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Exact WKB approach to 2-level adiabatic transition problems with a small spectral gap

by

Takuya WATANABE

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Exact WKB approach to 2-level adiabatic transition problems with a small spectral gap

A thesis presented

by

Takuya Watanabe

to

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for the degree of

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Contents

Chapter 0

Introduction

Let us consider the time-dependent Schrödinger equation

$$ih\partial_t \psi_h(t) = H(t)\psi_h(t), \quad t \in \mathbb{R}, \tag{0.0.1}$$

on a separable Hilbert space $\mathscr{H}$ along the paper by [HJ]. The Hamiltonian $H(t)$ is a family of bounded self-adjoint operators on $\mathscr{H}$ depending smoothly on t, and h is a small positive parameter called *adiabatic* parameter.

We assume that the spectrum of $H(t)$, included in $\mathbb{R}$, is decomposed into two components $\sigma_1(t)$ and $\sigma_2(t)$ with a positive gap ε:

$$\operatorname{Spec} H(t) = \sigma_1(t) \cup \sigma_2(t), \quad \inf_{t\in\mathbb{R}} \operatorname{dist}(\sigma_1(t), \sigma_2(t)) = \varepsilon > 0. \tag{0.0.2}$$

Under this hypothesis, there exists a self-adjoint spectral projection $\mathscr{P}(t)$ corresponding to $\sigma_1(t)$, which depends smoothly on t.

For $t, s \in \mathbb{R}$, let $U_h(t,s)$ be the propagator associated to (0.0.1) :

$$ih\partial_t U_h(t,s) = H(t)U_h(t,s), \quad U_h(s,s) = \mathrm{Id}.$$

It is known that for all (t,s),

$$||(\mathrm{Id} - \mathscr{P}(t))U_h(t,s)\mathscr{P}(s)|| = O(h) \tag{0.0.3}$$

as h tends to 0. The left hand side is the *transition amplitude* between the subspaces $\mathscr{P}(s)\mathscr{H}$ and $(\mathrm{Id} - \mathscr{P}(t))\mathscr{H}$ and this fact means that the quantum evolution follows the isolated spectral subspaces of the Hamiltonian up to an error of order h. This is called the *Adiabatic Theorem of Quantum Mechanics*. It has first been studied by Born and Fock [BF] in 1928 for matrix valued Hamiltonians and then generalized to self-adjoint operators by Kato [K], Nenciu [N1] etc.

In the scattering regime, where the initial and final times s and t tend to $-\infty$ and $+\infty$ respectively, we can define, under appropriate assumptions on $H(t)$ at infinity, the *transition probability $P(h)$* by

$$P(h) = \lim_{\substack{t\to+\infty \\ s\to-\infty}} ||(\mathrm{Id} - \mathscr{P}(t))U_h(t,s)\mathscr{P}(s)||^2. \tag{0.0.4}$$

Assume that there exists $\mu > 0$ such that $H(t)$ is analytic in a strip $\{t \in \mathbb{C}\ ;\ |\mathrm{Im}t| \le \mu\}$ as an $\mathscr{L}(\mathscr{H})$-valued function, and that there exist $\nu > 1$, two bounded self-adjoint operators $H_\pm$ independent of t and a constant c such that $\sup_{|s|\le\mu} ||H(t+is) - H_\pm|| \le c(1+t^2)^{-\nu/2}$ as $t \to \pm\infty$. Under these conditions, Joye and Pfister showed in [JP1] that there exist $C > 0$ and $\Gamma > 0$ such that for sufficiently small h

$$P(h) \le Ce^{-\Gamma/h} \tag{0.0.5}$$

(see also [N2], [S], [JP2], [M] for similar results).

It is an important and interesting problem to optimize the constants C and Γ or more precisely to obtain an asymptotic formula of $P(h)$ as h tends to 0.

This problem is not trivial even in a special case where $H(t)$ is a 2×2 real symmetric matrix (hence $\mathscr{H} = \mathbb{C}^2$):

$$H(t) = \begin{pmatrix} V(t) & \varepsilon(t) \\ \varepsilon(t) & -V(t) \end{pmatrix} \tag{0.0.6}$$

In this case, however, the WKB method enables us to construct and connect WKB solutions in complex domains in $\mathbb{C}_t$ and to compute the asymptotic formula of the scattering matrix as well as of the transition probability.

The *turning points* and the *Stokes lines* emanating from these points play crucial roles in the WKB method. Turning points are the zeros of $-\det H(t) = V(t)^2 + \varepsilon(t)^2$ and also the crossing points on the complex t-plane between positive and negative eigenvalues $\pm\sqrt{-\det H(t)}$ of $H(t)$. Stoke lines are the curves which are level sets of the real part of the phase function of the WKB solution. WKB solutions are singular at turning points and the so-called *Stokes phenomenon* occurs across the Stokes lines emanating from turning points.

Joye, Kunz and Pfister showed the asymptotic formula

$$P(h) = Ge^{-\gamma/h}(1+O(h)) \qquad \text{as} \quad h \to 0 \tag{0.0.7}$$

under a geometrical condition on the Stokes lines emanating from the nearest complex conjugate pair of turning points $\{x_0, \bar{x}_0\}$ ([JKP]). The exponential decay rate γ is given by $\gamma = 2|\mathrm{Im}A_0|$, where $A_0 = 2\int_0^{x_0} \sqrt{-\det H(t)}\,dt$ is the action integral, and G is also a positive constant determined by the local behavior of $H(t)$ near x_0.

The simplest example is the Landau-Zener model. In the particular case $V(t) = at$ $(a > 0)$, $\varepsilon(t) \equiv \varepsilon$, it is possible to compute explicitly the transition probability $P(\varepsilon, h)$ by making use of the asymptotic formula at infinity of the Weber function, and it is given by

$$P(\varepsilon, h) = \exp\left[-\frac{\pi\varepsilon^2}{ah}\right],$$

for all $\varepsilon > 0$, $h > 0$ ([L], [Z], see also Chapter 5 Appendix). This is called *Landau-Zener formula*.

We can easily check the formula (0.0.7):

$$\gamma = 4\left|\mathrm{Im}\int_0^{i\varepsilon/a}\sqrt{a^2t^2+\varepsilon^2}\,dt\right| = 4\,\mathrm{Im}\left(\frac{i\varepsilon^2}{a}\int_0^1\sqrt{1-s^2}\,ds\right) = \frac{\pi\varepsilon^2}{a}.$$

But the Landau-Zener model suggests a more precise study of the asymptotic formula (0.0.7). In this model, the spectral gap is $2\sqrt{a^2t^2+\varepsilon^2}$ and the minimum, 2ε, is attained at $t=0$, the zero of $V(t)=at$. If ε tends to 0, the spectral gap tends to 0 and one expects that the transition probability increases. In fact, this model implies that the exponential decay of the transition probability remains true if and only if h tends to 0 faster than ε^2.

In this thesis, we mainly study the Hamiltonian (0.0.6) with constant interaction $\varepsilon(t)\equiv\varepsilon$:

$$H(t,\varepsilon) = \begin{pmatrix} V(t) & \varepsilon \\ \varepsilon & -V(t) \end{pmatrix},$$

and assume that the real function $V(t)$ vanishes at least at one point. Our problems are the followings:

1. *What is the principal term of the asymptotic expansion of $P(\varepsilon,h)$ with respect to h for sufficiently small ε ?*

2. *Is the error uniform with respect to small ε ?*

3. *When $V(t)$ vanishes at more than one real point, which zeros make a major contribution to the principal term of $P(\varepsilon,h)$?*

As we will see in the next section where we suppose $V(t)$ vanishes at the origin only, if $V'(0)\neq 0$ as in the Landau-Zener model, the same asymptotic formula as (0.0.7) holds with $G=1$ and the error is uniform with respect to small ε (Theorem 1.2.1). But if $V(t)$ vanishes to higher order, then two action integrals appear in the principal term and the error is no longer uniform (Theorem 1.2.2). In fact, if V vanishes to order n, then there exist n pairs of complex conjugate turning points tending to 0 as ε tends to 0. The global behavior of the solutions on the real time axis are governed by the Stokes lines emanating from the closest two pairs of turning points from the real axis. Moreover, the asymptotic behavior of the principal term with respect to ε and h depends on the higher order derivatives of V than n at $t=0$ (Proposition1.2.1).

In the case where $V(t)$ vanishes at more than one real point, we will see that turning points around the lowest order zero make a major contribution to the asymptotic behavior of $P(\varepsilon,h)$ as ε, as well as h, tends to 0 (Theorem 4.2.2, Theorem 4.2.3). This is a natural result expected from Joye's indication in [J1].

The analysis of the problems including such a parameter ε as well as h is very delicate. This results from great changes of the geometrical structures of Stokes lines when turning points converge to one point on $\mathbb{R}$ as ε tends to 0. The fundamental tool we use is the theory of the exact WKB analysis developed by C. Gérard and A. Grigis for Schrödinger equations [GG] and extended by S. Fujiié, C. Lasser and L. Nedelec to a family of first order 2×2 systems [FLN]. This method gives a convergent resummation to a divergent power series solution in h and enables us to express the Wronskian of two exact WKB solutions as a convergent series defined inductively by integrations along a path. In particular, thanks to the expression of the kernel of the inductive integrations, we can see to what extent the asymptotic behavior of that Wronskian with respect to h is valid when ε tends to 0.

The contents of this thesis is organized as follows: In Chapter 1, we state the assumptions and our results. To apply the exact WKB method we define the scattering matrix and the transition probability by Jost solutions (§1.1). We state, as the main results, the asymptotic expansion of $P(\varepsilon,h)$ as $h/\varepsilon^{(n+1)/n} \to 0$ for any small ε and the principal term of $P(\varepsilon,h)$ with respect to sufficiently small ε as well as h (§1.2).

In Chapter 2, we explain the exact WKB method for a 2×2 system of first order differential equation along the following contents. We first construct locally WKB solutions as formal series solutions (§2.1). For any fixed h we prove their convergence and give the Wronskian formula between two exact WKB solutions (§2.2). We show that the series of the exact WKB symbol function themselves are asymptotic expansions as h goes to 0 in some complex domain (§2.3). We introduce the Stokes line which characterizes such a domain and illustrate some of its local geometrical properties (§2.4).

Chapter 3 is devoted to the proof of our results by the exact WKB method. We reduce the connection problem between the Jost solutions to the local problem around the turning points near the origin (§3.1). We study the local geometrical structures of the Stokes lines (§3.2). We express the scattering matrix by the product of some transfer matrices around the turning points and study the asymptotic behavior of them (§3.3). We show how the turning points which converge at the origin as ε tends to 0 cause the failure of the estimate of Wronskian formula (§3.4). To study the Stokes geometry and the distance between turning points we calculate the expansion of the action integral with respect to small ε (§3.5).

In Chapter 4, we consider the case where $V(t)$ vanishes at more than one real point. The scattering matrix is expressed as the product of transfer matrices between turning points associated to these zeros (§4.1). We consider a special case where $V(t)$ vanishes at two points and study the contribution of the vanishing order to the asymptotic behavior of $P(\varepsilon,h)$ (§4.2).

In Chapter 5, we give a proof of Landau-Zener formula, which can be performed by an exact calculus using the asymptotic formulae at infinity of the Weber function.

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Chapter 1

Results

1.1 Scattering matrix and transition probability

We consider the time-dependent Schrödinger equation:

$$ih\frac{d}{dt}\psi(t) = H(t,\varepsilon)\psi(t), \qquad H(t,\varepsilon) = \begin{pmatrix} V(t) & \varepsilon \\ \varepsilon & -V(t) \end{pmatrix} \tag{1.1.1}$$

on $\mathbb{R}$, where ε and h are small positive parameters and $V(t)$ is a real-valued function. $\psi(t)$ is a vector-valued function with complex components. This 2×2 real symmetric and trace-free matrix $H(\varepsilon,h)$ has two real eigenvalues $E_\pm(t,\varepsilon) = \pm\sqrt{V(t)^2+\varepsilon^2}$. The difference of these eigenvalues

$$E_+(t,\varepsilon) - E_-(t,\varepsilon) = 2\sqrt{V(t)^2+\varepsilon^2}$$

is strictly positive for all $t\in\mathbb{R}$ and has its minimum 2ε at the zeros of $V(t)$.

First we define the scattering matrix and the transition probability. We consider the asymptotic solutions at infinity under the following assumptions on $V(t)$:

(A) $V(t)$ is real-valued on $\mathbb{R}$ and there exist two real numbers $0<\theta_0<\pi/2$ and $\mu>0$ such that $V(t)$ is analytic in the complex domain:

$$\mathscr{S} = \{t\in\mathbb{C}\ ;\ |\mathrm{Im}\,t| < |\mathrm{Re}\,t|\tan\theta_0\} \cup \{|\mathrm{Im}\,t|<\mu\}.$$

(B) There exist two real non-zero constants E_r, E_l and $\sigma>1$ such that

$$V(t) = \begin{cases} E_r + O(|t|^{-\sigma}) & \text{as } \mathrm{Re}\,t\to+\infty \quad \text{in } \mathscr{S}, \\ E_l + O(|t|^{-\sigma}) & \text{as } \mathrm{Re}\,t\to-\infty \quad \text{in } \mathscr{S}. \end{cases}$$

Under the conditions **(A)** and **(B)**, there exist four solutions ψ_+^r, ψ_-^r, ψ_+^l, and ψ_-^l to (1.1.1) uniquely defined by the following asymptotic conditions:

$$\begin{aligned}
\psi_+^r(t) &\sim \exp\left[+\frac{i}{h}\sqrt{E_r^2+\varepsilon^2}\,t\right]\begin{pmatrix} -\sin\theta_r \\ \cos\theta_r \end{pmatrix}, \quad \text{as } \operatorname{Re} t \to +\infty \text{ in } \mathscr{S}, \\
\psi_-^r(t) &\sim \exp\left[-\frac{i}{h}\sqrt{E_r^2+\varepsilon^2}\,t\right]\begin{pmatrix} \cos\theta_r \\ \sin\theta_r \end{pmatrix}, \quad \text{as } \operatorname{Re} t \to +\infty \text{ in } \mathscr{S}, \\
\psi_+^l(t) &\sim \exp\left[+\frac{i}{h}\sqrt{E_l^2+\varepsilon^2}\,t\right]\begin{pmatrix} -\sin\theta_l \\ \cos\theta_l \end{pmatrix}, \quad \text{as } \operatorname{Re} t \to -\infty \text{ in } \mathscr{S}, \\
\psi_-^l(t) &\sim \exp\left[-\frac{i}{h}\sqrt{E_l^2+\varepsilon^2}\,t\right]\begin{pmatrix} \cos\theta_l \\ \sin\theta_l \end{pmatrix}, \quad \text{as } \operatorname{Re} t \to -\infty \text{ in } \mathscr{S},
\end{aligned} \tag{1.1.2}$$

where $\tan 2\theta_r = \varepsilon/E_r$ and $\tan 2\theta_l = \varepsilon/E_l$ $(0 < \theta_r, \theta_l < \pi/2)$. These solutions are called the *Jost solutions* to (1.1.1). We notice that the principal term of each Jost solution, for example $\exp[+\frac{i}{h}\sqrt{E_r^2+\varepsilon^2}\,t]\,{}^t(-\sin\theta_r\ \cos\theta_r)$, is a solution to the system with constant coefficient:

$$\begin{aligned}
ih\frac{d}{dt}\psi(t) &= \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & -E_r \end{pmatrix}\psi(t) \\
&= R(\theta_r)\begin{pmatrix} \sqrt{E_r^2+\varepsilon^2} & 0 \\ 0 & -\sqrt{E_r^2+\varepsilon^2} \end{pmatrix} R(\theta_r)^{-1}\psi(t),
\end{aligned}$$

where $R(\theta_r)$ is the following matrix.

$$R(\theta_r) = \begin{pmatrix} \cos\theta_r & -\sin\theta_r \\ \sin\theta_r & \cos\theta_r \end{pmatrix}.$$

The pairs of Jost solutions (ψ_+^r, ψ_-^r) and (ψ_+^l, ψ_-^l) are orthonormal bases on $\mathbb{C}^2$ for any fixed t.

Definition 1.1.1. *The scattering matrix S is defined as the change of bases of Jost solutions:*

$$\begin{pmatrix} \psi_+^l & \psi_-^l \end{pmatrix} = \begin{pmatrix} \psi_+^r & \psi_-^r \end{pmatrix} S(\varepsilon, h), \qquad S(\varepsilon, h) = \begin{pmatrix} s_{11}(\varepsilon, h) & s_{12}(\varepsilon, h) \\ s_{21}(\varepsilon, h) & s_{22}(\varepsilon, h) \end{pmatrix}.$$

S is a unitary matrix independent of t. Moreover the Jost solutions have the relations:

$$\psi_\pm^r(t) = \mp\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\overline{\psi_\mp^r(t)}, \qquad \psi_\pm^l(t) = \mp\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\overline{\psi_\mp^l(t)}. \tag{1.1.3}$$

Hence we have

$$|s_{11}(\varepsilon, h)|^2 + |s_{21}(\varepsilon, h)|^2 = 1,$$

$$s_{11}(\varepsilon, h) = \overline{s_{22}(\varepsilon, h)}, \qquad s_{12}(\varepsilon, h) = -\overline{s_{21}(\varepsilon, h)}.$$

Definition 1.1.2. *The transition probability $P(\varepsilon, h)$ is defined by*

$$P(\varepsilon, h) = |s_{21}(\varepsilon, h)|^2 = |s_{12}(\varepsilon, h)|^2.$$

Remark 1.1.1. *This definition is equivalent to* (0.0.4)*, that is*

$$|s_{12}(\varepsilon,h)|^2 = \lim_{\substack{t\to+\infty \\ s\to-\infty}} ||(Id-\mathscr{P}(t))U_h(t,s)\mathscr{P}(s)||^2,$$

$$|s_{21}(\varepsilon,h)|^2 = \lim_{\substack{t\to+\infty \\ s\to-\infty}} ||\mathscr{P}(t)U_h(t,s)(Id-\mathscr{P}(s))||^2,$$

where, for any fixed $t \in \mathbb{R}$, $\mathscr{P}(t)$ is the projection corresponding to the eigenvalue $\sqrt{V(t)^2+\varepsilon^2}$.

1.2 Results

As we have seen in Introduction, the vanishing points of $V(t)$ on real axis are important for our problems with small spectral gap. Throughout this thesis we assume

(C) $V(t)$ vanishes at least at one point on $\mathbb{R}$.

In particular we suppose in this section and in Chapter 3 that

(C1) $V(t)$ vanishes only at the origin on $\mathbb{R}$.

Then the spectral gap $2\sqrt{V(t)^2+\varepsilon^2}$ attains its minimum 2ε at $t=0$ (avoided crossing). Instead the analytic extension of $V(t)^2+\varepsilon^2$ has complex zeros near $t=0$, which we call *turning points*.

Notice that we do not assume any condition on the order of zero at $t=0$. Let $n \in \mathbb{N} = \{1,2,\cdots\}$ be the number such that $V^{(k)}(0)=0$ for $0 \le k \le n-1$ and $V^{(n)}(0) \neq 0$. We assume $V^{(n)}(0) > 0$ without loss of generality. Then there are $2n$ simple turning points $x_j(\varepsilon)$ and $\overline{x_j(\varepsilon)}$ $(j=1,\ldots,n)$ which behave like

$$x_j(\varepsilon) \sim \left(\frac{n!}{V^{(n)}(0)}\right)^{1/n} \exp\left[\frac{(2j-1)\pi i}{2n}\right]\varepsilon^{1/n} \qquad \text{as} \quad \varepsilon \to 0. \tag{1.2.1}$$

We define the action integral $A_j(\varepsilon)$ by

$$A_j(\varepsilon) = 2\int_0^{x_j(\varepsilon)} \sqrt{V(t)^2+\varepsilon^2}\,dt,$$

where the integration path is the complex segment from 0 to $x_j(\varepsilon)$ and the branch of the square root is ε at $t=0$. We put $V(t) = \frac{V^{(n)}(0)}{n!}t^n v(t)$, where $v(t)$ is holomorphic in a neighborhood of $t=0$ and $v(0)=1$. Then we obtain the asymptotic behavior of $A_j(\varepsilon)$ with respect to the small parameter ε.

Lemma 1.2.1. *$A_j(\varepsilon)$ is an analytic function of $\varepsilon^{1/n}$ at $t=0$ and has the following Maclaurin expansion:*

$$A_j(\varepsilon) = \sum_{k=1}^{\infty} C_k \exp\left[\frac{(2j-1)k\pi i}{2n}\right]\varepsilon^{\frac{n+k}{n}},$$

where $C_k = \dfrac{\sqrt{\pi}\,\Gamma(\frac{k}{2n})}{(n+k)\,\Gamma(k)\,\Gamma(\frac{n+k}{2n})}\left(\dfrac{n!}{V^{(n)}(0)}\right)^{\frac{k}{n}}\left[\dfrac{d^{k-1}}{dz^{k-1}}\left(v(z)^{-\frac{k}{n}}\right)\right]_{z=0}.$

Our main results are the following asymptotic formulae of $P(\varepsilon,h)$ when both ε and h are small. In the case $n=1$, we recover the uniform Landau-Zener type formula which has been shown by Joye [J2, Theorem 2.1].

Theorem 1.2.1. *Assume* **(A)**, **(B)**, **(C1)**, *and* $n=1$. *Then there exists* $\varepsilon_0>0$ *such that we have*

$$P(\varepsilon,h)=\exp\left[-\frac{2\mathrm{Im}A_1(\varepsilon)}{h}\right](1+O(h)) \qquad \text{as} \quad h\to 0$$

uniformly for $\varepsilon\in(0,\varepsilon_0)$.

In the case where $n\geq 2$, we have the following formula which is valid when $h/\varepsilon^{\frac{n+1}{n}}\to 0$:

Theorem 1.2.2. *Assume* **(A)**, **(B)**, **(C1)**, *and* $n\geq 2$. *Then there exists* $\varepsilon_0>0$ *such that, for all* $\varepsilon\in(0,\varepsilon_0)$, *we have*

$$P(\varepsilon,h)=\left|\exp\left[\frac{i}{h}A_1(\varepsilon)\right]+(-1)^{n+1}\exp\left[\frac{i}{h}A_n(\varepsilon)\right]\right|^2\left(1+O\left(\frac{h}{\varepsilon^{\frac{n+1}{n}}}\right)\right) \qquad \text{as} \quad \frac{h}{\varepsilon^{\frac{n+1}{n}}}\to 0.$$

Remark 1.2.1. *Note that* $h/\varepsilon^{(n+1)/n}$ *appears in an obvious way in the case* $V(t)=t^n$. *By a simple rescaling* $t=\varepsilon^{1/n}\tau$, (1.1.1) *is reduced to*

$$i\frac{h}{\varepsilon^{(n+1)/n}}\frac{d}{d\tau}\phi(\tau)=\begin{pmatrix}\tau^n & 1\\ 1 & -\tau^n\end{pmatrix}\phi(\tau),$$

where $\psi(\varepsilon^{1/n}\tau)=\phi(\tau)$.

Let us try to express the principal term

$$P_0(\varepsilon,h)=\left|\exp\left[\frac{i}{h}A_1(\varepsilon)\right]+(-1)^{n+1}\exp\left[\frac{i}{h}A_n(\varepsilon)\right]\right|^2$$

for $n\geq 2$ in the form (0.0.7). We rewrite it as

$$\begin{aligned}P_0(\varepsilon,h)=\exp&\left[-\frac{\mathrm{Im}(A_1(\varepsilon)+A_n(\varepsilon))}{h}\right]\\ &\times\left(\exp\left[\frac{\mathrm{Im}(A_1(\varepsilon)-A_n(\varepsilon))}{h}\right]+\exp\left[\frac{\mathrm{Im}(A_n(\varepsilon)-A_1(\varepsilon))}{h}\right]\right.\\ &\left.\qquad+(-1)^{n+1}2\cos\left[\frac{\mathrm{Re}(A_1(\varepsilon)-A_n(\varepsilon))}{h}\right]\right).\end{aligned}$$

Then by computing the asymptotic expansions of the action integrals $A_1(\varepsilon)$ and $A_n(\varepsilon)$ we have the following proposition:

Proposition 1.2.1.
1) *If* $V^{(n+2l-1)}(0)=0$ *for all* $l\in\mathbb{N}$*, then*

$$\mathrm{Im}A_1(\varepsilon)=\mathrm{Im}A_n(\varepsilon)$$

and

$$P_0(\varepsilon,h)=4\left(\sin^2\left[\frac{\mathrm{Re}(A_1(\varepsilon)-A_n(\varepsilon))}{2h}+\frac{n}{2}\pi\right]\right)\exp\left[-\frac{2\mathrm{Im}A_1(\varepsilon)}{h}\right].$$

2) *If there exists* $m\in\mathbb{N}$ *such that* $V^{(n+2l-1)}(0)=0$ $(l=0,\dots,m-1)$ *and* $V^{(n+2m-1)}(0)\neq 0$*, then for sufficiently small* ε

$$\mathrm{Im}(A_1(\varepsilon)-A_n(\varepsilon))=2C_{2m}\left(\sin\frac{m}{n}\pi\right)\varepsilon^{\frac{n+2m}{n}}+O\left(\varepsilon^{\frac{n+2m+2}{n}}\right), \tag{1.2.2}$$

where

$$C_{2m}=-\frac{2m\sqrt{\pi}\,\Gamma(\frac{m}{n})V^{(n+2m-1)}(0)}{n\Gamma(n+2m+1)\Gamma(\frac{n+2m}{2n})}\left(\frac{n!}{V^{(n)}(0)}\right)^{\frac{n+2m}{n}},$$

and the asymptotic behavior of $P_0(\varepsilon,h)$ *as* $(\varepsilon,h)\to(0,0)$ *is given by the following formulae:*

(i) *When* $\varepsilon^{(n+2m)/n}/h\to 0$*,* $P_0(\varepsilon,h)$ *is equal to*

$$4\left(\sin^2\left[\frac{\mathrm{Re}(A_1(\varepsilon)-A_n(\varepsilon))}{2h}+\frac{n}{2}\pi\right]\right)\exp\left[-\frac{\mathrm{Im}(A_1(\varepsilon)+A_n(\varepsilon))}{h}\right]\left(1+O\left(\frac{\varepsilon^{\frac{2(n+2m)}{n}}}{h^2}\right)\right).$$

(ii) *When* $h/\varepsilon^{(n+2m)/n}\to 0$*,* $P_0(\varepsilon,h)$ *is equal to*

$$\exp\left[-\frac{2\mathrm{Im}A_1(\varepsilon)}{h}\right]\left(1+O\left(\exp\left[\left(2C_{2m}\left(\sin\frac{m}{n}\pi\right)+\delta\right)\frac{\varepsilon^{\frac{n+2m}{n}}}{h}\right]\right)\right) \tag{1.2.3}$$

for any positive constant δ *if* $\frac{m}{n}\notin\mathbb{N}$ *and* $V^{(n+2m-1)}(0)\sin\frac{m}{n}\pi>0$ $($*i.e.* $C_{2m}\sin\frac{m}{n}\pi<0)$ *and* $P_0(\varepsilon,h)$ *is equal to*

$$\exp\left[-\frac{2\mathrm{Im}A_n(\varepsilon)}{h}\right]\left(1+O\left(\exp\left[-\left(2C_{2m}\left(\sin\frac{m}{n}\pi\right)-\delta\right)\frac{\varepsilon^{\frac{n+2m}{n}}}{h}\right]\right)\right) \tag{1.2.4}$$

for any positive constant δ *if* $\frac{m}{n}\notin\mathbb{N}$ *and* $V^{(n+2m-1)}(0)\sin\frac{m}{n}\pi<0$ $($*i.e.* $C_{2m}\sin\frac{m}{n}\pi>0)$.

Remark 1.2.2. *From the viewpoint of the Stokes geometry, a Stokes line emanating from* x_1 *is connected to one emanating from* x_n *in Case* 1)*, whereas it is not connected in Case* 2) (see §3.3 Figure 3.9, 3.10, 3.11.).

Chapter 2

Exact WKB method

2.1 Formal construction

We use as a basic tool the *exact WKB method* for 2×2 systems introduced in [FLN], which is a natural extension of the method in [GG] for Schrödinger equations. We first review it.

Let us consider the following $2{\times}2$ system of first order differential equations:

$$\frac{h}{i}\frac{d}{dt}\phi(t) = \begin{pmatrix} 0 & \alpha(t) \\ -\beta(t) & 0 \end{pmatrix}\phi(t). \tag{2.1.1}$$

The functions $\alpha(t)$ and $\beta(t)$ are assumed to be holomorphic in a simply connected domain $\Omega \subset \mathbb{C}$. Notice that any 2×2 symmetric system:

$$\frac{h}{i}\frac{d}{dt}\psi(t) = \begin{pmatrix} X(t) & Z(t) \\ Z(t) & Y(t) \end{pmatrix}\psi(t) \tag{2.1.2}$$

can be reduced to this anti-diagonal system (2.1.1) by $\psi(t) \mapsto \exp\left[\frac{i}{h}\int^t \frac{X(\tau)+Y(\tau)}{2}d\tau\right]\varphi(t)$ and $\varphi(t) \mapsto \frac{1}{2}\left(\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right)\phi(t)$. The first transformation reduces (2.1.2) to a trace-free system and the second to an anti-diagonal system. We also remark that, when $\alpha(t) = 1$ and $\beta(t) = V(t) - E$, (2.1.1) is equivalent to the Schrödinger equation: $-h^2\phi_1''(t) + (V(t) - E)\phi_1(t) = 0$, where $\phi(t) =^t (\phi_1(t), \phi_2(t))$.

First of all we make the change of variables $t \mapsto z$

$$z(t;t_0) = \int_{t_0}^{t} \sqrt{\alpha(\tau)\beta(\tau)}d\tau,$$

where t_0 is a fixed base point of Ω. If Ω_1 is a simply connected open subset of Ω in which $\alpha(t)\beta(t)$ does not vanish, the mapping z is bijective from Ω_1 to $z(\Omega_1)$ for a given determination of $(\alpha(t)\beta(t))^{1/2}$. Zeros of $\alpha(t)$ and $\beta(t)$ are called *turning points*. If $t = x$ is a simple turning point, we get

$$z(t) - z(x) = \frac{2i}{3}\left(\alpha(t)\beta(t)\right)'\Big|_{t=x}(t-x)^{\frac{3}{2}}\left(1 + g(t-x)\right), \tag{2.1.3}$$

where $g(t)$ is holomorphic near $t=0$ and $g(0)=0$.

We put $\phi(t) = e^{\pm z/h}\varphi_\pm(z)$ and reduce (2.1.1) to the next equation in the variable z:

$$\frac{h}{i}\frac{d}{dz}\varphi_\pm(z) = \begin{pmatrix} \pm i & K(z)^{-2} \\ -K(z)^2 & \pm i \end{pmatrix}\varphi_\pm(z), \tag{2.1.4}$$

where $K(z(t)) = \big(\beta(t)/\alpha(t)\big)^{1/4}$. We change unknown functions $\varphi_\pm(z) = M_\pm(z)w_\pm(z)$ with

$$M_\pm(z) = \begin{pmatrix} K(z)^{-1} & K(z)^{-1} \\ \mp iK(z) & \pm iK(z) \end{pmatrix}.$$

Consequently, we obtain the first order differential equation of $w_\pm(z)$:

$$\frac{d}{dz}w_\pm(z) = \begin{pmatrix} 0 & \frac{K'(z)}{K(z)} \\ \frac{K'(z)}{K(z)} & \mp\frac{2}{h} \end{pmatrix} w_\pm(z), \tag{2.1.5}$$

where $K'(z)$ stands for $\frac{d}{dz}K(z)$. We notice that $M_\pm(z(t))$ and $w_\pm(z(t))$ are independent of t_0.

We define the sequences of functions $\{w_{\pm,n}(z;z_1)\}_{n=0}^{\infty}$ by the following differential recurrent relations:

$$\begin{cases} w_{\pm,-1}(z) = 0, \qquad w_{\pm,0}(z) = 1, \\ \dfrac{d}{dz}w_{\pm,2k}(z) = \dfrac{K'(z)}{K(z)}w_{\pm,2k-1}(z) \qquad (k \geq 0), \\ \left(\dfrac{d}{dz} \pm \dfrac{2}{h}\right) w_{\pm,2k+1}(z) = \dfrac{K'(z)}{K(z)}w_{\pm,2k}(z) \qquad (k \geq 0). \end{cases} \tag{2.1.6}$$

The vector-valued functions $w_\pm(z(t)) = \begin{pmatrix} w_\pm^e(z(t)) \\ w_\pm^o(z(t)) \end{pmatrix}$ with

$$w_\pm^e(z(t)) = \sum_{k\geq 0} w_{\pm,2k}(z(t)), \qquad w_\pm^o(z(t)) = \sum_{k\geq 0} w_{\pm,2k-1}(z(t)),$$

satisfy (2.1.5) formally. Thus we get formal solutions to (2.1.1):

$$\phi_\pm(t,h;t_0) = e^{\pm z(t;t_0)}M_\pm(z(t))\sum_{k\geq 0}\begin{pmatrix} w_{\pm,2k}(z(t)) \\ w_{\pm,2k-1}(z(t)) \end{pmatrix}. \tag{2.1.7}$$

2.2 Convergence and Wronskian formula

In this section we show the convergence of symbol function $w_\pm(z(t))$ and define the exact WKB solution as the exact solution to (2.1.1). Moreover we give the Wronskian formula between two exact WKB solutions.

The singularities of $w_\pm(z(t))$ appear in $K'(z)/K(z)$. Actually $K'(z)/K(z)$ is expressed by, in terms of t,

$$\frac{\frac{d}{dz}K(z(t))}{K(z(t))} = \frac{\alpha(t)\beta'(t)-\alpha'(t)\beta(t)}{4(\alpha(t)\beta(t))^{3/2}}. \tag{2.2.1}$$

From (2.1.3) and (2.2.1), we see that $K'(z)/K(z)$ has a simple pole at $z = z(x)$ if x is a simple turning point.

We fix a point $z_1 = z(t_1)$ with $t_1 \in \Omega_1$ and take the initial conditions to $w_{\pm,n}(z_1) = 0$ for every $n \in \mathbb{N}$. Then the differential recurrent equations (2.1.6) are transformed to the integral recurrent relations:

$$\begin{cases} w_{\pm,0}(z;z_1) = 1, \\ w_{\pm,2k+1}(z;z_1) = \displaystyle\int_{z_1}^{z} e^{\pm\frac{2}{\hbar}(\zeta-z)}\frac{K'(\zeta)}{K(\zeta)} w_{\pm,2k}(\zeta;z_1)\,d\zeta \quad (k \geq 0), \\ w_{\pm,2k}(z;z_1) = \displaystyle\int_{z_1}^{z} \frac{K'(\zeta)}{K(\zeta)} w_{\pm,2k-1}(\zeta;z_1)\,d\zeta \qquad (k \geq 1). \end{cases} \tag{2.2.2}$$

From these integral representations, we obtain the following proposition on the convergence of these formal series.

Proposition 2.2.1. *The elements of the function $w_\pm(z;z_1)$:*

$$w^e_\pm(z;z_1) = \sum_{k\geq 0} w_{\pm,2k}(z;z_1), \qquad w^o_\pm(z;z_1) = \sum_{k\geq 0} w_{\pm,2k-1}(z;z_1) \tag{2.2.3}$$

converge absolutely and uniformly in a neighborhood of $z = z_1$.

Proof of Proposition 2.2.1 (2.2.2) can be written as

$$\begin{cases} w_{\pm,0}(z;z_1) &= 1, \\ w_{\pm,2k+1}(z;z_1) &= I_\pm[w_{\pm,2k}](z;z_1), \\ w_{\pm,2k}(z;z_1) &= J[w_{\pm,2k-1}](z;z_1), \end{cases} \tag{2.2.4}$$

where $I_\pm$ and J are integral operators defined by

$$I_\pm[f](z;z_1) = \int_{z_1}^{z} e^{\pm\frac{2}{\hbar}(\zeta-z)}\frac{K'(\zeta)}{K(\zeta)} f(\zeta)\,d\zeta, \tag{2.2.5}$$

$$J[f](z;z_1) = \int_{z_1}^{z} \frac{K'(\zeta)}{K(\zeta)} f(\zeta)\,d\zeta. \tag{2.2.6}$$

The convergence follows from the following lemma.

Lemma 2.2.1. *For all $z \in z(\Omega_1)$ there exists a finite curve $\Gamma(z;z_1)$ on complex z-plane which has the start point z_1 and the end point z. Put*

$$\max\left\{\sup_{\zeta\in\Gamma(z;z_1)}\left|e^{\pm\frac{2}{\hbar}(\zeta-z)}\frac{K'(\zeta)}{K(\zeta)}\right|, \sup_{\zeta\in\Gamma(z;z_1)}\left|\frac{K'(\zeta)}{K(\zeta)}\right|\right\} = A < \infty$$

and let L be the length of $\Gamma(z;z_1)$. Then there exists a positive constant C such that we have for all $n \in \mathbb{N}$

$$\left|w_{\pm,n}(z;z_1)\right| \leq \frac{C(AL)^n}{n!}. \tag{2.2.7}$$

Proof of Lemma 2.2.1 We prove this lemma by induction. In the case where $n = 0$, the statement is evident because $w_{\pm,0}(z;z_1) = 1$. We suppose the inequality (2.2.7) for n. In the case where n is even,

$$\left|w_{\pm,2k+1}(z;z_1)\right| = \left|\int_{\Gamma(z;z_1)} e^{\pm\frac{2}{\hbar}(\zeta-z)}\frac{K'(\zeta)}{K(\zeta)}w_{\pm,2k}(\zeta;z_1)\,d\zeta\right|.$$

We introduce an arc length parameter ξ to the integral path ζ.

$$\begin{aligned}\left|w_{\pm,2k+1}(z;z_1)\right| &= \left|\int_0^L e^{\pm\frac{2}{\hbar}(\zeta(\xi)-z)}\frac{K'(\zeta(\xi))}{K(\zeta(\xi))}w_{\pm,2k}(\zeta(\xi);z_1)\frac{d\zeta(\xi)}{d\xi}d\xi\right| \\ &\leq \int_0^L\left|e^{\pm\frac{2}{\hbar}(\zeta(\xi)-z)}\frac{K'(\zeta(\xi))}{K(\zeta(\xi))}\right|\left|w_{\pm,2k}(\zeta(\xi;z_1))\right|d\xi \\ &\leq A\int_0^L\left|w_{\pm,2k}(\zeta(\xi;z_1))\right|d\xi.\end{aligned}$$

With the assumption (2.2.7), we have $\left|w_{\pm,2k}(\zeta(\xi);z_1))\right| \leq \dfrac{C(A\xi)^{2k}}{(2k)!}$ in the arc length parameter ξ.

$$\left|W_{\pm,2k+1}(z;z_1)\right| \leq A\int_0^L\frac{C(A\xi)^{2k}}{(2k)!}d\xi = \frac{C(AL)^{2k+1}}{(2k+1)!}.$$

In the case where n is odd, we get similarly

$$\left|W_{\pm,2k+2}(z;z_1)\right| \leq \frac{C(AL)^{2k+2}}{(2k+2)!}.$$

Therefore we obtain the inequality (2.2.7).

□

Since the majorant series $\sum_{k=0}^{\infty} \frac{C(AL)^{2k}}{(2k)!}$ and $\sum_{k=0}^{\infty} \frac{C(AL)^{2k+1}}{(2k+1)!}$ are convergent, the formal series $\sum_{k=0}^{\infty} w_{\pm,2k}(\zeta(\xi);z_1))$ and $\sum_{k=0}^{\infty} w_{\pm,2k+1}(\zeta(\xi);z_1))$ converge absolutely and uniformly in a neighborhood of $z = z_1$.

□

This proposition claims that $w_\pm(z;z_1)$ are exact solutions to the equation (2.1.5). Let us define, for $t_0 \in \Omega$ and $t_1 \in \Omega_1$,

$$\phi_\pm(t,h;t_0,t_1) = e^{\pm z(t;t_0)/h} M_\pm(z(t)) w_\pm(z(t),h;z(t_1)), \tag{2.2.8}$$

then these are exact solutions to (2.1.1). We call $\phi_\pm(t,h;t_0,t_1)$ *exact WKB solutions*. The exact WKB solutions (2.2.8) are holomorphic in a neighborhood of $t = t_1$, and extended analytically to Ω because (2.2.8) satisfy (2.1.1) with the holomorphic coefficients in Ω. We call t_0 the base point of the phase and t_1 the base point of the symbol. We remark that the pair of exact WKB solutions $\phi_+(t,h;t_0,t_1)$, $\phi_-(t,h;t_0,t_1)$ are linearly independent.

The Wronskian between two exact WKB solutions $[\phi(t),\tilde{\phi}(t)] = \det\big(\phi(t)\ \tilde{\phi}(t)\big)$ is given by w_+^e:

Proposition 2.2.2. *Any exact WKB solutions $\phi_+(t,h;t_0,t_1)$ and $\phi_-(t,h;t_0,t_2)$ with the same base point t_0 of the phase satisfy the following Wronskian formula:*

$$[\phi_+(t,h;t_0,t_1),\phi_-(t,h;t_0,t_2)] = 2i\, w_+^e(z(t_2);z(t_1)).$$

Proof of Proposition 2.2.2.

$$\begin{aligned}
[\phi_+(t;t_0,t_1),\phi_-(t;t_0,t_2)] &= [e^{+z/h}M_+(z(t))w_+(z(t);z(t_1)), e^{-z/h}M_-(z(t))w_-(z(t);z(t_2))] \\
&= [M_+(z(t))w_+(z(t);z(t_1)), M_-(z(t))w_-(z(t);z(t_2))] \\
&= [M_+(z(t))w_+(z(t);z(t_1)), M_+(z(t))\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)w_-(z(t);z(t_2))] \\
&= \det M_+(z(t)) \det \begin{pmatrix} w_+^e(z(t);z(t_1)) & w_-^o(z(t);z(t_2)) \\ w_+^o(z(t);z(t_1)) & w_-^e(z(t);z(t_2)) \end{pmatrix} \\
&= 2i\big(w_+^e(z(t);z(t_1))w_-^e(z(t);z(t_2)) - w_+^o(z(t);z(t_1))w_-^o(z(t);z(t_2))\big) \\
&= 2i\, w_+^e(z(t_2);z(t_1)).
\end{aligned}$$

We notice that the Wronskian is independent of the variable t because the matrix of right side of (2.1.1) is trace-free.

□

2.3 Asymptotic property

In this section, we show that the convergent series (2.2.3) of the function $w_\pm(z(t),h;z(t_1))$ constructed in the previous section are also asymptotic expansions on h in the domains:

$$\Omega_\pm = \Big\{ t \in \Omega_1; \text{there exists a curve from } t_1 \text{ to } t \text{ along which } \pm\mathrm{Re}\, z(t) \text{ increases strictly} \Big\}.$$

Proposition 2.3.1. *There exist a positive integer N and a positive constant h_0 such that, for all $h \in (0,h_0)$, we have*

$$w^e_\pm(z(t),h;z(t_1)) - \sum_{k=0}^{N-1} w_{\pm,2k}(z(t),h;z(t_1)) \quad = O\left(h^N\right), \tag{2.3.1}$$

$$w^o_\pm(z(t),h;z(t_1)) - \sum_{k=0}^{N-1} w_{\pm,2k-1}(z(t),h;z(t_1)) = O\left(h^N\right), \tag{2.3.2}$$

uniformly in $\Omega_\pm$.

Proof of Proposition 2.3.1 To prove the asymptotic expansion (2.3.1), we show the inequality:

$$\left| w_{+,2k}(z;z_1) \right| \le Ch^k,$$

where C is some positive constant. Let $\| \cdot \|$ be the norm defined by

$$\| f \| = \sup_{\zeta \in \Gamma(z;z_1)} |f(\zeta)| \, + \, h \sup_{\zeta \in \Gamma(z;z_1)} |f'(\zeta)|,$$

where $\Gamma(z;z_1)$ is the same curve on the complex z-plane as Lemma 2.2.1. Similarly we put

$$\max \left\{ \sup_{\zeta \in \Gamma(z;z_1)} \left| \frac{K'(\zeta)}{K(\zeta)} \right|, \ \sup_{\zeta \in \Gamma(z;z_1)} \left| \frac{d}{d\zeta} \frac{K'(\zeta)}{K(\zeta)} \right| \right\} = A < \infty.$$

We estimate $\| I_+[f](h) \|$ in terms of $\| f \|$.

$$I_+[f](z,h;z_1) = \int_{z_1}^{z} e^{+\frac{2}{h}(\zeta - z)} \frac{K'(\zeta)}{K(\zeta)} f(\zeta) d\zeta.$$

We put $g(\zeta) = \frac{K'(\zeta)}{K(\zeta)} f(\zeta)$ and change the variables $s = \frac{\zeta - z}{h}$. Then we have

$$I_+[f](z,h;z_1) = h \int_{\frac{z_1 - z}{h}}^{0} e^{2s} g(hs+z) ds.$$

In developing $g(hs+z)$ in the neighborhood of $s=0$,

$$\begin{aligned} I_+[f](z,h;z_1) &= h \int_{\frac{z_1 - z}{h}}^{0} e^{2s} \Big(g(z) + hs \int_0^1 g'(hsy+z) dy \Big) ds \\ &= \frac{hg(z)}{2} \left(1 - e^{\frac{2}{h}(z_1 - z)} \right) + h^2 \int_{\frac{z_1 - z}{h}}^{0} se^{2s} \Big(\int_0^1 g'(hsy+z) dy \Big) ds. \end{aligned}$$

We estimate the absolute value of each term of the right-hand side.

$$\begin{aligned}\left|\frac{hg(z)}{2}\left(1-e^{\frac{2}{h}(z_1-z)}\right)\right| &\leq \frac{h}{2}\sup_{\zeta\in\Gamma(z;z_1)}|g(z)|\left|1-e^{\frac{2}{h}(z_1-z)}\right| \\ &\leq \frac{Ah}{2}\sup|f|\left|1-e^{\frac{2}{h}(z_1-z)}\right|.\end{aligned}$$

$$\begin{aligned}\left|h^2\int_{\frac{z_1-z}{h}}^{0} se^{2s}\Big(\int_0^1 g'(hsy+z)dy\Big)ds\right| &\leq h^2\left|\int_{\frac{z_1-z}{h}}^{0} se^{2s}\sup_{\zeta\in\Gamma(z;z_1)}|g'(\zeta)|ds\right| \\ &\leq h^2\sup\left|\left(\frac{d}{d\zeta}\frac{K'(\zeta)}{K(\zeta)}\right)f(\zeta)+\frac{K'(\zeta)}{K(\zeta)}f'(\zeta)\right|\left|\int_{\frac{z_1-z}{h}}^{0} se^{2s}ds\right| \\ &\leq Ah^2\left(\sup|f|+\sup|f'|\right)\left|\frac{1}{4}+\frac{2(z_1-z)-h}{4h}e^{\frac{2}{h}(z_1-z)}\right|.\end{aligned}$$

Hence we get

$$\begin{aligned}\Big|I_+[f](z,h;z_1)\Big| \leq \frac{Ah}{2}\sup|\,f\,|\left|1-e^{\frac{2}{h}(z_1-z)}\right| \\ +\frac{Ah^2}{4}\left(\sup|f|+\sup|f'|\right)\left|1+\left(\frac{2(z_1-z)}{h}-1\right)e^{\frac{2}{h}(z_1-z)}\right|.\end{aligned}$$

From $t\in\Omega_+$, we take a curve from t_1 to t along which $\mathrm{Re}\,z(t)$ increases strictly as the integral path, then $\exp[\frac{2}{h}(z_1-z)]$ decay exponentially as h goes to 0. We obtain

$$\Big|I_+[f](z,h;z_1)\Big| \leq Ch\parallel f\parallel,$$

where C is some positive constant.

We estimate $I'_+[f](z,h;z_1)$ in terms of $\parallel f\parallel$.

$$\begin{aligned}I'_+[f](z,h;z_1) &= \frac{d}{dz}\Big(e^{-\frac{2}{h}z}\int_{z_1}^{z} e^{\frac{2}{h}\zeta}\frac{K'(\zeta)}{K(\zeta)}f(\zeta)\,d\zeta\Big) \\ &= -\frac{2}{h}I_+[f](z,h;z_1)+\frac{K'(z)}{K(z)}f(z).\end{aligned}$$

$$\begin{aligned}\Big|I'_+[f](z,h;z_1)\Big| &\leq \frac{2}{h}\Big|I_+[f](z,h;z_1)\Big|+\left|\frac{K'(z)}{K(z)}f(z)\right| \\ &\leq 2C\parallel f\parallel+A\sup|f|.\end{aligned}$$

We can calculate $\| I_+[f] \|$ as follows.

$$\begin{aligned} \| I_+[f] \| &= \sup |I_+[f]| + h \sup |I'_+[f]| \\ &\leq Ch \| f \| + h \cdot 2C \| f \| + h \cdot A \sup |f| \\ &\leq Ch \| f \| . \end{aligned}$$

One sees that the integral operator I_+ is the operator of order h.

We study $\| J[f] \|$. We first estimate $|J[f](z;z_1)|$ and $|J'[f](z;z_1)|$.

$$\begin{aligned} \Big| J[f](z;z_1) \Big| &= \left| \int_{z_1}^{z} \frac{K'(\zeta)}{K(\zeta)} f(\zeta)\, d\zeta \right| \\ &\leq A \sup |f| \left| \int_{z_1}^{z} d\zeta \right| \leq C \sup |f|. \\ \Big| J'[f](z;z_1) \Big| &= \left| \frac{K'(z)}{K(z)} f(z) \right| \leq A \sup |f|. \end{aligned}$$

Hence one has

$$\begin{aligned} \| J[f] \| &= \sup |J[f]| + h \sup |J'[f]| \\ &\leq C \sup |f| + hA \sup |f| \leq C \| f \| . \end{aligned}$$

One also sees that the integral operator J is the operator of order 1. From the integral recurrent equations (2.2.4), we have

$$\| w_{+,2k} \| = \| J[w_{+,2k-1}] \| = \| JI_+[w_{+,2(k-1)}] \| = \| (JI_+)^k \| \leq Ch^k.$$

$$\left\| w_+^e - \sum_{n=0}^{N} w_{+,2k} \right\| = \left\| \sum_{n=N+1}^{\infty} w_{+,2n} \right\| \leq C h^{N+1} = O\left(h^{N+1}\right).$$

Hence we obtain the asymptotic expansion (2.3.1). About the asymptotic expansion (2.3.2), we consider for $t \in \Omega_-$ the integral path along which $-\mathrm{Re}\, z(t)$ increases strictly, and then we have it similarly.

□

2.4 Turning points and Stokes lines

In this section, we introduce the so-called *Stokes line*, which characterizes the asymptotic behavior of the exact WKB solution as h tends to 0. In particular we state some properties of the Stokes lines passing through turning points.

Definition 2.4.1 (Stokes line)**.** *The Stokes lines passing through $t = t_0$ in Ω are defined as the set:*

$$\left\{ t \in \Omega \; ; \; \mathrm{Re} \int_{t_0}^{t} \sqrt{\alpha(\tau)\beta(\tau)}\, d\tau = 0 \right\}.$$

A Stokes line is a level set of the real part of the WKB phase function $z(t;t_0)$.

If $\mathrm{Re}\, z(t)$ strictly increases along an oriented curve, such a curve is called *canonical curve*. In fact a canonical curve is transversal to Stokes lines. We can characterize the asymptotic behavior of the Wronskian between the linearly independent exact WKB solutions in terms of canonical curve.

Proposition 2.4.1. *If there exists a canonical curve from t_1 to t_2,*

$$[\phi_+(t,h;t_0,t_1), \phi_-(t,h;t_0,t_2)] = 2i\,(1 + O(h)) \qquad \text{as} \quad h \to 0.$$

Proof of Proposition 2.4.1 From Proposition 2.2.2, we have

$$[\phi_+(t,h;t_0,t_1), \phi_-(t,h;t_0,t_2)] = 2i\, w_+^e(z(t_2); z(t_1)).$$

Thanks to the existence of a canonical curve we apply Proposition 2.3.1, and then we obtain

$$[\phi_+(t,h;t_0,t_1), \phi_-(t,h;t_0,t_2)] = 2i\,(1 + O(h)) \qquad \text{as} \quad h \to 0.$$

□

Let us consider some geometrical local properties of Stokes lines. We state the local properties of Stokes lines near a fixed point $t_0 \in \Omega$.

(i) If t_0 is not a turning point, then $z(t;t_0)$ is conformal near $t = t_0$.

(ii) If t_0 is a turning point of order $r \in \mathbb{N}$, that is $\alpha(t)\beta(t) = (t - t_0)^r \gamma(t - t_0)$ with $\gamma(0) \neq 0$, then there exist $r + 2$ Stokes lines emanating from $t = t_0$ and every angle between two closest Stokes lines is $2\pi/(r+2)$ at $t = t_0$.

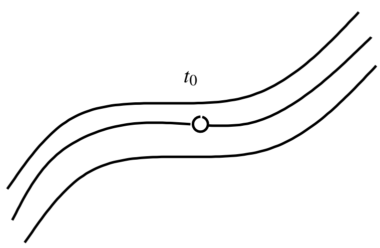

Figure 2.1: Regular

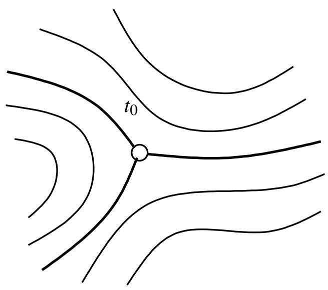

Figure 2.2: Simple

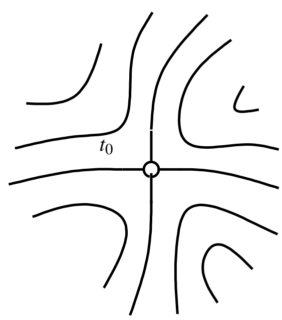

Figure 2.3: Double

Chapter 3

Connection of the exact WKB solutions

In this chapter, we calculate the asymptotic behavior of the scattering matrix $S(\varepsilon,h)$ and prove Theorem 1.2.1 and Theorem 1.2.2 making use of the exact WKB method of the previous chapter.

Recall that $S(\varepsilon,h)$ is the change of bases between Jost solutions at $-\infty$ and at $+\infty$. Hence its elements are expressed by Wronskians of Jost solutions. In order to apply the Wronskian formula Proposition 2.2.2, we first represent Jost solutions as exact WKB solutions (§3.1). According to Proposition 2.2.2, we know the asymptotic behavior of the Wronskian of two exact WKB solution only when there exists a canonical curve between the symbol base points. To investigate the existence of such a curve, we should know the global Stokes geometry near the real axis (§3.2). In general, there is no canonical curve connecting directly $-\infty$ and $+\infty$, and we should take some intermediate points so that we can find a canonical curve from one point to another. Then the scattering matrix is written as product of transfer matrices between exact WKB solutions which have their symbol base point at these intermediate points (§3.3). Looking carefully at the distance between the canonical curves and turning points, we will see to what extent the asymptotic formulae with respect to h are valid when ε tends to 0 §3.4. Finally in §3.5 we show the asymptotic behavior of the action integral (Lemma 1.2.1) to prove Proposition 1.2.1.

3.1 WKB expression of the Jost solutions

We express the Jost solutions as exact WKB solutions to (1.1.1). By the change of the unknown function $\psi(t) = Q\phi(t)$, $Q = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$, (1.1.1) is reduced to an equation of the anti-diagonal form (2.1.1):

(3.1.1)
$$\frac{h}{i}\frac{d}{dt}\phi(t) = \begin{pmatrix} 0 & -iV(t)-\varepsilon \\ iV(t)-\varepsilon & 0 \end{pmatrix}\phi(t).$$

The phase function $z(t;t_0)$ is

$$z(t;t_0) = i\int_{t_0}^{t} \sqrt{V(\tau)^2+\varepsilon^2}\,d\tau \qquad (t_0 \in \mathscr{S}). \tag{3.1.2}$$

Fix an $\varepsilon > 0$. Then there is no turning point in a neighborhood of the real axis $\mathbb{R}_t$. Hence $z(t;t_0)$ is a single-valued function there. Notice that $\mathbb{R}_t$ is itself a Stokes line. Recalling that we take the branch of $\sqrt{V(t)^2+\varepsilon^2}$ which is ε at $t=0$, we see that $\operatorname{Re} z(t)$ increases as $\operatorname{Im} t$ decreases, and $\operatorname{Im} z(t)$ increases as $\operatorname{Re} t$ increases. Similarly

$$K(z(t)) = \sqrt[4]{\frac{-iV(t)+\varepsilon}{-iV(t)-\varepsilon}}$$

has neither zero nor pole there and the branch of $K(z(t))$ is $e^{\pi i/4}$ at $t=0$.

We construct the exact WKB solutions which have the same behavior as Jost solutions as $|t| \to \infty$ as in [Ra]. Let $\mathscr{S}_R^r$, $\mathscr{S}_R^l$ be the unbounded simply connected domains

$$\mathscr{S}_R^r = \mathscr{S} \cap \{t \in \mathbb{C};\, \operatorname{Re} t > R\},$$
$$\mathscr{S}_R^l = \mathscr{S} \cap \{t \in \mathbb{C};\, \operatorname{Re} t < -R\},$$

for a positive constant R. For $t \in \mathscr{S}_R^r$ [resp. $t \in \mathscr{S}_R^l$], we define the phase functions $z^r(t)$ [resp. $z^l(t)$] with base points at infinity by

$$z^r(t) = i\int_{\infty}^{t}\left(\sqrt{V(\tau)^2+\varepsilon^2}-\lambda_r\right)d\tau + i\lambda_r t,$$

$$\left[\text{resp. } z^l(t) = i\int_{-\infty}^{t}\left(\sqrt{V(\tau)^2+\varepsilon^2}-\lambda_l\right)d\tau + i\lambda_l t,\right]$$

where $\lambda_{r,l} = \sqrt{E_{r,l}^2+\varepsilon^2}$. Note that these integrals are convergent thanks to the assumption **(B)**. These are also primitives of $i\sqrt{V(t)^2+\varepsilon^2}$ and satisfy for any $t_0 \in \mathscr{S}_R^{r,l}$

$$z^{r,l}(t) = z^{r,l}(t_0) + z(t;t_0). \tag{3.1.3}$$

Next we construct the symbol functions with base points at infinity. One sees that for all $t \in \mathscr{S}_R^l$, there exist infinite paths ending at t, $\gamma_\pm^l(t)$, which are asymptotic to the line $\operatorname{Im}\tau = \mp\delta \operatorname{Re}\tau\,(\delta > 0)$ as $\operatorname{Re}\tau \to -\infty$ and meet the Stokes line transversally. (Stokes lines are asymptotic to horizontal lines. See §2.4 and Figure 3.1.) For $t \in \mathscr{S}_R^r$ one similarly defines the paths $\gamma_\pm^r(t)$ which are asymptotic to the lines $\operatorname{Im}\tau = \pm\delta\operatorname{Re}\tau$ as $\operatorname{Re}\tau \to +\infty$.

We also denote by $\Gamma_\pm^r(z)$ [resp. $\Gamma_\pm^l(z)$] the infinite oriented paths $z^r(\gamma_\pm^r(t))$ [resp. $z^l(\gamma_\pm^l(t))$] ending at $z^r(t)$ [resp. $z^l(t)$]. We remark that $\Gamma_-^r(z)$ [resp. $\Gamma_+^l(z)$] is asymptotic to the line $\operatorname{Im}\zeta = \frac{1}{\delta}\operatorname{Re}\zeta$ as $\operatorname{Re}\zeta \to +\infty$ [resp. $\operatorname{Re}\zeta \to -\infty$], and similarly that $\Gamma_+^r(z)$ [resp. $\Gamma_-^l(z)$] is asymptotic to the line $\operatorname{Im}\zeta = -\frac{1}{\delta}\operatorname{Re}\zeta$ as $\operatorname{Re}\zeta \to -\infty$ [resp. $\operatorname{Re}\zeta \to +\infty$].

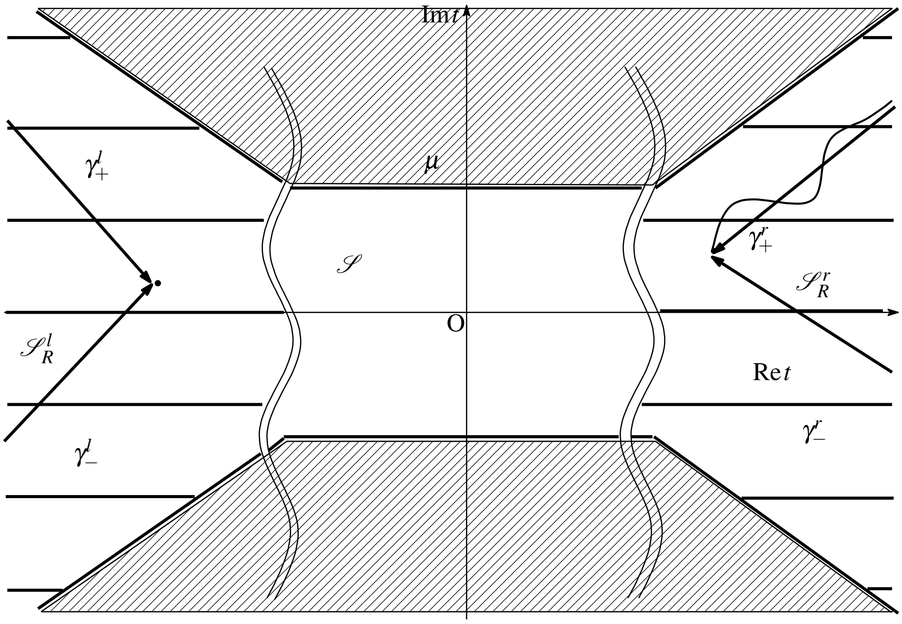

Figure 3.1: Global Stokes geometry

Let $\Gamma_\pm^{r,l}(z)$ be the paths defined above. The system of recurrence equations

$$
(3.1.4) \qquad \begin{cases} w_{\pm,0}^{r,l}(z) = 1, \\ w_{\pm,2k+1}^{r,l}(z) = \displaystyle\int_{\Gamma_\pm^{r,l}(z)} e^{\pm\frac{2}{\hbar}(\zeta - z)} \frac{K'(\zeta)}{K(\zeta)} w_{\pm,2k}^{r,l}(\zeta)\, d\zeta \quad (k \geq 0), \\ w_{\pm,2k}^{r,l}(z) = \displaystyle\int_{\Gamma_\pm^{r,l}(z)} \frac{K'(\zeta)}{K(\zeta)} w_{\pm,2k-1}^{r,l}(\zeta)\, d\zeta \qquad (k \geq 1), \end{cases}
$$

define the sequences of functions $\{w_{\pm,n}^{r,l}(z)\}_{n=0}^\infty$. We define

$$
w_{\pm;even}^{r,l}(z) = \sum_{k\geq 0} w_{\pm,2k}^{r,l}(z), \qquad w_{\pm;odd}^{r,l}(z) = \sum_{k\geq 0} w_{\pm,2k-1}^{r,l}(z),
$$

$$
w_\pm^{r,l}(z) = \begin{pmatrix} \sum_{k\geq 0} w_{\pm,2k}^{r,l}(z) \\ \sum_{k\geq 0} w_{\pm,2k-1}^{r,l}(z) \end{pmatrix}.
$$

Let us check the convergence of the integrals in (3.1.4) by induction with respect to k. Suppose $w_{\pm,2k}^r$ is bounded and analytic in $\mathscr{S}_R^r$. We can take the path as in Figure 3.1 from Cauchy's integral theorem, and then we get

$$
w_{+,2k+1}^r(z) = \int_{\Delta_+^r(z)} e^{+\frac{2}{\hbar}(\zeta - z)} \big(\partial_\zeta \log K(\zeta)\big) w_{+,2k}^r(\zeta)\, d\zeta,
$$

where the integration is now performed along the straight line $\Delta^r_+(z)$ ending at z given by $\mathrm{Im}\,(\zeta - z) = \frac{1}{\delta}\mathrm{Re}\,(\zeta - z)$. We obtain

$$w^r_{+,2k+1}(z) = \int_0^{\infty} e^{-\frac{2}{h}(1+\frac{i}{\delta})u} \left[\left(\partial_\zeta \log K(\zeta)\right) w^r_{+,2k}\right]\left(z - (1+i/\delta)u\right)\left(1+i/\delta\right) du, \tag{3.1.5}$$

where $u = -\mathrm{Re}\,(\zeta - z)$. This shows uniform convergence of the integral defining $w^r_{+,2k+1}(z)$ using the fact that

$$\frac{\frac{d}{dz}K(z(t))}{K(z(t))} = -\frac{\varepsilon V'(t)}{2(V(t)^2+\varepsilon^2)^{3/2}},$$

the behavior of V' at infinity in $\mathscr{S}$ from **(B)**, and Cauchy's inequality from **(A)**. This expression also shows that $w^r_{+,2k+1}(z)$ is a bounded and analytic function in $\mathscr{S}^r_R$. Furthermore the convergence of the integral defining $w^r_{+,2k+2}(z)$ is shown from the fact that $\partial_z \log K(z) \in L^1(\Gamma^r_+(z))$.

One sees that the convergence of $\sum_{k\geq 0} w^{r,l}_{\pm,2k}(z(t))$ and $\sum_{k\geq 0} w^{r,l}_{\pm,2k-1}(z(t))$ follows from the following lemma (see [Gr], Lemma 3.2, [Ra]):

Lemma 3.1.1 (Grigis)**.** *Suppose f is a function in $L^2((0,+\infty))$ and define, for all $n \geq 1$,*

$$I_{2n} = \int_{+\infty > s_1 > s_2 > \cdots > s_{2n} > 0} e^{-2(1+i/\delta)(s_1 - s_2 + \cdots - s_{2n})/h} f(s_1) f(s_2) \cdots f(s_{2n})\, ds_1 ds_2 \cdots ds_{2n},$$

$$I_{2n-1} = \int_{+\infty > s_1 > s_2 > \cdots > s_{2n-1} > 0} e^{-2(1+i/\delta)(s_1 - s_2 + \cdots + s_{2n-1})/h} f(s_1) f(s_2) \cdots f(s_{2n-1})\, ds_1 ds_2 \cdots ds_{2n-1}.$$

Then we have

$$|I_n(f)| \leq \left(\frac{h}{2}\right)^{\frac{n+1}{2}} \| f \|^n_{L^2}.$$

Moreover we see that

$$\lim_{t\to+\infty} w^r_{\pm,n}(t) = 0 \qquad \lim_{t\to-\infty} w^l_{\pm,n}(t) = 0 \qquad \forall n \in \mathbb{N}. \tag{3.1.6}$$

The corresponding WKB solutions $\phi^r_\pm(t)$ and $\phi^l_\pm(t)$ written by

$$\begin{aligned} \phi^r_\pm(t) &= \exp\left[\pm\frac{z^r(t)}{h}\right] M_\pm(z(t)) w^r_\pm(z(t)), \\ \phi^l_\pm(t) &= \exp\left[\pm\frac{z^l(t)}{h}\right] M_\pm(z(t)) w^l_\pm(z(t)), \end{aligned} \tag{3.1.7}$$

have the following relations with the Jost solutions.

Proposition 3.1.1. *For any fixed $h > 0$, the exact WKB solutions $\phi^r_\pm(t)$ and $\phi^l_\pm(t)$ have the asymptotic behaviors as t goes to $\pm\infty$:*

$$\phi^r_\pm(t) = \exp\left[\pm\frac{i}{h}\left(\lambda_r t + o(1)\right)\right] \begin{pmatrix} ie^{-i\theta_r} \\ \mp e^{i\theta_r} \end{pmatrix} \qquad \text{as } t \to +\infty,$$

$$\phi^l_\pm(t) = \exp\left[\pm\frac{i}{h}\left(\lambda_l t + o(1)\right)\right] \begin{pmatrix} ie^{-i\theta_l} \\ \mp e^{i\theta_l} \end{pmatrix} \qquad \text{as } t \to -\infty.$$

Consequently, we obtain the relations between the Jost solutions and the exact WKB solutions:

$$\psi_+^r(t) = -Q\phi_+^r(t), \qquad \psi_-^r(t) = -iQ\phi_-^r(t),$$
$$\psi_+^l(t) = -Q\phi_+^l(t), \qquad \psi_-^l(t) = -iQ\phi_-^l(t).$$

Remark 3.1.1. *Let $\tilde{S}(\varepsilon,h)$ be the change of bases between (ϕ_+^r,ϕ_-^r) and (ϕ_+^l,ϕ_-^l):*

$$\begin{pmatrix}\phi_+^l(t) & \phi_-^l(t)\end{pmatrix} = \begin{pmatrix}\phi_+^r(t) & \phi_-^r(t)\end{pmatrix}\tilde{S}(\varepsilon,h). \tag{3.1.8}$$

We express $\tilde{S}(\varepsilon,h)$ with the components of $S(\varepsilon,h)$ as

$$\tilde{S}(\varepsilon,h) = \begin{pmatrix} s_{11}(\varepsilon,h) & -is_{12}(\varepsilon,h) \\ is_{21}(\varepsilon,h) & s_{22}(\varepsilon,h)\end{pmatrix}. \tag{3.1.9}$$

Notice that (ϕ_+^r,ϕ_-^r) and (ϕ_+^l,ϕ_-^l) are orthonormal bases and $\tilde{S}(\varepsilon,h)$ is also a unitary matrix.

Proof of Proposition 3.1.1. The asymptotic behavior of the phase function $z^r(t)$ [resp. $z^l(t)$] is evident from the definition. That of the symbol functions is also obvious from (3.1.6) so that we have

$$\lim_{t\to+\infty} w_\pm^r(t) = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \lim_{t\to-\infty} w_\pm^l(t) = \begin{pmatrix}1\\0\end{pmatrix}.$$

We consider the asymptotic behaviors of $M_\pm(z(t))$. When $E_l > 0$, we get

$$\lim_{t\to+\infty}\big(-iV(t)-\varepsilon\big) = \lambda_r \exp\left[i\left(\frac{3}{2}\pi - 2\theta_r\right)\right], \quad \lim_{t\to+\infty}\big(-iV(t)+\varepsilon\big) = \lambda_r \exp\left[i\left(-\frac{\pi}{2}+2\theta_r\right)\right],$$

$$\lim_{t\to-\infty}\big(-iV(t)-\varepsilon\big) = \lambda_l \exp\left[i\left(\frac{3}{2}\pi - 2\theta_l\right)\right], \quad \lim_{t\to-\infty}\big(-iV(t)+\varepsilon\big) = \lambda_l \exp\left[i\left(-\frac{\pi}{2}+2\theta_l\right)\right].$$

Notice that in the case $E_l < 0$ these asymptotic behaviors as $t \to -\infty$ are the same as in the case $E_l > 0$, so we get

$$\lim_{t\to+\infty} K(z(t)) = -ie^{i\theta_r}, \qquad \lim_{t\to-\infty} K(z(t)) = -ie^{i\theta_l}.$$

Therefore we obtain

$$\lim_{t\to+\infty} M_\pm(z(t)) = \begin{pmatrix} ie^{-i\theta_r} & ie^{-i\theta_r} \\ \mp e^{i\theta_r} & \pm e^{i\theta_r}\end{pmatrix}, \quad \lim_{t\to-\infty} M_\pm(z(t)) = \begin{pmatrix} ie^{-i\theta_l} & ie^{-i\theta_l} \\ \mp e^{i\theta_l} & \pm e^{i\theta_l}\end{pmatrix}.$$

Hence we calculate the asymptotic behaviors of $\phi_\pm^r(t)$ and $\phi_\pm^l(t)$ from above consideration. In addition, we return $\phi_\pm^r(t)$ and $\phi_\pm^l(t)$ to the solutions to (1.1.1).

$$Q\phi_+^r(t) \sim \exp\left[+\frac{i}{h}\lambda_r t\right]\begin{pmatrix}\sin\theta_r \\ -\cos\theta_r\end{pmatrix}, \quad Q\phi_-^r(t) \sim i\exp\left[-\frac{i}{h}\lambda_r t\right]\begin{pmatrix}\cos\theta_r \\ \sin\theta_r\end{pmatrix} \qquad \text{as } t\to+\infty,$$

$$Q\phi_+^l(t) \sim \exp\left[+\frac{i}{h}\lambda_l t\right]\begin{pmatrix}\sin\theta_l \\ -\cos\theta_l\end{pmatrix}, \quad Q\phi_-^l(t) \sim i\exp\left[-\frac{i}{h}\lambda_l t\right]\begin{pmatrix}\cos\theta_l \\ \sin\theta_l\end{pmatrix} \qquad \text{as } t\to-\infty.$$

□

3.2 Stokes geometry around the avoided crossing

In this section we investigate the geometrical structures of the Stokes lines near the origin and define the exact WKB solutions with local base points.

If ε is sufficiently small, there exist $2n$ simple turning points $x_j(\varepsilon)$ and $\overline{x_j(\varepsilon)}$ $(j = 1, \ldots, n)$ near each root of $(V^{(n)}(0)/n!)^2 t^{2n} + \varepsilon^2 = 0$. We illustrate the Stokes lines passing through the turning points in the case $V(t) = t^n$ for $\varepsilon = 0$ and for ε positive and small.

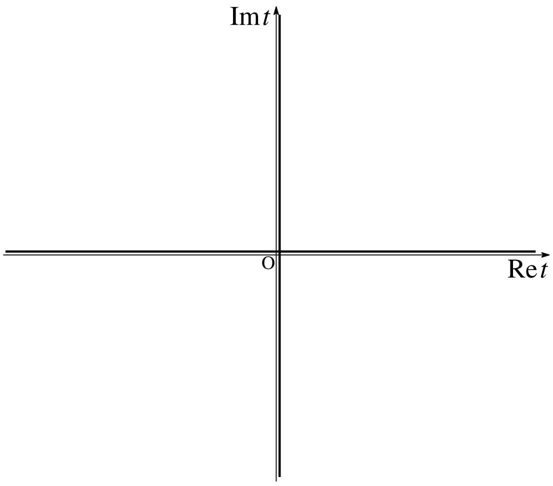

Figure 3.2: ($n = 1$, $\varepsilon = 0$)

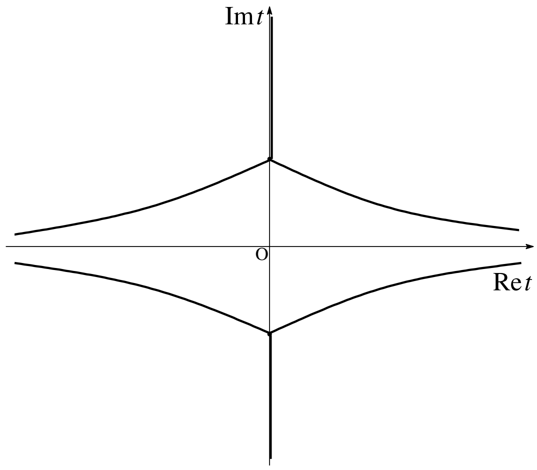

Figure 3.3: ($n = 1$, $\varepsilon > 0$)

It is important to see that, when ε is positive, the Stokes lines passing through the four turning points $\varepsilon^{1/n}\exp[\pm\frac{\pi}{2n}i]$, $\varepsilon^{1/n}\exp[\pm\frac{(2n-1)\pi}{2n}i]$ bound a domain containing the real axis and no turning point.

Let us return to our $V(t)$ satisfying **(A)**, **(B)**, **(C1)**. It is possible to take $\mu = \mu(\varepsilon)$ properly small, so that $\mathscr{S}$ includes only four turning points $x_1, x_n, \overline{x_1}$, and $\overline{x_n}$. The Stokes lines emanating from these turning points are not connected with those from the other $2n-4$ turning points. Indeed, by Lemma 1.2.1, we see that the principal terms of the action integrals for ε small enough satisfy:

$$\max\{|\mathrm{Re}\, z(x_1)|, |\mathrm{Re}\, z(x_n)|\} < \min\{|\mathrm{Re}\, z(x_2)|, \cdots, |\mathrm{Re}\, z(x_{n-1})|\}.$$

The larger the number n is, the more complicated the Stokes geometry becomes. However, if we restrict ourselves to a properly restricted domain $\mathscr{S}$, the geometrical structures of the Stokes lines emanating from four turning points x_1, x_n, $\bar{x}_1$ and $\bar{x}_n$ can be classified into three cases. One sees that, when $\mathrm{Re}\, z(x_1) = \mathrm{Re}\, z(x_n)$, the Stokes lines passing through the four turning points bound a domain containing the real axis and no turning point (see Figure 3.9). When $\mathrm{Re}\, z(x_1) > \mathrm{Re}\, z(x_n)$, the Stokes lines emanating from the two turning points x_1, $\bar{x}_1$ bound a domain containing the real axis and no turning point (see Figure 3.10) and when $\mathrm{Re}\, z(x_1) < \mathrm{Re}\, z(x_n)$, those emanating from the two turning points x_n, $\bar{x}_n$ also do such a domain (see Figure 3.11).

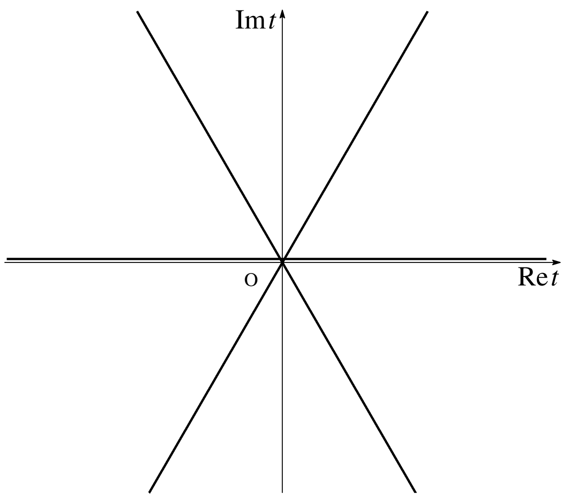

Figure 3.4: ($n = 2$, $\varepsilon = 0$)

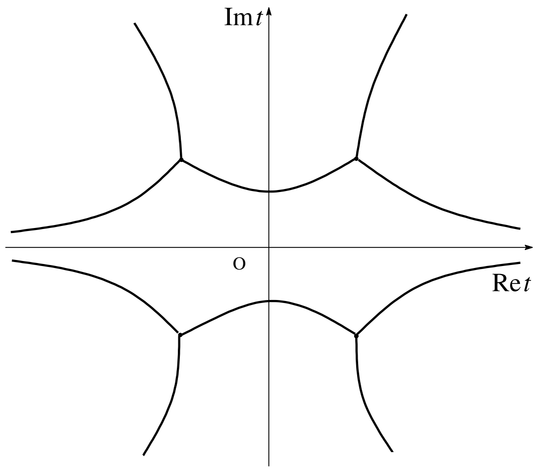

Figure 3.5: ($n = 2$, $\varepsilon > 0$)

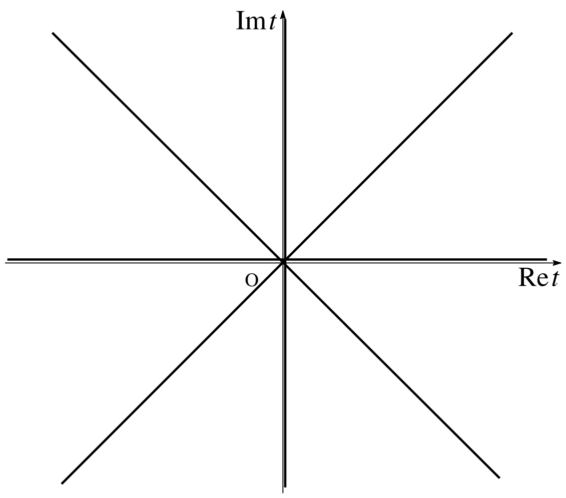

Figure 3.6: ($n = 3$, $\varepsilon = 0$)

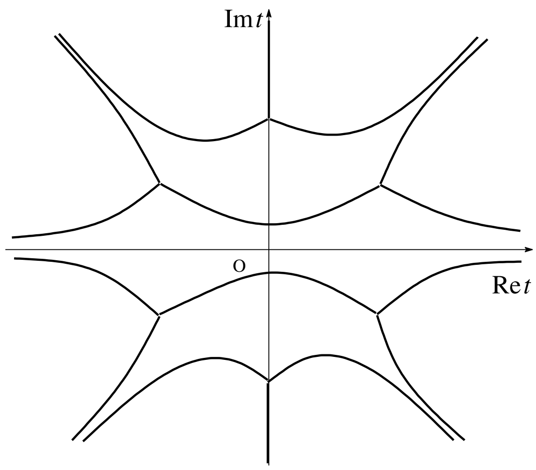

Figure 3.7: ($n = 3$, $\varepsilon > 0$)

We will separately discuss the cases where $V(t)$ has a simple zero or a zero of higher order. In the case $n=1$, let r, $\bar{r}$, l, and $\bar{l}$ be four base points of the symbol around the origin and we make the branch cuts dashed lines as in Figure 3.8. We define the exact WKB solutions $\phi_+(t;x_1,r)$, $\phi_-(t;\bar{x}_1,\bar{r})$, $\phi_+(t;x_1,l)$, $\phi_-(t;\bar{x}_1,\bar{l})$ as

$$\phi_+(t;x_1,r) = e^{+z(t;x_1)/h}M_+(z(t))w_+(z(t);z(r)),$$

$$\phi_-(t;\bar{x}_1,\bar{r}) = e^{-z(t;\bar{x}_1)/h}M_-(z(t))w_-(z(t);z(\bar{r})),$$

$$\phi_+(t;x_1,l) = e^{+z(t;x_1)/h}M_+(z(t))w_+(z(t);z(l)),$$

$$\phi_-(t;\bar{x}_1,\bar{l}) = e^{-z(t;\bar{x}_1)/h}M_-(z(t))w_-(z(t);z(\bar{l})).$$

Notice that each exact WKB solution has a valid asymptotic expansion on h in the direction toward its phase base point from its symbol base point.

In the case $n\geq 2$, we also put the symbol base points r, l and their complex conjugates and make the branch cuts dashed lines as in Figure 3.9. We can similarly define the exact WKB solutions $\phi_+(t;x_1,r)$, $\phi_-(t;\bar{x}_1,\bar{r})$, $\phi_+(t;x_n,l)$, $\phi_-(t;\bar{x}_n,\bar{l})$:

$$\phi_+(t;x_1,r) = e^{+z(t;x_1)/h}M_+(z(t))w_+(z(t);z(r)),$$

$$\phi_-(t;\bar{x}_1,\bar{r}) = e^{-z(t;\bar{x}_1)/h}M_-(z(t))w_-(z(t);z(\bar{r})),$$

$$\phi_+(t;x_n,l) = e^{+z(t;x_n)/h}M_+(z(t))w_+(z(t);z(l)),$$

$$\phi_-(t;\bar{x}_n,\bar{l}) = e^{-z(t;\bar{x}_n)/h}M_-(z(t))w_-(z(t);z(\bar{l})),$$

which have valid asymptotic expansions on h in the direction toward those phase base points from those symbol base points. Let δ and $\bar{\delta}$ be the intermediate symbol base points on the imaginary axis such that

$$\max\{|\mathrm{Re}\, z(x_1)|, |\mathrm{Re}\, z(x_n)|\} < |\mathrm{Re}\, z(\delta)| < \min\{|\mathrm{Re}\, z(x_2)|, |\mathrm{Re}\, z(x_{n-1})|\}$$

as in Figure 3.9, Figure 3.10, Figure 3.11. We consider the intermediate exact WKB solutions $\phi_+(t;x_1,\delta)$, $\phi_+(t;x_n,\delta)$, $\phi_-(t;\bar{x}_1,\bar{\delta})$ and $\phi_-(t;\bar{x}_n,\bar{\delta})$:

$$\phi_+(t;x_1,\delta) = e^{+z(t;x_1)/h}M_+(z(t))w_+(z(t);z(\delta)),$$

$$\phi_-(t;\bar{x}_1,\bar{\delta}) = e^{-z(t;\bar{x}_1)/h}M_-(z(t))w_-(z(t);z(\bar{\delta})),$$

$$\phi_+(t;x_n,\delta) = e^{+z(t;x_n)/h}M_+(z(t))w_+(z(t);z(\delta)),$$

$$\phi_-(t;\bar{x}_n,\bar{\delta}) = e^{-z(t;\bar{x}_n)/h}M_-(z(t))w_-(z(t);z(\bar{\delta})),$$

whose asymptotic expansions on h are valid in the direction toward four turning points from the symbol base points δ, $\bar{\delta}$.

We will connect these exact WKB solutions around the origin in next section.

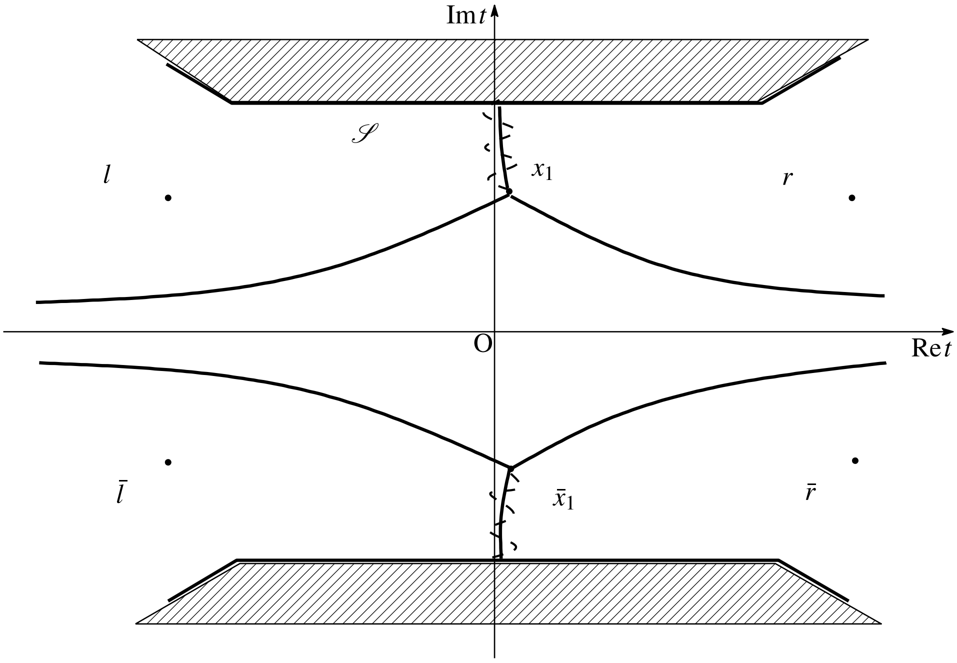

Figure 3.8: Stokes geometry $n = 1$

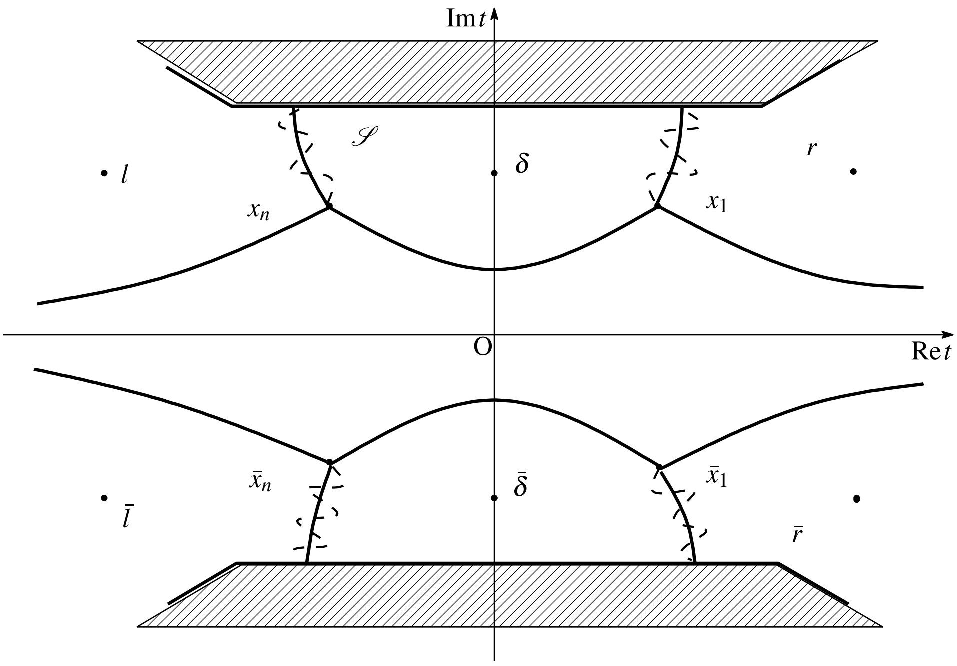

Figure 3.9: Stokes geometry $n \geq 2$

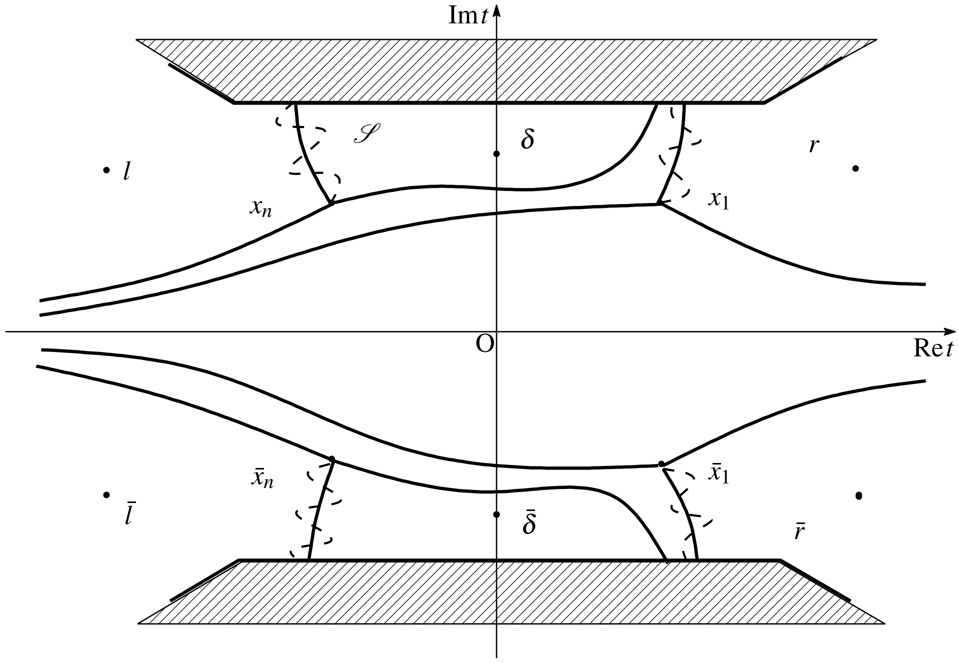

Figure 3.10: $\mathrm{Re}\,z(x_1) > \mathrm{Re}\,z(x_n)$

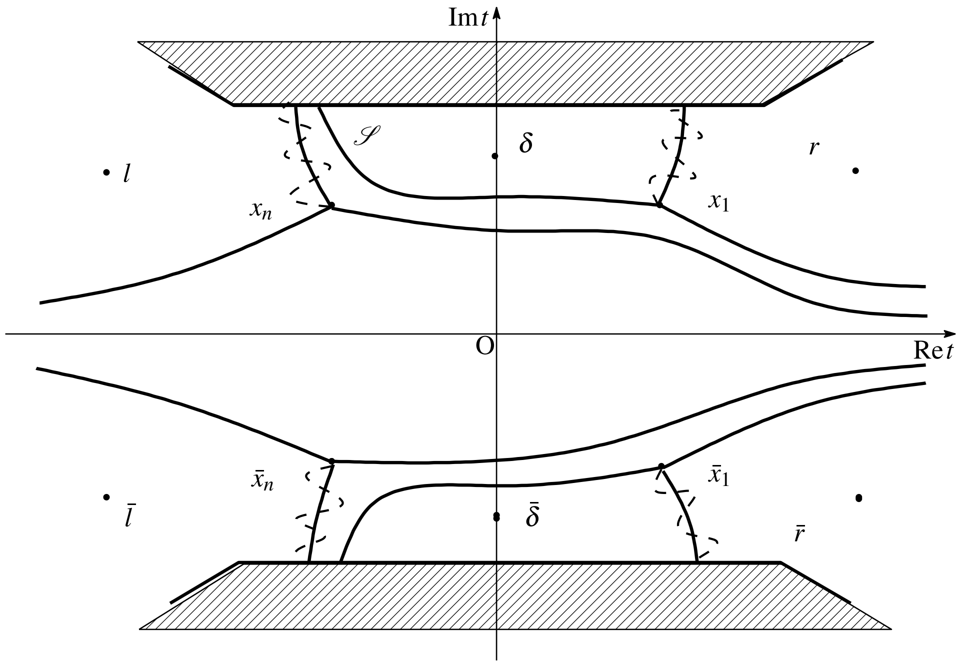

Figure 3.11: $\mathrm{Re}\,z(x_1) < \mathrm{Re}\,z(x_n)$

3.3 Transfer matrices

In this section we introduce transfer matrices, in terms of which we express the scattering matrix. We reduce the connection problem between the Jost solutions $\phi_\pm^r(t)$ and $\phi_\pm^l(t)$ to the local connection problem near the avoided crossing between the exact WKB solutions.

Lemma 3.3.1. *We have the following relations between the local exact WKB solutions and $\phi_\pm^{r,l}(t)$:*

$$\phi_+^r(t) = C_1(\varepsilon,h)\exp\left[+\frac{z^r(x_1)}{h}\right]\phi_+(t;x_1,r),$$

$$\phi_-^r(t) = C_2(\varepsilon,h)\exp\left[-\frac{z^r(\bar{x}_1)}{h}\right]\phi_-(t;\bar{x}_1,\bar{r}),$$

$$\phi_+^l(t) = C_3(\varepsilon,h)\exp\left[+\frac{z^l(x_n)}{h}\right]\phi_+(t;x_n,l),$$

$$\phi_-^l(t) = C_4(\varepsilon,h)\exp\left[-\frac{z^l(\bar{x}_n)}{h}\right]\phi_-(t;\bar{x}_n,\bar{l}),$$

where $C_k(\varepsilon,h)$ $(k=1,\dots,4)$ are some constants depending only on ε and h, and $C_k(\varepsilon,h) = 1+O(h)$ as h tends to 0 uniformly with respect to small ε.

Notice that this lemma is true in both cases $n=1$ and $n\geq 2$.

Proof of Lemma 3.3.1 We put $\tilde{\phi}_+^r(t;x_1) = e^{+z(t;x_1)/h}M_+(z(t))w_+^r(z(t))$, which is obtained by shifting the phase base point of $\phi_+^r(t)$ from the infinity in $\mathscr{S}_R^r$ to x_1 with (3.1.3). Actually we write $\tilde{\phi}_+^r(t;x_1)$ with the linear combination of $\phi_+(t;x_1,r)$ and $\phi_-(t;x_1,r)$ as

$$\tilde{\phi}_+^r(t;x_1) = C_1(\varepsilon,h)\phi_+(t;x_1,r) + \tilde{C}_1(\varepsilon,h)\phi_-(t;x_1,r).$$

When $|t|\to+\infty$ in the direction to the start point of γ_+^r, $\tilde{\phi}_+^r(t;x_1)$ and $\phi_+(t;x_1,r)$ decay exponentially and $\phi_-(t;x_1,r)$ grows exponentially. Hence $\tilde{C}_1(\varepsilon,h)$ is equal to 0. By the Wronskian formula (Proposition 2.2.2) and Proposition 2.4.1, we have

$$C_1(\varepsilon,h) = \frac{[\tilde{\phi}_+^r(t;x_1),\phi_-(t;x_1,r)]}{[\phi_+(t;x_1,r),\phi_-(t;x_1,r)]} = w_{+;even}^r(z(r)) = 1+O(h) \qquad \text{as} \quad h\to 0.$$

One sees that this asymptotic expansion is uniform for small ε because there exists no turning point in $\mathscr{S}_R^r$.

□

We define the transfer matrices $T_r(\varepsilon,h)$ and $T_l(\varepsilon,h)$ between the Jost solutions and the local exact WKB solutions by

$$\left(\phi_+^r(t)\ \phi_-^r(t)\right) = \left(\phi_+(t;x_1,r)\ \phi_-(t;\bar{x}_1,\bar{r})\right)T_r(\varepsilon,h), \tag{3.3.1}$$

$$\left(\phi_+^l(t)\ \phi_-^l(t)\right) = \left(\phi_+(t;x_n,l)\ \phi_-(t;\bar{x}_n,\bar{l})\right)T_l(\varepsilon,h). \tag{3.3.2}$$

From Lemma 3.3.1, we get

$$T_r(\varepsilon,h) = \begin{pmatrix} C_1(\varepsilon,h)e^{+z^r(x_1)/h} & 0 \\ 0 & C_2(\varepsilon,h)e^{-z^r(\bar{x}_1)/h} \end{pmatrix}, \tag{3.3.3}$$

$$T_l(\varepsilon,h) = \begin{pmatrix} C_3(\varepsilon,h)e^{+z^l(x_n)/h} & 0 \\ 0 & C_4(\varepsilon,h)e^{-z^l(\bar{x}_n)/h} \end{pmatrix}. \tag{3.3.4}$$

Moreover we define the transfer matrices $T(\varepsilon,h)$ around the zero of $V(t)$, that is $t=0$, by

$$\Big(\phi_+(t;x_n,l)\ \phi_-(t;\bar{x}_n,\bar{l})\Big) = \Big(\phi_+(t;x_1,r)\ \phi_-(t;\bar{x}_1,\bar{r})\Big)\, T(\varepsilon,h). \tag{3.3.5}$$

Then the scattering matrix $\tilde{S}(\varepsilon,h)$ is the product of these transfer matrices:

$$\tilde{S}(\varepsilon,h) = T_r^{-1}(\varepsilon,h)T(\varepsilon,h)T_l(\varepsilon,h). \tag{3.3.6}$$

Thus scattering problem is reduced to the calculations of the transfer matrix $T(\varepsilon,h)$. A lot of Wronskians of exact WKB solutions appear in connection coefficients of this calculations. We express them with exact WKB symbols from Proposition 2.2.2. We denote $w_+^e(z(t_1);z(t_2))$ by $\mathscr{W}(t_1;t_2)$ for short.

Let $t_{jk}(\varepsilon,h)$ be the components of $T(\varepsilon,h)$:

$$T(\varepsilon,h) = \begin{pmatrix} t_{11}(\varepsilon,h) & t_{12}(\varepsilon,h) \\ t_{21}(\varepsilon,h) & t_{22}(\varepsilon,h) \end{pmatrix}.$$

Proposition 3.3.1. *In the case $n=1$, $T(\varepsilon,h)$ is given by*

$$T(\varepsilon,h) = \begin{pmatrix} \dfrac{\mathscr{W}(\bar{r};l)}{\mathscr{W}(\bar{r};r)} & i\dfrac{\mathscr{W}(\bar{r};\hat{\bar{l}})}{\mathscr{W}(\bar{r};r)}e^{z(x_1;\bar{x}_1)/h} \\ i\dfrac{\mathscr{W}(\hat{l};r)}{\mathscr{W}(\bar{r};r)}e^{z(x_1;\bar{x}_1)/h} & \dfrac{\mathscr{W}(\bar{l};r)}{\mathscr{W}(\bar{r};r)} \end{pmatrix}, \tag{3.3.7}$$

where $\hat{l}$ is the same point as l but continued from r passing through the branch cut from x_1 and $\hat{\bar{l}}$ also the same as $\bar{l}$ but continued from $\bar{r}$ passing through the branch cut from $\bar{x}_1$.

Proposition 3.3.2. *In the case $n \geq 2$, the components of $T(\varepsilon, h)$ are given by*

$$t_{11}(\varepsilon,h) = \frac{\mathscr{W}(\bar{\delta};\delta)}{\mathscr{W}(\bar{r};r)}\left(\frac{\mathscr{W}(\bar{r};\delta)}{\mathscr{W}(\bar{\delta};\delta)}\frac{\mathscr{W}(\bar{\delta};l)}{\mathscr{W}(\bar{\delta};\delta)}e^{z(x_1;x_n)/h} + (-1)^n\frac{\mathscr{W}(\bar{\delta};\hat{\bar{r}})}{\mathscr{W}(\bar{\delta};\delta)}\frac{\mathscr{W}(\hat{l};\delta)}{\mathscr{W}(\bar{\delta};\delta)}e^{\{z(x_1;\bar{x}_1)-z(\bar{x}_1;x_n)\}/h}\right),$$

$$t_{12}(\varepsilon,h) = i\frac{\mathscr{W}(\bar{\delta};\delta)}{\mathscr{W}(\bar{r};r)}\left((-1)^{n+1}\frac{\mathscr{W}(\bar{\delta};\hat{\bar{l}})}{\mathscr{W}(\bar{\delta};\delta)}\frac{\mathscr{W}(\bar{r};\delta)}{\mathscr{W}(\bar{\delta};\delta)}e^{z(x_1;\bar{x}_n)/h} + \frac{\mathscr{W}(\bar{l};\delta)}{\mathscr{W}(\bar{\delta};\delta)}\frac{\mathscr{W}(\bar{\delta};\hat{\bar{r}})}{\mathscr{W}(\bar{\delta};\delta)}e^{\{z(x_1;\bar{x}_1)-z(\bar{x}_1;\bar{x}_n)\}/h}\right),$$

$$t_{21}(\varepsilon,h) = i\frac{\mathscr{W}(\bar{\delta};\delta)}{\mathscr{W}(\bar{r};r)}\left((-1)^{n+1}\frac{\mathscr{W}(\bar{\delta};r)}{\mathscr{W}(\bar{\delta};\delta)}\frac{\mathscr{W}(\hat{l};\delta)}{\mathscr{W}(\bar{\delta};\delta)}e^{z(x_n;\bar{x}_1)/h} + \frac{\mathscr{W}(\hat{r};\delta)}{\mathscr{W}(\bar{\delta};\delta)}\frac{\mathscr{W}(\bar{\delta};l)}{\mathscr{W}(\bar{\delta};\delta)}e^{\{z(x_1;\bar{x}_1)+z(x_1;x_n)\}/h}\right),$$

$$t_{22}(\varepsilon,h) = \frac{\mathscr{W}(\bar{\delta};\delta)}{\mathscr{W}(\bar{r};r)}\left(\frac{\mathscr{W}(\bar{\delta};r)}{\mathscr{W}(\bar{\delta};\delta)}\frac{\mathscr{W}(\bar{l};\delta)}{\mathscr{W}(\bar{\delta};\delta)}e^{z(\bar{x}_n;\bar{x}_1)/h} + (-1)^n\frac{\mathscr{W}(\hat{r};\delta)}{\mathscr{W}(\bar{\delta};\delta)}\frac{\mathscr{W}(\bar{\delta};\hat{\bar{l}})}{\mathscr{W}(\bar{\delta};\delta)}e^{\{z(x_1;\bar{x}_1)+z(x_1;\bar{x}_n)\}/h}\right),$$

where $\hat{r}$ [resp. $\hat{l}$] *is the same point as r* [resp. l] *but continued from δ passing through the branch cut from x_1* [resp. x_n] *and $\hat{\bar{r}}$* [resp. $\hat{\bar{l}}$] *the same point as $\bar{r}$* [resp. $\bar{l}$] *but continued from $\bar{\delta}$ passing through the branch cut from $\bar{x}_1$* [resp. $\bar{x}_n$].

Proof of Proposition 3.3.1. The Stokes lines passing through the turning points x_1, $\bar{x}_1$ are drawn in Figure 3.8. We write $\phi_+(t;x_1,l)$ and $\phi_-(t;\bar{x}_1,\bar{l})$ by linear combinations of the linearly independent exact WKB solutions $(\phi_+(t;x_1,r), \phi_-(t;x_1,\bar{r}))$:

$$\begin{aligned} \phi_+(t;x_1,l) &= F_1\phi_+(t;x_1,r) + F_2\phi_-(t;x_1,\bar{r}), \\ \phi_-(t;\bar{x}_1,\bar{l}) &= F_3\phi_+(t;\bar{x}_1,r) + F_4\phi_-(t;\bar{x}_1,\bar{r}), \end{aligned} \tag{3.3.8}$$

The coefficients are expressed by the Wronskians of exact WKB solutions:

$$F_1 = \frac{[\phi_+(t;x_1,l),\ \phi_-(t;x_1,\bar{r})]}{[\phi_+(t;x_1,r),\ \phi_-(t;x_1,\bar{r})]}, \qquad F_2 = \frac{[\phi_+(t;x_1,r),\ \phi_+(t;x_1,l)]}{[\phi_+(t;x_1,r),\ \phi_-(t;x_1,\bar{r})]},$$

$$F_3 = \frac{[\phi_-(t;\bar{x}_1,\bar{l}),\ \phi_-(t;\bar{x}_1,\bar{r})]}{[\phi_+(t;\bar{x}_1,r),\ \phi_-(t;\bar{x}_1,\bar{r})]}, \qquad F_4 = \frac{[\phi_+(t;\bar{x}_1,r),\ \phi_-(t;\bar{x}_1,\bar{l})]}{[\phi_+(t;\bar{x}_1,r),\ \phi_-(t;\bar{x}_1,\bar{r})]}.$$

In order to calculate the Wronskians $[\phi_+(t;x_1,r),\ \phi_+(t;x_1,l)]$ and $[\phi_-(t;\bar{x}_1,\bar{l}),\ \phi_-(t;\bar{x}_1,\bar{r})]$ along a canonical path, we need to go across a branch cut and to redefine one of the solutions on the other Riemann surface.

Since l is obtained from $\hat{l}$ after turning clockwise around x_1 and $\bar{l}$ is obtained from $\hat{\bar{l}}$ after turning anti-clockwise around $\bar{x}_1$ from the definitions of $\hat{l}$ and $\hat{\bar{l}}$, one sees

$$l = x_1 + \left(\hat{l} - x_1\right) e^{-2\pi i}, \qquad \bar{l} = \bar{x}_1 + \left(\hat{\bar{l}} - \bar{x}_1\right) e^{+2\pi i}, \tag{3.3.9}$$

if l is sufficiently close to x_1. Notice that x_1 is a zero of $V(t) - i\varepsilon$ and $\bar{x}_1$ is a zero of $V(t) + i\varepsilon$.

Lemma 3.3.2. *The following identities hold:*

$$\phi_+(t;x_1,l) = i\phi_-(t;x_1,\hat{l}), \qquad \phi_-(t;\bar{x}_1,\bar{l}) = i\phi_+(t;\bar{x}_1,\hat{\bar{l}}).$$

Proof of Lemma 3.3.2 We show the first equality. The phase function $z(t;x_1)$ corresponds to $-z(t;x_1)$ on the other Riemann sheet because the multiplicity of $z(t;x_1)$ is two. Hence the sign of the phase function changes.

We consider $K(z(t))$, which determine $M_\pm(z(t))$ and $w_\pm(z(t);z(l))$. In this case we must be careful whether tunning point x_1 is a zero of $V(t) - i\varepsilon$ or $V(t) + i\varepsilon$. Now x_1 is a zero of $V(t) - i\varepsilon$, so that $K(z(t))$ corresponds to

$$\sqrt[4]{\frac{V(t)+i\varepsilon}{e^{2\pi i}(V(t)-i\varepsilon)}} = -i\sqrt[4]{\frac{V(t)+i\varepsilon}{V(t)-i\varepsilon}} = -iK(z(t))$$

on the Riemann sheet continued from r passing through the branch cut from x_1. Hence $M_+(z(t))$ corresponds to

$$\begin{pmatrix} iK(z)^{-1} & iK(z)^{-1} \\ -K(z) & +K(z) \end{pmatrix} = i\begin{pmatrix} K(z)^{-1} & K(z)^{-1} \\ +iK(z) & -iK(z) \end{pmatrix} = iM_-(z(t)).$$

The symbol function on that Riemann sheet which corresponds to $w_+(z(t);z(l))$ satisfies

$$-\frac{d}{dz}f(z) = \begin{pmatrix} 0 & -\frac{K'(z)}{K(z)} \\ -\frac{K'(z)}{K(z)} & +\frac{2}{h} \end{pmatrix} f(z).$$

$$\frac{d}{dz}f(z) = \begin{pmatrix} 0 & +\frac{K'(z)}{K(z)} \\ +\frac{K'(z)}{K(z)} & -\frac{2}{h} \end{pmatrix} f(z).$$

This implies that $w_+(z(t);z(l))$ corresponds to $w_-(z(t);z(\hat{l}))$ on another Riemann sheet from the uniqueness of these differential equations. Hence we obtain $\phi_+(t;x_1,l) = i\phi_-(t;x_1,\hat{l})$. In the second equality case we pay attention to the fact that $\bar{x}_1$ is a zero of $V(t) + i\varepsilon$, so that $K(z(t))$ corresponds to

$$\sqrt[4]{\frac{e^{-2\pi i}(V(t)+i\varepsilon)}{V(t)-i\varepsilon}} = -i\sqrt[4]{\frac{V(t)+i\varepsilon}{V(t)-i\varepsilon}} = -iK(z(t))$$

on the Riemann sheet continued from $\bar{r}$ passing through the branch cut from x_1. Similarly we have $\phi_-(t;\bar{x}_1,\bar{l}) = i\phi_+(t;\bar{x}_1,\hat{\bar{l}})$.

□

We apply Lemma 3.3.2 to the Wronskian calculations of F_2 and F_3, then we have

$$F_1 = \frac{\mathscr{W}(\bar{r};l)}{\mathscr{W}(\bar{r};r)}, \qquad F_2 = i\,\frac{\mathscr{W}(\hat{l};r)}{\mathscr{W}(\bar{r};r)},$$

$$F_3 = i\,\frac{\mathscr{W}(\bar{r};\hat{\bar{l}})}{\mathscr{W}(\bar{r};r)}, \qquad F_4 = \frac{\mathscr{W}(\bar{l};r)}{\mathscr{W}(\bar{r};r)}.$$

We notice that each Wronskian has a canonical curve as in Proposition 2.4.1. Since

$$\phi_+(t;\bar{x}_1,r) = e^{z(x_1;\bar{x}_1)/h}\phi_+(t;x_1,r),$$

$$\phi_-(t;x_1,\bar{r}) = e^{-z(\bar{x}_1;x_1)/h}\phi_-(t;\bar{x}_1,\bar{r}),$$

we have

$$t_{11}(\varepsilon,h) = F_1 = \frac{\mathscr{W}(\bar{r};l)}{\mathscr{W}(\bar{r};r)},$$

$$t_{12}(\varepsilon,h) = F_3\,e^{z(x_1;\bar{x}_1)/h} = i\,\frac{\mathscr{W}(\bar{r};\hat{\bar{l}})}{\mathscr{W}(\bar{r};r)}\,e^{z(x_1;\bar{x}_1)/h},$$

$$t_{21}(\varepsilon,h) = F_2\,e^{-z(\bar{x}_1;x_1)/h} = i\,\frac{\mathscr{W}(\hat{l};r)}{\mathscr{W}(\bar{r};r)}\,e^{z(x_1;\bar{x}_1)/h},$$

$$t_{22}(\varepsilon,h) = F_4 = \frac{\mathscr{W}(\bar{l};r)}{\mathscr{W}(\bar{r};r)}.$$

□

Proof of Proposition 3.3.2. We introduce the intermediate symbol base points δ and $\bar{\delta}$ on the imaginary axis as in Figure 3.9, 3.10, 3.11. We consider the pairs of linearly independent intermediate exact WKB solutions $(\phi_+(t;x_1,\delta),\phi_-(t;x_1,\bar{\delta}))$, $(\phi_+(t;\bar{x}_1,\delta),\phi_-(t;\bar{x}_1,\bar{\delta}))$, $(\phi_+(t;x_n,\delta),\phi_-(t;x_n,\bar{\delta}))$, and $(\phi_+(t;\bar{x}_n,\delta),\phi_-(t;\bar{x}_n,\bar{\delta}))$. $\phi_+(t;x_1,r)$, $\phi_-(t;\bar{x}_1,\bar{r})$, $\phi_+(t;x_n,l)$ and $\phi_-(t;\bar{x}_n,\bar{l})$ are written as linear combinations of them:

$$\phi_+(t;x_1,r) = G_1\phi_+(t;x_1,\delta) + G_2\phi_-(t;x_1,\bar{\delta}),$$

$$\phi_-(t;\bar{x}_1,\bar{r}) = G_3\phi_+(t;\bar{x}_1,\delta) + G_4\phi_-(t;\bar{x}_1,\bar{\delta}),$$

$$\phi_+(t;x_n,l) = G_5\phi_+(t;x_n,\delta) + G_6\phi_-(t;x_n,\bar{\delta}),$$

$$\phi_-(t;\bar{x}_n,\bar{l}) = G_7\phi_+(t;\bar{x}_n,\delta) + G_8\phi_-(t;\bar{x}_n,\bar{\delta}).$$

Similarly we can express the coefficients with the Wronskians of them.

$$G_1 = \frac{[\phi_+(t;x_1,r),\ \phi_-(t;x_1,\bar{\delta})]}{[\phi_+(t;x_1,\delta),\ \phi_-(t;x_1,\bar{\delta})]}, \qquad G_2 = \frac{[\phi_+(t;x_1,\delta),\ \phi_+(t;x_1,r)]}{[\phi_+(t;x_1,\delta),\ \phi_-(t;x_1,\bar{\delta})]},$$

$$G_3 = \frac{[\phi_-(t;\bar{x}_1,\bar{r}),\ \phi_-(t;\bar{x}_1,\bar{\delta})]}{[\phi_+(t;\bar{x}_1,\delta),\ \phi_-(t;\bar{x}_1,\bar{\delta})]}, \qquad G_4 = \frac{[\phi_+(t;\bar{x}_1,\delta),\ \phi_-(t;\bar{x}_1,\bar{r})]}{[\phi_+(t;\bar{x}_1,\delta),\ \phi_-(t;\bar{x}_1,\bar{\delta})]}.$$

Because $\hat{r}$ is obtained from r after turning clockwise around x_1 and $\hat{\bar{r}}$ is obtained from $\bar{r}$ after turning anti-clockwise around $\bar{x}_1$, one sees

$$r = x_1 + (\hat{r} - x_1)\, e^{+2\pi i}, \qquad \bar{r} = \bar{x}_1 + \left(\hat{\bar{r}} - \bar{x}_1\right) e^{-2\pi i}, \tag{3.3.10}$$

if r is sufficiently close to x_1. Taking into account the fact that x_1 is a simple zero of $V(t) - i\varepsilon$ and $\bar{x}_1$ is a simple zero of $V(t) + i\varepsilon$, we have from (3.3.10) in the same way as Lemma 3.3.2

$$\phi_+(t; x_1, r) = -i\phi_-(t; x_1, \hat{r}) \qquad \phi_-(t; \bar{x}_1, \hat{\bar{r}}) \ = -i\phi_+(t; \bar{x}_1, \hat{\bar{r}}) \tag{3.3.11}$$

We apply (3.3.11) to the Wronskian calculations of G_2 and G_3.

$$G_1 = \frac{\mathscr{W}(\bar{\delta}; r)}{\mathscr{W}(\bar{\delta}; \delta)}, \qquad G_2 = -i\,\frac{\mathscr{W}(\hat{r}; \delta)}{\mathscr{W}(\bar{\delta}; \delta)},$$

$$G_3 = -i\,\frac{\mathscr{W}(\bar{\delta}; \hat{\bar{r}})}{\mathscr{W}(\bar{\delta}; \delta)}, \qquad G_4 = \frac{\mathscr{W}(\bar{r}; \delta)}{\mathscr{W}(\bar{\delta}; \delta)}.$$

We can similarly express the coefficients of $\phi_+(t, x_n, l)$ and $\phi_-(t, \bar{x}_n, \bar{l})$ with the Wronskians of the intermediate exact WKB solutions.

$$G_5 = \frac{[\phi_+(t; x_n, l),\ \phi_-(t; x_n, \bar{\delta})]}{[\phi_+(t; x_n, \delta),\ \phi_-(t; x_n, \bar{\delta})]}, \qquad G_6 = \frac{[\phi_+(t; x_n, \delta),\ \phi_+(t; x_n, l)]}{[\phi_+(t; x_n, \delta),\ \phi_-(t; x_n, \bar{\delta})]},$$

$$G_7 = \frac{[\phi_-(t; \bar{x}_n, \bar{l}),\ \phi_-(t; \bar{x}_n, \bar{\delta})]}{[\phi_+(t; \bar{x}_n, \delta),\ \phi_-(t; \bar{x}_n, \bar{\delta})]}, \qquad G_8 = \frac{[\phi_+(t; \bar{x}_n, \delta),\ \phi_-(t; \bar{x}_n, \bar{l})]}{[\phi_+(t; \bar{x}_n, \delta),\ \phi_-(t; \bar{x}_n, \bar{\delta})]}.$$

Since $\hat{l}$ is obtained from l after turning anti-clockwise around x_n and $\hat{\bar{l}}$ is obtained from $\bar{l}$ after turning clockwise around $\bar{x}_n$, one sees

$$l = x_n + \left(\hat{l} - x_n\right) e^{-2\pi i}, \qquad \bar{l} = \bar{x}_n + \left(\hat{\bar{l}} - \bar{x}_n\right) e^{+2\pi i},$$

if l is sufficiently close to x_n. Remarking that x_n is a simple zero of $V(t) - i\varepsilon$ if n is odd and $V(t) + i\varepsilon$ if n is even, we get in the same way as Lemma 3.3.2

$$\phi_+(t; x_n, l) = (-1)^{n+1} i\, \phi_-(t; x_n, \hat{l}) \qquad \phi_-(t; \bar{x}_n, \bar{l}) \ = (-1)^{n+1} i\, \phi_+(t; \bar{x}_n, \hat{\bar{l}}) \tag{3.3.12}$$

Therefore we have from (3.3.12)

$$G_5 = \frac{\mathscr{W}(\bar{\delta}; l)}{\mathscr{W}(\bar{\delta}; \delta)}, \qquad G_6 = (-1)^{n+1} i\,\frac{\mathscr{W}(\hat{l}; \delta)}{\mathscr{W}(\bar{\delta}; \delta)},$$

$$G_7 = (-1)^{n+1} i\,\frac{\mathscr{W}(\bar{\delta}; \hat{\bar{l}})}{\mathscr{W}(\bar{\delta}; \delta)}, \qquad G_8 = \frac{\mathscr{W}(\bar{l}; \delta)}{\mathscr{W}(\bar{\delta}; \delta)}.$$

From (3.3.5), the components of $T(\varepsilon,h)$ are also expressed by the Wronskian of the exact WKB solutions.

$$t_{11}(\varepsilon,h)=\frac{[\phi_+(t;x_n,l),\ \phi_-(t;\bar{x}_1,\bar{r})]}{[\phi_+(t;x_1,r),\ \phi_-(t;\bar{x}_1,\bar{r})]},\qquad t_{12}(\varepsilon,h)=\frac{[\phi_-(t;\bar{x}_n,\bar{l}),\ \phi_-(t;\bar{x}_1,\bar{r})]}{[\phi_+(t;x_1,r),\ \phi_-(t;\bar{x}_1,\bar{r})]},$$

$$t_{21}(\varepsilon,h)=\frac{[\phi_+(t;x_1,r),\ \phi_+(t;x_n,l)]}{[\phi_+(t;x_1,r),\ \phi_-(t;\bar{x}_1,\bar{r})]},\qquad t_{22}(\varepsilon,h)=\frac{[\phi_+(t;x_1,r),\ \phi_-(t;\bar{x}_n,\bar{l})]}{[\phi_+(t;x_1,r),\ \phi_-(t;\bar{x}_1,\bar{r})]}.$$

Each denominator is calculated as

$$\begin{aligned}[\phi_+(t;x_1,r),\ \phi_-(t;\bar{x}_1,\bar{r})]&=[\phi_+(t;x_1,r),\ e^{-z(x_1;\bar{x}_1)/h}\phi_-(t;x_1,\bar{r})]\\&=e^{-z(x_1;\bar{x}_1)/h}\mathscr{W}(\bar{r};r).\end{aligned}$$

We remark that there exists a canonical curve from r to $\bar{r}$.

Next let us study these numerators.

$$\begin{aligned}&[\phi_+(t;x_n,l),\ \phi_-(t;\bar{x}_1,\bar{r})]\\=&[G_5\phi_+(t;x_n,\delta)+G_6\phi_-(t;x_n,\bar{\delta}),\ G_3\phi_+(t;\bar{x}_1,\delta)+G_4\phi_-(t;\bar{x}_1,\bar{\delta})]\\=&[G_5\,e^{z(\bar{x}_1;x_n)/h}\phi_+(t;\bar{x}_1,\delta)+G_6\,e^{-z(\bar{x}_1;x_n)/h}\phi_-(t;\bar{x}_1,\bar{\delta}),G_3\phi_+(t;\bar{x}_1,\delta)+G_4\phi_-(t;\bar{x}_1,\bar{\delta})]\\=&G_5G_4\,e^{z(\bar{x}_1;x_n)/h}\,[\phi_+(t;\bar{x}_1,\delta),\ \phi_-(t;\bar{x}_1,\bar{\delta})]+G_6G_3\,e^{-z(\bar{x}_1;x_n)/h}\,[\phi_-(t;\bar{x}_1,\bar{\delta}),\ \phi_+(t;\bar{x}_1,\delta)]\\=&\mathscr{W}(\bar{\delta};\delta)\left(G_5G_4\,e^{z(\bar{x}_1;x_n)/h}-G_6G_3\,e^{-z(\bar{x}_1;x_n)/h}\right).\end{aligned}$$

Therefore we have

$$\begin{aligned}t_{11}(\varepsilon,h)&=e^{z(x_1;\bar{x}_1)/h}\,\frac{\mathscr{W}(\bar{\delta};\delta)}{\mathscr{W}(\bar{r};r)}\left(G_5G_4\,e^{z(\bar{x}_1;x_n)/h}-G_6G_3\,e^{-z(\bar{x}_1;x_n)/h}\right)\\&=\frac{\mathscr{W}(\bar{\delta};\delta)}{\mathscr{W}(\bar{r};r)}\left(\frac{\mathscr{W}(\bar{r};\delta)}{\mathscr{W}(\bar{\delta};\delta)}\frac{\mathscr{W}(\bar{\delta};l)}{\mathscr{W}(\bar{\delta};\delta)}\,e^{z(x_1;x_n)/h}\right.\\&\qquad\left.+(-1)^n\frac{\mathscr{W}(\bar{\delta};\hat{\bar{r}})}{\mathscr{W}(\bar{\delta};\delta)}\frac{\mathscr{W}(\hat{l};\delta)}{\mathscr{W}(\bar{\delta};\delta)}\,e^{\{z(x_1;\bar{x}_1)-z(\bar{x}_1;x_n)\}/h}\right).\end{aligned}$$

We can similarly calculate $t_{12}(\varepsilon,h)$, $t_{21}(\varepsilon,h)$ and $t_{22}(\varepsilon,h)$. Let us calculate $t_{12}(\varepsilon,h)$.

$$\begin{aligned}&[\phi_-(t;\bar{x}_n,l),\ \phi_-(t;\bar{x}_1,\bar{r})]\\=&[G_7\phi_+(t;\bar{x}_n,\delta)+G_8\phi_-(t;\bar{x}_n,\bar{\delta}),\ G_3\phi_+(t;\bar{x}_1,\delta)+G_4\phi_-(t;\bar{x}_1,\bar{\delta})]\\=&[G_7\,e^{z(\bar{x}_1;\bar{x}_n)/h}\phi_+(t;\bar{x}_1,\delta)+G_8\,e^{-z(\bar{x}_1;\bar{x}_n)/h}\phi_-(t;\bar{x}_1,\bar{\delta}),\ G_3\phi_+(t;\bar{x}_1,\delta)+G_4\phi_-(t;\bar{x}_1,\bar{\delta})]\\=&G_7G_4\,e^{z(\bar{x}_1;\bar{x}_n)/h}[\phi_+(t;\bar{x}_1,\delta),\ \phi_-(t;\bar{x}_1,\bar{\delta})]+G_8G_3\,e^{-z(\bar{x}_1;\bar{x}_n)/h}[\phi_-(t;\bar{x}_1,\bar{\delta}),\ \phi_+(t;\bar{x}_1,\delta)]\\=&\mathscr{W}(\bar{\delta};\delta)\left(G_7G_4\,e^{z(\bar{x}_1;\bar{x}_n)/h}-G_8G_3\,e^{-z(\bar{x}_1;\bar{x}_n)/h}\right).\end{aligned}$$

Hence we have

$$
\begin{aligned}
t_{12}(\varepsilon,h) &= e^{z(x_1;\bar{x}_1)/h}\frac{\mathscr{W}(\bar{\delta};\delta)}{\mathscr{W}(\bar{r};r)}\left(G_7G_4\,e^{z(\bar{x}_1;\bar{x}_n)/h}-G_8G_3\,e^{-z(\bar{x}_1;\bar{x}_n)/h}\right)\\
&= i\,\frac{\mathscr{W}(\bar{\delta};\delta)}{\mathscr{W}(\bar{r};r)}\Bigg((-1)^{n+1}\frac{\mathscr{W}(\bar{\delta};\hat{\bar{l}})}{\mathscr{W}(\bar{\delta};\delta)}\frac{\mathscr{W}(\bar{r};\delta)}{\mathscr{W}(\bar{\delta};\delta)}\,e^{z(x_1;\bar{x}_n)/h}\\
&\qquad\qquad +\frac{\mathscr{W}(\hat{l};\delta)}{\mathscr{W}(\bar{\delta};\delta)}\frac{\mathscr{W}(\bar{\delta};\hat{\bar{r}})}{\mathscr{W}(\bar{\delta};\delta)}\,e^{\{z(x_1;\bar{x}_1)-z(\bar{x}_1;\bar{x}_n)\}/h}\Bigg).
\end{aligned}
$$

We shall study the other component $t_{21}(\varepsilon,h)$.

$$
\begin{aligned}
&[\phi_+(t;x_1,r),\ \phi_+(t;x_n,l)]\\
=&[G_1\phi_+(t;x_1,\delta)+G_2\phi_-(t;x_1,\bar{\delta}),\ G_3\phi_+(t;x_n,\delta)+G_4\phi_-(t;x_n,\bar{\delta})]\\
=&[G_1\phi_+(t;x_1,\delta)+G_2\phi_-(t;x_1,\bar{\delta}),\ G_3\,e^{z(x_1;x_n)/h}\phi_+(t;x_1,\delta)+G_4\,e^{-z(x_1;x_n)/h}\phi_-(t;x_1,\bar{\delta})]\\
=&G_1G_4\,e^{-z(x_1;b_n)/h}[\phi_+(t;x_1,\delta),\ \phi_-(t;x_1,\bar{\delta})]+G_2G_3\,e^{z(x_1;x_n)/h}[\phi_-(t;x_1,\bar{\delta}),\ \phi_+(t;x_1,\delta)]\\
=&\mathscr{W}(\bar{\delta};\delta)\left(G_1G_4\,e^{-z(x_1;b_n)/h}-G_2G_3\,e^{z(x_1;x_n)/h}\right).
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
t_{21}(\varepsilon,h) &= e^{z(x_1;\bar{x}_1)/h}\frac{\mathscr{W}(\bar{\delta};\delta)}{\mathscr{W}(\bar{r};r)}\left(G_1G_4\,e^{-z(x_1;x_n)/h}-G_2G_3\,e^{z(x_1;x_n)/h}\right)\\
&= i\,\frac{\mathscr{W}(\bar{\delta};\delta)}{\mathscr{W}(\bar{r};r)}\Bigg((-1)^{n+1}\frac{\mathscr{W}(\bar{\delta};r)}{\mathscr{W}(\bar{\delta};\delta)}\frac{\mathscr{W}(\hat{l};\delta)}{\mathscr{W}(\bar{\delta};\delta)}\,e^{z(x_n;\bar{x}_1)/h}\\
&\qquad\qquad +\frac{\mathscr{W}(\hat{r};\delta)}{\mathscr{W}(\bar{\delta};\delta)}\frac{\mathscr{W}(\bar{\delta};l)}{\mathscr{W}(\bar{\delta};\delta)}\,e^{\{z(x_1;\bar{x}_1)+z(x_1;x_n)\}/h}\Bigg).
\end{aligned}
$$

Finally we calculate $t_{22}(\varepsilon,h)$.

$$
\begin{aligned}
&[\phi_+(t;x_1,r),\ \phi_-(t;\bar{x}_n,\bar{l})]\\
=&[G_1\phi_+(t;x_1,\delta)+G_2\phi_-(t;x_1,\bar{\delta}),\ G_7\phi_+(t;\bar{x}_n,\delta)+G_8\phi_-(t;\bar{x}_n,\bar{\delta})]\\
=&[G_1\phi_+(t;x_1,\delta)+G_2\phi_-(t;x_1,\bar{\delta}),\ G_7\,e^{z(\bar{x}_1;\bar{x}_n)/h}\phi_+(t;x_1,\delta)+G_8\,e^{-z(\bar{x}_1;\bar{x}_n)/h}\phi_-(t;x_1,\bar{\delta})]\\
=&G_1G_8\,e^{-z(x_1;\bar{x}_n)/h}[\phi_+(t;x_1,\delta),\ \phi_-(t;x_1,\bar{\delta})]+G_2G_7\,e^{z(x_1;\bar{x}_n)/h}[\phi_-(t;x_1,\bar{\delta}),\ \phi_+(t;x_1,\delta)]\\
=&\mathscr{W}(\bar{\delta};\delta)\left(G_1G_8\,e^{z(\bar{x}_n;x_1)/h}-G_2G_7\,e^{z(x_1;\bar{x}_n)/h}\right).
\end{aligned}
$$

Hence we have

$$t_{22}(\varepsilon,h) = e^{z(x_1;\bar{x}_1)/h}\frac{\mathscr{W}(\bar{\delta};\delta)}{\mathscr{W}(\bar{r};r)}\Big(G_1G_8\,e^{-z(x_1;\bar{x}_n)/h} - G_2G_7\,e^{z(x_1;\bar{x}_n)/h}\Big)$$

$$= \frac{\mathscr{W}(\bar{\delta};\delta)}{\mathscr{W}(\bar{r};r)}\Bigg(\frac{\mathscr{W}(\bar{\delta};r)}{\mathscr{W}(\bar{\delta};\delta)}\frac{\mathscr{W}(\bar{l};\delta)}{\mathscr{W}(\bar{\delta};\delta)}\,e^{z(\bar{x}_n;\bar{x}_1)/h}$$

$$+(-1)^n\frac{\mathscr{W}(\hat{r};\delta)}{\mathscr{W}(\bar{\delta};\delta)}\frac{\mathscr{W}(\bar{\delta};\hat{\bar{l}})}{\mathscr{W}(\bar{\delta};\delta)}\,e^{\{z(x_1;\bar{x}_1)+z(x_1;\bar{x}_n)\}/h}\Bigg).$$

□

The components of the matrix $\tilde{S}(\varepsilon,h)$ are expressed from (3.3.3), (3.3.4) and (3.3.6) as

$$\tilde{S}(\varepsilon,h) = \begin{pmatrix} C_1^{-1}C_3\,t_{11}(\varepsilon,h)\exp\left[\frac{-z^r(x_1)+z^l(x_n)}{h}\right] & C_1^{-1}C_4\,t_{12}(\varepsilon,h)\exp\left[\frac{-z^r(x_1)-z^l(\bar{x}_n)}{h}\right] \\ C_2^{-1}C_3\,t_{21}(\varepsilon,h)\exp\left[\frac{+z^r(\bar{x}_1)+z^l(x_n)}{h}\right] & C_2^{-1}C_4\,t_{22}(\varepsilon,h)\exp\left[\frac{+z^r(\bar{x}_1)-z^l(\bar{x}_n)}{h}\right] \end{pmatrix}.$$

Recall that $C_k = C_k(\varepsilon,h) = 1+O(h)$ as $h\to 0$ uniformly with respect to ε. In the remaining part of this section, we denote such constants simply by $1+O(h)$.

We study each exponential part of the components of $\tilde{S}(\varepsilon,h)$.

(3.3.13)
$$\begin{aligned}
-z^r(x_1)+z^l(x_n) &= \frac{i}{2}\Big(-A_1(\varepsilon)+A_n(\varepsilon)\Big)+\frac{i}{2}\Big(A_\infty(\varepsilon)-A_{-\infty}(\varepsilon)\Big),\\
-z^r(x_1)-z^l(\bar{x}_n) &= \frac{i}{2}\Big(-A_1(\varepsilon)-\overline{A_n(\varepsilon)}\Big)+\frac{i}{2}\Big(A_\infty(\varepsilon)+A_{-\infty}(\varepsilon)\Big),\\
z^r(\bar{x}_1)+z^l(x_n) &= \frac{i}{2}\Big(\overline{A_1(\varepsilon)}+A_n(\varepsilon)\Big)+\frac{i}{2}\Big(-A_\infty(\varepsilon)-A_{-\infty}(\varepsilon)\Big),\\
z^r(\bar{x}_1)-z^l(\bar{x}_n) &= \frac{i}{2}\Big(\overline{A_1(\varepsilon)}-\overline{A_n(\varepsilon)}\Big)+\frac{i}{2}\Big(-A_\infty(\varepsilon)+A_{-\infty}(\varepsilon)\Big).
\end{aligned}$$

where the action integrals $A_\infty(\varepsilon)$ and $A_{-\infty}(\varepsilon)$ are defined by

$$A_\infty(\varepsilon) = 2\int_0^\infty\left(\sqrt{V(t)^2+\varepsilon^2}-\lambda_r\right)dt,$$

$$A_{-\infty}(\varepsilon) = 2\int_0^{-\infty}\left(\sqrt{V(t)^2+\varepsilon^2}-\lambda_l\right)dt.$$

We remark that each second term of (3.3.13) is pure imaginary.

When $n=1$, we calculate by Proposition 3.3.1 the component $\tilde{s}_{21}(\varepsilon,h)$, which gives the transition probability.

$$\tilde{s}_{21}(\varepsilon,h) = i\,\frac{\mathscr{W}(\hat{l};r)}{\mathscr{W}(\bar{r};r)}\,e^{z(x_1;\bar{x}_1)/h}e^{\{z^r(\bar{x}_1)+z^l(x_1)\}/h}\Big(1+O(h)\Big)$$

(3.3.14)
$$= i\exp\left[\frac{i}{h}A_1(\varepsilon)-\frac{i}{2h}\Big(A_\infty(\varepsilon)+A_{-\infty}(\varepsilon)\Big)\right]\frac{\mathscr{W}(\hat{l};r)}{\mathscr{W}(\bar{r};r)}\Big(1+O(h)\Big).$$

One sees that there exist canonical curves from r to $\bar{r}$ and to $\hat{l}$. Similarly the others are calculated.

$$\tilde{s}_{11}(\varepsilon,h) = \exp\left[\frac{i}{2h}\Big(A_\infty(\varepsilon) - A_{-\infty}(\varepsilon)\Big)\right]\frac{\mathscr{W}(\bar{r};l)}{\mathscr{W}(\bar{r};r)}\Big(1+O(h)\Big),$$

$$\tilde{s}_{12}(\varepsilon,h) = i\exp\left[-\frac{i}{h}\overline{A_1(\varepsilon)} + \frac{i}{2h}\Big(A_\infty(\varepsilon) + A_{-\infty}(\varepsilon)\Big)\right]\frac{\mathscr{W}(\bar{r};\hat{\bar{l}})}{\mathscr{W}(\bar{r};r)}\Big(1+O(h)\Big),$$

$$\tilde{s}_{22}(\varepsilon,h) = \exp\left[\frac{i}{2h}\Big(-A_\infty(\varepsilon) + A_{-\infty}(\varepsilon)\Big)\right]\frac{\mathscr{W}(\bar{l};r)}{\mathscr{W}(\bar{r};r)}\Big(1+O(h)\Big). \tag{3.3.15}$$

When $n \geq 2$, we also study the component $\tilde{s}_{21}(\varepsilon,h)$.

$$\begin{aligned}\tilde{s}_{21}(\varepsilon,h) = {}& i\exp\left[\frac{i}{2h}\Big(\overline{A_1(\varepsilon)} + A_n(\varepsilon)\Big) - \frac{i}{2h}\Big(A_\infty(\varepsilon) + A_{-\infty}(\varepsilon)\Big)\right]\frac{\mathscr{W}(\bar{\delta};\delta)}{\mathscr{W}(\bar{r};r)}\Big(1+O(h)\Big)\\ &\times\Bigg((-1)^{n+1}\frac{\mathscr{W}(\bar{\delta};r)}{\mathscr{W}(\bar{\delta};\delta)}\frac{\mathscr{W}(\hat{l};\delta)}{\mathscr{W}(\bar{\delta};\delta)}\exp\left[\frac{i}{2h}\Big(A_n(\varepsilon) - \overline{A_1(\varepsilon)}\Big)\right]\\ &\qquad + \frac{\mathscr{W}(\hat{r};\delta)}{\mathscr{W}(\bar{\delta};\delta)}\frac{\mathscr{W}(\bar{\delta};l)}{\mathscr{W}(\bar{\delta};\delta)}\exp\left[\frac{i}{2h}\Big(2A_1(\varepsilon) - \overline{A_1(\varepsilon)} - A_n(\varepsilon)\Big)\right]\Bigg),\\ = {}& i\exp\left[-\frac{i}{2h}\Big(A_\infty(\varepsilon) + A_{-\infty}(\varepsilon)\Big)\right]\frac{\mathscr{W}(\bar{\delta};\delta)}{\mathscr{W}(\bar{r};r)}\Big(1+O(h)\Big)\\ &\times\Bigg((-1)^{n+1}\frac{\mathscr{W}(\bar{\delta};r)}{\mathscr{W}(\bar{\delta};\delta)}\frac{\mathscr{W}(\hat{l};\delta)}{\mathscr{W}(\bar{\delta};\delta)}\exp\left[\frac{i}{h}A_n(\varepsilon)\right] + \frac{\mathscr{W}(\hat{r};\delta)}{\mathscr{W}(\bar{\delta};\delta)}\frac{\mathscr{W}(\bar{\delta};l)}{\mathscr{W}(\bar{\delta};\delta)}\exp\left[\frac{i}{h}A_1(\varepsilon)\right]\Bigg).\end{aligned} \tag{3.3.16}$$

One sees that there exists a canonical curve for each Wronskian calculation. The other components are also calculated in the same way. However when h goes to 0 we must be careful in the dependence on sufficiently small ε of the asymptotic expansions of the Wronskians. We will discuss it in the next section.

3.4 Asymptotics of the Wronskians as $h \to 0$

In this section we study how the asymptotic expansions of the Wronskians in $\tilde{S}(\varepsilon,h)$ (see (3.3.14), (3.3.15), and (3.3.16)) as $h \to 0$ depend on small ε. We must pay attention to the distance between the canonical curve and the turning points on the complex z-plane because the turning points $z = z(x_j)$ are simple poles of the kernel $K'(z)/K(z)$ of inductive integral operators (see (2.2.1)).

First let us consider the case $n=1$, in particular $\tilde{s}_{22}(\varepsilon,h)$ and $\tilde{s}_{21}(\varepsilon,h)$. Because $\mathrm{Re}\,z(t)$ increases in the direction from the upper half-plane to the lower one near the real axis (see (3.1.2)), the denominator $\mathscr{W}(\bar{r};r)=w_+^e(z(\bar{r});z(r))$ has a canonical curve from r to $\bar{r}$ whose distance from the turning points is positive uniformly with respect to ε as in Figure 3.12. Hence

$$w_+^e(z(\bar{r});z(r))=1+O(h) \qquad \text{as} \quad h\to 0,$$

uniformly for sufficiently small ε. The numerator $\mathscr{W}(\hat{l};r)=w_+^e(z(\hat{l});z(r))$ of $\tilde{s}_{21}(\varepsilon,h)$, which give the transition probability, has a canonical curve from r to $\hat{l}$ through the branch cut as in Figure 3.12. This curve can also be taken so that the distance from turning points is bounded below by a positive constant independent of ε. We have

$$w_+^e(z(\hat{l});z(r))=1+O(h) \qquad \text{as} \quad h\to 0,$$

uniformly with respect to ε. Hence we have

$$\tilde{s}_{21}(\varepsilon,h)=i\exp\left[\frac{i}{h}A_1(\varepsilon)-\frac{i}{2h}\Big(A_\infty(\varepsilon)+A_{-\infty}(\varepsilon)\Big)\right]\Big(1+O(h)\Big),$$

as h tends to 0 uniformly for sufficiently small ε.

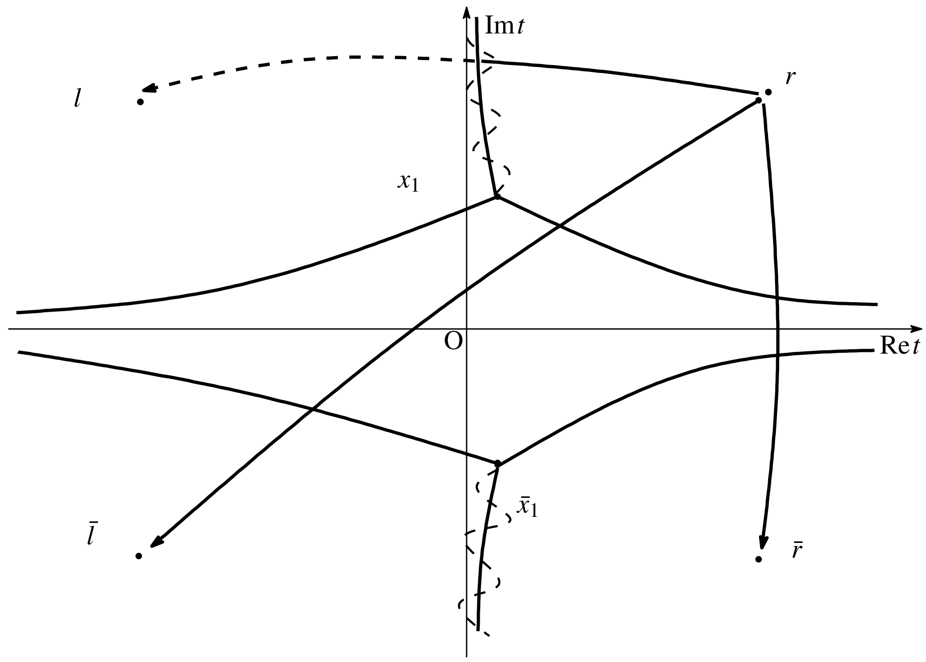

Figure 3.12: Canonical curves ($n=1$)

$\mathscr{W}(\bar{l};r)=w_+^e(z(\bar{l});z(r))$, that is the numerator of $\tilde{s}_{22}(\varepsilon,h)$, has a canonical curve from r to $\bar{l}$, which passes between x_1 and $\bar{x}_1$ as in Figure 3.12. In this case, however, the distance from this curve to the turning points $x_1,\bar{x}_1$ tends to 0 as $\varepsilon\to 0$ and we get

$$w_+^e(z(\bar{l});z(r))=1+O\left(\frac{h}{\mathrm{Re}\,z(\bar{x}_1)-\mathrm{Re}\,z(x_1)}\right) \qquad \text{as} \quad h\to 0.$$

By Lemma 1.2.1, we can take a canonical curve so that as $\varepsilon \to 0$,

$$\frac{1}{\operatorname{Re} z(\bar{x}_1) - \operatorname{Re} z(x_1)} = O\left(\varepsilon^{-2}\right).$$

Hence we obtain

$$\tilde{s}_{22}(\varepsilon, h) = \exp\left[\frac{i}{2h}\Big(-A_\infty(\varepsilon) + A_{-\infty}(\varepsilon)\Big)\right]\left(1 + O\left(\frac{h}{\varepsilon^2}\right)\right),$$

as both ε and h/ε^2 tend to 0.

In the case $n \geq 2$, there exit a lot of Wronskians in the components of $T(\varepsilon, h)$. In fact, these Wronskians are classified into three types: for example $\mathscr{W}(\bar{\delta}; \delta) = w_+^e(z(\bar{\delta}); z(\delta))$, $\mathscr{W}(\bar{\delta}; r) = w_+^e(z(\bar{\delta}); z(r))$ and $\mathscr{W}(\hat{r}; \delta) = w_+^e(z(\hat{r}); z(\delta))$. We draw these canonical curves under $\operatorname{Re} z(x_n) = \operatorname{Re} z(x_1)$ as in Figure 3.13. The canonical curve from δ to $\bar{\delta}$ passes between x_1 and x_n, that from r to $\bar{\delta}$ passes between x_1 and $\bar{x}_1$ and that from δ to $\hat{r}$ through the branch cut passes between x_1 and x_2 as in Figure 3.13. The distances from these canonical curves to those turning points goes to 0 as $\varepsilon \to 0$. We get

$$w_+^e(z(\bar{\delta}); z(\delta)) = 1 + O\left(\frac{h}{\operatorname{Im} z(x_1) - \operatorname{Im} z(x_n)}\right) \qquad \text{as} \quad h \to 0,$$

$$w_+^e(z(\bar{\delta}); z(r)) = 1 + O\left(\frac{h}{\operatorname{Re} z(\bar{x}_1) - \operatorname{Re} z(x_1)}\right) \qquad \text{as} \quad h \to 0,$$

$$w_+^e(z(\hat{r}); z(\delta)) = 1 + O\left(\frac{h}{\operatorname{Re} z(x_1) - \operatorname{Re} z(x_2)}\right) \qquad \text{as} \quad h \to 0.$$

On the other hand, by Lemma 1.2.1, we have as $\varepsilon \to 0$

$$\frac{1}{\operatorname{Im} z(x_1) - \operatorname{Im} z(x_n)} = O\left(\varepsilon^{-\frac{n+1}{n}}\right),$$

$$\frac{1}{\operatorname{Re} z(\bar{x}_1) - \operatorname{Re} z(x_1)} = O\left(\varepsilon^{-\frac{n+1}{n}}\right),$$

$$\frac{1}{\operatorname{Re} z(x_1) - \operatorname{Re} z(x_2)} = O\left(\varepsilon^{-\frac{n+1}{n}}\right).$$

We calculate the asymptotic expansions $w_+^e(z(\bar{\delta}); z(l))$ and $w_+^e(z(\hat{l}); z(\delta))$ in the same way as $w_+^e(z(\bar{\delta}); z(r))$ and $w_+^e(z(\hat{r}); z(\delta))$ respectively. Hence we obtain

$$\begin{aligned} \tilde{s}_{21}(\varepsilon, h) = {} & i \exp\left[-\frac{i}{2h}\Big(A_\infty(\varepsilon) + A_{-\infty}(\varepsilon)\Big)\right] \\ & \times \left((-1)^{n+1} \exp\left[\frac{i}{h} A_n(\varepsilon)\right] + \exp\left[\frac{i}{h} A_1(\varepsilon)\right]\right)\left(1 + O\left(\frac{h}{\varepsilon^{\frac{n+1}{n}}}\right)\right), \end{aligned}$$

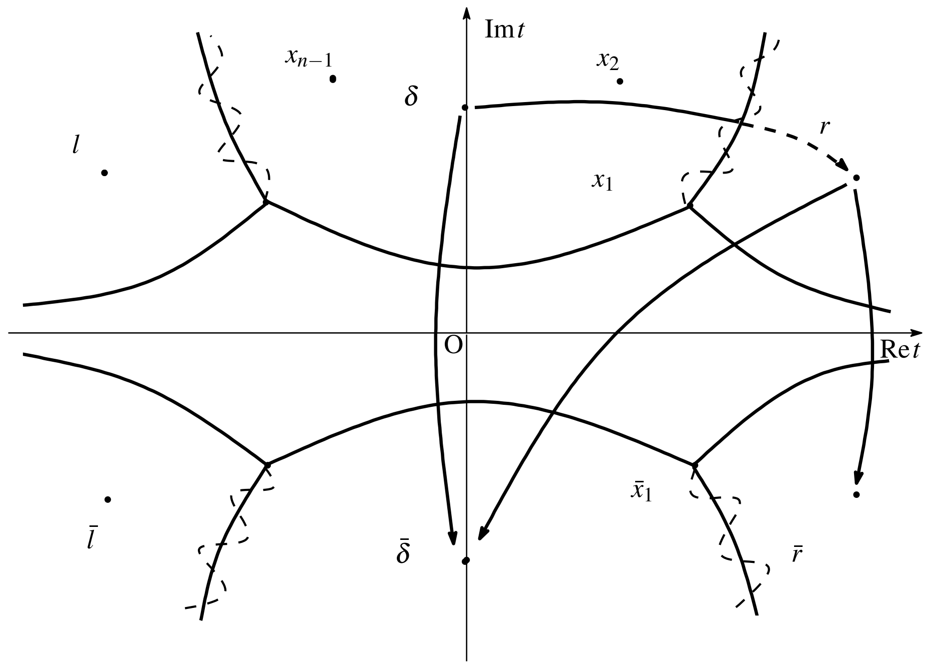

Figure 3.13: Canonical curves ($n \geq 2$)

as both ε and $h/\varepsilon^{n+1/n}$ tend to 0.

We remark that there exists a canonical curve from l to $\hat{r}$ in the case $\mathrm{Re}\, z(x_n) < \mathrm{Re}\, z(x_1)$ as in Figure 3.14.

The Wronskian can be calculated without intermediate exact WKB solutions as

$$\begin{aligned}[\phi_+(t;x_1,r), \phi_+(t;x_n,l)] &= \exp\left[\frac{i}{2h}(A_1(\varepsilon) - A_n(\varepsilon))\right] [\phi_+(t;x_1,r), \phi_+(t;x_1,l)] \\ &= -i \exp\left[\frac{i}{2h}(A_1(\varepsilon) - A_n(\varepsilon))\right] w_+^e(z(\hat{r}_+); z(l_+)).\end{aligned}$$

$$w_+^e(z(\hat{r}); z(l)) = 1 + O\left(\frac{h}{\mathrm{Re}\, z(x_1) - \mathrm{Re}\, z(x_n)}\right) \qquad \text{as} \quad h \to 0.$$

By Lemma 1.2.1 and Proposition 1.2.1, we have

$$\frac{1}{\mathrm{Re}\, z(x_1) - \mathrm{Re}\, z(x_n)} = O\left(\varepsilon^{-\frac{n+2m}{n}}\right) \qquad \text{as} \quad \varepsilon \to 0,$$

where m is positive integer in Proposition 1.2.1. The asymptotic expansions (1.2.3), (1.2.4) in Proposition 1.2.1 imply that $P(\varepsilon, h)$ in the case $n \geq 2$ can be calculated as in the case $n = 1$ when h goes to 0 faster than $\varepsilon^{(n+2m)/n}$ tends to 0 (see Figure 3.12).

Therefore, as the conclusion up to this section, we obtain the asymptotic expansion of the scattering matrix $S(\varepsilon, h)$:

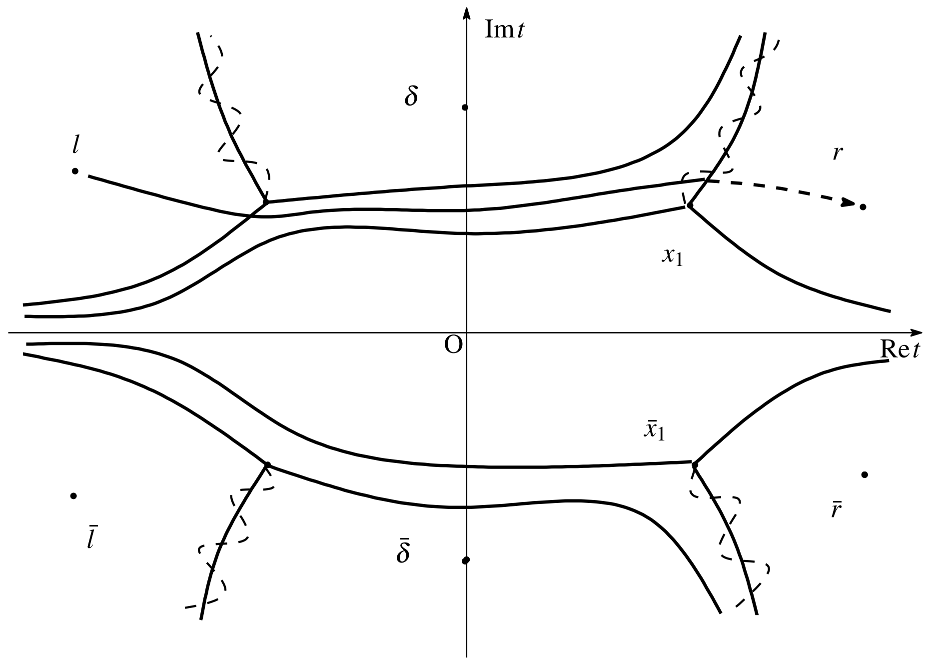

Figure 3.14: Canonical curve ($n \geq 2$ and $\mathrm{Re}\, z(x_n) < \mathrm{Re}\, z(x_1)$)

Proposition 3.4.1.

1. *In the case $n = 1$, We have*

$$s_{12}(\varepsilon,h) = -\exp\left[-\frac{i}{h}\overline{A_1(\varepsilon)} + \frac{i}{2h}\Big(A_\infty(\varepsilon) + A_{-\infty}(\varepsilon)\Big)\right]\Big(1+O(h)\Big), \quad \text{as} \quad h \to 0$$

$$s_{21}(\varepsilon,h) = \exp\left[\frac{i}{h}A_1(\varepsilon) - \frac{i}{2h}\Big(A_\infty(\varepsilon) + A_{-\infty}(\varepsilon)\Big)\right]\Big(1+O(h)\Big) \qquad \text{as} \quad h \to 0,$$

uniformly for sufficiently ε and

$$s_{11}(\varepsilon,h) = \exp\left[\frac{i}{2h}\Big(A_\infty(\varepsilon) - A_{-\infty}(\varepsilon)\Big)\right]\left(1+O\left(\frac{h}{\varepsilon^2}\right)\right) \qquad \text{as} \quad \frac{h}{\varepsilon^2} \to 0,$$

$$s_{22}(\varepsilon,h) = \exp\left[\frac{i}{2h}\Big(-A_\infty(\varepsilon) + A_{-\infty}(\varepsilon)\Big)\right]\left(1+O\left(\frac{h}{\varepsilon^2}\right)\right) \qquad \text{as} \quad \frac{h}{\varepsilon^2} \to 0,$$

for sufficiently small ε.

2. *In the case $n \geq 2$, we have*

$$s_{11}(\varepsilon,h) = \exp\left[\frac{i}{2h}\Big(A_\infty(\varepsilon) - A_{-\infty}(\varepsilon)\Big)\right]$$

$$\times\left(1+(-1)^n \exp\left[\frac{i}{h}\Big(-\overline{A_1(\varepsilon)}+A_n(\varepsilon)\Big)\right]\right)\left(1+O\left(\frac{h}{\varepsilon^{\frac{n+1}{n}}}\right)\right),$$

$$s_{12}(\varepsilon,h) = -\exp\left[\frac{i}{2h}\Big(A_\infty(\varepsilon) + A_{-\infty}(\varepsilon)\Big)\right]$$

$$\times\left(\exp\left[-\frac{i}{h}\overline{A_1(\varepsilon)}\right]+(-1)^{n+1} \exp\left[-\frac{i}{h}\overline{A_n(\varepsilon)}\right]\right)\left(1+O\left(\frac{h}{\varepsilon^{\frac{n+1}{n}}}\right)\right),$$

$$s_{21}(\varepsilon,h) = \exp\left[-\frac{i}{2h}\Big(A_\infty(\varepsilon) + A_{-\infty}(\varepsilon)\Big)\right]$$

$$\times\left(\exp\left[\frac{i}{h}A_1(\varepsilon)\right]+(-1)^{n+1} \exp\left[\frac{i}{h}A_n(\varepsilon)\right]\right)\left(1+O\left(\frac{h}{\varepsilon^{\frac{n+1}{n}}}\right)\right),$$

$$s_{22}(\varepsilon,h) = i\exp\left[\frac{i}{2h}\Big(-A_\infty(\varepsilon) + A_{-\infty}(\varepsilon)\Big)\right]$$

$$\times\left(1+(-1)^n \exp\left[\frac{i}{h}\Big(A_1(\varepsilon)-\overline{A_n(\varepsilon)}\Big)\right]\right)\left(1+O\left(\frac{h}{\varepsilon^{\frac{n+1}{n}}}\right)\right),$$

as $h/\varepsilon^{n+1/n}$ tends to 0 for any small ε.

3.5 Asymptotics of the action integral

We prove Lemma 1.2.1 which is important to study the decay rate of $P(\varepsilon,h)$ and the geometrical structures of the Stokes lines. Moreover we prove Proposition 1.2.1 by studying the relation between the asymptotic behavior of the action integral and the derivative of $V(t)$ at $t=0$.

Proof of Lemma 1.2.1

$$A_j(\varepsilon) = 2\int_0^{x_j(\varepsilon)} \sqrt{\left(\frac{V^{(n)}(0)}{n!}t^n v(t)\right)^2 + \varepsilon^2}\, dt.$$

By the change of variables $\varepsilon s^n = \frac{V^{(n)}(0)}{n!}t^n v(t)$, we get for small ε

$$A_j(\varepsilon) = 2\varepsilon\int_0^{\exp[\frac{2j-1}{2n}\pi i]} \sqrt{s^{2n}+\varepsilon^2}\ \left(\frac{dt}{ds}\right) ds.$$

By the Lagrange's formula, the Maclaurin expansion of t with respect to s is given by

$$t=\sum_{k=1}^{\infty}\frac{\varepsilon^{\frac{k}{n}}}{k!}\left(\frac{n!}{V^{(n)}(0)}\right)^{\frac{k}{n}}\left[\frac{d^{k-1}}{dz^{k-1}}\left(v(z)^{-\frac{k}{n}}\right)\right]_{z=0}s^k,$$

and hence

$$\frac{dt}{ds}=\sum_{k=1}^{\infty}\frac{\varepsilon^{\frac{k}{n}}}{(k-1)!}\left(\frac{n!}{V^{(n)}(0)}\right)^{\frac{k}{n}}\left[\frac{d^{k-1}}{dz^{k-1}}\left(v(z)^{-\frac{k}{n}}\right)\right]_{z=0}s^{k-1}.$$

Then the formula is obtained by term by term integrations and the identity

$$\int_0^{\exp[\frac{2j-1}{2n}\pi i]}s^{k-1}\sqrt{s^{2n}+1}\,ds=\frac{\sqrt{\pi}\,\Gamma(\frac{k}{2n})}{2(n+k)\,\Gamma(\frac{n+k}{2n})}\exp\left[\frac{(2j-1)k\pi i}{2n}\right].$$

□

Proof of Proposition 1.2.1 To prove Proposition 1.2.1, Let us calculate the principal term of $\mathrm{Im}(A_1(\varepsilon)-A_n(\varepsilon))$. From Lemma 1.2.1 we have

$$A_1(\varepsilon)=\sum_{k=1}^{\infty}C_k\exp\left[\frac{k\pi}{2n}i\right]\varepsilon^{\frac{n+k}{n}},\qquad A_n(\varepsilon)=\sum_{k=1}^{\infty}(-1)^kC_k\exp\left[-\frac{k\pi}{2n}i\right]\varepsilon^{\frac{n+k}{n}},$$

hence we get

$$\mathrm{Im}(A_1(\varepsilon)-A_n(\varepsilon))=2\sum_{k=1}^{\infty}C_{2k}\left(\sin\frac{k}{n}\pi\right)\varepsilon^{\frac{n+2k}{n}}.$$

The problem is reduced to studying whether C_{2k} vanishes or not. We make use of the following lemma:

Lemma 3.5.1. *Assume $v^{(2j-1)}(0)=0$ $(j=1,\dots,m)$ for any fixed $m\in\mathbb{N}$. Then we have for any positive rational number σ*

$$\left[\frac{d^{2j-1}}{dz^{2j-1}}\left(v(z)^{-\sigma}\right)\right]_{z=0}=0\qquad(j=1,\dots,m).\tag{3.5.1}$$

Proof of Lemma 3.5.1 We prove this lemma by induction on m. In the case where $m=1$, the statement (3.5.1) is evident. Assume that there exists $k\in\mathbb{N}$ such that (3.5.1) is true for all $m<k+1$.

By the Leibniz formula, we have

$$\left[\frac{d^{2k+1}}{dz^{2k+1}}\left(v(z)^{-\sigma}\right)\right]_{z=0}=-\sigma\left[\sum_{p=0}^{2k}\binom{2k}{p}v^{(2k+1-p)}(z)\frac{d^p}{dz^p}\left(v(z)^{-\sigma-1}\right)\right]_{z=0}$$

$$=-\sigma v^{(2k+1)}(0)-\sigma\sum_{q=1}^{k}\binom{2k}{2q-1}v^{(2k-2q+2)}(0)\left[\frac{d^{2q-1}}{dz^{2q-1}}\left(v(z)^{-\sigma-1}\right)\right]_{z=0}.$$

The second term vanishes from the assumption. If $v^{(2k+1)}(0) = 0$, the statement (3.5.1) with $m = k+1$ is also true.

$\square$

From this proof, if there exists $m \in \mathbb{N}$ such that $v^{(2j-1)}(0) = 0 \ (j = 1, \ldots, m-1)$ and $v^{(2m-1)}(0) \neq 0$, we obtain

$$\left[\frac{d^{2m-1}}{dz^{2m-1}}\left(v(z)^{-\sigma}\right)\right]_{z=0} = -\sigma v^{(2m-1)}(0).$$

If there exists $m \in \mathbb{N}$ such that $V^{(n+2l-1)}(0) = 0 \ (l = 0, \ldots, m-1)$ and $V^{(n+2m-1)}(0) \neq 0$, we get the following relation between derivatives of $V(t)$ and $v(t)$:

$$v^{(2m-1)}(0) = \frac{n!}{V^{(n)}(0)} \frac{(2m-1)!\,V^{(n+2m-1)}(0)}{(n+2m-1)!}$$

and moreover in the case where $m \geq 2$

$$v'(0) = v^{(3)}(0) = \cdots = v^{(2m-3)}(0) = 0.$$

Therefore, in this case, we obtain for sufficiently small ε

$$\mathrm{Im}(A_1(\varepsilon) - A_n(\varepsilon)) = 2C_{2m}\left(\sin\frac{m}{n}\pi\right)\varepsilon^{\frac{n+2m}{n}} + O\left(\varepsilon^{\frac{n+2m+2}{n}}\right),$$

where

$$C_{2m} = -\frac{2m\sqrt{\pi}\,\Gamma(\frac{m}{n})V^{(n+2m-1)}(0)}{n\Gamma(n+2m+1)\Gamma(\frac{n+2m}{2n})}\left(\frac{n!}{V^{(n)}(0)}\right)^{\frac{n+2m}{n}}.$$

$\square$

Chapter 4

Avoided crossings at several points

4.1 Scattering matrix in several avoided crossings

In this section we remove the assumption **(C1)** and give a formula of the scattering matrix as the product of transfer matrices.

Let $t_1 > t_2 > \cdots > t_N$ be the zeros of $V(t)$ and suppose $V(t_k) = V'(t_k) = \cdots = V^{(n_k-1)}(t_k) = 0$ and $V^{(n_k)}(t_k) \neq 0$. We can assume $V^{(n_1)}(t_1) > 0$ without loss of generality. The case $N = 1$ has been discussed in Chapter 3, hence in this section we consider the case $N \geq 2$.

There are $2n_k$ simple turning points around each $t = t_k$, which are denoted by $x_j^k(\varepsilon)$ and $\overline{x_j^k(\varepsilon)}$ $(j = 1, \ldots, n_k)$, and they behave like

$$x_j^k(\varepsilon) \sim t_k + \left(\frac{n_k!}{\left|V^{(n_k)}(t_k)\right|} \right)^{1/n_k} \exp\left[\frac{(2j-1)\pi i}{2n_k} \right] \varepsilon^{1/n_k} \qquad \text{as} \quad \varepsilon \to 0.$$

We also define the action integrals $A_j^k(\varepsilon)$ by

$$A_j^k(\varepsilon) = 2\int_{t_k}^{x_j^k(\varepsilon)} \sqrt{V(t)^2 + \varepsilon^2}\, dt,$$

where each integration path is the complex segment from t_k to $x_j^k(\varepsilon)$ and the branch of the square root is ε at $t = t_1$. We can express $V(t) = \frac{V^{(n_k)}(t_k)}{n_k!}(t-t_k)^{n_k} v_k(t-t_k)$, where $v_k(t)$ are holomorphic near $t = 0$ and $v_k(0) = 1$. Just like Lemma 1.2.1 we get

$$A_j^k(\varepsilon) = \sum_{q=1}^{\infty} C_q^k \exp\left[\frac{(2j-1)q\pi i}{2n_k} \right] \varepsilon^{\frac{n_k+q}{n_k}} \qquad (j = 1, \ldots, n_k), \tag{4.1.1}$$

where $C_q^k = \dfrac{\sqrt{\pi}\, \Gamma(\frac{q}{2n_k})}{(n_k+q)\, \Gamma(q)\, \Gamma(\frac{n_k+q}{2n_k})} \left(\dfrac{n_k!}{\left|V^{(n_k)}(t_k)\right|} \right)^{\frac{q}{n_k}} \left[\dfrac{d^{q-1}}{dz^{q-1}} \left(v_k(z)^{-\frac{q}{n_k}} \right) \right]_{z=0}.$

Similarly we define other action integrals by

$$A_\infty^1(\varepsilon) = 2\int_{t_1}^{\infty} \left(\sqrt{V(t)^2 + \varepsilon^2} - \lambda_r \right) dt, \qquad A_{-\infty}^N(\varepsilon) = 2\int_{t_N}^{-\infty} \left(\sqrt{V(t)^2 + \varepsilon^2} - \lambda_l \right) dt,$$

$$A_{k,k+1}(\varepsilon) = 2\int_{t_{k+1}}^{t_k} \sqrt{V(t)^2+\varepsilon^2}\,dt \quad (k=1,\dots,N-1).$$

We introduce the intermediate symbol base points $\delta_{k,k+1}$ $(k=1,\dots,N)$ and their complex conjugates as in Figure 4.1, Figure 4.2. In particular we put $r=\delta_{0,1}$, $l=\delta_{N,N+1}$. Then we consider exact WKB solutions:

$$(4.1.2)\quad \begin{aligned}
\phi_+(t;x_1^k,\delta_{k-1,k}) &= \exp\left[+\frac{z(t;x_1^k)}{h}\right] M_+(z(t))w_+(z(t);z(\delta_{k-1,k})),\\
\phi_-(t;\overline{x_1^k},\overline{\delta_{k-1,k}}) &= \exp\left[-\frac{z(t;\overline{x_1^k})}{h}\right] M_-(z(t))w_-(z(t);z(\overline{\delta_{k-1,k}})),\\
\phi_+(t;x_{n_k}^k,\delta_{k,k+1}) &= \exp\left[+\frac{z(t;x_{n_k}^k)}{h}\right] M_+(z(t))w_+(z(t);z(\delta_{k,k+1})),\\
\phi_-(t;\overline{x_{n_k}^k},\overline{\delta_{k,k+1}}) &= \exp\left[-\frac{z(t;\overline{x_{n_k}^k})}{h}\right] M_-(z(t))w_-(z(t);z(\overline{\delta_{k,k+1}})).
\end{aligned}$$

Notice that each exact WKB solution has a valid asymptotic expansion on h in the direction toward its phase base point from its symbol base point.

We define the transfer matrices $T_r(\varepsilon,h)$ and $T_l(\varepsilon,h)$ by

$$(4.1.3)\quad \Big(\phi_+^r(t)\ \phi_-^r(t)\Big) = \Big(\phi_+(t;x_1^1,r)\ \phi_-(t;\overline{x_1^1},\bar{r})\Big)\,T_r(\varepsilon,h),$$

$$(4.1.4)\quad \Big(\phi_+^l(t)\ \phi_-^l(t)\Big) = \Big(\phi_+(t;x_{n_N}^N,l)\ \phi_-(t;\overline{x_{n_N}^N},\bar{l})\Big)\,T_l(\varepsilon,h),$$

where $\phi_\pm^r(t)$ and $\phi_\pm^l(t)$ are the Jost solutions expressed by (3.1.7), and the transfer matrices $T_k(\varepsilon,h)$ around $t=t_k$:

$$(4.1.5)\quad \Big(\phi_+(t;x_{n_k}^k,\delta_{k,k+1})\ \phi_-(t;\overline{x_{n_k}^k},\overline{\delta_{k,k+1}})\Big) = \Big(\phi_+(t;x_1^k,\delta_{k-1,k})\ \phi_-(t;\overline{x_1^k},\overline{\delta_{k-1,k}})\Big)\,T_k(\varepsilon,h).$$

We also need the transfer matrices $T_{k,k+1}(\varepsilon,h)$ between t_k and t_{k+1} as

(4.1.6)
$$\Big(\phi_+(t;x_1^{k+1},\delta_{k,k+1})\ \phi_-(t;\overline{x_1^{k+1}},\overline{\delta_{k,k+1}})\Big) = \Big(\phi_+(t;x_{n_k}^k,\delta_{k,k+1})\ \phi_-(t;\overline{x_{n_k}^k},\overline{\delta_{k,k+1}})\Big)\,T_{k,k+1}(\varepsilon,h).$$

The transfer matrices $T_r(\varepsilon,h)$, $T_l(\varepsilon,h)$, and $T_{k,k+1}(\varepsilon,h)$ are diagonal matrices given by

(4.1.7)
$$T_r(\varepsilon,h) = \begin{pmatrix} \exp\left[\frac{i}{2h}\left(A_1^1 - A_\infty^1 + 2\lambda_r t_1\right)\right] & 0 \\ 0 & \exp\left[\frac{i}{2h}\left(A_\infty^1 - \overline{A_1^1} - 2\lambda_r t_1\right)\right] \end{pmatrix}\Big(1+O(h)\Big),$$

(4.1.8)

$$T_l(\varepsilon,h)=\begin{pmatrix} \exp\left[\frac{i}{2h}\left(A^N_{n_N}-A^N_{-\infty}+2\lambda_l t_N\right)\right] & 0 \\ 0 & \exp\left[\frac{i}{2h}\left(A^N_{-\infty}-\overline{A^N_{n_N}}-2\lambda_l t_N\right)\right] \end{pmatrix}\Big(1+O(h)\Big),$$

where $O(h)$ is uniform with respect to small ε, and

(4.1.9)

$$T_{k,k+1}(\varepsilon,h)=\begin{pmatrix} \exp\left[\frac{i}{2h}\left(A^k_{n_k}-A^{k+1}_1+A_{k,k+1}\right)\right] & 0 \\ 0 & \exp\left[-\frac{i}{2h}\left(\overline{A^k_{n_k}}-\overline{A^{k+1}_1}+A_{k,k+1}\right)\right] \end{pmatrix}.$$

Note that (4.1.7), (4.1.8) are obtained from the next relations:

$$\phi^r_+(t)=e^{+z^r(x^1_1)/h}\phi_+(t;x^1_1,r)\left(1+O(h)\right),\qquad \phi^r_-(t)=e^{-z^r(\overline{x^1_1})/h}\phi_-(t;\overline{x^1_1},\bar{r})\left(1+O(h)\right),$$

$$\phi^l_+(t)=e^{+z^l(x^N_{n_N})/h}\phi_+(t;x^N_{n_N},l)\left(1+O(h)\right),\quad \phi^l_-(t)=e^{-z^l(\overline{x^N_{n_N}})/h}\phi_-(t;\overline{x^N_{n_N}},\bar{l})\left(1+O(h)\right),$$

where each $O(h)$ is a constant (independent of t) depending on h and ε, which is of $O(h)$ as $h\to 0$ uniformly with respect to small ε.

We also denote the change of bases between (ϕ^r_+,ϕ^r_-) and (ϕ^l_+,ϕ^l_-) by $\tilde{S}(\varepsilon,h)$, whose components are expressed with the components of the scattering matrix $S(\varepsilon,h)$ as (3.1.9) and put

$$T_k(\varepsilon,h)=\begin{pmatrix} \tau^k_{11}(\varepsilon,h) & \tau^k_{12}(\varepsilon,h) \\ \tau^k_{21}(\varepsilon,h) & \tau^k_{22}(\varepsilon,h) \end{pmatrix},\qquad N_k=\sum_{j=1}^{k} n_j.$$

Then the asymptotic formula of $\tilde{S}(\varepsilon,h)$ as h tends to 0 is given by

Theorem 4.1.1. *The scattering matrix $\tilde{S}(\varepsilon,h)$ is the product of the 2×2 matrices $T_r(\varepsilon,h)$, $T_l(\varepsilon,h)$, $T_{k,k+1}(\varepsilon,h)$, and $T_k(\varepsilon,h)$:*

(4.1.10) $$\tilde{S}(\varepsilon,h)=T_r^{-1}(\varepsilon,h)T_1(\varepsilon,h)T_{1,2}(\varepsilon,h)T_2(\varepsilon,h)\cdots T_{N-1,N}(\varepsilon,h)T_N(\varepsilon,h)T_l(\varepsilon,h).$$

Moreover $T_k(\varepsilon,h)$ has the following asymptotic formulae:
In the case $n_k=1$, one has

(4.1.11) $$\tau^k_{11}(\varepsilon,h)=1+O\left(\frac{h}{\varepsilon^2}\right)\qquad \text{as}\quad \frac{h}{\varepsilon^2}\to 0,$$

(4.1.12) $$\tau^k_{12}(\varepsilon,h)=(-1)^{N_{k-1}}i\exp\left[-\frac{1}{h}\mathrm{Im}A^k_1\right]\Big(1+O(h)\Big)\qquad \text{as}\quad h\to 0,$$

(4.1.13) $$\tau^k_{21}(\varepsilon,h)=(-1)^{N_{k-1}}i\exp\left[-\frac{1}{h}\mathrm{Im}A^k_1\right]\Big(1+O(h)\Big)\qquad \text{as}\quad h\to 0,$$

(4.1.14) $$\tau^k_{22}(\varepsilon,h)=1+O\left(\frac{h}{\varepsilon^2}\right)\qquad \text{as}\quad \frac{h}{\varepsilon^2}\to 0.$$

In the case $n_k \geq 2$, one has

$$\tau_{11}^k(\varepsilon,h) = \left(\exp\left[\frac{i}{2h}\left(A_1^k - A_{n_k}^k\right)\right] + (-1)^{n_k}\exp\left[\frac{i}{2h}\left(A_1^k - 2\overline{A_1^k} + A_{n_k}^k\right)\right]\right)$$

$$\times\left(1+O\left(\frac{h}{\varepsilon^{\frac{n_k+1}{n_k}}}\right)\right) \qquad \text{as} \quad \frac{h}{\varepsilon^{\frac{n_k+1}{n_k}}} \to 0, \tag{4.1.15}$$

$$\tau_{12}^k(\varepsilon,h) = (-1)^{N_{k-1}} i \left((-1)^{n_k+1}\exp\left[\frac{i}{2h}\left(A_1^k - \overline{A_{n_k}^k}\right)\right] + \exp\left[\frac{i}{2h}\left(A_1^k - 2\overline{A_1^k} + \overline{A_{n_k}^k}\right)\right]\right)$$

$$\times\left(1+O\left(\frac{h}{\varepsilon^{\frac{n_k+1}{n_k}}}\right)\right) \qquad \text{as} \quad \frac{h}{\varepsilon^{\frac{n_k+1}{n_k}}} \to 0, \tag{4.1.16}$$

$$\tau_{21}^k(\varepsilon,h) = (-1)^{N_{k-1}} i \left((-1)^{n_k+1}\exp\left[\frac{i}{2h}\left(A_{n_k}^k - \overline{A_1^k}\right)\right] + \exp\left[\frac{i}{2h}\left(2A_1^k - \overline{A_1^k} - A_{n_k}^k\right)\right]\right)$$

$$\times\left(1+O\left(\frac{h}{\varepsilon^{\frac{n_k+1}{n_k}}}\right)\right) \qquad \text{as} \quad \frac{h}{\varepsilon^{\frac{n_k+1}{n_k}}} \to 0, \tag{4.1.17}$$

$$\tau_{22}^k(\varepsilon,h) = \left(\exp\left[\frac{i}{2h}\left(\overline{A_{n_k}^k} - \overline{A_1^k}\right)\right] + (-1)^{n_k}\exp\left[\frac{i}{2h}\left(2A_1^k - \overline{A_1^k} - \overline{A_{n_k}^k}\right)\right]\right)$$

$$\times\left(1+O\left(\frac{h}{\varepsilon^{\frac{n_k+1}{n_k}}}\right)\right) \qquad \text{as} \quad \frac{h}{\varepsilon^{\frac{n_k+1}{n_k}}} \to 0. \tag{4.1.18}$$

Notice that the turning points closest to the real axis are important in the calculation of the transfer matrix as in Chapter 3. The geometrical structures of Stokes lines are locally the same as in the case **(C1)** (see Figure 4.1, Figure 4.2). Hence the idea of this proof is also similar to Proposition 3.3.1 and Proposition 3.3.2; however we must be careful whether the turning points x_1^k and $x_{n_k}^k$ are zeros of $V(t)+\varepsilon$ or $V(t)-\varepsilon$. The sign of $V^{(n_k)}(t_k)$ is determined by the parity of N_{k-1}. One sees from $V^{(n_1)}(t_1) > 0$ that if $N_{k-1} = \sum_{j=1}^{k-1} n_j$ is even, then $V^{(n_k)}(t_k) > 0$ and if N_{k-1} is odd, then $V^{(n_k)}(t_k) < 0$. From this fact we can find a canonical curve through the branch cut like Lemma 3.3.2, hence we obtain the asymptotic behavior of $T_k(\varepsilon,h)$ as h tend to 0 from Proposition 2.4.1.

4.2 Avoided crossing at two points

In this section we study a special case where $V(t)$ vanishes at two points on $\mathbb{R}$. Joye indicated in [J1] that the asymptotic behavior of the transition probability as $h \to 0$ is determined by the geometrical structures generated by the Stokes lines closest to the real axis among those passing through some turning points. In our problem, when the spectral gap ε tends to 0, such turning points are determined by the asymptotic behavior of the imaginary part of the corresponding action integral (see (4.1.1)). More precisely, they are either x_1^k or $x_{n_k}^k$ if the vanishing order n_k of $V(t)$ at t_k is the minimum of $\{n_l\}_{l=1}^N$.

Here we restrict ourselves to the case $N=2$, that is,

(C2) $V(t)$ vanishes at two points $t=t_1,t_2$ $(t_1>t_2)$ on $\mathbb{R}$

and investigate which of the two makes the major contribution to the asymptotic behavior of the transition probability $P(\varepsilon,h)$ as ε and h tend to 0 according to each vanishing order.

For simplicity, we denote turning points $x_j^1(\varepsilon)$ $(j=1,\dots,n_1)$ and $x_j^2(\varepsilon)$ $(j=1,\dots,n_2)$ by $x_j(\varepsilon)$ $(j=1,\dots,n)$ and $y_j(\varepsilon)$ $(j=1,\dots,m)$ respectively, and the action integral $A_{1,2}(\varepsilon)$ by $B(\varepsilon)$. The Stokes lines passing through those turning points are drawn in Figure 4.1, Figure 4.2. We remark that Figure 4.1 is drawn under the condition $\mathrm{Im}A_1^1(\varepsilon)<\mathrm{Im}A_1^2(\varepsilon)$ and Figure 4.2 is done under the condition $\mathrm{Im}A_n^1(\varepsilon)<\mathrm{Im}A_1^1(\varepsilon)$ and $\mathrm{Im}A_n^1(\varepsilon)<\mathrm{Im}A_m^2(\varepsilon)<\mathrm{Im}A_1^2(\varepsilon)$. The dashed wave lines are always branch cuts.

In the case $n=m=1$, we obtain an analogous result to Theorem 1.2.2:

Theorem 4.2.1. *Assume* **(A)**, **(B)**, **(C2)**, *and* $n=m=1$. *Then there exists* $\varepsilon_0>0$ *such that we have for any* $\varepsilon\in(0,\varepsilon_0)$

$$P(\varepsilon,h)=\left|\exp\left[\frac{i}{h}\big(A_1^1(\varepsilon)+B(\varepsilon)\big)\right]-\exp\left[\frac{i}{h}A_1^2(\varepsilon)\right]\right|^2\left(1+O\left(\frac{h}{\varepsilon^2}\right)\right)\ \text{as}\ \frac{h}{\varepsilon^2}\to 0.$$

Remark 4.2.1. *Although the order of each zero of* $V(t)$ *is one as in the case* **(C1)**, *the error term is no longer uniform with respect to* ε.

Let $P_1(\varepsilon,h)$ be the principal term of this asymptotic expansion. Then we have

$$\begin{aligned}P_1(\varepsilon,h)&=\left|\exp\left[\frac{i}{h}\big(A_1^1(\varepsilon)+B(\varepsilon)\big)\right]-\exp\left[\frac{i}{h}A_1^2(\varepsilon)\right]\right|^2\\&=\exp\left[-\frac{\mathrm{Im}(A_1^1(\varepsilon)+A_1^2(\varepsilon))}{h}\right]\\&\qquad\left(\exp\left[\frac{\mathrm{Im}(A_1^1(\varepsilon)-A_1^2(\varepsilon))}{h}\right]+\exp\left[\frac{\mathrm{Im}(A_1^2(\varepsilon)-A_1^1(\varepsilon))}{h}\right]\right.\\&\qquad\qquad\left.-2\cos\left[\frac{\mathrm{Re}(A_1^1(\varepsilon)-A_1^2(\varepsilon))+B(\varepsilon)}{h}\right]\right).\end{aligned}$$

For each positive integer l, we consider the following condition on the derivative of $V(t)$ at t_1 and t_2:

$$(\mathbf{D}_l):\ V^{(2l-1)}(t_1)=-V^{(2l-1)}(t_2),\quad V^{(2l)}(t_1)=V^{(2l)}(t_2).$$

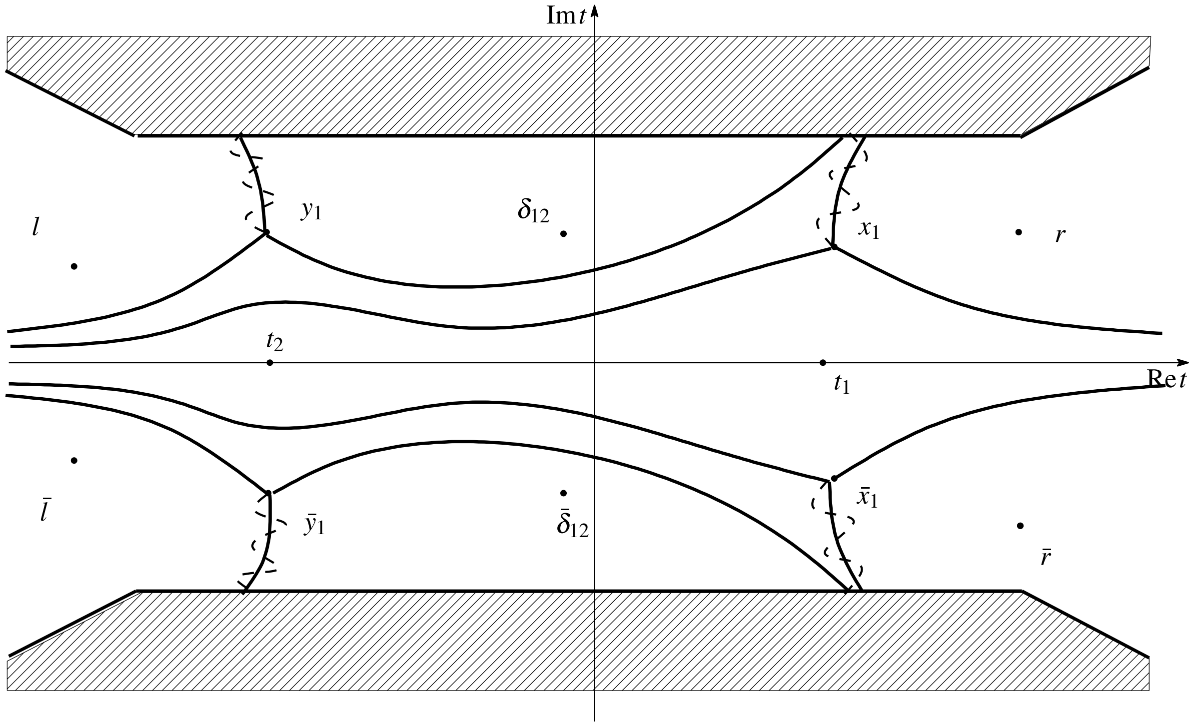

Figure 4.1: Stokes geometry $n = m = 1$

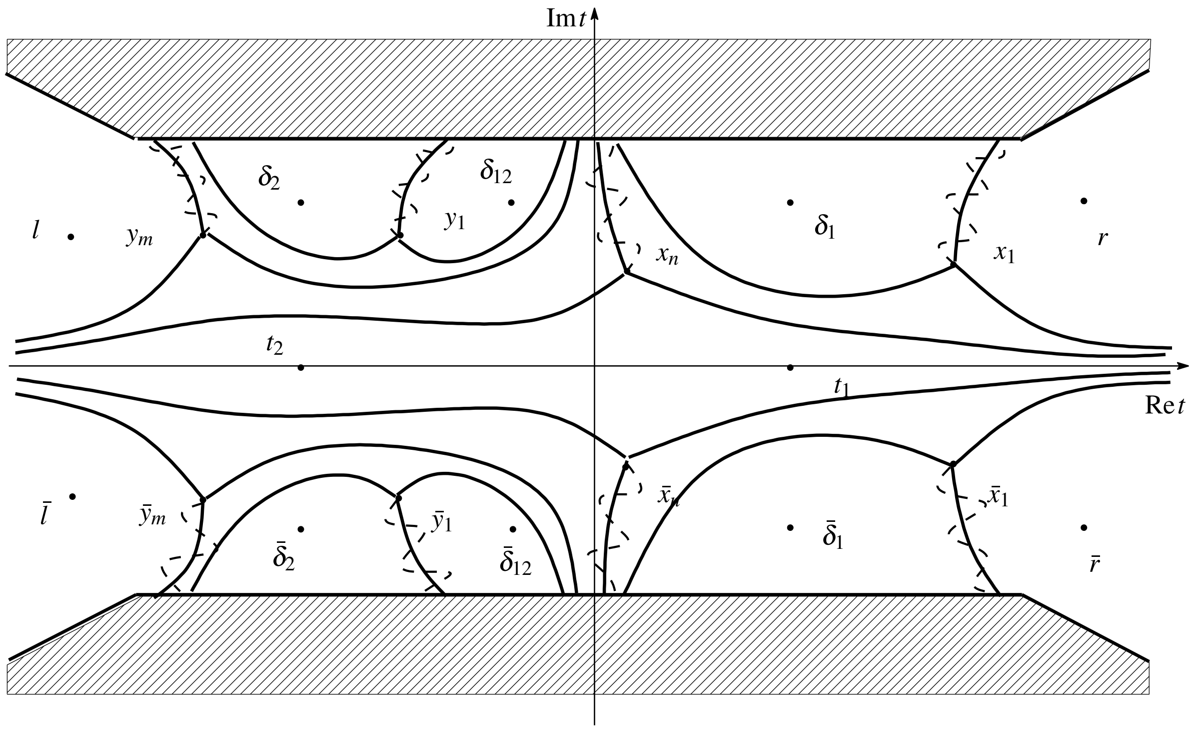

Figure 4.2: Stokes geometry $n \geq 2$, $m \geq 2$

Proposition 4.2.1.
1) *If* $(\mathbf{D}_l)$ *holds for all* $l \in \mathbb{N}$, *then one has*

$$\mathrm{Im} A_1^1(\varepsilon) = \mathrm{Im} A_1^2(\varepsilon)$$

and

$$P_1(\varepsilon,h) = 4\sin^2\left[\frac{\mathrm{Re}(A_1^1(\varepsilon)-A_1^2(\varepsilon))+B(\varepsilon)}{2h}\right]\exp\left[-\frac{2\mathrm{Im}A_1^1(\varepsilon)}{h}\right].$$

2) *If there exists a positive integer* κ *such that* $(\mathbf{D}_l)$ *holds for* l $(l=0,\dots,\kappa-1)$ *and* $(\mathbf{D}_\kappa)$ *does not hold, then*

$$\mathrm{Im}\big(A_1^1(\varepsilon)-A_1^2(\varepsilon)\big) = R_\kappa(t_1,t_2)\,\varepsilon^{2\kappa}+O\left(\varepsilon^{2\kappa+2}\right) \tag{4.2.1}$$

as $\varepsilon \to 0$, *where*

$$R_1(t_1,t_2) = \frac{\pi\left(|V'(t_2)|-V'(t_1)\right)}{2V'(t_1)|V'(t_2)|},$$

$$R_\kappa(t_1,t_2) = (-1)^{\kappa-1}\frac{\sqrt{\pi}\,\Gamma(\frac{2\kappa-1}{2})}{4\kappa\Gamma(2\kappa-1)\Gamma(\kappa+1)}(V'(t_1))^{-2\kappa-1}$$
$$\times\Bigg\{(\kappa+1)(4\kappa^2-1)V^{(2)}(t_1)\left(V^{(2\kappa)}(t_1)-V^{(2\kappa)}(t_2)\right)$$
$$-2\kappa V'(t_1)\left(V^{(2\kappa+1)}(t_1)+V^{(2\kappa+1)}(t_2)\right)\Bigg\}\quad(\kappa\geq 2).$$

The asymptotic behavior of $P_1(\varepsilon,h)$ *as* $(\varepsilon,h)\to(0,0)$ *is given by the following formulae:*

(i) *When* $\varepsilon^{2\kappa+2}/h\to 0$, $P_1(\varepsilon,h)$ *is equal to*

$$4\sin^2\left[\frac{\mathrm{Re}(A_1^1(\varepsilon)-A_1^2(\varepsilon))+B(\varepsilon)}{2h}\right]\exp\left[-\frac{\mathrm{Im}(A_1^1(\varepsilon)+A_1^2(\varepsilon))}{h}\right]\left(1+O\left(\frac{\varepsilon^{2(2\kappa+2)}}{h^2}\right)\right).$$

(ii) *If* $R_\kappa(t_1,t_2)$ *does not vanish, when* $\varepsilon^{2\kappa+2}/h\to\infty$, $P_1(\varepsilon,h)$ *is equal to*

$$\exp\left[-\frac{2}{h}\min\left\{\mathrm{Im}A_1^1(\varepsilon),\mathrm{Im}A_1^2(\varepsilon)\right\}\right]\left(1+O\left(\exp\left[-\left(\left|R_\kappa(t_1,t_2)\right|-\delta\right)\frac{\varepsilon^{2\kappa+2}}{h}\right]\right)\right)$$

for any positive constant δ.

Theorem 4.2.2. *Assume* **(A)**, **(B)**, **(C2)**, *and* $n=1$, $m\geq 2$. *Then there exists* $\varepsilon_0>0$ *such that we obtain for any* $\varepsilon\in(0,\varepsilon_0)$

$$P(\varepsilon,h) = \exp\left[-\frac{2}{h}\mathrm{Im}A_1^1(\varepsilon)\right]\left(1+O\left(\frac{h}{\varepsilon^{\frac{m+1}{m}}}\right)\right)\quad\text{as}\quad\frac{h}{\varepsilon^{\frac{m+1}{m}}}\to 0.$$

Remark 4.2.2. *This theorem implies that the contribution from t_2 is exponentially small with respect to that from t_1 and the principal term of $P(\varepsilon,h)$ is determined by the turning points around $t=t_1$. On the other hand, the estimate of the error term is determined by those around $t=t_2$.*

Theorem 4.2.3. *Assume* **(A)**, **(B)**, **(C2)**, *and $m \geq n \geq 2$. Then there exists $\varepsilon_0 > 0$ such that we obtain for any $\varepsilon \in (0,\varepsilon_0)$*

$$P(\varepsilon,h) = \left| \exp\left[\frac{i}{h}\left(A_1^1(\varepsilon)+B(\varepsilon)\right)\right] + (-1)^{n+1}\exp\left[\frac{i}{h}\left(A_n^1(\varepsilon)+B(\varepsilon)\right)\right] \right.$$
$$\left. + (-1)^n \exp\left[\frac{i}{h}A_1^2(\varepsilon)\right] + (-1)^{n+m+1}\exp\left[\frac{i}{h}A_m^2(\varepsilon)\right]\right|^2 \left(1+O\left(\frac{h}{\varepsilon^{\frac{n+1}{n}}}\right)\right)$$

as $h/\varepsilon^{\frac{n+1}{n}} \to 0$. In particular, when $m > n$ we have

$$P(\varepsilon,h) = \left|\exp\left[\frac{i}{h}A_1^1(\varepsilon)\right] + (-1)^{n+1}\exp\left[\frac{i}{h}A_n^1(\varepsilon)\right]\right|^2 \left(1+O\left(\frac{h}{\varepsilon^{\frac{n+1}{n}}}\right)\right) \tag{4.2.2}$$

as $h/\varepsilon^{\frac{n+1}{n}} \to 0$.

Remark 4.2.3. *The asymptotic expansion* (4.2.2) *is the same as that in Theorem* 1.2.2. *Even if $V(t)$ vanishes at two points $t=t_1,t_2$, we can presume such a case to be the case where it vanishes at one point $t=t_1$.*

We will prove the results of this section by using Theorem 4.1.1. From (4.1.10) the components of $\tilde{S}(\varepsilon,h)$ is expressed as follows:

$$\tilde{s}_{11} = \left\{\tau_{11}^1\tau_{11}^2 \exp\left[\frac{i}{2h}\left(A_n^1 - A_1^2 + B\right)\right] + \tau_{12}^1\tau_{21}^2\exp\left[\frac{i}{2h}\left(\overline{A_1^2} - \overline{A_n^1} - B\right)\right]\right\}$$
$$\exp\left[\frac{i}{2h}\left(-A_1^1 + A_m^2 + A_\infty^1 - A_{-\infty}^2 - 2\lambda_r t_1 + 2\lambda_l t_2\right)\right]\left(1+O(h)\right), \tag{4.2.3}$$

$$\tilde{s}_{12} = \left\{\tau_{11}^1\tau_{12}^2 \exp\left[\frac{i}{2h}\left(A_n^1 - A_1^2 + B\right)\right] + \tau_{12}^1\tau_{22}^2\exp\left[\frac{i}{2h}\left(\overline{A_1^2} - \overline{A_n^1} - B\right)\right]\right\}$$
$$\exp\left[\frac{i}{2h}\left(-A_1^1 - \overline{A_m^2} + A_\infty^1 + A_{-\infty}^2 - 2\lambda_r t_1 - 2\lambda_l t_2\right)\right]\left(1+O(h)\right), \tag{4.2.4}$$

$$\tilde{s}_{21} = \left\{ \tau_{21}^1 \tau_{11}^2 \exp\left[\frac{i}{2h}\Big(A_n^1 - A_1^2 + B\Big)\right] + \tau_{22}^1 \tau_{21}^2 \exp\left[\frac{i}{2h}\Big(\overline{A_1^2} - \overline{A_n^1} - B\Big)\right]\right\} \tag{4.2.5}$$
$$\exp\left[\frac{i}{2h}\Big(\overline{A_1^1} + A_m^2 - A_\infty^1 - A_{-\infty}^2 + 2\lambda_r t_1 + 2\lambda_l t_2\Big)\right]\Big(1 + O(h)\Big),$$

$$\tilde{s}_{22} = \left\{ \tau_{21}^1 \tau_{12}^2 \exp\left[\frac{i}{2h}\Big(A_n^1 - A_1^2 + B\Big)\right] + \tau_{22}^1 \tau_{22}^2 \exp\left[\frac{i}{2h}\Big(\overline{A_1^2} - \overline{A_n^1} - B\Big)\right]\right\} \tag{4.2.6}$$
$$\exp\left[\frac{i}{2h}\Big(\overline{A_1^1} - \overline{A_m^2} - A_\infty^1 + A_{-\infty}^2 + 2\lambda_r t_1 - 2\lambda_l t_2\Big)\right]\Big(1 + O(h)\Big).$$

Proof of Theorem 4.2.1. When $n = m = 1$, the component corresponding to the transition probability (4.2.5) is given by

$$\tilde{s}_{21} = \left\{ \tau_{21}^1 \tau_{11}^2 \exp\left[\frac{i}{2h}\Big(A_1^1 - A_1^2 + B\Big)\right] + \tau_{22}^1 \tau_{21}^2 \exp\left[\frac{i}{2h}\Big(\overline{A_1^2} - \overline{A_1^1} - B\Big)\right]\right\}$$
$$\exp\left[\frac{i}{2h}\Big(\overline{A_1^1} + A_1^2 - A_\infty^1 - A_{-\infty}^2 + 2\lambda_r t_1 + 2\lambda_l t_2\Big)\right]\Big(1 + O(h)\Big).$$

We put $B_\infty(\varepsilon) = -A_\infty^1(\varepsilon) - A_{-\infty}^2(\varepsilon) + 2\lambda_r t_1 + 2\lambda_l t_2$, where B_∞ is real-valued. We have, from the components of the transfer matrices (4.1.13) and (4.1.14) in the case $n = 1$, $m = 1$,

$$\begin{aligned}
\tilde{s}_{21}(\varepsilon, h) &= i \exp\left[\frac{i}{2h}\Big(\overline{A_1^1(\varepsilon)} + A_1^2(\varepsilon) + B_\infty(\varepsilon)\Big)\right]\Big(1 + O(h)\Big) \\
&\qquad \times \left\{ \exp\left[\frac{i}{2h}\Big(2A_1^1(\varepsilon) - \overline{A_1^1(\varepsilon)} - A_1^2(\varepsilon) + B(\varepsilon)\Big)\right]\left(1 + O\left(\frac{h}{\varepsilon^2}\right)\right)\right. \\
&\qquad\qquad \left. - \exp\left[\frac{i}{2h}\Big(A_1^2(\varepsilon) - \overline{A_1^1(\varepsilon)} - B(\varepsilon)\Big)\right]\left(1 + O\left(\frac{h}{\varepsilon^2}\right)\right)\right\}, \\
&= i \exp\left[\frac{i}{2h}\Big(B_\infty(\varepsilon) - B(\varepsilon)\Big)\right]\left(1 + O\left(\frac{h}{\varepsilon^2}\right)\right) \\
&\qquad \times \left(\exp\left[\frac{i}{h}\Big(A_1^1(\varepsilon) + B(\varepsilon)\Big)\right] - \exp\left[\frac{i}{h}\Big(A_1^2(\varepsilon)\Big)\right]\right)
\end{aligned}$$

as h/ε^2 tends to 0 for any small ε. Notice that $B(\varepsilon)$ as well as $B_\infty(\varepsilon)$ are real-valued. Hence we obtain

$$|\tilde{s}_{21}(\varepsilon, h)|^2 = \left|\exp\left[\frac{i}{h}\Big(A_1^1(\varepsilon) + B(\varepsilon)\Big)\right] - \exp\left[\frac{i}{h}\Big(A_1^2(\varepsilon)\Big)\right]\right|^2 \left(1 + O\left(\frac{h}{\varepsilon^2}\right)\right),$$

as h/ε^2 tends to 0 for any small ε. □

Proof of Proposition 4.2.1. The proof of this proposition is similar to that of Proposition 1.2.1 and we study the asymptotic behavior of $\mathrm{Im}\,(A_1^1(\varepsilon)-A_1^2(\varepsilon))$.

From (4.1.1), we have under $n=m=1$

$$\mathrm{Im}\left(A_1^1(\varepsilon)-A_1^2(\varepsilon)\right)=\sum_{q=0}^{\infty}(-1)^q\left(C_{2q+1}^1-C_{2q+1}^2\right)\varepsilon^{2q+2}.$$

When $V'(t_1)\neq|V'(t_2)|$, we get

$$C_{2q+1}^1-C_{2q+1}^2=\frac{\pi}{2}\left(\frac{1}{V'(t_1)}-\frac{1}{|V'(t_2)|}\right).$$

If $V'(t_1)=-V'(t_2)$, then $C_{2q+1}^1-C_{2q+1}^2$ is equal to

$$\frac{\sqrt{\pi}\,\Gamma(\frac{2q+1}{2})}{2\Gamma(2q+1)\Gamma(q+2)}\left(V'(t_1)\right)^{-(2q+1)}\left[\frac{d^{2q}}{dz^{2q}}\left((v_1(z))^{-(2q+1)}-(v_2(z))^{-(2q+1)}\right)\right]_{z=0}. \tag{4.2.7}$$

Hence we study whether the last factor vanishes or not. Here we give a lemma analogous to Lemma 3.5.1.

Lemma 4.2.1. *Assume for any fixed $\kappa\in\mathbb{N}$ the derivative condition*

$$(\mathbf{d}_l):\ v_1^{(2l-1)}(0)=-v_2^{(2l-1)}(0),\quad v_1^{(2l)}(0)=v_2^{(2l)}(0) \tag{4.2.8}$$

holds for l ($l=1,\dots,\kappa$). Then we have for any positive integer σ

$$\left[\frac{d^{2l-1}}{dz^{2l-1}}\left(v_1(z)^{-\sigma}+v_2(z)^{-\sigma}\right)\right]_{z=0}=0\qquad(l=1,\dots,\kappa), \tag{4.2.9}$$

$$\left[\frac{d^{2l}}{dz^{2l}}\left(v_1(z)^{-\sigma}-v_2(z)^{-\sigma}\right)\right]_{z=0}=0\qquad(l=1,\dots,\kappa). \tag{4.2.10}$$

Proof of Lemma 4.2.1 We prove this lemma by induction on κ. For $\kappa=1$, (4.2.9) and (4.2.10) are calculated as

$$\begin{aligned}\left[\frac{d}{dz}\left(v_1(z)^{-\sigma}+v_2(z)^{-\sigma}\right)\right]_{z=0}&=-\sigma\left[v_1(z)^{-(\sigma+1)}v_1'(z)+v_1(z)^{-(\sigma+1)}v_2'(z)\right]_{z=0}\\&=-\sigma(v_1'(0)+v_2'(0))=0,\end{aligned}$$

$$\begin{aligned}\left[\frac{d^2}{dz^2}\left(v_1(z)^{-\sigma}-v_2(z)^{-\sigma}\right)\right]_{z=0}&=-\sigma\frac{d}{dz}\left[v_1(z)^{-(\sigma+1)}v_1'(z)-v_1(z)^{-(\sigma+1)}v_2'(z)\right]_{z=0}\\&=\sigma\Big[(\sigma+1)\left(v_1(z)^{-(\sigma+2)}(v_1'(z))^2-v_1(z)^{-(\sigma+2)}(v_2'(z))^2\right)\\&\qquad-\left(v_1(z)^{-(\sigma+1)}v_1''(z)-v_1(z)^{-(\sigma+1)}v_2''(z)\right)\Big]_{z=0}\\&=\sigma\Big\{(\sigma+1)\left((v_1'(0))^2-(v_2'(0))^2\right)-(v_1''(0)-v_2''(0))\Big\}=0.\end{aligned}$$

Notice that $v_1(0) = v_2(0) = 1$. Hence the statement is true for $\kappa = 1$.

Assume that there exists $\kappa \in \mathbb{N}$ such that (4.2.9) and (4.2.10) is true for all $l < \kappa + 1$.

$$\left[\frac{d^{2\kappa+1}}{dz^{2\kappa+1}}\left(v_1(z)^{-\sigma} + v_2(z)^{-\sigma}\right)\right]_{z=0} = -\sigma\frac{d^{2\kappa}}{dz^{2\kappa}}\left[v_1(z)^{-(\sigma+1)}v_1'(z) + v_1(z)^{-(\sigma+1)}v_2'(z)\right]_{z=0}.$$

By the Leibniz formula,

$$\begin{aligned}
&-\sigma\left[\sum_{p=0}^{2\kappa}\binom{2k}{p}\left(v_1^{(2\kappa+1-p)}(z)\frac{d^p}{dz^p}v_1(z)^{-(\sigma+1)} + v_2^{(2\kappa+1-p)}(z)\frac{d^p}{dz^p}v_2(z)^{-(\sigma+1)}\right)\right]_{z=0} \\
=&-\sigma\left(v_1^{(2\kappa+1)}(0) + v_2^{(2\kappa+1)}(0)\right) \\
&-\sigma\sum_{q=1}^{\kappa}\binom{2k}{2q}v_1^{(2\kappa-2q+1)}(0)\left[\frac{d^{2q}}{dz^{2q}}\left(v_1(z)^{-(\sigma+1)} - v_2(z)^{-(\sigma+1)}\right)\right]_{z=0} \\
&-\sigma\sum_{q=0}^{\kappa-1}\binom{2k-1}{2q+1}v_1^{(2\kappa-2q)}(0)\left[\frac{d^{2q-1}}{dz^{2q-1}}\left(v_1(z)^{-(\sigma+1)} + v_2(z)^{-(\sigma+1)}\right)\right]_{z=0}.
\end{aligned}$$

From the assumption the second and third term are equal to 0.

$$\left[\frac{d^{2\kappa+1}}{dz^{2\kappa+1}}\left(v_1(z)^{-\sigma} + v_2(z)^{-\sigma}\right)\right]_{z=0} = -\sigma\left(v_1^{(2\kappa+1)}(0) + v_2^{(2\kappa+1)}(0)\right). \tag{4.2.11}$$

On the other hand, we can calculate (4.2.10) in the same way and obtain

$$\left[\frac{d^{2\kappa+2}}{dz^{2\kappa+2}}\left(v_1(z)^{-\sigma} - v_2(z)^{-\sigma}\right)\right]_{z=0} = -\sigma\frac{d^{2\kappa+1}}{dz^{2\kappa+1}}\left[v_1(z)^{-(\sigma+1)}v_1'(z) - v_1(z)^{-(\sigma+1)}v_2'(z)\right]_{z=0}.$$

By the Leibniz formula,

$$\begin{aligned}
&-\sigma\left[\sum_{p=0}^{2\kappa+1}\binom{2k+1}{p}\left(v_1^{(2\kappa+2-p)}(z)\frac{d^p}{dz^p}v_1(z)^{-(\sigma+1)} - v_2^{(2\kappa+2-p)}(z)\frac{d^p}{dz^p}v_2(z)^{-(\sigma+1)}\right)\right]_{z=0} \\
=&-\sigma\left(v_1^{(2\kappa+2)}(0) - v_2^{(2\kappa+2)}(0)\right) + \sigma(\sigma+1)(2\kappa+1)\left(v_1^{(2\kappa+1)}(0)v_1'(0) - v_2^{(2\kappa+1)}(0)v_2'(0)\right) \\
&-\sigma\sum_{q=1}^{\kappa}\binom{2k}{2q}v_1^{(2\kappa-2q+2)}(0)\left[\frac{d^{2q}}{dz^{2q}}\left(v_1(z)^{-(\sigma+1)} - v_2(z)^{-(\sigma+1)}\right)\right]_{z=0} \\
&-\sigma\sum_{q=1}^{\kappa}\binom{2k+1}{2q+1}v_1^{(2\kappa-2q+1)}(0)\left[\frac{d^{2q+1}}{dz^{2q+1}}\left(v_1(z)^{-(\sigma+1)} + v_2(z)^{-(\sigma+1)}\right)\right]_{z=0}.
\end{aligned}$$

By the assumption the summation terms are equal to 0.

(4.2.12)

$$\begin{aligned}
&\left[\frac{d^{2\kappa+2}}{dz^{2\kappa+2}}\left(v_1(z)^{-\sigma} + v_2(z)^{-\sigma}\right)\right]_{z=0} \\
=&-\sigma\left(v_1^{(2\kappa+2)}(0) - v_2^{(2\kappa+2)}(0)\right) + \sigma(\sigma+1)(2\kappa+1)v_1'(0)\left(v_1^{(2\kappa+1)}(0) + v_2^{(2\kappa+1)}(0)\right).
\end{aligned}$$

If the derivative condition ($\mathbf{d}_l$) holds also for the case $l = \kappa + 1$, then both (4.2.11) and (4.2.12) are equal to 0, so that (4.2.9) and (4.2.10) are also true for $l = \kappa + 1$. Therefore we complete the proof of the lemma.

□

From this proof, if there exists $\kappa \in \mathbb{N}$ such that ($\mathbf{d}_l$) holds for l $(l = 1, \ldots, \kappa - 1)$ and ($\mathbf{d}_\kappa$) does not hold, we obtain

$$\begin{aligned}&\left[\frac{d^{2\kappa}}{dz^{2\kappa}}\left(v_1(z)^{-(2\kappa+1)} + v_2(z)^{-(2\kappa+1)}\right)\right]_{z=0}\\ =&(2\kappa+1)\left\{(2\kappa-1)(2\kappa+2)v_1'(0)\left(v_1^{(2\kappa-1)}(0) + v_2^{(2\kappa-1)}(0)\right) - \left(v_1^{(2\kappa)}(0) - v_2^{(2\kappa)}(0)\right)\right\}.\end{aligned}$$

Observing, under $V'(t_1) = -V'(t_2)$, the relations

$$v_1^{(n)}(0) = \frac{V^{(n+1)}(t_1)}{(n+1)V'(t_1)}, \qquad v_2^{(n)}(0) = -\frac{V^{(n+1)}(t_2)}{(n+1)V'(t_1)},$$

we have

$$\begin{aligned}&\left[\frac{d^{2\kappa}}{dz^{2\kappa}}\left(v_1(z)^{-(2\kappa+1)} + v_2(z)^{-(2\kappa+1)}\right)\right]_{z=0}\\ =&\frac{1}{2\kappa(V'(t_1))^2}\left\{(\kappa+1)(2\kappa-1)(2\kappa+1)V^{(2)}(t_1)\left(V^{(2\kappa)}(t_1) - V^{(2\kappa)}(t_2)\right)\right.\\ &\left. - 2\kappa V'(t_1)\left(V^{(2\kappa+1)}(t_1) + V^{(2\kappa+1)}(t_2)\right)\right\}.\end{aligned}$$

□

Proof of Theorem 4.2.2 In the case $n = 1$, $m \geq 2$, we have, from the components of the transfer matrices (4.1.13), (4.1.14), (4.1.15), and (4.1.17),

$$\begin{aligned}\tilde{s}_{12} =& i\exp\left[\frac{i}{2h}\left(B_\infty + B\right)\right]\exp\left[\frac{i}{h}A_1^1\right]\\ &\left\{\left(1 + (-1)^m\exp\left[\frac{i}{h}\left(A_m^2 - \overline{A_1^2}\right)\right]\right)\left(1 + O\left(\frac{h}{\varepsilon^{\frac{m+1}{m}}}\right)\right)\right.\\ &\left. + \left((-1)^m\exp\left[\frac{i}{h}\left(A_m^2 - A_1^1 - B\right)\right] - \exp\left[\frac{i}{h}\left(A_1^2 - A_1^1 - B\right)\right]\right)\left(1 + O\left(\frac{h}{\varepsilon^2}\right)\right)\right\},\end{aligned}$$

$$
\begin{aligned}
=& i \exp\left[\frac{i}{2h}\left(B_\infty + B\right)\right] \exp\left[\frac{i}{h} A_1^1\right] \\
&\left\{ 1 + (-1)^m \exp\left[\frac{i}{h}\left(A_m^2 - \overline{A_1^2}\right)\right] + (-1)^m \exp\left[\frac{i}{h}\left(A_m^2 - A_1^1 - B\right)\right] \right. \\
&\quad - \exp\left[\frac{i}{h}\left(A_1^2 - A_1^1 - B\right)\right] + O\left(\frac{h}{\varepsilon^{\frac{m+1}{m}}}\right) + O\left(\frac{h}{\varepsilon^{\frac{m+1}{m}}} \exp\left[-\frac{1}{h}\mathrm{Im}\left(A_m^2 - \overline{A_1^2}\right)\right]\right) \\
&\quad \left. + O\left(\frac{h}{\varepsilon^2} \exp\left[-\frac{1}{h}\mathrm{Im}\left(A_m^2 - A_1^1\right)\right]\right) + O\left(\frac{h}{\varepsilon^2} \exp\left[-\frac{1}{h}\mathrm{Im}\left(A_1^2 - A_1^1\right)\right]\right) \right\}
\end{aligned}
$$

as $h/\varepsilon^2 \to 0$ for any $\varepsilon \in (0, \varepsilon_0)$. Since $\mathrm{Im}\,(A_m^2 - \overline{A_1^2})$, $\mathrm{Im}\,\left(A_m^2 - A_1^1\right)$, and $\mathrm{Im}\,\left(A_1^2 - A_1^1\right)$ are all positive for small ε by virtue of (4.1.1), every error term including the exponential factor is exponentially small. Hence

$$
\begin{aligned}
\tilde{s}_{12} =& i \exp\left[\frac{i}{2h}\left(B_\infty + B\right)\right] \exp\left[\frac{i}{h} A_1^1\right] \\
&\left\{ 1 + (-1)^m \exp\left[\frac{i}{h}\left(A_m^2 - \overline{A_1^2}\right)\right] + (-1)^m \exp\left[\frac{i}{h}\left(A_m^2 - A_1^1 - B\right)\right] \right. \\
&\quad \left. - \exp\left[\frac{i}{h}\left(A_1^2 - A_1^1 - B\right)\right] + O\left(\frac{h}{\varepsilon^{\frac{m+1}{m}}}\right) \right\}
\end{aligned} \tag{4.2.13}
$$

as $h/\varepsilon^{\frac{m+1}{m}} \to 0$ for any $\varepsilon \in (0, \varepsilon_0)$. To calculate $|\tilde{s}_{12}(\varepsilon, h)|^2$, we notice that the following identity holds: Let M be a positive integer and $z_j \in \mathbb{C}$ be a sequence. Then we have

$$
\left| \sum_{k=1}^{M} \exp\left[i z_k\right] \right|^2 = \sum_{k=1}^{M} \exp\left[-2\mathrm{Im}\, z_k\right] + 2\sum_{k<l}^{M} \exp\left[-\mathrm{Im}\left(z_k + z_l\right)\right] \cos\left[\mathrm{Re}\left(z_k - z_l\right)\right] \tag{4.2.14}
$$

By (4.2.14), we see that every exponential term in (4.2.13) is exponentially small. Hence we obtain for any small ε

$$
|\tilde{s}_{12}(\varepsilon, h)|^2 = \exp\left[-\frac{2}{h}\mathrm{Im} A_1^1\right] \left(1 + O\left(\frac{h}{\varepsilon^{\frac{m+1}{m}}}\right)\right) \quad \text{as} \quad \frac{h}{\varepsilon^{\frac{m+1}{m}}} \to 0.
$$

□

Proof of Theorem 4.2.3 In the case $m \geq n \geq 2$, we have, from the components of the transfer matrices (4.1.17), (4.1.18) and (4.1.15),

$$
\begin{aligned}
\tilde{s}_{21} =& i\left((-1)^{n+1}\exp\left[\frac{i}{2h}\left(A_n^1-\overline{A_1^1}\right)\right]+\exp\left[\frac{i}{2h}\left(2A_1^1-\overline{A_1^1}-A_n^1\right)\right]\right)\left(1+O\left(\frac{h}{\varepsilon^{\frac{n+1}{n}}}\right)\right)\\
&\times\left(\exp\left[\frac{i}{2h}\left(A_1^2-A_m^2\right)\right]+(-1)^m\exp\left[\frac{i}{2h}\left(A_1^2-2\overline{A_1^2}+A_m^2\right)\right]\right)\left(1+O\left(\frac{h}{\varepsilon^{\frac{m+1}{m}}}\right)\right)\\
&\times\exp\left[\frac{i}{2h}\left(A_n^1-A_1^2+\overline{A_1^1}+A_m^2+B_\infty+B\right)\right]\\
&+i\left((-1)^{n}\exp\left[\frac{i}{2h}\left(\overline{A_n^1}-\overline{A_1^1}\right)\right]+\exp\left[\frac{i}{2h}\left(2A_1^1-\overline{A_1^1}-\overline{A_n^1}\right)\right]\right)\left(1+O\left(\frac{h}{\varepsilon^{\frac{n+1}{n}}}\right)\right)\\
&\times\left((-1)^{m+1}\exp\left[\frac{i}{2h}\left(A_m^2-\overline{A_1^2}\right)\right]+\exp\left[\frac{i}{2h}\left(2A_1^2-\overline{A_1^2}-A_m^2\right)\right]\right)\left(1+O\left(\frac{h}{\varepsilon^{\frac{m+1}{m}}}\right)\right)\\
&\times\exp\left[\frac{i}{2h}\left(\overline{A_1^2}-\overline{A_n^1}+\overline{A_1^1}+A_m^2+B_\infty-B\right)\right],\\
=& i\exp\left[\frac{i}{2h}B_\infty\right]\left(1+O\left(\frac{h}{\varepsilon^{\frac{n+1}{n}}}\right)\right)\\
&\left[\left((-1)^{n+1}\exp\left[\frac{i}{2h}\left(2A_n^1-A_1^2+A_m^2+B\right)\right]+\exp\left[\frac{i}{2h}\left(2A_1^1-A_1^2+A_m^2+B\right)\right]\right)\right.\\
&\quad\times\left(\exp\left[\frac{i}{2h}\left(A_1^2-A_m^2\right)\right]+(-1)^m\exp\left[\frac{i}{2h}\left(A_1^2-2\overline{A_1^2}+A_m^2\right)\right]\right)\\
&\quad+\left((-1)^{n}\exp\left[\frac{i}{2h}\left(\overline{A_n^1}-\overline{A_1^1}\right)\right]+\exp\left[\frac{i}{2h}\left(2A_1^1-\overline{A_1^1}-\overline{A_n^1}\right)\right]\right)\\
&\quad\left.\times\left((-1)^{m+1}\exp\left[\frac{i}{2h}\left(\overline{A_1^1}-\overline{A_n^1}+2A_m^2-B\right)\right]+\exp\left[\frac{i}{2h}\left(\overline{A_1^1}-\overline{A_n^1}+2A_1^2-B\right)\right]\right)\right]\\
=& i\exp\left[\frac{i}{2h}B_\infty\right]\left(1+O\left(\frac{h}{\varepsilon^{\frac{n+1}{n}}}\right)\right)\\
&\left[\exp\left[\frac{i}{2h}\left(-A_1^2+A_m^2+B\right)\right]\left(\exp\left[\frac{i}{h}A_1^1\right]+(-1)^{n+1}\exp\left[\frac{i}{h}A_n^1\right]\right)\right.\\
&\quad\times\exp\left[\frac{i}{2h}\left(A_1^2-A_m^2\right)\right]\left(1+(-1)^m\exp\left[\frac{i}{h}\left(-\overline{A_1^2}+A_m^2\right)\right]\right)\\
&\quad+\exp\left[\frac{i}{2h}\left(\overline{A_1^1}-\overline{A_n^1}-B\right)\right]\left(\exp\left[\frac{i}{h}A_1^2\right]+(-1)^{m+1}\exp\left[\frac{i}{h}A_m^2\right]\right)\\
&\quad\left.\times(-1)^n\exp\left[\frac{i}{2h}\left(\overline{A_n^1}-\overline{A_1^1}\right)\right]\left(1+(-1)^n\exp\left[\frac{i}{h}\left(A_1^1-\overline{A_n^1}\right)\right]\right)\right]
\end{aligned}
$$

as $h/\varepsilon^{\frac{n+1}{n}} \to 0$ for any $\varepsilon \in (0, \varepsilon_0)$. The component $\tilde{s}_{21}$ is equal to

$$
\begin{aligned}
& i \exp\left[\frac{i}{2h}(B_\infty - B)\right]\left(1 + O\left(\frac{h}{\varepsilon^{\frac{n+1}{n}}}\right)\right) \\
& \times \left[\left(\exp\left[\frac{i}{h}\left(A_1^1 + B\right)\right] + (-1)^{n+1}\exp\left[\frac{i}{h}\left(A_n^1 + B\right)\right]\right)\left(1 + (-1)^m \exp\left[\frac{i}{h}\left(A_m^2 - \overline{A_1^2}\right)\right]\right)\right. \\
& \quad \left. + (-1)^n\left(\exp\left[\frac{i}{h}A_1^2\right] + (-1)^{m+1}\exp\left[\frac{i}{h}A_m^2\right]\right)\left(1 + (-1)^n \exp\left[\frac{i}{h}\left(A_1^1 - \overline{A_n^1}\right)\right]\right)\right]
\end{aligned}
$$

as $h/\varepsilon^{\frac{n+1}{n}} \to 0$ for any $\varepsilon \in (0, \varepsilon_0)$. Observing that $\operatorname{Im}\left(A_m^2 - \overline{A_1^2}\right) > 0$ and $\operatorname{Im}\left(A_1^1 - \overline{A_n^1}\right) > 0$, we have

$$
\begin{aligned}
\tilde{s}_{21} = & i \exp\left[\frac{i}{2h}(B_\infty - B)\right]\left(1 + O\left(\frac{h}{\varepsilon^{\frac{n+1}{n}}}\right)\right) \\
& \left(\exp\left[\frac{i}{h}\left(A_1^1 + B\right)\right] + (-1)^{n+1}\exp\left[\frac{i}{h}\left(A_n^1 + B\right)\right]\right. \\
& \quad \left. + (-1)^n\left(\exp\left[\frac{i}{h}A_1^2\right] + (-1)^{m+1}\exp\left[\frac{i}{h}A_m^2\right]\right)\right)
\end{aligned}
\tag{4.2.15}
$$

as $h/\varepsilon^{\frac{n+1}{n}} \to 0$ for any $\varepsilon \in (0, \varepsilon_0)$. Hence we obtain

$$
\begin{aligned}
|\tilde{s}_{21}(\varepsilon, h)|^2 = & \left|\exp\left[\frac{i}{h}\left(A_1^1 + B\right)\right] + (-1)^{n+1}\exp\left[\frac{i}{h}\left(A_n^1 + B\right)\right]\right. \\
& \quad \left. + (-1)^n \exp\left[\frac{i}{h}A_1^2\right] + (-1)^{n+m+1}\exp\left[\frac{i}{h}A_m^2\right]\right|^2\left(1 + O\left(\frac{h}{\varepsilon^{\frac{l+1}{l}}}\right)\right)
\end{aligned}
$$

as $h/\varepsilon^{\frac{n+1}{n}} \to 0$ for any $\varepsilon \in (0, \varepsilon_0)$.

In particular, when $m > n$, we can estimate (4.2.15) more precisely:

$$
\begin{aligned}
\tilde{s}_{21} = & i \exp\left[\frac{i}{2h}(B_\infty + B)\right]\left(\exp\left[\frac{i}{h}A_1^1\right] + (-1)^{n+1}\exp\left[\frac{i}{h}A_n^1\right]\right)\left(1 + O\left(\frac{h}{\varepsilon^{\frac{n+1}{n}}}\right)\right) \\
& \left\{1 + (-1)^n\left(\exp\left[\frac{i}{h}\left(A_1^1 + B\right)\right] + (-1)^{n+1}\exp\left[\frac{i}{h}\left(A_n^1 + B\right)\right]\right)^{-1}\right. \\
& \quad \left. \times \left(\exp\left[\frac{i}{h}A_1^2\right] + (-1)^{m+1}\exp\left[\frac{i}{h}A_m^2\right]\right)\right\}.
\end{aligned}
$$

Let us estimate the last term. We put for small $\varepsilon \in (0, \varepsilon_0)$

$$
\lambda(\varepsilon) = \max\left\{\operatorname{Im} A_1^1(\varepsilon),\ \operatorname{Im} A_n^1(\varepsilon)\right\}, \qquad \mu(\varepsilon) = \min\left\{\operatorname{Im} A_1^2(\varepsilon),\ \operatorname{Im} A_m^2(\varepsilon)\right\}.
$$

One sees that the exponential factors in the last term are estimated as follows:

$$\left(\exp\left[\frac{i}{h}\left(A_1^1+B\right)\right]+(-1)^{n+1}\exp\left[\frac{i}{h}\left(A_n^1+B\right)\right]\right)^{-1}=O\left(\exp\left[\frac{1}{h}\lambda(\varepsilon)\right]\right),$$

$$\exp\left[\frac{i}{h}A_1^2\right]+(-1)^{m+1}\exp\left[\frac{i}{h}A_m^2\right]=O\left(\exp\left[-\frac{1}{h}\mu(\varepsilon)\right]\right)$$

as $h/\varepsilon^{\frac{n+1}{n}}\to 0$ for any $\varepsilon\in(0,\varepsilon_0)$. Hence we get

$$\begin{aligned}\tilde{s}_{21}=&i\exp\left[\frac{i}{2h}(B_\infty+B)\right]\left(\exp\left[\frac{i}{h}A_1^1\right]+(-1)^{n+1}\exp\left[\frac{i}{h}A_n^1\right]\right)\\&\times\left(1+O\left(\frac{h}{\varepsilon^{\frac{n+1}{n}}}\right)\right)\left(1+O\left(\exp\left[-\frac{1}{h}\left(\mu(\varepsilon)-\lambda(\varepsilon)\right)\right]\right)\right)\end{aligned}$$

as $h/\varepsilon^{\frac{n+1}{n}}\to 0$ for any $\varepsilon\in(0,\varepsilon_0)$. The error term including the exponential factor is exponentially small as h tends to 0, so that $\tilde{s}_{21}$ has the following asymptotic expansion:

$$\tilde{s}_{21}=i\exp\left[\frac{i}{2h}(B_\infty+B)\right]\left(\exp\left[\frac{i}{h}A_1^1\right]+(-1)^{n+1}\exp\left[\frac{i}{h}A_n^1\right]\right)\left(1+O\left(\frac{h}{\varepsilon^{\frac{n+1}{n}}}\right)\right)$$

as $h/\varepsilon^{\frac{n+1}{n}}\to 0$ for any $\varepsilon\in(0,\varepsilon_0)$. Therefore we obtain Theorem 4.2.3.

$\square$

Chapter 5

Appendix

5.1 Landau-Zener formula

In this section we give a proof of the Landau-Zener formula. The Landau-Zener model is the following system:

$$ih\frac{d}{dt}\psi(t) = \begin{pmatrix} at & \varepsilon \\ \varepsilon & -at \end{pmatrix}\psi(t) \qquad \text{on} \quad t \in \mathbb{R}, \tag{5.1.1}$$

where ε, h are positive parameters. Although the diagonal components at and $-at$ do not satisfy the assumption **(B)** in this case, the modified Jost solutions $\Psi^r_+ \mathrm{C}\Psi^r_- \mathrm{C}\Psi^l_+ \mathrm{C}\Psi^l_-$ can be given by the asymptotic conditions:

$$\begin{aligned}
\Psi^r_+(t) &\sim \exp\left[+\frac{i}{h}\left(\frac{a}{2}t^2 + \frac{\varepsilon^2}{2a}\log t\right)\right]\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{as} \quad t \to +\infty, \\
\Psi^r_-(t) &\sim \exp\left[-\frac{i}{h}\left(\frac{a}{2}t^2 + \frac{\varepsilon^2}{2a}\log t\right)\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{as} \quad t \to +\infty, \\
\Psi^l_+(t) &\sim \exp\left[-\frac{i}{h}\left(\frac{a}{2}t^2 + \frac{\varepsilon^2}{2a}\log|t|\right)\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{as} \quad t \to -\infty, \\
\Psi^l_-(t) &\sim \exp\left[+\frac{i}{h}\left(\frac{a}{2}t^2 + \frac{\varepsilon^2}{2a}\log|t|\right)\right]\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{as} \quad t \to -\infty.
\end{aligned} \tag{5.1.2}$$

We can similarly define the scattering matrix $S(\varepsilon,h)$ by

$$\left(\Psi^l_+ \Psi^l_-\right) = \left(\psi^r_+ \psi^r_-\right) S(\varepsilon,h). \tag{5.1.3}$$

We denote its components by $S(\varepsilon,h) = \left(s_{kl}(\varepsilon,h)\right)_{1\le k,l\le 2}$. Then the transition probability $P(\varepsilon,h)$ is defined by

$$P(\varepsilon,h) = \left|s_{21}(\varepsilon,h)\right|^2 = \left|s_{12}(\varepsilon,h)\right|^2 \tag{5.1.4}$$

Theorem 5.1.1 (Landau, Zener)**.** *The transition probability is given by, for all ε and h,*

$$P(\varepsilon,h)=e^{-\pi\varepsilon^2/h}. \tag{5.1.5}$$

This formula were obtained in 1932 by L. D. Landau and C. Zener independently. We present a proof of the Landau-Zener formula along Zener's idea.

Proof of Theorem 5.1.1 The first order differential system (5.1.1) is essentially equal to the second order differential equation, in particular to Weber's differential equation [Z].

By the change of variables $t=h^{1/2}x\mathrm{C}\varepsilon=h^{1/2}v\mathrm{C}$ the system (5.1.1) can be reduced to the following system including only one parameter v.

$$i\frac{d}{dx}\phi(x)=\begin{pmatrix} ax & v \\ v & -ax \end{pmatrix}\phi(x),$$

where $\phi(x)=\psi(h^{1/2}x)$. Put $\phi(x)={}^t\,(\phi_1(x),\phi_2(x))$. This system can be reduced to the single equation:

$$\frac{d^2\phi_1(x)}{dx^2}+\left(ia+v^2+a^2x^2\right)\phi_1(x)=0, \tag{5.1.6}$$

and $\phi_2(x)$ is given by $L\phi_1(x)$, where $L=i\frac{d}{dx}-ax$. (5.1.6) is Weber's differential equation. Putting $x=(2a)^{-\frac{1}{2}}e^{-\frac{\pi i}{4}}z$, $v^2=2ai\lambda$ and $w(z)=\phi_1\left((2a)^{-\frac{1}{2}}e^{-\frac{\pi i}{4}}z\right)$, one has

$$\frac{d^2w(z)}{dz^2}+\left(\lambda+\frac{1}{2}-\frac{z^2}{4}\right)w(z)=0. \tag{5.1.7}$$

We make use of the Weber functions $D_\lambda(z), D_\lambda(-z), D_{-\lambda-1}(iz), D_{-\lambda-1}(-iz)$, which are solutions to (5.1.7). The Weber function $D_\lambda(z)$ is defined by the integral expression:

$$D_\lambda(z)=\frac{e^{-z^2/4}}{\Gamma(-\lambda)}\int_0^\infty e^{-zt-(t^2/2)}t^{-\lambda-1}\,dt \qquad \text{for} \quad \mathrm{Re}\,\lambda<0. \tag{5.1.8}$$

$D_\lambda(z)$ can be extended analytically in $\lambda\in\mathbb{C}\setminus\{0,1,2,\cdots\}$ by the recursion formula $\lambda D_\lambda=-zD_{\lambda+1}-D_{\lambda+2}$. The asymptotic expansion as $|z|$ goes to infinity is calculated from (5.1.8):

$$D_\lambda(z)=e^{-z^2/4}z^\lambda\left(1+O(z^{-2})\right) \qquad \text{as} \quad |z|\to\infty \quad \text{in} \ |\arg z|<\frac{3}{4}\pi, \tag{5.1.9}$$

$$D_\lambda(z)=e^{-z^2/4}z^\lambda\left(1+O(z^{-2})\right)-\frac{\sqrt{2\pi}\,e^{\lambda\pi i}}{\Gamma(-\lambda)}e^{z^2/4}z^{-\lambda-1}\left(1+O(z^{-2})\right) \tag{5.1.10}$$

$$\text{as} \quad |z|\to\infty \quad \text{in} \ \frac{\pi}{4}<\arg z<\frac{5}{4}\pi.$$

The difference of the asymptotic expansions between (5.1.9) and (5.1.10) the results from the irregular singularity $z=\infty$ of Weber's differential equation. This fact is called Stokes phenomenon.

In the case where the Stokes phenomenon happens, we need to study the connection coefficients between such sectorial domains. We can see, from the asymptotic expansions (5.1.9) and (5.1.10), the connection formula of the Weber functions:

$$\text{(5.1.11)} \qquad \begin{pmatrix} D_\lambda(-z) \\ D_{-\lambda-1}(iz) \end{pmatrix} = \begin{pmatrix} e^{-\lambda\pi i} & -i\dfrac{\sqrt{2\pi}}{\Gamma(-\lambda)}e^{-\frac{\lambda\pi i}{2}} \\ \dfrac{\sqrt{2\pi}}{\Gamma(\lambda+1)}e^{-\frac{\lambda\pi i}{2}} & -e^{-\lambda\pi i} \end{pmatrix} \begin{pmatrix} D_\lambda(z) \\ D_{-\lambda-1}(-iz) \end{pmatrix}.$$

We study the relations between the Jost solutions and the Weber functions. We see from (5.1.9)

$$\text{(5.1.12)} \qquad \begin{pmatrix} D_\lambda(z) \\ LD_\lambda(z) \end{pmatrix} = e^{-z^2/4}z^\lambda \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + O\left(z^{-1}\right) \right),$$

$$\text{(5.1.13)} \qquad \begin{pmatrix} D_\lambda(-z) \\ LD_\lambda(-z) \end{pmatrix} = e^{-z^2/4}(-z)^\lambda \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + O\left(z^{-1}\right) \right),$$

as $|z| \to \infty$ with $\arg z = \frac{\pi i}{4}$, and

$$\text{(5.1.14)} \qquad \begin{pmatrix} D_{-\lambda-1}(-iz) \\ LD_{-\lambda-1}(-iz) \end{pmatrix} = ice^{z^2/4}(-iz)^{-\lambda} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + O\left(z^{-1}\right) \right),$$

$$\text{(5.1.15)} \qquad \begin{pmatrix} D_{-\lambda-1}(iz) \\ LD_{-\lambda-1}(iz) \end{pmatrix} = -ice^{z^2/4}(iz)^{-\lambda} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + O\left(z^{-1}\right) \right),$$

as $|z| \to \infty$ with $\arg z = -\frac{\pi i}{4}$. Here $c = \frac{\sqrt{2a}}{v}e^{3\pi i/4}$ and $L = c(\frac{d}{dz} + \frac{z}{2})$. In fact the asymptotic conditions in (5.1.12), (5.1.14) correspond to $x \to +\infty$ and those in (5.1.13), (5.1.15) correspond to $x \to -\infty$. Consequently we obtain

$$\Psi^r_+ = \left(\frac{2a}{h}\right)^{\frac{\lambda}{2}} e^{-\frac{\lambda\pi i}{4}} e^{\frac{3\pi i}{4}} \frac{v}{\sqrt{2a}} \begin{pmatrix} D_{-\lambda-1}(-iz) \\ LD_{-\lambda-1}(-iz) \end{pmatrix}, \quad \Psi^r_- = \left(\frac{h}{2a}\right)^{\frac{\lambda}{2}} e^{-\frac{\lambda\pi i}{4}} \begin{pmatrix} D_\lambda(z) \\ LD_\lambda(z) \end{pmatrix},$$

$$\Psi^l_- = \left(\frac{2a}{h}\right)^{\frac{\lambda}{2}} e^{-\frac{\lambda\pi i}{4}} e^{-\frac{\pi i}{4}} \frac{v}{\sqrt{2a}} \begin{pmatrix} D_{-\lambda-1}(iz) \\ LD_{-\lambda-1}(iz) \end{pmatrix}, \quad \Psi^l_+ = \left(\frac{h}{2a}\right)^{\frac{\lambda}{2}} e^{-\frac{\lambda\pi i}{4}} \begin{pmatrix} D_\lambda(-z) \\ LD_\lambda(-z) \end{pmatrix}.$$

Hence we have, from (5.1.3), (5.1.11) and the above relations, the scattering matrix:

$$S(\varepsilon,h) = \begin{pmatrix} \dfrac{i}{\lambda^{\frac{1}{2}}}\left(\dfrac{h}{2a}\right)^\lambda \dfrac{\sqrt{2\pi}}{\Gamma(-\lambda)}e^{-\frac{\lambda\pi i}{2}} & e^{-\lambda\pi i} \\ e^{-\lambda\pi i} & \dfrac{1}{\lambda^{\frac{1}{2}}}\left(\dfrac{2a}{h}\right)^\lambda \dfrac{\sqrt{2\pi}}{\Gamma(\lambda)}e^{-\frac{\lambda\pi i}{2}} \end{pmatrix}.$$

From $v^2 = 2ai\lambda$ and $\varepsilon = h^{1/2}v$, we obtain

$$P(\varepsilon,h) = e^{-\pi\varepsilon^2/ah}.$$

□

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