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Concentration phenomena in singularly perturbed solutions of a spatially heterogeneous reaction-diffusion equation

by

Hiroko YAMAMOTO

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A thesis presented

by

Hiroko Yamamoto

to

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for the degree of

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Contents

Chapter 1

Introduction

1.1 Background

In 1952, A. M. Turing [21] proposed the idea that, when two chemicals with different diffusion coefficients react each other, a spatially uniform state may become unstable, and as a result a spatially non-uniform state emerges spontaneously. Nowadays his assertion is referred to as "diffusion-driven instability". In natural world, various patterns are observed, and it has been confirmed that there are many phenomena for which this principle can explain how those patterns are formed.

In 1972, A. Gierer and H. Meinhardt proposed the following activator-inhibitor system as a model to explain the head formation in *hydra*:

$$
\text{(GM)} \qquad \begin{cases} \dfrac{\partial A}{\partial t} = \varepsilon^2 \Delta A - \mu_a(x)A + \rho_a(x)\left(\dfrac{c_a A^p}{H^q} + \rho_0(x)\right) & \text{in } \Omega, \\ \dfrac{\partial H}{\partial t} = D\Delta H - \mu_h(x)H + \rho_h(x)\dfrac{c_h A^r}{H^s} & \text{in } \Omega, \\ \dfrac{\partial A}{\partial \nu} = \dfrac{\partial H}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}
$$

Here, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, ν denotes the unit outer normal to $\partial\Omega$, $\Delta = \sum_{j=1}^{n} \partial^2/\partial x_j^2$ is the Laplace operator, c_a, c_h, ε, D are positive constants, $\mu_a(x)$, $\rho_a(x)$, $\rho_0(x)$, $\mu_h(x)$, $\rho_h(x)$ are positive functions. They hypothesized that the head of hydra is formed at the place where the activator concentrates. Moreover, since the activator grows auto-catalytically, they assumed the inhibitor has the role of reducing the growth of activator to prevent the explosion of the activator concentration. In numerical situations, the system (GM) exhibits various type of patterns. Most typical one is the formation of spike-like patterns in which the activator concentrates in a very narrow region around finitely many points. Sometimes the activator concentrates around curves or surfaces. Some patterns are stationary, and others are nonstationary, depending on the parameters and initial data. From a mathematical point of view, it is very difficult to understand rigorously the process of the formation of pattern in (GM). For example, we do not know how to find all stationary solutions, and hence it is hopeless to understand the global behavior of a solution with an arbitrary initial data. Therefore, it is natural to consider a simplified system. Keener [10] proposed to take the limit of $D \to \infty$. Formally speaking, in this limit, $\Delta H \to 0$ and hence $H(x,t) \to \xi(t)$ because of the boundary condition. Here $\xi(t)$ is an unknown. To derive an equation for $\xi(t)$, we integrate the second equation of (GM) over Ω to obtain

$$\frac{\partial}{\partial t}\int_\Omega H(x,t)\,dx = -\int_\Omega \mu_h(x)H(x,t)\,dx + \int_\Omega \rho_h(x)c_h\frac{A(x,t)^r}{H(x,t)^s}\,dx.$$

Hence, as a formal limit, we are led to

$$\text{(SS)}\qquad \begin{cases} \dfrac{\partial A}{\partial t} = \varepsilon^2\Delta A - \mu_a(x)A + \rho_a(x)\Big(\dfrac{c_aA^p}{\xi^q} + \rho_0(x)\Big) & \text{for } x\in\Omega, t>0,\\ |\Omega|\dfrac{d\xi}{dt} = -\xi\displaystyle\int_\Omega \mu_h(x)\,dx + \frac{1}{\xi^s}\int_\Omega \rho_h(x)c_hA^r\,dx & \text{for } t>0,\\ \dfrac{\partial A}{\partial \boldsymbol{\nu}} = 0 & \text{for } x\in\partial\Omega, t>0, \end{cases}$$

which is called the *shadow system* for (GM). This shadow system is regarded to preserve some of the essential properties of the original system, and therefore the initial-boundary value problem for (SS) is an important one that should be investigated first in theoretical studies.

We note that ξ is an unknown constant if we consider the stationary problem for (SS). Therefore it is convenient to scale the activator as $A(x) = \xi^{q/(p-1)}u(x)$, which yields

$$\text{(SSS)}\qquad \begin{cases} \varepsilon^2\Delta u - \mu_a(x)u + \rho_a(x)c_au^p + \xi^{-q/(p-1)}\rho_a(x)\rho_0(x) = 0 & \text{in } \Omega,\\ c_h\displaystyle\int_\Omega \rho_h(x)u^r\,dx - \xi^{s+1-qr/(p-1)}\int_\Omega \mu_h(x)\,dx = 0, & \\ \dfrac{\partial u}{\partial \boldsymbol{\nu}} = 0 & \text{on } \partial\Omega. \end{cases}$$

If $\rho_0(x)\equiv 0$, then any (positive) solution of the Neumann problem for the single equation

$$\begin{cases} \varepsilon^2\Delta u - \mu_a(x)u + c_a\rho_a(x)u^p = 0 & \text{in } \Omega,\\ \dfrac{\partial u}{\partial \boldsymbol{\nu}} = 0 & \text{on } \partial\Omega \end{cases}$$

determines the value of ξ by the second equation of (SSS). A fundamental question is whether this Neumann problem has a nontrivial solution or not. There have been a huge amount of literature concerning this question in the case where $\mu_a(x)$ and $\rho_a(x)$ are constants. However, not much has been known about the case of variable coefficients.

The purpose of this thesis is to study the structure of nontrivial solutions of the boundary value problem for the following single equation with variable coefficients when the parameter $\varepsilon > 0$ is sufficiently small:

$$\text{(1.1)}\qquad \begin{cases} \varepsilon^2\mathcal{A}(x)u - a(x)u + b(x)u^p + \delta\sigma(x) = 0 & \text{in } \Omega,\\ \mathcal{B}(x)u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $\mathcal{A}(x)$ is a second order uniformly strictly elliptic operator; $a(x)$ and $b(x)$ are positive functions; $\sigma(x)$ is a nonnegative function; $\varepsilon > 0$ and $\delta \geq 0$ are sufficiently small constants; and the exponent $p > 1$ is subcritical in the sense of the Sobolev imbedding. For the detail, see Section 1.4.

1.2 Concentration phenomena for a homogeneous equation

In this section we review briefly some of the results on the Neumann problem for a single equation with constant coefficients:

$$
\begin{cases} \varepsilon^2 \Delta u - u + u^p = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}
$$

Clearly this problem has two nonnegative constant solutions $u \equiv 0$ and $u \equiv 1$ for any $\varepsilon > 0$ and $p > 1$. Therefore we are interested in the existence of nonconstant solutions. It is not difficult to check that bifurcation theory can be applied to obtain nonconstant solutions near the constant solution $u \equiv 1$ by choosing ε appropriately. However, Problem (1.2) is expected to have a large amplitude solution for $\varepsilon > 0$ sufficiently small. To find such a solution, the variational approach is more promising. In this case a standard method is the Mountain Pass Lemma (see Lemma 2.7 below) by Ambrosetti and Rabinowitz [1]. To apply this method we have to restrict p to the range $1 < p < (n+2)/(n-2)$ if $n \geq 3$. Then

$$
J_\varepsilon(u) = \frac{1}{2}\int_\Omega (\varepsilon^2 |\nabla u|^2 + u^2)\, dx - \frac{1}{p+1}\int_\Omega u_+^{p+1}\, dx
$$

defines a C^1-functional on $W^{1,2}(\Omega)$, where $u_+ = \max\{0, u\}$. It is well-known that u is a classical solution of (1.2) if and only if u is a critical point of J_ε. Indeed, $J_\varepsilon'(u)$ is a functional on $W^{1,2}(\Omega)$ defined by

$$
J_\varepsilon'(u)\phi = \int_\Omega (\varepsilon^2 \nabla u \cdot \nabla\phi + u\phi - u_+^p \phi)\, dx \qquad \text{for } \phi \in W^{1,2}(\Omega).
$$

Thus, $J_\varepsilon'(u) = 0$ means that $u \in W^{1,2}(\Omega)$ is a weak solution of (1.2). The smoothness of u is a consequence of the standard elliptic regularity theory.

In [12], Lin, Ni and Takagi proved the existence of a positive nonconstant solution of (1.2) by applying the Mountain Pass Lemma:

Theorem A ([12, Theorem 2]). *Under the assumption that* $1 < p < (n+2)/(n-2)$ *if* $n \geq 3$ *and* $1 < p < \infty$ *if* $n = 1, 2$, *there exists a positive nonconstant solution* u_ε *to* (1.2), *provided* ε *is sufficiently small. Moreover,* u_ε *satisfies*

$$
0 < J_\varepsilon(u_\varepsilon) \leq C_0 \varepsilon^n,
$$

where $C_0 > 0$ *depends only on* Ω *and* p.

The critical value $c_\varepsilon = J_\varepsilon(u_\varepsilon)$ is given by

$$
c_\varepsilon = \inf_{h\in\Gamma} \max_{0\leq t\leq 1} J_\varepsilon(h(t)),
$$

where $\Gamma = \{h \in C^0([0,1]; W^{1,2}(\Omega)) \mid h(0) = 0, h(1) = e\}$ and $e \in W^{1,2}(\Omega)$ satisfies $J_\varepsilon(e) < 0$. Furthermore, it turns out that c_ε is the smallest positive critical value of $J_\varepsilon(u)$. Hence the solution u_ε stated in Theorem A is called a *least-energy solution* of (1.2). The asymptotic behavior as $\varepsilon \downarrow 0$ of least-energy solutions was studied by Ni and Takagi:

Theorem B ([14, Theorem 2.1], [15, Theorem 1.2]). *Let u_ε be a least-energy solution to* (1.2). *Then u_ε has at most one local maximum in $\overline{\Omega}$ and it is attained exactly at one point P_ε which must lie on the boundary, provided that ε is sufficiently small. Moreover,* $\lim_{\varepsilon\downarrow 0} H(P_\varepsilon) = \max_{P\in\partial\Omega} H(P)$, *where $H(P)$ denotes the mean curvature of $\partial\Omega$ at P. Furthermore, $u_\varepsilon(x) \to 0$ as $\varepsilon \downarrow 0$ for $x \in \overline{\Omega} \setminus \{P_\varepsilon\}$.*

Therefore, as ε tends to zero, we see that $\{u_\varepsilon\}$ concentrates around only one point in the neighborhood of a maximum point of the mean curvature function of $\partial\Omega$.

A similar result was obtained by Ni and Wei for the Dirichlet boundary condition:

Theorem C ([16, Theorem 2.2]). *Let u_ε be a least-energy solution to*

$$\begin{cases} \varepsilon^2\Delta u - u + u^p = 0, \ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Then, for ε sufficiently small, we have

(i) *u_ε has at most one local maximum and it is achieved at exactly one point P_ε in Ω. Moreover, $u_\varepsilon(\cdot + P_\varepsilon) \to 0$ in $C^1_{\mathrm{loc}}((\Omega - P_\varepsilon) \setminus \{0\})$ where $\Omega - P_\varepsilon = \{x - P_\varepsilon \mid x \in \Omega\}$.*

(ii) $\operatorname{dist}(P_\varepsilon, \partial\Omega) \to \max_{P\in\Omega} \operatorname{dist}(P, \partial\Omega)$ *as $\varepsilon \to 0$.*

It is natural to ask whether this kind of concentration phenomenon occurs only to least-energy solutions. To be more precise, we define as follows:

Definition 1.1. A family $\{u_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ of solutions of (1.2) or (1.3) is said to exhibit a *point concentration phenomenon* if there exist M distinct points $\{P_{1,0}, \ldots, P_{M,0}\} \subset \overline{\Omega}$, a strictly decreasing sequence $\varepsilon_j \to 0$ $(j \to \infty)$ and M sequences $\{P_{k,\varepsilon_j}\}_{j=1}^\infty \subset \overline{\Omega}$ with $P_{k,\varepsilon_j} \to P_{k,0}$, $k = 1, \ldots, M$, such that (i) u_{ε_j} achieves *strict* local maxima at $x = P_{k,\varepsilon_j}$ and (ii) $u_{\varepsilon_j}(x) - W_k((x - P_{k,\varepsilon_j})/\varepsilon_j) \to 0$ as $j \to \infty$ in $B_\rho(P_{k,0}) \cap \overline{\Omega}$, where ρ is a positive number and $W_k \in C^2(\mathbb{R}^n)$ is a positive function satisfying $W_k(0) = \max_{y\in\mathbb{R}^n} W_k(y) > 1$ and $W_k(y) \to 0$ as $|y| \to \infty$. We say that $P_{k,0} \in \overline{\Omega}$ is a *concentration point* of $\{u_\varepsilon\}$ if there is a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ such that $\varepsilon_j \downarrow 0$ and $P_{k,\varepsilon_j} \to P_{k,0}$ as $j \to \infty$ for some $1 \le k \le M$.

Notice that the definition above does not rule out the possibility of the coexistence of point concentration and, say, surface concentration.

If a concentration point P_0 is on the boundary, it must be a critical point of the mean curvature function $H(P)$. The proof of this fact is essentially due to Wei who considered a slightly different question:

Theorem D ([24, Theorem 1.1]). *If u_ε a solution of* (1.2) *and* $\lim_{\varepsilon\to 0} \varepsilon^{-n} J_\varepsilon(u_\varepsilon) = I(w)/2$, *then for ε sufficiently small u_ε has only one local (hence global) maximum point, P_ε, and $P_\varepsilon \in \partial\Omega$. Moreover, $\nabla_{\tau_{P_\varepsilon}} H(P_\varepsilon) \to 0$ as $\varepsilon \to 0$ where $\nabla_{\tau_{P_\varepsilon}}$ is the tangential derivative at P_ε.*

Here $w \in W^{1,2}(\mathbb{R}^n)$ is a unique positive solution of $\Delta w - w + w^p = 0$ in $\mathbb{R}^n$ with $w(0) = \max_{y\in\mathbb{R}^n} w(y)$ and $I(w) = 2^{-1} \int_{\mathbb{R}^n} (|\nabla w|^2 + w^2)\, dy - (p+1)^{-1} \int_{\mathbb{R}^n} w^{p+1}\, dy$.

Also, Wei proved that a nondegenerate critical point of the mean curvature function is indeed a concentration point:

Theorem E ([24, Theorem 1.2])**.** *Let $P_0 \in \partial\Omega$. Suppose that P_0 is a nondegenerate critical point of the mean curvature function $H(P)$. Then for ε sufficiently small there exists a solution u_ε to* (1.2) *such that $\varepsilon^{-n} J_\varepsilon(u_\varepsilon) \to I(w)/2$, u_ε has only one local maximum point P_ε, and $P_\varepsilon \in \partial\Omega$. Moreover, $P_\varepsilon \to P_0$.*

So far we have mentioned solutions concentrating at only one point. There are many results on the existence of solutions which concentrate at more than one points. For instant, Gui and Wei [9] established the existence of solutions concentrating at K_1 points in the interior of the domain and K_2 points of the boundary. It is to be emphasized that other types of concentration phenomena occur in solutions of (1.2). For example, Malchiodi and Montenegro [13] obtained a family of solutions concentrating on the entire boundary of the domain.

1.3 Concentration phenomena for a heterogeneous equation

The diffusion-driven instability asserts that patterns can emerge spontaneously even in spatially uniform environments. However, biological pattern formation takes place usually in spatially heterogeneous environments. As a matter of fact, in numerical simulations for head-transplantation experiments on hydra, Gierer and Meinhardt allowed strong spatial dependence of the coefficient $\rho_a(x)$ (see Fig. 2 in [7]). This suggests that spatial heterogeneity also plays an important role in pattern formation, and there should be a systematic and quantitative study on the effect of spatial heterogeneity. This thesis is motivated by these observations.

Contrary to the spatially homogeneous case, only a few works from a view point of pattern formation have appeared so far in this direction. The first result seems to have been given by Ren, who considered

$$(1.4) \qquad \begin{cases} \varepsilon^2 \Delta u - u + K(x) u = 0 & \text{in } \Omega, \\ B(x) u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $K(x)$ is a sufficiently smooth positive function on $\overline{\Omega}$, ε is a positive constant, and the boundary operator $B(x)$ denotes either the identity operator or the differential operator in the normal direction $\partial/\partial\nu$. Instead of J_ε in Section 1.2, he introduces an energy functional

$$J_{K,\varepsilon}(u) = \frac{1}{2} \int_\Omega (\varepsilon^2 |\nabla u|^2 + u^2)\, dx - \frac{1}{p+1} \int_\Omega K(x) u_+^{p+1}\, dx.$$

By applying the Mountain Pass Lemma, one obtains a least-energy solution u_ε again both for the Dirichlet problem (working in the space $W_0^{1,2}(\Omega)$) and for the Neumann problem (working in the space $W^{1,2}(\Omega)$).

Theorem F ([19, Theorem 1.1])**.** *Let u_ε be a least-energy solution of* (1.4) *under the homogeneous Dirichlet boundary condition. Then we have*

1. *There exist positive constants C_1 and C_2 independent of ε such that $C_1 \le \|u_\varepsilon\|_{L^\infty(\Omega)} \le C_2$.*

2. *For ε small enough u_ε has only one local maximum point P_ε with* $\lim_{\varepsilon \to 0} \varepsilon^{-1} \operatorname{dist}(P_\varepsilon, \partial\Omega) = \infty$.

3. *If P is a limit point of $\{P_\varepsilon\}$ as $\varepsilon \to 0$, then $K(P) = \max_{x\in\overline{\Omega}} K(x)$.*

Theorem G ([19, Theorems 1.2–1.3])**.** *Let u_ε be a least-energy solution of* (1.4) *under the homogeneous Neumann boundary condition. Assuming*

$$(i)\ \max_{\overline{\Omega}} K(x) > 2^{(p-1)/2} \max_{\partial\Omega} K(x) \qquad or \qquad (ii)\ \max_{\overline{\Omega}} K(x) < 2^{(p-1)/2} \max_{\partial\Omega} K(x),$$

we have the following:

1. *There exists positive constants C_1 and C_2 independent of ε such that $C_1 \le \|u_\varepsilon\|_{L^\infty(\Omega)} \le C_2$.*
2. *For ε small enough, u_ε possesses only one local maximum point P_ε. Moreover, as $\varepsilon \downarrow 0$, P_ε stays away from the boundary of Ω if (i) holds, whereas P_ε stays on the boundary if (ii) holds.*
3. *Every limit point of $\{P_\varepsilon\}$ as $\varepsilon \downarrow 0$ must be a maximum point of $K(x)$ in the interior of Ω if (i) holds, while it must be a maximum point of $K(x)$ restricted on $\partial\Omega$ if (ii) holds.*

On the other hand, in the case $\Omega = \mathbb{R}^n$ there are many works on concentration phenomena in bound states of nonlinear Schrödinger equations initiated by Floer and Weinstein [4]. See, e.g., Wang [22] and references there in for fundamental results on the equation

$$h^2\Delta u - V(x)u + u^p = 0 \quad \text{in } \mathbb{R}^n. \tag{1.5}$$

Under some mild assumptions on $V(x)$, Wang proved that the ground state of (1.5) concentrates at a global minimum point of $V(x)$. This result was generalizend to nonlinear Schrödinger equation with competing potential functions

$$h^2\Delta u - V(x)u + K(x)|u|^{p-1}u + Q(x)|u|^{q-1}u = 0 \quad \text{in } \mathbb{R}^n \tag{1.6}$$

by Wang and Zeng [23] and they presented the method of locating the maximum point of a ground state. In fact, our approach is based on theirs.

All the results mentioned above treats the case where the basic production term $\sigma(x)$ in (1.1) vanishes identically. In this thesis, we are interested in the nontrivial basic production term. For, in the activator-inhibitor system (GM) the basic production term $\rho_a(x)\sigma_0(x)$ plays an important role: Without the basic production term, patterns may collapse, that is, there are solutions of the initial-boundary value problem which form nontrivial patterns for a while, but eventually they converge to $(0, 0)$ uniformly as $t \to \infty$. It is also proved that collapse of patterns never occurs if $\rho_a(x)\sigma_0(x) \not\equiv 0$. For a proof of this fact, see Suzuki and Takagi [20].

1.4 Statement of results

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and p a number satisfying $1 < p < (n+2)/(n-2)$ if $n \ge 3$, $1 < p < \infty$ if $n = 1, 2$. We are concerned with the following boundary value problem:

$$\text{(P)} \qquad \begin{cases} \varepsilon^2 \mathcal{A}(x)u - a(x)u + b(x)u^p + \delta\sigma(x) = 0,\ u > 0 & \text{in } \Omega, \\ \mathcal{B}(x)u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\varepsilon > 0$ and $\delta \geq 0$ are sufficiently small constants, $\mathcal{A}(x) = \sum_{i,j=1}^{n}(\partial/\partial x_i)a_{ij}(x)(\partial/\partial x_j)$ is a strictly and uniformly elliptic operator with $a_{ij} \in C^{1,\alpha}(\overline{\Omega})$; $a_{ij} = a_{ji}$, both of a and b are of class C^2 on $\overline{\Omega}$ and bounded from below by positive constants; and the coefficient σ is a nonnegative C^2-function on $\overline{\Omega}$ with $\|\sigma\|_{L^\infty(\Omega)} = 1$. Moreover, $\mathcal{B}(x) = \sum_{i,j=1}^{n} \nu_i a_{ij}(x)(\partial/\partial x_j)$ is the co-normal differential operator, and $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$ is the unit outward normal to $\partial\Omega$.

We are interested in point condensation phenomena, or point concentration phenomena, observed in solutions of the problem (P) which mean that as $\varepsilon \downarrow 0$, the distribution of a solution concentrates around a finitely many points on $\overline{\Omega}$. In this thesis, we consider the case of only one concentration point. Problem (P) is a generalization of [14], [15] and [19], and we would like to know the effect of the spatial heterogeneity on the concentration point, especially in the case of the inhomogeneous term $\delta\sigma(x) \not\equiv 0$, i.e., $\delta > 0$ by $\|\sigma\|_{L^\infty(\Omega)} = 1$.

First, we introduce an energy functional $J_\varepsilon(u)$ corresponding to (P):

$$
\begin{aligned}
J_\varepsilon(u) := \frac{1}{2}\int_\Omega \Big(\varepsilon^2 \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + a(x)u^2\Big)\,dx \\
- \frac{1}{p+1}\int_\Omega b(x)u_+^{p+1}\,dx - \delta\int_\Omega \sigma(x)u\,dx,
\end{aligned}
\tag{1.7}
$$

for $u \in W^{1,2}(\Omega)$, where $u_+(x) = \max\{u(x), 0\}$. Then we can prove the following

Proposition 1.1 (Minimal Solution)**.** *There exists a positive number δ_* such that for each $\delta \in [0, \delta_*)$ the functional $J_\varepsilon(u)$ has a unique local minimizer $u_{m,\varepsilon}$ in $W^{1,2}(\Omega)$, regardless of the size of $\varepsilon > 0$. Moreover, if $\delta = 0$, then $u_{m,\varepsilon}(x) \equiv 0$, while if $\delta > 0$, then*

$$
0 < u_{m,\varepsilon}(x) \leq \frac{\delta}{\min_{x\in\overline{\Omega}} a(x)} \quad \textit{for all } x \in \overline{\Omega}.
$$

Definition 1.2. We call the solution obtained in Proposition 1.1 the *minimal solution* for the problem (P).

Next, we put

$$
I_\varepsilon(v) := J_\varepsilon(u_{m,\varepsilon} + v) - J_\varepsilon(u_{m,\varepsilon}) \quad \text{for } v \in W^{1,2}(\Omega). \tag{1.8}
$$

We can apply the Mountain Pass Lemma [1], [18, Theorem 2.2] to this functional I_ε and conclude as follows:

Lemma 1.2 (Mountain Pass Solution)**.** *Let δ_* be the positive constant given by Proposition 1.1 and $0 \leq \delta < \delta_*$. Then zero is a local minimum of I_ε in $W^{1,2}(\Omega)$ for each $\varepsilon > 0$. In addition, there exists an $e \in W^{1,2}(\Omega)$ such that $I_\varepsilon(e) < 0$. Let $\Gamma = \{h \in C^0([0,1]; W^{1,2}(\Omega)) \mid h(0) = 0, h(1) = e\}$. Then*

$$
c_\varepsilon = \inf_{h\in\Gamma}\max_{t\in[0,1]} I_\varepsilon(h(t))
$$

is a positive critical point of I_ε. Moreover, c_ε is the smallest positive critical value of I_ε.

We remark here that a critical point $u_c \in W^{1,2}(\Omega)$ of J_ε is a weak solution of Problem (P). Then by the elliptic regularity theory we conclude that u_c is a classical solution of (P). In particular, $u_c \in C^{2,\alpha}(\overline{\Omega})$ (see [8, Theorem 6.31 and the remark immediately after its proof in p.130]). Clearly, a classical solution of (P) gives rise to a critical point of J_ε. Hence, finding a solution of (P) is equivalent to finding a critical point of J_ε. On the other hand, $v_c \in W^{1,2}(\Omega)$ is a critical point of I_ε if and only if $u_{m,\varepsilon} + v_c$ is a critical point of J_ε. Consequently our problem is reduced to finding a critical point of I_ε.

Now let v_ε be a critical point of I_ε corresponding to c_ε: $I_\varepsilon(v_\varepsilon) = c_\varepsilon$ and $I'_\varepsilon(v_\varepsilon) = 0$. Then

$$u_\varepsilon = u_{m,\varepsilon} + v_\varepsilon$$

is a solution of (P). We call u_ε a *ground-state solution* of (P).

To be precise, we state the definition of "point concentration" for the problem (P).

Definition 1.3. A family $\{u_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ of solutions of (P) is said to exhibit a *point concentration phenomenon* if there exist M distinct points $\{P_{1,0}, \ldots, P_{M,0}\} \subset \overline{\Omega}$, a strictly decreasing sequence $\varepsilon_j \to 0$ ($j \to \infty$) and M sequences $\{P_{k,\varepsilon_j}\}_{j=1}^\infty \subset \overline{\Omega}$ with $P_{k,\varepsilon_j} \to P_{k,0}$, $k = 1, \ldots, M$, such that (i) u_{ε_j} achieves *strict* local maxima at $x = P_{k,\varepsilon_j}$ and (ii) $u_{\varepsilon_j}(x) - u_{m,\varepsilon_j}(x) - W_k((x - P_{k,\varepsilon_j})/\varepsilon_j) \to 0$ as $j \to \infty$ in $B_\rho(P_{k,0}) \cap \overline{\Omega}$, where ρ is a positive number and $W_k \in C^2(\mathbb{R}^n)$ is a positive function satisfying $W_k(0) = \max_{y\in\mathbb{R}^n} W_k(y) > 0$ and $W_k(y) \to 0$ as $|y| \to \infty$. We say that $P_{k,0} \in \overline{\Omega}$ is a *concentration point* of $\{u_\varepsilon\}$ if there is a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ such that $\varepsilon_j \downarrow 0$ and $P_{k,\varepsilon_j} \to P_{k,0}$ as $j \to \infty$ for some $1 \le k \le M$.

It will be shown in Appendix A that if there exist two positive constants c_0 and C_0 ($> c_0$) such that

$$c_0\varepsilon^n \le J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_{m,\varepsilon}) \le C_0\varepsilon^n, \tag{1.9}$$

then $\{u_\varepsilon\}$ concentrates at finitely many points on $\overline{\Omega}$.

A point $P_0 \in \overline{\Omega}$ is called a concentration point if there is a sequence $\{\varepsilon_j\}$ such that $\varepsilon_j \downarrow 0$ and $P_{k,\varepsilon_j} \to P_0$ for some $1 \le k \le M$.

The purpose of this thesis is (i) to show that the ground-state solutions $\{u_\varepsilon\}$ exhibit a point-condensation phenomenon, and they concentrate at exactly one point $P_0 \in \overline{\Omega}$; and (ii) to give a method to locate P_0 by introducing a *locator function*. We remark here that this type of function was introduced first by Wang and Zeng [23] when they considered a point concentration phenomenon for (1.6).

Definition 1.4. For any $Q \in \overline{\Omega}$, let

$$\Phi(Q) := a(Q)^{1-n/2+2/(p-1)} b(Q)^{-2/(p-1)} (\det A_Q)^{1/2},$$
$$\text{where} \quad A_Q := (a_{ij}(Q))_{1\le i,j\le n}.$$

We call $\Phi(Q)$ *the primary locator function*.

Let $u_m(Q)$ denote the smaller of the two non-negative roots of the algebraic equation

$$-a(Q)\zeta + b(Q)\zeta^p + \delta\sigma(Q) = 0. \tag{1.10}$$

Put

$$\gamma_0(Q) := \left\{ \frac{b(Q)}{a(Q)} \right\}^{1/(p-1)} u_m(Q). \tag{1.11}$$

Finally we define for $\gamma \in [0, \gamma_*)$ an important integral as follows:

$$\begin{aligned} \mathcal{I}(\gamma) &:= \mathcal{I}(\gamma; w) \\ &= \frac{1}{2} \int_{\mathbb{R}^n} \left(|\nabla w|^2 + w^2 \right) dy - \frac{1}{p+1} \int_{\mathbb{R}^n} \left\{ (\gamma + w)^{p+1} - \gamma^{p+1} - (p+1)\gamma^p w \right\} dy \end{aligned} \tag{1.12}$$

where $\gamma_* > 0$ is a sufficiently small constant (see Proposition 2.9), and $w = w_\gamma$ is a unique positive solution of the following boundary value problem:

$$\begin{cases} \Delta w - w + (\gamma + w)^p - \gamma^p = 0 & \text{in } \mathbb{R}^n, \\ \lim\limits_{|y| \to \infty} w(y) = 0, \ w(0) = \max\limits_{y \in \mathbb{R}^n} w(y). \end{cases} \tag{GS-γ}$$

Definition 1.5. For each $Q \in \overline{\Omega}$, let

$$\Lambda(Q) := \Phi(Q) \mathcal{I}(\gamma_0(Q)).$$

We call $\Lambda(Q)$ the *locator function* for the boundary value problem (P).

A few remarks are in order here. First, w_γ is known to be spherically symmetric with respect to the origin, and decays exponentially as $|y| \to \infty$ (see [5]). Second, in Section 2.3 we shall prove that (GS-γ) has at most one solution if δ is sufficiently small by making use of the Implicit Function Theorem and the uniqueness of solution of $\Delta w - w + w^p = 0$ (due to, e.g., [11]). Third, note that $\gamma_0(Q)$ is constant on $\overline{\Omega}$ if and only if either (i) $\delta = 0$ or (ii) $\sigma(x) = Ca(x)^{p/(p-1)} b(x)^{-1/(p-1)}$ where C is a constant. In the case where $\gamma_0(Q)$ is a constant function, the locator function $\Lambda(Q)$ reduces to a constant multiple of the primary locator function $\Phi(Q)$.

Note also that in the case of $\delta > 0$, we do not know what the upper bound of γ depends on since we use the Implicit Function Theorem to prove the uniqueness of solution of (GS-γ). However, by the shooting argument for ordinary differential equations as in [3] and [11], we can obtain an upper bound on γ_* depending only on p and n in the cases (a) $1 < p < \infty$ if $n = 1$, (b) $1 < p \leq 2$ if $n = 2$ and (c) $1 < p \leq n/(n-2)$ if $n \geq 3$.

The main results of this thesis are stated as follows.

Theorem 1.3. *Suppose that $P_0 \in \overline{\Omega}$ is a concentration point of a family $\{u_\varepsilon\}_{\varepsilon > 0}$ of ground-state solutions. Then, the following holds:*

(i) *If $\min_{Q \in \partial\Omega} \Lambda(Q) < 2 \min_{Q \in \overline{\Omega}} \Lambda(Q)$, then $P_0 \in \partial\Omega$. Moreover, P_0 is a minimum point of the locator function $\Lambda(Q)$ over $\partial\Omega$.*

(ii) *If $\min_{Q \in \partial\Omega} \Lambda(Q) > 2 \min_{Q \in \overline{\Omega}} \Lambda(Q)$, then $P_0 \in \Omega$. Moreover, P_0 is a minimum point of $\Lambda(Q)$ over $\overline{\Omega}$.*

Corollary 1.4. *Assume either* (i) *that $\delta = 0$ or* (ii) *that $\sigma(x) = Ca(x)^{p/(p-1)} b(x)^{-1/(p-1)}$ where C is a constant. Suppose that $P_0 \in \overline{\Omega}$ is a concentration point of a family $\{u_\varepsilon\}_{\varepsilon > 0}$ of ground-state solutions of* (P)*. Then, the following holds:*

(I) *If* $\min_{Q\in\partial\Omega}\Phi(Q) < 2\min_{Q\in\overline{\Omega}}\Phi(Q)$*, then* $P_0 \in \partial\Omega$*. Moreover,* P_0 *is a minimal point of the primary locator function* $\Phi(Q)$ *over* $\partial\Omega$*.*

(II) *If* $\min_{Q\in\partial\Omega}\Phi(Q) > 2\min_{Q\in\overline{\Omega}}\Phi(Q)$*, then* $P_{0\in\Omega}$*. Moreover,* P_0 *is a minimal point of* $\Phi(Q)$ *over* $\overline{\Omega}$*.*

Although we can locate the concentration point P_0 by finding the minimum points of Λ over $\overline{\Omega}$ and $\partial\Omega$, it is in general very difficult to calculate these minimum points. For, we must solve the boundary value problem (GS-γ) in $\mathbb{R}^n$ and know the dependence of the energy $\mathcal{I}(\gamma_0(Q);\mathbb{R}^n)$ on Q explicitly. However, if δ is sufficiently small, then the minimal points of the primary locator function Φ gives us a first approximation:

Theorem 1.5. *Suppose that* $P_0 \in \overline{\Omega}$ *is a concentration point of a family* $\{u_\varepsilon\}_{\varepsilon>0}$ *of ground-state solutions. Then, the following holds if* δ *is sufficiently small:*

(I) *If* $\min_{Q\in\partial\Omega}\Phi(Q) < 2\min_{Q\in\overline{\Omega}}\Phi(Q)$*, then* $P_0 \in \partial\Omega$*. Moreover, if all the minimum points of* $\Phi|_{\partial\Omega}$ *on* $\partial\Omega$ *are nondegenerate (as a critical point), then there exists a minimum point* Q_0 *of* Φ *over* $\partial\Omega$ *such that* $|P_0 - Q_0| = O(\delta)$ *as* $\delta \downarrow 0$*.*

(II) *If* $\min_{Q\in\partial\Omega}\Phi(Q) > 2\min_{Q\in\overline{\Omega}}\Phi(Q)$*, then* $P_0 \in \Omega$*. Moreover, if all the minimum points of* Φ *in* Ω *are nondegenerate, then there exists a minimum point* Q_0 *of* Φ *over* $\overline{\Omega}$ *such that* $|P_0 - Q_0| = O(\delta)$ *as* $\delta \downarrow 0$*.*

Consequently, we know the location of P_0 by calculating the minimum of Φ over $\overline{\Omega}$ and that over $\partial\Omega$. Moreover, we find that if the inhomogeneous term $\delta\sigma$ is sufficiently small, then $\delta\sigma$ does not affect much the location of the concentration point.

So far, we have been concerned with a concentration phenomena observed in ground-state solutions whose existence is guranteed by the Mountain Pass Lemma. However, it is quite possible that solutions with higher energy $J_\varepsilon(u) > c_\varepsilon$ exist and exhibit a point-concentration phenomenon, as in the case of spatially homogeneous equations. The following result reveals the role of the primary locator function $\Phi(Q)$ in locating the concentration point.

Theorem 1.6. *Let* $\{u_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ *be a family of positive solutions of the following Neumann problem:*

$$\begin{cases} \varepsilon^2\Delta u - a(x)u + b(x)u^p = 0,\ u(x) > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{\nu}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.13}$$

Assume that there exists a positive constant C_0 *such that* $0 < J_\varepsilon(u_\varepsilon) \le C_0\varepsilon^n$ *for* $0 < \varepsilon < \varepsilon_0$ *and that* u_ε *attains a local maximum at* $P_\varepsilon \in \Omega$ *and* $P_\varepsilon \to P_0 \in \Omega$ *as* $\varepsilon \downarrow 0$*. Then* P_0 *is a critical point of the primary locator function* Φ*, that is,* $\nabla\Phi(P_0) = 0$*. Moreover, for any* $R > 0$*,*

$$u_\varepsilon(P_\varepsilon + \varepsilon z) = v_{P_\varepsilon}(z) + O(\varepsilon) \quad \text{in } C^2(\overline{B}_R(0)) \quad \text{as } \varepsilon \downarrow 0,$$

where $v_Q(z) = (a(Q)/b(Q))^{1/(p-1)}w(a(Q)^{1/2}z)$ *and* w *is a unique positive solution of the boundary value problem*

$$\begin{cases} \Delta w - w + w^p = 0 & \text{in } \mathbb{R}^n, \\ \lim\limits_{|y|\to\infty} w(y) = 0,\ w(0) = \max\limits_{y\in\mathbb{R}^n} w(y). \end{cases} \tag{GS-0}$$

This theorem says that any solution u_ε with $0 < J_\varepsilon(u_\varepsilon) \le C_0\varepsilon^n$ looks like $v_{\mathbb{P}_\varepsilon}((x - P_\varepsilon)/\varepsilon)$ near a local maximum point P_ε as long as P_ε stays away from the boundary.

This thesis is organized as follows: In Chapter 2, we construct the minimal solution $u_{m,\varepsilon}$ and then prove the existence of mountain-pass solution stated in Lemma 1.2. Moreover, we prove the uniqueness of entire solution which appears as the first approximation of ground-state solutions. In the last section of Chapter 2 we derive an upper bound of energy of a ground-state solution, which is crucial in proving Theorem 1.3. Chapter 3 is concerned with the asymptotic behavior of ground-state solutions as $\varepsilon \downarrow 0$. In Chapter 4 we prove Theorem 1.3, Corollary 1.4 and Theorem 1.5. Finally in Chapter 5 we consider the boundary value problem (1.12) and prove Theorem 1.6.

Chapter 2

Minimal solution and ground-state solutions

2.1 Existence and local convergence of the minimal solution

In this section we prove the existence of the minimal solution $u_{m,\varepsilon}$ of (P) stated in Proposition 1.1 and then consider its behavior as $\varepsilon \downarrow 0$.

We begin with the existence of $u_{m,\varepsilon}(x)$. First, as an approximate function, we choose the solution of the boundary value problem for the linear equation

$$\begin{cases} \varepsilon^2 \mathcal{A}(x)u - a(x)u + \delta\sigma(x) = 0 & \text{in } \Omega, \\ \mathcal{B}(x)u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

which is known to have a unique solution $u_{0,\varepsilon} \in C^2(\overline{\Omega})$ (see [8, Theorem 6.31 and also the remark immediately after the end of the proof in p.130]). By the maximum principle and the assumptions $\sigma \geq 0$ and $\max \sigma = 1$, we see that

$$0 < u_{0,\varepsilon}(x) \leq \frac{\delta \|\sigma\|_{L^\infty(\Omega)}}{\min_{x\in\overline{\Omega}} a(x)} = \frac{\delta}{\min_{x\in\overline{\Omega}} a(x)} \quad \text{for all } x \in \overline{\Omega}, \tag{2.2}$$

provided that $\delta > 0$. Clearly, if $\delta = 0$, then $u_{0,\varepsilon} \equiv 0$.

In order to construct the minimal solution, we need the following maximum principle for weak solutions:

Lemma 2.1. *Let $f \in L^2(\Omega) \cap L^\infty(\Omega)$ and $u \in W^{1,2}(\Omega)$ satisfy*

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j}\, dx + \int_\Omega a(x) u \varphi\, dx = \int_\Omega f \varphi\, dx$$

for any $\varphi \in W^{1,2}(\Omega)$. Then

$$\inf_\Omega \frac{f(x)}{a(x)} \leq u(x) \leq \sup_\Omega \frac{f(x)}{a(x)} \qquad \text{for almost every } x \in \Omega.$$

In particular, if $f(x) \geq 0$, then

$$\frac{1}{\max_{x\in\overline{\Omega}} a(x)} \inf_{x\in\Omega} f(x) \leq u(x) \leq \frac{1}{\min_{x\in\overline{\Omega}} a(x)} \sup_{x\in\Omega} f(x) \qquad \text{for almost every } x \in \Omega.$$

This lemma can be proved by Stampacchia's truncation function method as in the proof of [2, Theorem 9.29].

Put $u = u_{0,\varepsilon} + \phi_\varepsilon$ and substitute this in the equation of (P). Then our problem reduces to finding a ϕ_ε which satisfies

$$\text{(2.3)} \qquad \varepsilon^2 \mathcal{A}(x)\phi_\varepsilon - a(x)\phi_\varepsilon + b(x)\{u_{0,\varepsilon} + \phi_\varepsilon\}_+^p = 0 \quad \text{in } \Omega \qquad \text{and} \qquad \mathcal{B}(x)u = 0 \quad \text{on } \partial\Omega.$$

We construct ϕ_ε by using the contraction mapping principle. For this purpose we introduce a function space X and an operator $\mathcal{F} : X \to C^0(\overline{\Omega})$ as follows:

$$X := \left\{ \phi \in C^0(\overline{\Omega}) \;\middle|\; \|\phi\|_{L^\infty(\Omega)} \le \delta \;\middle/\; \min_{x\in\overline{\Omega}} a(x) \right\},$$

$$\mathcal{F}\phi := -\left\{\varepsilon^2 \mathcal{A}(x) - a(x)\right\}^{-1} b(x)(u_{0,\varepsilon} + \phi)_+^p.$$

Here, the operator $\mathcal{F}$ is interpreted as follows: For $h \in L^q(\Omega)$ with $q \in (1, \infty)$, the boundary value problem

$$\varepsilon^2 \mathcal{A}(x)v - a(x)v + h = 0 \quad \text{in } \Omega, \qquad \mathcal{B}(x)v = 0 \quad \text{on } \partial\Omega$$

has a unique strong solution $v_q \in W^{2,q}(\Omega)$ and v_q satisfies the estimate $\|v_q\|_{W^{2,q}(\Omega)} \le C\|h\|_{L^q(\Omega)}$. Let $(\varepsilon^2 \mathcal{A}(x) - a(x))_q^{-1}$ denote the inverse operator $h \mapsto v_q$. Note that if $1 < q_1 < q_2 < \infty$ and $h \in L^{q_2}(\Omega)$, then $h \in L^{q_1}(\Omega)$ and hence $v_{q_2} = v_{q_1}$ by virtue of the uniqueness of strong solution. By the Sobolev imbedding theorem ([8, Theorem 7.26]) $W^{2,q}(\Omega) \subset C^0(\overline{\Omega})$ if $q > n/2$. Thus, if $h \in C^0(\overline{\Omega})$, then $h \in L^q(\Omega)$ and $(\varepsilon^2 \mathcal{A}(x) - a(x))_q^{-1} h$ defines a unique function $v \in C^0(\overline{\Omega})$, independent of the choice of $q > n/2$. Let us denote this operator $h \mapsto v$ by $(\varepsilon^2 \mathcal{A}(x) - a(x))_q^{-1}$, which is a bounded operator on $C^0(\overline{\Omega})$. Consequently, $\mathcal{F}$ maps X into $C^0(\overline{\Omega})$.

Let us now show that $\mathcal{F}$ is a contraction mapping on X for sufficiently small δ. First, put $v = \mathcal{F}\phi$ for $\phi \in X$. Then v may be regarded as a strong solution of

$$\begin{cases} \varepsilon^2 \mathcal{A}(x)v - a(x)v + b(x)(u_{0,\varepsilon} + \phi)_+^p = 0 & \text{in } \Omega, \\ \mathcal{B}(x)v = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, by Lemma 2.1 we see that

$$\frac{\inf_\Omega b(x)(u_{0,\varepsilon} + \phi)_+^p}{\sup_\Omega a(x)} \le v(x) \le \frac{\sup_\Omega b(x)(u_{0,\varepsilon} + \phi)_+^p}{\inf_\Omega a(x)};$$

since $u_{0,\varepsilon}, \phi \in X$, we have

$$0 \le v(x) \le \frac{\sup_\Omega b(x)}{\inf_\Omega a(x)} \cdot \frac{(2\delta)^p}{(\inf_\Omega a(x))^p}.$$

Therefore, if we put

$$\delta_1 = \left(\frac{(\inf_\Omega a(x))^p}{2^p \sup_\Omega b(x)} \right)^{1/(p-1)},$$

then $\mathcal{F}\phi \in X$ as long as $0 < \delta \leq \delta_1$. Next, we observe that for any $\phi_1, \phi_2 \in X$,

$$
\begin{aligned}
&\mathcal{F}\phi_1 - \mathcal{F}\phi_2 \\
&= -\left\{\varepsilon^2\mathcal{A}(x) - a(x)\right\}^{-1} b(x)(u_{0,\varepsilon} + \phi_1)_+^p + \left\{\varepsilon^2\mathcal{A}(x) - a(x)\right\}^{-1} b(x)(u_{0,\varepsilon} + \phi_2)_+^p \\
&= -\left\{\varepsilon^2\mathcal{A}(x) - a(x)\right\}^{-1} b(x)\left\{(u_{0,\varepsilon} + \phi_1)_+^p - (u_{0,\varepsilon} + \phi_2)_+^p\right\} \\
&= -\left\{\varepsilon^2\mathcal{A}(x) - a(x)\right\}^{-1} b(x)\left\{p(u_{0,\varepsilon} + \phi_2 + \theta(\phi_1 - \phi_2))_+^{p-1}(\phi_1 - \phi_2)\right\} \\
&\qquad\qquad\qquad\qquad \text{(for some } \theta \in (0,1) \text{ depending on } x \text{ by the mean value theorem)} \\
&= -\left\{\varepsilon^2\mathcal{A}(x) - a(x)\right\}^{-1} pb(x)(u_{0,\varepsilon} + \theta\phi_1 + (1-\theta)\phi_2)_+^{p-1}(\phi_1 - \phi_2).
\end{aligned}
$$

By Lemma 2.1, it follows that

$$
\|\mathcal{F}\phi_1 - \mathcal{F}\phi_2\|_{L^\infty(\Omega)} \leq C_1 \|u_{0,\varepsilon} + \theta\phi_1 + (1-\theta)\phi_2\|_{L^\infty(\Omega)}^{p-1} \|\phi_1 - \phi_2\|_{L^\infty(\Omega)},
$$

where $C_1 = \max_{\overline{\Omega}} b(x)/\min_{\overline{\Omega}} a(x)$. From $\phi_j \in X$ and (2.2), we see that $\|u_{0,\varepsilon} + \theta\phi_1 + (1 - \theta)\phi_2\|_{L^\infty(\Omega)} \leq (2\delta)/\min_{\overline{\Omega}} a(x)$. Therefore,

$$
\|\mathcal{F}\phi_1 - \mathcal{F}\phi_2\|_{L^\infty(\Omega)} \leq \frac{2^{p-1}\sup_\Omega b(x)}{(\inf_\Omega a(x))^p}\delta^{p-1}\|\phi_1 - \phi_2\|_{L^\infty(\Omega)}.
$$

Note that $(2\delta)^{p-1}\sup_\Omega b(x)/(\inf_\Omega a(x))^p \leq 1/2$ if $0 \leq \delta \leq \delta_1$. Hence, it is shown that $\mathcal{F}$ is a contraction mapping in X, provided that $0 \leq \delta \leq \delta_1$. Therefore, there exists a unique $\phi_\varepsilon \in X$ such that $\mathcal{F}\phi_\varepsilon = \phi_\varepsilon$, that is, ϕ_ε satisfies $-\{\varepsilon^2\mathcal{A}(x) - a(x)\}\phi_\varepsilon = b(x)(u_{0,\varepsilon} + \phi_\varepsilon)_+^p$. By the regularity theory for elliptic equations, ϕ_ε turns out to be a C^2-function on $\overline{\Omega}$; and hence $u_{m,\varepsilon} := u_{0,\varepsilon} + \phi_\varepsilon$ is a classical solution of (P) with $\|u_{m,\varepsilon}\|_{L^\infty(\Omega)} = O(\delta)$. As a matter of fact, $\phi_\varepsilon(x)$ is a nonnegative function since $b(x)(u_{0,\varepsilon} + \phi_\varepsilon)_+^p \geq 0$; therefore $0 \leq u_{0,\varepsilon}(x) \leq u_{m,\varepsilon}(x) \leq \delta/\min_{\overline{\Omega}} a(x)$ on $\overline{\Omega}$ for any $\varepsilon > 0$. Consequently, we have proved all the assertions of Proposition 1.1. Q.E.D.

Next, we consider the limiting behavior of $\{u_{m,\varepsilon}\}_{\varepsilon>0}$ as $\varepsilon \downarrow 0$. As a preliminary we prove the following

Lemma 2.2. *For each $Q \in \overline{\Omega}$, it holds that*

$$
u_{0,\varepsilon}(Q + \varepsilon z) \to \frac{\delta\sigma(Q)}{a(Q)} \quad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^n) \quad \text{as } \varepsilon \downarrow 0.
$$

Precisely speaking, if $Q \in \partial\Omega$, then we have to extend $u_{0,\varepsilon}(x)$ outside of Ω. This will be done in the proof of Proposition 3.1 in Chapter 3. The conclusion of the Lemma 2.2 applies to this extended function.

Proof. We put $v_{0,\varepsilon}(z) := u_{0,\varepsilon}(x)$, $z := (x - Q)/\varepsilon$, $\Omega_{\varepsilon,Q} := \{z \in \mathbb{R}^n \mid x = Q + \varepsilon z \in \Omega\}$ and $u_0 := \delta\sigma(Q)/a(Q)$.

Case 1): $Q \in \Omega$. From (2.1) we derive the equation satisfied by $v_{0,\varepsilon}$:

$$
\mathcal{A}(Q + \varepsilon z)v_{0,\varepsilon} - a(Q + \varepsilon z)v_{0,\varepsilon} + \delta\sigma(Q + \varepsilon z) = 0 \quad \text{in } \Omega_{\varepsilon,Q},
$$

where $\mathcal{A}(Q+\varepsilon z)$ is an operator with respect to z, i.e., $\mathcal{A}(Q+\varepsilon z) = \sum_{i,j=1}^{n}(\partial/\partial z_i)a_{ij}(Q+\varepsilon z)(\partial/\partial z_j)$. Since u_0 satisfies $\mathcal{A}(Q)u_0 - a(Q)u_0 + \delta\sigma(Q) = 0$ in Ω, subtracting the equation for u_0 from that for $v_{0,\varepsilon}$, we obtain for all $z \in \Omega_{\varepsilon,Q}$ that

$$\mathcal{A}(Q)(v_{0,\varepsilon} - u_{0,\varepsilon}) - a(Q)(v_{0,\varepsilon} - u_{0,\varepsilon}) = f_\varepsilon$$

with

$$f_\varepsilon(z) := -\{\mathcal{A}(Q + \varepsilon z) - \mathcal{A}(Q)\}v_{0,\varepsilon} - \{a(Q + \varepsilon z) - a(Q)\}v_{0,\varepsilon} - \{\delta\sigma(Q + \varepsilon z) - \sigma(Q)\}.$$

Put $\psi_\varepsilon := v_{0,\varepsilon} - u_0$. Then ψ_ε satisfies

$$\mathcal{A}(Q)\psi_\varepsilon - a(Q)\psi_\varepsilon = f_\varepsilon \quad \text{in } \Omega_{\varepsilon,Q}. \tag{2.4}$$

For any $R > 0$, there is an $\varepsilon_R > 0$ such that $B_{3R}(0) \Subset \Omega_{\varepsilon,Q}$ for $0 < \varepsilon < \varepsilon_R$. Since $\|v_{0,\varepsilon}\|_{L^\infty(\Omega_{\varepsilon,Q})} = \|u_{0,\varepsilon}\|_{L^\infty(\Omega)}$ is bounded in ε, there exists a positive constant C_R independent of ε such that $\|\psi_\varepsilon\|_{L^r(B_{3R}(0))} \le C_R$ holds and f_ε converges to zero in $C^0_{\text{loc}}(\mathbb{R}^n)$ as $\varepsilon \downarrow 0$. From the regularity estimate for elliptic equations, $\|\psi_\varepsilon\|_{W^{2,r}(B_{2R}(0))}$ is bounded as $\varepsilon \downarrow 0$. Let $r > n$ and apply the Sobolev imbedding theorem. Then there exists a constant $\beta \in (0,1)$ such that ψ_ε is bounded in $C^{1,\beta}(\overline{B}_{2R}(0))$. Moreover, by the interior Schauder estimate, ψ_ε is bounded in $C^{2,\beta}(\overline{B}_R(0))$. By the Ascoli-Arzelà theorem, for any sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ such that $\varepsilon_j \downarrow 0$, there exists a subsequence $\{\varepsilon_{j_k}\}_{k\in\mathbb{N}} \subset \{\varepsilon_j\}_{j\in\mathbb{N}}$ and a function $\psi_0^{(R)} \in C^2(\overline{B}_R(0))$ satisfying

$$\psi_{\varepsilon_{j_k}} \to \psi_0^{(R)} \quad \text{in } C^2(\overline{B}_R(0)) \quad \text{as } k \to \infty.$$

Note that the choice of the subsequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ depends on R. Next, we choose a strictly increasing sequence $\{R_k\}_{k\in\mathbb{N}}$ such that $R_k \to \infty$ as $k \to \infty$, and use the diagonal argument to obtain a subsequence of $\{\varepsilon_{j_k}\}$ which we denote by the same symbol $\{\varepsilon_{j_k}\}_{k\in\mathbb{N}}$ such that

$$\psi_{\varepsilon_{j_k}} \to \psi_0 \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^n) \quad \text{as } k \to \infty,$$

where $\psi_0 \in C^2(\mathbb{R}^n)$. Note that f_ε converges to zero locally uniformly as $\varepsilon \downarrow 0$. From (2.4), it therefore follows that ψ_0 satisfies the equation $\mathcal{A}(Q)\psi_0 - a(Q)\psi_0 = 0$ in $\mathbb{R}^n$. Since ψ_0 satisfies an equation with constant coefficients, we can scale ψ_0 appropriately so that the scaled function $\tilde{\psi}_0$ satisfies the equation $\Delta\tilde{\psi}_0 - \tilde{\psi}_0 = 0$. Here, we recall that the operator $\Delta - 1$ is invertible in $\mathcal{S}'(\mathbb{R}^n)$ (the space of tempered distributions). Note also that $\|\tilde{\psi}_0\|_{L^\infty(\mathbb{R}^n)}$ is bounded because of the boundedness of $\|\psi_\varepsilon\|_{L^\infty(\Omega_{\varepsilon,Q})}$. Hence $\tilde{\psi}_0 \in \mathcal{S}'(\mathbb{R}^n)$, and hence we have $\tilde{\psi}_0 = \psi_0 \equiv 0$ in $\mathbb{R}^n$. Now, the limit ψ_0 is unique and therefore the entire sequence $\{\psi_{\varepsilon_j}\}$ converges to $\psi_0 \equiv 0$. Hence, $\psi_\varepsilon \to 0$ as $\varepsilon \downarrow 0$ because $\{\varepsilon_j\}$ is arbitrary. Recall that $\psi_\varepsilon = v_{0,\varepsilon} - \delta\sigma(Q)/a(Q)$; therefore, $u_{0,\varepsilon_j}(Q + \varepsilon_j y)$ converges to $\delta\sigma(Q)/a(Q)$ in $C^2_{\text{loc}}(\mathbb{R}^n)$ as $j \to \infty$.

Case 2): $Q \in \partial\Omega$. It suffices to show that $v_{0,\varepsilon}(z) = u_{0,\varepsilon}(S(\varepsilon z', \varepsilon|z_n|))$ converges to $\delta\sigma(Q)/a(Q)$ in $B_R(0)$ for each $R > 0$. For details, see the proof of Proposition 3.1, for example the definition (3.10) of S and how to extend $u_{0,\varepsilon}$ outside Ω along the conormal vector. Then, the equation of $\psi_\varepsilon := v_{0,\epsilon} - u_0$ converges to an elliptic equation similar to (2.4). From L^∞-boundedness of ψ_ε, similarly to the case of $Q \in \Omega$, there exists $\psi_0 \in C^2(\overline{B}_R(0))$ such that ψ_ε converges to ψ_0 in $C^2(\overline{B}_R(0))$ and ψ_0 satisfies an elliptic equation similar to (2.4). Since $\psi_0 \equiv 0$, we see that v_{0,ε_j} converges to $\delta\sigma(Q)/a(Q)$ in $C^2_{\text{loc}}(\mathbb{R}^n)$ as $\varepsilon_j \downarrow 0$. Q.E.D.

Lemma 2.3. *For $Q \in \overline{\Omega}$, let $u_m(Q)$ denote the smallest nonnegative root of the algebraic equation $-a(Q)\zeta + b(Q)\zeta^p + \delta\sigma(Q) = 0$. Then, there exists a subsequence $\{\varepsilon_j\}_{j\in\mathbb{N}} \subset \{\varepsilon\}_{\varepsilon>0}$ converging to zero as $j \to \infty$ such that*

$$u_{m,\varepsilon_j}(Q + \varepsilon_j z) \to u_m(Q) \quad \textit{in } C^2_{\mathrm{loc}}(\mathbb{R}^n) \quad \textit{as } j \to \infty.$$

Precisely speaking, if $Q \in \partial\Omega$, then we have to extend $u_{m,\varepsilon}(x)$ outside of Ω. This will be done in the proof of Proposition 3.1 in Chapter 3. The conclusion of Lemma 2.3 applies to this extended function.

Proof. As in the proof of Lemma 2.2, we treat only the case of $Q \in \Omega$. Put $v_{m,\varepsilon}(z) := u_{m,\varepsilon}(x)$, $z := (x - Q)/\varepsilon$ and $\Omega_{\varepsilon,Q} := \{z \in \mathbb{R}^n \mid x = Q + \varepsilon z \in \Omega\}$. Let $\psi_\varepsilon(z) := v_{m,\varepsilon}(z) - u_m(Q)$. Since $u_m(Q)$ is the smaller of the two nonnegative roots of $-a(Q)\zeta + b(Q)\zeta^p + \delta\sigma(Q) = 0$ and is independent of z, we have

$$\mathcal{A}(Q)\psi_\varepsilon - a(Q)\psi_\varepsilon + b(Q)\{(u_m(Q) + \psi_\varepsilon)^p - u_m(Q)^p\} =: f_\varepsilon,$$

where we have defined

$$\begin{aligned} f_\varepsilon(z) = &-\{\mathcal{A}(Q + \varepsilon z) - \mathcal{A}(Q)\}v_{m,\varepsilon} + \{a(Q + \varepsilon z) - a(Q)\}v_{m,\varepsilon} \\ &- \{b(Q + \varepsilon z) - b(Q)\}v^p_{m,\varepsilon} - \delta\{\sigma(Q + \varepsilon z) - \sigma(Q)\}. \end{aligned}$$

From the uniform boundedness of $\|v_{m,\varepsilon}\|_{L^\infty(\Omega_{\varepsilon,Q})} = \|u_{m,\varepsilon}\|_{L^\infty(\Omega)}$ with respect to ε, in a manner similar to that in the proof of Lemma 2.2, we see that f_ε converges to zero in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$ as $\varepsilon \downarrow 0$. Moreover, $b(Q)\{(u_m(Q) + \psi_{m,\varepsilon})^p - u_m(Q)^p\}$ remains bounded as $\varepsilon \to 0$. Hence for any sequence of positive numbers $\{\varepsilon_j\}$ converging to zero, we can find a subsequence, which is denoted by $\{\varepsilon_j\}$ again, and a function $\psi_0 \in C^2(\mathbb{R}^n)$ such that

$$\psi_{\varepsilon_j} \to \psi_0 \quad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^n) \quad \text{as } \varepsilon_j \downarrow 0.$$

Therefore, it suffices to show $\psi_0 \equiv 0$. Note that the boundedness of ψ_ε implies that ψ_0 is a bounded function and satisfies the equation $\mathcal{A}(Q)\psi_0 - a(Q)\psi_0 + b(Q)\{(u_m(Q) + \psi_0)^p - u_m(Q)^p\} = 0$ in $\mathbb{R}^n$. Since the coefficients $\mathcal{A}(Q)$, $a(Q)$, $b(Q)$ and $u_m(Q)$ are independent of z, we get the equation $\Delta\tilde{\psi}_0 - \tilde{\psi}_0 + (\tilde{u}_m(Q) + \tilde{\psi}_0)^p - \tilde{u}_m(Q)^p = 0$ by a suitable change of variables $z \mapsto \tilde{z}$, where $\tilde{\psi}_0(\tilde{z}) = (b(Q)/a(Q))\psi_0(z)$ and $\tilde{u}_m(Q) = (b(Q)/a(Q))^{1/(p-1)}u_m(Q)$. Let G be the Green's function for $1 - \Delta$ on $\mathbb{R}^n$. Then

$$\tilde{\psi}_0(z) = \int_{\mathbb{R}^n} G(z - \zeta)\left\{(\tilde{u}_m(Q) + \tilde{\psi}_0(\zeta))^p - \tilde{u}_m(Q)^p\right\} d\zeta.$$

By the boundedness of $\|\tilde{\psi}_0\|_{L^\infty(\mathbb{R}^n)}$ and the positivity of G, we see that

$$\begin{aligned} |\tilde{\psi}_0(z)| &\le \int_{\mathbb{R}^n} G(z - \zeta)\left|(\tilde{u}_m(Q) + \tilde{\psi}_0(\zeta))^p - \tilde{u}_m(Q)^p\right| d\zeta \\ &\le p\left(\tilde{u}_m(Q) + \|\tilde{\psi}_0\|_{L^\infty(\mathbb{R}^n)}\right)^{p-1} \|\tilde{\psi}_0\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} G(z - \zeta)\, d\zeta. \end{aligned}$$

Notice that $U(z) := \int_{\mathbb{R}^n} G(z-\zeta)\cdot 1\, d\zeta$ satisfies the equation $\Delta U - U + 1 = 0$ and therefore $U \equiv 1$. Hence we get the following inequality:

$$|\tilde{\psi}_0(z)| \le p\left(\tilde{u}_m(Q) + \|\tilde{\psi}_0\|_{L^\infty(\mathbb{R}^n)}\right)^{p-1} \|\tilde{\psi}_0\|_{L^\infty(\mathbb{R}^n)}, \quad z \in \mathbb{R}^n.$$

Since $z \in \mathbb{R}^n$ is arbitrary, this implies

$$\|\tilde{\psi}_0\|_{L^\infty(\mathbb{R}^n)} \le p\left(\tilde{u}_m(Q) + \|\tilde{\psi}_0\|_{L^\infty(\mathbb{R}^n)}\right)^{p-1} \|\tilde{\psi}_0\|_{L^\infty(\mathbb{R}^n)},$$

and we obtain

$$\left(1 - p\left(\tilde{u}_m(Q) + \|\tilde{\psi}_0\|_{L^\infty(\mathbb{R}^n)}\right)^{p-1}\right)\|\tilde{\psi}_0\|_{L^\infty(\mathbb{R}^n)} \le 0.$$

Here, recall that $\tilde{u}_m(Q) = (b(Q)/a(Q))^{1/(p-1)} u_m(Q) = O(\delta)$ and $\|\tilde{\psi}_0\|_{L^\infty(\mathbb{R}^n)} = O(\delta)$ by Proposition 1.1. Therefore $\tilde{u}_m(Q) + \|\tilde{\psi}_0\|_{L^\infty(\mathbb{R}^n)} = O(\delta)$ for sufficiently small δ, so that

$$1 - p\left(\tilde{u}_m(Q) + \|\tilde{\psi}_0\|_{L^\infty(\mathbb{R}^n)}\right)^{p-1} > 0$$

whenever δ is sufficiently small. Consequently, by the two previous inequalities, we have $\|\tilde{\psi}_0\|_{L^\infty(\mathbb{R}^n)} \le 0$, i.e., $\tilde{\psi}_0 \equiv 0$. Q.E.D.

2.2 Existence of a ground-state solution u_ε

In view of the definition of the energy functionals (1.7) and (1.8), we notice the following:

***Remark* 2.4.** *Since $u_{m,\varepsilon}$ is a solution of* (P), $J'_\varepsilon(u_{m,\varepsilon})v = 0$ *holds for any* $v \in W^{1,2}(\Omega)$. *Thus,* $I_\varepsilon(v)$ *may be arranged as follows:*

$$\begin{aligned} I_\varepsilon(v) = &\frac{1}{2}\int_\Omega \left(a_{ij}(x)\frac{\partial v}{\partial x_j}\frac{\partial v}{\partial x_i} + a(x)v^2\right)dx \\ &- \frac{1}{p+1}\int_\Omega b(x)\left((u_{m,\varepsilon}+v)_+^{p+1} - u_{m,\varepsilon}^{p+1} - (p+1)u_{m,\varepsilon}^p v\right)dx, \end{aligned}$$

where we adopt the Einstein convention (i.e., $a_{ij}\xi_i\xi_j$ means $\sum_{i,j=1}^n a_{ij}\xi_i\xi_j$).

Definition 2.1. Let $\alpha(x) := a(x) - pb(x)u_{m,\varepsilon}(x)^{p-1}$ and $g(\gamma, v) := (\gamma+v)_+^p - \gamma^p - p\gamma^{p-1}v$ for $\gamma \ge 0$, $v \in \mathbb{R}$. For any $u, v \in W^{1,2}(\Omega)$, we define an inner product $\langle\cdot,\cdot\rangle_{E_\varepsilon}$ on $W^{1,2}(\Omega)$ by

$$\begin{aligned} \langle u, v\rangle_{E_\varepsilon} &:= \int_\Omega \left(\varepsilon^2 a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + (a(x) - pb(x)u_{m,\varepsilon}^{p-1})uv\right)dx \\ &= \int_\Omega \left(\varepsilon^2 a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + \alpha(x)uv\right)dx \end{aligned}$$

and denote by E_ε the space $W^{1,2}(\Omega)$ equipped with this inner product. Moreover, we define as follows:

$$M_\varepsilon := \left\{v \in E_\varepsilon\backslash\{0\} \;\middle|\; \|v\|_{E_\varepsilon}^2 = \int_\Omega b(x)g(u_{m,\varepsilon}, v)v\,dx\right\} : \text{ Nehari manifold},$$

$$c_\varepsilon := \inf_{h\in\Gamma_\varepsilon}\max_{0\le t\le 1} I_\varepsilon(h(t)), \quad \text{where } \Gamma_\varepsilon := \left\{h \in C^0([0,1]; E_\varepsilon) \;\middle|\; h(0), h(1) \not\equiv 0, I_\varepsilon(h(1)) \le 0\right\},$$

$$c_\varepsilon^* := \inf_{v\in E_\varepsilon\cap\{v\ge 0, \not\equiv 0\}} \max_{t\ge 0} I_\varepsilon(tv),$$

$$c_\varepsilon^{**} := \inf_{v\in M_\varepsilon} I_\varepsilon(v).$$

Since the coefficients a_{ij}, a, b and σ are bounded, the norm $\|\cdot\|_{E_\varepsilon}$ is equivalent to the standard norm $\|u\|_{W^{1,2}(\Omega)} = (\int_\Omega |\nabla u|^2\,dx + \int_\Omega u^2\,dx)^{1/2}$ on $W^{1,2}(\Omega)$. By the properties of $g(u_{m,\varepsilon}, v)$ with respect to v, the following lemma holds:

Lemma 2.5. $c_\varepsilon = c_\varepsilon^* = c_\varepsilon^{**}$.

Since this lemma is verified in a fashion similar to Lemma 2.1 in [23], we omit the proof here.

Lemma 2.6. *The quantity c_ε defined in Definition 2.1 is a positive critical value of $I_\varepsilon(v)$.*

This lemma is a consequence of the Mountain Pass Lemma ([1], [18, Theorem 2.2]), which is stated as follows:

Lemma 2.7 (Mountain pass lemma)**.** *Let E_ε be a real Banach space and $I_\varepsilon \in C^1(E_\varepsilon; \mathbb{R}^n)$ satisfying (PS). Suppose $I_\varepsilon(0) = 0$ and*

(i) *there are constants $\rho, \alpha > 0$ such that $I_\varepsilon|_{\partial B_\rho} \geq \alpha$, and*

(ii) *there is an $v_* \in E_\varepsilon \setminus \overline{B}_\rho$ such that $I_\varepsilon(v_*) \leq 0$.*

Then I_ε possesses a critical value $c_\varepsilon \geq \alpha$. Moreover c_ε can be characterized as

$$c_\varepsilon = \inf_{\eta \in \Gamma} \max_{t \in [0,1]} I_\varepsilon(\eta(t))$$

where $\Gamma = \{\eta \in C^1([0,1], E_\varepsilon) \mid \eta(0) = 0, \eta(1) = e\}$.

To check the (PS) condition for I_ε, we prove the following claim.

Claim 2.8. *If $\theta \in (\max\{1/3, 1/(p+1)\}, 1/2)$, then for any $x \in \overline{\Omega}$, $v \in \mathbb{R}$,*

$$G(u_{m,\varepsilon}(x), v) \leq \theta g(u_{m,\varepsilon}(x), v)|v|$$

where

$$G(\gamma, v) := \int_0^v g(\gamma, t)\,dt = \frac{1}{p+1}\Big\{(\gamma + v)_+^{p+1} - \gamma^{p+1} - (p+1)\gamma^p v - \frac{p(p+1)}{2}\gamma^{p-1}v^2\Big\}.$$

Proof of Claim 2.8. Let $\varphi(v) := \theta g(u_{m,\varepsilon}, v)|v| - G(u_{m,\varepsilon}, v)$. We prove that $\varphi(v) \geq 0$ for any $v \in \mathbb{R}^n$. In the following we suppress $u_{m,\varepsilon}$ in g and write $g(v) = g(u_{m,\varepsilon}, v)$.

Case 1: $v \geq 0$. By differentiating φ, we have

$$\varphi'(v) = \theta g'(v)v - (1-\theta)g(v),$$
$$\varphi''(v) = \theta g''(v)v - (1-2\theta)g'(v),$$
$$\varphi'''(v) = \theta g'''(v)v - (1-3\theta)g''(v).$$

Recalling $g(v) = (u_{m,\varepsilon} + v)_+^p - u_{m,\varepsilon}^p - pu_{m,\varepsilon}^{p-1}v$, we obtain that

$$\varphi'''(v) = p(p-1)\Big[(3\theta - 1)u_{m,\varepsilon} + \{\theta(p+1) - 1\}v\Big](u_{m,\varepsilon} + v)^{p-3}.$$

Since, $u_{m,\varepsilon}$ is strictly positive by Proposition 1.1, so is $(3\theta - 1)u_{m,\varepsilon} + \{\theta(p+1) - 1\}v$ for any $\theta > \max\{1/3, 1/(p+1)\}$ and $v \geq 0$. Hence, $\varphi''' > 0$. Therefore, φ'' is strictly increasing. Thus $\varphi''(v) > \varphi''(0)$ is obtained for $v > 0$, and we have $\varphi'' > 0$. Since $\varphi''(0) = -(1 - 2\theta)g'(0) = p\{(u_{m,\varepsilon}+0)^{p-1} - u_{m,\varepsilon}^{p-1}\} = 0$, we see that φ' is non-decreasing. From $\varphi'(v) \geq \varphi'(0) = -(1-\theta)g(0) = 0$ for $v \geq 0$, it follows that φ is non-decreasing and $\varphi(v) \geq -G(u_{m,\varepsilon}, 0) = 0$ for $v \geq 0$.

Case 2: $-u_{m,\varepsilon} < v \leq 0$. By differentiating φ for $v \leq 0$, we see that

$$\begin{aligned}\varphi'(v) &= -\theta g'(v)v - (1+\theta)g(v),\\ \varphi''(v) &= -\theta g''(v)v - (1+2\theta)g'(v),\\ \varphi'''(v) &= -\theta g'''(v)v - (1+3\theta)g''(v).\end{aligned}$$

By the definition of g, it holds that

$$\begin{aligned}\varphi''(v) &= -p(p-1)\theta(u_{m,\varepsilon} + v)^{p-2}v - p(1+2\theta)\{(u_{m,\varepsilon} + v)^{p-1} - u_{m,\varepsilon}^{p-1}\}\\ &= p\Big[(p-1)\theta(u_{m,\varepsilon} + v)^{p-2}(-v) - (1+2\theta)\{(u_{m,\varepsilon} + v)^{p-1} - u_{m,\varepsilon}^{p-1}\}\Big]\\ &= p\Big[(u_{m,\varepsilon} + v)^{p-2}\Big\{\Big(1 + (p+1)\theta\Big)(-v) - (1+2\theta)u_{m,\varepsilon}\Big\} + (1+2\theta)u_{m,\varepsilon}^{p-1}\Big].\end{aligned}$$

In the case $p \geq 2$, we have for $-u_{m,\varepsilon} < v \leq 0$ that

$$\begin{aligned}\varphi''(v) &\geq p\Big[(u_{m,\varepsilon} + v)^{p-2}\Big\{-(1+2\theta)u_{m,\varepsilon}\Big\} + (1+2\theta)u_{m,\varepsilon}^{p-1}\Big]\\ &= p(1+2\theta)u_{m,\varepsilon}\{u_{m,\varepsilon}^{p-2} - (u_{m,\varepsilon} + v)^{p-2}\} \geq 0,\end{aligned}$$

and $\varphi'(v)$ is non-decreasing for $-u_{m,\varepsilon} < v \leq 0$. Since $\varphi'(v) \leq \varphi'(0) = -(1+\theta)g(0) = 0$ holds, $\varphi(v)$ is non-increasing for $-u_{m,\varepsilon} < v \leq 0$. Hence $\varphi(v) \geq \varphi(0) = -G(u_{m,\varepsilon}, 0) = 0$.

In the case $p < 2$, we calculate φ''' to see that

$$\begin{aligned}\varphi'''(v) &= p(u_{m,\varepsilon} + v)^{p-3}\\ &\quad \times \Big[(p-2)\Big\{(1 + (p+1)\theta)(-v) - (1+2\theta)u_{m,\varepsilon}\Big\} - (1 + (p+1)\theta)(u_{m,\varepsilon} + v)\Big]\\ &= p(u_{m,\varepsilon} + v)^{p-3}\Big\{(p-1)(1 + (p+1)\theta)(-v) - ((p-2)(1+2\theta) + 1 + (p+1)\theta)u_{m,\varepsilon}\Big\}\\ &= p(u_{m,\varepsilon} + v)^{p-3}\Big\{(p-1)(1 + (p+1)\theta)(-v) - (p-1)(1+3\theta)u_{m,\varepsilon}\Big\}\\ &= p(p-1)(u_{m,\varepsilon} + v)^{p-3}\Big\{(1 + (p+1)\theta)(-v) - (1+3\theta)u_{m,\varepsilon}\Big\}.\end{aligned}$$

Here, for $-v < u_{m,\varepsilon}$ we observe that

$$\begin{aligned}\varphi'''(v) &\leq p(p-1)(u_{m,\varepsilon} + v)^{p-3}\Big\{(1 + (p+1)\theta)u_{m,\varepsilon} - (1+3\theta)u_{m,\varepsilon}\Big\}\\ &= p(p-1)(u_{m,\varepsilon} + v)^{p-3}(p-2)\theta u_{m,\varepsilon} < 0,\end{aligned}$$

and hence $\varphi''(v)$ is decreasing for $-u_{m,\varepsilon} < v \leq 0$. Since $\varphi''(v) \geq \varphi''(0) = 0$, we see that $\varphi'(v)$ is non-decreasing for $-u_{m,\varepsilon} < v \leq 0$. Hence $\varphi'(v) \leq \varphi'(0) = 0$ and φ is non-increasing. Consequently, $\varphi(v) \geq \varphi(0) = -G(u_{m,\varepsilon}, 0) = 0$.

Case 3: $v \le -u_{m,\varepsilon}$. By virtue of $(u_{m,\varepsilon}+v)_+ = 0$, we calculate φ' as follows:

$$\begin{aligned}\varphi'(v) &= -\theta g'(v)v - (1+\theta)g(v)\\ &= -p\theta\{(u_{m,\varepsilon}+v)_+^{p-1} - u_{m,\varepsilon}^{p-1}\}v - (1+\theta)\{(u_{m,\varepsilon}+v)_+^p - u_{m,\varepsilon}^p - pu_{m,\varepsilon}^{p-1}v\}\\ &= -p\theta\{-u_{m,\varepsilon}^{p-1}\}v - (1+\theta)\{-u_{m,\varepsilon}^p - pu_{m,\varepsilon}v\} = (1+2\theta)pu_{m,\varepsilon}^{p-1}v + (1+\theta)u_{m,\varepsilon}^p\\ &\le (1+2\theta)pu_{m,\varepsilon}^{p-1}(-u_{m,\varepsilon}) + (1+\theta)u_{m,\varepsilon}^p = -u_{m,\varepsilon}^p\{(1+2\theta)p - (1+\theta)\} < 0.\end{aligned}$$

Thus, $\varphi(v)$ is decreasing for $v \le -u_{m,\varepsilon}$. Hence $\varphi(v) \ge \varphi(-u_{m,\varepsilon})$ holds. Here, from

$$\begin{aligned}&\varphi(-u_{m,\varepsilon}) = \theta g(-u_{m,\varepsilon})u_{m,\varepsilon} - G(-u_{m,\varepsilon})\\ &= \theta\Big(-u_{m,\varepsilon}^p + pu_{m,\varepsilon}^p\Big)u_{m,\varepsilon} - \frac{1}{p+1}\Big\{-u_{m,\varepsilon}^{p+1} + (p+1)u_{m,\varepsilon}^{p+1} - \frac{p(p+1)}{2}u_{m,\varepsilon}^{p+1}\Big\}\\ &= \frac{1}{2(p+1)}u_{m,\varepsilon}^{p+1}\Big\{(p-1)(p+1)2\theta - 2p + p(p+1)\Big\} = \frac{p-1}{2(p+1)}u_{m,\varepsilon}^{p+1}\Big\{(p+1)2\theta + p\Big\} > 0,\end{aligned}$$

it follows that $\varphi(v) > 0$ for $v \le -u_{m,\varepsilon}$.

We therefore have proved that $\varphi(v) \ge 0$ for all $v \in \mathbb{R}$. Q.E.D.

To use the Mountain Pass Lemma, we verify only the (PS) condition by using Claim 2.8, since the other conditions are verified easily (see, e.g., [12]).

Verification of the (PS) condition. Suppose that $\{v_k\}_{k\in\mathbb{N}} \subset E_\varepsilon$ is any sequence such that $I_\varepsilon(v_k)$ is bounded and $I'_\varepsilon(v_k) \to 0$ as $k \to \infty$. By $I'_\varepsilon(v_k)v_k = \|v_k\|_{E_\varepsilon}^2 - \int_\Omega b(x)g(u_{m,\varepsilon}, v_k)v_k\,dx$, we can calculate $I(v_k)$ as follows:

$$\begin{aligned}I_\varepsilon(v_k) &= \frac{1}{2}\int_\Omega\Big(a_{ij}(x)\frac{\partial v_k}{\partial x_j}\frac{\partial v_k}{\partial x_i} + \Big\{a(x) - pb(x)u_{m,\varepsilon}(x)^{p-1}\Big\}v_k^2\Big)\,dx\\ &\quad - \frac{1}{p+1}\int_\Omega b(x)\Big\{(u_{m,\varepsilon}+v_k)_+^{p+1} - u_{m,\varepsilon}^{p+1} - (p+1)u_{m,\varepsilon}^p v_k - \frac{p(p+1)}{2}u_{m,\varepsilon}^{p-1}v_k^2\Big\}\,dx\\ &= \frac{1}{2}\|v_k\|_{E_\varepsilon}^2 - \int_\Omega b(x)G(u_{m,\varepsilon}, v_k)\,dx\\ &= \frac{1}{2}I'_\varepsilon(v_k)v_k + \frac{1}{2}\int_\Omega b(x)g(u_{m,\varepsilon}, v_k)v_k\,dx - \int_\Omega b(x)G(u_{m,\varepsilon}, v_k)\,dx\\ &= \frac{1}{2}I'_\varepsilon(v_k)v_k + \int_\Omega b(x)\Big\{\frac{1}{2}g(u_{m,\varepsilon}, v_k)v_k - G(u_{m,\varepsilon}, v_k)\Big\}\,dx.\end{aligned}$$

We verify the (PS) condition in the two cases. *Case 1*: $\liminf_{k\to\infty}\|v_k\|_{E_\varepsilon} = 0$, and *Case 2*: $\liminf_{k\to\infty}\|v_k\|_{E_\varepsilon} > 0$; we may assume that there exists $c_0 > 0$ such that $c_0 \le \|v_k\|_{E_\varepsilon}$ for any $k \in \mathbb{N}$.

Case 1: By the assumption, there exists a subsequence $\{k_j\}_{j\in\mathbb{N}}$ such that $\|v_{k_j}\|_{E_\varepsilon} \to 0$ as $j \to \infty$. This $\{v_{k_j}\}_{j\in\mathbb{N}}$ is a convergent subsequence in E_ε.

Case 2: By the above identity, it holds that

$$\begin{aligned}
&\int_\Omega b(x)\left\{\frac{1}{2}g(v_k)\frac{|v_k|}{\|v_k\|_{E_\varepsilon}} - \frac{G(v_k)}{\|v_k\|_{E_\varepsilon}}\right\}dx \\
&= \frac{1}{2}\int_\Omega b(x)g(v_k)\frac{|v_k| - v_k}{\|v_k\|_{E_\varepsilon}}\,dx + \int_\Omega b(x)\left\{\frac{1}{2}g(v_k)\frac{v_k}{\|v_k\|_{E_\varepsilon}} - G(v_k)\frac{1}{\|v_k\|_{E_\varepsilon}}\right\}dx \\
&= \int_{\Omega\cap\{v_k<0\}} b(x)g(v_k)\frac{-v_k}{\|v_k\|_{E_\varepsilon}}\,dx + \frac{I_\varepsilon(v_k)}{\|v_k\|_{E_\varepsilon}} - \frac{1}{2}I'_\varepsilon(v_k)\frac{v_k}{\|v_k\|_{E_\varepsilon}}.
\end{aligned}$$

Since $g(v_k) \le -pu_{m,\varepsilon}^{p-1}v_k$ holds for $v_k < 0$ and $\|v_k\|_{E_\varepsilon} \ge c_0$, we see that

$$\begin{aligned}
&\int_\Omega b(x)\left\{\frac{1}{2}g(v_k)\frac{|v_k|}{\|v_k\|_{E_\varepsilon}} - \frac{G(v_k)}{\|v_k\|_{E_\varepsilon}}\right\}dx \\
&\le \int_{\Omega\cap\{v_k<0\}} pb(x)u_{m,\varepsilon}^{p-1}\frac{v_k^2}{\|v_k\|_{E_\varepsilon}}\,dx + \frac{I_\varepsilon(v_k)}{c_0} - \frac{1}{2}I'_\varepsilon(v_k)\frac{v_k}{\|v_k\|_{E_\varepsilon}} \\
&\le p\|b\|_{L^\infty}\|u_{m,\varepsilon}\|_{L^\infty}^{p-1}\frac{1}{\|v_k\|_{E_\varepsilon}}\|v_k\|_{L^2}^2 + \frac{I_\varepsilon(v_k)}{c_0} - \frac{1}{2}I'_\varepsilon(v_k)\frac{v_k}{\|v_k\|_{E_\varepsilon}}.
\end{aligned}$$

Here, we note that $\|I'_\varepsilon(v_k)\|_{\mathcal{L}(E_\varepsilon,\mathbb{R})} \to 0$ as $k \to \infty$ and $I_\varepsilon(v_k)$ is bounded by the assumption, therefore $I_\varepsilon(v_k)/c_0 - I'_\varepsilon(v_k)v_k/(2\|v_k\|_{E_\varepsilon})$ is bounded, that is, there exists a constant $M > 0$ such that $|I_\varepsilon(v_k)/c_0 - I'_\varepsilon(v_k)v_k/(2\|v_k\|_{E_\varepsilon})| < M$. By $\|v_k\|_{L^2}^2 \le (1/\inf_\Omega \alpha(x))\|v_k\|_{E_\varepsilon}^2$, we obtain

$$\begin{aligned}
&\int_\Omega b(x)\left\{\frac{1}{2}g(v_k)\frac{|v_k|}{\|v_k\|_{E_\varepsilon}} - \frac{G(v_k)}{\|v_k\|_{E_\varepsilon}}\right\}dx \\
&\le \frac{p\|b\|_{L^\infty}}{\min_{x\in\overline{\Omega}}\alpha(x)}\|u_{m,\varepsilon}\|_{L^\infty}^{p-1}\frac{1}{\|v_k\|_{E_\varepsilon}}\|v_k\|_{E_\varepsilon}^2 + M = C\delta^{p-1}\frac{1}{\|v_k\|_{E_\varepsilon}}\|v_k\|_{E_\varepsilon}^2 + M.
\end{aligned}$$

Substituting $\|v_k\|_{E_\varepsilon}^2 = I'_\varepsilon(v_k)v_k + \int_\Omega b(x)g(u_{m,\varepsilon}, v_k)v_k\,dx$ in the right-hand side of the above inequality, we have

$$\begin{aligned}
&\int_\Omega b(x)\left\{\frac{1}{2}g(v_k)\frac{|v_k|}{\|v_k\|_{E_\varepsilon}} - \frac{G(v_k)}{\|v_k\|_{E_\varepsilon}}\right\}dx \\
&\le C\delta^{p-1}\frac{1}{\|v_k\|_{E_\varepsilon}}\left\{I'_\varepsilon(v_k)v_k + \int_\Omega b(x)g(v_k)v_k\,dx\right\} + M \\
&\le C\delta^{p-1}\left\{I'_\varepsilon(v_k)\frac{v_k}{\|v_k\|_{E_\varepsilon}} + \int_\Omega b(x)g(v_k)\frac{|v_k|}{\|v_k\|_{E_\varepsilon}}\,dx\right\} + M \\
&\le C\delta^{p-1}\int_\Omega b(x)g(v_k)\frac{|v_k|}{\|v_k\|_{E_\varepsilon}}\,dx + M'.
\end{aligned}$$

Therefore, we get

$$\int_\Omega b(x)\left\{\left(\frac{1}{2} - C\delta^{p-1}\right)g(v_k)\frac{|v_k|}{\|v_k\|_{E_\varepsilon}} - G(v_k)\frac{1}{\|v_k\|_{E_\varepsilon}}\right\}dx \le M'.$$

By using Claim 2.8, we have $(1/2 - C\delta^{p-1} - \theta)\int_\Omega b(x)g(v_k)|v_k|/\|v_k\|_{E_\varepsilon}\,dx \le M'$. For small δ, we can choose a θ such that $\max\{1/3, 1/(p+1)\} < \theta < 1/2 - C\delta^{p-1}$, and hence

$$\int_\Omega b(x)g(v_k)\frac{|v_k|}{\|v_k\|_{E_\varepsilon}}\,dx \le M'' < \infty.$$

Since $g(v) \ge 0$ for $v \in \mathbb{R}$, it follows that

$$\left|\int_\Omega b(x)g(v_k)\frac{v_k}{\|v_k\|_{E_\varepsilon}}\,dx\right| \le \int_\Omega b(x)g(v_k)\frac{|v_k|}{\|v_k\|_{E_\varepsilon}}\,dx \le M'' < \infty.$$

Therefore, $\|v_k\|_{E_\varepsilon}$ is bounded since $\|v_k\|_{E_\varepsilon} \le M'' + I'_\varepsilon(v_k)v_k/\|v_k\|_{E_\varepsilon} \to M''$ as $k \to \infty$. By the equivalence of $\|\cdot\|_{E_\varepsilon}$ and $\|\cdot\|_{W^{1,2}(\Omega)}$, the sequence $\{v_k\}_{k\in\mathbb{N}}$ is bounded in $W^{1,2}(\Omega)$. Hence, there exists a subsequence $\{v^{(1)}_{k_j}\}_{j\in\mathbb{N}} \subset \{v_k\}_{k\in\mathbb{N}}$ and $v_0 \in W^{1,2}(\Omega)$ such that $v^{(1)}_{k_j}$ converges to v_0 weakly in $W^{1,2}(\Omega)$. By the compactness of the imbedding $W^{1,2}(\Omega) \hookrightarrow L^{p+1}(\Omega)$, there exists a subsequence $\{v^{(2)}_{k_j}\}_{j\in\mathbb{N}} \subset \{v^{(1)}_{k_j}\}_{j\in\mathbb{N}}$ and $\tilde{v}_0 \in L^2 \cap L^{p+1}(\Omega)$ such that $v^{(2)}_{k_j}$ converges to $\tilde{v}_0$ strongly in $L^2 \cap L^{p+1}(\Omega)$. By the uniqueness of the weak limit, $\tilde{v}_0 = v_0$ in $L^2 \cap L^{p+1}(\Omega)$. Since $I'_\varepsilon(v^{(2)}_{k_j}) \to 0$ and $v^{(2)}_{k_j} \rightharpoonup v_0$ weakly in $W^{1,2}(\Omega)$ as $j \to \infty$, it follows that

$$\langle v^{(2)}_{k_j}, v_0\rangle_{E_\varepsilon} - \int_\Omega b(x)g(v^{(2)}_{k_j})v_0\,dx = I'_\varepsilon(v^{(2)}_{k_j})v_0 \to 0,$$

$$\langle v^{(2)}_{k_j}, v_0\rangle_{E_\varepsilon} \to \|v_0\|^2_{E_\varepsilon}, \quad \text{and} \quad \int_\Omega b(x)g(v^{(2)}_{k_j})v_0\,dx \to \int_\Omega b(x)g(v_0)v_0\,dx.$$

Thus, $v_0 \in M_\varepsilon$ is obtained. Since $|I'_\varepsilon(v^{(2)}_{k_j})v^{(2)}_{k_j}| \to 0$ and $\int_\Omega b(x)g(v^{(2)}_{k_j})v^{(2)}_{k_j}\,dx$ converges to $\int_\Omega b(x)g(v_0)v_0\,dx$ as $j \to \infty$, we obtain

$$\|v^{(2)}_{k_j}\|^2_{E_\varepsilon} = I'_\varepsilon(v^{(2)}_{k_j})v^{(2)}_{k_j} + \int_\Omega b(x)g(v^{(2)}_{k_j})v^{(2)}_{k_j}\,dx \to \int_\Omega b(x)g(v_0)v_0\,dx = \|v_0\|^2_{E_\varepsilon} \text{as } j \to \infty.$$

From $\|v^{(2)}_{k_j} - v_0\|^2_{E_\varepsilon} = \|v^{(2)}_{k_j}\|^2_{E_\varepsilon} - 2\langle v^{(2)}_{k_j}, v_0\rangle_{E_\varepsilon} + \|v_0\|^2_{E_\varepsilon}$, by taking the limit as $k \to \infty$, it is shown that $v^{(2)}_{k_j}$ converges to v_0 in E_ε as $j \to \infty$. Therefore the sequence $\{v_k\}_{k\in\mathbb{N}}$ has a convergent subsequence $\{v^{(2)}_{k_j}\}_{j\in\mathbb{N}}$ in E_ε. Hence, the (PS) condition is satisfied.

Consequently, c_ε in Definition 2.1 is a positive critical value of $I_\varepsilon(v)$. Q.E.D.

Definition 2.2. Let v_ε be a critical point of $I_\varepsilon(v)$ corresponding to c_ε. We call $u_\varepsilon = u_{m,\varepsilon} + v_\varepsilon$ a *ground-state solution* of (P).

2.3 Entire solution on $\mathbb{R}^n$ to approximate ground-state solutions

In this section we consider a slightly general nonlinearity including $f(u) = u_+^p$, $1 < p < (n+2)/(n-2)$. Let f be a $C^{1,\beta}$-function on $\mathbb{R}^n$ satisfying $f(t) > 0$ if $t > 0$ and $f(t) = 0$ if $t \le 0$. Moreover, we assume that the boundary value problem

$$(\mathrm{P})_0 \qquad \begin{cases} \Delta w - w + f(w) = 0 \quad \text{in } \mathbb{R}^n, \\ w(0) = \max_{y\in\mathbb{R}^n} w(y), \quad \lim_{|y|\to\infty} w(y) = 0 \end{cases}$$

has a unique positive solution w_0 and the linearized operator

$$L = \Delta - 1 + f'(w_0) : W^{2,p}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

satisfies

$$\operatorname{Ker} L = \operatorname{span}\left\{ \frac{\partial w_0}{\partial y_j} \;\middle|\; j = 1, \dots, n \right\}.$$

Proposition 2.9. *There exists a constant $\gamma_* > 0$ such that for any $\gamma \in [0, \gamma_*)$, the boundary value problem*

$$(\mathrm{P})_\gamma \qquad \begin{cases} \Delta w - w + f(w+\gamma) - f(\gamma) = 0 & \text{in } \mathbb{R}^n, \\ w(0) = \max\limits_{y \in \mathbb{R}^n} w(y), \quad \lim\limits_{|y| \to \infty} w(y) = 0 \end{cases}$$

has a unique positive solution $w = w_\gamma$. Moreover, w_γ is symmetric with respect to the origin and decays exponentially at infinity.

The symmetry and exponential decay of w_0 have been proved by [5]. (Notes that, for sufficiently small $\gamma \geq 0$, $\sqrt{1 - f'(\gamma)}$ is bounded below with a positive constant.)

We outline the proof of uniqueness:

0°) Assume that such a positive number γ_* does not exist. Then there is a sequence $\{\gamma_j\}_{j \in \mathbb{N}}$ such that $\gamma_j \downarrow 0$ as $j \to \infty$ and $(\mathrm{P})_{\gamma_j}$ has two distinct positive solutions $w^{(1)}_{\gamma_j}$ and $w^{(2)}_{\gamma_j}$.

1°) $w^{(1)}_{\gamma_j} \to w^{(1)}$ and $w^{(2)}_{\gamma_j} \to w^{(2)}$. Moreover both $w^{(1)}$ and $w^{(2)}$ are solutions of $(\mathrm{P})_0$.

2°) There exists a neighborhood $\mathcal{N}$ of w_0 such that $(\mathrm{P})_\gamma$ has a unique solution in $\mathcal{N}$ if $\gamma \geq 0$ is sufficiently small.

3°) By 2°), $w^{(1)} \neq w^{(2)}$. A contradiction with the uniqueness assumption for $\gamma = 0$.

To prove 1°) we begin by noting that $(f(t+\gamma) - f(\gamma))/t^p \to c_0 > 0$ as $t \to \infty$. We point out also that w_{γ_j} attains its maximum only at the origin. By the same method as in the proof of Theorem 3 of [12, pp.18-20] (or more precisely, [6]) we can prove that there exists a positive constant M such that $\|w^{(1)}_{\gamma_j}\|_{L^\infty(\mathbb{R}^n)} + \|w^{(2)}_{\gamma_j}\|_{L^\infty(\mathbb{R}^n)} \leq M$ for any $j = 1, 2, 3, \dots$. Moreover, they decay exponentially as $|y| \to \infty$ (see, e.g., [5]). Hence for any $\kappa \in (0, \sqrt{1 - f'(\gamma)})$ there exists a positive constant C_κ such that $0 < w^{(i)}_{\gamma_j}(y) = w^{(i)}_{\gamma_j}(|y|) \leq C_\kappa e^{-\kappa|y|}$ and $|D^\alpha w^{(i)}_{\gamma_j}(y)| \leq C_\kappa e^{-\kappa|y|}$ for $|\alpha| \leq 2$ and $i = 1, 2$. Therefore $\{w^{(i)}_{\gamma_j}\}$ and $\{\nabla w^{(i)}_{\gamma_j}\}$ are uniformly bounded and equicontinuous, hence it contains a subsequence $\{w^{(i)}_{\gamma_{j_k}}\}$ convergent in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$. Let $w^{(i)}$ be its limit. Clearly, it satisfies $(\mathrm{P})_\gamma$ for $i = 1, 2$.

2°) Let G be the Green's function for $1 - \Delta$ on $\mathbb{R}^n$. Then, for a bounded and continuous function h on $\mathbb{R}^n$, the solution of the equation $\Delta u - u + h = 0$ in $\mathbb{R}^n$ is expressed as

$$u(y) = \int_{\mathbb{R}^n} G(|y - z|) h(z)\, dz \left(= \int_{\mathbb{R}^n} G(y, z) h(z)\, dz \right).$$

We call the standard implicit function theorem on Banach spaces. The following version is found in [17, Theorem 2.7.2].

Lemma 2.10 (Implicit function theorem)**.** *Let X, Y and Z be Banach spaces and $\mathcal{F}$ a continuous mapping of an open set $U \subset X \times Y \to Z$. Assume that $\mathcal{F}$ has a Frechét derivative with respect to x, $\mathcal{F}_x(x, y)$ which is continuous in U. Let $(x_0, y_0) \in U$ and $\mathcal{F}(x_0, y_0) = 0$. If $A = \mathcal{F}_x(x_0, y_0)$ is an isomorphism of X onto Z, then*

(i) *there is a ball $B_r(y_0) = \{y : \|y - y_0\| < r\}$ and a unique continuous map $u : B_r(y_0) \to X$ such that $u(y_0) = x_0$ and $\mathcal{F}(u(y), y) \equiv 0$.*

(ii) *If $\mathcal{F}$ is of class C^1, then $u(y)$ is of class C^1 and*

$$u_y(y) = -\{\mathcal{F}_x(u(y), y)\}^{-1} \circ f_y(u(y), y).$$

(iii) *$u_y(y)$ belongs to C^p if $\mathcal{F}$ is in C^p, $p > 1$.*

To use the Implicit Function Theorem, we formulate our problem as follows: Let $X_r := \{w \in C^0(\mathbb{R}^n) \mid w(y) = w(|y|), \lim_{|y|\to\infty} w(y) = 0\}$ and $\delta_0 > 0$ be sufficiently small. We define a mapping $\mathcal{F}$ from $X_r \times (-\delta_0, \delta_0)$ into X_r by

$$\mathcal{F}(w, \gamma) := w - \int_{\mathbb{R}^n} G(y, z)\Big\{f(\gamma + w(z)) - f(\gamma)\Big\}\, dz.$$

It is easy to verify that $\mathcal{F} \in C^1(X_r \times (-\delta_0, \delta_0), X_r)$, and $\mathcal{F}(w_0, 0) = 0$. Moreover, we can prove that $D_w\mathcal{F}(w_0, 0)$ is an isomorphism from X_r onto X_r. Indeed, $D_w\mathcal{F}(w_0, 0)\phi = \phi - \int_{\mathbb{R}^n} G(y, z) f'(w_0(z))\phi(z)\, dz$ is a Fredholm operator and $\operatorname{Ker} D_w\mathcal{F}(w_0, 0) = \{0\}$ on X_r by the assumption that $\operatorname{Ker} L = \operatorname{span}\{\partial w/\partial y_j \mid j = 1, \ldots, n\}$. This implies the assertion. Therefore by the Implicit Function Theorem there exists a C^1-function $W : (-\delta_1, \delta_1) \to X_r$ such that $W(0) = w_0$ and $\mathcal{F}(W(\gamma), \gamma) = 0$ for $\gamma \in (-\delta_1, \delta_1)$, where $\delta_1 > 0$ is a sufficiently small constant. Moreover, in the ball $\|w - w_0\|_{L^\infty(\mathbb{R}^n)} < \delta_1$ of X_r, there is no solution of $\mathcal{F}(w, \gamma) = 0$ in $\{w \in X_r \mid \|w - w_0\|_{L^\infty(\mathbb{R}^n)} < \delta_1\} \times (-\delta_0, \delta_0)$ other than $W(\gamma)$.

2.4 Upper bound of the energy for ground-state solutions

By the Mountain Pass Lemma, we know that $c_\varepsilon > 0$ for any $\varepsilon > 0$. Moreover, v_ε belong to $C^2(\Omega) \cap C^1(\overline{\Omega})$ and is positive on $\overline{\Omega}$ by the regularity theory for elliptic equations and by the maximum principle (see, e.g., [12, p.9]). Recall that the energy $J_\varepsilon(u_\varepsilon)$ of a ground-state solution $u_\varepsilon = u_{m,\varepsilon} + v_\varepsilon$ is given by $J_\varepsilon(u_\varepsilon) = J_\varepsilon(u_{m,\varepsilon}) + I_\varepsilon(v_\varepsilon)$ and that $J_\varepsilon(u_{m,\varepsilon})$ is a unique fixed constant once $\varepsilon > 0$ is given. Therefore, we are interested in the behavior of $c_\varepsilon = I_\varepsilon(v_\varepsilon)$ as $\varepsilon \downarrow 0$. It is convenient to introduce the following notation.

Definition 2.3. Let $w = w_\gamma$ be the positive solution of $(\mathrm{P})_\gamma$ stated in Chapter 1. We define

$$I_\delta(Q; R_Q) := \frac{1}{2}\int_{R_Q} (|\nabla w|^2 + w^2)\, dz - \int_{R_Q} G(\gamma, w)\, dy \quad \text{where } \gamma = \gamma_0(Q).$$

Here, R_Q is given by

$$R_Q := \begin{cases} \mathbb{R}^n & \text{if} \quad Q \in \Omega, \\ \mathbb{R}^n_+ = \{y \in \mathbb{R}^n \mid y_n > 0\} & \text{if} \quad Q \in \partial\Omega. \end{cases} \tag{2.5}$$

***Remark* 2.11.** *(i) By Proposition 1.1 and* (1.11), $0 \le \gamma_0(Q) \le \{b(Q)/a(Q)\}^{1/(p-1)}\delta/\min_{\overline{\Omega}} a(x)$. *Therefore,* $\sup_{Q\in\overline{\Omega}} \|w_{\gamma_0(Q)}\|_{L^r(\mathbb{R}^n)} < \infty$ *for any* $r \in [1,\infty]$. *(ii) The function* $I_\delta(Q;R_Q)$ *depends only on the value of* $\gamma_0(Q)$ *and the domain of integration* R_Q*. Moreover, by the symmetry of* w

$$I_\delta(Q;\mathbb{R}^n_+) = \frac{1}{2}I_\delta(Q;\mathbb{R}^n) = \frac{1}{2}\mathcal{I}(\gamma_0(Q)). \tag{2.6}$$

The goal of this section is to prove the following estimate.

Proposition 2.12. *For sufficiently small* ε*,* $c_\varepsilon/\varepsilon^n$ *is bounded, that is, the following holds:*

$$\limsup_{\varepsilon\downarrow 0} \frac{c_\varepsilon}{\varepsilon^n} \le \min\left\{\min_{Q\in\overline{\Omega}} \Phi(Q)I_\delta(Q;\mathbb{R}^n),\ \frac{1}{2}\min_{Q\in\partial\Omega} \Phi(Q)I_\delta(Q;\mathbb{R}^n)\right\}. \tag{2.7}$$

To prove this result we use the characterization $c_\varepsilon = c_\varepsilon^*$ (see Lemma 2.5). Hence

$$c_\varepsilon \le \max_{t\ge 0} I_\varepsilon(tv)$$

for any $v \ge 0$ with $v \not\equiv 0$. We choose an approximately scaled entire solution $w_{\gamma_0(Q)}$ as v. To do so, we need a few definitions.

Definition 2.4. For each $Q \in \overline{\Omega}$ and $k = 1,\dots,n$, let $\lambda_k(Q)$ denote an eigenvalue of the symmetric matrix $A_Q = (a_{ij}(Q))_{1\le i,j\le n}$ which is numbered so that $\lambda_l(Q) \le \lambda_{l+1}(Q)$ for $l = 1,\dots,n-1$. Put

$$D_Q := \begin{pmatrix} \lambda_1(Q) & & 0 \\ & \ddots & \\ 0 & & \lambda_n(Q), \end{pmatrix} = \mathrm{diag}(\lambda_1(Q),\dots,\lambda_n(Q))$$

and let B_Q be the orthogonal matrix which diagonalizes A_Q: $B_Q A_Q {}^t B_Q = D_Q$.

With these notations, we define v as follows:

$$v(x) = \left(\frac{a(Q)}{b(Q)}\right)^{1/(p-1)} w_{\gamma_0(Q)}\left(\sqrt{a(Q)}\sqrt{D_Q^{-1}}B_Q\frac{x-Q}{\varepsilon}\right) = \left(\frac{a(Q)}{b(Q)}\right)^{1/(p-1)} w_{\gamma_0(Q)}(y) \tag{2.8}$$

and its scaled version V by

$$V(z) := v(x), \quad z := \frac{x-Q}{\varepsilon}, \quad \Omega_{\varepsilon,Q} := \{z\in\mathbb{R}^n \mid x = Q+\varepsilon z \in \Omega\}. \tag{2.9}$$

Notice that V is no longer dependent on ε.

The following lemma is crucial.

Lemma 2.13. *For each* $Q \in \overline{\Omega}$*,*

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon^n}\max_{t\ge 0} I_\varepsilon(tv) = \frac{1}{2}\int_{R_Q}\left\{a_{ij}(Q)\frac{\partial V}{\partial z_i}\frac{\partial V}{\partial z_j} + \alpha(Q)V^2\right\}dz - \int_{R_Q} b(Q)G(u_m(Q),V)\,dz. \tag{2.10}$$

Proof. Let $\phi(t) = I_\varepsilon(tv)$. Then $\phi(0) = 0$ and $\phi(t) \to -\infty$ as $t \to \infty$. Moreover, $\phi(t) > 0$ for $t > 0$ sufficiently small. Therefore, $\phi(t)$ attains the global maximum, which is positive.

Step 1. Let t_ε be the maximum point of the function $\phi(t)$. We prove that $t_\varepsilon \to 1$ as $\varepsilon \downarrow 0$. Since $\phi'(t) = I'_\varepsilon(tv)v$ and $\phi'(t_\varepsilon) = 0$, it is easily seen that

$$\|t_\varepsilon v\|^2_{E_\varepsilon} = \int_\Omega b(x)g(u_{m,\varepsilon}(x), t_\varepsilon v)t_\varepsilon v\, dx,$$

hence

$$\|v\|^2_{E_\varepsilon} = \int_\Omega \frac{1}{t_\varepsilon} b(x)g(u_{m,\varepsilon}(x), t_\varepsilon v)v\, dx.$$

After performing the change of integration variable $x \to z$ and dividing by ε^n, we obtain

$$\begin{aligned}(2.11)\qquad &\int_{\Omega_{\varepsilon,Q}} \left\{a_{ij}(Q+\varepsilon z)\frac{\partial V}{\partial z_i}\frac{\partial V}{\partial z_j} + \alpha(Q+\varepsilon z)V^2\right\} dz\\ &= \int_{\Omega_{\varepsilon,Q}} \frac{1}{t_\varepsilon} b(Q+\varepsilon z)g(u_{m,\varepsilon}(Q+\varepsilon z), t_\varepsilon V)V\, dz.\end{aligned}$$

We observe that if we put

$$\psi(t) = \int_{\Omega_{\varepsilon,Q}} \frac{1}{t} b(Q+\varepsilon z)g(u_{m,\varepsilon}(Q+\varepsilon z), tV)V\, dz,$$

then

$$\psi'(t) = \frac{1}{t^2}\int_{\Omega_{\varepsilon,Q}} b(Q+\varepsilon z)\Big\{g'(u_{m,\varepsilon}(Q+\varepsilon z), tV)tV - g(u_{m,\varepsilon}(Q+\varepsilon z), tV)\Big\}V\, dz.$$

On the other hand, $g'(u_{m,\varepsilon}(Q+\varepsilon z), \xi)\xi - g(u_{m,\varepsilon}(Q+\varepsilon z), \xi) > 0$ for all $\xi > 0$ since the left-hand side is zero for $\xi = 0$ and its derivative with respect to ξ is equal to $p(p-1)(u_{m,\varepsilon} + \xi)_+^{p-2}\xi > 0$ for $\xi > 0$. Therefore ψ is strict increasing in $t > 0$, and hence (2.11) determines t_ε uniquely.

Now recall that $w_{\gamma_0(Q)}$ decays exponentially together with its derivatives up to order 2 as $|y| \to \infty$:

$$|D^\alpha w_{\gamma_0(Q)}(y)| \le C_0 e^{-\mu_0|y|} \qquad \text{for all } y \in \mathbb{R}^n,\ |\alpha| \le 2.$$

Therefore,

$$(2.12)\qquad |D^\alpha V(z)| \le C_1 e^{-\mu_1|y|} \qquad \text{for all } z \in \mathbb{R}^n,\ |\alpha| \le 2.$$

Let us consider the case $Q \in \Omega$. Then there exist an $R > 0$ such that $\overline{B}_R(Q) \subset \Omega$. By (2.12) it is readily seen that

$$\begin{aligned}(2.13)\qquad &\int_{\Omega_{\varepsilon,Q}} \left\{a_{ij}(Q+\varepsilon z)\frac{\partial V}{\partial z_i}\frac{\partial V}{\partial z_j} + \alpha(Q+\varepsilon z)V^2\right\} dz\\ &= \int_{|z|>R/\varepsilon} \left\{a_{ij}(Q+\varepsilon z)\frac{\partial V}{\partial z_i}\frac{\partial V}{\partial z_j} + \alpha(Q+\varepsilon z)V^2\right\} dz + O(e^{-\mu_2/\varepsilon})\end{aligned}$$

and

$$\begin{aligned}(2.14)\qquad &\int_{\Omega_{\varepsilon,Q}} b(Q+\varepsilon z)g(u_{m,\varepsilon}(Q+\varepsilon z), t_\varepsilon V)V\, dz\\ &= \int_{|z|<R/\varepsilon} b(Q+\varepsilon z)g(u_{m,\varepsilon}(Q+\varepsilon z), t_\varepsilon V)V\, dz + O(e^{-\mu_2/\varepsilon}).\end{aligned}$$

Note that $a_{ij}(Q+\varepsilon z) - a_{ij}(Q) = \varepsilon \nabla a_{ij}(Q+\theta\varepsilon z)\cdot z = O(\varepsilon|z|)$. Also, $\alpha(Q+\varepsilon z)-\alpha(Q) = a(Q+\varepsilon z)-a(Q)-p\{b(Q+\varepsilon z)-b(Q)\}u_{m,\varepsilon}(Q)^{p-1}+b(Q+\varepsilon z)\{u_{m,\varepsilon}(Q+\varepsilon z)^{p-1}-u_{m,\varepsilon}(Q)^{p-1}\}$ and $u_{m,\varepsilon}(Q+\varepsilon z)^{p-1}-u_{m,\varepsilon}(Q)^{p-1} = O((\varepsilon|z|))^{\min\{1,p-1\}}$, and hence $\alpha(Q+\varepsilon z)-\alpha(Q) = O((\varepsilon|z|))^{\min\{1,p-1\}})$. Therefore,

$$\int_{|z|>R/\varepsilon}\left\{(a_{ij}(Q+\varepsilon z)-a_{ij}(Q))\frac{\partial V}{\partial z_i}\frac{\partial V}{\partial z_j}+(\alpha(Q+\varepsilon z)-\alpha(Q))V^2\right\}dz = O(\varepsilon^{\min\{1,p-1\}}). \tag{2.15}$$

Likewise, we get

$$\int_{\Omega_{\varepsilon,Q}}\left\{b(Q+\varepsilon z)g(u_{m,\varepsilon}(Q+\varepsilon z),t_\varepsilon V)V-b(Q)g(u_{m,\varepsilon}(Q+\varepsilon z),t_\varepsilon V)V\right\}dz = O(\varepsilon^{\min\{1,p-1\}}). \tag{2.16}$$

Putting (2.13), (2.14), (2.15) and (2.16) together, we obtain from (2.11) that

$$\begin{aligned} &t_\varepsilon\left[\int_{|z|<R/\varepsilon}\left\{a_{ij}(Q)\frac{\partial V}{\partial z_i}\frac{\partial V}{\partial z_j}+\alpha(Q)V^2\right\}dz+O(\varepsilon^{\min\{1,p-1\}})\right]\\ &=\int_{|z|<R/\varepsilon} b(Q)g(u_{m,\varepsilon}(Q),t_\varepsilon V)V\,dz+O(\varepsilon^{\min\{1,p-1\}}). \end{aligned} \tag{2.17}$$

We notice here that V is a solution of the boundary value problem

$$\mathcal{A}(Q)V-\alpha(Q)V+b(Q)g(u_m(Q),V)=0.$$

Hence

$$\int_{\mathbb{R}^n}\left\{a_{ij}(Q)\frac{\partial V}{\partial z_i}\frac{\partial V}{\partial z_j}+\alpha(Q)V^2\right\}dz = \int_{\mathbb{R}^n} b(Q)g(u_{m,\varepsilon}(Q),V)V\,dz.$$

Therefore, from (2.17) it follows that

$$\lim_{\varepsilon\downarrow 0}\frac{1}{t_\varepsilon}\int_{|z|<R/\varepsilon} b(Q)g(u_{m,\varepsilon}(Q),t_\varepsilon V)V\,dz = \int_{\mathbb{R}^n} b(Q)g(u_m(Q),V)V\,dz. \tag{2.18}$$

We observe that t_ε remains bounded as $\varepsilon\downarrow 0$. Indeed, if $t_{\varepsilon_j}\to\infty$ along a sequence $\varepsilon_j\downarrow 0$, then $g(u_{m,\varepsilon}(Q),t_\varepsilon V)/(t_\varepsilon V)^p\to 1$ for each z. This means that

$$\frac{1}{t_\varepsilon^p}\int_{|z|<R/\varepsilon_j} g(u_{m,\varepsilon_j}(Q),t_{\varepsilon_j}V)V\,dz\to\int_{\mathbb{R}^n}V^{p+1}\,dz.$$

Hence the left-hand side of (2.18) diverges, a contradiction. Now $\{t_\varepsilon\}$ turned out to be bounded, and hence for any $\{\varepsilon_j\}$, $\varepsilon_j\downarrow 0$, $\{t_{\varepsilon_j}\}$ has a convergent subsequence $\{t_{j_k}\}$, and $t_{\varepsilon_{j_k}}\to t_*$. Then

$$\frac{1}{t_{\varepsilon_{j_k}}}\int_{|z|<R/\varepsilon_{j_k}} b(Q)g(u_{m,\varepsilon_{j_k}}(Q),t_{\varepsilon_{j_k}}V)V\,dz\to\frac{1}{t_*}\int_{\mathbb{R}^n} b(Q)g(u_m(Q),t_*V)V\,dz,$$

thus from (2.17)

$$\frac{1}{t_*}\int_{\mathbb{R}^n} b(Q)g(u_m(Q),t_*V)V\,dz = \int_{\mathbb{R}^n} b(Q)g(u_m(Q),V)V\,dz.$$

Clearly $t_* = 1$ satisfies this condition, and there is no other t_* satisfying this condition. Hence $t_* = 1$. Because the limit $t_* = 1$ is independent of choice of the subsequence $\{t_{j_k}\}$, $\{t_{\varepsilon_j}\}$ must converge to 1. This proves our assertion $t_\varepsilon \to 1$ as $\varepsilon \downarrow 0$.

Step 2. Recall that

$$\begin{aligned}\frac{1}{\varepsilon^n}\max_{t\geq 0} I_\varepsilon(tv) = \frac{1}{2}\int_{\Omega_{\varepsilon,Q}} t_\varepsilon^2 \left\{ a_{ij}(Q+\varepsilon z)\frac{\partial V}{\partial z_i}\frac{\partial V}{\partial z_j} + \alpha(Q+\varepsilon z)V^2 \right\} dz \\ - \int_{\Omega_{\varepsilon,Q}} b(Q+\varepsilon z)G(u_{m,\varepsilon}(Q+\varepsilon z), t_\varepsilon V)\, dz.\end{aligned}$$

By the same reasoning as above, we obtain (2.10) in the case $Q \in \Omega$.

Step 3. In the case $Q \in \partial\Omega$, we have to show that

$$\int_{\Omega_{\varepsilon,Q}} \left\{ a_{ij}(Q+\varepsilon z)\frac{\partial V}{\partial z_i}\frac{\partial V}{\partial z_j} + \alpha(Q+\varepsilon z)V^2 \right\} dz \to \int_{\mathbb{R}^n_+} \left\{ a_{ij}(Q)\frac{\partial V}{\partial z_i}\frac{\partial V}{\partial z_j} + \alpha(Q)V^2 \right\} dz,$$

$$\int_{\Omega_{\varepsilon,Q}} b(Q+\varepsilon z)g(u_{m,\varepsilon}(Q+\varepsilon z), t_\varepsilon V)V\, dz \to \int_{\mathbb{R}^n_+} b(Q)g(u_m(Q), V)V\, dz,$$

$$\int_{\Omega_{\varepsilon,Q}} b(Q+\varepsilon z)G(u_{m,\varepsilon}(Q+\varepsilon z), t_\varepsilon V)\, dz \to \int_{\mathbb{R}^n_+} b(Q)G(u_m(Q), V)\, dz.$$

These will be done in Chapter 3. Q.E.D.

Proof of Proposition 2.12. First of all, we express the right-hand side of (2.10) in terms of $w_{\gamma_0(Q)}$. Since

$$\frac{\partial}{\partial z_j} = \sum_{l=1}^n \frac{\partial y_l}{\partial z_j}\frac{\partial}{\partial y_l} = \sqrt{a(Q)}\sum_{l=1}^n \left(\sqrt{D_Q^{-1}}B_Q\right)_{lj}\frac{\partial}{\partial y_l},$$

we compute

$$\begin{aligned}&\sum_{i,j=1}^n a_{ij}(Q)\frac{\partial V}{\partial z_i}\frac{\partial V}{\partial z_j} \\ &= \left(\frac{a(Q)}{b(Q)}\right)^{\frac{2}{p-1}} a(Q)\sum_{i,j=1}^n a_{ij}(Q)\left\{\sum_{l=1}^n\left(\sqrt{D_Q^{-1}}B_Q\right)_{lj}\frac{\partial w_{\gamma_0(Q)}}{\partial y_l}\right\}\left\{\sum_{k=1}^n\left(\sqrt{D_Q^{-1}}B_Q\right)_{ki}\frac{\partial w_{\gamma_0(Q)}}{\partial y_k}\right\} \\ &= \left(\frac{a(Q)}{b(Q)}\right)^{\frac{2}{p-1}} a(Q)\sum_{k,l=1}^n\sum_{i,j=1}^n\left(\sqrt{D_Q^{-1}}B_Q\right)_{ki}a_{ij}(Q)\left(\sqrt{D_Q^{-1}}B_Q\right)_{lj}\frac{\partial w_{\gamma_0(Q)}}{\partial y_k}\frac{\partial w_{\gamma_0(Q)}}{\partial y_l} \\ &= \left(\frac{a(Q)}{b(Q)}\right)^{\frac{2}{p-1}} a(Q)\sum_{k,l=1}^n\sum_{i,j=1}^n\left(\sqrt{D_Q^{-1}}B_Q\right)_{ki}a_{ij}(Q)\,{}^t\!\left(\sqrt{D_Q^{-1}}B_Q\right)_{jl}\frac{\partial w_{\gamma_0(Q)}}{\partial y_k}\frac{\partial w_{\gamma_0(Q)}}{\partial y_l} \\ &= \left(\frac{a(Q)}{b(Q)}\right)^{\frac{2}{p-1}} a(Q)\sum_{k,l=1}^n\left\{\sqrt{D_Q^{-1}}B_QA_Q\,{}^tB_Q\sqrt{D_Q^{-1}}\right\}_{kl}\frac{\partial w_{\gamma_0(Q)}}{\partial y_k}\frac{\partial w_{\gamma_0(Q)}}{\partial y_l}.\end{aligned}$$

By Definition 2.4 we have $B_QA_Q{}^tB_Q = D_Q$, and hence

$$\sqrt{D_Q^{-1}}B_QA_Q{}^t\left(\sqrt{D_Q^{-1}}B_Q\right) = \sqrt{D_Q^{-1}}\left(B_QA_Q\,{}^tB_Q\right){}^t\sqrt{D_Q^{-1}} = \sqrt{D_Q^{-1}}D_Q\sqrt{D_Q^{-1}} = E_n$$

where E_n denotes the n-dimensional unit matrix. Therefore,

$$\sum_{i,j=1}^{n} a_{ij}(Q)\frac{\partial V}{\partial z_j}\frac{\partial V}{\partial z_i} = \left(\frac{a(Q)}{b(Q)}\right)^{\frac{2}{p-1}} a(Q)\sum_{k=1}^{n}\left(\frac{\partial w_{\gamma_0(Q)}}{\partial y_k}\right)^2 = \left(\frac{a(Q)}{b(Q)}\right)^{\frac{2}{p-1}} a(Q)|\nabla w_{\gamma_0(Q)}|^2.$$

Note also that $dz = \{\det\sqrt{a(Q)^{-1}D_Q}\}dy = a(Q)^{-n/2}(\det A_Q)^{1/2}dy$ and

$$\begin{aligned}
&\frac{1}{2}\int_{R_Q}\left\{a_{ij}(Q)\frac{\partial V}{\partial z_i}\frac{\partial V}{\partial z_j} + \alpha(Q)V^2\right\}dz - \int_{R_Q} b(Q)G(u_m(Q),V)\,dz \\
&= \frac{1}{2}\int_{R_Q}\left\{a_{ij}(Q)\frac{\partial V}{\partial z_i}\frac{\partial V}{\partial z_j} + a(Q)V^2\right\}dz \\
&\quad - \frac{1}{p+1}\int_{R_Q} b(Q)\Big\{(u_m(Q)+V)^{p+1} - u_m(Q)^{p+1} - (p+1)u_m(Q)^pV\Big\}\,dz.
\end{aligned}$$

We thus have

$$\begin{aligned}
&\frac{1}{2}\int_{R_Q}\left\{a_{ij}(Q)\frac{\partial V}{\partial z_i}\frac{\partial V}{\partial z_j} + a(Q)V^2\right\}dz \\
&= a(Q)^{-n/2}(\det A_Q)^{1/2}\,\frac{1}{2}\int_{R_Q}\left\{\left(\frac{a(Q)}{b(Q)}\right)^{\frac{2}{p-1}} a(Q)|\nabla w_{\gamma_0(Q)}|^2 + a(Q)\left(\frac{a(Q)}{b(Q)}\right)^{\frac{2}{p-1}} w_{\gamma_0(Q)}^2\right\}dy \\
&= a(Q)^{1-n/2+2/(p-1)}b(Q)^{-2/(p-1)}(\det A_Q)^{1/2}\,\frac{1}{2}\int_{R_Q}\Big(|\nabla w_{\gamma_0(Q)}|^2 + w_{\gamma_0(Q)}^2\Big)\,dy
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{1}{p+1}\int_{R_Q} b(Q)\Big\{(u_m(Q)+V)^{p+1} - u_m(Q)^{p+1} - (p+1)u_m(Q)^pV\Big\}\,dz \\
&= -\frac{1}{p+1}a(Q)^{-n/2}(\det A_Q)^{1/2} \\
&\quad\times b(Q)\left(\frac{a(Q)}{b(Q)}\right)^{\frac{p+1}{p-1}}\int_{R_Q}\Big\{(\gamma_0(Q)+w_{\gamma_0(Q)})^{p+1} - \gamma_0(Q)^{p+1} - (p+1)\gamma_0(Q)^p w_{\gamma_0(Q)}\Big\}\,dy \\
&= a(Q)^{1-n/2+2/(p-1)}b(Q)^{-2/(p-1)}(\det A_Q)^{1/2} \\
&\quad\times\left[-\frac{1}{p+1}\int_{R_Q}\Big\{(\gamma_0(Q)+w_{\gamma_0(Q)})^{p+1} - \gamma_0(Q)^{p+1} - (p+1)\gamma_0(Q)^p w_{\gamma_0(Q)}\Big\}\,dy\right].
\end{aligned}$$

Hence, the right-hand side of (2.10) is equal to

$$\begin{aligned}
&a(Q)^{1-n/2+2/(p-1)}b(Q)^{-2/(p-1)}(\det A_Q)^{1/2} \\
&\times\left[\frac{1}{2}\int_{R_Q}\Big(|\nabla w_{\gamma_0(Q)}|^2 + w_{\gamma_0(Q)}^2\Big)\,dy\right. \\
&\quad\left.-\frac{1}{p+1}\int_{R_Q}\Big\{(\gamma_0(Q)+w_{\gamma_0(Q)})^{p+1} - \gamma_0(Q)^{p+1} - (p+1)\gamma_0(Q)^p w_{\gamma_0(Q)}\Big\}\,dy\right] \\
&= \Phi(Q)I_\delta(Q;R_Q).
\end{aligned}$$

Therefore

$$\limsup_{\varepsilon\downarrow 0} \frac{c_\varepsilon}{\varepsilon^n} \le \Phi(Q) I_\delta(Q; R_Q) \qquad \text{for all } Q \in \overline{\Omega}.$$

Taking the minimum of the right-hand side over $\overline{\Omega}$, we obtain (2.7) due to (2.6). Q.E.D.

By the boundedness of $c_\varepsilon = I_\varepsilon(v_\varepsilon)$, the following proposition holds for a critical point v_ε.

Proposition 2.14. *For any $r \in [1, \infty)$, there exists a constant $C_r > 0$ such that for sufficiently small $\varepsilon > 0$,*

$$\int_\Omega v_\varepsilon^r \, dx \le C_r \varepsilon^n \tag{2.19}$$

holds and $C_r^{1/r}$ is bounded in r.

We can prove this proposition by $\|v_\varepsilon\|^2_{E_\varepsilon} = \int_\Omega b(x) g(u_{m,\varepsilon}, v_\varepsilon)\, dx$, Claim 2.8, the Sobolev inequality, and the estimate $c_\varepsilon = \|v_\varepsilon\|^2_{E_\varepsilon}/2 - \int_\Omega b(x) G(u_{m,\varepsilon}, v_\varepsilon)\, dx = O(\varepsilon^n)$ (for more detail, see the proof of Lemma 2.3 in [12]).

Chapter 3

Asymptotic form of ground-state solutions

In this chapter, we investigate the asymptotic form of a ground-state solution as $\varepsilon \downarrow 0$ around its (local) maximum point.

Recall that a ground-state solution u_ε is the sum of the minimal solution $u_{m,\varepsilon}$ and a mountain-pass solution v_ε: $u_\varepsilon = u_{m,\varepsilon} + v_\varepsilon$. In Lemma 2.3 we have shown that $u_{m,\varepsilon}(Q + \varepsilon y) \to u_m(Q)$ as $\varepsilon \downarrow 0$ in $C^2(K)$ in each compact set of $\mathbb{R}^n$. Therefore in this chapter we focus on the asymptotic behavior of the mountain-pass solution v_ε, which satisfies

$$\begin{cases} \varepsilon^2 \mathscr{A}(x)v - a(x)v + b(x)(u_{m,\varepsilon} + v)_+^p - u_{m,\varepsilon}^p = 0, \; v > 0 & \text{in } \Omega, \\ \mathscr{B}(x)v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

The main results of this chapter hold under the assumption that δ is sufficiently small. We begin by describing how small δ must be. First of all, we require that the algebraic equation (1.10) has exactly two nonnegative roots for any $Q \in \overline{\Omega}$. Since the function $-a(Q)\zeta + b(Q)\zeta^p$ achieves the global minimum $-(p-1)a(Q)^{p/(p-1)}/\{p(pb(Q))^{1/(p-1)}\}$ at $\zeta = \{a(Q)/(pb(Q))\}^{1/(p-1)}$, this condition is satisfied if and only if

$$\delta\sigma(Q) < \frac{p-1}{p^{p/(p-1)}}\left(\frac{a(Q)^p}{b(Q)}\right)^{1/(p-1)} \quad \text{for all } Q \in \overline{\Omega}.$$

By virtue of $\max_{\overline{\Omega}} \sigma = 1$, we see that

$$\delta < \frac{p-1}{p^{p/(p-1)}}\left(\frac{\min a(x)^p}{\max b(x)}\right)^{1/(p-1)}$$

is a sufficient condition for the first requirement. Moreover, we require the algebraic equation

$$-a(Q)\eta + b(Q)\eta^p + a(Q)u_{m,\varepsilon}(Q) - b(Q)u_{m,\varepsilon}(Q)^p = 0 \tag{3.2}$$

has exactly two nonnegative roots for any $Q \in \overline{\Omega}$ and $\varepsilon > 0$, which is equivalent to the condition

$$a(Q)u_{m,\varepsilon}(Q) - b(Q)u_{m,\varepsilon}(Q)^p < \frac{p-1}{p^{p/(p-1)}}\left(\frac{a(Q)^p}{b(Q)}\right)^{1/(p-1)}$$

for all $Q \in \overline{\Omega}$ and $\varepsilon > 0$. Since there exists a positive constant C_m such that $0 \le u_{m,\varepsilon}(x) \le C_m\delta$ for any $x \in \overline{\Omega}$ and $\varepsilon > 0$, for the second requirement it is sufficient to assume that δ satisfies

$$\max_{\overline{\Omega}} a(x)C_m\delta < \frac{p-1}{p^{p/(p-1)}}\left(\frac{a(Q)^p}{b(Q)}\right)^{1/(p-1)}. \tag{3.3}$$

Proposition 3.1. *Given a family of ground-state solutions* $\{u_{\varepsilon_j}\}_{0<\varepsilon<\varepsilon_0}$ *of* (P)*, let* P_ε *be a local maximum point of* u_ε*. Assume that there is a sequence* $\{\varepsilon_j\}_{j\in\mathbb{N}}$ *converging to zero such that* $P_{\varepsilon_j} \to P_0 \in \overline{\Omega}$*. Then, as* $j \to \infty$*,*

$$\left.\begin{aligned} &u_{m,\varepsilon_j}(P_{\varepsilon_j} + \varepsilon_j z) \longrightarrow u_m(P_0) \\ &v_{\varepsilon_j}(P_{\varepsilon_j} + \varepsilon_j z) \longrightarrow (a(P_0)/b(P_0))^{1/(p-1)} w_\gamma\left(\sqrt{a(P_0)}\,\sqrt{D_{P_0}}^{-1} B_{P_0} z\right) \end{aligned}\right\} \text{ in } C^2(K) \tag{3.4}$$

on each compact set K *of* $\mathbb{R}^n$*, where* w_γ *is a unique positive solution of* (GS-γ)*,* $\gamma = \gamma_0(P_0)$ *and the function* γ_0 *on* $\overline{\Omega}$ *is defined by* (1.11)*.*

***Remark* 3.2.** *We note that* (3.4) *is not the precise expression in the case* P_{ε_j} *sufficiently close to the boundary (e.g.,* $P_0 \in \partial\Omega$*). In such a case, we have to extend the functions* u_{m,ε_j} *and* u_{ε_j} *to the outside of* Ω *because the point* $P_{\varepsilon_j} + \varepsilon_j z$ *may be in the outside of* Ω*. See Case (II) in the following proof for the precise expression in the case where* P_{ε_j} *is close to* $\partial\Omega$*.*

Proof. We treat the two cases (I) $P_0 \in \Omega$ and (II) $P_0 \in \partial\Omega$ separately.

Case (I): $P_0 \in \Omega$. Since Ω is an open set in $\mathbb{R}^n$, there exists a positive constant r_0 such that $B_{3r_0}(P_0) \subset \Omega$. By $P_{\varepsilon_j} \to P_0$ and $\varepsilon_j \downarrow 0$ as $j \to \infty$, we see that $\varepsilon_j \in (0, r_0)$ and $P_{\varepsilon_j} \in B_{r_0}(P_0)$ for sufficiently large j. Let

$$V_\varepsilon(z) = v_\varepsilon(x) \quad \text{with } z = (x - P_\varepsilon)/\varepsilon$$

and

$$\Omega_{\varepsilon,P_\varepsilon} = \{z \in \mathbb{R}^n \mid x = P_\varepsilon + \varepsilon z \in \Omega\}.$$

Note that if $z \in B_{2r_0/\varepsilon_j}(0)$, then $|x - P_0| = |P_{\varepsilon_j} + \varepsilon_j z - P_0| < r_0 + 2r_0 = 3r_0$, and hence $z \in \Omega_{\varepsilon_j,P_{\varepsilon_j}}$. Since v_ε satisfies (3.1) in Ω, we see that V_ε satisfies

$$\begin{aligned} &\mathcal{A}(P_\varepsilon + \varepsilon z)V_\varepsilon - a(P_\varepsilon + \varepsilon z)V_\varepsilon \\ &\qquad + b(P_\varepsilon + \varepsilon z)\left\{(u_{m,\varepsilon}(P_\varepsilon + \varepsilon z) + V_\varepsilon)^p - u_{m,\varepsilon}(P_\varepsilon + \varepsilon z)^p\right\} = 0 \quad \text{in } \Omega_{\varepsilon,P_\varepsilon}, \end{aligned} \tag{3.5}$$

where $\mathcal{A}(P_\varepsilon + \varepsilon z)V_\varepsilon = \sum_{k,l=1}^n (\partial/\partial z_k)\{a_{kl}(P_\varepsilon + \varepsilon z)(\partial V_\varepsilon/\partial z_l)\}$. Recall also that

$$\|V_\varepsilon\|_{L^r(\Omega_{\varepsilon,P_\varepsilon})} = \left(\int_{\Omega_{\varepsilon,P_\varepsilon}} V_\varepsilon^r\,dz\right)^{1/r} = \left(\varepsilon^{-n}\int_\Omega v_\varepsilon^r\,dx\right)^{1/r} \le C_r^{1/r} \quad \text{for } r \in [1,\infty)$$

by Proposition 2.14. In particular, for each $R > 0$,

$$\sup_{0<\varepsilon_j<r_0} \|V_{\varepsilon_j}\|_{L^r(B_{3R}(0))} \le \sup_{0<\varepsilon<r_0} C_r^{1/r} < \infty, \tag{3.6}$$

which implies that the nonlinear term $b(P_{\varepsilon_j} + \varepsilon_j z)\{(u_{m,\varepsilon_j}(P_{\varepsilon_j} + \varepsilon_j z) + V_{\varepsilon_j})_+^p - u_{m,\varepsilon_j}(P_{\varepsilon_j} + \varepsilon_j z)^p\}$ is bounded in $L^r(B_{3R}(0))$ for $0 < \varepsilon_j < r_0$. By the interior elliptic estimate, we have $\|V_{\varepsilon_j}\|_{W^{2,r}(B_{2R}(0))} \le C$, so that $\|V_{\varepsilon_j}\|_{C^{1,\alpha}(\overline{B}_{2R}(0))} \le C$ by the Sobolev imbedding theorem if we choose $r > n$. Finally by the interior Schauder estimate, we conclude that $\|V_{\varepsilon_j}\|_{C^{2,\beta}(\overline{B}_R(0))} \le C$. Therefore, the Ascoli-Arzelà theorem allow us to select a subsequence, which we denote again by $\{V_{\varepsilon_j}\}$, converging to a C^2-function V^r in the topology of $C^2(\overline{B}_R(0))$. Now we take a strictly increasing sequence of positive numbers $\{R_k\}$ divergent to ∞, and apply the diagonal argument to obtain a subsequence

$\{V_{\varepsilon_{j_k}}\}$ converging to $V_0 \in C^2(\mathbb{R}^n)$ in $C^2_{\rm loc}(\mathbb{R}^n)$. For simplicity we write $\{\varepsilon_k\}$ instead of $\{\varepsilon_{j_k}\}$. From (3.5) we see that V_0 satisfies

$$\mathcal{A}(P_0)V_0 - a(P_0)V_0 + b(P_0)\Big\{(u_m(P_0) + V_0)^p - u_m(P_0)^p\Big\} = 0 \tag{3.7}$$

by virtue of Lemma 2.3, where $\mathcal{A}(P_0)V_0 = \sum_{k,l=1}^n a_{kl}(P_0)(\partial^2 V_0/\partial z_k \partial z_l)$. Since $\|V_{\varepsilon_j}\|_{L^r(B_{3R}(0))}$ is bounded by a constant $C_r^{1/r}$ independent of R, the limit V_0 belongs to $L^r(\mathbb{R}^n)$ for any $r \in [1, \infty)$. In addition, $\limsup_{r\to\infty} C_r^{1/r} < \infty$ implies $V_0 \in L^\infty(\mathbb{R}^n)$.

Furthermore, we claim that $\max V_0(z) > 0$. To prove this we notice that $\sum_{k,l=1}^n (\partial/\partial z_k) a_{kl}(P_{\varepsilon_j} + \varepsilon_j z)(\partial V_{\varepsilon_j}/\partial z_l)|_{z=0} \le 0$ since V_{ε_j} attains a local maximum at $z = 0$, and hence

$$-a(P_{\varepsilon_j})V_{\varepsilon_j}(0) + b(P_{\varepsilon_j})\Big\{\Big(u_{m,\varepsilon_j}(P_{\varepsilon_j}) + V_{\varepsilon_j}(0)\Big)^p - u_{m,\varepsilon_j}(P_{\varepsilon_j})^p\Big\} \ge 0.$$

On the other hand, we know that

$$\mathcal{A}(P_{\varepsilon_j})u_{m,\varepsilon_j}(P_{\varepsilon_j}) - a(P_{\varepsilon_j})u_{m,\varepsilon_j}(P_{\varepsilon_j}) + b(P_{\varepsilon_j})u_{m,\varepsilon_j}(P_{\varepsilon_j})^p + \delta\sigma(P_{\varepsilon_j}) = 0, \tag{3.8}$$

hence, combining these two together, we obtain

$$\mathcal{A}(P_{\varepsilon_j})u_{m,\varepsilon_j}(P_{\varepsilon_j}) - a(P_{\varepsilon_j})\Big(u_{m,\varepsilon_j}(P_{\varepsilon_j}) + V_{\varepsilon_j}(0)\Big) + b(P_{\varepsilon_j})\Big(u_{m,\varepsilon_j}(P_{\varepsilon_j}) + V_{\varepsilon_j}(0)\Big)^p + \delta\sigma(P_{\varepsilon_j}) \ge 0.$$

Put $\eta = u_{m,\varepsilon_j}(P_{\varepsilon_j}) + V_{\varepsilon_j}(0)$ and eliminate $\mathcal{A}(P_{\varepsilon_j})u_{m,\varepsilon_j}(P_{\varepsilon_j}) + \delta\sigma(P_{\varepsilon_j})$ by using (3.8). Then we have

$$-a(P_{\varepsilon_j})\eta + b(P_{\varepsilon_j})\eta^p + a(P_{\varepsilon_j})u_{m,\varepsilon_j}(P_{\varepsilon_j}) - b(P_{\varepsilon_j})u_{m,\varepsilon_j}(P_{\varepsilon_j}) \ge 0. \tag{3.9}$$

By (3.3), the equation $-a(P_{\varepsilon_j})\eta + b(P_{\varepsilon_j})\eta^p + a(P_{\varepsilon_j})u_{m,\varepsilon_j}(P_{\varepsilon_j}) - b(P_{\varepsilon_j})u_{m,\varepsilon_j}(P_{\varepsilon_j})^p = 0$ has two nonnegative roots $0 \le \eta_1(\varepsilon_j) < \eta_2(\varepsilon_j)$ and they satisfy the inequality

$$0 \le \eta_1(\varepsilon_j) < \left\{\frac{a(P_{\varepsilon_j})}{pb(P_{\varepsilon_j})}\right\}^{1/(p-1)} < \eta_2(\varepsilon_j).$$

Note also that (3.9) holds if $\eta \le \eta_1(\varepsilon_j)$ or $\eta \ge \eta_2(\varepsilon_j)$. But if $\eta \le \eta_1(\varepsilon_j)$, then $u_{m,\varepsilon_j}(P_{\varepsilon_j}) + V_{\varepsilon_j}(0) \le C'\delta$; this means that $\max V_{\varepsilon_j}(z) = O(\delta)$ and therefore $a(P_{\varepsilon_j} + \varepsilon_j z) - b(P_{\varepsilon_j} + \varepsilon_j z)\{(u_{m,\varepsilon_j}(P_{\varepsilon_j} + \varepsilon_j z) + V_{\varepsilon_j})^p - u_{m,\varepsilon_j}(P_{\varepsilon_j} + \varepsilon_j z)^p\}/V_{\varepsilon_j} = a(P_{\varepsilon_j} + \varepsilon_j z) - pb(P_{\varepsilon_j} + \varepsilon_j z)(u_{m,\varepsilon_j}(P_{\varepsilon_j} + \varepsilon_j z) + \theta V_{\varepsilon_j})^{p-1} > 0$ for all $z \in B_R(0)$. Hence in this case we have $\mathcal{A}(P_{\varepsilon_j} + \varepsilon_j z)V_{\varepsilon_j} > 0$ on $B_R(0)$, which is a contradiction since $V_{\varepsilon_j}(0)$ is a local maximum. Therefore,

$$V_{\varepsilon_j}(0) \ge \eta_2(\varepsilon_j) - u_{m,\varepsilon_j}(P_{\varepsilon_j}) > \{a(P_{\varepsilon_j})/(pb(P_{\varepsilon_j})\}^{1/(p-1)} - u_{m,\varepsilon_j}(P_{\varepsilon_j}),$$

and this proves our assertion $V_0(0) > 0$ because $u_{m,\varepsilon_j}(P_{\varepsilon_j}) = O(\delta)$.

Finally we put

$$w(y) = \left\{\frac{b(P_0)}{a(P_0)}\right\}^{1/(p-1)} V_0(z) \quad \text{with } y = \sqrt{a(P_0)}\sqrt{D_{P_0}^{-1}}B_{P_0}z.$$

It is easily checked that w is a positive solution of $\Delta w - w + (\gamma_0(P_0) + w)^p - \gamma_0(P_0)^p = 0$ in $\mathbb{R}^n$, and $w \in W^{2,r}(\mathbb{R}^n)$ for any $r \in [0, \infty)$. From $w \in W^{2,r}(\mathbb{R}^n)$ with $r > n$ it follows that $w(y) \to 0$ as $|y| \to \infty$. Hence, $w = w_{\gamma_0(P_0)}$ by the uniqueness of solution of (GS-γ).

Case (II): $P_0 \in \partial\Omega$. There are two possibilities to be considered.

(a) $\liminf_{j\to\infty} \mathrm{dist}(P_{\varepsilon_j}, \partial\Omega)/\varepsilon_j = \infty$,

(b) $\limsup_{j\to\infty} \mathrm{dist}(P_{\varepsilon_j}, \partial\Omega)/\varepsilon_j < \infty$,

where $\mathrm{dist}(Q, \partial\Omega) = \inf_{P\in\partial\Omega} |Q - P|$.

First we rule out the possibility of (a). Suppose that (a) occurs. Then for any $R > 0$ we have $\overline{B}_{3\varepsilon_j R}(P_{\varepsilon_j}) \subset \Omega$, provided that ε_j is sufficiently small. This is the same situation as in Case (I),

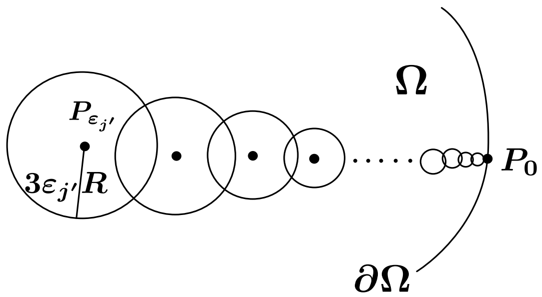

Figure 3.1: A typical situation of case (a)

and we can argue in the same way to prove that $V_{\varepsilon_j}(z) = v_{\varepsilon_j}(P_{\varepsilon_j} + \varepsilon_j z)$ converges to $V_0 \in C^2(\mathbb{R}^n)$ as $j \to \infty$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, V_0 is a positive solution of (3.7). Therefore

$$
\begin{aligned}
\frac{c_{\varepsilon_j}}{\varepsilon_j^n} &> \frac{1}{2}\int_{B_R(0)} \left\{ a_{kl}(P_{\varepsilon_j} + \varepsilon_j z)\frac{\partial V_{\varepsilon_j}}{\partial z_k}\frac{\partial V_{\varepsilon_j}}{\partial z_l} + \alpha(P_{\varepsilon_j} + \varepsilon_j z)V_{\varepsilon_j}^2 \right\} dz \\
&\quad - \int_{B_R(0)} b(P_{\varepsilon_j} + \varepsilon_j z) G(u_{m,\varepsilon_j}(P_{\varepsilon_j} + \varepsilon_j z), V_{\varepsilon_j})\, dz \\
&= \frac{1}{2}\int_{B_R(0)} \left\{ a_{kl}(P_0)\frac{\partial V_0}{\partial z_k}\frac{\partial V_0}{\partial z_l} + \alpha(P_0)V_0^2 \right\} dz - \int_{B_R(0)} b(P_0)G(u_{m,0}, V_0)\, dz \\
&\quad + \frac{1}{2}\int_{B_R(0)} \left\{ \left(a_{kl}(P_{\varepsilon_j} + \varepsilon_j z)\frac{\partial V_{\varepsilon_j}}{\partial z_k}\frac{\partial V_{\varepsilon_j}}{\partial z_l} - a_{kl}(P_0)\frac{\partial V_0}{\partial z_k}\frac{\partial V_0}{\partial z_l} \right) \right. \\
&\qquad\qquad \left. + \left(\alpha(P_{\varepsilon_j} + \varepsilon_j z)V_{\varepsilon_j}^2 - \alpha(P_0)V_0^2 \right) \right\} dz \\
&\quad - \int_{B_R(0)} \left\{ b(P_{\varepsilon_j} + \varepsilon_j z)G(u_{m,\varepsilon_j}(P_{\varepsilon_j} + \varepsilon_j z), V_{\varepsilon_j}) - b(P_0)G(u_m(P_0), V_0) \right\} dz.
\end{aligned}
$$

Letting $\varepsilon_j \downarrow 0$, we see that

$$
\begin{aligned}
\limsup \frac{c_{\varepsilon_j}}{\varepsilon_j^n} &\geq \frac{1}{2}\int_{B_R(0)} \left\{ a_{kl}(P_0)\frac{\partial V_0}{\partial z_k}\frac{\partial V_0}{\partial z_l} + \alpha(P_0)V_0^2 \right\} dz - \int_{B_R(0)} b(P_0)G(u_m(P_0), V_0)\, dz \\
&\geq \frac{1}{2}\int_{\mathbb{R}^n} \left\{ a_{kl}(P_0)\frac{\partial V_0}{\partial z_k}\frac{\partial V_0}{\partial z_l} + \alpha(P_0)V_0^2 \right\} dz - \int_{\mathbb{R}^n} b(P_0)G(u_m(P_0), V_0)\, dz \\
&\quad - O(e^{-\kappa R}) \quad (\text{with } \kappa > 0)
\end{aligned}
$$

because of the exponential decay of V_0 together with its derivatives of order up to 2 as $|z| \to \infty$. Therefore, by Proposition 2.12, we obtain

$$\Phi(P_0)I_\delta(P_0;\mathbb{R}^n) - O(e^{-\kappa R}) \le \min\left\{\min_{Q\in\overline{\Omega}} \Phi(Q)I_\delta(Q;\mathbb{R}^n), \frac{1}{2}\min_{Q\in\partial\Omega} \Phi(Q)I_\delta(Q;\mathbb{R}^n)\right\}.$$

Since $P_0 \in \partial\Omega$, this results in

$$\Phi(P_0)I_\delta(P_0;\mathbb{R}^n) \le \frac{1}{2}\Phi(P_0)I_\delta(P_0;\mathbb{R}^n) + O(e^{-\kappa R}).$$

This is a contradiction because $\Phi(P_0)I_\delta(P_0;\mathbb{R}^n)$ is a positive number, while we can take $R > 0$ as large as we wish. Therefore, (a) cannot occur.

Note that (b) is the only possible case, and hence, there exists an $R_* > 0$ such that $B_{\varepsilon_j R}(P_{\varepsilon_j}) \cap \partial\Omega \ne \emptyset$ whenever $R \ge R_*$ and j is sufficiently large. Since $B_{\varepsilon_j R}(P_{\varepsilon_j})$ protrudes from the domain Ω, we cannot argue as in the case (I) or (a) above and we have to take the effect of the boundary $\partial\Omega$ into consideration. We therefore introduce a diffeomorphism flattening the boundary portion around P_0, and extend the solution v_ε outside Ω along the conormal vector.

Step 1: Diffeomorphism. By translation and rotation of the coordinate system, we may assume that P_0 is the origin and the outer normal to $\partial\Omega$ at P_0 points in the negative direction of the x_n-axis. We write $x' = (x_1, \dots, x_{n-1})$ and $x = (x', x_n)$. Then there exists a smooth function $\psi(x')$ defined in $|x'| < \kappa_0$, where κ_0 is a small positive number, such that

(i) $\psi(0) = 0$ and $(\partial\psi/\partial x_j)(0) = 0$ for $j = 1, \dots, n-1$,

(ii) $\Omega \cap \mathcal{N} = \{(x', x_n) \mid x_n > \psi(x')\}$, $\partial\Omega \cap \mathcal{N} = \{(x', \psi(x')) \mid |x| < \kappa\}$.

Here, $\mathcal{N}$ is a neighborhood of the origin. Put $p_i(x) = (\partial\psi/\partial x_i)(x')$ for $i = 1, \dots, n-1$, and $p_n(x) = -1$. Then $\boldsymbol{p} = (p_1(x), \dots, p_n(x))$ gives an outer normal at point $(x', \psi(x')) \in \partial\Omega$, and $\boldsymbol{\nu} = \boldsymbol{p}(x)/|\boldsymbol{p}(x)|$ is the unit outer normal.

In what follows (until the proof of Proposition 3.1 is completed), we make a convention that $\varXi$ denotes a generic point with coordinate $\xi \in \mathbb{R}^n$, $\varXi'$ denotes a point on the hyperplane $\xi_n = 0$, i.e., $\varXi' = (\xi', 0)$ with $\xi' \in \mathbb{R}^{n-1}$. Likewise, X denotes a point with coordinate $x \in \mathbb{R}^n$ and X' denotes a point on the boundary, i.e., $X' = (x', \psi(x'))$. This notation will be used especially when we designate the point at which the differential of a mapping is computed.

Now we are ready to define the first diffeomorphism $\mathcal{S}(\xi) = (\mathcal{S}_1(\xi), \dots, \mathcal{S}_n(\xi))$ by

$$\begin{cases} \mathcal{S}_i(\xi) := \xi_i - \xi_n p_i(\xi') \quad \text{for} \quad i = 1, 2, \dots, n-1, \\ \mathcal{S}_n(\xi) := \xi_n + \psi(\xi'). \end{cases} \tag{3.10}$$

We observe that

$$D\mathcal{S}(\varXi) = \left(\frac{\partial \mathcal{S}_i}{\partial \xi_j}(\xi)\right)_{1\le i,j\le n} = \left(\begin{array}{c|c} \delta_{ij} - \xi_n(\partial p_i/\partial \xi_j) & -p_i \\ \hline p_j & -p_n \end{array}\right)$$

and $D\mathcal{S}(O) = E_n$ where E_n is the unit matrix of dimension n. Hence, there exists a $\kappa_0 > 0$ such that $\mathcal{S}$ is the inverse mapping for $|\xi| \le 3\kappa_0$. Let us define $\mathcal{T}(x) = \mathcal{S}^{-1}(x)$ with $\mathcal{T}(x) = (\mathcal{T}_1(x), \dots, \mathcal{T}_n(x))$. It is to be noted that $\mathcal{T}$ maps $\partial\Omega \cap \mathcal{N}_0$ into the hyperplane $\{\xi_n = 0\}$ and

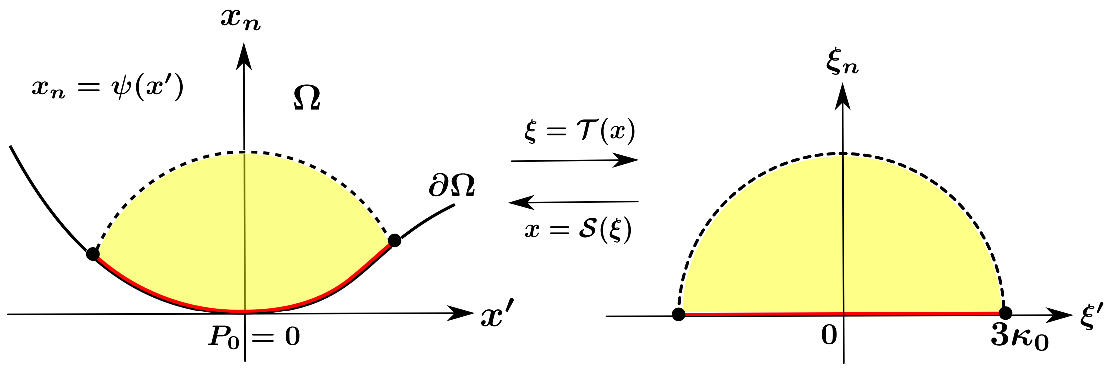

Figure 3.2: Diffeomorphism to make the boundary near P_0 flat

$\Omega \cap \mathcal{N}_0$ into the upper half space $\{\xi_n > 0\}$, where $\mathcal{N}_0 = \{x = \mathcal{S}(\xi) \mid |\xi| \le 3\kappa_0\}$. Indeed, since $D\mathcal{S}(O) = E_n$, by continuity we may assume that $\det(D\mathcal{S}(\Xi)) > 0$ for $|\xi| \le 3\kappa_0$. Hence, if $\xi_n > 0$ then $\mathcal{S}_n(\xi) > \psi(\xi')$, which means that $\mathcal{S}(B^+_{3\kappa_0}(0)) \subset \Omega$, and this proves $\mathcal{T}(\overline{\Omega \cap \mathcal{N}_0}) = \overline{B^+_{3\kappa_0}}(0)$. If we put $\boldsymbol{\mu}(\xi) = \boldsymbol{\mu}(\xi') := (D\mathcal{T}(x)A_x\boldsymbol{\nu})|_{x=\mathcal{S}(\Xi')}$, then $\boldsymbol{\mu}$ is the vector in the ξ-space corresponding to the conormal vector $A_x\boldsymbol{\nu}$. Moreover, we have $\det(D\mathcal{S}(\Xi')) = |\boldsymbol{p}|^2 = |\boldsymbol{p}(\xi', 0)|^2$ and

$$D\mathcal{T}(\mathcal{S}(\Xi')) = D\mathcal{S}(\Xi')^{-1} = \frac{1}{|\boldsymbol{p}|^2}\left(\begin{array}{cccc|c} |\boldsymbol{p}|^2 - p_1^2 & -p_1p_2 & \cdots & -p_1p_{n-1} & p_1 \\ -p_1p_2 & |\boldsymbol{p}|^2 - p_2^2 & \cdots & -p_2p_{n-1} & p_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -p_1p_{n-1} & -p_2p_{n-1} & \cdots & |\boldsymbol{p}|^2 - p_{n-1}^2 & p_{n-1} \\ \hline -p_1 & -p_2 & \cdots & -p_{n-1} & -p_n \end{array}\right) \tag{3.11}$$

$$= \frac{1}{|\boldsymbol{p}|^2}\left(\begin{array}{c|c} \delta_{ij}|\boldsymbol{p}|^2 - p_ip_j & p_i \\ \hline -p_j & -p_n \end{array}\right).$$

Now, we define the second diffeomorphism $\mathcal{U}(\zeta) = (\mathcal{U}_1(\zeta), \ldots, \mathcal{U}_n(\zeta))$ by

$$\begin{cases} \mathcal{U}_i(\zeta) = \zeta_i - \mu_i(\zeta')\zeta_n & \text{for} \quad i = 1, \ldots, n-1, \\ \mathcal{U}_n(\zeta) = -\mu_n(\zeta')\zeta_n. \end{cases}$$

Then, we obtain that

$$D\mathcal{U}(\zeta) = \left(\begin{array}{c|c} \delta_{ij} - \zeta_n(\partial\mu_i/\partial\zeta_j) & -\mu_i(\zeta') \\ \hline 0 & -\mu_n(\zeta') \end{array}\right),$$

$$\det(D\mathcal{U}(O)) = -\mu_n(0) = (D\mathcal{S}(O)^{-1}A_{P_0}\boldsymbol{\nu})_n = a_{nn}(P_0) > 0.$$

Hence there exists a $\tilde{\kappa}_0 > 0$ such that $\mathcal{U}$ is the inverse mapping for $|\zeta| \le 3\tilde{\kappa}_0$. We write κ_0 instead of $\min\{\kappa_0, \tilde{\kappa}_0\}$, and put $\mathcal{V}(\xi) = \mathcal{U}^{-1}(\xi)$ with $\mathcal{V}(\xi) = (\mathcal{V}_1(\xi), \ldots, \mathcal{V}_n(\xi))$. In similar way to $\mathcal{S}$, by $\det(D\mathcal{U}(O)) > 0$, we may suppose that $\det(D\mathcal{U}(\zeta)) > 0$ for $|\zeta| \le 3\kappa_0$. Hence, if $\zeta_n > 0$ then $\mathcal{U}_n(\zeta) > 0$, which means that $\mathcal{V}(\overline{B^+_{3\kappa_0}}(0)) = \overline{B^+_{3\kappa_0}}(0)$. Therefore we see that

$$x \underset{\mathcal{S}}{\overset{\mathcal{T}}{\rightleftarrows}} \xi \underset{\mathcal{U}}{\overset{\mathcal{V}}{\rightleftarrows}} \zeta\ .$$

Step 2: Extension of functions. Given a smooth function $v(x)$ defined on $\overline{\Omega}$, we obtain a function $\phi(\zeta)$ defined on the upper semiball $\overline{B^+_{3\kappa_0}}(0) = \{\zeta \in \mathbb{R}^n \mid |\zeta| \leq 3\kappa_0 \text{ and } \zeta_n \geq 0\}$ by putting $\phi(\zeta) = v(\mathcal{SU}(\zeta))$, which we call the pull back of v by $\mathcal{SU}$ where $\mathcal{SU} = \mathcal{S} \circ \mathcal{U}$. We can extend ϕ into the lower semiball $B^-_{3\kappa_0}(0)$ by defining

$$\tilde{\phi}(\zeta', \zeta_n) = \phi(\zeta', -\zeta_n) \quad \text{for } \zeta_n < 0.$$

Therefore

$$\tilde{\phi}(\zeta) = \phi(\zeta', |\zeta_n|) \quad \text{for } (\zeta', \zeta_n) \in \overline{B_{3\kappa_0}}(0).$$

Hence, we extend v to the outside of Ω near P_0 as in the following figure:

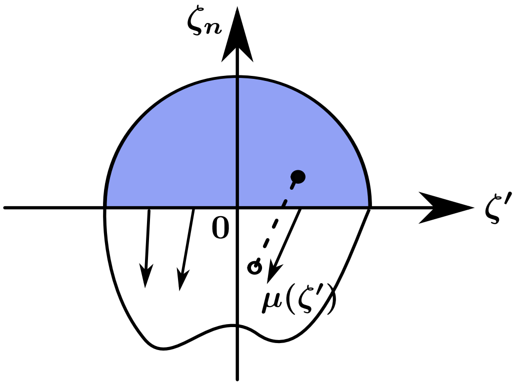

Figure 3.3: extension of $v \circ \mathcal{SU}(\zeta)$ along the vector $\boldsymbol{\mu}$.

Next, let us examine the smoothness of the extended function when v satisfies the boundary condition $\mathcal{B}(x)v = 0$ on $\partial\Omega$. In terms of ξ-variables, the boundary operator is expressed as follows:

$$\begin{aligned}\mathcal{B}(x)v &= \boldsymbol{\nu} \cdot A_x \nabla v(x)|_{x_n=\psi(x')} = A_x \boldsymbol{\nu}|_{x=\mathcal{SU}(\zeta',0)} \cdot (D(\mathcal{SU})(\zeta', 0))^{-1} \nabla_\zeta (v(\mathcal{SU}(\zeta)))(\zeta', 0) \\ &= (D(\mathcal{SU})(\zeta', 0))^{-1} A_{\mathcal{SU}(\zeta',0)} \boldsymbol{\nu} \cdot \nabla\phi(\zeta', 0) \\ &= D\mathcal{U}(\zeta', 0)^{-1} \boldsymbol{\mu}(\zeta') \cdot \nabla\phi(\zeta)|_{\zeta_n=0},\end{aligned}$$

where $\boldsymbol{a} \cdot \boldsymbol{b}$ denotes the Euclidean inner product of the vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ and

$$D\mathcal{U}(\zeta', 0)^{-1} = \frac{1}{-\mu_n(\zeta')} \left(\begin{array}{c|c} -\mu_n(\zeta')\delta_{ij} & \mu_i(\zeta') \\ \hline 0 & 1 \end{array} \right). \tag{3.12}$$

Therefore, for $\phi = v \circ \mathcal{SU}$, $\mathcal{B}(x)v = 0$ is equivalent to $D\mathcal{U}(\zeta)^{-1}\boldsymbol{\mu} \cdot \nabla\phi|_{\zeta_n=0} = 0$, where it is to be noted that $(\xi', 0) = \mathcal{U}(\zeta', 0) = (\zeta', 0)$ and $D\mathcal{U}(\zeta)^{-1}\boldsymbol{\mu}$ is the vector in the ζ-space corresponding to $\boldsymbol{\mu}(\xi')$.

Step 3: Equation of v_ε. Let $\varphi(\zeta) := v_\varepsilon \circ \mathcal{SU}(\zeta)$ and $\tilde{\varphi}(\zeta) := \varphi(\zeta', |\zeta_n|)$. Then, we calculate derivatives of φ as follows:

$$
\begin{aligned}
&\frac{\partial\tilde\varphi}{\partial\zeta_k}(\zeta)=\frac{\partial\varphi}{\partial\zeta_k}(\zeta',|\zeta_n|)\quad(1\le k\le n-1),\qquad \frac{\partial\tilde\varphi}{\partial\zeta_n}(\zeta)=\frac{\zeta_n}{|\zeta_n|}\frac{\partial\varphi}{\partial\zeta_n}(\zeta',|\zeta_n|),\\
&\frac{\partial^2\tilde\varphi}{\partial\zeta_k\partial\zeta_l}(\zeta)=\frac{\partial^2\varphi}{\partial\zeta_k\partial\zeta_l}(\zeta',|\zeta_n|)\quad(1\le k,l\le n-1),\qquad \frac{\partial^2\tilde\varphi}{\partial\zeta_n^2}(\zeta)=\frac{\partial^2\varphi}{\partial\zeta_n^2}(\zeta',|\zeta_n|),\\
&\frac{\partial^2\tilde\varphi}{\partial\zeta_k\partial\zeta_n}(\zeta)=\frac{\partial^2\tilde\varphi}{\partial\zeta_n\partial\zeta_k}(\zeta)=\frac{\zeta_n}{|\zeta_n|}\frac{\partial^2\varphi}{\partial\zeta_k\partial\zeta_n}(\zeta',|\zeta_n|),\quad(1\le k\le n-1).
\end{aligned}
\tag{3.13}
$$

Now, to derive the equation for $\varphi(\zeta)$ $(=v_\varepsilon(x))$, we first substitute $\tilde\varphi$ to $\mathcal{A}(x)v_\varepsilon$. From $\zeta=\mathcal{VT}(x)$, noting that

$$
\frac{\partial}{\partial x_j}=\sum_{l=1}^n\frac{\partial(\mathcal{VT})_l}{\partial x_j}(x)\frac{\partial}{\partial\zeta_l}\Big|_{x=\mathcal{SU}(\zeta)}=\sum_{l=1}^n(D(\mathcal{VT})(x))_{lj}\frac{\partial}{\partial\zeta_l}\Big|_{x=\mathcal{SU}(\zeta)},
$$

we have

$$
\begin{aligned}
\mathcal{A}(x)v_\varepsilon&=\sum_{i,j=1}^n\frac{\partial}{\partial x_i}a_{ij}(x)\frac{\partial}{\partial x_j}v_\varepsilon(x)=\sum_{i,j=1}^n\frac{\partial}{\partial x_i}a_{ij}(x)\sum_{l=1}^n(D(\mathcal{VT})(x))_{lj}\Big|_{x=\mathcal{SU}(\zeta)}\frac{\partial\tilde\varphi}{\partial\zeta_l}\\
&=\sum_{k,l=1}^n\sum_{i,j=1}^n(D(\mathcal{VT})(x))_{ki}a_{ij}(x)(D(\mathcal{VT})(x))_{lj}\Big|_{x=\mathcal{SU}(\zeta)}\frac{\partial^2\tilde\varphi}{\partial\zeta_k\partial\zeta_l}\\
&\quad+\sum_{i,j=1}^n\sum_{l=1}^n\frac{\partial}{\partial x_i}\Big\{a_{ij}(x)(D(\mathcal{VT})(x))_{lj}\Big\}\Big|_{x=\mathcal{SU}(\zeta)}\frac{\partial\tilde\varphi}{\partial\zeta_l}.
\end{aligned}
$$

In view of (3.13), we define

$$
\alpha_{kl}(\zeta):=\begin{cases}\displaystyle\sum_{i,j=1}^n(D(\mathcal{VT})(x))_{ki}a_{ij}(x)(D(\mathcal{VT})(x))_{lj}\Big|_{x=\mathcal{SU}(\zeta)} & \text{for } 1\le k,l\le n-1 \text{ or } k=l=n,\\[2ex]
\displaystyle\frac{\zeta_n}{|\zeta_n|}\sum_{i,j=1}^n(D(\mathcal{VT})(x))_{ki}a_{ij}(x)(D(\mathcal{VT})(x))_{nj}\Big|_{x=\mathcal{SU}(\zeta)} & \text{for } 1\le k\le n-1,\end{cases}
$$

$$
\beta_l(\zeta):=\begin{cases}\displaystyle\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\Big\{a_{ij}(x)(D(\mathcal{VT})(x))_{lj}\Big\}\Big|_{x=\mathcal{SU}(\zeta)} & \text{for } 1\le l\le n-1,\\[2ex]
\displaystyle\frac{\zeta_n}{|\zeta_n|}\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\Big\{a_{ij}(x)(D(\mathcal{VT})(x))_{nj}\Big\}\Big|_{x=\mathcal{SU}(\zeta)} & \text{for } l=n.\end{cases}
\tag{3.14}
$$

Then, we have $\alpha_{kl}=\alpha_{lk}$ and

$$
\mathcal{A}(x)v_\varepsilon=\sum_{k,l=1}^n\alpha_{kl}(\zeta)\frac{\partial^2\varphi}{\partial\zeta_k\partial\zeta_l}(\zeta',|\zeta_n|)+\sum_{l=1}^n\beta_l(\zeta)\frac{\partial\varphi}{\partial\zeta_l}(\zeta',|\zeta_n|).
\tag{3.15}
$$

Since v_ε satisfies (3.1) and $\partial\Omega \cap \mathcal{U}_0 = \{\xi_n = 0\} \cap \overline{B_{3\kappa_0}}(0) = \{\zeta_n = 0\} \cap \overline{B_{3\kappa_0}}(0)$, it follows that

$$\begin{aligned}(3.16)\qquad &\varepsilon^2\left\{\sum_{k,l=1}^{n}\alpha_{kl}(\zeta)\frac{\partial^2\varphi}{\partial\zeta_k\partial\zeta_l}(\zeta',|\zeta_n|)+\sum_{l=1}^{n}\beta_l(\zeta)\frac{\partial\varphi}{\partial\zeta_l}(\zeta',|\zeta_n|)\right\}-a(\mathcal{SU}(\zeta',|\zeta_n|))\varphi(\zeta',|\zeta_n|)\\ &+b(\mathcal{SU}(\zeta',|\zeta_n|))\left\{\left(u_{m,\varepsilon}(\mathcal{SU}(\zeta',|\zeta_n|))+\varphi(\zeta',|\zeta_n|)\right)^p-u_{m,\varepsilon}(\mathcal{SU}(\zeta',|\zeta_n|))^p\right\}=0,\end{aligned}$$

where $\zeta \in B_{3\kappa_0}(0) \setminus \{\zeta_n = 0\}$.

Now, we note that (3.16) holds at $\zeta_n = 0$. Actually, by (3.14), the continuity of $D(\mathcal{VT})$, and $a_{ij} \in C^2(\overline{\Omega})$, the coefficient α_{kl} is Lipschitz continuous for $1 \le k,l \le n-1$ or $k = l = n$. Moreover, α_{kn} is Lipschitz continuous except $\zeta_n = 0$. Therefore, it remains to verify the Lipschitz continuity of α_{kn} at $\zeta_n = 0$. Since

$$\begin{aligned}\frac{\zeta_n}{|\zeta_n|}\alpha_{kn} &= (D(\mathcal{VT})(x))_{ki}a_{ij}(x)(D(\mathcal{VT})(x))_{nj}\Big|_{x=\mathcal{SU}(\zeta)}\\ (3.17)\qquad &= (D(\mathcal{VT})(x))_{ki}a_{ij}(x)(D(\mathcal{VT})(x))_{nj}\Big|_{x=\mathcal{SU}(\zeta',0)}\\ &\quad+\Big[(D(\mathcal{VT})(x))_{ki}a_{ij}(x)(D(\mathcal{VT})(x))_{nj}\Big]_{x=\mathcal{SU}(\zeta',0)}^{x=\mathcal{SU}(\zeta)},\end{aligned}$$

it suffices to prove that when $\zeta_n = 0$, $\sum_{i,j=1}^{n}(D(\mathcal{VT})(x))_{ki}a_{ij}(x)(D(\mathcal{VT})(x))_{nj}|_{x=\mathcal{SU}(\zeta',\zeta_n)} = 0$. In fact, the second term of the right-hand side of (3.17) is Lipschitz continuous by the continuity of $D(\mathcal{VT})$ and a_{ij}. Hence, one obtains

$$\begin{aligned}\alpha_{kn}(\zeta) &= \frac{\zeta_n}{|\zeta_n|}\sum_{i,j=1}^{n}(D(\mathcal{VT})(x))_{ki}a_{ij}(x)(D(\mathcal{VT})(x))_{nj}(\zeta)\Big|_{x=\mathcal{SU}(\zeta)}\\ (3.18)\qquad &= \frac{\zeta_n}{|\zeta_n|}\sum_{i,j=1}^{n}(D(\mathcal{VT})(x))_{ki}a_{ij}(x)(D(\mathcal{VT})(x))_{nj}\Big|_{x=\mathcal{SU}(\zeta',0)}+O(|\zeta_n|).\end{aligned}$$

Since it holds that $D(\mathcal{VT})(x) = D\mathcal{V}(\Xi')D\mathcal{T}(x) = D\mathcal{U}(\zeta',0)^{-1}D\mathcal{S}(\Xi')^{-1}$ at $x = \mathcal{SU}(\zeta',0)$, by (3.11) and (3.12), we calculate as follows:

$$\begin{aligned}(D(\mathcal{VT})(x))_{ki}\Big|_{x=\mathcal{SU}(\zeta',0)} &= 1/|\boldsymbol{p}|^2\left[\begin{pmatrix}\delta_{ij} & -\mu_i/\mu_n\\ 0 & -1/\mu_n\end{pmatrix}\begin{pmatrix}\delta_{ij}|\boldsymbol{p}|^2-p_ip_j & p_i\\ -p_j & -p_n\end{pmatrix}\right]_{ki}\\ (3.19)\qquad &= 1/|\boldsymbol{p}|^2\begin{cases}\delta_{ki}|\boldsymbol{p}|^2-p_kp_i-p_i(\mu_k/\mu_n) & \text{for } 1\le i\le n-1,\\ p_k+p_n(\mu_k/\mu_n) & \text{for } i=n,\end{cases}\\ (D(\mathcal{VT})(x))_{nj}\Big|_{x=S\circ U(\zeta',0)} &= p_j/(\mu_n|\boldsymbol{p}|^2)\qquad \text{for } 1\le j\le n.\end{aligned}$$

Then, the following holds for the second term of the right-hand side of (3.18):

$$\begin{aligned}&\frac{\xi_n}{|\xi_n|}\sum_{i,j=1}^{n}(D(\mathcal{VT})(x))_{ki}a_{ij}(x)(D(\mathcal{VT})(x))_{nj}\\ &=\frac{\zeta_n}{|\zeta_n|}\frac{1}{|\boldsymbol{p}|^4}\sum_{j=1}^{n}\frac{p_j}{\mu_n}\left\{\sum_{i=1}^{n-1}\Big(\delta_{ki}|\boldsymbol{p}|^2-p_kp_i-p_i\frac{\mu_k}{\mu_n}\Big)a_{ij}+\Big(p_k+p_n\frac{\mu_k}{\mu_n}\Big)a_{nj}\right\} =: \Gamma_{kn}\end{aligned}$$

where $a_{ij} = a_{ij}(\mathcal{SU}(\zeta', 0))$. Calculating Γ_{kn} and making use of $p_n = -1$, we have

$$\begin{aligned}
\frac{\zeta_n}{|\zeta_n|}\mu_n|\boldsymbol{p}|^4\Gamma_{kn} &= \sum_{j=1}^{n} p_j\Big\{|\boldsymbol{p}|^2 a_{kj} - \Big(p_k + \frac{\mu_k}{\mu_n}\Big)\sum_{i=1}^{n-1} a_{ij}(x)p_i + \Big(p_k + p_n\frac{\mu_k}{\mu_n}\Big)a_{nj}\Big\} \\
&= \sum_{j=1}^{n} p_j\Big\{|\boldsymbol{p}|^2 a_{kj} - \Big(p_k + p_n\frac{\mu_k}{\mu_n}\Big)\Big(\sum_{i=1}^{n-1} a_{ij}p_i - a_{nj}(x)\Big)\Big\} \\
&= \sum_{j=1}^{n} p_j\Big\{|\boldsymbol{p}|^2 a_{kj} - \Big(p_k + p_n\frac{\mu_k}{\mu_n}\Big)\sum_{i=1}^{n} a_{ij}p_i\Big\} \\
&= |\boldsymbol{p}|^2\sum_{j=1}^{n} a_{kj}p_j - \Big(p_k + p_n\frac{\mu_k}{\mu_n}\Big)\sum_{i,j=1}^{n} a_{ij}p_ip_j.
\end{aligned} \tag{3.20}$$

Substituting $\boldsymbol{\mu}(\xi') = D\mathcal{S}(\xi', 0)^{-1}A_{\mathcal{S}(\xi',0)}\boldsymbol{p}(\mathcal{S}(\xi', 0))$ where $\xi = \mathcal{U}(\zeta)$, we verify that (3.20) is equal to zero. By the definition of $\boldsymbol{\mu}$, it holds that

$$\begin{aligned}
\mu_k &= \frac{1}{|\boldsymbol{p}|^2}\left[\begin{pmatrix} \delta_{ij}|\boldsymbol{p}|^2 - p_ip_j & p_i \\ \hline -p_j & -p_n \end{pmatrix} A_{S(\zeta',0)}{}^t\boldsymbol{p}\right]_k = \frac{1}{|\boldsymbol{p}|^2}\sum_{j=1}^{n}\Big\{\sum_{i=1}^{n-1}(\delta_{ki}|\boldsymbol{p}|^2 - p_kp_i)a_{ij}p_j + p_ka_{nj}p_j\Big\} \\
&= \frac{1}{|\boldsymbol{p}|^2}\sum_{j=1}^{n}\Big\{|\boldsymbol{p}|^2 a_{kj}p_j - p_k\sum_{i=1}^{n-1} a_{ij}p_ip_j + p_ka_{nj}p_j\Big\} = \sum_{j=1}^{n} a_{kj}p_j - \frac{p_k}{|\boldsymbol{p}|^2}\sum_{i,j=1}^{n} a_{ij}p_ip_j, \\
\mu_n &= \frac{1}{|\boldsymbol{p}|^2}\sum_{j=1}^{n}(-p_i)a_{ij}p_j = -\frac{1}{|\boldsymbol{p}|^2}\sum_{j=1}^{n} a_{ij}p_ip_j.
\end{aligned}$$

Now, by using $p_n = -1$, these identities and (3.20), we obtain

$$\begin{aligned}
\frac{\zeta_n}{|\zeta_n|}\mu_n|\boldsymbol{p}|^4\Gamma_{kn} &= |\boldsymbol{p}|^2\sum_{j=1}^{n} a_{kj}p_j - \frac{1}{\mu_n}\sum_{i,j=1}^{n} a_{ij}p_ip_j\big(p_k\mu_n + p_n\mu_k\big) \\
&= |\boldsymbol{p}|^2\sum_{j=1}^{n} a_{kj}p_j - \frac{1}{\mu_n}(-\mu_n|\boldsymbol{p}|^2)\Big\{-\frac{p_k}{|\boldsymbol{p}|^2}\sum_{j=1}^{n} a_{ij}p_ip_j + p_n\Big(\sum_{j=1}^{n} a_{kj}p_j - \frac{p_k}{|\boldsymbol{p}|^2}\sum_{i,j=1}^{n} a_{ij}p_ip_j\Big)\Big\} \\
&= |\boldsymbol{p}|^2\sum_{j=1}^{n} a_{kj}p_j + |\boldsymbol{p}|^2 p_n\sum_{j=1}^{n} a_{kj}p_j = 0.
\end{aligned}$$

Thus, we have $\alpha_{kn} = O(|\xi_n|)$ and proved that α_{kn} is Lipschitz continuous at $\xi_n = 0$. Consequently, (3.16) holds at $\xi_n = 0$, and we see that $\tilde{\varphi}$ satisfies the following in the sense of weak solution:

$$\begin{aligned}
&\varepsilon^2\left\{\sum_{\substack{1\le k,l\le n-1 \\ \text{or } k=l=n}} \alpha_{kl}\frac{\partial^2\tilde{\varphi}}{\partial\xi_k\partial\xi_l} + \sum_{l=1}^{n-1}\beta_l\frac{\partial\tilde{\varphi}}{\partial\xi_l} + \beta_n\frac{\xi_n}{|\xi_n|}\frac{\partial\tilde{\varphi}}{\partial\xi_n}\right\} - a(S(\xi', |\xi_n|))\tilde{\varphi} \\
&\quad + b(S(\xi', |\xi_n|))\Big\{\Big(u_{m,\varepsilon}(S(\xi', |\xi_n|)) + \tilde{\varphi}\Big)^p - u_{m,\varepsilon}(S(\xi', |\xi_n|))^p\Big\} = -2\varepsilon^2\sum_{k=1}^{n-1}\alpha_{kn}\frac{\xi_n}{|\xi_n|}\frac{\partial^2\tilde{\varphi}}{\partial\xi_k\partial\xi_n}.
\end{aligned} \tag{3.21}$$

By using the regularity theory, we know that $\tilde{\varphi}$ is a classical solution of (3.21) in $B_{3\kappa_0}(0)$.

Step 4: Convergence of v_ε. For a while, we suppress j in ε_j for simplicity. We let $\zeta := Q_\varepsilon + \varepsilon z$ and $V_\varepsilon(z) := \varphi(\zeta)$ where $Q_\varepsilon := \mathcal{VT}(P_\varepsilon)$. Then by (3.21), V_ε satisfies the following elliptic equation for $Q_\varepsilon + \varepsilon z \in B_{3\kappa_0}(0)$

$$\begin{aligned}&\sum_{\substack{1\le k,l\le n-1\\ \text{or } k=l=n}} \alpha_{kl}(Q_\varepsilon+\varepsilon z)\frac{\partial^2 V_\varepsilon}{\partial z_k \partial z_l} + 2\sum_{k=1}^{n-1}\alpha_{kn}(Q_\varepsilon+\varepsilon z)\frac{Q_{\varepsilon,n}+\varepsilon z_n}{|Q_{\varepsilon,n}+\varepsilon z_n|}\frac{\partial^2 V_\varepsilon}{\partial z_k\partial z_n}\\ &+\varepsilon\left\{\sum_{l=1}^{n-1}\beta_l(Q_\varepsilon+\varepsilon z)\frac{\partial V_\varepsilon}{\partial z_l}+\beta_n(Q_\varepsilon+\varepsilon z)\frac{Q_{\varepsilon,n}+\varepsilon z_n}{|Q_{\varepsilon,n}+\varepsilon z_n|}\frac{\partial V_\varepsilon}{\partial z_n}\right\}\\ &-(a\circ\mathcal{SU})V_\varepsilon+(b\circ\mathcal{SU})\left\{\Big((u_{m,\varepsilon}\circ\mathcal{SU})+V_\varepsilon\Big)^p-(u_{m,\varepsilon}\circ\mathcal{SU})^p\right\}=0.\end{aligned}$$

Since $Q_\varepsilon \to \mathcal{VT}(P_0) = O$ as $\varepsilon \downarrow 0$ and $\alpha_{kn} = O(|\zeta_n|)$, for each $R > 0$, we have $B_{3\varepsilon R}(0) \subset B_{2\kappa_0}(Q_\varepsilon) \subset B_{3\kappa_0}(0)$ and $|Q_\varepsilon+\varepsilon z| = |\mathcal{VT}(P_0)|+o(\varepsilon(1+R))$ by taking ε sufficiently small. Hence, since $\varepsilon z \in B_{3\varepsilon R}(0)$ is equivalent to $z \in B_{3R}(0)$, we see that $V_\varepsilon(z)$ satisfies the following equation in $B_{3R}(0)$:

$$\begin{aligned}&\sum_{\substack{1\le k,l\le n-1\\ \text{or } k=l=n}} \alpha_{kl}(T(P_0))\frac{\partial^2 V_\varepsilon}{\partial z_k\partial z_l} - a(P_0)V_\varepsilon + b(P_0)\{(u_m(P_0)+V_\varepsilon)_+^p - u_m(P_0)^p\}\\ &+\{o(|P_\varepsilon-P_0|)+O(\varepsilon R)\}\\ &\times\left\{\sum_{k,l=1}^{n-1}\frac{\partial^2 V_\varepsilon}{\partial z_k\partial z_l}+\varepsilon\sum_{l=1}^{n}\frac{\partial V_\varepsilon}{\partial z_l}+V_\varepsilon+\Big((u_{m,\varepsilon}\circ\mathcal{SU})+V_\varepsilon\Big)^p-(u_{m,\varepsilon}\circ\mathcal{SU})^p\right\}=0.\end{aligned}\tag{3.22}$$

To prove the convergence of V_ε, we begin by verifying the boundedness of $\|V_\varepsilon\|_{L^r(B_{3R}(0))}$. By using Proposition 2.14, we have

$$C_r\varepsilon^n \ge \int_{\mathcal{U}_0\cap\Omega} v_\varepsilon^r\,dx \ge \int_{B^+_{3\kappa_0}(0)} \varphi^r\left|\det(D(\mathcal{SU})(\zeta))\right|d\zeta = \frac{1}{2}\int_{B_{3\kappa_0}(0)}\tilde{\varphi}^r\left|\det(D(\mathcal{SU})(\zeta))\right|d\zeta.$$

Since $\zeta = Q_\varepsilon + \varepsilon z$, $B_{3\varepsilon R}(0) \subset B_{2\kappa_0}(Q_\varepsilon) \subset B_{3\kappa_0}(0)$, $V_\varepsilon(z) = \tilde{\varphi}(\zeta)$ and $d\zeta = \varepsilon^n dz$, one obtains

$$\int_{B_{3R}(0)} V_\varepsilon^r\left|\det(D(\mathcal{SU})(Q_\varepsilon+\varepsilon z))\right|dz \le 2C_r.$$

Moreover, noting $|\det(D(\mathcal{SU})(Q_\varepsilon+\varepsilon z))| = a_{nn}(P_0)+o(|P_\varepsilon-P_0|)+O(\varepsilon R)$ for sufficiently small ε, we have

$$\int_{B_{3R}(0)} V_\varepsilon^r\Big\{a_{nn}(P_0)+o(|P_\varepsilon-P_0|)+O(\varepsilon R)\Big\}dz \le 2C_r.$$

From the assumption of (b), since $\limsup_{\varepsilon\downarrow 0}|P_\varepsilon-P_0|/\varepsilon \le C$, as $\varepsilon\downarrow 0$, we get

$$a_{nn}(P_0)\int_{B_{3R}(0)} V_\varepsilon^r\,dz \le 2C_r.$$

Thus,

$$\int_{B_{3R}(0)} V_\varepsilon^r\,dz \le 2C_r/a_{nn}(P_0).$$

Noting that $C_r^{1/r}$ is bounded in $r \in [1, \infty)$ by Proposition 2.14, in a way similar to (3.6), we see that $\|V_\varepsilon\|_{L^r(B_{3R}(0))}$ is bounded for any $r \in [1, \infty)$ as $\varepsilon \downarrow 0$. Therefore, using the regularity theory for the equation (3.22), we see that $\|V_\varepsilon\|_{W^{2,r}(B_{2R}(0))}$ is bounded. From the Sobolev imbedding theorem and the interior Schauder estimate, V_ε is bounded in $C^{2,\beta}(\overline{B}_R(0))$. By using the Ascoli-Arzelà theorem, there exists $\{\varepsilon_{j_k}^{(R)}\}_{k\in\mathbb{N}} \subset \{\varepsilon_j\}$ and $V^r \in C^2(\overline{B}_R(0))$ such that

$$V_{\varepsilon_{j_k}^{(R)}} \to V^r \quad \text{in } C^2(\overline{B}_R(0)) \qquad \text{as } k \to \infty.$$

Then, taking a sequence $\{R_k\}$ with $R_k \uparrow \infty$ and applying the diagonal argument, we obtain that a subsequence $\{V_{\varepsilon_{j_k}}\}$ converges to $V_0 \in C^2(\mathbb{R}^n)$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$. From (3.21), V_0 satisfies

$$\begin{aligned}(3.23)\qquad &\sum_{\substack{1\le k,l\le n-1\\ \text{or } k=l=n}} \alpha_{kl}(\mathcal{VT}(P_0))\frac{\partial^2 V_0}{\partial z_k \partial z_l}\\ &\quad - a(P_0)V_0 + b(P_0)\{(u_m(P_0) + V_0)^p - u_m(P_0)^p\} = 0 \quad \text{in } \mathbb{R}^n.\end{aligned}$$

Here, by (3.14),

$$(3.24)\qquad \begin{aligned}\alpha_{kl}(\mathcal{VT}(P_0)) &= \begin{cases} \displaystyle\sum_{i,j=1}^n (D(\mathcal{VT})(P_0))_{ki} a_{ij}(P_0)(D(\mathcal{VT})(P_0))_{lj} & \\ & \text{for } 1 \le k, l \le n-1 \text{ or } k = l = n,\\ 0 & \text{for } 1 \le k \le n-1, l = n\end{cases}\\ &= (D(\mathcal{VT})(P_0) A_{P_0} {}^t D(\mathcal{VT})(P_0))_{kl}\end{aligned}$$

where we note that

$$D(\mathcal{VT})(P_0) = \frac{1}{a_{nn}(P_0)}\left(\begin{array}{c|c} a_{nn}(P_0)\delta_{ij} & -a_{in}(P_0) \\ \hline 0 & 1\end{array}\right)$$

by virtue of $\boldsymbol{p} = -\boldsymbol{e}_n$, $\boldsymbol{\mu} = -A_{P_0}\boldsymbol{e}_n$, $\boldsymbol{p}\cdot\boldsymbol{\mu} = a_{nn}(P_0)$ at $x = P_0$, and (3.11).

Step 5: Limit of the energy $c_\varepsilon/\varepsilon^n$. Let

$$w(y) := \left\{\frac{b(P_0)}{a(P_0)}\right\}^{1/(p-1)} V_0(z) \quad \text{with } y := \sqrt{a(P_0)}\,\sqrt{D_{P_0}^{-1}} B_{P_0} D(\mathcal{SU})(O) z$$

where $D(\mathcal{SU})(O) = (D(\mathcal{VT})(P_0))^{-1}$. Then, since

$$\frac{\partial}{\partial z_k} = \sqrt{a(P_0)} \sum_{m=1}^n \left(\sqrt{D_{P_0}^{-1}} B_{P_0} D(\mathcal{SU})(O)\right)_{mk} \frac{\partial}{\partial y_m},$$

we obtain by (3.23) and (3.24) that

$$\begin{aligned}&\sum_{\substack{1\le k,l\le n-1\\ \text{or } k=l=n}} \alpha_{kl}(\mathcal{VT}(P_0))\frac{\partial^2}{\partial z_k\partial z_l} = \sum_{k,l=1}^n (D(\mathcal{VT})(P_0)A_{P_0}{}^tD(\mathcal{VT})(P_0))_{kl}\frac{\partial^2}{\partial z_k \partial z_l}\\ &= \sqrt{a(P_0)}^2 \sum_{m,m'=1}^n \left(\sqrt{D_{P_0}^{-1}} B_{P_0} A_{P_0} {}^t B_{P_0} \sqrt{D_{P_0}^{-1}}\right)_{mm'} \frac{\partial^2}{\partial y_m \partial y_{m'}} = a(P_0) \sum_{m,m'=1}^n \delta_{mm'} \frac{\partial^2}{\partial y_m \partial y_{m'}}\\ &= a(P_0)\Delta_y.\end{aligned}$$

Therefore, by substituting $V_0(z) = (a(P_0)/b(P_0))^{1/(p-1)}w(y)$ in (3.23), we see that w satisfies

$$\Delta w - w + \{(\gamma_0(P_0) + w)^p - \gamma_0(P_0)^p\} = 0 \quad \text{in } \mathbb{R}^n$$

where it is to be noted that $\gamma_0(P_0) = (b(P_0)/a(P_0))^{1/(p-1)}u_m(P_0)$. Moreover, in a way similar to the latter half of the proof of Case (I), we have $\max V_0(z) > 0$. In addition, since $\|V_{\varepsilon_j}\|_{L^r(B_{3R}(0))}$ is bounded by a constant $C_r^{1/r}$ independent of R, we obtain that $\|V_0\|_{L^r(\mathbb{R}^n)}$ is bounded for any $r \in [1, \infty)$. Thus, we see that $w \to 0$ as $|y| \to \infty$ because $V_0 \to 0$ as $|z| \to \infty$. Hence, $w = w_{\gamma_0(P_0)}$ holds by the uniqueness of solutions of (GS-γ). Using these results and noting that $V_0(z) = \lim_{\varepsilon\downarrow 0} \varphi(Q'_\varepsilon + \varepsilon z', |Q_{\varepsilon,n} + \varepsilon z_n|)$, $Q_\varepsilon \to O$, and $|\det(D(\mathcal{SU})(Q_\varepsilon + \varepsilon z))| = a_{nn}(P_0) + o(\varepsilon)$ as $\varepsilon \downarrow 0$, we then obtain that $V_0(z) = V_0(z', |z_n|)$ and the energy $c_\varepsilon/\varepsilon^n$ of v_ε converges to

$$\begin{aligned}(3.25)\qquad &\frac{a_{nn}(P_0)}{2}\left[\frac{1}{2}\int_{\mathbb{R}^n}\left\{\sum_{k,l=1}^{n}(D(\mathcal{VT})(P_0)A_{P_0}{}^tD(\mathcal{VT})(P_0))_{kl}\frac{\partial V_0}{\partial z_k}\frac{\partial V_0}{\partial z_l} + a(P_0)V_0^2\right\}dz\right.\\ &\left.-\frac{1}{p+1}\int_{\mathbb{R}^n} b(P_0)\{(u_m(P_0)+V_0)^{p+1} - u_m(P_0)^{p+1} - (p+1)u_m(P_0)^pV_0\}\,dz\right].\end{aligned}$$

Recall that we let $y = \sqrt{a(P_0)}\sqrt{D_{P_0}^{-1}}B_{P_0}D(\mathcal{SU})(O)z$. Then, by $D(\mathcal{SU})(O) = D(\mathcal{VT})(P_0)^{-1}$, it holds that

$$dz = \det\left(\sqrt{a(P_0)}^{-1}D(\mathcal{VT})(P_0)^tB_{P_0}\sqrt{D_{P_0}}\right)dy = \frac{a(P_0)^{-n/2}(\det A_{P_0})^{1/2}}{a_{nn}(P_0)}.$$

Therefore, (3.25) becomes

$$\begin{aligned}&\frac{a_{nn}(P_0)}{2}\frac{a(P_0)^{-n/2}(\det A_{P_0})^{1/2}}{a_{nn}(P_0)}\\ &\times\left[\frac{a(P_0)}{2}\left(\frac{a(P_0)}{b(P_0)}\right)^{2/(p-1)}\int_{\mathbb{R}^n}(|\nabla w|^2 + w^2)\,dy\right.\\ &\left.-\frac{b(P_0)}{p+1}\left(\frac{a(P_0)}{b(P_0)}\right)^{(p+1)/(p-1)}\int_{\mathbb{R}^n}\left\{(\gamma_0(P_0)+w)^{p+1} - \gamma_0(P_0)^{p+1} - (p+1)\gamma_0(P_0)^{p-1}w\right\}dy\right]\\ &= \frac{a(P_0)^{1-n/2}(\det A_{P_0})^{1/2}}{2}\left(\frac{a(P_0)}{b(P_0)}\right)^{2/(p-1)}I_\delta(P_0;\mathbb{R}^n) = \frac{1}{2}\Phi(P_0)I_\delta(P_0;\mathbb{R}^n).\end{aligned}$$

Finally, we verify that $v_\varepsilon(P_\varepsilon + \varepsilon q)$ converges to $(a(P_0)/b(P_0))^{1/(p-1)}w_{\gamma_0(P_0)}(\sqrt{a(P_0)}\sqrt{D_{P_0}}^{-1}B_{P_0}q)$ in $C^2(K \cap \mathbb{R}^n_+)$ where K is a compact set in $\mathbb{R}^n$. Let $P_0 = 0$ and rotate the coordinate system so that $\boldsymbol{\nu}(P_0) = -\boldsymbol{e_n} = (0, \dots, 0, -1)$. We also let $q \in B_R(0)\ (= K)$ for any $R > 0$. From the relation of y and z, it follows that

$$\begin{aligned}y &= \sqrt{a(P_0)}\sqrt{D_{P_0}^{-1}}B_{P_0}D(\mathcal{SU})(O)z = \sqrt{a(P_0)}\sqrt{D_{P_0}^{-1}}B_{P_0}D(\mathcal{SU})(O)\frac{\zeta - Q_\varepsilon}{\varepsilon}\\ &= \sqrt{a(P_0)}\sqrt{D_{P_0}^{-1}}B_{P_0}D(\mathcal{SU})(O)\frac{\mathcal{VT}(P_\varepsilon + \varepsilon q) - \mathcal{VT}(P_\varepsilon)}{\varepsilon}\\ &= \sqrt{a(P_0)}\sqrt{D_{P_0}^{-1}}B_{P_0}D(\mathcal{SU})(O)\frac{1}{\varepsilon}\left(D(\mathcal{VT})(P_0)(\varepsilon q) + O(\varepsilon^2)\right) = \sqrt{a(P_0)}\sqrt{D_{P_0}^{-1}}B_{P_0}q + O(\varepsilon).\end{aligned}$$

Hence, for any $q \in \mathbb{R}^n_+ \cap B_R(0)$ and $R > 0$, it holds that

$$\begin{aligned} v_\varepsilon(P_\varepsilon + \varepsilon q) &= \left(\frac{a(P_0)}{b(P_0)}\right)^{\frac{1}{p-1}} w_{\gamma_0(P_0)}\Big(\sqrt{a(P_0)}\sqrt{D^{-1}_{P_0}}B_{P_0}q + O(\varepsilon)\Big) + o(1) \\ &= \left(\frac{a(P_0)}{b(P_0)}\right)^{\frac{1}{p-1}} w_{\gamma_0(P_0)}\Big(\sqrt{a(P_0)}\sqrt{D^{-1}_{P_0}}B_{P_0}q\Big) + o(1) \quad \text{as } \varepsilon \downarrow 0. \end{aligned}$$

Thus, $v_\varepsilon(P_\varepsilon + \varepsilon q) \to (a(P_0)/b(P_0))^{1/(p-1)} w_{\gamma_0(P_0)}(\sqrt{a(P_0)}\sqrt{D^{-1}_{P_0}}B_{P_0}q)$ in $C^2(\mathbb{R}^n_+ \cap B_R(0))$. Since $R > 0$ is arbitrary, we see that

$$v_\varepsilon(P_\varepsilon + \varepsilon q) \to \left(\frac{a(P_0)}{b(P_0)}\right)^{\frac{1}{p-1}} w_{\gamma_0(P_0)}\Big(\sqrt{a(P_0)}\sqrt{D^{-1}_{P_0}}B_{P_0}q\Big) \quad \text{in } C^2(K \cap \mathbb{R}^n_+) \quad \text{as } \varepsilon \downarrow 0.$$

Q.E.D.

By Proposition 3.1, we know the limit of the energy $c_{\varepsilon_j}/\varepsilon_j^n$ as $\varepsilon_j \downarrow 0$:

Proposition 3.3. *Let $P_0 \in \overline{\Omega}$ be a concentration point of a family $\{u_\varepsilon\}_{\varepsilon>0}$ of ground-state solutions. Then, there exists a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ tending to zero as $j \to \infty$ such that*

$$\lim_{j\to\infty} \frac{I_{\varepsilon_j}(v_{\varepsilon_j})}{\varepsilon_j^n} = \begin{cases} \Phi(P_0)\mathcal{I}(\gamma_0(P_0);\mathbb{R}^n) & \text{if } P_0 \in \Omega, \\ \frac{1}{2}\Phi(P_0)\mathcal{I}(\gamma_0(P_0);\mathbb{R}^n) & \text{if } P_0 \in \partial\Omega \end{cases}$$

where $v_{\varepsilon_j} = u_{\varepsilon_j} - u_{m,\varepsilon_j}$ and

$$\begin{aligned} &\mathcal{I}(\gamma_0(P_0);\mathbb{R}^n) \\ &= \frac{1}{2}\int_{\mathbb{R}^n}(|\nabla w|^2 + w^2)\,dy - \frac{1}{p+1}\int_{\mathbb{R}^n}\Big\{(\gamma_0(P_0)+w)^{p+1} - \gamma_0(P_0)^{p+1} - (p+1)\gamma_0(P_0)^p w\Big\}\,dy \end{aligned}$$

with $w = w_{\gamma_0(P_0)}$.

Chapter 4

Point concentration for ground-state solutions

In this section, we prove Theorems 1.3 and 1.5 stated in Section 1.4.

Proof of Theorem1.3. First we verify that any ground-state solution u_ε has exactly one local maximum point on $\overline{\Omega}$, provided that $\varepsilon > 0$ is sufficiently small. Assume that there exists a sequence $\{\varepsilon_j\}$ such that $\varepsilon_j \downarrow 0$ and u_{ε_j} attains local maxima at two distinct points P_{ε_j} and Q_{ε_j}. We may assume that $P_{\varepsilon_j} \to P_0 \in \overline{\Omega}$ and $Q_{\varepsilon_j} \to Q_0 \in \overline{\Omega}$ as $j \to \infty$. Then by Proposition 3.1, $v_{\varepsilon_j} = u_{\varepsilon_j} - u_{m,\varepsilon_j}$ is approximated by

$$v_{\varepsilon_j}(P_{\varepsilon_j} + \varepsilon_j z) = \left\{\frac{a(P_0)}{b(P_0)}\right\}^{1/(p-1)} w_{\gamma_0(P_0)}\left(\sqrt{a(P_0)}\sqrt{D_{P_0}^{-1}} B_{P_0} z\right) + o(1)$$

and

$$v_{\varepsilon_j}(Q_{\varepsilon_j} + \varepsilon_j z) = \left\{\frac{a(Q_0)}{b(Q_0)}\right\}^{1/(p-1)} w_{\gamma_0(Q_0)}\left(\sqrt{a(Q_0)}\sqrt{D_{Q_0}^{-1}} B_{Q_0} z\right) + o(1)$$

in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$ at the same time. Therefore,

$$\frac{c_{\varepsilon_j}}{\varepsilon_j^n} > \Phi(P_0) I_\delta(P_0; R_{P_0}) + \Phi(Q_0) I_\delta(Q_0; R_{Q_0}) + o(1),$$

so that

$$\liminf_{j\to\infty} \frac{c_{\varepsilon_j}}{\varepsilon_j^n} \geq \Phi(P_0) I_\delta(P_0; R_{P_0}) + \Phi(Q_0) I_\delta(Q_0; R_{Q_0}).$$

Then by Proposition 2.12 we obtain

$$\begin{aligned}
&\Phi(P_0) I_\delta(P_0; R_{P_0}) + \Phi(Q_0) I_\delta(Q_0; R_{Q_0}) + o(1) \\
&\leq \min\left\{\min_{Q\in\overline{\Omega}} \Phi(Q) I_\delta(Q; \mathbb{R}^n), \frac{1}{2} \min_{Q\in\partial\Omega} \Phi(Q) I_\delta(Q; \mathbb{R}^n)\right\} \\
&\leq \min\left\{\Phi(P_0) I_\delta(P_0; R_{P_0}), \Phi(Q_0) I_\delta(Q_0; R_{Q_0})\right\}.
\end{aligned}$$

This is clearly a contradiction. Hence, u_ε has at most one local maximum point. Since it is continuous on $\overline{\Omega}$, it must have a maximum point in $\overline{\Omega}$. Therefore, u_ε has exactly one local maximum point, hence the maximum point.

We now proceed to the proof of assertions (a) and (b). From Propositions 2.12 and 3.3, we know that

$$\left.\begin{array}{ll} \Lambda(P_0) & \text{if } P_0 \in \Omega \\ \frac{1}{2}\Lambda(P_0) & \text{if } P_0 \in \partial\Omega \end{array}\right\} \leq \min\left\{\min_{Q\in\overline{\Omega}} \Lambda(Q), \frac{1}{2} \min_{Q\in\partial\Omega} \Lambda(Q)\right\}.$$

Therefore, if $P_0 \in \Omega$, then P_0 must be a minimum point of Λ over $\overline{\Omega}$, while if $P_0 \in \partial\Omega$, then it is necessarily a minimum point of Λ over $\partial\Omega$. Hence, it suffices to know whether $P_0 \in \Omega$ or $P_0 \in \partial\Omega$.

(i) Suppose that $\min_{\partial\Omega} \Lambda(Q) < 2 \min_{\overline{\Omega}} \Lambda(Q)$. If $P_0 \in \Omega$, then

$$\Lambda(P_0) \leq \min\left\{\min_{Q\in\overline{\Omega}} \Lambda(Q), \frac{1}{2} \min_{Q\in\partial\Omega} \Lambda(Q)\right\} = \frac{1}{2} \min_{Q\in\partial\Omega} \Lambda(Q) < \min_{Q\in\overline{\Omega}} \Lambda(Q),$$

which is a contradiction. Therefore, $P_0 \in \partial\Omega$, and it is a minimum point of Λ over $\partial\Omega$.

(ii) Suppose that $2 \min_{\overline{\Omega}} \Lambda(Q) < \min_{\partial\Omega} \Lambda(Q)$. If $P_0 \in \partial\Omega$, then

$$\frac{1}{2}\Lambda(P_0) \leq \min\left\{\min_{Q\in\overline{\Omega}} \Lambda(Q), \frac{1}{2} \min_{Q\in\partial\Omega} \Lambda(Q)\right\} = \min_{Q\in\overline{\Omega}} \Lambda(Q) < \frac{1}{2} \min_{Q\in\partial\Omega} \Lambda(Q),$$

and this is a contradiction. Therefore, $P_0 \in \Omega$, and it is a minimum point of Λ over $\overline{\Omega}$. Q.E.D.

To prove Theorem 1.5, we investigate the dependence of $\mathcal{I}(\gamma_0(Q); \mathbb{R}^n)$ on $\delta \geq 0$ sufficiently small.

Proposition 4.1. *The function $\mathcal{I}(\gamma_0(Q); \mathbb{R}^n)$ belongs to $C^2(\overline{\Omega})$ and*

$$\max_{1\leq j\leq n}\left\|\frac{\partial}{\partial Q_j}\mathcal{I}(\gamma_0(Q))\right\|_{L^\infty(\Omega)} = O(\delta), \quad \max_{1\leq i,j\leq n}\left\|\frac{\partial^2}{\partial Q_i \partial Q_j}\mathcal{I}(\gamma_0(Q))\right\|_{L^\infty(\Omega)} = O(\delta) \quad \textit{as } \delta \downarrow 0,$$

where $\mathcal{I}(\gamma_0(Q)) = \mathcal{I}(\gamma_0(Q); \mathbb{R}^n)$. Moreover,

$$\mathcal{I}(\gamma_0(Q)) = I_0 - \gamma_0(Q)\left\{\int_{\mathbb{R}^n} w_0\, dy + o(1)\right\} \quad \textit{with } I_0 = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^n} w_0^{p+1}\, dy, \tag{4.1}$$

where w_0 is the unique positive solution of $\Delta w - w + w^p = 0$ in $\mathbb{R}^n$ satisfying $\lim_{|y|\to\infty} w(y) = 0$ and $w(0) = \max_{y\in\mathbb{R}^n} w(y)$.

Proof. First, we prove $\|(\partial/\partial Q_j)\mathcal{I}(\gamma_0(Q))\|_{L^\infty(\Omega)} = O(\delta)$. We show that w_γ is a C^1-function with respect to γ. Let us compute the derivative of $I(\gamma)$:

$$\begin{aligned} \frac{d\mathcal{I}}{d\gamma}(\gamma) &= \frac{d}{d\gamma}\left[\frac{1}{2}\|w_\gamma\|^2_{W^{1,2}(\mathbb{R}^n)} - \frac{1}{p+1}\int_{\mathbb{R}^n}\left\{(\gamma + w_\gamma)^{p+1} - \gamma^{p+1} - (p+1)\gamma^p w_\gamma\right\} dy\right] \\ &= \left\langle w_\gamma, \frac{\partial w_\gamma}{\partial\gamma}\right\rangle_{W^{1,2}(\mathbb{R}^n)} - \int_{\mathbb{R}^n}\left\{(\gamma + w_\gamma)^p\left(1 + \frac{\partial w_\gamma}{\partial\gamma}\right) - \gamma^p - p\gamma^{p-1}w_\gamma - \gamma^p\frac{\partial w_\gamma}{\partial\gamma}\right\} dy. \end{aligned}$$

However, since w_γ satisfies $\Delta w - w + (\gamma + w)^p - \gamma^p = 0$ in $\mathbb{R}^n$ and $\lim_{|y|\to\infty} w(y) = 0$, we have

$$\left\langle w_\gamma, \frac{\partial w_\gamma}{\partial\gamma}\right\rangle_{W^{1,2}(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\left\{(\gamma + w_\gamma)^p - \gamma^p\right\}\frac{\partial w_\gamma}{\partial\gamma}\, dy, \quad \int_{\mathbb{R}^n}\left\{(\gamma + w_\gamma)^p - \gamma^p\right\} dy = \int_{\mathbb{R}^n} w_\gamma\, dy.$$

From these two identities we obtain

$$\begin{aligned}\frac{d\mathcal{I}}{d\gamma}(\gamma) &= \int_{\mathbb{R}^n}\left[\left\{(\gamma+w_\gamma)^p-\gamma^p\right\}\frac{\partial w_\gamma}{\partial\gamma}-(\gamma+w_\gamma)^p\Big(1+\frac{\partial w_\gamma}{\partial\gamma}\Big)+\gamma^p\frac{\partial w_\gamma}{\partial\gamma}+\gamma^p+p\gamma^{p-1}w_\gamma\right]dy\\ &= -\int_{\mathbb{R}^n}\left\{(\gamma+w_\gamma)^p-\gamma^p-p\gamma^{p-1}w_\gamma\right\}dy = -\int_{\mathbb{R}^n}\left\{w_\gamma-p\gamma^{p-1}w_\gamma\right\}dy\\ &= -(1-p\gamma^{p-1})\int_{\mathbb{R}^n}w_\gamma\,dy.\end{aligned}$$

Hence,

$$\frac{\partial}{\partial Q_j}\mathcal{I}(\gamma_0(Q)) = -\Big(1-p\gamma_0(Q)^{p-1}\Big)\frac{\partial\gamma_0}{\partial Q_j}(Q)\int_{\mathbb{R}^n}w_{\gamma_0(Q)}\,dy. \tag{4.2}$$

From $(\partial\gamma_0/\partial Q_j)=O(\delta)$, we obtain $\|(\partial/\partial Q_j)\mathcal{I}(\gamma_0(Q))\|_{L^\infty(\Omega)}=O(\delta)$.

Next, we prove $\|(\partial^2/\partial Q_i\partial Q_j)\mathcal{I}(\gamma_0(Q))\|_{L^\infty(\Omega)}=O(\delta)$. By (4.2) and partial differentiability of $w_{\gamma_0(Q)}$ with respect to Q, it suffices to prove only the partial differentiability of $(1-p\gamma_0(Q)^{p-1})(\partial\gamma_0/\partial Q_j)$ with respect to Q. In particular, we note that $(1-p\gamma_0(Q)^{p-1})(\partial\gamma_0/\partial Q_j)$ is differentiable even when $p<2$. Indeed, from the algebraic equation for $u_m(Q)$ and the definition of $\gamma_0(Q)$, we see that $\gamma_0(Q)$ satisfies the following equation in $\overline{\Omega}$:

$$-\gamma_0(Q)+\gamma_0(Q)^p+\delta\sigma(Q)a(Q)^{-p/(p-1)}b(Q)^{1/(p-1)}=0.$$

Differentiating this equation by Q_j, we have

$$-\Big(1-p\gamma_0(Q)^{p-1}\Big)\frac{\partial\gamma_0}{\partial Q_j}+\delta\frac{\partial}{\partial Q_j}\Big(\sigma(Q)a(Q)^{-p/(p-1)}b(Q)^{1/(p-1)}\Big)=0. \tag{4.3}$$

Since $a,b,\sigma\in C^2(\overline{\Omega})$ and $a(Q)$ is strictly positive, we see that $\sigma(Q)a(Q)^{-p/(p-1)}b(Q)^{1/(p-1)}$ is a C^2-function on $\overline{\Omega}$. Hence, by (4.3), we obtain that $(1-p\gamma_0(Q)^{p-1})(\partial\gamma_0/\partial Q_j)$ is differentiable on $\overline{\Omega}$, and it holds that

$$-\frac{\partial}{\partial Q_i}\left\{\Big(1-p\gamma_0(Q)^{p-1}\Big)\frac{\partial\gamma_0}{\partial Q_j}(Q)\right\}=-\delta\frac{\partial^2}{\partial Q_i\partial Q_j}\Big(\sigma(Q)a(Q)^{-p/(p-1)}b(Q)^{1/(p-1)}\Big). \tag{4.4}$$

Therefore, we have $(\partial/\partial Q_i)\{(1-p\gamma_0(Q)^{p-1})(\partial\gamma_0/\partial Q_j)\}=O(\delta)$. Noting $(\partial\gamma_0/\partial Q_j)=O(\delta)$, we see that $(1-p\gamma_0(Q)^{p-1})(\partial\gamma_0/\partial Q_j)=O(\delta)$, so that $\|(\partial^2/\partial Q_i\partial Q_j)\mathcal{I}(\gamma_0(Q))\|_{L^\infty(\Omega)}=O(\delta)$. Q.E.D.

Proof of Theorem 1.5. First, we investigate whether $P_0\in\Omega$ or $P_0\in\partial\Omega$. By (4.1) and $\gamma_0(Q)=O(\delta)$ uniformly in Q, we see that $\Lambda(Q)=\Phi(Q)I_0-O(\delta)$ holds uniformly in $Q\in\overline{\Omega}$.

(I): Suppose $\min_{Q\in\partial\Omega}\Phi(Q)<2\min_{Q\in\overline{\Omega}}\Phi(Q)$. Since I_0 is a constant and $\delta\geq 0$ is small,

$$\min_{Q\in\partial\Omega}\Lambda(Q)\leq\min_{Q\in\partial\Omega}\Phi(Q)I_0<2\min_{Q\in\overline{\Omega}}\{\Phi(Q)I_0\}-O(\delta)\leq 2\min_{Q\in\overline{\Omega}}\Lambda(Q).$$

Hence, $P_0\in\partial\Omega$ is a minimum point of $\Lambda(Q)$ over $\partial\Omega$ by Theorem 1.3.

(II): Suppose $\min_{Q\in\partial\Omega}\Phi(Q)>2\min_{Q\in\overline{\Omega}}\Phi(Q)$. Similarly to (I), we have

$$\min_{Q\in\partial\Omega}\Lambda(Q)\geq\min_{Q\in\partial\Omega}\{\Phi(Q)I_0\}-O(\delta)>2\min_{Q\in\overline{\Omega}}\{\Phi(Q)I_0\}\geq 2\min_{Q\in\overline{\Omega}}\Lambda(Q),$$

and $P_0 \in \Omega$ is a minimum point of $\Lambda(Q)$ over $\overline{\Omega}$ by Theorem 1.3.

Next, we prove that P_0 must be close to a nondegenerate minimum point Q_0 of $\Phi(Q)$.

(I): Let P_0 be a minimal point of the locator function $\Lambda(Q)$ restricted to $\partial\Omega$. As in the proof of Proposition 3.1, we introduce the coordinate system with the origin at P_0, and $\boldsymbol{\nu}(P_0) = -\boldsymbol{e_n} := (0, \dots, 0, -1)$. We express the boundary $\partial\Omega$ near P_0 as a graph $x_n = \psi(x')$. For a smooth function $f(x)$ defined on $\overline{\Omega}$, we denote its restriction on the boundary portion near P_0 by $f_b(x') := f(x', \psi(x'))$. Moreover $\nabla' := ((\partial/\partial x_1), \dots, (\partial/\partial x_{n-1}))$, $\mathrm{Hess}' := (\partial^2/\partial x_i \partial x_j)_{1\le i,j\le n-1}$. Hence,

$$\nabla' f_b(x') = \left(\frac{\partial f}{\partial x_j}(x', \psi(x')) + \frac{\partial f}{\partial x_n}(x', \psi(x'))\frac{\partial \psi}{\partial x_j}(x')\right)_{1\le j\le n-1}.$$

Recalling that P_0 is a minimum point of Λ_b, we obtain $\nabla'\Lambda_b(P_0) = 0$, that is,

$$0 = \nabla'\Lambda_b(P_0) = \mathcal{I}(\gamma_0(P_0))\nabla'\Phi_b(P_0) + \Phi(P_0)\nabla'(\mathcal{I}\circ\gamma_b)(P_0),$$

where $\gamma_b(Q) = \gamma_0|_{\partial\Omega}(Q)$. From this follows that

$$\nabla'\Phi_b(P_0) = -\frac{\Phi(P_0)}{\mathcal{I}(\gamma_0(P_0))}\nabla'(\mathcal{I}\circ\gamma_b)(P_0) \tag{4.5}$$

by Proposition 4.1, we have

$$|\nabla'\Phi_b(P_0)| \le \frac{\Phi(P_0)}{I_0 - O(\delta)}O(\delta) = O(\delta).$$

Suppose that Q_0 is a critical point of Φ_b. Since $\nabla'\Phi_b(Q_0) = 0$, by the mean value theorem, we see that

$$\nabla'\Phi_b(P_0) = \mathrm{Hess}'\Phi_b(Q_0)(P_0' - Q_0') + o(1)(P_0' - Q_0').$$

Substituting this in (4.5) and multiplying by the inverse matrix $(\mathrm{Hess}'\Phi_b(Q_0))^{-1}$, we see that

$$|P_0' - Q_0'| = \frac{\Phi(P_0)}{I_0}\left|(\mathrm{Hess}'\Phi_b(P_0))^{-1}\nabla'(\mathcal{I}\circ\gamma_b)(P_0)\right| + o(|P_0' - Q_0'|) + O(\delta^2).$$

Since $(\mathrm{Hess}'(\Phi(Q)|_{\partial\Omega})(P_0))^{-1}$ is bounded, we obtain $|P_0' - Q_0'| = O(\delta) + o(|P_0' - Q_0'|)$. Hence, $|P_0' - Q_0'| = O(\delta)$ is shown. By the Lipschitz continuity of ψ, we see that $|P_0 - Q_0| = O(\delta)$. Although we assumed that Q_0 is a critical point of Φ, Q_0 is actually a minimum point of Φ over $\partial\Omega$ because of $\Lambda(P_0) = \min_{Q\in\partial\Omega}\Phi(Q)I_0 - O(\delta) = \Phi(Q_0)I_0 - O(\delta)$.

(II): Suppose that all minimum points Φ in Ω are nondegenerate. Then the Hesse matrix $\mathrm{Hess}(\Phi(Q))$ is invertible. Since P_0 is the minimum point of Λ,

$$\nabla\Phi(P_0) = -\frac{\Phi(P_0)}{\mathcal{I}(\gamma_0(P_0))}\nabla\Big(\mathcal{I}(\gamma_0(Q))\Big)(P_0). \tag{4.6}$$

Now, suppose that Q_0 is a critical point of Φ. We expand $\nabla\Phi$ at Q_0 to see that, by $\nabla\Phi(Q_0) = 0$,

$$\nabla\Phi(P_0) = \mathrm{Hess}\,\Phi(Q_0)\,(P_0 - Q_0) + o(1)(P_0 - Q_0).$$

Substituting this in (4.6) and multiplying by the inverse matrix $(\mathrm{Hess}\,\Phi(Q_0))^{-1}$, we obtain

$$|P_0 - Q_0| = \frac{\Phi(P_0)}{I_0}\left|\Big(\mathrm{Hess}\,\Phi(P_0)\Big)^{-1}\nabla\Big(\mathcal{I}(\gamma_0(Q))\Big)(P_0)\right| + o(|P_0 - Q_0|) + O(\delta^2).$$

Noting that $(\mathrm{Hess}\,\Phi(P_0))^{-1}$ is bounded thanks to $\Phi \in C^2(\overline{\Omega})$, we have $|P_0 - Q_0| = O(\delta) + o(|P_0 - Q_0|)$. Hence, $|P_0 - Q_0| = O(\delta)$ is shown. Although we have assumed Q_0 to be a critical point of Φ, Q_0 is actually a minimum point of Φ over $\overline{\Omega}$ since $\Lambda(P_0) = \min_{Q\in\overline{\Omega}} \Phi(Q)I_0 - O(\delta) = \Phi(Q_0)I_0 - O(\delta)$.

Q.E.D.

Chapter 5

Point concentration and the primary locator function

In this chapter we study the relation between the primary locator function $\Phi(Q)$ and the location of a concentration point of a solution exhibiting a point-condensation phenomenon. Here, we consider any concentrating solution u_ε with $J(u_\varepsilon) = O(\varepsilon^n)$, hence, u_ε may not necessarily be a ground-state solution.

Let u_ε be a solution of the Neumann problem (1.13). We note that this equation is the case of $\mathcal{A} = \Delta$ and $\delta = 0$ in $\mathcal{A}(x)u - a(x)u + b(x)u^p + \delta\sigma(x) = 0$.

Theorem 1.6 asserts that the condition "$\nabla\Phi(P_0) = 0$" is a necessary condition for a point P_0 to be a concentration point of a family of solutions $\{u_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ of (1.1). This gives us an important clue when we try to construct solutions concentrating at some point.

The objective of this chapter is to prove Theorem 1.6. The proof relies on Lemma 5.1 below, which claims that the second term in the asymptotic expansion of u_ε in the neighborhood of the concentration point is of the order of ε when $\varepsilon \to 0$. Since we consider only the case of $P_0 \in \Omega$, there exists a positive constant R such that $\overline{B}_{3R}(P_0) \subset \Omega$. For each $R_0 \in (0, R)$, taking any point $Q_\varepsilon \in \overline{B}_{R_0}(P_0)$, we have

$$B_{2R_0}(Q_\varepsilon) \subset B_{3R}(P_0) \Subset \Omega. \tag{5.1}$$

To prove Theorem 1.6, we would like to approximate u_ε around Q_ε. Hence, we take a cut-off function $\chi \in C_0^\infty(\mathbb{R}^n)$ satisfying $0 \le \chi \le 1$ and

$$\chi(\zeta) = \chi(|\zeta|) := \begin{cases} 1 & (|\zeta| \le R_0), \\ 0 & (|\zeta| \ge 2R_0) \end{cases} \tag{5.2}$$

and put

$$\chi(x - Q_\varepsilon)u_\varepsilon(x) = v_{Q_\varepsilon}\left(\frac{x - Q_\varepsilon}{\varepsilon}\right) + \varepsilon\phi\left(\frac{x - Q_\varepsilon}{\varepsilon}\right).$$

Here $v_Q(z) = (a(Q)/b(Q))^{1/(p-1)}w(y)$, $y = a(Q)^{1/2}z$ and w is a unique positive solution of the boundary value problem (GS-0) stated immediately after (1.12). Let $\tilde{u}_\varepsilon$ be a function in $\mathbb{R}^n$ which we extend u_ε by putting $u_\varepsilon = 0$ outside Ω. In what follows, we denote $\tilde{u}_\varepsilon$ by the same symbol u_ε. Note that $\chi(x - Q_\varepsilon)u_\varepsilon(x) = 0$ on $|x - Q_\varepsilon| \ge 2R_0$ by (5.2). Hence, $\chi(\cdot - Q_\varepsilon)u_\varepsilon$ is a C^2-function in $\mathbb{R}^n$ from (5.1) and $\chi(\cdot - Q_\varepsilon)u_\varepsilon = 0$ on $\mathbb{R}^n \setminus B_{2R_0}(Q_\varepsilon)$.

To have an approximation of $\chi(\cdot - Q_\varepsilon)u_\varepsilon$, we need to prove the boundedness of ϕ:

Lemma 5.1. *Assume that for any $R_0 > 0$, $\chi(x - Q_\varepsilon)u_\varepsilon(x)$ decays exponentially as $|(x - Q_\varepsilon)/\varepsilon| \to \infty$. If $\varepsilon\phi((x - Q_\varepsilon)/\varepsilon)$ converges to zero in $W^{2,r}(\Omega)$ and $C^2_{\mathrm{loc}}(\Omega)$ as $\varepsilon \downarrow 0$, then $\phi((x - Q_\varepsilon)/\varepsilon)$ is bounded in $W^{1,2}(\Omega)$ as $\varepsilon \downarrow 0$.*

To simplify the notation we write Q instead of Q_ε.

Proof. Putting $z := (x - Q)/\varepsilon$, we have

$$\chi(\varepsilon z)u_\varepsilon(Q + \varepsilon z) = v_Q(z) + \varepsilon\phi(z).$$

By the definitions of v_Q and w, v_Q is a unique positive solution of the boundary value problem

$$\begin{cases} \Delta v_Q - a(Q)v_Q + b(Q)v_Q^p = 0 & \text{in } \mathbb{R}^n, \\ \lim\limits_{|z|\to\infty} v_Q(z) = 0, \ v_Q(0) = \max\limits_{z\in\mathbb{R}^n} v_Q(z). \end{cases}$$

Since $w(y) = w(|y|)$, w decays exponentially as $|y| \to \infty$, and $y = a(Q)^{1/2}z$, we see that $v_Q(z) = v_Q(|z|)$ and v_Q decays exponentially as $|z| \to \infty$. Moreover, since $\chi(\varepsilon z)u_\varepsilon(Q + \varepsilon z)$ decays exponentially as $|z| \to \infty$ by the assumption, there exist positive constants r_0, C_1 and C_2 such that

$$|\chi(\varepsilon z)u_\varepsilon(Q + \varepsilon z)| + |v_Q(z)| \le C_1 e^{-2C_2|z|} \quad \text{for all } |z| > r_0.$$

If we take a sufficiently small $\varepsilon > 0$ again satisfying $R_0/\varepsilon > r_0$, then we obtain

$$\varepsilon|\phi(z)| = |\chi(\varepsilon z)u_\varepsilon(Q + \varepsilon z) - v_Q(z)| \le C_1 e^{-2C_2R_0/\varepsilon} \quad \text{for all } |z| \ge R_0/\varepsilon.$$

Since $e^{-C_2R_0/\varepsilon} \le \varepsilon$ is satisfied by taking $\varepsilon > 0$ even smaller, it holds that

$$|\phi(z)| \le C_1 e^{-C_2R_0/\varepsilon} \quad \text{for all } |z| \ge R_0/\varepsilon.$$

Now, we derive an equation that ϕ satisfies. First, we calculate $\Delta_z(\chi(\varepsilon z)u_\varepsilon(Q + \varepsilon z))$:

$$\begin{aligned} &\Delta_z(\chi(\varepsilon z)u_\varepsilon(Q + \varepsilon z)) \\ &= \varepsilon^2\Delta u_\varepsilon(Q + \varepsilon z)\chi(\varepsilon z) + 2\varepsilon^2\nabla\chi(\varepsilon z)\cdot\nabla u_\varepsilon(Q + \varepsilon z) + \varepsilon^2 u_\varepsilon(Q + \varepsilon z)\Delta\chi(\varepsilon z) \\ &= \chi(\varepsilon z)\Big\{a(Q + \varepsilon z)u_\varepsilon(Q + \varepsilon z) - b(Q + \varepsilon z)u_\varepsilon(Q + \varepsilon z)^p\Big\} \\ &\quad + \varepsilon^2\Big\{2\nabla u_\varepsilon(Q + \varepsilon z)\cdot\nabla\chi(\varepsilon z) + u_\varepsilon(Q + \varepsilon z)\Delta\chi(\varepsilon z)\Big\} \\ &= a(Q + \varepsilon z)(v_Q + \varepsilon\phi) - b(Q + \varepsilon z)(v_Q + \varepsilon\phi)^p \\ &\quad - b(Q + \varepsilon z)\chi(\varepsilon z)(1 - \chi(\varepsilon z)^{p-1})u_\varepsilon(Q + \varepsilon z)^p \\ &\quad + \varepsilon^2\Big\{2\nabla u_\varepsilon(Q + \varepsilon z)\cdot\nabla\chi(\varepsilon z) + u_\varepsilon(Q + \varepsilon z)\Delta\chi(\varepsilon z)\Big\}. \end{aligned}$$

Putting

$$\begin{aligned} \mathcal{E}(\chi(\varepsilon z)) := &- b(Q + \varepsilon z)\chi(\varepsilon z)(1 - \chi(\varepsilon z)^{p-1})u_\varepsilon(Q + \varepsilon z)^p \\ &+ \varepsilon^2\Big\{2\nabla u_\varepsilon(Q + \varepsilon z)\cdot\nabla\chi(\varepsilon z) + u_\varepsilon(Q + \varepsilon z)\Delta\chi(\varepsilon z)\Big\}, \end{aligned} \tag{5.3}$$

and noting $\Delta_z(\chi(\varepsilon z)u_\varepsilon(Q + \varepsilon z)) = \Delta v_Q + \varepsilon\Delta\phi$, we get that

$$\Delta v_Q + \varepsilon\Delta\phi - a(Q + \varepsilon z)(v_Q + \varepsilon\phi) + b(Q + \varepsilon z)(v_Q + \varepsilon\phi)^p + \mathcal{E}(\chi(\varepsilon z)) = 0.$$

Note that $\mathcal{E}(\chi(\varepsilon z)) = O(e^{-C_2R_0/\varepsilon})$ since $\mathcal{E}(\chi(\varepsilon z)) = 0$ on $|z| \le R_0/\varepsilon$ from (5.2) and (5.3). Moreover, from $\Delta v_Q - a(Q)v_Q + b(Q)v_Q^p = 0$ in $\mathbb{R}^n$, the last equation becomes

$$a(Q)v_Q - b(Q)v_Q^p + \varepsilon\Delta\phi - a(Q+\varepsilon z)(v_Q+\varepsilon\phi) + b(Q+\varepsilon z)(v_Q+\varepsilon\phi)^p + O(e^{-C_2R_0/\varepsilon}) = 0. \tag{5.4}$$

In deriving the equation above, we have computed as follows:

$$\begin{aligned}
&\varepsilon\Delta\phi - a(Q+\varepsilon z)\varepsilon\phi \\
&\quad - (a(Q+\varepsilon z) - a(Q))v_Q - b(Q)v_Q^p + b(Q+\varepsilon z)(v_Q+\varepsilon\phi)^p + O(e^{-C_2R_0/\varepsilon}) = 0,
\end{aligned}$$

$$\begin{aligned}
&\varepsilon\Delta\phi - a(Q)\varepsilon\phi + pb(Q)v_Q^{p-1}\varepsilon\phi - (a(Q+\varepsilon z) - a(Q))\varepsilon\phi - pb(Q)v_Q^{p-1}\varepsilon\phi \\
&\quad - (a(Q+\varepsilon z) - a(Q))v_Q - b(Q)v_Q^p + b(Q+\varepsilon z)\{v_Q^p + pv_Q^{p-1}\varepsilon\phi\} \\
&\quad + b(Q+\varepsilon z)\{(v_Q+\varepsilon\phi)^p - v_Q^p - pv_Q^{p-1}\varepsilon\phi\} + O(e^{-C_2R_0/\varepsilon}) = 0,
\end{aligned}$$

$$\begin{aligned}
&\varepsilon\Delta\phi - a(Q)\varepsilon\phi + pb(Q)v_Q^{p-1}\varepsilon\phi \\
&\quad + \left\{-(a(Q+\varepsilon z) - a(Q)) + (b(Q+\varepsilon z) - b(Q))pv_Q^{p-1}\right\}\varepsilon\phi \\
&\quad + \left\{-(a(Q+\varepsilon z) - a(Q)) + (b(Q+\varepsilon z) - b(Q))v_Q^{p-1}\right\}v_Q \\
&\quad + b(Q+\varepsilon z)\left\{(v_Q+\varepsilon\phi)^p - v_Q^p - pv_Q^{p-1}\varepsilon\phi\right\} + O(e^{-C_2R_0/\varepsilon}) = 0.
\end{aligned}$$

Dividing (5.4) by ε, we see that ϕ satisfies

$$\Delta\phi - a(Q)\phi + pb(Q)v_Q^{p-1}\phi + f\varepsilon\phi + gv_Q + h(\phi) = 0,$$

where

$$\begin{aligned}
f(z) &:= -\frac{a(Q+\varepsilon z) - a(Q)}{\varepsilon} + \frac{b(Q+\varepsilon z) - b(Q)}{\varepsilon}pv_Q^{p-1}, \\
g(z) &:= -\frac{a(Q+\varepsilon z) - a(Q)}{\varepsilon} + \frac{b(Q+\varepsilon z) - b(Q)}{\varepsilon}v_Q^{p-1} + O(e^{-C_2R_0/\varepsilon}), \\
h(\phi(z)) &:= b(Q+\varepsilon z)\frac{1}{\varepsilon}\left\{(v_Q+\varepsilon\phi)^p - v_Q^p - pv_Q^{p-1}\varepsilon\phi\right\}.
\end{aligned}$$

Next, we prove the boundedness of $\|\phi\|_{W^{1,2}(\mathbb{R}^n)}$ by using that of f and g as $\varepsilon \downarrow 0$. If this assertion is proved, then we see that the principal term of $u_\varepsilon(Q+\varepsilon z)$ is v_Q and the second term of $u_\varepsilon(Q+\varepsilon z)$ is of the order of ε. Let

$$L_Q := \Delta - a(Q) + pb(Q)v_Q^{p-1}.$$

Then the equation for ϕ is written as

$$L_Q\phi + f\varepsilon\phi + gv_Q + h(\phi) = 0. \tag{5.5}$$

From $\varepsilon\phi = \chi(\varepsilon z)u_\varepsilon(Q+\varepsilon z) - v_Q(z)$, for $\varepsilon > 0$ fixed, ϕ is a C^2-function in $\mathbb{R}^n$. Then, we decompose ϕ as follows:

$$\phi = \beta_0 v_Q + \sum_{l=1}^{n} \beta_l \frac{\partial v_Q}{\partial z_l} + \psi. \tag{5.6}$$

Here, β_0, β_l, and ψ in (5.6) satisfy

$$\beta_0 = \langle \phi, v_Q \rangle_{W^{1,2}} \|v_Q\|_{W^{1,2}}^{-2}, \qquad \beta_l = \left\langle \phi, \frac{\partial v_Q}{\partial z_l} \right\rangle_{W^{1,2}} \left\| \frac{\partial v_Q}{\partial z_l} \right\|_{W^{1,2}}^{-2},$$
$$\psi \in E := \left\{ \psi \in W^{1,2}(\mathbb{R}^n) \;\middle|\; \langle \psi, v_Q \rangle_{W^{1,2}} = \left\langle \psi, \frac{\partial v_Q}{\partial z_k} \right\rangle_{W^{1,2}} = 0, \ k = 1, \ldots, n \right\} \tag{5.7}$$

where $\langle u, v \rangle_{W^{1,2}} := \int_{\mathbb{R}^n} \{\nabla u \cdot \nabla v + a(Q)uv\}\, dz$ and $\|u\|_{W^{1,2}} = (\langle u, u \rangle_{W^{1,2}})^{1/2}$. From (5.6) and the boundedness of v_Q and $(\partial v_Q / \partial z_l)$, as $\varepsilon \downarrow 0$, if β_0, β_l and $\|\psi\|_{W^{1,2}}$ are bounded, then ϕ is bounded. Therefore, the proof reduces to showing the boundedness of β_0, β_l, and $\|\psi\|_{W^{1,2}}$.

By taking the $L^2(\mathbb{R}^n)$-inner product between (5.5) and v_Q, we have

$$(L_Q \phi, v_Q)_{L^2} + (f \varepsilon \phi, v_Q)_{L^2} + (g v_Q, v_Q)_{L^2} + (h(\phi), v_Q)_{L^2} = 0.$$

Let us calculate $(L_Q \phi, v_Q)_{L^2}$. Since L_Q is a self-adjoint operator on $L^2(\mathbb{R}^n)$, recalling $L_Q v_Q = (p-1)b(Q)v_Q^p$, (5.6), and $v_Q(z) = v_Q(|z|)$, we get

$$\begin{aligned}
(L_Q \phi, v_Q)_{L^2} &= (\phi, L_Q v_Q)_{L^2} = (\phi, (p-1)b(Q)v_Q^p)_{L^2} = (p-1)b(Q)\,(\phi, v_Q^p)_{L^2} \\
&= (p-1)b(Q) \left\{ \beta_0 \int_{\mathbb{R}^n} v_Q^{p+1}\, dz + \sum_{l=1}^n \beta_l \int_{\mathbb{R}^n} v_Q^p \frac{\partial v_Q}{\partial z_l}\, dz + \int_{\mathbb{R}^n} v_Q^p \psi\, dz \right\} \\
&= \beta_0 (p-1) b(Q) \int_{\mathbb{R}^n} v_Q^{p+1}\, dz + (p-1) \int_{\mathbb{R}^n} b(Q) v_Q^p \psi\, dz.
\end{aligned}$$

Now, by $\psi \in E$, we have $\int_{\mathbb{R}^n} b(Q)v_Q^p \psi\, dz = \int_{\mathbb{R}^n} (\Delta v_Q - a(Q)v_Q)\psi\, dz = \langle v_Q, \psi \rangle_{W^{1,2}} = 0$, so that

$$(L_Q \phi, v_Q)_{L^2} = \beta_0 (p-1) b(Q) \int_{\mathbb{R}^n} v_Q^{p+1}\, dz.$$

Therefore, we obtain

$$\beta_0 (p-1) b(Q) \int_{\mathbb{R}^n} v_Q^{p+1}\, dz + (h(\phi), v_Q)_{L^2} = -(f \varepsilon \phi, v_Q)_{L^2} - (g v_Q, v_Q)_{L^2}. \tag{5.8}$$

Note that $(h(\phi), v_Q)_{L^2}$ is estimated as

$$\begin{aligned}
|(h(\phi), v_Q)_{L^2}| &= \left| \int_{\mathbb{R}^n} pb(Q + \varepsilon z) \left\{ (v_Q + \theta \varepsilon \phi)^{p-1} - v_Q^{p-1} \right\} \phi v_Q\, dz \right| \quad (\text{ for some } \theta \in (0,1)) \\
&\leq p \|b\|_{L^\infty(\mathbb{R}^n)} \left\| \left\{ (v_Q + \theta \varepsilon \phi)^{p-1} - v_Q^{p-1} \right\} v_Q \right\|_{L^{(p+1)/p}(\mathbb{R}^n)} \|\phi\|_{L^{p+1}(\mathbb{R}^n)}.
\end{aligned}$$

Since we assume that $\varepsilon \phi$ converges to zero as $\varepsilon \downarrow 0$ in $W^{2,r}(\mathbb{R}^n)$, it follows that

$$|(h(\phi), v_Q)_{L^2}| = o(1) \|\phi\|_{L^{p+1}(\mathbb{R}^n)} \quad \text{as } \varepsilon \downarrow 0.$$

Therefore, $|(h(\phi), v_Q)_{L^2}| = o(1)\|\phi\|_{W^{1,2}(\mathbb{R}^n)}$ as $\varepsilon \downarrow 0$ by the Sobolev inequality. Moreover, from (5.6) and definition of $\langle \cdot, \cdot \rangle_{W^{1,2}}$, we have

$$\|\phi\|_{W^{1,2}}^2 = \beta_0^2 \|v_Q\|_{W^{1,2}}^2 + \sum_{l=1}^n \beta_l^2 \left\| \frac{\partial v_Q}{\partial z_l} \right\|_{W^{1,2}}^2 + \|\psi\|_{W^{1,2}}^2.$$

By these observations and (5.8), we obtain

$$|\beta_0|(p-1)b(Q)\int_{\mathbb{R}^n} v_Q^{p+1}\,dz - o(1)\left\{\beta_0^2\|v_Q\|_{W^{1,2}}^2 + \sum_{l=1}^n \beta_l^2\left\|\frac{\partial v_Q}{\partial z_l}\right\|_{W^{1,2}}^2 + \|\psi\|_{W^{1,2}}^2\right\}^{1/2}$$
$$\le \Big(\|f\varepsilon\phi\|_{L^2} + \|g v_Q\|_{L^2}\Big)\|v_Q\|_{L^2} \qquad \text{as } \varepsilon \downarrow 0.$$

Clearly, since

$$\left\{\beta_0^2\|v_Q\|_{W^{1,2}}^2 + \sum_{l=1}^n \beta_l^2\left\|\frac{\partial v_Q}{\partial z_l}\right\|_{W^{1,2}}^2 + \|\psi\|_{W^{1,2}}^2\right\}^{1/2} \le |\beta_0|\|v_Q\|_{W^{1,2}} + \sum_{l=1}^n |\beta_l|\left\|\frac{\partial v_Q}{\partial z_l}\right\|_{W^{1,2}} + \|\psi\|_{W^{1,2}},$$

it follows that

$$|\beta_0|\left\{(p-1)b(Q)\int_{\mathbb{R}^n} v_Q^{p+1}\,dz - o(1)\|v_Q\|_{W^{1,2}}\right\}$$
$$\le (\|f\varepsilon\phi\|_{L^2} + \|g\|_{L^2})\|v_Q\|_{L^2} + o(1)\left(\sum_{l=1}^n |\beta_l|\left\|\frac{\partial v_Q}{\partial z_l}\right\|_{W^{1,2}} + \|\psi\|_{W^{1,2}}\right) \qquad \text{as } \varepsilon \downarrow 0.$$

We note that $(\|f\varepsilon\phi\|_{L^2} + \|g\|_{L^2})\|v_Q\|_{L^2}$ and $\|(\partial v_Q/\partial z_l)\|_{W^{1,2}}$ are bounded, so that we have

(5.9) $$|\beta_0| \le C_3 + o(1)\left(\sum_{l=1}^n |\beta_l| + \|\psi\|_{W^{1,2}}\right) \qquad \text{as } \varepsilon \downarrow 0.$$

Next, by taking the inner product in $L^2(\mathbb{R}^n)$ between (5.5) and ψ, we have

(5.10) $$(L_Q\phi, \psi)_{L^2} + (f\varepsilon\phi, \psi)_{L^2} + (g v_Q, \psi)_{L^2} + (h(\phi), \psi)_{L^2} = 0.$$

Note that $(b(Q)v_Q^p, \psi)_{L^2} = 0$ and

$$L_Q\phi = L_Q\left(\beta_0 v_Q + \sum_{l=1}^n \beta_l \frac{\partial v_Q}{\partial z_l} + \psi\right) = (p-1)\beta_0 b(Q) v_Q^p + L_Q\psi.$$

Therefore, we see that

$$(L_Q\phi, \psi)_{L^2} = (\beta_0(p-1)b(Q)^p + L\psi, \psi)_{L^2} = \beta_0(p-1)b(Q)(v_Q^p, \psi)_{L^2} + (L\psi, \psi)_{L^2}$$
$$= (L\psi, \psi)_{L^2}.$$

Hence, we have

$$(L_Q\psi, \psi)_{L^2} + (h(\phi), \psi)_{L^2} + (f\varepsilon\phi + g v_Q, \psi)_{L^2} = 0.$$

Here, we divide $(h(\phi), \psi)_{L^2}$ into two integrals as follows:

(5.11)
$$\begin{aligned}(h(\phi), \psi)_{L^2} &= \int_{\mathbb{R}^n} pb(Q+\varepsilon z)\Big\{(v_Q + \theta\varepsilon\phi)^{p-1} - v_Q^{p-1}\Big\}\phi\psi\,dz \quad (\text{ for some } \theta \in (0,1)) \\ &= \int_{\mathbb{R}^n} pb(Q+\varepsilon z)\Big\{(v_Q + \theta\varepsilon\phi)^{p-1} - v_Q^{p-1}\Big\}(\psi + \phi - \psi)\psi\,dz \\ &= \int_{\mathbb{R}^n} pb(Q+\varepsilon z)\Big\{(v_Q + \theta\varepsilon\phi)^{p-1} - v_Q^{p-1}\Big\}\psi^2\,dz \\ &\quad + \int_{\mathbb{R}^n} pb(Q+\varepsilon z)\Big\{(v_Q + \theta\varepsilon\phi)^{p-1} - v_Q^{p-1}\Big\}\left(\beta_0 v_Q + \sum_{l=1}^n \beta_l \frac{\partial v_Q}{\partial z_l}\right)\psi\,dz.\end{aligned}$$

Substituting (5.11) in (5.10), we find

$$
\begin{aligned}
&(L_Q\psi, \psi)_{L^2} + \int_{\mathbb{R}^n} pb(Q+\varepsilon z)\Big\{(v_Q + \theta\varepsilon\phi)^{p-1} - v_Q^{p-1}\Big\}\psi^2\, dz \\
&+ \int_{\mathbb{R}^n} pb(Q+\varepsilon z)\Big\{(v_Q + \theta\varepsilon\phi)^{p-1} - v_Q^{p-1}\Big\}\left(\beta_0 v_Q + \sum_{l=1}^{n}\beta_l \frac{\partial v_Q}{\partial z_l}\right)\psi\, dz \\
&+ (f\varepsilon\phi + g v_Q, \psi)_{L^2} = 0.
\end{aligned}
$$

By Hölder's inequality, we have the following inequality:

$$
\begin{aligned}
&(-L_Q\psi, \psi)_{L^2} - p\|b\|_{L^\infty}\left\|(v_Q + \theta\varepsilon\phi)^{p-1} - v_Q^{p-1}\right\|_{L^{(p+1)/(p-1)}}\|\psi^2\|_{L^{(p+1)/2}} \\
&\le p\|b\|_{L^\infty}\left\|(v_Q + \theta\varepsilon\phi)^{p-1} - v_Q^{p-1}\right\|_{L^{(p+1)/(p-1)}}\left(\|\beta_0 v_Q\|_{L^{p+1}} + \sum_{l=1}^{n}\left\|\beta_l\frac{\partial v_Q}{\partial z_l}\right\|_{L^{p+1}}\right)\|\psi\|_{L^{p+1}} \\
&\quad + (\|f\varepsilon\phi\|_{L^2} + \|g v_Q\|_{L^2})\|\psi\|_{L^2}.
\end{aligned}
$$

By $\|\psi^2\|_{L^{(p+1)/2}} = \|\psi\|_{L^{p+1}}^2$ and using the Sobolev inequality,

$$
\begin{aligned}
&(-L_Q\psi, \psi)_{L^2} - C_{S,p}p\|b\|_{L^\infty}\left\|(v_Q + \theta\varepsilon\phi)^{p-1} - v_Q^{p-1}\right\|_{L^{(p+1)/(p-1)}}\|\psi\|_{W^{1,2}}^2 \\
&\le C_{S,p}p\|b\|_{L^\infty}\left\|(v_Q + \theta\varepsilon\phi)^{p-1} - v_Q^{p-1}\right\|_{L^{(p+1)/(p-1)}}\left(|\beta_0| + \sum_{l=1}^{n}|\beta_l|\right)\|\psi\|_{W^{1,2}} \\
&\quad + (\|f\varepsilon\phi\|_{L^2} + \|g v_Q\|_{L^2})\|\psi\|_{W^{1,2}}
\end{aligned}
$$

where $C_{S,p}$ is the Sobolev imbedding constant for $W^{1,2}(\mathbb{R}^n) \hookrightarrow L^{p+1}(\mathbb{R}^n)$. On the other hand, when we define $Q(u,v) := (-L_Q u, v)_{L^2}$ for any $u,\ v \in E \subset W^{1,2}(\mathbb{R}^n)$, then Q is a bounded, coercive bilinear form on E (see Appendix B). Hence there exists a constant $c_0 > 0$ such that $c_0\|\psi\|_{W^{1,2}}^2 \le (-L_Q\psi, \psi)_{L^2}$. Since we assume $\varepsilon\phi \to 0$ in $W^{2,r}(\mathbb{R}^n)$ as $\varepsilon \downarrow 0$,

$$
\{c_0 - o(1)\}\|\psi\|_{W^{1,2}}^2 \le \left\{o(1)\left(|\beta_0| + \sum_{l=1}^{n}|\beta_l|\right) + \|f\varepsilon\phi\|_{L^2} + \|g v_Q\|_{L^2}\right\}\|\psi\|_{W^{1,2}} \quad \text{as } \varepsilon \downarrow 0.
$$

Since $c_0 - o(1) > 0$ as $\varepsilon \downarrow 0$, dividing the above inequality by $\{c_0 - o(1)\}\|\psi\|_{W^{1,2}}$, we see that

$$
\|\psi\|_{W^{1,2}} \le o(1)\left(|\beta_0| + \sum_{l=1}^{n}|\beta_l|\right) + \|f\varepsilon\phi\|_{L^2} + \|g v_Q\|_{L^2} \quad \text{as } \varepsilon \downarrow 0.
$$

Moreover, by using (5.9) for $|\beta_0|$ in the right-hand side, we obtain

$$
\|\psi\|_{W^{1,2}} \le o(1)C_3 + o(1)\left(\sum_{l=1}^{n}|\beta_l| + \|\psi\|_{W^{1,2}}\right) + o(1)\sum_{l=1}^{n}|\beta_l| + \|f\varepsilon\phi\|_{L^2} + \|g v_Q\|_{L^2} \quad \text{as } \varepsilon \downarrow 0.
$$

Hence, we conclude that

$$
\|\psi\|_{W^{1,2}} \le C_4 + o(1)\sum_{l=1}^{n}|\beta_l| \quad \text{as } \varepsilon \downarrow 0, \tag{5.12}
$$

where $\|f\varepsilon\phi\|_{L^2} = o(1)$ as $\varepsilon \downarrow 0$ and $\|gv_Q\|_{L^2}$ is bounded. We use (5.12) to estimate $\|\psi\|_{W^{1,2}}$ in (5.9), so that (5.9) yields follows:

$$|\beta_0| \le C_6 + o(1)\sum_{l=1}^{n}|\beta_l| \quad \text{as } \varepsilon \downarrow 0. \tag{5.13}$$

Therefore, if $|\beta_k|$ is bounded for all $k = 1, \dots, n$, then the boundedness of $\|\phi\|_{W^{1,2}}$ is proved by (5.6).

Now, we turn to the proof of the boundedness of $|\beta_k|$ for all $k = 1, \dots, n$. Let $|\beta| := \max|\beta_k|$, and assume that $|\beta|$ is not bounded as $\varepsilon \downarrow 0$. Then, there exists a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ with $\varepsilon_j \downarrow 0$ such that $|\beta| \to \infty$ as $j \to \infty$. Recall that we put $u_\varepsilon(Q+\varepsilon z)\chi(\varepsilon z) = v_Q(z)+\varepsilon\phi(z)$ and decompose ϕ as (5.6). Then, it follows that

$$u_\varepsilon(Q + \varepsilon z)\chi(\varepsilon z) = (1 + \varepsilon\beta_0)v_Q(z) + \varepsilon\sum_{l=1}^{n}\beta_l\frac{\partial v_Q}{\partial z_l}(z) + \varepsilon\psi(z).$$

Differentiating both sides with respect to z_k, we see that

$$\begin{aligned}&\varepsilon\left\{\frac{\partial u_\varepsilon}{\partial x_k}(Q + \varepsilon z)\chi(\varepsilon z) + u_\varepsilon(Q + \varepsilon z)\frac{\partial\chi}{\partial z_k}(\varepsilon z)\right\}\\ &= (1 + \varepsilon\beta_0)\frac{\partial v_Q}{\partial z_k}(z) + \varepsilon\sum_{l=1}^{n}\beta_l\frac{\partial^2 v_Q}{\partial z_k\partial z_l}(z) + \varepsilon\frac{\partial\psi}{\partial z_k}(z).\end{aligned}$$

We evaluate this identity at $z = 0$ and note $\chi(0) = 1$, $(\partial\chi/\partial z_k)(0) = 0$, and $(\partial v_Q/\partial z_k)(0) = 0$, so that we have

$$\varepsilon\frac{\partial u_\varepsilon}{\partial x_k}(Q) = \varepsilon\sum_{l=1}^{n}\beta_l\frac{\partial^2 v_Q}{\partial z_k\partial z_l}(0) + \varepsilon\frac{\partial\psi}{\partial z_k}(0).$$

Dividing both sides by ε, we obtain

$$\frac{\partial u_\varepsilon}{\partial x_k}(Q) = \sum_{l=1}^{n}\beta_l\frac{\partial^2 v_Q}{\partial z_k\partial z_l}(0) + \frac{\partial\psi}{\partial z_k}(0).$$

Multiplying this by $(\beta_k/|\beta|^2)$ and summing the resulting identity from $k = 1$ to $k = n$, we obtain

$$\sum_{l=1}^{n}\frac{\beta_k}{|\beta|^2}\frac{\partial u_\varepsilon}{\partial x_k}(Q) = \sum_{k,l=1}^{n}\frac{\beta_k}{|\beta|}\frac{\beta_l}{|\beta|}\frac{\partial^2 v_Q}{\partial z_k\partial z_l}(0) + \sum_{k=1}^{n}\frac{\beta_k}{|\beta|^2}\frac{\partial\psi}{\partial z_k}(0).$$

By $|\beta| \to \infty$ and the boundedness of $(\partial u_\varepsilon/\partial x_k)(Q)$,

$$\sum_{k=1}^{n}\frac{\beta_k}{|\beta|^2}\frac{\partial u_\varepsilon}{\partial x_k}(Q) \to 0 \quad \text{as } j \to \infty.$$

Moreover, by the equation (5.5) of ϕ, we note that ψ satisfies the equation $L_Q\psi + \beta_0(p-1)b(Q)v_Q^p + f\varepsilon\phi + gv_Q + h(\phi) = 0$. Then, by (5.12), we see that ψ is a weak solution of the elliptic equation $\Delta\psi - a(Q)\psi = \tilde{f}$ for each $\varepsilon > 0$ where $-\tilde{f} := pb(Q)v_Q^{p-1}\psi + \beta_0(p-1)b(Q)v_Q^p +$

$f\varepsilon\phi + gv_Q + h(\phi)$. Since $0 \le |\beta_k|/|\beta| \le 1$ is bounded and $|\beta| \to \infty$ as $\varepsilon \downarrow 0$, by using (5.12) again, we have

$$\frac{1}{|\beta|}\|\psi\|_{W^{1,2}} \le \frac{C_4}{|\beta|} + o(1)\sum_{l=1}^{n}\frac{|\beta_l|}{|\beta|} < \infty \quad \text{as } j \to \infty. \tag{5.14}$$

Therefore, $|\beta|^{-1}\|\tilde{f}\|_{L^q}$ is bounded independent of ε for any $q \in [1, \infty)$. Applying the regularity estimate to the solution $|\beta|^{-1}\psi$ of the elliptic equation $\Delta(|\beta|^{-1}\psi) - a(Q)(|\beta|^{-1}\psi) = |\beta|^{-1}\tilde{f}$, we have

$$\frac{1}{|\beta|}\|\psi\|_{W^{2,q}} \le \frac{C_4'}{|\beta|}\left(\|\tilde{f}\|_{L^q} + \|\psi\|_{L^q}\right). \tag{5.15}$$

Hence, $|\beta|^{-1}\psi$ is a strong solution of this equation and we obtain that $|\beta|^{-1}\|\psi\|_{W^{2,r}}$ is bounded as $j \to \infty$ from (5.15) and the boundedness of $\tilde{f}$ and ψ proved by (5.14). By using the Sobolev imbedding theorem for $q > n$, it holds that $W^{2,q}(B_r(0)) \hookrightarrow C^{1,\alpha}(\overline{B}_r(0))$ for any $r > 0$, that is, $\|u\|_{C^{1,\alpha}(\overline{B}_r(0))} \le C_S\|u\|_{W^{2,q}(B_r(0))}$ for any $u \in W^{2,q}(B_r(0))$ where C_S is a constant. Therefore, since the estimates of (5.14) and (5.15), $\|u\|_{C^{1,\alpha}(\overline{B}_r(0))} \le C_S\|u\|_{W^{2,q}(B_r(0))}$ at $u = |\beta|^{-1}\psi$ and $1/|\beta| \to 0$ as $j \to \infty$, we see that

$$\frac{1}{|\beta|}\|\psi\|_{C^1(\overline{B}_r(0))} \le \frac{C_5'}{|\beta|} + o(1)\sum_{l=1}^{n}\frac{|\beta_l|}{|\beta|} \to 0 \quad \text{as } j \to \infty,$$

hence $\sum_{k=1}^{n}(\beta_k/|\beta|^2)(\partial\psi/\partial z_k)(0) \to 0$ as $j \to \infty$. Moreover, for each $k = 1, \ldots, n$ from $|\beta_k|/|\beta| \in [0, 1]$, $\max_{0\le k\le 1}|\beta_k|/|\beta| = 1 \ne 0$, there exists $\gamma_k \in [-1, 1]$ such that

$$\frac{\beta_k}{|\beta|} \to \gamma_k \quad \text{as } j \to \infty \quad \text{and } \max_{1\le k\le n}|\gamma_k| = 1.$$

Hence, as $j \to \infty$, we have $\sum_{k,l=1}^{n}\gamma_k\gamma_l(\partial^2 v_Q/\partial z_k\partial z_l)(0) = 0$. From $v_Q(z) = v_Q(|z|)$ and $v_Q'(0) = 0$, we calculate $(\partial^2 v_Q/\partial z_k\partial z_l)$ as follows:

$$\begin{aligned}\frac{\partial^2 v_Q}{\partial z_k\partial z_l}(z) &= \frac{\partial}{\partial z_k}\left\{\frac{z_l}{|z|}v_Q'(|z|)\right\} = \delta_{kl}\frac{v_Q'(|z|)}{|z|} - \frac{z_k z_l}{|z|^3}v_Q'(|z|) + \frac{z_k z_l}{|z|^2}v_Q''(|z|)\\ &= \delta_{kl}v_Q''(0) + \delta_{kl}\left\{\frac{v_Q'(|z|) - v_Q'(0)}{|z| - 0} - v_Q''(0)\right\} - \frac{z_k z_l}{|z|^2}\left\{\frac{v_Q'(|z|) - v_Q'(0)}{|z| - 0} - v_Q''(|z|)\right\}.\end{aligned}$$

As $|z| \to 0$, we see that $(\partial^2 v_Q/\partial z_k\partial z_l)(0) = \delta_{kl}v_Q''(0)$. Hence it holds that

$$|\gamma|^2 v''(0) = \sum_{k,l=1}^{n}\gamma_k\gamma_l\delta_{kl}v''(0) = 0$$

where $\gamma = (\gamma_1, \ldots, \gamma_n)$ and $|\gamma|^2 = \sum_{k=1}^{n}\gamma_k^2$. Note that $v_Q''(0) < 0$ is proved by the assertion $w''(0) < 0$ which is obtained from $w(y) = w(|y|)$ and (GS-0). However, $\max|\gamma_k| = 1$, that is, $\gamma \ne 0$ and $v_Q''(0) < 0$ is in contradiction to $|\gamma|^2 v_Q''(0) = 0$. Therefore, $|\beta| = \max|\beta_k|$ is bounded as $\varepsilon \downarrow 0$. Consequently, the boundedness of $\|\phi\|_{W^{1,2}}$ is proved by (5.12) and (5.13). Q.E.D.

Proof of Theorem 1.6. First, we assume that (i): $Q \in \overline{B}_{R_0}(P_0)$, (ii): $\chi(\varepsilon z)u_\varepsilon(Q+\varepsilon z)$ decays exponentially as $|z| \to \infty$, and (iii): $\chi(\varepsilon z)u_\varepsilon(Q+\varepsilon z) - v_Q(z) =: \varepsilon\phi \to 0$ in $W^{2,r}(\mathbb{R}^n)$ as $\varepsilon \downarrow 0$. Taking the inner product in $L^2(\mathbb{R}^n)$ between (5.5) and $(\partial v_Q/\partial z_k)$, we have

$$\left(f\varepsilon\phi, \frac{\partial v_Q}{\partial z_k}\right)_{L^2} + \left(g v_Q, \frac{\partial v_Q}{\partial z_k}\right)_{L^2} + \left(h(\phi), \frac{\partial v_Q}{\partial z_k}\right)_{L^2} = 0.$$

Now, we estimate $(h(\phi), (\partial v_Q/\partial z_k))_{L^2}$ and $(f\varepsilon\phi, (\partial v_Q/\partial z_k))_{L^2}$ as follows:

$$\begin{aligned}
\left|\left(h(\phi), \frac{\partial v_Q}{\partial z_k}\right)_{L^2}\right| &\le p\|b\|_{L^\infty} \int_{\mathbb{R}^n} \{(v_Q + |\varepsilon\phi|)^{p-1} - v_Q^{p-1}\}|\phi|\left|\frac{\partial v_Q}{\partial z_k}\right| dz \\
&\le p\|b\|_{L^\infty} \left\| \{(v_Q + |\varepsilon\phi|)^{p-1} - v_Q^{p-1}\}\left|\frac{\partial v_Q}{\partial z_k}\right| \right\|_{L^{(p+1)/p}} \|\phi\|_{L^{p+1}} \\
&\le Cp\|b\|_{L^\infty}\|\phi\|_{W^{1,2}}\, o(1) \quad \text{ as } \varepsilon \downarrow 0, \\
\left|\left(f\varepsilon\phi, \frac{\partial v_Q}{\partial z_k}\right)_{L^2}\right| &\le \|f\|_{L^\infty}\|\varepsilon\phi\|_{L^2}\left\|\frac{\partial v_Q}{\partial z_k}\right\|_{L^2} = o(1) \quad \text{ as } \varepsilon \downarrow 0.
\end{aligned}$$

Therefore

$$\left(g v_Q, \frac{\partial v_Q}{\partial z_k}\right)_{L^2} = o(1) \quad \text{ as } \varepsilon \downarrow 0.$$

Then, we calculate $(g v_Q, \partial v_Q/\partial z_k)_{L^2}$ as follows:

$$\begin{aligned}
o(1) &= \left(g v_Q, \frac{\partial v_Q}{\partial z_k}\right)_{L^2} \\
&= -\int_{\mathbb{R}^n} \frac{a(Q+\varepsilon z) - a(Q)}{\varepsilon} v_Q \frac{\partial v_Q}{\partial z_k}\, dz + \int_{\mathbb{R}^n} \frac{b(Q+\varepsilon z) - b(Q)}{\varepsilon} v_Q^p \frac{\partial v_Q}{\partial z_k}\, dz \\
&= -\int_{\mathbb{R}^n} \nabla a(Q)\cdot z v_Q \frac{\partial v_Q}{\partial z_k}\, dz + \int_{\mathbb{R}^n} \nabla b(Q)\cdot z v_Q^p \frac{\partial v_Q}{\partial z_k}\, dz + O(\varepsilon).
\end{aligned}$$

Now suppose that $Q_\varepsilon \to Q$ as $\varepsilon \downarrow 0$. Then in the limit of $\varepsilon \downarrow 0$,

$$-\int_{\mathbb{R}^n} \nabla a(Q)\cdot z v_Q \frac{\partial v_Q}{\partial z_k}\, dz + \int_{\mathbb{R}^n} \nabla b(Q)\cdot z v_Q^p \frac{\partial v_Q}{\partial z_k}\, dz = 0.$$

Since $(\partial v_Q/\partial z_k)$ is an odd function with respect to z_k, we obtain

$$-\frac{\partial a}{\partial x_k}(Q)\int_{\mathbb{R}^n} z_k v_Q \frac{\partial v_Q}{\partial z_k}\, dz + \frac{\partial b}{\partial x_k}(Q)\int_{\mathbb{R}^n} z_k v_Q^p \frac{\partial v_Q}{\partial z_k}\, dz = 0.$$

Now, for $r = 2,\ p+1$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} z_k v_Q^r \frac{\partial v_Q}{\partial z_k}\, dz &= \int_{\mathbb{R}^{n-1}} \left\{\left[\frac{1}{r+1} v_Q^{r+1} z_k\right]_{z_k=-\infty}^{z_k=\infty} - \frac{1}{r+1}\int_{\mathbb{R}} v_Q^{r+1}\, dz_k\right\} dz' \\
&= -\frac{1}{r+1}\int_{\mathbb{R}^n} v_Q^{r+1}\, dz.
\end{aligned}$$

We therefore see that

$$\frac{1}{2}\frac{\partial a}{\partial x_k}(Q)\int_{\mathbb{R}^n} v_Q^2\, dz - \frac{1}{p+1}\frac{\partial b}{\partial x_k}(Q)\int_{\mathbb{R}^n} v_Q^{p+1}\, dz = 0. \tag{5.16}$$

Second, we prove that (5.16) holds at $Q = P_0$. Recall that we are given a family $\{u_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ of solutions of (5.12) such that $J_\varepsilon(u_\varepsilon) = O(\varepsilon^n)$ and $P_\varepsilon \to P_0 \in \Omega$ as $\varepsilon \downarrow 0$, where P_ε is a local maximum point of u_ε. Now we take $Q_\varepsilon = P_\varepsilon$. Obviously $P_\varepsilon \in \overline{B}_{R_0}(P_0)$ holds for $\varepsilon > 0$ sufficiently small, and hence the assumption (i) made at the beginning of this proof is satisfied. Next, in a way similar to the proof of approximation of ground-state solutions (see Proposition 3.1), we can verify that $u_\varepsilon(P_\varepsilon + \varepsilon z) \to v_{P_0}(z)$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$ as $\varepsilon \downarrow 0$ for any $r \in [1, \infty)$ by using the estimate $J_\varepsilon(u_\varepsilon) = O(\varepsilon^n)$ and the equation satisfied by u_ε. Put $\varepsilon\phi(x) = \chi(z)u_\varepsilon(P_\varepsilon + \varepsilon z) - v_{P_\varepsilon}(z)$ with $z = (x - P_\varepsilon)/\varepsilon$. Since v_{P_ε} decays exponentially as $|z| \to \infty$, there exist constants $r_1 > 0$ and $c_1 > 0$ such that $v_{P_\varepsilon}(z) = O(e^{-c_1|z|})$ for any $|z| > r_1$, uniformly in ε. Also, we know that $u_\varepsilon(P_\varepsilon + \varepsilon z)$ for any z satisfying $r_1 < |z| < R_0$, where R_0 is an arbitrarily fixed constant. Therefore, we have checked that the assumption (ii) is also satisfied. Moreover, by a computation similar to that in deriving (5.5), we see that $\varepsilon\phi$ satisfies the elliptic equation

$$L_{P_\varepsilon}(\varepsilon\phi) + \varepsilon(f|_{Q=P_\varepsilon})\varepsilon\phi + \varepsilon(g|_{Q=P_\varepsilon})v_{P_\varepsilon} + \varepsilon h(\phi)|_{Q=P_\varepsilon} = 0.$$

From $\varepsilon\phi \to 0$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$ as $j \to \infty$, applying the elliptic L^r-estimates to $\varepsilon\phi$, we get $\varepsilon\phi \to 0$ in $W^{2,r}(\mathbb{R}^n)$. Hence, the assumption (iii) is satisfied. Consequently, we have proved that (5.16) at $Q = P_0$ holds for any $k = 1, \ldots, n$, since $P_\varepsilon \to P_0$.

Finally, we put

$$I_k(Q) = \frac{1}{2}\frac{\partial a}{\partial x_k}(Q)\int_{\mathbb{R}^n} v_Q^2\,dz - \frac{1}{p+1}\frac{\partial b}{\partial x_k}(Q)\int_{\mathbb{R}^n} v_Q^{p+1}\,dz$$

and prove

$$I_k(P_0) = \frac{p-1}{2(p+1)}\int_{\mathbb{R}^n} w^{p+1}dy\,\frac{\partial\Phi}{\partial x_k}(P_0). \tag{5.17}$$

From the definition of v_{P_0}, i.e., $v_{P_0}(z) = (a(P_0)/b(P_0))^{1/(p-1)}w(\sqrt{a(P_0)}z)$, $y = \sqrt{a(P_0)}z$, we calculate the left-hand side of (5.16) on $Q = P_0$ as follows:

$$\begin{aligned}
I_k(P_0) &= \frac{1}{2}\frac{\partial a}{\partial x_k}(P_0)\left(\frac{a(P_0)}{b(P_0)}\right)^{2/(p-1)} a(P_0)^{-n/2}\int_{\mathbb{R}^n} w^2\,dz \\
&\quad - \frac{1}{p+1}\frac{\partial b}{\partial x_k}(P_0)\left(\frac{a(P_0)}{b(P_0)}\right)^{(p+1)/(p-1)} a(P_0)^{-n/2}\int_{\mathbb{R}^n} w^{p+1}\,dz \\
&= a(P_0)^{1-n/2+2/(p-1)}b(P_0)^{-2/(p-1)} \\
&\quad \times\left\{\frac{1}{2}\frac{1}{a(P_0)}\frac{\partial a}{\partial x_k}(P_0)\int_{\mathbb{R}^n} w^2\,dz - \frac{1}{p+1}\frac{1}{b(P_0)}\frac{\partial b}{\partial x_k}(P_0)\int_{\mathbb{R}^n} w^{p+1}\,dz\right\}.
\end{aligned}$$

Note that the primary locator function $\Phi = a^{1-n/2+2/(p-1)}b^{-2/(p-1)}$. Then, the left-hand side of $(5.16)|_{Q=P_0}$ becomes

$$\Phi(P_0)\left\{\frac{1}{2}\frac{1}{a(P_0)}\frac{\partial a}{\partial x_k}(P_0)\int_{\mathbb{R}^n} w^2\,dy - \frac{1}{p+1}\frac{1}{b(P_0)}\frac{\partial b}{\partial x_k}(P_0)\int_{\mathbb{R}^n} w^{p+1}\,dy\right\}. \tag{5.18}$$

On the other hand, since w is symmetric with respect to the origin and satisfies $\Delta w - w + w^p = 0$, we have

$$w'' + \frac{n-1}{r}w' - w + w^p = 0 \qquad (r = |y|).$$

Multiplying this equation by $r^n w'$ and integrating it on $[0, \infty)$, we see that

$$\int_0^\infty w'w''r^n\,dr + (n-1)\int_0^\infty (w')^2 r^{n-1}\,dr - \int_0^\infty ww'r^n\,dr + \int_0^\infty w^p w' r^n\,dr = 0.$$

We calculate each term of the above identity:

$$\begin{aligned}\int_0^\infty w'w''r^n\,dr &= \int_0^\infty \frac{1}{2}\frac{d}{dr}(w')^2 r^n\,dr = \left[\frac{1}{2}r^n(w')^2\right]_0^\infty - \frac{n}{2}\int_0^\infty (w')^2 r^{n-1}\,dr \\ &= -\frac{n}{2}\int_0^\infty (w')^2 r^{n-1}\,dr,\end{aligned}$$

for $q = 2$ and $q = p+1$,

$$\begin{aligned}\int_0^\infty w^r w' r^n\,dr &= \int_0^\infty \frac{1}{q+1}\frac{d}{dr}(w^{q+1})r^n\,dr \\ &= \left[\frac{1}{q+1}w^{q+1}r^n\right]_0^\infty - \frac{n}{q+1}\int_0^\infty w^{q+1}r^{n-1}\,dr \\ &= -\frac{n}{q+1}\int_0^\infty w^{q+1}r^{n-1}\,dr.\end{aligned}$$

Hence, the above identity becomes

$$\left(-\frac{n}{2} + n - 1\right)\int_0^\infty (w')^2 r^{n-1}\,dr + \frac{n}{2}\int_0^\infty w^2 r^{n-1}\,dr - \frac{n}{p+1}\int_0^\infty w^{p+1}r^{n-1}\,dr = 0.$$

Multiplying this identity by $n\omega_n$ and transforming the integration of r to that of y,

$$\left(\frac{n}{2} - 1\right)\int_{\mathbb{R}^n} |\nabla w|^2\,dy + \frac{n}{2}\int_{\mathbb{R}^n} w^2\,dy - \frac{n}{p+1}\int_{\mathbb{R}^n} w^{p+1}\,dy = 0, \tag{5.19}$$

where ω_n is volume of unit ball in $\mathbb{R}^n$ and note $w' = |\nabla w|$. Moreover, multiplying $\Delta w - w + w^p = 0$ by w and integrating it over $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} |\nabla w|^2\,dy = -\int_{\mathbb{R}^n} w^2\,dy + \int_{\mathbb{R}^n} w^{p+1}\,dy. \tag{5.20}$$

Substituting (5.20) in (5.19), we see

$$\left(\frac{n}{2} - 1\right)\left\{-\int_{\mathbb{R}^n} w^2\,dy + \int_{\mathbb{R}^n} w^{p+1}\,dy\right\} + \frac{n}{2}\int_{\mathbb{R}^n} w^2\,dy - \frac{n}{p+1}\int_{\mathbb{R}^n} w^{p+1}\,dy = 0.$$

Therefore, we have

$$\int_{\mathbb{R}^n} w^2\,dy = \left(1 - \frac{n}{2} + \frac{n}{p+1}\right)\int_{\mathbb{R}^n} w^{p+1}\,dy.$$

Substituting this to (5.18), we calculate as follows:

$$
\begin{aligned}
&\Phi(P_0)\left\{\frac{1}{2}\frac{1}{a(P_0)}\frac{\partial a}{\partial x_k}(P_0)\int_{\mathbb{R}^n} w^2\,dy - \frac{1}{p+1}\frac{1}{b(P_0)}\frac{\partial b}{\partial x_k}(P_0)\int_{\mathbb{R}^n} w^{p+1}\,dy\right\}\\
&= \Phi(P_0)\left\{\frac{1}{2}\left(1-\frac{n}{2}+\frac{n}{p+1}\right)\frac{1}{a(P_0)}\frac{\partial a}{\partial x_k}(P_0) - \frac{1}{p+1}\frac{1}{b(P_0)}\frac{\partial b}{\partial x_k}(P_0)\right\}\int_{\mathbb{R}^n} w^{p+1}\,dy\\
&= \Phi(P_0)\frac{1}{2}\left[\left\{1-\frac{n(p-1)}{2(p+1)}\right\}\frac{1}{a(P_0)}\frac{\partial a}{\partial x_k}(P_0) - \frac{2}{p+1}\frac{1}{b(P_0)}\frac{\partial b}{\partial x_k}(P_0)\right]\int_{\mathbb{R}^n} w^{p+1}\,dy\\
&= \Phi(P_0)\frac{p-1}{2(p+1)}\left\{\left(\frac{p+1}{p-1}-\frac{n}{2}\right)\frac{1}{a(P_0)}\frac{\partial a}{\partial x_k}(P_0) - \frac{2}{p-1}\frac{1}{b(P_0)}\frac{\partial b}{\partial x_k}(P_0)\right\}\int_{\mathbb{R}^n} w^{p+1}\,dy\\
&= \Phi(P_0)\frac{p-1}{2(p+1)}\int_{\mathbb{R}^n} w^{p+1}\,dy\\
&\quad\times\left\{\left(1-\frac{n}{2}+\frac{2}{p-1}\right)\frac{\partial}{\partial x_k}(\log a(x))\Big|_{x=P_0} - \frac{2}{p-1}\frac{\partial}{\partial x_k}(\log b(x))\Big|_{x=P_0}\right\}\\
&= \frac{p-1}{2(p+1)}\int_{\mathbb{R}^n} w^{p+1}\,dy\left[\Phi(P_0)\frac{\partial}{\partial x_k}\left\{\log(a(P_0)^{1-n/2+2/(p-1)}b(P_0)^{-2/(p-1)})\right\}\right]\\
&= \frac{p-1}{2(p+1)}\int_{\mathbb{R}^n} w^{p+1}\,dy\left\{\Phi(P_0)\frac{\partial}{\partial x_k}(\log\Phi(P_0))\right\}\\
&= \frac{p-1}{2(p+1)}\int_{\mathbb{R}^n} w^{p+1}\,dy\,\frac{\partial\Phi}{\partial x_k}(P_0).
\end{aligned}
$$

We thus obtain (5.17). Recall that the left-hand side of (5.16) vanishes at $Q = P_0$. Hence, we have

$$\frac{p-1}{2(p+1)}\int_{\mathbb{R}^n} w^{p+1}\,dy\,\frac{\partial\Phi}{\partial x_k}(P_0) = 0.$$

Now, note that $(p-1)/2(p+1) > 0$ and $\int_{\mathbb{R}^n} w^{p+1}\,dy > 0$ since w is a positive solution of (GS-0). Consequently, $(\partial\Phi/\partial x_k)(P_0) = 0$ holds. Since $(\partial\Phi/\partial x_k)(P_0) = 0$ for all $k = 1, \ldots, n$, we conclude that $\nabla\Phi(P_0) = 0$. Q.E.D.

Appendix A

In this appendix, we prove the following

Lemma A.1. *Let* $\{u_\varepsilon\}$ *be a solution of* (P) *satisfying*

(1.9) $$c_0\varepsilon^n \le J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_{m,\varepsilon}) \le C_0\varepsilon^n,$$

where c_0 *and* C_0 *are some positive constants with* $c_0 \le C_0$ *and* $u_{m,\varepsilon}$ *is the minimal solution of* (P). *Then,* $\{u_\varepsilon\}$ *concentrates at finitely many points on* $\overline{\Omega}$.

Proof. We suppose that u_ε is a solution of (P) and satisfies (1.9). By the positivity of $u_{m,\varepsilon}$ and the maximal principle applied to (3.1), we see that $v_\varepsilon := u_\varepsilon - u_{m,\varepsilon}$ is positive on $\overline{\Omega}$. Recalling the definition of J_ε (see (1.7)) and I_ε (see (1.8)), we see that

$$I_\varepsilon(v_\varepsilon) = J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_{m,\varepsilon}) = \frac{1}{2}\|v_\varepsilon\|^2_{E_\varepsilon} - \int_\Omega b(x)G(u_{m,\varepsilon}(x), v_\varepsilon(x))\,dx,$$

where $\|\cdot\|_{E_\varepsilon}$ is defined by Definition 2.1. Hence, by Claim 2.8, we see that for any $\theta \in (\max\{1/3, 1/(p+1)\}, 1/2)$

(A.1) $$I_\varepsilon(v_\varepsilon) \ge \frac{1}{2}\|v_\varepsilon\|^2_{E_\varepsilon} - \theta\int_\Omega b(x)g(u_{m,\varepsilon}(x), v_\varepsilon(x))v_\varepsilon(x)\,dx,$$

where we have used the fact that θ is a constant. Moreover, since v_ε satisfies the equation $\varepsilon^2\mathcal{A}(x)v - a(x)v + b(x)\{(u_{m,\varepsilon}(x)+v)^p_+ - u_{m,\varepsilon}(x)^p\} = 0$, we have

(A.2) $$\int_\Omega b(x)g(u_{m,\varepsilon}(x), v_\varepsilon(x))v_\varepsilon(x)\,dx = \|v_\varepsilon\|^2_{E_\varepsilon}.$$

Substituting (A.2) in (A.1), one obtains that

(A.3) $$I_\varepsilon(v_\varepsilon) \ge \left(\frac{1}{2} - \theta\right)\|v_\varepsilon\|^2_{E_\varepsilon}.$$

Therefore, from the assumption (1.9), we have $(1/2-\theta)\|v_\varepsilon\|^2_{E_\varepsilon} \le C_0\varepsilon^n$, and hence

$$\|v_\varepsilon\|^2_{E_\varepsilon} \le C_0'\varepsilon^n$$

with $C_0' = C_0/(1/2-\theta)$. Starting with this estimate, we can prove, as in Proposition 2.14, that there exists a positive constant C_r for every $r \ge 1$ such that

(A.4) $$\int_\Omega v_\varepsilon^r\,dx \le C_r\varepsilon^n$$

and $\sup_{1\le r} C_r^{1/r} < \infty$.

Now, let P_ε be a local minimum point of u_ε. In a way similar to that in the proof of Proposition 3.1, we can check that there exists a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ with $\varepsilon_j \downarrow 0$, a point $P_0 \in \overline{\Omega}$ and a function $V_\varepsilon(z)$ in $\mathbb{R}^n$ such that $P_{\varepsilon_j} \to P_0$ and

$$v_\varepsilon(x) = V_\varepsilon(z) \to V_0(z) \quad \text{in } C^2(K), \tag{A.5}$$

where $z = (x - P_\varepsilon)/\varepsilon$ and K is any compact set in $\mathbb{R}^n$. We prove in the case of $P_0 \in \Omega$ only. The case $P_0 \in \partial\Omega$ can be treated in the same line as in the proof of Proposition 3.1 Case (II). Hence, by (A.4) and (A.5), we have $\|V_0\|_{L^r(\mathbb{R}^n)} \le C_r^{1/r}$ for any $r \ge 1$. Since $u_{m,\varepsilon}$ is the minimal solution of (P), we can use Lemma 2.3, and hence $u_{m,\varepsilon_j}(P_{\varepsilon_j} + \varepsilon_j z)$ converges to $u_m(P_0)$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$. Recalling that $v_\varepsilon(x)$ satisfies the equation $\varepsilon^2\mathcal{A}(x)v - a(x)v + b(x)\{(u_{m,\varepsilon}(x) + v)_+^p - u_{m,\varepsilon}(x)^p\} = 0$, we obtain that V_{ε_j} satisfies

$$\mathcal{A}(P_{\varepsilon_j} + \varepsilon_j z)V - a(P_{\varepsilon_j})V + b(P_{\varepsilon_j} + \varepsilon_j z)\{(u_{m,\varepsilon_j} + V)_+^p - u_{m,\varepsilon_j}^p\} = 0 \quad \text{in } \Omega_{\varepsilon_j, P_{\varepsilon_j}},$$

where $\mathcal{A}(P_{\varepsilon_j} + \varepsilon_j z)V = \sum_{i,j=1}^n (\partial/\partial z_i)(a_{ij}(P_{\varepsilon_j} + \varepsilon_j z)\partial V/\partial z_j)$ and $\Omega_{\varepsilon_j, P_{\varepsilon_j}} = \{z \in \mathbb{R}^n \mid x = P_{\varepsilon_j} + \varepsilon_j z \in \Omega\}$. Since $\Omega_{\varepsilon_j, P_{\varepsilon_j}}$ tends to $\mathbb{R}^n$ as $j \to \infty$, we see that V_0 satisfies

$$\mathcal{A}(P_0)V - a(P_0)V - b(P_0)\{(u_m(P_0) + V)_+^p - u_m(P_0)^p\} = 0 \quad \text{in } \mathbb{R}^n,$$

where $\mathcal{A}(P_0)V = \sum_{i,j=1}^n a_{ij}(P_0)(\partial^2 V/\partial z_i \partial z_j)$. Noting that V_0 satisfies the same equation as (3.7) and $\|V_0\|_{L^r(\mathbb{R}^n)} \le C_r^{1/r}$ for $r \ge 1$, by the uniqueness of the solution of (GS-γ), we have

$$V_0(z) = \left\{\frac{a(P_0)}{b(P_0)}\right\}^{1/(p-1)} w_\gamma(y) \quad \text{with } y = \sqrt{a(P_0)}\sqrt{D_{P_0}^{-1}}B_{P_0} z \text{ and } \gamma = \gamma_0(P_0).$$

Therefore, we see that

$$u_{\varepsilon_j}(x) = u_m(P_0) + \left\{\frac{a(P_0)}{b(P_0)}\right\}^{1/(p-1)} w_{\gamma_0(P_0)}\left(\sqrt{a(P_0)}\sqrt{D_{P_0}^{-1}}B_{P_0}\frac{x - P_0}{\varepsilon_j}\right) + o(1) \quad \text{as } \varepsilon_j \downarrow 0. \tag{A.6}$$

Recalling that w_γ is symmetric with respect to the origin and $w_\gamma(|y|)$ is decreasing in $0 \le |y| < \infty$, for ε_j small enough, P_{ε_j} becomes a strict local maximum point of u_{ε_j}. In view of the expression (A.6) and noting that w_γ decays exponentially at infinity, we see that $v_{\varepsilon_j}(x) = u_{\varepsilon_j}(x) - u_{m,\varepsilon_j}(x)$ converges to zero for $x \in B_\rho(P_{\varepsilon_j}) \cap \overline{\Omega} \setminus \overline{\{P_{\varepsilon_j} \mid 0 < \varepsilon_j < \varepsilon_0\}}$ where ρ is some positive constant. Consequently, we obtain that $\{u_\varepsilon\}$ concentrates at P_0. Q.E.D.

Appendix B

In this appendix we give a detailed proof of the coercivity of the quadratic form $Q(u, v)$ on E due to Wei [24, 25].

Lemma B.1. *The eigenvalue problem*

$$\text{(EVP)} \qquad \begin{cases} \Delta u - a(Q)u + \mu b(Q)v_Q^{p-1}u = 0 & \text{in } \mathbb{R}^n, \\ u \in W^{1,2}(\mathbb{R}^n) \end{cases}$$

has the family of real eigenvalues $\{\mu_j\}_{j\in\mathbb{N}}$ which diverge to infinity, and $L^2(\mathbb{R}^n)$ is spanned by a family of eigenfunctions $\{\phi_{j,k}\}_{k\in\{1,\dots,m(j)\},j\in\mathbb{N}}$ where $\phi_{j,k}$ is an eigenfunction belonging to the eigenvalue μ_j and $m(j)$ is the multiplicity of μ_j. Moreover, the principal eigenvalue is $\mu_1 = 1$ and the second eigenvalue is $\mu_2 = p$.

Proof. We define an operator G_Q on $L^2(\mathbb{R}^n)$ by

$$G_Q[\phi] := (a(Q) - \Delta)^{-1}\phi,$$

so that from the regularity estimate of elliptic equation, we have $\|G_Q[\phi]\|_{W^{2,2}(\mathbb{R}^n)} \leq C\|\phi\|_{L^2(\mathbb{R}^n)}$. Moreover, define T_Q by

$$T_Q\phi := G_Q[b(Q)v_Q^{p-1}\phi] \quad \text{for any } \phi \in L^2(\mathbb{R}^n).$$

Then, it holds that $T_Q : L^2(\mathbb{R}^n) \to W^{2,2}(\mathbb{R}^n)(\subset L^2(\mathbb{R}^n))$ and T_Q becomes a compact operator on $L^2(\mathbb{R}^n)$ by $v_Q(z)$ decaying exponentially $|z| \to \infty$. Note that T_Q is a symmetric operator on L^2, i.e., $(T_Q\phi, \psi)_{L^2} = (\phi, T_Q\psi)_{L^2}(\mathbb{R}^n)$. Hence, the spectrums of T_Q are comprised of the eigenvalues and the multiplicity of each eigenvalue is finite. Put the eigenvalues in order and let the set of these eigenvalues be $\{\lambda_j\}_{j\in\mathbb{N}}$. Then, $\{\lambda_j\}_{j\in\mathbb{N}}$ does not have an accumulation point except 0, and λ_j have to be a real number from the symmetry of T_Q. Moreover, a family of the eigenfunctions $\{\phi_{j,k}\}_{k\in\{1,\dots,m(j)\},j\in\mathbb{N}}$ spans $L^2(\mathbb{R}^n)$. By $(a(Q) - \Delta)^{-1}(b(Q)v_Q^{p-1}\phi_j) = T_Q\phi_j = \lambda_j\phi_j$ and $\lambda_j \neq 0$ for each j, ϕ_j satisfies

$$\Delta\phi_j - a(Q)\phi_j + \frac{1}{\lambda_j}b(Q)v_Q^{p-1}\phi_j = 0.$$

Comparing this equation with (EVP), we see that the eigenvalue μ of (EVP) corresponds to $1/\lambda$. Therefore, putting $\mu_j = 1/\lambda_j$, since $\lambda_j \to 0$, we obtain that $\mu_j \to \infty$ as $j \to \infty$.

First, we prove $\mu_j > 0$ for all $j \in \mathbb{N}$. Since the eigenfunction $\phi_{j,k}$ of μ_j satisfies $\Delta\phi_{j,k} - a(Q)\phi_{j,k} + \mu_j b(Q)v_Q^{p-1}\phi_{j,k} = 0$, multiplying this by $\phi_{j,k}$ and integrating it in $\mathbb{R}^n$, we have

$$\mu_j b(Q)\int_{\mathbb{R}^n} v_Q^{p-1}\phi_{j,k}^2\,dz = \int_{\mathbb{R}^n}\left(|\nabla\phi_{j,k}|^2 + a(Q)\phi_{j,k}^2\right)dz > 0.$$

By $b(Q)\int_{\mathbb{R}^n} v_Q^{p-1}\phi_{j,k}^2\,dz > 0$, hence μ_j is positive.

Second, it is proved that we choose $\{\phi_{j,k}\}_{k\in\{1,\dots,m(j)\},j\in\mathbb{N}}$ such that ϕ_{j_1,k_1} and ϕ_{j_2,k_2} are orthogonal in $L^2(\mathbb{R}^n)$ with the weight $b(Q)v_Q^{p-1}$ where ϕ_{j_1,k_1} and ϕ_{j_2,k_2} are different. Assume $\mu_{j_1} < \mu_{j_2}$. Let ϕ_{j_i} be an eigenfunction of μ_{j_i} for $i = 1, 2$. Since ϕ_{j_i} satisfies (EVP) on $\mu = \mu_{j_i}$, it follows that

$$\begin{aligned}
&\phi_{j_2}\Delta\phi_{j_1} - \phi_{j_1}\Delta\phi_{j_2} + (\mu_{j_1} - \mu_{j_2})b(Q)v_Q^{p-1}\phi_{j_1}\phi_{j_2}\\
&= \phi_{j_2}(\Delta\phi_{j_1} - a(Q)\phi_{j_1} + \mu_{j_1}b(Q)v_Q^{p-1}\phi_{j_1}) - \phi_{j_1}(\Delta\phi_{j_2} - a(Q)\phi_{j_2} + \mu_{j_2}b(Q)v_Q^{p-1}\phi_{j_2}) = 0.
\end{aligned}$$

Integrating this in $\mathbb{R}^n$, by $\int_{\mathbb{R}^n}(\phi_{j_2}\Delta\phi_{j_1} - \phi_{j_1}\Delta\phi_{j_2})\,dz = -\int_{\mathbb{R}^n}(\nabla\phi_{j_2}\cdot\nabla\phi_{j_1} - \nabla\phi_{j_1}\cdot\nabla\phi_{j_2})\,dz = 0$, we obtain

$$(\mu_{j_1} - \mu_{j_2})\int_{\mathbb{R}^n} b(Q)v_Q^{p-1}\phi_{j_1}\phi_{j_2}\,dz = 0.$$

Since we assume $\mu_{j_1} \neq \mu_{j_2}$, it holds that $\int_{\mathbb{R}^n} b(Q)v_Q^{p-1}\phi_{j_1}\phi_{j_2}\,dz = 0$. In the case of $\mu_{j_1} = \mu_{j_2}$, there exists linearly independent functions $\phi_{j_1,1}, \dots, \phi_{j_1,m(j_1)}$. Hence, it suffices to orthogonalize those functions in $L^2(\mathbb{R}^n)$ with weight $b(Q)v_Q^{p-1}$.

Third, we start to prove $\mu_1 = 1$ where μ_1 is the principal eigenvalue of (EVP). Taking $\phi = v_Q$, by the definition of v_Q, it follows that $\Delta\phi - a(Q)\phi + b(Q)v_Q^{p-1}\phi = 0$. Hence, there exists $j \geq 1$ and $k \in \{1, \dots, m(j)\}$ such that $\mu_j = 1$. By $v_Q > 0$, v_Q must be the principal eigenfunction of (EVP) where we note that the principal eigenfunction is a definite sign. Moreover, by the variational characterization of eigenvalues, the following holds:

$$\begin{aligned}
\mu_1 &= \inf_{\phi\in W^{1,2}(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n}(|\nabla\phi|^2 + a(Q)\phi^2)\,dz}{\int_{\mathbb{R}^n} b(Q)v_Q^{p-1}\phi^2\,dz},\\
\mu_2 &= \inf_{\phi\in W^{1,2}(\mathbb{R}^n)\setminus\{0\},\ \phi\perp\phi_1} \frac{\int_{\mathbb{R}^n}(|\nabla\phi|^2 + a(Q)\phi^2)\,dz}{\int_{\mathbb{R}^n} b(Q)v_Q^{p-1}\phi^2\,dz},
\end{aligned}$$

where μ_j is the j-th eigenvalue of (EVP). Now, from $\Delta v_Q - a(Q)v_Q + b(Q)v_Q^p = 0$, we have

$$\Delta\frac{\partial v_Q}{\partial z_l} - a(Q)\frac{\partial v_Q}{\partial z_l} + pb(Q)v_Q^{p-1}\frac{\partial v_Q}{\partial z_l} = 0.$$

Hence, p is an eigenvalue and $(\partial v_Q/\partial z_l)$ is the eigenfunction of (EVP). Now, by $v_Q(z) = v_Q(|z|)$, since $(\partial v_Q/\partial z_l) = v_Q'(|z|)z_l/|z|$ holds, the nodal set becomes $\{z \in \mathbb{R}^n \mid z_l = 0\}$. Therefore, $(\partial v_Q/\partial z_l)$ is the second eigenfunction. Q.E.D.

Lemma B.2. *Let $L_Q := \Delta - a(Q)v_Q + b(Q)v_Q^{p-1}$, and define a bilinear form $\mathcal{Q}$ on $W^{1,2}(\mathbb{R}^n)$ by*

$$\mathcal{Q}(u, v) := \int_{\mathbb{R}^n}\{\nabla u\cdot\nabla v + a(Q)v_Q uv - b(Q)v_Q^{p-1}uv\}\,dz.$$

Then $\mathcal{Q}$ is bounded and coercive on E defined by (5.7) *where $\langle\psi,\phi\rangle_{W^{1,2}} := \int_{\mathbb{R}^n}(\nabla\psi\cdot\nabla\phi + a(Q)\psi\phi)\,dz$ and $\|\phi\|_{W^{1,2}} := \sqrt{\langle\phi,\phi\rangle_{W^{1,2}}}$.*

Proof. The boundedness of Q is easy to check. Indeed, by the Cauchy-Schwartz inequality, it follows that

$$|Q(u,v)| \le \int_{\mathbb{R}^n} \Big\{ |\nabla u|\,|\nabla v| + \Big| a(Q)v_Q - b(Q)v_Q^{p-1} \Big|\,|uv| \Big\}\,dz$$
$$\le \max\Big[1, \max_{z\in\mathbb{R}^n}\{v_Q(z) - b(Q)v_Q(z)^{p-1}/a(Q)\}\Big]\,\|u\|_{W^{1,2}(\mathbb{R}^n)}\|v\|_{W^{1,2}(\mathbb{R}^n)}.$$

Next, taking any $\psi \in E$, from $E \subset W^{1,2}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) = \mathrm{span}\{\phi_{j,k} \mid k = 1,\dots,m(j),\ j \in \mathbb{N}\}$, we have

$$\psi = \sum_{j=1}^{\infty}\sum_{k=1}^{m(j)} c_{j,k}\,\phi_{j,k}.$$

To simplify the description, denote $\sum_j^{\infty}\sum_{k=1}^{m(j)}$ by $\sum_j$. By the definition of E, we see also that

$$\int_{\mathbb{R}^n} b(Q)v_Q^p\psi\,dz = \int_{\mathbb{R}^n}(-\Delta v_Q + a(Q)v_Q)\psi\,dz = \int_{\mathbb{R}^n}(\nabla v_Q\cdot\nabla\psi + a(Q)v_Q\psi)\,dz = 0,$$
$$mu_2\int_{\mathbb{R}^n} b(Q)v_Q^{p-1}\frac{\partial v_Q}{\partial z_l}\psi\,dz = \int_{\mathbb{R}^n}\Big(-\Delta\frac{\partial v_Q}{\partial z_l} + a(Q)\frac{\partial v_Q}{\partial z_l}\Big)\psi\,dz$$
$$= \int_{\mathbb{R}^n}\Big(\nabla\frac{\partial v_Q}{\partial z_l}\cdot\nabla\psi + a(Q)\frac{\partial v_Q}{\partial z_l}\psi\Big)\,dz = 0.$$

Hence, ψ is orthogonal in $L^2(\mathbb{R}^n)$ with weight $b(Q)v_Q^{p-1}$. Since $\phi_{1,1} = v_Q$ and $\phi_{2,k} = (\partial v_Q/\partial z_k)$, we have $c_{j,k} = 0$ for $j = 1, 2$. From now, we calculate $Q(\psi,\psi)$:

$$Q(\psi,\psi) = \int_{\mathbb{R}^n}(\nabla\psi\cdot\nabla\psi + a(Q)\psi^2 - pb(Q)v_Q\psi^2)\,dz = \int_{\mathbb{R}^n}\psi\,L_Q\psi\,dz$$
$$= \int_{\mathbb{R}^n}\sum_j c_{j,k}\phi_{j,k}L_Q\Big\{\sum_{j'} c_{j',k'}\phi_{j',k'}\Big\}\,dz = \int_{\mathbb{R}^n}\sum_j c_{j,k}\phi_{j,k}\sum_{j'} c_{j',k'}L_Q\phi_{j',k'}\,dz$$
$$= \int_{\mathbb{R}^n}\sum_j c_{j,k}\phi_{j,k}\Big\{\sum_{j'}(\mu_{j'} - p)b(Q)v_Q^{p-1}c_{j',k'}\phi_{j',k'}\Big\}\,dz$$
$$= \sum_{j,j'=1}^{\infty} c_{j,k}c_{j',k'}(\mu_{j'} - p)\int_{\mathbb{R}^n} b(Q)v_Q^{p-1}\phi_{j,k}\phi_{j',k'}\,dz,$$

where we note that $L_Q\phi_{j',k'} = (\mu_{j'} - p)b(Q)v_Q^{p-1}\phi_{j',k'}$. Recall that $\phi_{j,k}$ and $\phi_{j',k'}$ are orthogonal in L^2 with the weight $b(Q)v_Q^{p-1}$. Then, it follows that

$$\int_{\mathbb{R}^n} b(Q)v_Q^{p-1}\phi_{j,k}\phi_{j',k'}\,dz = \delta_{jj'}\delta_{kk'}\int_{\mathbb{R}^n} b(Q)v_Q^{p-1}\phi_{j,k}^2\,dz.$$

Therefore, by $c_{j,k} = 0$ for $j = 1, 2$, we obtain

$$Q(\psi,\psi) = \sum_j c_{j,k}^2(\mu_j - p)\int_{\mathbb{R}^n} b(Q)v_Q^{p-1}\phi_{j,k}^2\,dz$$
$$= \sum_{j\neq 1,2} c_{j,k}^2(\mu_j - p)\int_{\mathbb{R}^n} b(Q)v_Q^{p-1}\phi_{j,k}^2\,dz$$
$$= \sum_{j\neq 1,2}\Big(1 - \frac{p}{\mu_j}\Big)\mu_j\int_{\mathbb{R}^n} b(Q)v_Q^{p-1}(c_{j,k}\phi_{j,k})^2\,dz.$$

Now, from $\mu_3 \le \mu_j$ for $j \ge 3$, we see that

$$\begin{aligned}
\mathcal{Q}(\psi,\psi) &\ge \left(1-\frac{p}{\mu_3}\right)\sum_{j\neq 1,2}\int_{\mathbb{R}^n}\mu_j b(Q) v_Q^{p-1}(c_{j,k}\phi_{j,k})^2\,dz \\
&= \left(1-\frac{p}{\mu_3}\right)\sum_{j,j'}\int_{\mathbb{R}^n}\Big\{\mu_j b(Q) v_Q^{p-1}(c_{j,k}\phi_{j,k})\Big\}(c_{j',k'}\phi_{j',k'})\,dz \\
&= \left(1-\frac{p}{\mu_3}\right)\int_{\mathbb{R}^n}\sum_j\Big\{-\Delta(c_{j,k}\phi_{j,k}) + a(Q)(c_{j,k}\phi_{j,k})\Big\}\psi\,dz \\
&= \left(1-\frac{p}{\mu_3}\right)\int_{\mathbb{R}^n}(|\nabla\psi|^2 + a(Q)\psi^2)\,dz = \|\psi\|^2_{W^{1,2}(\mathbb{R}^n)}.
\end{aligned}$$

Noting that $p = \mu_2 < \mu_3$, we obtain $1 - p/\mu_3 > 0$. Consequently, there exists a positive constant $c_0 = 1 - p/\mu_3$ such that

$$\mathcal{Q}(\psi,\psi) \ge c_0\|\psi\|^2_{W^{1,2}(\mathbb{R}^n)} \quad \text{for any } \psi \in E.$$

Q.E.D.

We emphasize that c_0 depends on $Q \in \Omega$, but it is uniformly bounded away from zero on each compact subset since the eigenvalue μ_3 depends continuously on Q.

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