GAKUTO International Series
Mathematical Sciences and Applications, Vol.(****)**
Current Advances in Applied Nonlinear Analysis and
Mathematical Modelling Issues, pp.0–3

Maximality for the sum of two monotone operators in L^p-spaces

Goro Akagi
Department of Machinery and Control Systems,
School of Systems Engineering and Sciences, Shibaura Institute of Technology
307 Fukasaku, Minuma-ku, Saitama-shi, Saitama 337-8570 JAPAN
E-mail: g-akagi@sic.shibaura-it.ac.jp

Abstract. This paper is devoted to providing a sufficient condition for the maximality of the sum of subdifferential operators defined on reflexive Banach spaces (cf. Brézis, Crandall and Pazy [?]) and finally proving the maximal monotonicity in $L^p(\Omega) \times L^{p'}(\Omega)$ of the nonlinear elliptic operator $u \mapsto -\Delta_m u + \beta(u(\cdot))$ with a maximal monotone graph β.

AMS Subject Classification: 47H05, 47J05

1 Introduction

Let E and E^* be a reflexive Banach space and its dual space, respectively, and let $\phi_1, \phi_2 : E \to (-\infty, \infty]$ be proper (i.e., $\phi_1, \phi_2 \not\equiv \infty$) lower semicontinuous convex functionals with the effective domains $D(\phi_i) := \{u \in E;\ \phi_i(u) < \infty\}$ for $i = 1, 2$. Then the subdifferential operator $\partial_E \phi_i : E \to 2^{E^*}$ of ϕ_i is defined by

$$\partial_E \phi_i(u) := \{\xi \in E^*;\ \phi_i(v) - \phi_i(u) \geq \langle \xi, v - u \rangle_E \ \text{ for all } \ v \in D(\phi_i)\},$$

where $\langle \cdot, \cdot \rangle_E$ denotes the duality pairing between E and E^*, with the domain $D(\partial_E \phi_i) = \{u \in D(\phi_i);\ \partial_E \phi_i(u) \neq \emptyset\}$ for $i = 1, 2$. This paper provides a new sufficient condition for the maximality of the sum $\partial_E \phi_1 + \partial_E \phi_2$ in $E \times E^*$ and an application to nonlinear elliptic operators in L^p-spaces.

This paper is motivated by the question of whether the following operator $\mathcal{M}$ is maximal monotone in $L^p(\Omega) \times L^{p'}(\Omega)$ for $p \in [2, \infty)$, $p' = p/(p-1)$ and a bounded domain Ω of $\mathbb{R}^N$:

$$\mathcal{M} : D(\mathcal{M}) \subset L^p(\Omega) \to L^{p'}(\Omega);\ u \mapsto -\Delta_m u + \beta(u(\cdot)), \tag{1.1}$$

where β is a maximal monotone graph in $\mathbb{R}$ such that $\beta(0) \ni 0$, and Δ_m is a modified Laplacian given by

$$\Delta_m u = \nabla \cdot \left(|\nabla u|^{m-2} \nabla u\right), \quad 1 < m < \infty$$

equipped with the homogeneous Dirichlet boundary condition, i.e., $u|_{\partial\Omega} = 0$. Set $E = L^p(\Omega)$ and put

$$\phi_1(u) \ := \begin{cases} \dfrac{1}{m} \displaystyle\int_\Omega |\nabla u(x)|^m dx & \text{if } \ u \in W_0^{1,m}(\Omega), \\ \infty & \text{otherwise}, \end{cases} \tag{1.2}$$

$$\phi_2(u) \ := \begin{cases} \displaystyle\int_\Omega j(u(x)) dx & \text{if } \ j(u(\cdot)) \in L^1(\Omega), \\ \infty & \text{otherwise}, \end{cases} \tag{1.3}$$

where $j : \mathbb{R} \to (-\infty, \infty]$ is a proper lower semicontinuous convex function such that $\partial j = \beta$. Then $\partial_E \phi_1(u)$ and $\partial_E \phi_2(u)$ coincide with $-\Delta_m u$ equipped with $u|_{\partial\Omega} = 0$ and $\beta(u(\cdot))$ respectively. Moreover, the maximal monotonicity of $\mathcal{M}$ ensures $\mathcal{M} = \partial_E(\phi_1 + \phi_2)$.

The maximality for the sum of two maximal monotone operators was well studied in Hilbert space settings (see [?], [?]). As for Banach space settings, a couple of sufficient conditions are proposed by Brézis, Crandall and Pazy [?]. Let A and B be maximal monotone operators from E into E^*. Their results ensure the maximal monotonicity of $A + B$ in $E \times E^*$ if one of the following conditions is at least satisfied:

(i) $D(A) \cap (\mathrm{Int} D(B)) \neq \emptyset$,

(ii) B is dominated by A, i.e., $D(A) \subset D(B)$ and $\|B(u)\|_{E^*} \leq k \|A(u)\|_{E^*} + \ell(|u|_E)$ for all $u \in D(A)$ with $k \in (0, 1)$ and a non-decreasing function ℓ in $\mathbb{R}$,

(iii) $B = \partial_E \phi$ with a proper, lower semicontinuous convex function $\phi : E \to (-\infty, +\infty]$, and

$$\phi(J_\lambda u) \leq \phi(u) + C\lambda \quad \text{for } \ u \in D(\phi) \text{ and } \lambda > 0, \tag{1.4}$$

where J_λ denotes the resolvent of A in E.

Here we write $\|C\|_{E^*} := \inf\{|c|_{E^*};\ c \in C\}$ for each non-empty subset C of E^*, and furthermore, the resolvent $J_\lambda : E \to D(A)$ is given such that $u_\lambda := J_\lambda u$ is a unique solution of $F_E(u_\lambda - u) + A(u_\lambda) \ni 0$, where F_E stands for the duality mapping between E and E^*, for each $u \in E$.

However, these results could not be applicable directly to our setting for (1.1). As for (i), neither $D(\partial_E\phi_1)$ nor $D(\partial_E\phi_2)$ might have any interior points in E $(= L^p(\Omega))$. Condition (ii) cannot be checked unless we impose an appropriate growth condition on β. Condition (iii) is available for the case that $p = 2$, because the duality mapping F_E of $E = L^2(\Omega)$ is the identity and the resolvent J_λ for $\partial_E\phi_2$ has a simple representation formula,

$$(J_\lambda u)(x) = (1 + \lambda\beta)^{-1}(u(x)) \quad \text{for a.e. } x \in \Omega, \tag{1.5}$$

which enables us to check (1.4). However, it is somewhat difficult to check (1.4) for the case that $p \neq 2$. Actually, the relation between the resolvents of $\partial_E\phi_2$ and β is unclear, since the duality mapping F_E is severely nonlinear whenever $p \neq 2$ (see (**??**)).

In this paper we propose a new sufficient condition for the maximality of $\partial_E\phi_1 + \partial_E\phi_2$ in $E \times E^*$ such that the representation formula (1.5) in $L^2(\Omega)$ can be effectively used in applications to nonlinear elliptic operators such as (1.1). More precisely, we introduce a Hilbert space H as a pivot space of the triplet $E \hookrightarrow H \equiv H^* \hookrightarrow E^*$ and an extension ϕ_2^H of ϕ_2 to H, and moreover, we give a sufficient condition for the maximality in terms of the resolvent and the Yosida approximation in H for $\partial_H\phi_2^H$.

The treatment of the operator $\mathcal{M}$ in $L^p(\Omega)$ with $p \neq 2$ is often required from severely nonlinear problems such as generalized Allen-Cahn equations of the form

$$|u_t|^{p-2}u_t - \Delta_m u + \beta(u) + g(u) \ni f \quad \text{in } \Omega \times (0, \infty), \tag{1.6}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, \infty), \tag{1.7}$$

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega \tag{1.8}$$

with a non-monotone function $g : \mathbb{R} \to \mathbb{R}$. In [**?**], the base space is chosen as $E = L^p(\Omega)$ to avoid a difficulty arising from the nonlinearity in u_t, and moreover, (1.6) is reduced to an evolution equation,

$$\partial_E\psi(u'(t)) + \partial_E\phi(u(t)) + g(u(\cdot, t)) \ni f(t) \ \text{ in } E^*,$$

by putting

$$\psi(u) = \frac{1}{p}\int_\Omega |u(x)|^p dx$$

and setting $\phi = \phi_1 + \phi_2$ with ϕ_1, ϕ_2 defined by (1.2), (1.3). One can check $\partial_E\psi$ becomes coercive and bounded in $E \times E^*$, and the existence of global solutions, the formation of a generalized semiflow and the existence of a global attractor can be discussed.

In §**??**, we first propose an abstract framework on the maximality for the sum of two subdifferential operators in Banach spaces. Moreover, in §**??**, we also establish an estimate for the nonlinear elliptic operator $\mathcal{M}$ in $L^r(\Omega)$ with $r \in (1, \infty)$ to check a sufficient condition presented in §**??**. The final section is devoted to an application to the nonlinear elliptic operator $\mathcal{M}$.

References

[1] J. R. Cannon, E. DiBenedetto and G. H. Knightly, The bidimensional Stefan problem with convection: The time dependent case, Comm. Partial Differential Equations, **14**(1983), 1549–1604.

[2] E. Casella and M. Giangi, An analytical and numerical study of the Stefan problem with convection by means of an enthalpy method, Math. Methods Appl. Sci., **24**(2001), 623–639.

[3] A. Damlamian, Some results on the multi-phase Stefan problem. Comm. Partial Differential Equations, **2**(1977),1017–1044.

[4] E. DiBenedetto and M. O'Leary, Three-dimensional conduction-convection problems with change of phase, Arch. Ration. Mech. Anal., **123**(1993), 99–116.