Stability of stationary solutions for semilinear heat equations with concave nonlinearity

Goro Akagi Ryuji Kajikiya

Abstract

This paper is concerned with the stability analysis of stationary solutions of the Cauchy-Dirichlet problem for some semilinear heat equation with concave nonlinearity. The instability of sign-changing solutions is verified under some variational assumption. Moreover, the exponential stability of the positive stationary solution at an optimal rate is proved by exploiting a super-subsolution method as well as the parabolic regularity theory. The base of our analysis relies on the linearization of the equation at each stationary solution and spectral analysis of the corresponding linearized operator. The main difficulties reside in the singularity of the linearized operator due to the concave nonlinearity.

Keywords. Sublinear heat equation; asymptotic behavior; stability; linearized problem; eigenvalue problem

1 Introduction and main results

Semilinear heat equation has been one of the most active fields in the study of nonlinear partial differential equation during the last half century. It is related to the broad range of topics involving *reaction-diffusion equation* and *blow-up problem*, and various phenomena particular to nonlinear nature have been discovered so far. In this paper, we address ourselves to such semilinear heat equations with *concave nonlinearity*. More precisely, we are concerned with the following Cauchy-Dirichlet problem with an exponent $0 < p < 1$,

$$\partial_t u = \Delta u + |u|^{p-1}u \quad \text{in } \Omega \times (0,\infty), \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{1.2}$$

$$u(\cdot,0) = u_0 \quad \text{in } \Omega, \tag{1.3}$$

G. Akagi (corresponding author): Graduate School of System Informatics, Kobe University, 1-1 Rokkodai-cho, Nada-ku, Kobe 657-8501 Japan; e-mail: akagi@port.kobe-u.ac.jp

R. Kajikiya: Department of Mathematics, Faculty of Science and Engineering, Saga University, Saga 840-8502, Japan; e-mail: kajikiya@ms.saga-u.ac.jp

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where $\partial_t = \partial/\partial t$ and Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. There are vast amount of contributions to the blow-up case, $p > 1$ (see a pioneer work in [16] and also a survey [30] with references therein); however, the case $0 < p < 1$ has not been fully pursued yet because of some difficulties peculiar to the concavity of $|u|^{p-1}u$. As contrasted with the blow-up case, solutions can exist globally in time for any data, e.g., $u_0 \in L^2(\Omega)$. The most peculiar feature of this case would be the nonuniqueness of (nonnegative) solution. In [17], the nonuniqueness is pointed out, and moreover, the comparison principle is established only for positive solutions. The long-time behavior of solutions is also investigated by using the principle in [8] and [33], where semilinear heat equations with convex-concave nonlinearity are treated (see also related works [2, 12, 18, 24, 3, 25, 14, 13]). The nonuniqueness of solution for (1.1)–(1.3) and major difficulties in this issue are arising from the concave nonlinearity, in particular, the fact that the nonlinear term is more singular at $u = 0$ than other semilinear heat equations that have been vigorously studied so far (e.g., reaction-diffusion equation and blow-up case).

Long-time behaviors of solutions for (1.1)–(1.3) are closely related to the corresponding stationary problem, which can be written as

$$-\Delta v = |v|^{p-1}v \quad \text{in } \Omega, \tag{1.4}$$

$$v = 0 \quad \text{on } \partial\Omega. \tag{1.5}$$

Equation (1.4) is called the *sublinear Lane-Emden* (or *Emden-Fowler*) *equation*. Let us focus on the stability issue of stationary solutions for (1.1)–(1.3). It is proved by Brézis and Oswald [7] that the positive (negative) stationary solution $\phi(x)$ of (1.4), (1.5) is unique and it takes the minimum of a Lyapunov energy corresponding to (1.1)–(1.3). So one may expect that $\phi(x)$ is asymptotically stable. In this paper, we take one step further; that is to investigate the *exponential stability* of $\phi(x)$. We discuss the stability in the Sobolev space $H^1_0(\Omega)$, where the Lyapunov method (energy method) works well. Note that any open subset of $H^1_0(\Omega)$ contains sign changing functions because $C^\infty_0(\Omega)$ is dense in $H^1_0(\Omega)$. Therefore we must take account of sign-changing initial data as well as positive ones in any neighborhood of $\phi(x)$. The notions of stability, asymptotic stability and exponential stability of stationary solutions will be explicitly defined in Definition 2.5 below. We shall prove in Proposition 4.1 of §4 that for the unique positive solution $\phi(x)$ of (1.4), (1.5), the linearized operator $-\Delta - p\phi^{p-1}$ is well-defined and its principal eigenvalue is positive. Denote it by $\mu > 0$. Main results on the stability are stated as follows:

Theorem 1.1 (Stability of the positive stationary solution). (i) *The positive solution ϕ and the negative solution $-\phi$ of* (1.4), (1.5) *are asymptotically stable* (*under flows of possibly sign-changing solutions as well*).

(ii) *The trivial solution $v \equiv 0$ of* (1.4), (1.5) *is unstable.*

(iii) *Let* $u_0 \in L^\infty(\Omega)$ *be nonnegative and let* $u(x,t)$ *be the positive solution of* (1.1)–(1.3)*. Then there exists a constant* $C > 0$ *such that*

$$\|u(\cdot,t) - \phi\|_{L^\infty(\Omega)} \le Ce^{-\mu t} \quad \text{for all } t \ge 0. \tag{1.6}$$

Here $\mu > 0$ *is the first eigenvalue of the linearized operator* $-\Delta - p\phi^{p-1}$*. Moreover, if* $0 < u_0(x) \le (1-\delta)\phi(x)$ (*or* $(1+\delta)\phi(x) \le u_0(x)$) *for a.e.* $x \in \Omega$ *with some* $0 < \delta < 1$*, then there exist constants* $c, C >$ *such that*

$$ce^{-\mu t} \le \|u(\cdot,t) - \phi\|_{L^\infty(\Omega)} \le Ce^{-\mu t} \quad \text{for } t \ge 0. \tag{1.7}$$

(iv) *Let* $u(x,t)$ *be any positive solution of* (1.1)–(1.3) *and* $t_0 > 0$*. Then there exists a constant* $C > 0$ *such that*

$$\left\| \frac{u(\cdot,t)}{\phi} - 1 \right\|_{L^\infty(\Omega)} \le Ce^{-\mu t} \quad \text{for } t \ge t_0.$$

(v) *Let* $u(x,t)$ *be any solution of* (1.1)–(1.3) *with initial data* $u_0 \in H_0^1(\Omega)$*. Assume that* $\|u_0 - \phi\|_{H_0^1(\Omega)}$ *is small enough or* $u_0(x) > 0$ *in* Ω*. Then* (a) *and* (b) *below hold.*

(a) *For any* $t_0 > 0$ *and* $\theta \in (0,1)$ *there exists a constant* $C(t_0,\theta) > 0$ *such that*

$$\|u(\cdot,t) - \phi\|_{C^{1,\theta}(\overline{\Omega})} \le C(t_0,\theta)e^{-\mu t} \quad \text{for } t \ge t_0. \tag{1.8}$$

Here $\|\cdot\|_{C^{1,\theta}(\overline{\Omega})}$ *denotes the norm of the space* $C^{1,\theta}(\overline{\Omega})$*, which consists of all* $C^1(\overline{\Omega})$ *functions whose derivatives are Hölder continuous with exponent* θ*.*

(b) *There exists a constant* $C > 0$ *such that*

$$\|u(\cdot,t) - \phi\|_{H_0^1(\Omega)} \le Ce^{-\mu t} \quad \text{for } t \ge 0.$$

The result above along with (i) *ensures that* ϕ *is exponentially stable with the optimal exponent* μ*.*

On the other hand, sign-changing stationary solutions take larger energies. So it is conjectured that they are not asymptotically stable. Another main interest of this paper is to rigorously prove this conjecture by seeking an explicit perturbation to the sign-changing solutions in such a way as to decrease their Lyapunov energy. We here define the Lagrangian functional (or the Lyapunov energy) $I : H_0^1(\Omega) \to \mathbb{R}$ by

$$I(v) := \int_\Omega \left(\frac{1}{2}|\nabla v|^2 - \frac{1}{p+1}|v|^{p+1} \right) dx \quad \text{for } v \in H_0^1(\Omega). \tag{1.9}$$

Note that $I(v) = -((1-p)/(2(p+1))\|\nabla v\|_{L^2}^2 = -((1-p)/(2(p+1))\|v\|_{L^{p+1}}^{p+1}$ for any solution v of (1.4), (1.5), because $\|\nabla v\|_{L^2}^2 = \|v\|_{L^{p+1}}^{p+1}$. Then another main result is as follows.

Theorem 1.2 (Instability of sign-changing stationary solutions)**.** *Let v be a sign-changing solution of* (1.4)*,* (1.5) *which satisfies*

$$\|v\|_{L^{p+1}(\Omega)} < p^{1/(1-p)}\|\phi\|_{L^{p+1}(\Omega)}, \quad \text{equivalently } I(v) > p^{\frac{p+1}{1-p}} I(\phi), \tag{1.10}$$

where ϕ denotes the unique positive solution of (1.4)*,* (1.5)*. Then v is not asymptotically stable. Moreover, let $\mathcal{S}$ denote the set of all solutions for* (1.4)*,* (1.5)*. If v is isolated from all $w \in \mathcal{S}$ satisfying $I(w) < I(v)$, then it is unstable.*

Note that there exist infinitely many sign-changing solutions of (1.4), (1.5) which satisfy (1.10), since there exists a sequence of sign-changing solutions converging to zero (see [5], [29], [11], [20]).

For the one-dimensional case, we can determine the stability and instability of all stationary solutions. Let $N = 1$ and $\Omega = (0, 1)$. Then (1.4) and (1.5) are reduced to

$$v'' + |v|^{p-1}v = 0 \quad \text{in } (0,1), \tag{1.11}$$

$$v(0) = v(1) = 0, \tag{1.12}$$

which possesses a unique positive solution $\phi_1(x)$ (see [7]). Moreover, ϕ_1 can be extended onto $\mathbb{R}$ as a 2-periodic odd solution of (1.11) with the interval $(0, 1)$ replaced by $\mathbb{R}$. Furthermore, (1.11) is invariant under the scaling $v(x) \mapsto \lambda^{-2/(1-p)}v(\lambda x)$. Therefore the functions

$$\phi_k(x) := k^{-2/(1-p)}\phi_1(kx) \quad \text{for } k \in \mathbb{N} \tag{1.13}$$

become solutions of (1.11), (1.12) having exactly $(k-1)$ zeros in $(0, 1)$. Furthermore, the set of $\pm\phi_k$ for all $k \in \mathbb{N}$ covers all nontrivial solutions of (1.11), (1.12).

Theorem 1.3 (Stability classification in $N = 1$)**.** *The positive and negative stationary solutions $\pm\phi_1$ are exponentially stable, and sign-changing ones $\pm\phi_k$* (*for all $k \geq 2$*) *are unstable.*

The methods of analysis in the present paper are mainly based on the linearization of the equation at corresponding stationary solutions. However, due to the concave nonlinearity and the Dirichlet boundary condition, it is delicate to treat the corresponding linearized operators as well as to formulate it in a rigorous way. Indeed, the linearized operators have potentials with some singularity on the nodal set (including the boundary $\partial\Omega$) of each stationary solution.

This paper consists of seven sections. In Section 2, we collect preliminary facts on the stationary problem and the long-time behavior of solutions for (1.1)–(1.3). Moreover, we also discuss stability analysis of stationary solutions; in particular, the asymptotic stability of the positive (negative) stationary solution is rigorously proved (Proofs for (i) and (ii) of Theorem 1.1). In Section 3, we shall prove the instability of sign-changing stationary solutions (Proof of Theorem 1.2). Here, we handle a Schrödinger operator with singularity on the nodal set and derive the negativity of its principal eigenvalue in a proper fashion. Moreover, some result on asymptotic profiles for porous medium equations will be also given as a by-product. In Section 4, we investigate several properties of the linearized operator at the positive stationary solution for a latter use. In Section 5, we shall verify the convergence of positive solutions of (1.1)–(1.3) to the positive stationary solution at an optimal rate by constructing some super- and subsolutions and by using comparison principle (Proofs for (iii) and (iv) of Theorem 1.1). Here the construction of the super- and subsolutions is closely related to the eigenvalue problem for the linearized operator. In Section 6, the results obtained in Section 5 will be extended to general evolutionary solutions (Proof for (v) of Theorem 1.1). Section 7 is presented for the one-dimensional case (Proof of Theorem 1.3).

Notation. We denote the $L^p(\Omega)$-norm by $\|\cdot\|_p$ and the $H^1_0(\Omega)$-norm by $\|\cdot\|_{1,2}$, which is defined by $\|u\|_{1,2} := \|u\|_{H^1_0(\Omega)} = \|\nabla u\|_2$.

2 Preliminaries

This section is devoted to preliminary facts on the stationary problem and the asymptotic behavior of solutions for (1.1)–(1.3). Moreover, (i) and (ii) of Theorem 1.1 will be verified.

2.1 Stationary problem

The Dirichlet problem (1.4), (1.5) has been studied from various view points, and here, we particularly focus on its variational aspect. Brézis and Oswald [7] proved the existence of least energy solutions (the minimizers of I defined by (1.9) in $H^1_0(\Omega)$), which are positive and negative solutions, and they also verified the uniqueness of positive (negative) solution. Furthermore, by the symmetric mountain pass lemma due to Ambrosetti-Rabinowitz [5], it has been proved that (1.4), (1.5) has a sequence of nontrivial solutions converging to zero in $H^1_0(\Omega)$. Then the convergence is valid in $C^2(\overline{\Omega})$ also by the elliptic regularity theorem. These solutions must change their signs because of the uniqueness of the positive solution. See also Clark [11], Rabinowitz [29] and Kajikiya [20]. Furthermore, each nontrivial solution of (1.4), (1.5) has a negative energy because of the identity after (1.9).

2.2 Long-time behavior of evolutionary solutions

Let us start with the definition of solution for (1.1)–(1.3).

Definition 2.1. A function $u \in C([0,\infty); L^2(\Omega))$ is called a *solution* of (1.1)–(1.3) if the following conditions hold true:

- $u \in C^1((0,\infty); L^2(\Omega)) \cap C((0,\infty); H^2(\Omega) \cap H_0^1(\Omega))$,
- it satisfies (1.1) a.e. in $\Omega \times (0,\infty)$ and $u(\cdot,0) = u_0$ a.e. in Ω.

For $u_0 \in L^2(\Omega)$, the existence of (global in time) solutions for (1.1)–(1.3) is well known (see also Proposition 2.3 below). The functional I defined by (1.9) becomes a Lyapunov energy for (1.1)–(1.3). Indeed, $I(u(t))$ is nonincreasing in t. We also define *super-* and *subsolutions* for (1.1).

Definition 2.2. We call $u \in C([0,T); L^2(\Omega))$ a *supersolution* of (1.1) in $\Omega \times (0,T)$ if the following conditions hold true:

- $u \in C^1((0,T); L^2(\Omega)) \cap C((0,T); H^2(\Omega))$,
- $\partial_t u \geq \Delta u + |u|^{p-1}u$ a.e. in $\Omega \times (0,T)$.

If the reverse inequality holds, then u is called a *subsolution*.

Then let us recall the following known results:

Proposition 2.3. (i) *For* $u_0 \in H_0^1(\Omega)$ (*or* $u_0 \in L^\infty(\Omega)$)*, the Cauchy-Dirichlet problem* (1.1)–(1.3) *admits at least one time-global solution* $u \in C([0,\infty); H_0^1(\Omega))$ (*or* $u \in L^\infty(\Omega \times (0,\infty))$*, respectively*). Moreover, $\sup_{0 \leq t < \infty} \|u(t)\|_{1,2} < \infty$ when $u_0 \in H_0^1(\Omega)$.

(ii) Let $u(x,t)$ be a (possibly sign-changing) solution of (1.1)–(1.3). Let (t_n) be an arbitrary positive sequence diverging to infinity. Then there exist a subsequence (n') of (n) and a solution v of (1.4), (1.5) such that

$$u(t_{n'}) \to v \quad \text{strongly in } H_0^1(\Omega) \text{ as } n' \to \infty.$$

(iii) If $u_0 \in L^\infty(\Omega)$ is nonnegative, (1.1)–(1.3) admits a unique time-global positive classical solution $u \in L^\infty(\Omega \times (0,\infty))$.

(iv) Let u^- be a subsolution of (1.1) in the parabolic domain $Q = \Omega \times (0,T)$ and let u^+ be a supersolution of (1.1) in Q such that either u^- is negative or u^+ is positive a.e. in Q and they belong to $L^\infty(Q)$. If $u^- \leq u^+$ a.e. on the parabolic boundary $\partial_p Q = (\partial\Omega \times (0,T)) \cup (\overline{\Omega} \times \{0\})$ of Q, then $u^- \leq u^+$ a.e. in Q.

Remark 2.4. (i) The assertions (i) and (ii) of Proposition 2.3 are well known (see, e.g., [26, 27, 28], where more general equations are treated). The comparison principle (iv) is due to [8] (see also [33]). As for (iii), by [8], (1.1)–(1.3) admits the unique positive solution $u(x,t)$ on a maximal time interval $[0,T)$. Then u is bounded in $\Omega \times [0,T)$ by a positive constant independent of T. Indeed, by Theorem 6.2 and Remark 6.1 of [8], for each nonnegative data $u_0 \in L^\infty(\Omega)$, there exists a unique positive classical solution u of (1.1)–(1.3) on $[0,T)$. Let $t_0 \in (0,T)$ and let ϕ be the positive stationary solution. Then by (vi) of Remark 6.1 of [8] and the fact that $\partial\phi/\partial\nu < 0$ on $\partial\Omega$, one can take a constant $C_0 > 1$ such that $u(x,t_0) \le C_0\phi(x)$ for all $x \in \Omega$. Moreover, $U^+(x,t) = C_0\phi(x)$ becomes a positive supersolution of (1.1) in $\Omega \times (0,\infty)$. Since $u \le U^+$ on the parabolic boundary of $\Omega \times (t_0,T)$, the comparison principle ensures that $0 \le u(x,t) \le C_0\phi(x) \le C_0\|\phi\|_\infty$ for all $(x,t) \in \Omega \times (t_0,T)$. This implies that $T = \infty$ and $u \in L^\infty(\Omega \times (0,\infty))$; hence (iii) of Proposition 2.3 follows.

(ii) If the initial data u_0 is indefinite, solutions of (1.1)–(1.3) need not be unique. Indeed, the nonuniqueness is pointed out for the case that $u_0 \equiv 0$ in [17] and [8]. More precisely, for the initial data $u_0 \equiv 0$, there exists a (unique) positive solution $\hat{u} > 0$ of (1.1)–(1.3) (see Remark 6.4 of [8]).

2.3 Stability of stationary solutions

We first define the notions of stability and instability of stationary solutions for (1.1)–(1.3) in the following. It is noteworthy that we here do not assume the positivity of solutions for (1.1)–(1.3); so the uniqueness of solution may fail.

Definition 2.5 (Stability of stationary solutions)**.** Let v be a stationary solution of (1.1)–(1.3), that is, a solution of (1.4), (1.5).

(i) v is said to be *stable* if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that *every* solution u of (1.1), (1.2) satisfies

$$\sup_{0 \le t < \infty} \|u(t) - v\|_{1,2} < \varepsilon \quad \text{whenever } \|u(0) - v\|_{1,2} < \delta.$$

(ii) v is said to be *unstable* if v is not stable.

(iii) v is said to be *asymptotically stable* if it is stable, and moreover, there exists a $\delta_0 > 0$ such that *every* solution u of (1.1), (1.2) satisfies

$$\lim_{t \to \infty} \|u(t) - v\|_{1,2} = 0 \quad \text{whenever } \|u(0) - v\|_{1,2} < \delta_0.$$

(iv) v is said to be *exponentially stable* if it is stable, and moreover, there exist positive constants C, λ, δ_1 such that *every* solution u of (1.1), (1.2) satisfies

$$\|u(t) - v\|_{1,2} \le Ce^{-\lambda t} \text{ for all } t \ge 0 \text{ whenever } \|u(0) - v\|_{1,2} < \delta_1.$$

The uniqueness of the positive stationary solution leads to the next proposition.

Proposition 2.6. *Every positive solution* $u(x,t)$ *of* (1.1)–(1.3) *converges to the positive stationary solution* ϕ *strongly in* $H_0^1(\Omega)$ *as* $t \to \infty$.

Proof. Let $u(t)$ be as in the theorem. In case $u_0 := u(0) \not\equiv 0$, choose $\lambda \in (0,1)$ so small that

$$I(\lambda u_0) = \frac{\lambda^2}{2}\|\nabla u_0\|_2^2 - \frac{\lambda^{p+1}}{p+1}\|u_0\|_{p+1}^{p+1} < 0.$$

Denote by $u_\lambda(t)$ the positive solution of (1.1)–(1.3) with $u(0) = \lambda u_0$. Since I is a Lyapunov functional, we have $-(1/(p+1))\|u_\lambda(t)\|_{p+1}^{p+1} \le I(u_\lambda(t)) \le I(\lambda u_0) < 0$ for $t \ge 0$. Therefore $\|u_\lambda(t)\|_{p+1}$ cannot converge to zero as $t \to \infty$. Since $\lambda < 1$, we have $\lambda u_0 \le u_0$, and therefore, $0 \le u_\lambda(t) \le u(t)$ by the comparison theorem (see (iv) of Proposition 2.3). Hence $u(t)$ never converges to zero strongly in $L^{p+1}(\Omega)$ and in $H_0^1(\Omega)$ also. Since the positive stationary solution ϕ is unique by [7], $u(t)$ converges to ϕ strongly in $H_0^1(\Omega)$ by (ii) of Proposition 2.3. The case $u_0 \equiv 0$ is reduced to the former case, since $u(t)$ is positive in Ω for any $t > 0$ and one can take $u(t)$ as an initial data and repeat the argument above. □

To prove (i) of Theorem 1.1, we recall that the unique positive solution ϕ of (1.4), (1.5) is a minimizer of I in $H_0^1(\Omega)$. Define d by

$$d := I(\phi) = \inf_{v \in H_0^1(\Omega)} I(v) < 0. \tag{2.1}$$

By [7], if $I(v) = d$, then $v = \phi$ or $-\phi$.

Proofs of (i) *and* (ii) *of Theorem* 1.1. The assertion (ii) follows immediately from Proposition 2.6. So, let us prove (i). Since the unique positive solution ϕ is isolated from all the other solutions of (1.4), (1.5) in the $H_0^1(\Omega)$ topology (see [4, Lemma 4]), one can choose $\varepsilon_0 > 0$ so small that (1.4), (1.5) has no solution in a ball $B(\phi;\varepsilon_0)$ except for ϕ, where $B(\phi;\varepsilon_0)$ is defined by

$$B(\phi;\varepsilon_0) := \{v \in H_0^1(\Omega) \colon \|v - \phi\|_{1,2} < \varepsilon_0\}. \tag{2.2}$$

Fix $\varepsilon \in (0, \varepsilon_0)$ arbitrarily. We first claim that

$$c_\varepsilon := \inf\left\{I(v) \colon v \in H_0^1(\Omega),\ \|v - \phi\|_{1,2} = \varepsilon\right\} > d.$$

Assume on the contrary that $c_\varepsilon = d$, i.e., there exists a sequence (v_n) in $H_0^1(\Omega)$ such that $\|v_n - \phi\|_{1,2} = \varepsilon$ and $I(v_n) \to d$. Then (v_n) is bounded in $H_0^1(\Omega)$, and therefore, up to a subsequence, $v_n \to v$ weakly in $H_0^1(\Omega)$ and strongly in $L^{p+1}(\Omega)$, since $H_0^1(\Omega)$ is compactly embedded in $L^{p+1}(\Omega)$. Moreover, we see that

$$\begin{aligned}\frac{1}{2}\|\nabla v_n\|_2^2 &= I(v_n) + \frac{1}{p+1}\|v_n\|_{p+1}^{p+1} \\ &\to d + \frac{1}{p+1}\|v\|_{p+1}^{p+1} \le I(v) + \frac{1}{p+1}\|v\|_{p+1}^{p+1} = \frac{1}{2}\|\nabla v\|_2^2,\end{aligned}$$

which leads to $\limsup_{n\to\infty} \|\nabla v_n\|_{1,2} \le \|\nabla v\|_{1,2}$.

This fact with the uniform convexity of $H_0^1(\Omega)$ implies that $v_n \to v$ strongly in $H_0^1(\Omega)$. Therefore $\|v - \phi\|_{1,2} = \varepsilon$ and $I(v) = d$. As mentioned after (2.1), the identity $I(v) = d$ implies that $v = \phi$ or $v = -\phi$. However, this contradicts the choice of ε_0. Hence $c_\varepsilon > d$.

From the continuity of I in $H_0^1(\Omega)$, one can take $\delta \in (0, \varepsilon)$ so small that

$$I(u_0) < \frac{d + c_\varepsilon}{2} \quad \text{for all } u_0 \in B(\phi; \delta).$$

Let $u_0 \in B(\phi; \delta)$ and $u(t)$ be any solution of (1.1)–(1.3) with $u(0) = u_0$. Here u_0 may change its sign. Then we claim that

$$u(t) \in B(\phi; \varepsilon) \quad \text{for all } t > 0,$$

which means that ϕ is stable. Assume on the contrary that $u(t_0) \in \partial B(\phi; \varepsilon)$ at some $t_0 > 0$. Then by the definition of c_ε, it follows that $I(u(t_0)) \ge c_\varepsilon$. On the other hand, the nonincrease of the energy implies that

$$I(u(t_0)) \le I(u_0) < \frac{d + c_\varepsilon}{2}.$$

However, since $c_\varepsilon > d$, it yields a contradiction. Therefore ϕ is stable.

Moreover, $u(t)$ converges to a certain solution of (1.4), (1.5) along a divergent subsequence of time by (ii) of Proposition 2.3. Recall that there is no stationary solution in $B(\phi, \varepsilon_0)$ except for ϕ. Therefore $u(t)$ converges to ϕ (without extracting any subsequence), and hence, ϕ is asymptotically stable. □

Remark 2.7 (Connectedness of domain and asymptotic stability)**.** The connectedness of the domain Ω is necessary for the asymptotic stability of the positive (negative) stationary solution. Indeed, let us consider a disconnected bounded domain Ω consisting of a sequence of disjoint balls $B_n \subset \mathbb{R}^N$ whose radii r_n rapidly converge to zero. Moreover, by using a scale invariance of solution for (1.4), due to the fact that $0 < p < 1$, one can construct a positive solution $\phi \in H_0^1(\Omega)$ and a sequence of nonnegative stationary solutions $\phi_n \in H_0^1(\Omega)$ which is positive on B_j for $1 \le j \le n$ and vanishes elsewhere. Then the positive solution ϕ is an accumulation point in $H_0^1(\Omega)$ of the sequence (ϕ_n). Hence ϕ is never asymptotically stable.

3 Instability of sign-changing stationary solutions

The purpose of this section is to prove Theorem 1.2. To this end, we employ the following lemma.

Lemma 3.1. *If v is a sign-changing solution of* (1.4), (1.5) *and satisfies* (1.10)*, then for any $\varepsilon > 0$, there exists a $w \in H_0^1(\Omega)$ such that $I(w) < I(v)$ and $\|v - w\|_{1,2} < \varepsilon$.*

Before proving this lemma, we further bring

Lemma 3.2. *Let ϕ and v be the unique positive solution and a sign-changing solution of* (1.4), (1.5)*, respectively. For each $\eta > 0$, define $\Omega(\eta) \subset \Omega$ by*

$$\Omega(\eta) := \{x \in \Omega \colon |v(x)| > \eta\}.$$

If (1.10) *holds, then*

$$\frac{1}{p}\int_\Omega |\nabla\phi|^2 dx < \lim_{\eta\to 0}\int_{\Omega(\eta)} |v|^{p-1}\phi^2 dx. \tag{3.1}$$

Here the right-hand side could be divergent.

Roughly speaking, Lemma 3.2 means that the linearized operator $-\Delta - p|v|^{p-1}$ has a negative eigenvalue. Indeed, the first eigenvalue λ_1 is formally represented as

$$\lambda_1 = \inf_{\|w\|_2=1}\int_\Omega (|\nabla w|^2 - p|v|^{p-1}w^2)dx.$$

Then (3.1) implies $\lambda_1 < 0$. However, we cannot use this expression, because it is unclear whether the Schrödinger operator $-\Delta - p|v|^{p-1}$ is well defined in $H_0^1(\Omega)$ due to the singularity of the potential on the nodal set of v.

Proof of Lemma 3.2. Recall the reverse Hölder inequality (see [1, Theorem 2.12]),

$$\left(\int_\Omega |f|^q dx\right)^{1/q}\left(\int_\Omega |g|^{q/(q-1)} dx\right)^{(q-1)/q} \le \int_\Omega |fg|\, dx \quad \text{for } 0 < q < 1,$$

provided that $|f|^q$, $|g|^{q/(q-1)}$ and $|fg|$ are integrable over Ω. Setting $q = (p+1)/2$, $f = \phi^2$ and $g = |v|^{p-1}$, we obtain

$$\int_{\Omega(\eta)} |v|^{p-1}\phi^2 dx \ge \|v\|^{p-1}_{L^{p+1}(\Omega(\eta))}\|\phi\|^2_{L^{p+1}(\Omega(\eta))},$$

since the integrands in both sides are integrable over $\Omega(\eta)$. Moreover, note that the left-hand side is nondecreasing as η is decreasing. Then letting $\eta \to 0$, we have

$$\lim_{\eta\to 0}\int_{\Omega(\eta)} |v|^{p-1}\phi^2 dx \ge \|v\|^{p-1}_{p+1}\|\phi\|^2_{p+1}.$$

By assumption (equivalently, $p\|v\|_{p+1}^{p-1} > \|\phi\|_{p+1}^{p-1}$), it follows that

$$\lim_{\eta\to 0}\int_{\Omega(\eta)} |v|^{p-1}\phi^2 dx > \frac{1}{p}\|\phi\|_{p+1}^{p+1} = \frac{1}{p}\|\nabla\phi\|_2^2.$$

Here we used the fact that $\|\phi\|_{p+1}^{p+1} = \|\nabla\phi\|_2^2$. This completes the proof. □

Proof of Lemma 3.1. Let v be a solution of (1.4), (1.5) which satisfies (1.10). Let $\varepsilon > 0$ be fixed and let us show that $w = v + \varepsilon\phi$ satisfies the assertion of Lemma 3.1. We put $f(t) := |t|^{p-1}t$ and

$$F(t) := \int_0^t f(s)ds = \frac{1}{p+1}|t|^{p+1}.$$

Since v is a solution of (1.4), (1.5), it follows that

$$\begin{aligned} I(v+\varepsilon\phi) - I(v) &= \varepsilon\int_\Omega \nabla v\cdot\nabla\phi dx + \frac{\varepsilon^2}{2}\|\nabla\phi\|_2^2 - \int_\Omega \big(F(v+\varepsilon\phi) - F(v)\big)dx \\ &= \frac{\varepsilon^2}{2}\|\nabla\phi\|_2^2 - \int_\Omega \big(F(v+\varepsilon\phi) - F(v) - \varepsilon f(v)\phi\big)dx \\ &= \frac{\varepsilon^2}{2}\|\nabla\phi\|_2^2 - \int_\Omega\int_0^\varepsilon (f(v+t\phi) - f(v))\phi\, dtdx. \end{aligned}$$

Since $\phi > 0$ in Ω, the integrand is nonnegative by the increase of f. For $\eta > 0$, set $\Omega(\eta) := \{x\in\Omega\colon |v(x)| > \eta\}$. Since $\Omega(\eta)\subset\Omega$, we find that

$$I(v+\varepsilon\phi) - I(v) \le \frac{\varepsilon^2}{2}\|\nabla\phi\|_2^2 - \int_{\Omega(\eta)}\int_0^\varepsilon (f(v+t\phi) - f(v))\phi\, dtdx.$$

Divide both sides by ε^2 and let $\varepsilon\to 0$. In the right-hand side, it follows that

$$\begin{aligned} &\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{\Omega(\eta)}\int_0^\varepsilon (f(v+t\phi) - f(v))\,\phi\, dtdx \\ &= \lim_{\varepsilon\to 0}\frac{p}{\varepsilon^2}\int_{\Omega(\eta)}\int_0^\varepsilon \left(\int_0^1 |v+\theta t\phi|^{p-1}d\theta\right) t\phi^2\, dtdx \\ &\ge \lim_{\varepsilon\to 0}\frac{p}{2}\int_{\Omega(\eta)} \frac{\phi^2}{|v|^{1-p} + \varepsilon^{1-p}|\phi|^{1-p}}dx = \frac{p}{2}\int_{\Omega(\eta)} |v|^{p-1}\phi^2 dx. \end{aligned}$$

Then we obtain

$$\limsup_{\varepsilon\to 0}\frac{I(v+\varepsilon\phi) - I(v)}{\varepsilon^2} \le \frac{1}{2}\|\nabla\phi\|_2^2 - \frac{p}{2}\int_{\Omega(\eta)}|v|^{p-1}\phi^2 dx.$$

Letting $\eta\to 0$ and using Lemma 3.2, we have

$$\limsup_{\varepsilon\to 0}\frac{I(v+\varepsilon\phi) - I(v)}{\varepsilon^2} \le \frac{1}{2}\|\nabla\phi\|_2^2 - \frac{p}{2}\lim_{\eta\to 0}\int_{\Omega(\eta)}|v|^{p-1}\phi^2 dx < 0.$$

Therefore, for $\varepsilon > 0$ small enough, we obtain $I(v+\varepsilon\phi) < I(v)$. □

Proof of Theorem 1.2. Let $\varepsilon > 0$ be fixed. By Lemma 3.1, we can choose $u_0 \in H_0^1(\Omega)$ such that $I(u_0) < I(v)$ and $\|v - u_0\|_{1,2} < \varepsilon$. Every solution $u(t)$ of (1.1)–(1.3) with the initial data $u(0) = u_0$ satisfies $I(u(t)) \le I(u(0)) < I(v)$ for $t > 0$. Then $u(t)$ cannot converge to v. Consequently, v is not asymptotically stable.

In addition, we assume that v is isolated from any $w \in \mathcal{S}$ satisfying $I(w) < I(v)$. Then there is an $r > 0$ such that

$$\overline{B(v;r)} \cap \{w \in \mathcal{S} \colon I(w) < I(v)\} = \emptyset, \tag{3.2}$$

where $B(v;r)$ has been defined by (2.2). Let u_0 and $u(t)$ be as above. We assume that u_0 is in $B(v;r)$. Then $u(t) \notin B(v;r)$ for all t large enough. Indeed, if $u(t_n) \in B(v;r)$ with a sequence t_n diverging to ∞, then $u(t_{n'})$ converges to a stationary solution u_∞ strongly in $H_0^1(\Omega)$ along a subsequence $t_{n'}$ (see (ii) of Proposition 2.3). Then $u_\infty \in \overline{B(v;r)}$. Since $I(u(t)) < I(v)$, it holds that $I(u_\infty) < I(v)$. This contradicts (3.2). Therefore $u(t) \notin B(v;r)$ for all t large enough, and hence, v is unstable. □

Remark 3.3 (An application to the Porous Medium Equation)**.** By applying Lemma 3.1 developed here, one can also verify the *instability of sign-changing asymptotic profiles* of solutions to the Cauchy-Dirichlet problem for the *Porous Medium Equation*

$$\left.\begin{array}{c} \partial_s \left(|u|^{p-1}u\right) = \Delta u \ \text{ in } \ \Omega \times (0,\infty), \\ u = 0 \ \text{ on } \ \partial\Omega \times (0,\infty), \quad u(\cdot,0) = u_0 \ \text{ in } \ \Omega, \end{array}\right\} \tag{3.3}$$

where $0 < p < 1$ and Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. It is well known that u decays at the power rate $s^{-1/(1-p)}$ as $s \to \infty$, and moreover, for any sequence $s_n \to \infty$, up to a subsequence, there exists $w \in H_0^1(\Omega) \cap L^\infty(\Omega) \setminus \{0\}$ such that

$$(s_n + 1)^{1/(1-p)} u(\cdot, s_n) \to w \quad \text{ strongly in } H_0^1(\Omega) \cap L^\infty(\Omega),$$

where w is called an *asymptotic profile* of u as $s \to \infty$ (see, e.g., [31]).

By change of variables, $z(x,t) = (s+1)^{1/(1-p)} u(\cdot, s)$, $t = \log(s+1)$, Equation (3.3) is transformed to

$$\left.\begin{array}{c} \partial_t \left(|z|^{p-1}z\right) = \Delta z + a|z|^{p-1}z \ \text{ in } \ \Omega \times (0,\infty), \\ z = 0 \ \text{ on } \ \partial\Omega \times (0,\infty), \quad z(\cdot,0) = u_0 \ \text{ in } \ \Omega, \end{array}\right\} \tag{3.4}$$

with a constant $a := p/(1-p) > 0$. Then every asymptotic profile w turns out to be a nontrivial solution of the elliptic problem

$$-\Delta w = a|w|^{p-1}w \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega.$$

Hence $v = a^{-1/(1-p)} w$ solves (1.4), (1.5). The notions of stability and instability of profiles are defined by those of stationary solutions for (3.4) in a similar way to Definition 2.5.

Due to Lemma 3.1, we assure that

Theorem 3.4 (Instability of sign-changing asymptotic profiles)**.** *Let w be a sign-changing asymptotic profile of a solution of the Cauchy-Dirichlet problem for* (3.3) (*hence $w = a^{1/(1-p)}v$ with some nontrivial solution v of* (1.4), (1.5)). *Assume that* (1.10) *holds for v. Then w is not asymptotically stable. Moreover, if v is isolated from all $u \in \mathcal{S}$ satisfying $I(u) < I(v)$, then w is unstable.*

Proof. We first remark that (3.4) has a Lyapunov functional $E : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$E(u) := \frac{1}{2}\|\nabla u\|_2^2 - \frac{a}{p+1}\|u\|_{p+1}^{p+1} = a^{2/(1-p)} I(a^{-1/(1-p)}u) \quad \text{for } u \in H_0^1(\Omega).$$

Then by using Lemma 3.1, for any $\varepsilon > 0$, one can take $v_\varepsilon \in H_0^1(\Omega)$ such that $\|v - v_\varepsilon\|_{1,2} < \varepsilon$ and $I(v_\varepsilon) < I(v)$. Set $w_\varepsilon := a^{1/(1-p)}v_\varepsilon$. Then it follows that $\|w - w_\varepsilon\|_{1,2} < \varepsilon a^{1/(1-p)}$ and $E(w_\varepsilon) < E(w)$. Let z be a solution of the Cauchy-Dirichlet problem for (3.4) with the initial data w_ε. Then since $E(z(t))$ is nonincreasing in t, z never converges to w strongly in $H_0^1(\Omega)$ as $t \to \infty$. Therefore w is not asymptotically stable. The instability part can be also proved as in Theorem 1.2. □

4 Linearized operator and its eigenvalue problem

In what follows, we denote the unique positive solution of (1.4), (1.5) by ϕ. The purpose of this section is to prove that the first eigenvalue of the linearized operator $-\Delta - p\phi^{p-1}$ is positive and the corresponding eigenfunction belongs to $C^2(\overline{\Omega})$. This result will be employed to prove the exponential stability of ϕ in later sections.

Proposition 4.1. *The linearized operator $-\Delta - p\phi^{p-1}$ is self-adjoint and has a compact resolvent in $L^2(\Omega)$, and moreover, its first eigenvalue is positive.*

Proof. We remark that the potential $p\phi^{p-1}$ has singularity on the boundary because $p < 1$ and $\phi = 0$ on $\partial\Omega$. Define the distance function $\rho(x)$ by

$$\rho(x) := \text{dist}(x, \partial\Omega) = \inf\{|x - y| : y \in \partial\Omega\}. \tag{4.1}$$

By the strong maximum principle, $\phi > 0$ in Ω and the outward normal derivative $\partial\phi/\partial\nu$ is strictly negative on $\partial\Omega$. Furthermore, since $\partial\Omega$ is smooth, $\rho(x)$ is smooth near the boundary and $\partial\rho/\partial\nu = -1$ on $\partial\Omega$. Then there exists a $C > 0$ such that $\rho(x) \le C\phi(x)$ in Ω. Therefore by Hardy's inequality (see, e.g., [6, p.313] or [10, 23, 32]), one has

$$\|u/\phi\|_2 \le C\|\nabla u\|_2 \quad \text{for all } u \in H_0^1(\Omega) \tag{4.2}$$

for some constant $C > 0$ independent of u. Define three operators A, B, T in $L^2(\Omega)$ by

$$T := A + B, \quad Au := -\Delta u, \quad Bu := -p\phi(x)^{p-1}u$$

with domains $D(T) = D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $D(B) = \{u \in L^2(\Omega) \colon Bu \in L^2(\Omega)\}$. Then $T = -\Delta - p\phi^{p-1}$. By (4.2), we have

$$\|Bu\|_2^2 = p^2 \int_\Omega \phi^{-2(1-p)} u^2 dx \le p^2 \|\phi\|_\infty^{2p} \int_\Omega u^2 \phi^{-2} dx \le C^2 \|\nabla u\|_2^2 \tag{4.3}$$

for all $u \in H_0^1(\Omega)$, where $C > 0$ is independent of u. Hence one can deduce that $D(A) \subset D(B)$. It is well known that A is self-adjoint, and moreover, B is obviously symmetric. We use the the perturbation theory due to Kato [22, Theorem 4.3, Chap. V], that is, if there exist constants $\varepsilon \in [0, 1)$ and $C > 0$ such that

$$\|Bu\|_2 \le \varepsilon \|Au\|_2 + C\|u\|_2 \quad \text{for } u \in D(A), \tag{4.4}$$

then $A + B$ is self-adjoint. Denote the $L^2(\Omega)$-inner product of u and v by (u, v). Then we use the Schwartz inequality to get

$$\|\nabla u\|_2 = \sqrt{(Au, u)} \le \|Au\|_2^{1/2} \|u\|_2^{1/2} \le (\varepsilon/2)\|Au\|_2 + (\varepsilon^{-1}/2)\|u\|_2,$$

for any $\varepsilon > 0$ and $u \in D(A)$. Therefore by (4.3) we obtain

$$\|Bu\|_2 \le (C/2)(\varepsilon \|Au\|_2 + \varepsilon^{-1}\|u\|_2) \quad \text{for } u \in D(A).$$

Fix $\varepsilon > 0$ satisfying $\varepsilon C/2 < 1$. Then B satisfies (4.4) and $T = A + B$ is self-adjoint.

We next claim that the operator T has a compact resolvent. Indeed, let $\lambda < 0$ be a number to be determined. Let $u \in D(T)$ and set $v = (T - \lambda)u$. Multiply both sides by u and use (4.3) and the Poincaré inequality. Then we have

$$\begin{aligned} -\lambda \|u\|_2^2 + \|\nabla u\|_2^2 &\le \|Bu\|_2 \|u\|_2 + \|v\|_2 \|u\|_2 \\ &\le C\|\nabla u\|_2 \|u\|_2 + C\|\nabla u\|_2 \|v\|_2 \le \frac{1}{2}\|\nabla u\|_2^2 + C_0 \left(\|u\|_2^2 + \|v\|_2^2\right) \end{aligned}$$

with some constant $C_0 > 0$ independent of u and v. Take λ so small that $\lambda \le -C_0$. Then

$$(-\lambda - C_0)\|u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 \le C_0 \|v\|_2^2. \tag{4.5}$$

Particularly, if $v = 0$, then $u = 0$. Thus $T - \lambda$ is injective. Since $T - \lambda$ is also self-adjoint, it is surjective. Moreover, $(T - \lambda)^{-1}$ is bounded in $L^2(\Omega)$. Therefore $(-\infty, -C_0]$ is included in the resolvent set of T. As for the compactness of $(T - \lambda)^{-1}$ for any $\lambda \le -C_0$, let (v_n) be a bounded sequence in $L^2(\Omega)$ and let $u_n := (T - \lambda)^{-1} v_n$. Then by (4.5), (u_n) is bounded in $H_0^1(\Omega)$. Since $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, (u_n) is precompact in $L^2(\Omega)$. Hence $(T - \lambda)^{-1}$ is compact in $L^2(\Omega)$ when $\lambda \le -C_0$. Thus the spectrum of T consists only of discrete real eigenvalues and each eigenspace is finite dimensional.

We rewrite (1.4) for the positive solution ϕ as $(-\Delta - \phi^{p-1})\phi = 0$. Since $\phi > 0$, the operator $-\Delta - \phi^{p-1}$ has the zero first eigenvalue. By the order relation $-p\phi^{p-1} > -\phi^{p-1}$ of the potentials, the first eigenvalue of $-\Delta - p\phi^{p-1}$ is strictly greater than that of $-\Delta - \phi^{p-1}$. Consequently, the first eigenvalue of $-\Delta - p\phi^{p-1}$ is positive. □

We denote the first eigenvalue of $-\Delta - p\phi^{p-1}$ by μ and the corresponding eigenfunction by $\psi(x)$. To determine ψ uniquely, we impose the condition, $\psi > 0$ in Ω and $\|\nabla\psi\|_2 = 1$. Therefore we have

$$(-\Delta - p\phi^{p-1})\psi = \mu\psi \quad \text{in } \Omega, \qquad \psi = 0 \quad \text{on } \partial\Omega, \tag{4.6}$$

$$\psi > 0 \quad \text{in } \Omega, \qquad \|\nabla\psi\|_2 = 1. \tag{4.7}$$

For each $k \in \mathbb{N}$ and $\theta \in (0,1)$, let $C^{k,\theta}(\overline{\Omega})$ denote the set of all $C^k(\overline{\Omega})$ functions whose k-th derivatives are Hölder continuous with exponent θ. Although the potential $p\phi^{p-1}$ in (4.6) is singular on $\partial\Omega$, ψ is of class $C^{2,p}(\overline{\Omega})$. We shall prove it in the next proposition.

Proposition 4.2. *The principal eigenfunction ψ belongs to $C^{2,p}(\overline{\Omega})$.*

To prove Proposition 4.2, we consider the linear problem with a singular weight function $h(x)$,

$$-\Delta u = hu \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{4.8}$$

Lemma 4.3. *Let $0 < a < 1$, $q \in (N, \infty)$ and $h(x)$ be a function satisfying $h\rho^a \in L^q(\Omega)$, where $\rho(x)$ has been defined by* (4.1). *Then there exist constants $C, \xi > 0$ such that*

$$\|u\|_{W^{2,q}(\Omega)} \le C\|h\rho^a\|_q^{\xi}\|\nabla u\|_2,$$

provided that u is in $W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ and satisfies (4.8). *Here C and ξ are independent of u and h.*

To prove the lemma above, recall the Gagliardo-Nirenberg inequality (see [15, p.27, Theorem 10.1]) of the form,

$$\|D^j u\|_p \le C\|u\|_{W^{m,r}(\Omega)}^{\alpha}\|u\|_q^{1-\alpha}, \tag{4.9}$$

if $m - j - N/r$ is not nonnegative integer, j, m are integers and α is a real number such that $0 \le j < m$, $j/m \le \alpha \le 1$ and

$$\frac{1}{p} = \frac{j}{N} + \alpha\left(\frac{1}{r} - \frac{m}{N}\right) + (1-\alpha)\frac{1}{q}.$$

Proof of Lemma 4.3. We use the same idea as in [21, Proposition 3.9]. Since $u \in W^{2,q}(\Omega)$ with $q > N$, u belongs to $C^1(\overline{\Omega})$. Using [21, Lemma 3.8], by (4.8), we get

$$\|\Delta u\|_q = \|hu\|_q = \|h\rho^a|u|^{(1-a)}(|u|/\rho)^a\|_q \le \|h\rho^a\|_q\|u\|_\infty^{(1-a)}\|\nabla u\|_\infty^a. \tag{4.10}$$

Hereafter C denotes various positive constants independent of u and h. Since $\|u\|_\infty \le C\|\nabla u\|_\infty$ by the Poincaré inequality and $\|u\|_{W^{2,q}(\Omega)} \le C\|\Delta u\|_q$ by the elliptic regularity theorem, (4.10) is reduced to

$$\|u\|_{W^{2,q}(\Omega)} \le C\|h\rho^a\|_q\|\nabla u\|_\infty. \tag{4.11}$$

By (4.9), we have $\|v\|_\infty \le C\|v\|_{W^{1,q}(\Omega)}^{\alpha}\|v\|_2^{1-\alpha}$ for all $v \in W^{1,q}(\Omega)$, with $\alpha = Nq/(Nq+2q-2N)$. Substituting $\partial u/\partial x_i$ for v, we have

$$\|\nabla u\|_\infty \le C\|u\|_{W^{2,q}(\Omega)}^{\alpha}\|\nabla u\|_2^{1-\alpha}. \tag{4.12}$$

Combining (4.11) with (4.12), we obtain $\|u\|_{W^{2,q}(\Omega)} \le C\|h\rho^a\|_q\|u\|_{W^{2,q}(\Omega)}^{\alpha}\|\nabla u\|_2^{1-\alpha}$, or equivalently $\|u\|_{W^{2,q}(\Omega)} \le C\|h\rho^a\|_q^{\xi}\|\nabla u\|_2$, where $\xi := \frac{1}{1-\alpha} = \frac{(N+2)q-2N}{2(q-N)}$. □

Proof of Proposition 4.2. We first show that $\psi \in W^{2,q}(\Omega)$ for any $1 \le q < \infty$ by using the approximation of the eigenvalue. Let us consider

$$-\Delta u = p(\phi(x)+\varepsilon)^{p-1}u + \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{4.13}$$

with a parameter $\varepsilon > 0$. Let $\lambda_1(\varepsilon)$ denote the first eigenvalue of (4.13). Let us prove that $\lambda_1(\varepsilon) \to \mu$ as $\varepsilon \to 0$. Since $(\phi+\varepsilon)^{p-1} < \phi^{p-1}$, it holds that $\lambda_1(\varepsilon) > \mu$. Since ψ is an eigenfunction, it belongs to $D(T) = H^2(\Omega) \cap H_0^1(\Omega)$. Furthermore, we use (4.6) to get

$$\mu\|\psi\|_2^2 = \int_\Omega \left(|\nabla\psi|^2 - p\phi^{p-1}\psi^2\right) dx.$$

Since $\lambda_1(\varepsilon)$ is the infimum of the Rayleigh quotient, we have

$$\lambda_1(\varepsilon) \le \|\psi\|_2^{-2}\int_\Omega \left(|\nabla\psi|^2 - p(\phi+\varepsilon)^{p-1}\psi^2\right) dx. \tag{4.14}$$

Since $0 \le p(\phi+\varepsilon)^{p-1}\psi^2 \le p\phi^{p-1}\psi^2$ and $p\phi^{p-1}\psi^2 \in L^1(\Omega)$, the Lebesgue dominated convergence theorem shows that the right-hand side of (4.14) converges to

$$\|\psi\|_2^{-2}\int_\Omega \left(|\nabla\psi|^2 - p\phi^{p-1}\psi^2\right) dx = \mu.$$

Thus $\lambda_1(\varepsilon) \to \mu$ as $\varepsilon \to 0$.

Let ψ_ε denote the eigenfunction corresponding to $\lambda_1(\varepsilon)$ such that

$$\psi_\varepsilon(x) > 0, \quad \|\nabla\psi_\varepsilon\|_2 = 1. \tag{4.15}$$

Then ψ_ε is uniquely determined and satisfies $-\Delta\psi_\varepsilon = h_\varepsilon\psi_\varepsilon$ in Ω and $\psi_\varepsilon = 0$ on $\partial\Omega$, where $h_\varepsilon(x) := p(\phi(x)+\varepsilon)^{p-1} + \lambda_1(\varepsilon)$. Let $q \in (N,\infty)$ be arbitrarily fixed. Let $\rho(x)$ be the distance function defined by (4.1). Recall that $\rho(x) \le C\phi(x)$ in Ω for some constant $C > 0$. This shows that $\|h_\varepsilon(x)\rho^{1-p}\|_\infty$ is bounded as $\varepsilon \to 0$. Using (4.15) and Lemma 4.3 with $a = 1-p$, we conclude that $\|\psi_\varepsilon\|_{W^{2,q}(\Omega)}$ is bounded as $\varepsilon \to 0$. Therefore, up to a subsequence, ψ_ε converges to a certain function ψ_0 weakly in $W^{2,q}(\Omega)$ and strongly in $C^{1,\theta}(\overline{\Omega})$ with $\theta \in [0, 1-N/q)$. Since $\|\nabla\psi_0\|_2 = 1$ by (4.15), ψ_0 does not identically vanish. Hence it becomes an eigenfunction corresponding to the first eigenvalue μ, that is, ψ_0 coincides with ψ defined by (4.6), (4.7). Thus $\psi \in W^{2,q}(\Omega)$ for $q \in (N,\infty)$. This assertion is also valid for all $q \in [1, \infty)$, because Ω is bounded. Therefore $\psi \in C^{1,\theta}(\overline{\Omega})$ for all $\theta \in (0, 1)$ by the Sobolev embedding.

We finally show that $\psi \in C^{2,p}(\overline{\Omega})$. To this end, we first claim that $f(x) := \phi(x)^{p-1}\psi(x)$ belongs to $C^{0,p}(\overline{\Omega})$, that is, there exists a constant $C > 0$ such that $|f(x) - f(y)| \le C|x - y|^p$ for all $x, y \in \overline{\Omega}$. We first note by the Hopf maximum principle and the $C^1(\overline{\Omega})$-regularity of ϕ, ψ that $c_1\rho(x) \le \phi(x)$, $\psi(x) \le c_2\rho(x)$ in $\overline{\Omega}$ for some $c_1, c_2 > 0$. In the rest of this proof, we denote by C a constant which is independent of x, y and may vary from line to line. Let $x, y \in \overline{\Omega}$ be fixed. In case $\rho(x) \le |x - y|$, we observe that $\rho(y) \le |x - y| + \rho(x) \le 2|x - y|$. It follows that

$$|f(x) - f(y)| \le |f(x)| + |f(y)| \le c_1^{p-1}c_2 \left(\rho(x)^p + \rho(y)^p\right) \le c_1^{p-1}c_2(1 + 2^p)|x - y|^p.$$

One can obtain the same inequality for the case $\rho(y) \le |x - y|$ as well. In case $|x - y| \le \rho(x)$ and $|x - y| \le \rho(y)$, if $\phi(x) = \phi(y)$, then the $C^1(\overline{\Omega})$-regularity of ψ shows

$$|f(x) - f(y)| = \phi(x)^{p-1}|\psi(x) - \psi(y)| \le C\rho(x)^{p-1}|x - y| \le C|x - y|^p.$$

If $\phi(y) < \phi(x)$, then the $C^1(\overline{\Omega})$-regularity of ϕ, ψ and Mean-Value Theorem imply

$$\begin{aligned}|f(x) - f(y)| &\le \phi(x)^{p-1}|\psi(x) - \psi(y)| + |\phi(x)^{p-1} - \phi(y)^{p-1}|\psi(y) \\ &\le C\rho(x)^{p-1}|x - y| + (1 - p)\xi^{p-2}|\phi(x) - \phi(y)|\psi(y),\end{aligned}$$

where ξ is a constant satisfying $\phi(y) < \xi < \phi(x)$. Since $\xi > \phi(y) \ge c_1\rho(y)$, we obtain

$$|f(x) - f(y)| \le C\rho(x)^{p-1}|x - y| + C\rho(y)^{p-1}|x - y| \le 2C|x - y|^p.$$

The inequality above also holds if $\phi(x) < \phi(y)$. Thus f belongs to $C^{0,p}(\overline{\Omega})$. Therefore $-\Delta\psi = (p\phi^{p-1} + \mu)\psi \in C^{0,p}(\overline{\Omega})$. By the Schauder estimate, ψ belongs to $C^{2,p}(\overline{\Omega})$. □

5 Exponential convergence of positive solutions

This section is devoted to proving (iii) and (iv) of Theorem 1.1. The method of proofs relies on a super-subsolution method along with comparison principle as well as the results obtained in Section 4 for the linearized operator.

In order to construct super- and subsolutions, recall the eigenvalue problem (4.6), (4.7) for the linearized operator $-\Delta - p\phi^{p-1}$ and denote by $\mu > 0$ and $\psi > 0$ the first eigenvalue and its eigenfunction, respectively (see Propositions 4.1 and 4.2). Since ψ satisfies $-\Delta\psi = (p\phi^{p-1} + \mu)\psi \ge 0$ in Ω, the Hopf maximum principle with $\psi \in C^2(\overline{\Omega})$ implies that $\partial\psi/\partial\nu$ is negative on $\partial\Omega$. Let us first construct a supersolution. For $c \in \mathbb{R}$, we define

$$U(x, t; c) := \phi(x) + ce^{-\mu t}\psi(x).$$

Since $\phi, \psi > 0$ in Ω, $\phi, \psi \in C^2(\overline{\Omega})$ and $\partial\phi/\partial\nu, \partial\psi/\partial\nu < 0$ on $\partial\Omega$, there exists a constant $c_0 > 0$ such that

$$c_0 \leq \frac{\phi(x)}{\psi(x)} \quad \text{in } \Omega. \tag{5.1}$$

Lemma 5.1. *$U(x,t;c)$ is a positive supersolution of* (1.1), (1.2) *if $-c_0 < c < \infty$.*

Proof. Let $-c_0 < c < \infty$. Then $U(x,t;c) > 0$. We compute $U_t = -\mu c e^{-\mu t}\psi$ and $-\Delta U = \phi^p + ce^{-\mu t}(p\phi^{p-1} + \mu)\psi$. Hence it follows that $U_t - \Delta U - U^p = \phi^p + cpe^{-\mu t}\phi^{p-1}\psi - (\phi + ce^{-\mu t}\psi)^p$. Let us show that the right-hand side is nonnegative. Put $a := \phi$, $s := ce^{-\mu t}\psi$ and define $f(s) := a^p + pa^{p-1}s - (a+s)^p$. From an easy computation, it follows that $f(0) = f'(0) = 0$ and $f''(s) > 0$ in $(-a, \infty)$. Thus $f(s) > 0$ for $s \in (-a,\infty) \setminus \{0\}$, which shows $U_t - \Delta U - U^p \geq 0$. The proof is complete. $\square$

Let λ_1 be the first eigenvalue of $-\Delta$ and ϕ_1 an eigenfunction, that is,

$$-\Delta\phi_1 = \lambda_1\phi_1, \quad \phi_1 > 0 \quad \text{in } \Omega, \qquad \phi_1 = 0 \quad \text{on } \partial\Omega.$$

From now on, we fix ϕ_1. One can derive the following lemma from a direct computation.

Lemma 5.2. *For $\alpha, \beta \in \mathbb{R}$ and $\xi \in C^1[0,\infty)$, put*

$$v(x,t) := \phi(x) + \alpha\xi(t)\psi(x) + \beta e^{-2\mu t}\phi_1(x),$$

where $|\alpha|$ and $|\beta|$ are assumed to be so small that $v(x,t) > 0$ in $\Omega \times (0,\infty)$. Then

$$\begin{aligned} v_t - \Delta v - v^p = {} & \alpha(\xi' + \mu\xi)\psi + \beta(\lambda_1 - 2\mu)e^{-2\mu t}\phi_1 \\ & + \phi^p + \alpha p\xi\phi^{p-1}\psi - (\phi + \alpha\xi\psi + \beta e^{-2\mu t}\phi_1)^p. \end{aligned}$$

Since ϕ, ψ and ϕ_1 are positive and belong to $C^2(\overline{\Omega})$, and furthermore, their normal derivatives are negative on $\partial\Omega$, there exist constants $c_1, c_2 > 0$ such that

$$c_1 \leq \frac{\psi(x)}{\phi(x)}, \frac{\phi_1(x)}{\phi(x)} \leq c_2 \quad \text{in } \Omega. \tag{5.2}$$

To make a subsolution, we set

$$\xi(t) := \mu(e^{\mu t} + 1)^{-1}, \tag{5.3}$$

which is a solution of the Bernoulli differential equation,

$$\xi' + \mu\xi = \xi^2, \quad \xi(0) = \mu/2. \tag{5.4}$$

Then, we find by (5.3) that

$$(\mu/2)e^{-\mu t} \leq \xi(t) \leq \mu e^{-\mu t}. \tag{5.5}$$

For $\varepsilon > 0$, we define

$$V(x,t;\varepsilon) := \phi(x) - \varepsilon^2\xi(t)\psi(x) + \varepsilon^3 e^{-2\mu t}\phi_1(x). \tag{5.6}$$

Lemma 5.3. *$V(x,t;\varepsilon)$ is a positive subsolution of* (1.1), (1.2) *for $\varepsilon > 0$ small enough.*

To show the lemma above, we prepare a couple of elementary inequalities.

Lemma 5.4. *Let $0 < p < 1$ and $a > 0$. Then it holds that*

$$a^p + pa^{p-1}x - (a+x+y)^p < -pa^{p-1}y + \frac{1}{2}p(1-p)a^{p-2}(x+y)^2, \tag{5.7}$$

when $x, y > 0$. Moreover,

$$a^p - pa^{p-1}x - (a-x+y)^p < -pa^{p-1}y + 2^{1-p}p(1-p)a^{p-2}(x-y)^2, \tag{5.8}$$

when $-a/2 < -x+y < 0$.

Proof. First, we define $f(t) := a^p + pa^{p-1}t - (a+t)^p - \frac{1}{2}p(1-p)a^{p-2}t^2$. We easily compute that $f(0) = f'(0) = 0$ and $f''(t) < 0$ for $t > 0$. Therefore $f(t) < 0$ in $(0,\infty)$. Then $f(x+y) < 0$ means (5.7). Next, we put $g(t) := a^p + pa^{p-1}t - (a+t)^p - 2^{1-p}p(1-p)a^{p-2}t^2$. Computing the second derivative of g, we see that $g(t) < 0$ for $-a/2 < t < 0$. Putting $t = -x+y$, we get (5.8). The proof is complete. □

Proof of Lemma 5.3. By (5.2) and (5.5), we have $V(x,t;\varepsilon) \geq (1 - \varepsilon^2\mu c_2)\,\phi(x) > 0$, if $0 < \varepsilon < (\mu c_2)^{-1/2}$. Applying Lemma 5.2 with $\alpha = -\varepsilon^2$, $\beta = \varepsilon^3$ and using (5.4), we have

$$\begin{aligned} V_t - \Delta V - V^p = &-\varepsilon^2\xi^2\psi + \varepsilon^3(\lambda_1 - 2\mu)e^{-2\mu t}\phi_1 + \phi^p \\ &- \varepsilon^2 p\xi\phi^{p-1}\psi - (\phi - \varepsilon^2\xi\psi + \varepsilon^3 e^{-2\mu t}\phi_1)^p. \end{aligned}$$

Employing (5.8) with $a = \phi$, $x = \varepsilon^2\xi\psi$ and $y = \varepsilon^3 e^{-2\mu t}\phi_1$ and noting by (5.2) and (5.5) that $-x+y = -\varepsilon^2(\xi\psi - \varepsilon e^{-2\mu t}\phi_1) \in (-a/2, 0)$ for $\varepsilon > 0$ enough close to 0, we have

$$\begin{aligned} V_t - \Delta V - V^p \leq &-\varepsilon^2\xi^2\psi + \varepsilon^3|\lambda_1 - 2\mu|e^{-2\mu t}\phi_1 - p\varepsilon^3 e^{-2\mu t}\phi^{p-1}\phi_1 \\ &+ 2^{1-p}p(1-p)\phi^{p-2}(\varepsilon^2\xi\psi - \varepsilon^3 e^{-2\mu t}\phi_1)^2. \end{aligned}$$

Using this inequality, (5.2) and (5.5), we get

$$\begin{aligned} V_t - \Delta V - V^p &\leq -(\mu^2/4)\varepsilon^2 c_1 e^{-2\mu t}\phi + \varepsilon^3 c_2|\lambda_1 - 2\mu|e^{-2\mu t}\phi \\ &\quad - pc_1\varepsilon^3 e^{-2\mu t}\phi^p + 2^{1-p}p(1-p)\varepsilon^4\phi^p e^{-2\mu t}(\mu c_2 - \varepsilon c_1 e^{-\mu t})^2 \\ &\leq -\varepsilon^2 e^{-2\mu t}\phi\left\{\mu^2 c_1/4 - \varepsilon c_2|\lambda_1 - 2\mu|\right\} \\ &\quad - \varepsilon^3 p e^{-2\mu t}\phi^p\left\{c_1 - 2^{1-p}\varepsilon(1-p)(\mu c_2)^2\right\}) < 0, \end{aligned}$$

provided that $\varepsilon > 0$ is small enough. Consequently, V is a subsolution. □

Remark 5.5. In (5.6), we cannot replace the coefficients ε^2 and ε^3 by ε and ε^2, respectively. Such a replacement makes Lemma 5.3 invalid. Indeed, instead of $V(x,t;\varepsilon)$, let us use

$$\widetilde{V}(x,t;\alpha,\beta) := \phi(x) - \alpha\xi(t)\psi(x) + \beta e^{-2\mu t}\phi_1(x),$$

with $\alpha, \beta > 0$ and $\xi(t)$ defined by (5.3). From the same computation as in the proof of Lemma 5.3, we find that

$$\begin{aligned}\widetilde{V}_t - \Delta\widetilde{V} - \widetilde{V}^p \le& -\alpha e^{-2\mu t}\phi\left\{\mu^2 c_1/4 - (\beta/\alpha)c_2|\lambda_1 - 2\mu|\right\} \\ &- p\beta e^{-2\mu t}\phi^p\left\{c_1 - 2^{1-p}(1-p)(\alpha^2/\beta)\left(\mu c_2 - (\beta/\alpha)e^{-\mu t}c_1\right)^2\right\}.\end{aligned}$$

Hence the right-hand side is negative if both β/α and α^2/β are small enough. So, we cannot choose $\alpha = \varepsilon$ and $\beta = \varepsilon^2$. On the other hand, this condition is fulfilled if we take $\alpha = \varepsilon$ and $\beta = \varepsilon^\nu$ with $1 < \nu < 2$ and $\varepsilon > 0$ is small enough. In the proof of Lemma 5.3 above, we simply chose $\alpha = \varepsilon^2$, $\beta = \varepsilon^3$ and defined $V(x,t;\varepsilon)$ by the one parameter ε.

Let $\varepsilon_0 > 0$ be so small that $V(x,t;\varepsilon)$ is a positive subsolution for $\varepsilon \in (0, \varepsilon_0)$.

Proposition 5.6. *Let $u_0(x)$ satisfy*

$$0 < \phi(x) - (\mu/2)\varepsilon^2\psi(x) + \varepsilon^3\phi_1(x) \le u_0(x) \le \phi(x) + c\psi(x) \quad \text{in } \Omega, \tag{5.9}$$

with some $\varepsilon \in (0, \varepsilon_0)$ and $c \in (-c_0, \infty)$. Here c_0 is defined by (5.1). *Then the positive solution $u(x,t)$ of* (1.1)–(1.3) *with the initial data $u_0(x)$ satisfies*

$$0 < \phi(x) - \varepsilon^2\xi(t)\psi(x) + \varepsilon^3 e^{-2\mu t}\phi_1(x) \le u(x,t) \le \phi(x) + ce^{-\mu t}\psi(x), \tag{5.10}$$

for $t \ge 0$ and $x \in \Omega$. Therefore, there exists a $C > 0$ such that

$$\|u(t) - \phi\|_\infty \le Ce^{-\mu t}. \tag{5.11}$$

Proof. Let $U(x,t;c)$ and $V(x,t;\varepsilon)$ be defined as before. Then (5.9) means that $V(x,0;\varepsilon) \le u(x,0) \le U(x,0;c)$. By the comparison theorem ((iv) of Proposition 2.3), we have $V(x,t;\varepsilon) \le u(x,t) \le U(x,t;c)$ for $t \ge 0$, which is just (5.10). By (5.10) with (5.5), we get (5.11). □

Even if we remove assumption (5.9) from Proposition 5.6, the conclusion (5.11) remains true. To prove this, we define

$$a := \|\phi\|_\infty^{p-1}, \quad \zeta(t) := \left(1 - e^{-(1-p)at}\right)^{1/(1-p)},$$

which solves the Bernoulli differential equation, $\zeta' = a(\zeta^p - \zeta)$ and $\zeta(0) = 0$.

We shall show that $\zeta(t)\phi(x)$ becomes a positive subsolution of (1.1)–(1.3), which is smaller than any positive solution of (1.1)–(1.3). In particular, it holds that $\zeta(t)\phi(x) \le \hat{u}(x,t)$, where $\hat{u}(x,t)$ is the unique positive solution with initial data $\hat{u}(x,0) \equiv 0$ (see Remark 2.4 (ii)).

Lemma 5.7. *For any positive solution* $u(x,t)$ *of* (1.1) *and* (1.2)*, it holds that*

$$\zeta(t)\phi(x) \le u(x,t) \quad \textit{in } \Omega \times (0,\infty).$$

Proof. Put $v(x,t) := \zeta(t)\phi(x)$. Then we have

$$\begin{aligned} v_t - \Delta v - v^p &= \zeta'\phi + \zeta\phi^p - \zeta^p\phi^p \\ &= a(\zeta^p - \zeta)\phi + (\zeta - \zeta^p)\phi^p = (\zeta^p - \zeta)\phi(a - \phi^{p-1}) \le 0, \end{aligned}$$

because $0 < \zeta(t) < 1$ and $\phi(x)^{p-1} \ge \|\phi\|_\infty^{p-1} = a$. Therefore v is a subsolution. Since $v(x,0) \equiv 0 \le u_0(x)$, the comparison theorem ensures that $v(x,t) \le u(x,t)$. □

We are now ready to prove (iii) of Theorem 1.1.

Proof of (iii) *of Theorem* 1.1. Let $V(x,t;\varepsilon)$ be defined by (5.6). Let $\varepsilon_0 > 0$ be so small that $V(x,t;\varepsilon)$ is a positive subsolution for $\varepsilon \in (0,\varepsilon_0)$. Fix $\varepsilon > 0$ so small that $0 < (\mu/2)\varepsilon^2 c_1 - \varepsilon^3 c_2 < 1$ and $0 < \varepsilon < \varepsilon_0$. Put $\varepsilon_1 := (\mu/2)\varepsilon^2 c_1 - \varepsilon^3 c_2 < 1$. Then we have

$$V(x,0;\varepsilon) = \phi - \varepsilon^2(\mu/2)\psi + \varepsilon^3\phi_1 \le \phi - \varepsilon^2(\mu/2)c_1\phi + \varepsilon^3 c_2\phi = (1-\varepsilon_1)\phi.$$

Since $\zeta(t) \to 1$ as $t \to \infty$, we can choose $T > 0$ so large that $\zeta(T) \ge 1 - \varepsilon_1$. Then by Lemma 5.7, $u(x,T) \ge \zeta(T)\phi(x) \ge (1-\varepsilon_1)\phi(x)$. Therefore $V(x,0;\varepsilon) \le u(x,T)$.

By the parabolic regularity theorem (see the appendix of [8]), $u(x,T)$ belongs to $C^2(\overline{\Omega})$. Since $\phi \in C^2(\overline{\Omega})$, $\phi(x) > 0$ in Ω and $\partial\phi/\partial\nu < 0$ on $\partial\Omega$, there exists a $C > 0$ such that $u(x,T) \le C\phi(x)$ in Ω. Then there is a constant $C' > 0$ such that $u(x,T) \le C\phi(x) \le \phi(x) + C'\psi(x)$. Accordingly, we have

$$V(x,0;\varepsilon) \le u(x,T) \le \phi(x) + C'\psi(x) = U(x,0;C'). \tag{5.12}$$

Thus condition (5.9) holds with $u_0(x)$ replaced by $u(x,T)$. By Proposition 5.6, we have

$$\|u(t) - \phi\|_\infty \le Ce^{-\mu t} \quad \text{for } t \ge T.$$

Since $\|u(t) - \phi\|_\infty$ is bounded on $[0,T]$, (1.6) holds for all $t \ge 0$.

Concerning the optimality of the exponent μ (i.e., the estimate from below in (1.7)), assume that $0 < u_0(x) \le (1-\delta)\phi(x)$ with a $\delta \in (0,1)$. Let us show (1.7). By (5.1), we have $u_0(x) \le (1-\delta)\phi(x) \le \phi - \delta c_0\psi = U(x,0;-\delta c_0)$. Then Lemma 5.1 shows that $u(x,t) \le U(x,t;-\delta c_0) = \phi - \delta c_0 e^{-\mu t}\psi$, which leads to

$$\delta c_0 e^{-\mu t}\|\psi\|_\infty \le \|u(t) - \phi\|_\infty.$$

This inequality with (1.6) yields (1.7). As for the case $u_0 \ge (1+\delta)\phi$ a.e. in Ω, take a solution $\eta(t) := \mu(2e^{\mu t} - 1)^{-1}$ of the Bernoulli differential equation $\eta' + \mu\eta = -\eta^2$, and moreover, for $\varepsilon > 0$, we define

$$W(x,t;\varepsilon) := \phi(x) + \varepsilon^2\eta(t)\psi(x) + \varepsilon^3 e^{-2\mu t}\phi_1(x).$$

As in the proof of Lemma 5.3 with (5.8) replaced by (5.7), one can prove that W is a subsolution. Thus choose $\varepsilon > 0$ so small that $0 < W(x, 0; \varepsilon) \leq (1+\delta)\phi(x) \leq u_0(x)$ to obtain the conclusion. □

Remark 5.8. According to the proofs above, one can also derive the same conclusions as in (iii) of Theorem 1.1 with $\|\cdot\|_\infty$ replaced by $\|\cdot\|_p$ for any $1 \leq p < \infty$.

Let us move on to a proof for (iv) of Theorem 1.1.

Proof of (iv) *of Theorem* 1.1. Let $u(x,t)$ be any positive solution of (1.1)–(1.3). By (5.12) and Proposition 5.6, we have

$$\phi(x) - \varepsilon^2 \xi(t)\psi(x) + \varepsilon^3 e^{-2\mu t}\phi_1(x) \leq u(x, t+T) \leq \phi(x) + C' e^{-\mu t}\psi(x)$$

for $t \geq 0$. We use (5.5) and (5.2) to get

$$-\varepsilon^2 \mu c_2 e^{-\mu t}\phi \leq u(x, t+T) - \phi(x) \leq C' c_2 e^{-\mu t}\phi.$$

Dividing both sides by $\phi > 0$, we obtain

$$-Ce^{-\mu t} \leq \frac{u(x,t)}{\phi(x)} - 1 \leq Ce^{-\mu t} \quad \text{for } x \in \Omega,\ t \geq T,$$

for some $C > 0$. The above inequality remains valid for $t \geq t_0$ with any $t_0 > 0$ after replacing C by a larger constant. Indeed, since $u(t) \in C^1(\overline{\Omega})$ and $\partial u(t)/\partial \nu < 0$ on $\partial\Omega$, $\|u(t)/\phi\|_\infty$ is bounded for t on compact subintervals in $(0, \infty)$. □

6 Exponential stability of the positive stationary solution

In this section, we shall verify (v) of Theorem 1.1. To do so, we first extend the convergence result (iii) of Theorem 1.1 proved in the last section to *more general* (i.e., possibly sign-changing) solutions for (1.1)–(1.3) and eventually prove the exponential stability of the positive stationary solution ϕ at the optimal rate under flows of such general solutions. Moreover, we also derive the exponential convergence with the norm of $C^{1,\theta}(\overline{\Omega})$ as well.

We shall utilize a fractional power of the Dirichlet Laplacian. Let $1 < q < \infty$ and define the operator A with the domain $D(A)$ by

$$Au := -\Delta u, \quad D(A) := W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega). \tag{6.1}$$

Then the fractional power A^α with $\alpha > 0$ is well defined (see [19, Chapter 1]). We denote the domain of A^α by $X(\alpha, q)$, i.e., $X(\alpha, q) := \{u \in L^q(\Omega) : A^\alpha u \in L^q(\Omega)\}$, which is equipped with the norm defined by $\|u\|_{X(\alpha,q)} := \|A^\alpha u\|_q$ for $u \in X(\alpha, q)$. In the next lemma, we investigate the embedding of $X(\alpha, q)$.

Lemma 6.1. *Let $0 < \alpha \le 1$ and $1 < q < \infty$. The following compact embeddings hold.*

(i) $X(\alpha, q) \hookrightarrow C^{1,\theta}(\overline{\Omega})$ *if* $0 < \theta < 2\alpha - N/q - 1$.

(ii) $X(\alpha, q) \hookrightarrow C^{\theta}(\overline{\Omega})$ *if* $0 < \theta < 2\alpha - N/q \le 1$.

(iii) $X(\alpha, q) \hookrightarrow W^{k,r}(\Omega)$ *if* $k - N/r < 2\alpha - N/q,\ r \ge q$.

Proof. It is proved in [19, Theorem 1.6.1 and Exercise 10] that (i)–(iii) are continuous embeddings. Since Ω is a bounded domain, A has a compact resolvent. Then the embedding $X(\alpha, q) \hookrightarrow X(\beta, q)$ is compact when $\alpha > \beta \ge 0$ by [19, Theorem 1.4.8]. Let α satisfy the assumption of (i). Then we choose β slightly less than α such that $0 < \theta < 2\beta - N/q - 1 < 2\alpha - N/q - 1$. Then we have $X(\alpha, q) \hookrightarrow X(\beta, q) \hookrightarrow C^{1,\theta}(\overline{\Omega})$, which shows that $X(\alpha, q)$ is compactly embedded in $C^{1,\theta}(\overline{\Omega})$. In the same way as above, we obtain (ii) and (iii). □

We shall show the $C^{1,\theta}(\overline{\Omega})$ regularity of solutions and the convergence to ϕ in $C^{1,\theta}(\overline{\Omega})$.

Lemma 6.2. *Let $u = u(x, t)$ be any (possibly sign-changing) solution of* (1.1)–(1.3) *with initial data $u_0 \in H_0^1(\Omega)$. Then $u(\cdot, t)$ belongs to $C^{1,\theta}(\overline{\Omega})$ for any $t > 0$ and $\theta \in (0, 1)$. Moreover, the $C^{1,\theta}(\overline{\Omega})$ norm of $u(\cdot, t)$ is bounded in $[t_0, \infty)$ for any $t_0 > 0$. In addition, if $\|u_0 - \phi\|_{1,2}$ is small enough or $u_0 > 0$ in Ω, then $\|u(t) - \phi\|_{C^{1,\theta}(\overline{\Omega})} \to 0$ as $t \to \infty$. In particular, $u(x, t) > 0$ in Ω for $t > 0$ large enough.*

Proof. Let $q \in (1, \infty)$ be determined later on and A be as in (6.1). For $q \in [1, 2^*]$ with the Sobolev critical exponent $2^* := 2N/(N - 2)_+$, using the contraction semigroup e^{-tA} generated by A in $L^q(\Omega)$, we rewrite (1.1)–(1.3) as

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(u(s))ds \quad \text{in } L^q(\Omega), \tag{6.2}$$

where $f(u) = |u|^{p-1}u$. Indeed, by Definition 2.1 and Lemma 4.1.1 of [9], (if necessary, by replacing u_0 by $u(t_0) \in H^2(\Omega) \cap H_0^1(\Omega)$ for any $t_0 > 0$) one has (6.2) for $q = 2$, since $f(u(\cdot))$ belongs to $C([0, \infty); L^2(\Omega))$. Since u_0 and $f(u(s))$ belong to $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, (6.2) follows for $q \in [1, 2^*]$.

We use a fractional power A^α of the operator A with $\alpha \in (0, 1)$. Let λ_1 be the first eigenvalue of the Dirichlet Laplacian and fix any $0 < \lambda < \lambda_1$. Then it is known (see [19, Theorem 1.4.3]) that

$$\|A^\alpha e^{-tA} v\|_q \le C_{\alpha,q} t^{-\alpha} e^{-\lambda t} \|v\|_q \quad \text{for } v \in L^q(\Omega), \tag{6.3}$$

where $C_{\alpha,q}$ is a positive constant depending only on α and q. Applying A^α to (6.2) and taking the $L^q(\Omega)$ norm, we obtain

$$\|A^\alpha u(t)\|_q \le C_{\alpha,q} t^{-\alpha} e^{-\lambda t} \|u_0\|_q + C_{\alpha,q} \sup_{0<s<\infty} \|f(u(s))\|_q \int_0^t (t - s)^{-\alpha} e^{-\lambda(t-s)} ds.$$

Putting $\tau = t - s$, we estimate the last integral as

$$\int_0^t \tau^{-\alpha} e^{-\lambda\tau} d\tau \le \int_0^\infty \tau^{-\alpha} e^{-\lambda\tau} d\tau =: c_0 < \infty$$

by $\alpha \in (0,1)$. Recalling the definition of the norm $\|\cdot\|_{X(\alpha,q)}$, we find that

$$\|u(t)\|_{X(\alpha,q)} \le C_{\alpha,q} t^{-\alpha} e^{-\lambda t} \|u_0\|_q + C_{\alpha,q} c_0 |\Omega|^{(1-p)/q} \sup_{0<s<\infty} \|u(s)\|_q^p.$$

We rewrite the maximum of $C_{\alpha,q}$ and $C_{\alpha,q} c_0 |\Omega|^{(1-p)/q}$ as $C_{\alpha,q}$ again. Then

$$\|u(t)\|_{X(\alpha,q)} \le C_{\alpha,q} t^{-\alpha} e^{-\lambda t} \|u_0\|_q + C_{\alpha,q} \sup_{0<s<\infty} \|u(s)\|_q^p. \tag{6.4}$$

We shall show the lemma for $N \ge 3$ only; however, the argument below also works well for $N = 1, 2$ with slight modifications. By (i) of Proposition 2.3, there exists a $C > 0$ such that $\|u(t)\|_{1,2} \le C$ for $t \in [0, \infty)$.

We use a bootstrap argument to derive the L^∞-boundedness (and eventually $C^{1,\theta}$-boundedness) of $u(\cdot, t)$ for t apart from 0. Put $q_1 := 2N/(N-2)$. By the boundedness of $\|u(t)\|_{1,2}$ and the Sobolev embedding, $\|u(t)\|_{q_1}$ is bounded for $t \in [0,\infty)$. From (6.4) with $q = q_1$, it follows that $u(t) \in X(\alpha, q_1)$ for $0 < t < \infty$ and $\sup_{t_0 \le t < \infty} \|u(t)\|_{X(\alpha,q_1)} < \infty$ for any $t_0 > 0$. Fix $r \in (1, N/(N-2))$. If $2\alpha/N - 1/q_1 < 0$ for all $\alpha \in (0,1)$, we define $q_2 := r q_1$. Then $-1/q_2 < 2\alpha/N - 1/q_1$ for some $\alpha \in (0,1)$ sufficiently close to 1. In Lemma 6.1 (iii), we choose $k = 0$, $r = q_2$ and $q = q_1$. Then $u(t)$ belongs to $L^{q_2}(\Omega)$ for $0 < t < \infty$ and $\|u(t)\|_{q_2}$ is bounded in $[t_0, \infty)$ for any $t_0 > 0$. Hence one has (6.2) with $q = q_2$, u_0 replaced by $u(t_0)$ and obvious modification. Moreover, (6.4) holds with $u(t_0)$ and $q = q_2$. Then for any $\alpha \in (0,1)$, it follows that

$$\|u(t)\|_{X(\alpha,q_2)} \le C_{\alpha,q_2} (t-t_0)^{-\alpha} e^{-\lambda(t-t_0)} \|u(t_0)\|_{q_2} + C_{\alpha,q_2} \sup_{t_0 \le t < \infty} \|u(s)\|_{q_2}^p.$$

Thus $\|u(t)\|_{X(\alpha,q_2)}$ is bounded in $[t_1, \infty)$ for any $t_1 > t_0$. Since t_0 is arbitrary, so is t_1. If $2\alpha/N - 1/q_2 < 0$ for all $\alpha \in (0,1)$, we define $q_3 := r q_2$. Then $-1/q_3 < 2\alpha/N - 1/q_2$ for some $\alpha \in (0,1)$ sufficiently close to 1. By Lemma 6.1, $u(t)$ belongs to $L^{q_3}(\Omega)$ for $0 < t < \infty$ and $\|u(t)\|_{q_3}$ is bounded in $[t_0, \infty)$ for any $t_0 > 0$. Repeating a similar discussion, we see that $\|u(t)\|_{X(\alpha,q_3)}$ is bounded in $[t_0, \infty)$ for any $t_0 > 0$.

Put $q_k := r^{k-1} q_1$ and repeat the argument above. Since $q_k \to \infty$ as $k \to \infty$, there exists a positive integer m such that $2\alpha/N - 1/q_m < 0$ for all $\alpha \in (0,1)$ and $2\alpha/N - 1/q_{m+1} > 0$ for some $\alpha \in (0,1)$. Then by Lemma 6.1, $u(t)$ belongs to $X(\alpha, q_{m+1})$, which is (compactly) embedded in $C^\theta(\overline{\Omega})$ or $C^{1,\theta}(\overline{\Omega})$ for some $\theta \in (0,1)$. In both cases, $\|u(t)\|_\infty$ is bounded in $[t_0, \infty)$ for any $t_0 > 0$. Then (6.4) implies that

$$\|u(t)\|_{X(\alpha,q)} \le C(\alpha, q, t_0) \quad \text{for all } \alpha \in (0,1),\ q \in (1,\infty), t \in [t_0, \infty)$$

for some constant $C(\alpha, q, t_0) > 0$. Give $\theta \in (0,1)$ arbitrarily. Choose $\alpha \in (0,1)$ sufficiently close to 1 and take q large enough such that $X(\alpha, q)$ is compactly embedded in $C^{1,\theta}(\overline{\Omega})$ by Lemma 6.1. Therefore the orbit $u(t)$ is relatively compact in $C^{1,\theta}(\overline{\Omega})$ and $\|u(t)\|_{C^{1,\theta}(\overline{\Omega})}$ is bounded in $[t_0, \infty)$ for any $t_0 > 0$.

In case $\|u_0 - \phi\|_{1,2}$ is small enough, $\|u(t) - \phi\|_{1,2}$ converges to zero as $t \to \infty$ by (i) of Theorem 1.1. Since the orbit $u(t)$ is relatively compact in $C^{1,\theta}(\overline{\Omega})$, $\|u(t) - \phi\|_{C^{1,\theta}(\overline{\Omega})}$ also converges to zero. Since $\phi > 0$ in Ω and $\partial\phi/\partial\nu < 0$ on $\partial\Omega$, it holds that $u(\cdot, t) > 0$ in Ω for $t > 0$ large enough. In case $u_0 > 0$ in Ω, since $u(t_0) \in C(\overline{\Omega})$ and $u(\cdot, t_0) > 0$ for any $t_0 > 0$, $\|u(t) - \phi\|_{1,2}$ converges to zero by Proposition 2.6. Then $\|u(t) - \phi\|_{C^{1,\theta}(\overline{\Omega})} \to 0$ as $t \to \infty$ because of the relative compactness of the orbit. The proof is complete. □

Proof of (v) *of Theorem* 1.1. The assertion (b) follows from (a) along with the stability of ϕ in $H_0^1(\Omega)$. So we prove only (a). By Lemma 6.2, $u(x,t) > 0$ for $t \geq T$ with some $T > 0$, and $\|u(t)\|_{C^{1,\theta}(\overline{\Omega})}$ is bounded in $[t_0, \infty)$ for any $t_0 > 0$ and $\theta \in (0,1)$. Hence it is enough to show (1.8) for $t \geq T$. Rewriting $u(x,T)$ as $u_0(x)$, we may assume that $u(x,t) > 0$ for all $t \geq 0$ and $u_0 \in X(\alpha, q)$ for any $\alpha \in (0,1)$ and $q \in (1, \infty)$. Put $v(x,t) := u(x,t) - \phi(x)$, $v_0 := u_0 - \phi$ and $g(x,t) := u(x,t)^p - \phi(x)^p$. Then v satisfies

$$v_t - \Delta v = g(x,t), \quad v|_{\partial\Omega} = 0, \quad v(\cdot, 0) = v_0,$$

which is rewritten as

$$v(t) = e^{-tA} v_0 + \int_0^t e^{-(t-s)A} g(s) ds \quad \text{in } L^q(\Omega), \tag{6.5}$$

since g belongs to $C([0,\infty); L^2(\Omega)) \cap L^\infty(0,\infty; L^\infty(\Omega))$ (see (6.7) below). Recall that λ_1 and μ are the first eigenvalue of $-\Delta$ and $-\Delta - p\phi^{p-1}$, respectively. Since $p\phi^{p-1} > 0$, it holds that $\lambda_1 > \mu$. Hence one can take a constant λ such that $\mu < \lambda < \lambda_1$. Applying A^α to both sides of (6.5), taking the $L^q(\Omega)$ norm and using (6.3), we have

$$\|v(t)\|_{X(\alpha,q)} \leq e^{-\lambda t} \|v_0\|_{X(\alpha,q)} + C_{\alpha,q} \int_0^t (t-s)^{-\alpha} e^{-\lambda(t-s)} \|g(s)\|_q ds. \tag{6.6}$$

Let us estimate $\|g(s)\|_q$. Observe that

$$0 \leq \frac{t^p - s^p}{t - s} \leq s^{p-1} \quad \text{for any } t, s > 0$$

to get

$$|g(x,s)| \leq \left| \frac{u^p - \phi^p}{u - \phi} (u - \phi) \right| \leq \phi^{p-1} |u - \phi|.$$

Using (iv) of Theorem 1.1, we get

$$\|g(s)\|_\infty \leq \|\phi^p((u/\phi) - 1)\|_\infty \leq Ce^{-\mu s}. \tag{6.7}$$

Putting $\tau = t - s$ and using $\lambda > \mu$, we estimate the integral in (6.6) as

$$\int_0^t (t-s)^{-\alpha} e^{-\lambda(t-s)} \|g(s)\|_\infty ds \le C e^{-\mu t} \int_0^\infty \tau^{-\alpha} e^{-(\lambda-\mu)\tau} d\tau.$$

Then (6.6) is rewritten as

$$\|v(t)\|_{X(\alpha,q)} \le e^{-\lambda t} \|v_0\|_{X(\alpha,q)} + \tilde{C}_{\alpha,q} e^{-\mu t}$$

for some constant $\tilde{C}_{\alpha,q}$. Let $\theta \in (0,1)$. Choose $\alpha \in (0,1)$ sufficiently close to 1 and q large enough such that $X(\alpha,q) \hookrightarrow C^{1,\theta}(\overline{\Omega})$ compactly to obtain the convergence. □

7 One-dimensional case

In this section, we address ourselves to the one-dimensional case, i.e., $N = 1$ and $\Omega = (0,1)$. Recall that ϕ_k defined by (1.13) is a solution of (1.11), (1.12) which has exactly $(k-1)$ zeros in $(0,1)$. By the relation (1.13), we have

$$\|\phi_k\|_{L^{p+1}(0,1)} = k^{-2/(1-p)} \|\phi_1\|_{L^{p+1}(0,1)}. \tag{7.1}$$

Noting that $\|\phi_k'\|_2^2 = \|\phi_k\|_{p+1}^{p+1}$ and using the identity above, we have

$$I(\phi_k) = -\frac{1-p}{2(1+p)} \|\phi_k\|_{p+1}^{p+1} = -\frac{1-p}{2(1+p)} k^{-2(1+p)/(1-p)} \|\phi_1\|_{p+1}^{p+1},$$

which also implies $I(\pm\phi_1) < I(\pm\phi_2) < I(\pm\phi_3) < \cdots \nearrow 0$. Therefore all nontrivial solutions of (1.11), (1.12) are distinct to each other. Thus (ii) of Proposition 2.3 yields

Theorem 7.1. *In case $N = 1$ and $\Omega = (0,1)$, let $u(x,t)$ be a solution of* (1.1)–(1.3). *Then $u(x,t)$ converges to a solution v of* (1.11), (1.12) *strongly in $H_0^1(0,1)$ as $t \to \infty$. Hence the ω-limit set of $u(x,t)$ consists only of v.*

If $k^{-2} < p$, then ϕ_k satisfies (1.10) because of (7.1). Hence it is unstable by Theorem 1.2. In particular, when $p \in (1/4, 1)$, all ϕ_k with $k \ge 2$ are unstable. Next, for all $p \in (0,1)$, we shall show that all sign-changing stationary solutions, i.e., $\pm\phi_k$ with $k \ge 2$, are unstable. To this end, we prepare the following lemma:

Lemma 7.2. *Let $k \ge 2$. For $\lambda \in (0,2)$, we define*

$$\Phi_\lambda(x) := \begin{cases} \lambda^{2/(1-p)} \phi_k(\lambda^{-1} x) & \text{if } x \in [0, \lambda/k], \\ \alpha^{-2/(1-p)} \phi_k(\alpha x + \beta) & \text{if } x \in (\lambda/k, 2/k], \\ \phi_k(x) & \text{if } x \in (2/k, 1], \end{cases}$$

with $\alpha := 1/(2-\lambda)$ and $\beta := (1-\lambda\alpha)/k$. Then it follows that

(i) $\Phi_\lambda(x) \in H_0^1(0,1)$ *and* $\Phi_\lambda \to \phi_k$ *in* $H_0^1(0,1)$ *as* $\lambda \to 1$.

(ii) $I(\Phi_\lambda) < I(\phi_k) \quad$ *if* $\lambda \neq 1$.

Proof. Note that all the zeros of ϕ_k on $[0,1]$ are j/k with $j = 0, 1, \dots, k$. Then the assertion (i) is obvious. So, we prove only (ii). Φ_λ solves (1.11) in each section $(0, \lambda/k)$, $(\lambda/k, 2/k)$, $(2/k, 1)$. For $0 \leq a < b \leq 1$, let us write

$$I(u;(a,b)) := \int_a^b \left(\frac{1}{2}|u'|^2 - \frac{1}{p+1}|u|^{p+1} \right) dx.$$

Then $I(\Phi_\lambda) = I(\Phi_\lambda;(0,\lambda/k)) + I(\Phi_\lambda;(\lambda/k,2/k)) + I(\Phi_\lambda;(2/k,1))$. Since Φ_λ solves (1.11) in $(0,\lambda/k)$ and $\Phi_\lambda(x) = 0$ at $x = 0, \lambda/k$, we have

$$I(\Phi_\lambda;(0,\lambda/k)) = -\frac{1-p}{2(1+p)} \int_0^{\lambda/k} \Phi_\lambda^{p+1} dx.$$

For $0 < x < \lambda/k$, we see $\Phi_\lambda(x) = \lambda^{2/(1-p)} \phi_k(\lambda^{-1}x) = (k^{-1}\lambda)^{2/(1-p)} \phi_1(k\lambda^{-1}x)$. Setting $y = k\lambda^{-1}x$, we have

$$I(\Phi_\lambda;(0,\lambda/k)) = -c_k \lambda^{(3+p)/(1-p)} \tag{7.2}$$

and

$$c_k := \frac{1-p}{2(1+p)} k^{-(3+p)/(1-p)} \int_0^1 \phi_1(y)^{p+1} dy.$$

Next, noting $\Phi_\lambda(x) = (\alpha k)^{-2/(1-p)} \phi_1(k(\alpha x + \beta))$ for $\lambda/k < x < 2/k$ and setting $y = k(\alpha x + \beta)$, we get

$$\begin{aligned} I(\Phi_\lambda;(\lambda/k,2/k)) &= -\frac{1-p}{2(1+p)} (\alpha k)^{-(3+p)/(1-p)} \int_1^2 |\phi_1(y)|^{p+1} dy \\ &= -c_k (2-\lambda)^{(3+p)/(1-p)}. \end{aligned} \tag{7.3}$$

Here we have used the fact that the integrals of $|\phi_1|^{p+1}$ on $[0,1]$ and on $[1,2]$ are equal to each other because $\phi_1(1-x) = -\phi_1(1+x)$ for $x \in \mathbb{R}$. Since $I(\Phi_\lambda;(2/k,1)) = I(\phi_k;(2/k,1))$, we claim that

$$I(\Phi_\lambda;(0,\lambda/k)) + I(\Phi_\lambda;(\lambda/k,2/k)) < I(\phi_k;(0,2/k)) \quad \text{if } \lambda \neq 1, \tag{7.4}$$

which implies (ii). Indeed, note that

$$I(\phi_k;(0,2/k)) = -\frac{1-p}{2(1+p)} \int_0^{2/k} |\phi_k|^{p+1} dx = -2c_k.$$

By (7.2), (7.3), one can rewrite (7.4) as $\lambda^{(3+p)/(1-p)} + (2-\lambda)^{(3+p)/(1-p)} > 2$ if $\lambda \neq 1$, which can be checked by a standard calculation. Thus (7.4) follows. $\square$

We conclude this paper by proving Theorem 1.3.

Proof of Theorem 1.3. Fix $k \geq 2$. We prove the instability of ϕ_k only. Let $u(x,t)$ be any solution of (1.1)–(1.3) with an initial data $u_0 = \Phi_\lambda$ for $\lambda \neq 1$, where Φ_λ is obtained in Lemma 7.2. Then by (ii) of Lemma 7.2, $I(u(t)) \leq I(\Phi_\lambda) < I(\phi_k)$ for all $t \geq 0$. By Theorem 7.1, $u(t)$ converges to a stationary solution u_∞ strongly in $H_0^1(0,1)$ as $t \to \infty$. Obviously, $u_\infty \neq \phi_k$, and moreover, $u(0)$ is sufficiently close to ϕ_k in $H_0^1(0,1)$ when $\lambda \neq 1$ is sufficiently close to 1. From the isolation of stationary solutions in $H_0^1(0,1)$, one observes that $c := \inf\{\|\phi_k - \phi\|_{1,2} : \ \phi \in \mathcal{S} \setminus \{\phi_k\}\} > 0$, where $\mathcal{S}$ denotes the set of solutions for (1.11), (1.12). Then it holds that $\lim_{t\to\infty} \|u(t) - \phi_k\|_{1,2} = \|u_\infty - \phi_k\|_{1,2} \geq c$. Therefore ϕ_k is unstable. One can also prove the instability of $-\phi_k$ in a similar way. The exponential stability of $\pm\phi_1$ has already been proved by (v) of Theorem 1.1. □

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