_\lambda(t)|_H/h \leq |u_\lambda(t+h) - u_\lambda(t)|_H/h$ for all $h \in \mathbb{R}$ with $t+h \in [0,T]$, we have

$$\int_0^T \left|\frac{d}{dt}J_\lambda u_\lambda(t)\right|_H^2 dt \leq \int_0^T \left|\frac{du_\lambda}{dt}(t)\right|_H^2 dt,$$

which together with (8) implies (16). □

Lemma 3. *There exists a constant* M_3 *such that*

$$\sup_{t\in[0,T]} \left|\partial_H \tilde{\varphi}^2_\lambda(u_\lambda(t))\right|_{V^*} \leq M_3, \tag{17}$$

$$\int_0^T \left|g^1_\lambda(t)\right|^2_{V^*} dt \leq M_3, \tag{18}$$

where $g^1_\lambda(\cdot) = f_\lambda(\cdot) - du_\lambda(\cdot)/dt + \partial_H \tilde{\varphi}^2_\lambda(u_\lambda(\cdot)) \in \partial_H \varphi^1_H(u_\lambda(\cdot))$.

Proof of Lemma 3. Since $J_\lambda u_\lambda(t) \in D(\partial_H \tilde{\varphi}^2) \cap V$ for all $t \in [0,T]$, we get $\partial_H \tilde{\varphi}^2(J_\lambda u_\lambda(t)) \subset \partial \varphi^2(J_\lambda u_\lambda(t))$ for all $t \in [0,T]$. Furthermore, since $\partial_H \tilde{\varphi}^2_\lambda(u_\lambda(\cdot)) \in \partial_H \tilde{\varphi}^2(J_\lambda u_\lambda(\cdot))$ (see [5, p104]), it follows from (A2), (13), (14) and (16) that

$$\{\partial_H \tilde{\varphi}^2_\lambda(u_\lambda(\cdot))\} \text{ forms a precompact subset of } C([0,T];V^*), \tag{19}$$

which yields (17).

Since f_λ is bounded in $W^{1,p'}(0,T;V^*)$ and $g^1_\lambda(t) = f_\lambda(t) - du_\lambda(t)/dt + \partial_H \tilde{\varphi}^2_\lambda(u_\lambda(t))$ for a.e. $t \in (0,T)$, (8) and (17) imply (18). □

From Lemmas 1–3, we can extract a sequence λ_n such that $\lambda_n \to 0$ and the following lemma hold.

Lemma 4. *There exists* $u \in C_w([0,T];V) \cap W^{1,2}(0,T;H)$ *such that*

$$u_{\lambda_n} \to u \quad \text{weakly in } L^2(0,T;V) \cap W^{1,2}(0,T;H), \tag{20}$$

$$u_{\lambda_n}(t) \to u(t) \quad \text{weakly in } H \text{ for all } t \in [0,T], \tag{21}$$

$$J_{\lambda_n} u_{\lambda_n} \to u \quad \text{weakly in } L^2(0,T;V) \cap W^{1,2}(0,T;H). \tag{22}$$

Moreover we have $u(t) \to u_0$ *strongly in* H *as* $t \to +0$.

Proof of Lemma 4. Since H and V are reflexive, (8) and (9) imply (20). Moreover, let $q \in [1,+\infty)$. Then by (6), we can extract a subsequence λ^q_n of λ_n depending on q such that $u_{\lambda^q_n} - u_0 \to u - u_0$ weakly in $L^q(0,T;H)$. Hence since $u_{\lambda^q_n}(0) = u_0$, it follows from (8) that

$$\begin{aligned} \|u - u_0\|_{L^q(0,t;H)} &\leq \liminf_{\lambda^q_n \to 0} \|u_{\lambda^q_n} - u_0\|_{L^q(0,t;H)} \\ &\leq \liminf_{\lambda^q_n \to 0} \left\{ \int_0^t \left(\int_0^\tau \left| \frac{du_{\lambda^q_n}}{ds}(s) \right|^2_H ds \right)^{q/2} \tau^{q/2} d\tau \right\}^{1/q} \\ &\leq M_1^{1/2} \left(\frac{2}{q+2} \right)^{1/q} t^{(1/2+1/q)}. \end{aligned}$$

Thus we have

$$|u(t) - u_0|_H \leq \sup_{\tau\in[0,t]} |u(\tau) - u_0|_H = \lim_{q\to+\infty} \|u - u_0\|_{L^q(0,t;H)} \leq M_1^{1/2} t^{1/2},$$

which implies $u(t) \to u_0$ strongly in H as $t \to +0$.

Now let $t \in [0,T]$ be fixed. Since $u_{\lambda_n}(0) = u(0) = u_0$, (20) shows that

$$(u_{\lambda_n}(t) - u(t), \phi)_H = \int_0^t \left(\frac{du_{\lambda_n}}{d\tau}(\tau) - \frac{du}{d\tau}(\tau), \phi \right)_H d\tau \to 0 \text{ for any } \phi \in H,$$

which yields (21).

By (9) and (21), we can take a subsequence λ^t_n of λ_n depending on t such that

$$u_{\lambda^t_n}(t) \to u(t) \quad \text{weakly in } V. \tag{23}$$

Hence, by (9), it follows that $|u(t)|_V \le \liminf_{\lambda_n^t \to 0} |u_{\lambda_n^t}(t)|_V \le M_1$, where M_1 is independent of t. Therefore we conclude that $\sup_{t\in[0,T]} |u(t)|_V \le M_1 < +\infty$. Moreover, since $u \in L^\infty(0,T;V) \cap C([0,T];H)$, by Lemma 8.1 of [10, Chap.3, §8.4], we can deduce that $u \in C_w([0,T];V)$.

By (14) and (16), we find that $J_{\lambda_n} u_{\lambda_n} \to v$ weakly in $L^2(0,T;V) \cap W^{1,2}(0,T;H)$. Here, by (17), we notice that

$$|u_{\lambda_n}(t) - J_{\lambda_n} u_{\lambda_n}(t)|_{V^*} = \lambda_n |\partial_H \tilde{\varphi}^2_{\lambda_n}(u_{\lambda_n}(t))|_{V^*} \le \lambda_n M_3$$

for all $t \in [0,T]$. Hence since $u_{\lambda_n} - J_{\lambda_n} u_{\lambda_n} \to 0$ strongly in $C([0,T];V^*)$ as $\lambda_n \to 0$, it follows that $v = u$. □

Lemma 5. *There exists $g^2 \in C([0,T];V^*)$ such that*

$$\partial_H \tilde{\varphi}^2_{\lambda_n}(u_{\lambda_n}(\cdot)) \to g^2 \text{ strongly in } C([0,T];V^*) \quad (24)$$
$$\text{and} \quad g^2(t) \in \partial\varphi^2(u(t)) \text{ for a.e. } t \in (0,T).$$

Proof of Lemma 5. By (19), there exists $g^2 \in C([0,T];V^*)$ such that $\partial_H \tilde{\varphi}^2_{\lambda_n}(u_{\lambda_n}(\cdot)) \to g^2$ strongly in $C([0,T];V^*)$. Hence Lemma 1.3 of [3, Chap.II] and Proposition 1.1 of [7] imply $g^2(t) \in \partial\varphi^2(u(t))$ for a.e. $t \in (0,T)$. □

Lemma 6. *There exists $g^1 \in L^2(0,T;V^*)$ such that*

$$g^1_{\lambda_n} \to g^1 \text{ weakly in } L^2(0,T;V^*) \quad (25)$$
$$\text{and} \quad g^1(t) = f(t) + g^2(t) - \frac{du}{dt}(t) \in \partial\varphi^1(u(t)) \text{ for a.e. } t \in (0,T).$$

Proof of Lemma 6. By (18), it is obvious that $g^1_{\lambda_n} \to g^1$ weakly in $L^2(0,T;V^*)$. Moreover, by $(\mathrm{CP})_{\lambda_n}$, it follows from (20) and (24) that $g^1 = f + g^2 - du/dt$.

Hence it remains to prove that $f(t) + g^2(t) - du(t)/dt \in \partial\varphi^1(u(t))$ for a.e. $t \in (0,T)$. Multiplying $g^1_{\lambda_n}(t)$ by $u_{\lambda_n}(t)$ and integrating over $(0,T)$, we get

$$\int_0^T \langle g^1_{\lambda_n}(t), u_{\lambda_n}(t)\rangle dt = \int_0^T \langle f_{\lambda_n}(t), u_{\lambda_n}(t)\rangle dt + \int_0^T \langle \partial_H \tilde{\varphi}^2_{\lambda_n}(u_{\lambda_n}(t)), u_{\lambda_n}(t)\rangle dt - \frac{1}{2}|u_{\lambda_n}(T)|_H^2 + \frac{1}{2}|u_0|_H^2.$$

Since $f_{\lambda_n} \to f$ strongly in $W^{1,p'}(0,T;V^*)$, it follows from (20), (21) and (24) that

$$\begin{aligned}\limsup_{\lambda_n\to 0} \int_0^T \langle g^1_{\lambda_n}(t), u_{\lambda_n}(t)\rangle dt &\le \int_0^T \langle f(t), u(t)\rangle dt + \int_0^T \langle g^2(t), u(t)\rangle dt \\ &\quad - \frac{1}{2}|u(T)|_H^2 + \frac{1}{2}|u_0|_H^2 \\ &= \int_0^T \left\langle f(t) + g^2(t) - \frac{du}{dt}(t), u(t)\right\rangle dt.\end{aligned} \quad (26)$$

By Lemma 1.3 of [3, Chap.II] and Proposition 1.1 of [7], we conclude that $g^1(t) = f(t) + g^2(t) - du(t)/dt \in \partial\varphi^1(u(t))$ for a.e. $t \in (0,T)$. □

Now, let $t \in [0,T]$ be fixed. Then since $\varphi^1 \in \Phi(V)$, (7) and (23) imply $\varphi^1(u(t)) \le \liminf_{\lambda_n^t \to 0} \varphi^1(u_{\lambda_n^t}(t)) \le M_1$, where M_1 is independent of t. Hence we conclude that $\sup_{t\in[0,T]} \varphi^1(u(t)) \le M_1 < +\infty$. Moreover let $t_n \in [0,T]$ be such that $t_n \to t$. From the fact that $u \in C_w([0,T];V)$, it follows that $u(t_n) \to u(t)$ weakly in V. Since $\varphi^1(u(t_n)) \le \sup_{t\in[0,T]} \varphi^1(u(t)) \le M_1$, where M_1 is independent of n, Proposition 1 assures that there exists a subsequence $t_{n'}$ of t_n such that $\varphi^2(u(t_{n'})) \to \varphi^2(u(t))$.

Hence, since this argument does not depend on the choice of subsequence $t_{n'}$, we deduce that $\varphi^2(u(\cdot)) \in C([0,T])$.

3.2. **Proof of Theorem 2.** We first introduce an auxiliary problem. Let $r \in \mathbb{R}$ be such that $r > \varphi^2(u_0)$ and define the cut off function of φ^1 by

$$\varphi^{1,r}(u) = \begin{cases} \varphi^1(u) & \text{if } \varphi^2(u) \le r, \\ +\infty & \text{otherwise.} \end{cases}$$

Then, it follows that $\varphi^{1,r} \in \Phi(V)$ and $D(\varphi^{1,r}) = D(\varphi^1) \cap \{u \in V; \varphi^2(u) \le r\}$. Let us consider the following Cauchy problem:

$$(\mathrm{CP})^r \quad \frac{du}{dt}(t) + \partial\varphi^{1,r}(u(t)) - \partial\varphi^2(u(t)) \ni f(t) \quad \text{in } V^*, \quad u(0) = u_0.$$

We then easily find that (A1) and (A2) with φ^1 replaced by $\varphi^{1,r}$ hold. Moreover since $\varphi^2(u) \le r$ for all $u \in D(\varphi^{1,r})$, (A4) is satisfied with $k = 0$, $C_4 = 0$, $C_5 = r$ and $\varphi^1 = \varphi^{1,r}$.

Let $u \in D(\varphi^{1,r})$. Then since Proposition of [4] says that $\varphi^2(J_\lambda u) \le \varphi^2(u) \le r$ and (A3) implies $J_\lambda u \in D(\varphi^1)$, it follows that $\varphi^{1,r}(J_\lambda u) = \varphi^1(J_\lambda u)$. Hence (A3) with φ^1 replaced by $\varphi^{1,r}$ holds true. Therefore, since the fact that $\varphi^2(u_0) < r$ and $u_0 \in D(\varphi^1)$ yields $u_0 \in D(\varphi^{1,r})$, Theorem 1 assures the existence of strong solutions of $(\mathrm{CP})^r$ as follows:

Lemma 7. *Assume that* (A1), (A2) *and* (A3) *are satisfied. Then for all* $u_0 \in D(\varphi^1)$, $f \in W^{1,p'}(0,T;V^*)$ *and* $r \in \mathbb{R}$ *with* $r > \varphi^2(u_0)$, $(\mathrm{CP})^r$ *has a strong solution* u *on* $[0,T]$ *satisfying* (5) *with* φ^1 *replaced by* $\varphi^{1,r}$.

In order to show that $u(t)$ becomes a strong solution of (CP) time-locally, it is sufficient to prove that there exists a number $T_0 \in (0,T]$ such that $\partial\varphi^{1,r}(u(t)) = \partial\varphi^1(u(t))$ for all $t \in [0,T_0)$. To this end, we prepare a couple of lemma.

Lemma 8. *There exists a number* $T_0 \in (0,T]$ *such that* $\varphi^2(u(t)) < r$ *for all* $t \in [0,T_0)$.

Proof of Lemma 8. We here remark that $\varphi^2(u(\cdot)) \in C([0,T])$. Hence, for the case where $\max_{t\in[0,T]} \varphi^2(u(t)) < r$, we can take $T_0 = T$. For the case where $\max_{t\in[0,T]} \varphi^2(u(t)) \ge r$, since $\varphi^2(u_0) < r$, there exists a number $T_0 \in (0,T]$ such that $\varphi^2(u(t))$ attains r at $t = T_0$ for the first time. □

Lemma 9. *If* $u \in D(\partial\varphi^{1,r})$ *and* $\varphi^2(u) < r$, *then* $\partial\varphi^{1,r}(u) = \partial\varphi^1(u)$.

Proof of Lemma 9. It is obvious that $\partial\varphi^1(u) \subset \partial\varphi^{1,r}(u)$ for all $u \in D(\partial\varphi^{1,r})$. Hence it suffices to show that $\partial\varphi^{1,r}(u) \subset \partial\varphi^1(u)$ when $u \in D(\partial\varphi^{1,r})$ and $\varphi^2(u) < r$. Let $[u,\xi] \in \partial\varphi^{1,r}$ be such that $\varphi^2(u) < r$ and take an arbitrary element $v \in D(\varphi^1)$. Then since $u_s := (1-s)u + sv \to u$ strongly in V as $s \to 0$ and $\varphi^1(u_s) \le (1-s)\varphi^1(u) + s\varphi^1(v) \le \varphi^1(u) + \varphi^1(v)$, where $\varphi^1(u) + \varphi^1(v)$ is independent of s, for all $s \in [0,1]$, Proposition 1 assures that there exists a sequence s_n such that $\varphi^2(u_{s_n}) \to \varphi^2(u)$ as $s_n \to 0$. Hence from the fact that $\varphi^2(u) < r$, it follows that there exists a number $s_0 \in (0,1)$ such that $\varphi^2(u_{s_0}) < r$. Since $u_{s_0} \in D(\varphi^{1,r})$, we get $\varphi^1(u_{s_0}) - \varphi^1(u) = \varphi^{1,r}(u_{s_0}) - \varphi^{1,r}(u) \ge \langle \xi, u_{s_0} - u\rangle$. Hence, by the convexity of φ^1, we have $s_0(\varphi^1(v) - \varphi^1(u)) \ge \langle \xi, s_0(v-u)\rangle$. By dividing both sides by $s_0 > 0$, we deduce $\varphi^1(v) - \varphi^1(u) \ge \langle \xi, v-u\rangle$ for all $v \in D(\varphi^1)$, whence follows $\xi \in \partial\varphi^1(u)$. □

By Lemmas 7, 8 and 9, there exists a number $T_0 \in (0,T]$ such that $u(t) \in D(\partial\varphi^1)$ and $\partial\varphi^{1,r}(u(t)) = \partial\varphi^1(u(t))$ for a.e. $t \in (0,T_0)$. Consequently we deduce that u becomes a strong solution of (CP) on $[0,T_0]$.

4. **Application.** In this section, we apply the preceding abstract theory to (NHE) and give sufficient conditions for the existence of solutions.

The sufficient conditions for the existence of solutions in [11] are weaker than those in [13] except for the case where $p < q$ and $p < N$. For this case, [13] requires $q < 2p/(N+p)$ for the local existence and that $q < p^*$ for the global existence for small data, where p^* is the so-called Sobolev's critical exponent given by $p^* = Np/(N-p)$ if $p < N$; and $p^* = +\infty$ if $p \geq N$. The latter condition is called subcritical growth condition. On the other hand, [11] requires $q < p^*/2 + 1$ for the local existence and the global existence for small data. Moreover, [12] succeeded in relaxing the sufficient condition for the local existence to subcritical growth condition when $p = 2$.

Thus the sufficient conditions for the local existence in [11] and [12] are strictly weaker than those in [13]. However, as for the global existence, the sufficient conditions in [11] and [12] are more restrictive than those in [13] when $p \neq 2$.

Moreover when $p \neq 2$, there still exists a considerable gap between the subcritical growth condition and the sufficient conditions for local existence in [13], [11] and [12]. It is remarkable to note that our abstract theory enables us to relax the sufficient condition for the local existence of weak solutions to the subcritical growth condition. This fact has been conjectured but left as an open problem for long time.

To apply our abstract theory, we set $V = W_0^{1,p}(\Omega)$ and $H = L^2(\Omega)$ with norms $|\cdot|_V := |\nabla \cdot|_{L^p(\Omega)}$ and $|\cdot|_H := |\cdot|_{L^2(\Omega)}$ respectively. Here we assume that $2N/(N+2) \leq p < +\infty$ and $1 < q < p^*$. We then find that V, V^* and H satisfy (1). Moreover, by Sobolev's embedding theorem, the injection $V \subset L^q(\Omega)$ is compact (see [1]). Now let $\varphi_p,\ \psi_q \in \Phi(V)$ be given by

$$\varphi_p(u) = \frac{1}{p}\int_\Omega |\nabla u(x)|^p \, dx, \quad \psi_q(u) = \frac{1}{q}\int_\Omega |u(x)|^q \, dx \quad \forall u \in V.$$

Then it is obvious that $\partial\varphi_p$ and $\partial\psi_q$ coincide with $-\Delta_p u$ and $|u|^{q-2}u$ in the distribution sense, where $D(\varphi_p) = D(\partial\varphi_p) = V$ and $D(\psi_q) = D(\partial\psi_q) = V$ respectively. Therefore (NHE) can be reduced to (CP) with $\partial\varphi^1$ and $\partial\varphi^2$ replaced by $\partial\varphi_p$ and $\partial\psi_q$ respectively.

Lemma 10. (A1), (A2) *and* (A3) *with* φ^1 *and* φ^2 *replaced by* φ_p *and* ψ_q *hold.*

Proof of Lemma 10. We see that $\varphi_p(u) = (1/p)|u|_V^p$, which implies (A1) with $C_1 = C_2 = 0$, $C_3 = p$. From the definition of φ_p and ψ_q, it is obvious that $D(\varphi_p) \subset D(\partial\psi_q)$. Let u_n be such that $\sup_{t\in[0,T]}\{\varphi_p(u_n(t)) + |u_n(t)|_H\} + \int_0^T |du_n(t)/dt|_H^2 dt$ is bounded. Now since V is embedded in $L^q(\Omega)$ with compact injection, it follows that $\{u_n(t)\}$ is compact in $L^q(\Omega)$ for all $t \in [0,T]$. Moreover since $q \in (1,p^*)$ and Ω is bounded, we observe that there exists a number $\theta \in (0,1]$ such that $|u|_{L^q(\Omega)} \leq C|u|_H^\theta |u|_V^{1-\theta}$ for all $u \in V$, whence follows that $u_n(\cdot)$ is equi-continuous in $C([0,T];L^q(\Omega))$. Hence, by Ascoli's lemma, there exists a subsequence n' of n such that $u_{n'} \to u$ strongly in $C([0,T];L^q(\Omega))$. Then we can easily check that

$$|u_{n'}|^{q-2}u_{n'}(\cdot) \to |u|^{q-2}u(\cdot) \quad \text{strongly in } C([0,T];L^{q'}(\Omega)).$$

Therefore we deduce that $\partial\psi_q(u_{n'}(\cdot)) \to \partial\psi_q(u(\cdot))$ strongly in $C([0,T];V^*)$. Moreover (A3) also holds true (see [5, Proof of Corollary 16]). □

4.1. **The case where $p \leq q$ and $u_0 \in W_0^{1,p}(\Omega)$.**

Theorem 3. (*Local existence*) *Suppose that* $p \leq q < p^*$. *Then, for all* $u_0 \in W_0^{1,p}(\Omega)$ *and* $f \in W^{1,p'}(0,T;W^{-1,p'}(\Omega))$, *there exists a number* $T_0 \in (0,T]$ *such that* (NHE) *has a solution* u *on* $[0,T_0]$ *satisfying*:

$$u \in C_w([0,T_0];W_0^{1,p}(\Omega)) \cap C([0,T_0];L^q(\Omega)) \cap W^{1,2}(0,T_0;L^2(\Omega)).$$

Proof of Theorem 3. By Lemma 10, we infer that (A1), (A2) and (A3) hold. Hence, by Theorem 2, there exists a number $T_0 \in (0,T]$ such that (NHE) has a solution u on $[0,T_0]$. Moreover since $\psi_q(u(\cdot)) \in C([0,T_0])$, the uniformly convexity of $L^q(\Omega)$ implies $u \in C([0,T_0];L^q(\Omega))$. □

4.2. **The case where** $p > q$ **and** $u_0 \in W_0^{1,p}(\Omega)$**.**

Theorem 4. (*Global existence*) *Suppose that* $p > q$. *Then, for all* $u_0 \in W_0^{1,p}(\Omega)$ *and* $f \in W^{1,p'}(0,T;W^{-1,p'}(\Omega))$, (NHE) *has a solution* u *on* $[0,T]$ *satisfying*:

$$u \in C_w([0,T];W_0^{1,p}(\Omega)) \cap C([0,T];L^q(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega)).$$

Proof of Theorem 4. By Lemma 10, we see that (A1), (A2) and (A3) hold. Moreover, since $p > q$, we find

$$\psi_q(u) = \frac{1}{q}|u|^q_{L^q(\Omega)} \leq C|u|^q_V \leq \frac{1}{2}\varphi_p(u) + C \quad \forall u \in V,$$

which implies (A4). Therefore, by Theorem 1, (NHE) has a global solution on $[0,T]$. □

Remark 2. In Theorem 3, if we assume the smallness of given data, we can assure the global existence of solutions.

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E-mail address: goro@toki.waseda.jp