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Makoto SAKAGAITO

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A thesis presented

by

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**Abstract**

Harder proved that the Hasse principle for the Brauer group of a purely transcendental extension field in one variable over an arbitrary field. In this thesis we prove this result by an alternative method. At first we characterize the edge map of the Grothendieck spectral sequence and then prove that a certain sequence is exact. Harder's result is proved by using this exact sequence. As another application of the exact sequence, we prove the Hasse principle for the Brauer group of any algebraic function field in one variable over a separably closed field.

# Contents

# 1 Introduction

For a field $k$, let $k_s$ be the separable closure of $k$ and $\bar{k}$ the algebraic closure of $k$. Let $K$ be a global field (i.e., an algebraic number field or an algebraic function field of transcendental degree one over a finite field), $S$ the set of all primes of $K$ and $\widehat{K}_{\mathfrak{p}}$ the completion of $K$ at $\mathfrak{p} \in S$. For a ring $A$, let $\mathrm{Br}(A)$ be the Brauer group of $A$ (see [Me, p.141, IV, §2]). The Brauer groups play an important role to define the reciprocity map of the class field theory.

It is known that the local-global map

$$\mathrm{Br}(K) \to \prod_{\mathfrak{p} \in S} \mathrm{Br}(\widehat{K}_{\mathfrak{p}})$$

is injective (see [K-K-S, Theorem 8.42 (2)]). We call a statement of this form the Hasse principle. In terms of central simple algebras, the Hasse principle is expressed as follows.

Let $A$ be a central simple algebra over $K$. Then

$$A \simeq M_n(K)$$

if and only if

$$A \otimes_K \widehat{K}_{\mathfrak{p}} \simeq M_n(\widehat{K}_{\mathfrak{p}})$$

for all $\mathfrak{p} \in S$. Moreover, suppose that $K = \mathbb{Q}$ and $A$ is a quaternion algebra. Then the above equivalence means that for $a, b \in K^*$, there exist $x, y \in K$ such that

$$ax^2 + by^2 = 1$$

if and only if there exist $x_{\mathfrak{p}}, y_{\mathfrak{p}} \in \widehat{K}_{\mathfrak{p}}$ such that

$$ax_{\mathfrak{p}}^2 + by_{\mathfrak{p}}^2 = 1 \tag{1.1}$$

for all $\mathfrak{p} \in S$. Moreover, let $(\ ,\ )_{\mathfrak{p}} : \mathbb{Q}_{\mathfrak{p}} \times \mathbb{Q}_{\mathfrak{p}} \to \{\pm 1\}$ be the Hilbert symbol. Then (1.1) is equivalent to

$$(a, b)_{\mathfrak{p}} = 1.$$

For a group $G$, let $X(G)$ be the group of characters of $G$. Then the local-global map for the group of characters of $G(K_s/K)$

$$\mathrm{X}(G(K_s/K)) \to \prod_{\mathfrak{p} \in S} \mathrm{X}(G((\widehat{K}_{\mathfrak{p}})_s/\widehat{K}_{\mathfrak{p}}))$$

is also injective. On the other hand, suppose that $m$ is a positive integer which is prime to the characteristic of $k$. Then

$$\mathrm{X}(G(k_s/k))_m = \mathrm{H}^1(k, \mathbb{Z}/m\mathbb{Z}), \quad \mathrm{Br}(k)_m = \mathrm{H}^2(k, \mu_m)$$

where $\mu_m$ is the group of $m$-th roots of unity. So we consider the following conjecture.

**Conjecture 5.4.** Let $k$ be a finitely generated field over a prime field, $m$ an odd prime with $(m, \mathrm{ch}(k)) = 1$, $p$ any positive integer and $X$ a normal complete curve over $k$. Let

$R(X)$ be the function field of $X$, $\mathcal{O}_{X,\mathfrak{p}}$ the local ring at $\mathfrak{p}$ of $X$, $\widetilde{\mathcal{O}}_{X,\mathfrak{p}}$ the Henselization of $\mathcal{O}_{X,\mathfrak{p}}$ and $\widetilde{R(X)}_{\mathfrak{p}}$ its quotient field. Then the local-global map

$$\mathrm{H}^p(R(X), \mu_m) \to \prod_{\substack{\mathfrak{p}\in X \\ \dim(\overline{\mathfrak{p}})=0}} \mathrm{H}^p(\widetilde{R(X)}_{\mathfrak{p}}, \mu_m)$$

is injective.

We claim that this conjecture is true if $k$ is an algebraic number field and $p = 1$ (in preparation). On the other hand, Harder proved that the local-global map of the Brauer group is injective in the case where $k$ is arbitrary field and $X = \mathbb{P}^1_k$ as follows.

**Theorem 7.27.[Ha]** For an arbitrary field $k$, let $k(t)$ be the purely transcendental extension field in one variable $t$ over $k$. Then, the local-global map

$$\mathrm{Br}(k(t)) \to \prod_{\substack{\mathfrak{p}\in \mathbf{P}^1_k \\ \mathrm{ht}(\mathfrak{p})=1}} \mathrm{Br}(\widehat{k(t)}_{\mathfrak{p}})$$

is injective.

In this thesis, we prove Theorem 7.27 by an alternative method. In Section 6, we characterize the edge maps of the Grothendieck spectral sequence.

Suppose that $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are abelian categories which have enough injective objects. Let $G : \mathcal{C} \to \mathcal{B}, F : \mathcal{B} \to \mathcal{A}$ be left exact functors such that $G$ takes injective objects of $\mathcal{C}$ to $F$-acyclic objects. Then the Grothendieck spectral sequence is

$$(R^pG)(R^qF)(A) \Rightarrow R^n(GF)(A)$$

and we show that the edge maps $r^p_{F,G}$ and $l^p_{F,G}$ of the Grothendieck spectral sequence satisfy the following properties.

Property 6.2. The functor $r^p_{F,G}(A)$: $R^pF(G(A)) \to R^pFG(A)$ is characterized by the following properties.

(1) If $p = 0$, $r^0_{F,G}(A) = \mathrm{id}_{FG(A)}$.

(2) Suppose that $p > 0$ and assume that $r^i_{F,G}$ is defined for $i \leqq p-1$. Let $0 \to A \to I \to M \to 0$ be an exact sequence and $I$ an injective object. Then the following diagram

$$\begin{tikzcd}
R^{p-1}F(G(I)/G(A)) \arrow[r] \arrow[d, "\delta_1^{p-1}"'] & R^{p-1}F(G(M)) \arrow[r, "r^{p-1}_{F,G}(M)"] & R^{p-1}(FG)(M) \arrow[d, "\delta_2^{p-1}"] \\
R^pF(G(A)) \arrow[rr, "r^p_{F,G}(A)"'] & & R^p(FG)(A)
\end{tikzcd}$$

is commutative where $\delta_1^{p-1}, \delta_2^{p-1}$ are the connecting homomorphisms.

Property 6.23. The functor $l^p_{F,G}(A)\colon R^p(FG)(A) \to F(R^pG(A))$ is characterized by the following properties.

(1) $l^0_{F,G}(A) = \mathrm{id}_{FG(A)}$.

(2) If $0 \to A \to B \to C \to 0$ is an exact sequence, then $l^p_{F,G}$ satisfies the following commutative diagram

$$\begin{tikzcd} R^p(FG)(C) \arrow[r, "\delta"] \arrow[d, "l^p_{F,G}(C)"'] & R^{p+1}(FG)(A) \arrow[d, "l^{p+1}_{F,G}(A)"] \\ F(R^pG(C)) \arrow[r, "F(\delta)"'] & F(R^{p+1}(G)(A)) \end{tikzcd} .$$

where $\delta$ is the connecting homomorphism.

Moreover, we can determine the form of the edge map of the Grothendieck spectral sequence by the above properties.

In Section 7 we prove several results by using the results which we prove in Section 6. For example, by the above properties, we shall show the following lemma.

**Lemma 7.4** Let $X$ be a regular quasi-compact scheme, $K = R(X)$ and $g\colon \operatorname{Spec} K \to X$ the generic point of $X$. Then

$$\mathrm{H}^2(X, g_*(\mathbb{G}_{m,K})) = \operatorname{Ker}\left(\mathrm{Br}(K) \overset{\mathrm{Res}}{\to} \prod_{x \in X_{(0)}} \mathrm{Br}(K_{\bar{x}})\right).$$

Moreover, for a ring $A$ and $x \in \operatorname{Spec}(A)$, let $\bar{x}$ be the geometric point which corresponds to $x$, $A_{\bar{x}}$ the strict Henselization at $\bar{x}$ (see [Me, p.38, I, §4]) and $K_{\bar{x}}$ the quotient field of $A_{\bar{x}}$. Then we prove the following proposition.

**Proposition 7.14.** Let $A$ be a ring such that $\operatorname{Spec}(A)$ is smooth over a field $k$ (see [Me, pp.30-31, I, §3] for the definition) and $K$ the quotient field of $A$. Then, the following sequence

$$0 \to \mathrm{Br}(A) \overset{(*)}{\to} \operatorname{Ker}\left(\mathrm{Br}(K) \overset{\mathrm{Res}}{\to} \prod_{\substack{x \in \operatorname{Spec}(A) \\ \mathrm{ht}(x)=1}} \mathrm{Br}(K_{\bar{x}})\right) \overset{(**)}{\to} \prod_{\substack{x \in \operatorname{Spec}(A) \\ \mathrm{ht}(x)=1}} \mathrm{X}(G(\kappa(x)_s/\kappa(x)))$$

where $\kappa(x)_s$ is the separable closure of $\kappa(x)$, Res is the restriction map and homomorphisms $(*), (**)$ are the natural maps, is exact.

More generally, if $X$ is regular quasi-compact scheme, we have the exact sequence as follows.

$$0 \to \mathrm{Br}(X) \overset{(*)}{\to} \operatorname{Ker}\left(\mathrm{Br}(R(X)) \overset{\mathrm{Res}}{\to} \prod_{\substack{x \in X \\ \dim \overline{\{x\}}=0}} \mathrm{Br}(R(\mathcal{O}_{X,\bar{x}}))\right) \overset{(**)}{\to} \bigoplus_{\substack{x \in X \\ \dim(\mathcal{O}_{X,x})=1}} \mathrm{X}(G_{\kappa(x)}).$$

(see Remark 7.18). We use results in Section 6 to prove that homomorphisms $(*), (**)$ are the natural maps. By using Proposition 7.14, we can prove Theorem 7.27 and the Hasse principle for the Brauer group of any algebraic function fields in one variable over a separably closed field (Corollary 7.23). Moreover we can prove Corollary 4.30 by Remark 7.18.

# 2 Notation

In this section, we define basic notations used throughout this thesis. More specialized notations will be introduced in each section.

For a field $k$ and a Galois extension field $k'$ of $k$, $G(k'/k)$ denotes the Galois group of $k'/k$. We denote $G(k_s/k)$ by $G_k$ and the category of discrete modules on which $G_k$ acts continuously by $G_k$-mod. We call such modules simply discrete $G_k$-modules. For a discrete $G(k'/k)$-module $A$ and a positive integer $q$, $\mathrm{H}^q(k'/k, A)$ denotes the $q$-th cohomology group of $G(k'/k)$ with coefficients in $A$ (see [S1, p.10, I, §2]). We put $\mathrm{H}^q(k, A) = \mathrm{H}^q(k_s/k, A)$. $\mathrm{Res} : \mathrm{H}^p(k, A) \to \mathrm{H}^p(k', A)$ denotes the restriction homomorphism. For an abelian group $G$, we put $G_q = \{g \in G \mid g^q = 1\}$.

For a scheme $X$, $X^{(i)}$ is the set of points of codimension $i$ and $X_{(i)}$ is the set of points of dimension $i$. We denote the étale site on $X$ by $X_{et}$ and the category of sheaves over $X_{et}$ by $\mathbf{S}_{X_{et}}$. For $\mathcal{F} \in \mathbf{S}_{X_{et}}$, we denote the $q$-th cohomology group of $X_{et}$ with values in $\mathcal{F}$ by $\mathrm{H}^q(X, \mathcal{F})$. For an integral scheme $X$ and $\mathfrak{p} \in X^{(1)}$, let $R(X)$ be the function field of $X$, $\mathcal{O}_{X,\mathfrak{p}}$ the local ring at $\mathfrak{p}$ of $X$, $\widehat{\mathcal{O}}_{X,\mathfrak{p}}$ the completion of $\mathcal{O}_{X,\mathfrak{p}}$, $\widehat{R(X)}_{\mathfrak{p}}$ its quotient field , $\widetilde{\mathcal{O}}_{X,\mathfrak{p}}$ the Henselization of $\mathcal{O}_{X,\mathfrak{p}}$ , $\widetilde{R(X)}_{\mathfrak{p}}$ its quotient field , $\mathcal{O}_{X,\bar{\mathfrak{p}}}$ the strictly Henselization of $\mathcal{O}_{X,\mathfrak{p}}$ and $R(X)_{\bar{\mathfrak{p}}}$ its quotient field. When we consider schemes over a field $k$, we also use the notation such as $k(X), \widehat{k(X)}_{\mathfrak{p}}$, etc. We denote the category of abelian groups by **Ab**.

# 3 Galois cohomology

## 3.1 The definition of Galois cohomology

Let $\mathbb{A}$ be an abelian category. An object $I$ of $\mathbb{A}$ is *injective* if the functor

$$M \mapsto \mathrm{Hom}_{\mathbb{A}}(M, I) : \ \mathbb{A} \to \mathbf{Ab}$$

is exact. $\mathbb{A}$ has *enough injective objects* if, for every $M$ in $\mathbb{A}$, there is a monomorphism from $M$ into an injective object. If $\mathbb{A}$ is an abelian category which has enough injective objects and $f : \mathbb{A} \to \mathbb{B}$ is a left exact functor from $\mathbb{A}$ into another abelian category $\mathbb{B}$, then there is an essentially unique sequence of functors $R^i f : \mathbb{A} \to \mathbb{B}$, $i \geqq 0$, called the *right derived functors* of $f$ with the following properties.

(a) $R^0 f = f$.

(b) $R^i f(I) = 0$ if $I$ is injective and $i > 0$.

(c) For any exact sequence

$$0 \to M^{'} \to M \to M^{''} \to 0$$

in $\mathbb{A}$, there are morphisms

$$\partial^i : R^i f(M^{''}) \to R^{i+1} f(M^{'}),\ i \geqq 0,$$

such that the sequence

$$\cdots \to R^i f(M) \to R^i f(M^{''}) \xrightarrow{\partial^i} R^{i+1} f(M^{'}) \to R^{i+1} f(M) \to \cdots$$

is exact.

(d) The association in (c) of the long exact sequence to the short exact sequence is functorial.

### 3.1.1 Cohomology of finite groups

Here we review the standard complex which is used to compute cohomology of finite groups.

Let $G$ be a finite group and $A$ a $G$-module. Then the functor $A \to A^G$ is left exact. The $q$-th right derived functor of the functor $A^G$ is denoted by $\mathrm{H}^q(G, A)$, $q \geqq 0$. We also denote the $q$-th right derived functor of the functor $\mathrm{Hom}_G$ by $\mathrm{Ext}_G^q$. First note that $A^G$ can be identified with $\mathrm{Hom}_G(\mathbb{Z}, A)$, the group $\mathbb{Z}$ being considered as a $G$-module with trivial action ($s \cdot n = n$ for all $s \in G$). Hence

$$\mathrm{H}^q(G, A) = \mathrm{Ext}_G^q(\mathbb{Z}, A),$$

since $\mathrm{Ext}_G^q$ is the derived functor of the functor $\mathrm{Hom}_G$ for all $q$. To compute $\mathrm{Hom}_G(\mathbb{Z}, A)$, there is a well-known complex called *the standard complex*, which can be described as follows.

Choose a resolution of the $G$-module $\mathbb{Z}$ by projective $G$-modules, i.e., an exact sequence

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0$$

where $P_i$ is projective for all $i$. Putting $K^i = \mathrm{Hom}_G(P_i, A)$, we see that the $K^i$ form a cochain complex $K$, and

$$\mathrm{H}^q(G, A) = \mathrm{H}^q(K)$$

which gives a method of computing these groups;

A free resolution of $\mathbb{Z}$ can be obtained by taking $P_i$ to be the free $\mathbb{Z}$-module $L_i$ having basis which consists of the systems $(g_0, \cdots, g_i)$ of $i+1$ elements of $G$, and making $G$ operate on $L_i$ by translations:

$$s \cdot (g_0, \cdots, g_i) = (sg_0, \cdots, sg_i).$$

We define a homomorphism $\mathrm{d} : L_i \to L_{i-1}$ by the formula

$$\mathrm{d}(g_0, \cdots, g_i) = \sum_{j=1}^{j=i} (-1)^j (g_0, \cdots, \hat{g}_j, \cdots, g_i)$$

where the symbol $\hat{g}_i$ means that $g_i$ does not appear.

An element of $K^i = \mathrm{Hom}(L_i, A)$ can then be identified with a function $f(g_0, \cdots, g_i)$ having values in $A$, and satisfying the covariance condition:

$$f(s \cdot g_0, \cdots, s \cdot g_i) = s \cdot f(g_0, \cdots, g_i).$$

A covariant cochain $f$ is uniquely determined by its restriction to systems of the form $(1, g_1, g_1 g_2, \cdots, g_1 \cdots g_i)$. This leads us to interpret elements of $K^i$ as inhomogeneous cochains, i.e., as functions $f(g_0, \cdots, g_i)$ of $i$ arguments, with values in $A$, whose coboundary is given by:

$$\begin{aligned}(\mathrm{d}\, f)(g_1, \cdots, g_{i+1}) =& g_1 \cdot f(g_2, \cdots, g_{i+1}) \\ &= \sum_{j=1}^{j=i} (-1)^j f(g_1, \cdots, g_j g_{j+1}, \cdots, g_{i+1}) \\ &+ (-1)^{i+1} f(g_1, \cdots, g_i).\end{aligned} \tag{3.1}$$

Especially, if $G = \{1\}$, $\mathrm{H}^p(G, A) = 0$ for $p > 0$.

### 3.1.2 Cohomology of profinite groups

Here we define the cohomology of profinite groups. For example, Galois groups for infinite extensions are profinite groups. The definition of a profinite group is the following.

**Definition 3.2.** A topological group which is the projective limit of finite groups, each given the discrete topology, is called a *profinite group*. If $G = \varprojlim H$ is a profinite group, the quotient of $G$ obtained by taking the inverse limit of the finite groups $H$ which are $p$-groups is called the pro-$p$-part of $G$ and is denoted by $G(p)$.

Let $G$ be a profinite group. The discrete abelian groups on which $G$ acts continuously form an abelian category $C_G$, which is a full subcategory of the category of all $G$-modules. To say that a $G$-module $A$ belongs to $C_G$ means that the stabilizer of each element of $A$ is open in $G$, or, that one has

$$A = \bigcup A^U = \varinjlim A^U$$

where $U$ runs through all open subgroups of $G$ (as usual, $A^U$ denotes the largest subgroup of $A$ fixed by $U$). An element $A$ of $C_G$ is called a *discrete $G$-module*.

If $A \in C_G$, define $\mathrm{H}^q(G, A)$ by the formula

$$\mathrm{H}^q(G, A) = \varinjlim \mathrm{H}^q(G/H, A^H)$$

where $H$ runs through open normal subgroups of $G$.

We denote by $C^n(G, A)$ the set of all *continuous* maps from $G^n$ to $A$. One defines the coboundary

$$\mathrm{d}^{n+1} : C^n(G, A) \to C^{n+1}(G, A)$$

by (3.1). We now set

$$Z^n(G,A) = \mathrm{Ker}(C^n(G,A) \overset{\mathrm{d}^{n+1}}{\to} C^{n+1}(G,A)),$$
$$B^n(G,A) = \mathrm{Im}(C^{n-1}(G,A) \overset{\mathrm{d}^{n}}{\to} C^{n}(G,A))$$

and $B^0(G,A) = 0$. The elements of $Z^n(G,A)$ and $B^n(G,A)$ are called the *$n$-cocycles* and *$n$-coboundaries* respectively.

Then one obtains a complex $C^*(G,A)$ whose cohomology groups $Z^q(G,A)/B^q(G,A)$ are $\mathrm{H}^q(G,A)$. It is easy to see that $\mathrm{H}^0(G,A) = A^G$ and $\mathrm{H}^1(G,A)$ is the group of classes of *continuous* crossed homomorphisms of $G$ into $A$.

## 3.2 Functoriality

In this subsection we describe functors such as the restriction and the inflation regarding the group cohomology.

Let $G$ and $G'$ be two profinite groups, and let $f : G \to G'$ be a homomorphism. Assume $A \in C_G$ and $A' \in C_{G'}$. Let $h : A' \to A$ be a homomorphism which is *compatible* with $f$ (this is a $G$-homomorphism, if one regards $A'$ as a $G$-module via $f$). Such a pair $(f,h)$ defines, by passing to cohomology, the homomorphisms

$$\mathrm{H}^q(G',A') \to \mathrm{H}^q(G,A), \;\; q \geqq 0.$$

If we apply this consideration where $H$ is a closed subgroup of $G$, and $A = A'$ is a discrete $G$-module, we obtains the *restriction* homomorphisms

$$\mathrm{Res}: \;\; \mathrm{H}^q(G,A) \to \mathrm{H}^q(H,A), \;\; q \geqq 0.$$

When $H$ is a normal subgroup of $G$ and $A$ is a discrete $G$-module, the group $A^H$ is a $G/H$-module and one obtains the *inflation* homomorphisms

$$\mathrm{Inf}: \;\; \mathrm{H}^q(G/H, A^H) \to \mathrm{H}^q(G,A) \;\; q \geqq 0.$$

When $H$ is open in $G$, with index $n$, it is possible to define the *Corestriction* homomorphisms

$$\mathrm{Cor}: \;\; \mathrm{H}^q(H,A) \to \mathrm{H}^q(G,A), \;\; q \geqq 0.$$

Then $\mathrm{Cor} \circ \mathrm{Res} = n$.

**Proposition 3.3.** The groups $\mathrm{H}^p(G,A)$ are torsion.

Let $H$ be a closed subgroup of a profinite group $G$, and let $A \in C_H$. The induced module $A^* = M_G^H(A)$ is defined as the group of continuous maps $a^*$ from $G$ to $A$ such that a $a^*(hx) = h \cdot a^*(x)$ for $h \in H$, $x \in G$. The group $G$ acts on $A^*$ by

$$(ga^*)(x) = a^*(xg).$$

If to each $a^* \in M_G^H(A)$ one associates its value at the point 1, one obtains a homomorphism $M_G^H(A) \to A$ which is compatible with the injection of $H$ into $G$, hence the homomorphisms

$$\mathrm{H}^q(G, M_G^H(A)) \to \mathrm{H}^q(H,A).$$

**Proposition 3.4.** [W, p.171, Shapiro's Lemma 6.3.2] The homomorphisms

$$\mathrm{H}^q(G, M_G^H(A)) \to \mathrm{H}^q(H, A)$$

defined above are isomorphisms.

## 3.3 The cohomological dimension

Let $p$ be a prime number, and $G$ a profinite group. One calls the *p-cohomological dimension* of $G$, and uses the notation $\mathrm{cd}_p(G)$ for the lower bound of the integers $n$ which satisfy the following condition:

> (*) for every discrete torsion $G$-module $A$, and for every $q > n$, the $p$-primary component of $\mathrm{H}^q(G, A)$ is null.

**Proposition 3.5.** [S1, p.83, II, §4, Proposition 11] Let $k'$ be an extension of $k$, of transcendence degree $N$. If $p$ is a prime, we have

$$\mathrm{cd}_p(G_{k'}) \leqq N + \mathrm{cd}_p(G_k).$$

This is an equality when $k'$ is finitely generated over $k$, $\mathrm{cd}_p(G_k) < \infty$, and $p$ is distinct from the characteristic of $k$.

**Proposition 3.6.** [S1, p.85, II, §4, Proposition 12] Let $K$ be a complete field with respect to a discrete valuation with residue field $k$. For any prime $p$, we have:

$$\mathrm{cd}_p(G_K) \leqq 1 + \mathrm{cd}_p(G_k).$$

This is an equality when $\mathrm{cd}_p(G_k) < \infty$ and $p$ is different from the characteristic of $K$.

**Proposition 3.7.** [S1, p.79, II, §3, Proposition 7] If $k$ is a field of characteristic $p > 0$, we have $\mathrm{cd}_p(G_k) \leqq 1$ and $\mathrm{cd}(G_k(p)) \leqq 1$.

## 3.4 The Brauer group of a field

In this subsection we define and discuss properties of the Brauer group.

For a field $k$, the group $\mathrm{H}^2(G(k_s/k), k_s^*)$ is called the *Brauer group* of the field $k$ and is denoted by $\mathrm{Br}(k)$. By Proposition 3.3, this is a torsion group. At first, we consider the Brauer group of a Henselian field. The definition of a Henselian ring is the following.

**Definition 3.8.** Let $A$ be a local ring with maximal ideal $\mathfrak{m}$ and residue field $k$. Let the homomorphism $A[T] \to k[T]$ be written as $(f \mapsto \bar{f})$. Then a ring $A$ which satisfies the following condition $(*)$ is called a *Henselian* ring.

> (*) If $f$ is a monic polynomial with coefficients in $A$ such that $\bar{f}$ factors as $\bar{f} = g_0 h_0$ with $g_0$ and $h_0$ monic and coprime, then $f$ itself factors as $f = gh$ with $g$ and $h$ monic and such that $\bar{g} = g_0, \bar{h} = h_0$.

**Proposition 3.9.** Let $A$ be a Henselian discrete valuation ring, $K$ its quotient field and $\widehat{K}$ its completion. Moreover, let $A$ be an excellent, i.e., $K$ be a Henselian discrete valuation field such that the completion $\widehat{K}$ is separable over $K$. Then

$$\mathrm{Br}(K) = \mathrm{Br}(\widehat{K}).$$

*Proof.* Let $k$ be the residue field of $A$. If $(\mathrm{ch}(k), l) = 1$, then we have the Kummer sequence

$$0 \to \mu_l \to K_s^* \xrightarrow{l} K_s^* \to 0.$$

So

$$\mathrm{Br}(K)_l = \mathrm{H}^2(K, \mu_l)$$

and

$$\mathrm{Br}(K)_l = \mathrm{Br}(\widehat{K})_l$$

because $G(K_s/K) \simeq G(\widehat{K}_s/\widehat{K})$. By the exact sequence

$$0 \to K_s^* \xrightarrow{p} K_s^* \to K_s^*/(K_s^*)^p \to 0,$$

we have the exact sequence

$$\{0\} = \mathrm{H}^1(K, K_s^*) \to \mathrm{H}^1(K, K_s^*/(K_s^*)^p) \to \mathrm{H}^2(K, K_s^*) \xrightarrow{p} \mathrm{H}^2(K, K_s^*).$$

So

$$\mathrm{Br}(K)_p = \mathrm{H}^1(K, K_s^*/(K_s^*)^p).$$

Therefore it is sufficient to show that $K_s^*/(K_s^*)^p \simeq \widehat{K}_s^*/(\widehat{K}_s^*)^p$. Let $\mathfrak{m}$ be the maximal ideal of $A$ and $U_K^n = 1 + \mathfrak{m}^n$. Then $K^*/U_K^l = \widehat{K}^*/U_{\widehat{K}}^l$ and there exists $N$ such that $U_{\widehat{K}^n} \subset (\widehat{K}^*)^p$ for any $N \leqq n$. Therefore $K_s^*/(K_s^*)^p \to \widehat{K}_s^*/(\widehat{K}_s^*)^p$ is surjective. Moreover Henselian discrete valuation ring $K$ is algebraically separably closed in $\widehat{K}$. Therefore $K_s^*/(K_s^*)^p \to \widehat{K}_s^*/(\widehat{K}_s^*)^p$ is injective. So the statement is holds. ☐

**Definition 3.10.** Assume $p = \mathrm{ch}(k) > 0$. Let $\Omega_k^i$ be the $i$-th exterior product over $k$ of the absolute differential module $\Omega^1_{k/\mathbb{Z}}$ , and let $\mathrm{H}_p^{i+1}(k)$ be the cokernel of the homomorphism $\mathfrak{p} : \Omega_k^i \to \Omega_k^i/\,\mathrm{d}(\Omega_k^{i-1})$

$$\mathfrak{p}(x\frac{\mathrm{d}\, y_1}{y_1} \wedge \cdots \wedge \frac{\mathrm{d}\, y_i}{y_i}) = (x^p - x)\frac{\mathrm{d}\, y_1}{y_1} \wedge \cdots \wedge \frac{\mathrm{d}\, y_i}{y_i} \ \mathbf{mod}\, \mathrm{d}(\Omega_k^{i-1})$$

$(x \in k,\ y_1, \cdots y_i \in k^*)$

**Proposition 3.11.** [K-K, proof of Theorem 1] Let $A$ be an excellent Henselian discrete valuation ring, $K$ the quotient field and $\widehat{K}$ its completion. Then the natural homomorphism

$$\mathrm{H}_p^q(K) \to \mathrm{H}_p^q(\hat{K})$$

is isomorphism.

**Definition 3.12.** Let $k$ be a field with $\mathrm{ch}(k) = p > 0$. Let $q \geqq 0$ and $n \geqq 0$. We define the group $P_n^q(k)$ by

$$P_n^q(k) = (W_n(k)) \otimes \overbrace{k^* \otimes \cdots \otimes k^*}^{q \text{ times}} / J,$$

where $W_n(k)$ denotes the group of all $p$-Witt vectors of length $n$ over $k$ and $J$ denotes the subgroup of $(W_n(k)) \otimes k^* \otimes \cdots \otimes k^*$ generated by all elements of the following forms (i) (ii) (iii).

(i) $(\overbrace{0, \cdots, 0}^{i \text{ times}}, a, 0, \cdots, 0) \otimes b_1 \otimes \cdots \otimes b_{q-1}$ $(0 \leqq i < n, a, b_1, \cdots, b_{q-1} \in k^*)$.

(ii) $(\mathfrak{F}(\omega) - \omega) \otimes b_1 \otimes \cdots \otimes b_q$ $(\omega \in W_n(k), b_1, \cdots, b_q \in k^*)$,
where $\mathfrak{F}$ denotes the homomorphism $W_n(k) \to W_n(k)$;

$$(a_0, \cdots, a_{n-1}) \to (a_0^p, \cdots, a_{n-1}^p).$$

(iii) $\omega \otimes b_1 \otimes \cdots \otimes b_q$ such that $b_i = b_j$ for some $i \neq j$.

Then we have the following fact.

**Proposition 3.13.** [K, p.674, §3.4, Lemma 16] Let $k$ be a field with $\mathrm{ch}(k) = p$, and let $n \geqq 0$. Then there is a canonical isomorphism

$$P_n^1(k) \simeq \mathrm{Br}(k)_{p^n}; \ \{\chi, a\} \mapsto (\chi, a) \ \ (\chi \in P_n^0(k), a \in k^*),$$

where we identify $P_n^0(k)$ with $(X_k)_{p^n}$ via Witt theory.

*An alternative proof of Proposition 3.9.* For the proof of Proposition 3.9, it is sufficient to show that $\mathrm{Br}(k)_p \simeq \mathrm{Br}(\widehat{k})_p$. Let $F$ be a field. Then we have an isomorphism $P_1^q(F) \simeq \Omega_F^q/(1-\gamma)\Omega_F^q$ where $\gamma$ denotes the Cartier operator by [K, Corollary to Lemma 5 in §1.3]. On the other hand, $\Omega_F^1/(1-\gamma)\Omega_F^1 = \mathrm{H}_p^1(F)$ by the definition and $P_1^1(F) \simeq \mathrm{Br}(F)_p$ by Proposition 3.13. So the statement follows from Proposition 3.11. □

# 4 Étale cohomology

We shall be concerned with classes $E$ of morphisms of schemes satisfying the following conditions.

**Condition 4.1.**
1. all isomorphism are in $E$,
2. the composite of two morphisms in $E$ is in $E$,
3. any base change of a morphism in $E$ is in $E$.

A morphism in such a class $E$ will be referred to as an *$E$-morphism*. Now fix a base scheme $X$, a class $E$ as above, and a full subcategory $\mathbb{C}/X$ of $\mathbf{Sch}/X$ which is closed under fiber products and for any $Y \to X$ in $\mathbb{C}/X$ and any $E$-morphism $U \to Y$, the composite $U \to X$ is in $\mathbb{C}/X$.

An *$E$-covering* of an object $Y$ of $\mathbb{C}/X$ is a family $(U_i)_{i\in I}$ of $E$-morphisms such that $Y = \bigcup g_i(Y_i)$. The class of all such coverings of all such objects is the *$E$-site* over $X$, and will be written $(\mathbb{C}/X)_E$.

**Definition 4.2.** A *presheaf* $P$ of abelian groups on a site $(\mathbb{C}/X)_E$ is a contravariant functor $(\mathbb{C}/X)^\circ \to \mathbf{Ab}$.

Thus $P$ associates with each $U$ in $\mathbb{C}/X$ an abelian group $P(U)$, which we shall sometimes write as $\Gamma(U, P)$.

**Definition 4.3.** $P$ is a *sheaf* if the sequences

$$P(U) \to \prod_i P(U_i) \rightrightarrows \prod_{i,j} P(U_i \times_U U_j)$$

is exact for all coverings $(U_i \to U)$.

Let $P$ be a presheaf on $X_{et}$. The *stalk* $P_{\bar{x}}$ of $P$ at $\bar{x}$ is the abelian group $(u_x^p P)(\bar{x})$. More explicitly,

$$P_{\bar{x}} = \varinjlim P(U)$$

where the limit runs through all commutative triangles

$$\begin{tikzcd} \bar{x} \arrow[r] \arrow[dr, "u_x"'] & U \arrow[d] \\ & X \end{tikzcd}$$

with $U$ étale over $X$.

**Proposition 4.4.** [Me, p.60, II, §2] Let $F$ be a sheaf on $X_{et}$. If $s \in F(U)$ is nonzero, then there is an $x \in X$ and an $\bar{x}$-point of $U$ such that $s_{\bar{x}}$ is nonzero.

## 4.1 Étale morphisms

**Definition 4.5.** A morphism $f : Y \to X$ that is locally of finite-type is said to be *unramified* at $y \in Y$ if $\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y}$ is a field and is a finite separable extension of $\kappa(x)$, where $x = f(y)$.

**Definition 4.6.** A morphism of schemes is defined to be *étale* if it is flat and unramified.

**Theorem 4.7.** [Me, p.29, I, Theorem 3.21] Let $X$ be a connected normal scheme, and let $K = R(X)$. Let $X'$ be the normalization of $X$ in $L$, and $U$ any open subscheme of $X'$ that is disjoint from the support of $\Omega^1_{X'/X}$. Then $U \to X$ is étale, and conversely any separated étale morphism $Y \to X$ of finite-type can be written

$$Y = \coprod U_i \to X$$

where each $U_i \to X$ is of this form.

Consider sites $(\mathbb{C}'/X')_{E'}$ and $(\mathbb{C}/X)_E$. A morphism $\pi : X' \to X$ of schemes defines a *morphism of sites* $(\mathbb{C}'/X')_{E'} \to (\mathbb{C}/X)_E$ if :

1. for any $Y$ in $\mathbb{C}/X$, $Y_{(X')}$ is in $\mathbb{C}'/X'$;
2. for any $E$-morphism $U \to Y$ in $\mathbb{C}/X$, $U_{(X')} \to Y_{(X')}$ is an $E'$-morphism.

Since the base change of a surjective family of morphisms is again surjective, $\pi$ defines a functor

$$\pi^{\cdot} = (Y \mapsto Y_{X'}) : \mathbb{C}/X \to \mathbb{C}'/X'$$

which takes coverings to coverings. Suppose that $\pi : X'_{E'} \to X_E$ is continuous. If $P'$ is a presheaf on $X'_{E'}$ $\pi_p(P') = P' \circ \pi^{\cdot}$ is a presheaf on $X_E$. Explicitly, $\pi_p(P')$ is the presheaf on $X_E$ such that $\Gamma(U, \pi_p(P')) = \Gamma(U(X'), P')$. The presheaf $\pi_p(P')$ is called the *direct image* of $P'$. Then $\pi_p$ defines a functor $\mathbb{P}(X'_{E'}) \to \mathbb{P}(X_E)$, and we define the *inverse image* functor $\pi^p : \mathbb{P}(X_E) \to \mathbb{P}(X'_{E'})$ to be the left adjoint of $\pi_p$, that is, $\pi^p$ is such that

$$\mathrm{Hom}_{\mathbb{P}(X')}(\pi^p P, P') = \mathrm{Hom}_{\mathbb{P}(X)}(P, \pi_p P')$$

by the following proposition.

**Proposition 4.8.** [Me, p.56, II, Proposition 2.2] Let $\mathbb{C}$ and $\mathbb{C}'$ be small categories ( i.e., its class of objects is a set), and $p$ a functor $\mathbb{C} \to \mathbb{C}'$. Let $\mathbb{A}$ be a category equipped with direct limits, and write $\mathrm{Fun}(\mathbb{C}, \mathbb{A})$ and $\mathrm{Fun}(\mathbb{C}', \mathbb{A})$ for the categories of functors $\mathbb{C} \to \mathbb{A}$ and $\mathbb{C}' \to \mathbb{A}$. Then the functor

$$(f \mapsto f \circ p) : \mathrm{Fun}(\mathbb{C}', \mathbb{A}) \to \mathrm{Fun}(\mathbb{C}, \mathbb{A})$$

has a left adjoint.

The inverse image presheaf $\pi^p P$ can be expressed explicitly by the proof of Proposition 4.8. We put

$$(\pi^p P)(U') = \varinjlim P(U)$$

where the limit runs through all commutative squares

$$(g, U) \quad = \quad \begin{array}{ccc} U' & \xrightarrow{g} & U \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

with $U \to X$ in $\mathbb{C}/X$.

## 4.2 Henselian rings

We defined the notion of Henselian rings in Section 3.4. There exist equivalent relations as follows.

**Theorem 4.9.** [Me, p.32, I, §4] Let $x$ be the closed point of $X = \mathrm{Spec}\, A$. The following are equivalent.

1. $A$ is Henselian.

2. Any finite $A$-algebra $B$ is a direct product of local rings $B = \prod B_i$ ($B_i$ is then isomorphic to the ring $B_{\mathfrak{m}_i}$, where $\mathfrak{m}_i$ is the maximal ideal of $B$).

3. If $f\colon Y \to X$ is étale and there is a point $y \in Y$ such that $f(y) = x$ and $\kappa(y) = \kappa(x)$, then $f$ has a section $s\colon X \to Y$.

**Corollary 4.10.** [Me, p.34, I, §4] If $A$ is Henselian, then so is any finite local $A$-algebra $B$.

By definition, $A$ being Henselian means that it has no finite étale extensions with trivial residue field extension, except those of the form $A \to A^r$. Thus if the residue field of $A$ is separably algebracally closed, then $A$ has no finite étale extensions at all. Such a Henselian ring is called *strictly Henselian* or *strictly local.*

Let $i\colon A \to A^h$ be a local homomorphism of local rings. Then $A^h$ is the *Henselization* of $A$ if it is a Henselian local ring and if any other local homomorphism from $A$ into a Henselian local ring factors uniquely through $i$.

The *strict Henselization* $(A^{sh}, h)$ of $A$ is defined as the definition of the Henselization of $A$.

Let $X$ be a scheme and let $x \in X$. An *étale neighborhood* of $x$ is a pair $(Y, y)$ where $Y$ is an étale $X$-scheme and $y$ is a point of $Y$ which maps to $x$ and $\kappa(x) = \kappa(y)$. The connected étale neighborhoods of $x$ form a filtered system and the limit

$$\varinjlim \Gamma(Y, \mathcal{O}) = \mathcal{O}^h_{X,x}.$$

Let $X$ be a scheme and $\bar{x} \to X$ a geometric point of $X$. An *étale neighborhood* of $\bar{x}$ is a commutative diagram:

$$\begin{tikzcd} \bar{x} \arrow[r] \arrow[dr] & U \arrow[d] \\ & X \end{tikzcd}$$

with $U \to X$ étale. Then

$$\mathcal{O}^{sh}_{X,x} = \lim \Gamma(U, \mathcal{O}_U)$$

where the limit is taken over all étale neighborhoods of $\bar{x}$. We write $\mathcal{O}^{sh}_{X,x}$, or simply $\mathcal{O}_{X,x}$ for this limit.

## 4.3 Cohomology

We consider right derived functors of left exact functors in the following cases.

1. The functor $\Gamma(X, -) : \mathbb{S}(X_E) \to \mathbf{Ab}$ with

$$\Gamma(X, F) = \mathrm{H}^i(X, -) = \mathrm{H}^i(X_E, -),$$

is left exact and its right derived functors are written as

$$R^i\Gamma(X, -) = \mathrm{H}^i(X, -) = \mathrm{H}^i(X_E, -).$$

The group $\mathrm{H}^i(X_E, F)$ is called the *$i$-cohomology group* of $X_E$ with values in $F$.

2. For any continuous morphism $\pi : X'_{E'} \to X_E$, the right derived functors $R^i\pi_*$ of the functor $\pi_* : \mathbb{S}(X'_{E'}) \to \mathbb{S}(X_E)$ are defined. The sheaves $R^i\pi_*F$ are called the *higher direct images* of $F$.

**Proposition 4.11.** [Me, p.116, III, Remark 3.11 (a)] For any smooth quasi-projective group scheme $G$ over a Henselian ring $A$,

$$\mathrm{H}^i(X, G) \simeq \mathrm{H}^i(X_0, G_0)$$

for $i \geq 1$, where $X = \operatorname{Spec} A$, $X_0$ is the closed point of $X$, and $G_0 = G \times_X X_0$ is the closed fiber of $G/X$.

**Proposition 4.12.** [Me, p.111, III, Proposition 3.3] Let $E_1 \supset E_2$ be classes of morphisms satisfying Condition 4.1, let $\mathbb{C}_2/X$ be a subcategory of $\mathbb{C}_1$, and let $f : (\mathbb{C}_1/X)_{E_1} \to (\mathbb{C}_2/X)_{E_2}$ be the morphism induced by the identity map on $X$. Assume that for every $U$ in $\mathbb{C}_2/X$ and every covering of $U$ in the $E_1$-topology, there is a covering of $U$ in the $E_2$-topology that refines it. Then $f_* : \mathbb{S}(X_{E_1}) \to \mathbb{S}(X_{E_2})$ is exact and hence

$$\mathrm{H}^i(X_{E_2}, f_*(F)) \simeq \mathrm{H}^i(X_{E_1}, F)$$

for any sheaf $F$ on $X_{E_1}$.

## 4.4 The definition of Čech cohomology

Let $X$ be a scheme. Then we define Čech cohomology groups over $X$ in this subsection. Let $\mathcal{U} = (U_i \xrightarrow{\phi_i} X)_{i \in I}$ be a covering of $X$ in the $E$-topology on $X$. For any $(p-1)$-tuple $(i_0, \cdots, i_p)$ with $i_j$ in $I$ we write $U_{i_0} \times_X \cdots \times_X U_{i_p} = U_{i_0 \cdots i_p}$. Let $P$ be a presheaf on $X_E$. The canonical projection

$$U_{i_0 \cdots i_p} \to U_{i_0 \cdots \hat{i}_j \cdots i_p} = U_{i_0} \times \cdots \times U_{i_{j-1}} \times U_{i_j} \times \cdots \times U_{i_p}$$

induces a restriction morphism

$$P(U_{i_0 \cdots \hat{i}_j \cdots i_p}) \to P(U_{i_0 \cdots i_p})$$

which we write ambiguously as $\mathrm{res}_j$. We put

$$C^{\cdot}(\mathcal{U}, P) = (C^p(\mathcal{U}, P), \mathrm{d}^p)_p,$$
$$C^p(\mathcal{U}, P) = \prod_{I^{p+1}} P(U_{i_0 \cdots i_p}).$$

We define a homomorphism

$$\mathrm{d}^p : C^p(\mathcal{U}, P) \to C^{p+1}(\mathcal{U}, P)$$

so that if $s = (s_{i_0 \dots i_p}) \in C^p(\mathcal{U}, P)$, then

$$(\mathrm{d}^p\, s)_{i_0 \cdots i_p} = \sum_{j=0}^{p+1} (-1)^j \,\mathrm{res}_j(s_{i_0 \cdots \check{i}_j \cdots i_{p+1}}).$$

The usual argument shows that $\mathrm{d}^{p+1}\,\mathrm{d}^p = 0$. The cohomology groups of $(C^p(\mathcal{U}, P), \mathrm{d}^p)$ are called the *Čech cohomology groups* with respect to the covering $\mathcal{U}$ of $X$ and is denoted by $\check{\mathrm{H}}^p(\mathcal{U}, P)$. A covering $\mathcal{V} = (V_j \xrightarrow{\psi_j} X)_{j\in J}$ is called a *refinement* of $\mathcal{U}$ if there is a map $\tau : J \to I$ such that for each $j$, $\psi_j$ factors through $\phi_{\tau j}$, that is, $\psi_j = \phi_{\tau j}\eta_j$ for some $\eta_j : V_j \to U_{\tau j}$.

The map $\tau$, together with the family $\eta_j$, induces maps $\tau^p : C^p(\mathcal{U}, P) \to C^p(\mathcal{V}, P)$ as follows. If $s = (s_{i_0} \cdots s_{i^p}) \in C^p(\mathcal{U}, P)$, then

$$(\tau^p s)_{j_0\cdots j_p} = \mathrm{res}_{\eta_{j_0}\times\eta_{j_1}\times\cdots\times\eta_{j_p}}(s_{\tau_{j_0}\cdots\tau_{j_p}}).$$

These maps $\tau^p$ commute with d and hence induce maps on the cohomology,

$$\rho(\mathcal{V},\mathcal{U},\tau) : \check{\mathrm{H}}^p(\mathcal{U}, P) \to \check{\mathrm{H}}^p(\mathcal{V}, P)$$

The map $\rho(\mathcal{V},\mathcal{U},\tau)$ does not depend on $\tau$ or $\eta_j$. Hence, if $\mathcal{V}$ is a refinement of $\mathcal{U}$, we get a homomorphism

$$\rho(\mathcal{V},\mathcal{U}) : \check{\mathrm{H}}^p(\mathcal{U}, P) \to \check{\mathrm{H}}^p(\mathcal{V}, P)$$

depending only on $\mathcal{V}$ and $\mathcal{U}$. It follows that if $\mathcal{U}, \mathcal{V}, \mathcal{W}$ are three coverings of $X$ such that $\mathcal{W}$ is a refinement of $\mathcal{V}$ and $\mathcal{V}$ is a refinement of $\mathcal{U}$, then

$$\rho(\mathcal{W},\mathcal{U}) = \rho(\mathcal{W},\mathcal{V})\rho(\mathcal{V},\mathcal{U}).$$

Thus we may define the *Čech cohomology groups* of $P$ over $X$ to be

$$\check{\mathrm{H}}^p(X_E, P) = \varinjlim \check{\mathrm{H}}^p(\mathcal{U}, P),$$

where the limit is taken over all coverings $\mathcal{U}$ of $X$.

**Theorem 4.13.** [Me, p.104, III, Theorem 2.17] Let $X$ be a quasi-compact scheme such that every finite subset of $X$ is contained in an affine open set (for example, $X$ quasi-projective over an affine scheme), and let $F$ be a sheaf on $X_{et}$. Then there are canonical isomorphisms

$$\check{\mathrm{H}}^p(X_{et}, F) \simeq \mathrm{H}^p(X_{et}, F)$$

for all $p$.

## 4.5 The relation between Galois cohomology and étale cohomology

Let $K$ be a field, $G = G(K_s/K)$ and $X = \mathrm{Spec}\,(K)$. In this subsection we consider the relation between a sheaf on $X_{et}$ and a discrete $G$-module.

Let $P$ be a presheaf on $X_{et}$. If $K'$ is a finite separable field extension of $K$, then we write $P(K') = P(\mathrm{Spec}\,K')$. Define $M_P = \varinjlim P(K')$ where the limit is taken over all subfields $K'$ of $K_s$ which are finite over $K$. Then $G$ acts on $P(K')$ on the left through its action on $K'$ whenever $K'/K$ is Galois, and it follows that $G$ acts on the limit $M_P$. Clearly $M_P = \bigcup M_P^H$, where $H$ runs through the open subgroups of $G$, and so $M_P$ is a discrete $G$-module.

Conversely, given a discrete $G$-module $M$, we can define a presheaf $F_M$ so that

1. $F_M(K') = M^H$, if $H = \mathrm{Gal}(K_s/K)$,

2. $F_M(\prod K_i) = \coprod F_M(K_i)$.

**Theorem 4.14.** [Me, p.53, II, Theorem 1.9] The correspondences $F \leftrightarrow M_F$, $M \leftrightarrow F_M$ induce an equivalence between the category $\mathbb{S}(X_{et})$ and the category $G$-mod of discrete $G$-modules.

## 4.6 Cohomology with support

We define cohomology groups which is different from in Section 4.3.

Let $i$: $Z \to X$ be a closed immersion and $j$: $U \to X$ an open immersion such that $X$ is the disjoint union of $i(Z)$ and $j(U)$. For any sheaf $F$ on $X_{et}$, $i_* i^! F$ is the largest subsheaf of $F$ that is zero outside $Z$. The functor $F \mapsto \Gamma(Z, i^! F)$ is left exact, and its right derived functors, $\mathrm{H}^p_Z(X, F)$, are called the *cohomology groups of $F$ with support on $Z$*. The functor $\mathrm{H}^p_Z(X, F)$, are contravariant in $(X, U)$.

**Proposition 4.15.** [Me, p.92, III, Proposition 1.25] For any sheaf $F$ on $X_{et}$ there is a long exact sequence

$$\begin{aligned} 0 \to (i^! F)(Z) \to F(X) \to F(U) \to \cdots \to \mathrm{H}^p(X, F) \\ \to \mathrm{H}^{p+1}_Z(X, F) \to \mathrm{H}^{p+1}(X, F) \to \cdots . \end{aligned}$$

**Proposition 4.16.** [Me, p.92, III, proposition 1.27] Let $Z \subset X$ and $Z' \subset X'$ be closed subschemes, and $\pi : X' \to X$ an étale morphism such that the restriction of $\pi$ to $Z'$ is an isomorphism $\pi|Z'$: $Z' \simeq Z$ and $\pi(X' - Z') \subset X - Z$. Then

$$\mathrm{H}^p_Z(X, F) \to \mathrm{H}^p_{Z'}(X', \pi^* F)$$

is an isomorphism for all $p \geqq 0$ and all shaeves $F$ on $X_{et}$.

**Lemma 4.17.** [Me, p.88, III, Lemma 1.16] Let $I$ be a filtered category and $(i \mapsto X_i)$ a contravariant functor from $I$ to schemes over $X$. Assume that all schemes $X_i$ are quasi-compact and that the maps $X_i$ are affine. Let $X_\infty = \varprojlim X_i$ , and, for a sheaf $F$ on $X_{et}$, let $F_i$ and $F_\infty$ be their inverse images on $X_i$ and $X_\infty$ respectively. Then

$$\varinjlim \mathrm{H}^p((X_i)_{et}, F_i) \simeq \mathrm{H}^p((X_\infty)_{et}, F_\infty).$$

**Corollary 4.18.** [Me, p.93, III, Corollary 1.28] Let $z$ be a closed point of $X$. Then

$$\mathrm{H}^p_z(X, F) \simeq \mathrm{H}^p_z(\mathrm{Spec}\, \mathcal{O}^h_{X,z}, F).$$

*Proof.* By Proposition 4.16,

$$\mathrm{H}^p_z(X, \mathcal{F}) = \mathrm{H}^p_y(Y, \mathcal{F})$$

for any étale neighborhood $(Y, y)$ of $z$ such that only $y$ maps to $z$. Moreover,

$$\varinjlim \mathrm{H}^p_y(Y, \mathcal{F}) = \mathrm{H}^p_z(\mathrm{Spec}\, \mathcal{O}^h_{X,z}, \mathcal{F}) \tag{4.19}$$

by Propositions 4.15 and 4.17. So the statement follows. ☐

**Theorem 4.20.** [Me, p.88, III, Theorem 1.15] Let $\pi : Y \to X$ be a quasi-compact morphism and $F$ a sheaf on $Y_{et}$. Let $\bar{x}$ be a geometric point of $X$ such that $\kappa(\bar{x})$ is the separable closure of $\kappa(x)$. Let $\widetilde{X} = \operatorname{Spec} \mathcal{O}_{X,\bar{x}}$, $\widetilde{Y} = Y \times_X \widetilde{X}$, and let $\widetilde{F}$ be the inverse image of $F$ on $\widetilde{Y}$:

$$\begin{tikzcd} Y \arrow[d] & \widetilde{Y} \arrow[l] \arrow[d] \\ X & \widetilde{X} \arrow[l] \end{tikzcd} .$$

Then $R^p\pi_*(F)_{\bar{x}} \simeq \mathrm{H}^p(\widetilde{Y}, \widetilde{F})$.

## 4.7 Spectral sequences of étale cohomology

Here we list a few spectral sequences which will be needed later. We discuss the details in Section 6

**Theorem 4.21.** [W, p.150, The Grothendieck spectral sequence] Let $\mathbb{A}, \mathbb{B}$, and $\mathbb{C}$ be abelian categories. Assume that $\mathbb{A}$ and $\mathbb{B}$ have enough injective objects, and let $f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{B} \to \mathbb{C}$ be left exact functors. If $f$ takes injective objects to $g$-acyclic objects (i.e, $(R^i g)(f(I)) = 0$ for all $i > 0$ if $I$ is an injective object of $\mathbb{A}$), then there is a spectral sequence

$$E_2^{p,q} = (R^p g)(R^q f)(A) \Rightarrow R^n(gf)(A)$$

for any object $A$ of $\mathbb{A}$.

The following theorems follow from Theorem 4.21.

**Theorem 4.22.** [Leray spectral sequence] [Me, p.89, III, Theorem 1.18 (a)] Let $(\mathbb{C}/X)_E$, $(\mathbb{C}'/X')_{E'}$ be sites. Then for any morphism of sites $\pi : (\mathbb{C}'/X')_{E'} \to (\mathbb{C}/X)_E$ , there is a spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(X_E, R^p\pi_*F) \Rightarrow \mathrm{H}^{p+q}(X'_{E'}, F)$$

where $F$ is a sheaf on $X'_{E'}$.

**Theorem 4.23.** [Hochschild-Serre spectral sequence] [Me, p.105, III, Theorem 2.20] Let $\pi : X' \to X$ be a finite Galois covering with Galois group $G$, and let $F$ be a sheaf for the étale topology on $X$. Then there is a spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(G, \mathrm{H}^q(X'_{et}, F)) \Rightarrow \mathrm{H}^{p+q}(X_{et}, F).$$

## 4.8 The definition of Azumaya algebra

In this subsection we define Azumaya algebras over a scheme. By definition, we see that Azumaya algebras over a field correspond with central simple algebras over a field.

Let $R$ be a local ring. For an $R$-algebra $A$, we denote the opposite algebra of $A$ by $A^\circ$. This is the algebra with the order of the multiplication reversed. $A$ is called an *Azumaya algebra* over $R$ if it is free of finite rank as an $R$-module and if the map

$A \otimes_R A^\circ \to \mathrm{End}_{R\text{-mod}}(A)$ which sends $a \otimes a'$ to the endomorphism $(x \mapsto axa')$ is an isomorphism.

Let $X$ be a scheme. An $\mathcal{O}_X$-algebra $A$ is called an *Azumaya algebra* over $X$ if it is coherent as an $\mathcal{O}_X$-module and if, for all closed points $x$ of $X$, $A_x$ is an Azumaya algebra over the local ring $\mathcal{O}_{X,x}$. For a finitely generated $A$-module, the following is known.

**Theorem 4.24.** [Me, p.11, I, Theorem 2.9] Let $M$ be a finitely generated $A$-module. The followings are equivalent.

1. $M$ is flat.
2. $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m}$ of $A$.
3. $\widetilde{M}$ is a locally free sheaf on $\mathrm{Spec}\, A$.
4. $M$ is a projective $A$-module.

*Remark* 4.25. Let $X = \mathrm{Spec}\, R$ be affine. An Azumaya algebra over $X$ corresponds to an $R$-algebra $A$. Moreover $A$ is projective and finitely generated as an $R$-module and that the canonical map $A \otimes_R A^\circ \to \mathrm{End}_{R\text{-}mdl}(A)$ is an isomorphism by Theorem 4.24.

Two Azumaya algebras $A$ and $A'$ over $X$ are said to be *similar* if there exist locally free $\mathcal{O}_X$-modules $E$ and $E'$ of finite rank over $\mathcal{O}_X$, such that

$$A \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}_{\mathcal{O}_X}(E) \simeq A' \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}_{\mathcal{O}_X}(E').$$

The set of similarity classes of Azumaya algebras on $X$ becomes a group under the operation $[A][A'] = [A \otimes A']$: the identity element is $[\mathcal{O}_X]$ and $[A]^{-1} = [A^\circ]$. This is the *Brauer group* $\mathrm{Br}(X)$ of $X$. Br is a functor from schemes to abelian group.

**Theorem 4.26.** [Me, p.142, IV, Theorem 2.5] There is a canonical injective homomorphism

$$i: \ \mathrm{Br}(X) \to \mathrm{H}^2(X_{et}, \mathbb{G}_m).$$

We write $\mathrm{Br}'(X)$ for $\mathrm{H}^2(X, \mathbb{G}_m)$ and call it the *cohomological Brauer group.*

**Proposition 4.27.** [Me, IV, §2, Proposition 2.11] Let $X = \mathrm{Spec}\, R$, where $R$ is a local ring, and let $r \in \mathrm{Br}'(X)$. Then the followings are equivalent.

1. $r \in \mathrm{Br}(X)$.
2. There is a finite étale surjective map $Y \to X$ such that $r$ maps to zero in $\mathrm{Br}'(Y)$.

**Theorem 4.28.** [H, Theorem] Let $X = \mathrm{Spec}\,(A)$ be a regular scheme. If $c \in \mathrm{Br}'(X)$ and $c_y = i([\Lambda_y])$ in $\mathrm{Br}'(A_{\mathfrak{m}_y})$ for all closed points $y \in X$, then $c = i([\Lambda])$.

**Proposition 4.29.** Let $A$ be a regular local ring, $X = \mathrm{Spec}\,(A)$, $K = R(A)$ and $g$: $\mathrm{Spec}\,(K) \to X$ the generic point. Put

$$K_{un} = \varinjlim_{X'} R(X')$$

where $X'$ runs through finite étale morphism over $X$. Then

$$\mathrm{H}^2(X, g_*(\mathbb{G}_m)) = \mathrm{Ker}\,(\mathrm{Br}(K) \to \mathrm{Br}(K_{un}))$$

*Proof.* Let $x$ be the closed point of $X$. Then

$$\mathrm{H}^2(X, g_*(\mathbb{G}_m)) = \mathrm{Ker}\,(\mathrm{Br}(K) \to \mathrm{Br}(K_{\bar{x}}))$$

by Lemma 7.4. So it is sufficient to prove that $K_{\bar{x}} = K_{un}$. If $X' \to X$ is an étale neighborhood of $x$, then there exists an étale neighborhood of $x$ $X'' \to X$ such that $R(X'')/R(X)$ is a Galois extension and $R(X') \subset R(X'')$. □

The following fact is proved by Gabber (c.f, [Go]).

**Corollary 4.30.** Let $A$ be a regular integral ring. Then

$$\mathrm{Br}(A) = \mathrm{Br}'(A).$$

*Proof.* If $A$ is a local ring, there exists a finite étale morphism $Y \to X$ for any $r \in \mathrm{Br}'(X)$ such that

$$r \in \mathrm{Ker}\,(\mathrm{Br}(K) \to \mathrm{Br}(R(Y)))\,.$$

So

$$r \in \mathrm{Ker}\,(\mathrm{Br}'(X) \to \mathrm{Br}(Y))$$

because $\mathrm{Br}'(Y) \subset \mathrm{Br}(R(Y))$ and

$$\begin{array}{ccc} \mathrm{Br}(X) & \longrightarrow & \mathrm{Br}(R(X)) \\ \downarrow & & \downarrow \\ \mathrm{Br}(Y) & \longrightarrow & \mathrm{Br}(R(Y)) \end{array}$$

is commutative. Therefore $r \in \mathrm{Br}(Y)$ by Proposition 4.27. So the statement follows from Theorem 4.28. □

**Definition 4.31.** If $S$ is a Galois extension of $R$ relative to $G$, and $f$ is a 2-cocycle of $G$ in $U(S)$, i.e., $f \in Z^2(G, U(S))$, we can define *a crossed-product algebra* $\Delta(f; S; G)$ as follows. $\Delta(f; S; G)$ is a free (left) $S$-module with free generators $\{u_\delta\}$ indexed by elements of $G$, with multiplication defined by

$$(au_\rho)(bu_\delta) = a\rho(b)f(\rho, \delta)u_{\rho\delta}.$$

**Theorem 4.32.** [A-G, p.406, Theorem A.12] Let $S$ be a Galois extension of $R$ relative to $G$. If $f \in Z^2(G, U(S))$, then $\Delta(f; S; G)$ is a central separable algebra over $R$ which contains $S$ as a maximal commutative subring and is split by $S$. Moreover the map

$$f \to \Delta(f; S; G)$$

induces a homomorphism

$$\mathrm{H}^2(G, U(S)) \to \mathrm{Br}(R). \tag{4.33}$$

**Theorem 4.34.** [A-G, p.408, Theorem A.15] Let $S$ be a Galois extension of $R$ relative to $G$, $U(S)$ the group of units of the commutative ring $S$ and suppose that $S$ has the property that every finitely generated projective $S$-module of rank one is free. Then the sequence

$$0 \to \mathrm{H}^2(G, U(S)) \to \mathrm{Br}(R) \to \mathrm{Br}(S)$$

is exact.

Let $G$ be a cyclic group of order $n$, and choose a generator $s$ of $G$. The choice of $s$ defines a character $\chi_s$: $G \to \mathbb{Q}/\mathbb{Z}$ such that $\chi_s(s) = 1/n$, and the coboundary of the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

transforms $\chi_s$ into an element $\theta_s = \delta\chi_s$ of $\mathrm{H}^2(G, \mathbb{Z})$.

**Theorem 4.35.** [S2, VIII, §4] Let $G$ be a cyclic group of order $n$, and choose a generator $s$ of $G$. Let $N = \sum_{t \in G} t$. Then the homomorphism

$$A^G/NA \to \mathrm{H}^2(G, A) \tag{4.36}$$

which is given by the cup product with $\theta_s$ is an isomorphism. This isomorphism depends on the choice of $s$.

*Remark* 4.37. Let $G$ be a cyclic group of order $n$ and $S$ a Galois extension of $R$ relative to $G$. Let

$$\Omega_{G,s} : \ U(S)^G/NU(S) \to \mathrm{Br}(R)$$

be the composition of the maps (4.36) and (4.33). Then, for $b \in U(S)$, $\Omega_{G,s}(\bar{b})$ corresponds to $\Delta_b$ which is generated over $S$ by element $f$ with the relation:

$$f^n = b, \quad f \cdot y = s(y) \cdot f.$$

**Theorem 4.38.** [A-G, p.389, Theorem 7.5] $\mathrm{Br}(k) = \mathrm{Br}(k[x])$ if and only if $k$ is perfect.

*Proof.* It is sufficient to show that $\mathrm{Br}(k) \neq \mathrm{Br}(k[x])$ if $k$ is not perfect. We shall show that $\mathrm{Br}(k[x]) \to \mathrm{Br}(k)$ is not injective. Let $p = \mathrm{ch}(k) > 0$, $y$ a root of $y^p - y - x = 0$, and $\Omega = k(x)(y)$. Then $\Omega$ is a cyclic extension of $k(x)$ of degree $p$ and $k[x][y] = k[y]$ is the integral closure of $k[x]$ in $\Omega$. Moreover, $k[x][y]$ is a ring Galois extension of $k[x]$ relative to $G(\Omega/k(x))$. Let $c \in k$ with $c \notin k$. Then $c$ as an element of $k(x)$ is not a norm from $\Omega$. Thus, $c$ determines a non-trivial crossed-product $\Omega_{G(\Omega/k(x)),s}(c)$ where $s \in G(\Omega/k(x))$ with $s(y) = y + 1$. Let

$$\Lambda = \Omega_{G(k[y]/k[x]),s}(c).$$

Then $\Lambda \otimes k(x) = \Omega_{G(\Omega/k(x)),s}$ and $\Omega_{G(\Omega/k(x)),s}$ is not the trivial algebra. So $\Lambda$ is not the trivial algebra over $k[x]$.

We see that $\Lambda$ is the kernel of the map $\mathrm{Br}(k[x]) \to \mathrm{Br}(k)$. Now $\Lambda$ is generated over $k[x]$ by elements $\alpha$ and $\beta$ with the relations:

$$\alpha^p - \alpha = x, \ \beta^p = c, \ \beta\alpha = (\alpha + 1)\beta.$$

The image $\bar{\Lambda}$ of $\Lambda$ under the map $k[x] \to k$ is generated over $k$ by elements $\bar{\alpha}$ and $\bar{\beta}$ with the relations:

$$\bar{\alpha}^p - \bar{\alpha} = 0,\ \bar{\beta}^p = c,\ \bar{\beta}\bar{\alpha} = (\bar{\alpha}+1)\bar{\beta}.$$

Let $L = k(t)$ with $t^p = c$ and let elements $\alpha'$, $\beta'$ of the endomorphism ring of $k$-module $L$ $\mathrm{Hom}_k(L, L)$ be defined as follows: $\alpha'$ is the derivation of $L$ over $k$ given by $\alpha'(t^n) = -nt^n$ and $\beta'$ is the multiplication by $t$. Then $\alpha'$ and $\beta'$ satisfy the same relations over $k$ as do $\bar{\alpha}$ and $\bar{\beta}$, i.e,

$$(\alpha')^p - \alpha' = 0,\ (\beta')^p = c,\ \beta'\alpha' = (\alpha'+1)\beta'.$$

So there is a homomorphism from $\bar{\Lambda}$ to $\mathrm{Hom}_k(L, L)$ which maps $\bar{\alpha}, \bar{\beta}$ to $\alpha', \beta'$ respectively. Since $\bar{\Lambda}$ is a central simple algebra over $k$ of dimension $p^2$ and $\mathrm{Hom}_k(L, L)$ also has dimension $p^2$ over $k$, it follows that $\bar{\Lambda} \simeq \mathrm{Hom}_k(L, L)$. So the proof is complete. □

**Proposition 4.39.** Let $k$ be the separable closure of an imperfect field, $K$ an algebraic function field of $k$ in one variable. Then

$$\mathrm{Br}(K) \neq 0.$$

For the proof of Proposition 4.39, we prove the following lemmas.

**Lemma 4.40.** Suppose that $K$ is a field with $\mathrm{ch}(K) = p > 0$ and $\mathrm{Br}(K)_p \neq 0$. Then

$$\mathrm{Br}(K)_{p^{i+1}} \setminus \mathrm{Br}(K)_{p^i} \neq \emptyset.$$

*Proof.* Let $K_s$ be the separable closure of $K$. Then, we have the exact sequence

$$0 \to K_s^*/K_s^{*p^i} \xrightarrow{p} K_s^*/K_s^{*p^{i+1}} \to K_s^*/K_s^{*p} \to 0.$$

Therefore, we have the exact sequence

$$\mathrm{H}^1(K, K_s^*/K_s^{*p^i}) \xrightarrow{p} \mathrm{H}^1(K_s^*/K_s^{*p^{i+1}}) \to \mathrm{H}^1(K_s^*/K_s^{*p}) \to \mathrm{H}^2(K, K_s^*/K_s^{*p^i}). \tag{4.41}$$

Then the diagram

$$\begin{CD} \mathrm{H}^1(K, K_s^*/K_s^{*p^i}) @>p>> \mathrm{H}^1(K_s^*/K_s^{*p^{i+1}}) \\ @VVV @VVV \\ \mathrm{Br}(K)_{p^i} @>>> \mathrm{Br}(K)_{p^{i+1}} \end{CD} \tag{4.42}$$

is commutative and the top right arrow of (4.42) is injective because the bottom right arrow of (4.42) is the natural inclusion map.

Moreover, $\mathrm{H}^2(K, K_s^*/K_s^{*p^i}) = 0$ by Proposition 3.7. Therefore the sequence

$$0 \to \mathrm{Br}(K)_{p^i} \to \mathrm{Br}(K)_{p^{i+1}} \to \mathrm{Br}(K)_p \to 0$$

is exact by (4.41) and $\mathrm{H}^1(K, K_s^*/K_s^{*p^i}) \simeq \mathrm{Br}(K)_{p^i}$. So the statement follows by the induction. □

**Lemma 4.43.** Suppose that $K$ satisfies the assumption of Lemma 4.40 and $L$ is a finite extension field of $K$. Then $\mathrm{Br}(L) \neq 0$.

*Proof.* It is sufficient to prove the statement in the case where $[L:K]$ is a prime number. Suppose that $([L:K], p) = 1$. Then the homomorphism $\mathrm{Res}: \mathrm{Br}(K) \to \mathrm{Br}(L)$ is injective when restricted to $\mathrm{Br}(K)_p$ ([S1, I, §2, p.12, Corollary of Proposition 9]). So $\mathrm{Br}(L) \neq 0$. Suppose that $[L:K] = p$. Then, by Lemma 4.40, we have $x \in \mathrm{Br}_{p^2}(K) \setminus \mathrm{Br}_p(K)$. On the other hand, $\mathrm{Cor} \circ \mathrm{Res} = p$. Therefore, $\mathrm{Res}(x) \neq 0$. So the statement follows. □

*Proof of Proposition 4.39.* $\mathrm{Br}(k(x))_p \neq 0$ by Proposition 4.38. Therefore the statement follows from Lemma 4.43. □

# 5 Hasse principle

In this section we discuss a known result on the Hasse principle in Subsection 5.1 and give a certain counter-example to the Hasse principle in the case of function fields of curves in Subsection 5.2.

## 5.1 Classical results and Conjectures

**Theorem 5.1.** [K-K-S, Theorem 8.40 and Theorem 8.42] Let $k$ be a global field. Then the following $1, 2$ holds.

1. The local-global map
$$\mathrm{X}(G_k) \to \prod_{\mathfrak{p}} \mathrm{X}(G_{k_{\mathfrak{p}}})$$
where $\mathfrak{p}$ runs through almost all primes of $k$ is injective.

2. The local-global map
$$\mathrm{Br}(k) \to \prod_{\mathfrak{p}} \mathrm{Br}(k_{\mathfrak{p}})$$
where $\mathfrak{p}$ runs through all primes of $k$ is injective.

Moreover, let $m$ be a positive integer which is prime to $\mathrm{ch}(k)$. Then
$$\mathrm{X}(G(k_s/k))_m = \mathrm{H}^1(k, \mathbb{Z}/m\mathbb{Z}), \quad \mathrm{Br}(k)_m = \mathrm{H}^2(k, \mu_m).$$

On the other hand, for the cohomology groups of higher dimension, the following fact is known.

**Proposition 5.2.** [S1, p.87, II, §4, Proposition 13] Let $k$ be an algebraic number field. If $p \neq 2$, or if $k$ is totally imaginary, we have $\mathrm{cd}_p(G_k) \leqq 2$.

Let $m$ be a positive integer which is prime to $\mathrm{ch}(k)$. Then
$$k^*/k^{*m} \simeq \mathrm{H}^1(k, \mu_m)$$
by Hilbert theorem 90. The following fact on cohomology groups is known.

**Theorem 5.3.** [N-S-W, p.530, IX, (9.1.11) Theorem] Let $k$ be a global field, $m$ a natural number prime to $\mathrm{ch}(k)$ and $T$ a set of primes of $k$ of density

$$\delta(T) = \lim_{s \to 1} \frac{\sum_{\mathfrak{p} \in T} N\mathfrak{p}^{-s}}{\sum_{\text{all } \mathfrak{p}} N\mathfrak{p}^{-s}} = 1.$$

Then the localization homomorphism

$$k^*/k^{*m} \to \prod_{\mathfrak{p} \in T} k_{\mathfrak{p}}^*/k_{\mathfrak{p}}^{*m}$$

is injective except in the case: $(*)$ $k$ is a number field, $m = 2^r m'$, $m'$ odd, $r \geqq 3$, $k(\mu_{2^r})/k$ is not cyclic, and all primes $\mathfrak{p}$ dividing 2 in $T$ decompose in $k(\mu_{2^r})/k$, where the kernel is cyclic of order 2.

Because of the above results and results in Section 3.3, we consider the following conjecture.

**Conjecture 5.4.** Let $k$ be a finitely generated field over a prime field, $m$ an odd prime with $(m, \mathrm{ch}(k)) = 1$, $p$ any positive integer and $X$ a normal complete curve over $k$. Then the local-global map

$$\mathrm{H}^p(R(X), \mu_m) \to \prod_{\mathfrak{p} \in X} \mathrm{H}^p(\widetilde{R(X)}_{\mathfrak{p}}, \mu_m)$$

is injective.

We claim that Conjecture 5.4 is true if $k$ is an algebraic number field and $p = 1$ (in preparation). On the other hand, it is proved that the local-global map of the Brauer group is injective in the case where $k$ is an arbitrary field and $X = \mathbb{P}^1_k$ by Harder ([Ha]). In this thesis, we prove this result by an alternative method in Section 7.

## 5.2 A counterexample for the Hasse principle

In this section, we show that Conjecture 5.4 does not hold in the case where $X$ is a normal complete curve over an algebraically closed field. We start with the following lemma.

**Lemma 5.5.** Suppose that $X$ is a regular scheme. Then

$$\mathrm{H}^1(X, \mathbb{Q}/\mathbb{Z}) = \bigcap_{\mathfrak{p} \in X^{(1)}} \mathrm{H}^1(\mathrm{Spec}\,(\mathcal{O}_{X,\mathfrak{p}}), \mathbb{Q}/\mathbb{Z}).$$

For the proof of Lemma 5.5, we use the following fact.

**Theorem 5.6.** [Zariski-Nagata purity theorem] Let $\phi$: $X \to S$ be a finite surjective morphism of integral schemes with $X$ normal and $S$ regular. Assume that the fiber $X_p$ of $\phi$ above each codimension 1 point $p$ of $S$ is étale over $\kappa(p)$. Then $\phi$ is a finite étale cover.

*Proof of Lemma 5.5.* Let $K$ be a field, $G = G(K_s/K)$ and $G_\lambda$ $(\lambda \in \Lambda)$ a normal closed subgroup of $G$. Suppose that $M_\lambda$ is the Galois extension field over $K$ which corresponds to $G_\lambda$. Then, according to the infinite Galois theory, $\bigcap_{\lambda\in\Lambda} M_\lambda$ corresponds to the closure of the group which is generated by $G_\lambda$ $(\lambda \in \Lambda)$. So the statement follows from Theorem 5.6. $\square$

**Lemma 5.7.** Let $X$ be a regular scheme. Then

$$\begin{aligned}&\operatorname{Ker}\left(\mathrm{H}^1(R(X),\mathbb{Q}/\mathbb{Z}) \to \prod_{\mathfrak{p}\in X^{(1)}} \mathrm{H}^1(\widetilde{R(X)_{\mathfrak{p}}},\mathbb{Q}/\mathbb{Z})\right)\\ &= \operatorname{Ker}\left(\mathrm{H}^1(X,\mathbb{Q}/\mathbb{Z}) \to \prod_{\mathfrak{p}\in X^{(1)}} \mathrm{H}^1(\kappa(\mathfrak{p}),\mathbb{Q}/\mathbb{Z})\right).\end{aligned}$$

*Proof.* Let $\mathcal{O}$ be a discrete valuation ring and $\mathfrak{m}$ its maximal ideal. Then the sequence

$$0 \to \mathrm{H}^1(\mathrm{Spec}\,(\mathcal{O}),\mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(R(\mathrm{Spec}\,(\mathcal{O})),\mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(R(\mathrm{Spec}\,(\mathcal{O}_{\bar{\mathfrak{m}}})),\mathbb{Q}/\mathbb{Z}) \tag{5.8}$$

is exact. So, the sequence

$$0 \to \mathrm{H}^1(X,\mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(R(X),\mathbb{Q}/\mathbb{Z}) \to \prod_{\mathfrak{p}\in X^{(1)}} \mathrm{H}^1(R(\mathrm{Spec}\,(\mathcal{O}_{X,\bar{\mathfrak{p}}})),\mathbb{Q}/\mathbb{Z})$$

is exact by Lemma 5.5 and (5.8).

On the other hand, the sequence

$$\begin{aligned}0 \to \prod_{\mathfrak{p}\in X^{(1)}} \mathrm{H}^1(\mathrm{Spec}\,(\widetilde{\mathcal{O}}_{X,\mathfrak{p}}),\mathbb{Q}/\mathbb{Z}) &\to \prod_{\mathfrak{p}\in X^{(1)}} \mathrm{H}^1(\widetilde{R(X)_{\mathfrak{p}}},\mathbb{Q}/\mathbb{Z})\\ &\to \prod_{\mathfrak{p}\in X^{(1)}} \mathrm{H}^1(R(\mathrm{Spec}\,(\mathcal{O}_{X,\bar{\mathfrak{p}}})),\mathbb{Q}/\mathbb{Z})\end{aligned}$$

is also exact by (5.8).

Let

$$M = \prod_{\mathfrak{p}\in X^{(1)}} \mathrm{H}^1(\mathrm{Spec}\,(\widetilde{\mathcal{O}}_{X,\mathfrak{p}}),\mathbb{Q}/\mathbb{Z}),\ \ M' = \prod_{\mathfrak{p}\in X^{(1)}} \mathrm{H}^1(\widetilde{R(X)_{\mathfrak{p}}},\mathbb{Q}/\mathbb{Z}).$$

Then the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^1(X,\mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{H}^1(R(X),\mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d] & \prod_{\mathfrak{p}\in X^{(1)}} \mathrm{H}^1(R(\mathrm{Spec}\,(\mathcal{O}_{X,\bar{\mathfrak{p}}})),\mathbb{Q}/\mathbb{Z}) \arrow[d] \\
0 \arrow[r] & M \arrow[r] & M' \arrow[r] & \prod_{\mathfrak{p}\in X^{(1)}} \mathrm{H}^1(R(\mathrm{Spec}\,(\mathcal{O}_{X,\bar{\mathfrak{p}}})),\mathbb{Q}/\mathbb{Z})
\end{tikzcd}$$

is commutative. By applying the snake lemma to the above diagram, we see that

$$\begin{aligned}&\operatorname{Ker}\left(\mathrm{H}^1(R(X),\mathbb{Q}/\mathbb{Z}) \to \prod_{\mathfrak{p}\in X^{(1)}} \mathrm{H}^1(\widetilde{R(X)_{\mathfrak{p}}},\mathbb{Q}/\mathbb{Z})\right)\\ &= \operatorname{Ker}\left(\mathrm{H}^1(X,\mathbb{Q}/\mathbb{Z}) \to \prod_{\mathfrak{p}\in X^{(1)}} \mathrm{H}^1(\mathrm{Spec}\,(\widetilde{\mathcal{O}}_{X,\mathfrak{p}}),\mathbb{Q}/\mathbb{Z})\right)\end{aligned}$$

because $\mathcal{O}_{X,\bar{\mathfrak{p}}} = \widetilde{\mathcal{O}}_{X,\bar{\mathfrak{p}}}$. Moreover,

$$\begin{aligned}
&\mathrm{Ker}\left(\mathrm{H}^1(X, \mathbb{Q}/\mathbb{Z}) \to \prod_{\mathfrak{p}\in X^{(1)}} \mathrm{H}^1(\mathrm{Spec}\,(\widetilde{\mathcal{O}}_{X,\mathfrak{p}}), \mathbb{Q}/\mathbb{Z})\right) \\
= &\mathrm{Ker}\left(\mathrm{H}^1(X, \mathbb{Q}/\mathbb{Z}) \to \prod_{\mathfrak{p}\in X^{(1)}} \mathrm{H}^1(\kappa(\mathfrak{p}), \mathbb{Q}/\mathbb{Z})\right)
\end{aligned}$$

by Proposition 4.11. So the statement follows. □

If $X$ is an algebraic curve of genus $g$ over an algebraically closed field,

$$\mathrm{H}^1(X, \mathbb{Q}/\mathbb{Z}) = \mathrm{Ker}\left(\mathrm{H}^1(R(X), \mathbb{Q}/\mathbb{Z}) \to \prod_{\mathfrak{p}\in X_{(0)}} \mathrm{H}^1(\widetilde{R(X)}_{\mathfrak{p}}, \mathbb{Q}/\mathbb{Z})\right)$$

because of the fact that $\kappa(\mathfrak{p})$ is an algebraically closed field and Lemma 5.7.

Moreover,

$$\begin{aligned}
\mathrm{H}^1(X, \mathbb{Q}/\mathbb{Z})_m &= \mathrm{H}^1(X, \mathbb{Z}/m\mathbb{Z}) \\
&= (\mathbb{Z}/m\mathbb{Z})^{2g}.
\end{aligned}$$

So if $g \neq 0$, the Hasse principle for function fields of curves of genus $g$ and the group of characters does not hold.

# 6 The edge maps of the Grothendieck spectral sequence

In this section, we characterize the edge maps of the Grothendieck spectral sequence. We often consider the following assumption in this section.

**Assumption 6.1.** $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are abelian categories which have enough injective objects. $G : \mathcal{C} \to \mathcal{B}, F : \mathcal{B} \to \mathcal{A}$ are left exact functors such that $G$ takes injective objects of $\mathcal{C}$ to $F$-acyclic objects.

We assume that Assumption 6.1 is satisfied. We define a homomorphism of functors $r^p_{F,G}(A) : R^pF(G(A)) \to R^p(FG)(A)$ (for any object $A$ of $\mathcal{C}$) by induction on $p$ so that it satisfies the following properties.

Property 6.2. (1) If $p = 0$, $r^0_{F,G}(A) = \mathrm{id}_{FG(A)}$.

(2) Suppose that $p > 0$ and assume that $r^i_{F,G}$ is defined for $i \leqq p-1$. Let $0 \to A \to I \to M \to 0$ be an exact sequence and $I$ an injective object. Then the following

commutative diagram

$$
\text{(6.3)} \qquad
\begin{tikzcd}
 & R^{p-1}F(G(I)) \arrow[rr, "r^{p-1}_{F,G}(I)"] \arrow[d] \arrow[dr] & & R^{p-1}(FG)(I) \arrow[d] \\
 & R^{p-1}F(G(I)/G(A)) \arrow[r] \arrow[d, "\delta_1^{p-1}"'] & R^{p-1}F(G(M)) \arrow[r, "r^{p-1}_{F,G}(M)"] & R^{p-1}(FG)(M) \arrow[d, "\delta_2^{p-1}"] \\
 & R^pF(G(A)) \arrow[rr, "r^p_{F,G}(A)"'] & & R^p(FG)(A)
\end{tikzcd}
$$

is commutative where $\delta_1^{p-1}, \delta_2^{p-1}$ are the connecting homomorphisms.

Note that the vertical sequences are exact. Since $G(I)$ is acyclic, $R^pF(G(I)) = 0$ by assumption. So the connecting homomorphism $\delta_1^{p-1}$ is surjective if $p > 0$. Therefore, $r^p_{F,G}(A)$ is well-defined.

**Proposition 6.4.** If $s^p(A) : R^pF(G(A)) \to R^p(FG)(A)$ $(p \geq 0)$ is a homomorphism of functors satisfying Property 6.2 (1), (2) above (replacing $r^p_{F,G}$ by $s^p$), then $s^p$ coincides with $r^p_{F,G}$ for all $p \geq 0$.

*Proof.* Suppose that $A$ is an object of $\mathcal{C}$. If $p = 0$ then $s^0(A) = \mathrm{id}_{FG(A)} = r^0_{F,G}(A)$.

We assume that $p > 0$ and $s^q = r^q_{F,G}$ for all $q < p$. Take an exact sequence

$$0 \to A \to I \to M \to 0$$

with $I$ injective. Then we have the commutative diagram of the above (1) (replacing $r^{p-1}_{F,G}, r^p_{F,G}$ by $s^{p-1}, s^p$). As we pointed out above, $\delta_1^{p-1}$ is surjective. The homomorphism $R^{p-1}F(G(I)/G(A)) \to R^{p-1}F(G(M))$ and the connecting homomorphisms

$$\delta_1^{p-1} : R^{p-1}F(G(I)/G(A)) \to R^pF(G(A)), \qquad \delta_2^{p-1} : R^{p-1}(FG)(M) \to R^p(FG)(A)$$

are independent of $s^{p-1}, s^p$. By assumption $s^{p-1}(M) = r^{p-1}_{F,G}(M)$. Therefore, $s^p(A) = r^p_{F,G}(A)$. This proves the proposition. ☐

Moreover if $p > 1$, $\delta_l$ is an isomorphism. Since the image of the map from $R^{p-1}F(G(I))$ to $R^p(FG)(A)$ in the above diagram is 0, $r^p_{F,G}(A)$ is well-defined. This definition is independent of the choice of the exact sequence $0 \to A \to I \to M \to 0$ ($I$ is an injective object).

**Lemma 6.5.** (a) Even if $I$ is not an injective object, the diagram (6.3) is commutative.

(b) Suppose that $\mathcal{D}$ is an abelian category, $H : \mathcal{D} \to \mathcal{C}$ is a left exact functor which takes injective objects to $FG$-acyclic objects, and that $GH$ takes injective objects to $F$-acyclic objects. Then the following diagram

$$
\begin{tikzcd}
R^pF(GH(A)) \arrow[d, "r^p_{F,G}(H(A))"'] \arrow[dr, "r^p_{F,GH}(A)"] & \\
R^p(FG)(H(A)) \arrow[r, "r^p_{FG,H}(A)"'] & R^p(FGH)(A)
\end{tikzcd}
$$

is commutative.

*Proof.* (a) For an exact sequence $0 \to A \to I \to M \to 0$ ( $I$ is not always an injective object), there exists an exact sequence $0 \to A \to J \to M' \to 0$ ($J$ is an injective object) such that a diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}"'] & I \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & J \arrow[r] & M' \arrow[r] & 0 \end{tikzcd}$$

is commutative. As $R^pF, R^p(FG)$ are $\delta$-functors,

$$\begin{tikzcd} R^{p-1}F(G(I)/G(A)) \arrow[r, "\delta"] \arrow[d] & R^pF(G(A)) \arrow[d, equal] \\ R^{p-1}F(G(J)/G(A)) \arrow[r, "\delta"'] & R^pF(G(A)) \end{tikzcd} \qquad \begin{tikzcd} R^{p-1}(FG)(M) \arrow[r, "\delta"] \arrow[d] & R^p(FG)(A) \arrow[d, equal] \\ R^{p-1}(FG)(M') \arrow[r, "\delta"'] & R^p(FG)(A) \end{tikzcd}$$

are commutative diagrams. Moreover, the following diagram is commutative:

$$\begin{tikzcd} R^{p-1}F(G(I)/G(A)) \arrow[r] \arrow[d] & R^{p-1}F(G(M)) \arrow[r, "r^{p-1}_{F,G}(M)"] \arrow[d] & R^{p-1}(FG)(M) \arrow[d] \\ R^{p-1}F(G(J)/G(A)) \arrow[r] \arrow[d, "\delta"'] & R^{p-1}F(G(M')) \arrow[r, "r^{p-1}_{F,G}(M')"'] & R^{p-1}(FG)(M') \arrow[d, "\delta"] \\ R^pF(G(A)) \arrow[rr, "r^p_{F,G}(A)"'] & & R^p(FG)(A). \end{tikzcd}$$

The statement (a) follows from the above consideration.

(b) Let $0 \to A \to I \to M \to 0$ be an exact sequence with $I$ injective. Then

$$0 \to H(A) \to H(I) \to H(I)/H(A) \to 0$$

is exact. We put $N = H(I)/H(A)$. By the statement $(a)$, the diagram

$$\begin{tikzcd} R^{p-1}F(GH(I)/GH(A)) \arrow[r] \arrow[d] & R^{p-1}F(G(H(I)/H(A))) \arrow[r, "r^{p-1}_{F,G}(N)"] & R^{p-1}FG(H(I)/H(A)) \arrow[d] \\ R^pF(GH(A)) \arrow[rr, "r^p_{F,GH}(A)"'] & & R^p(FG)(H(A)) \end{tikzcd}$$

is commutative. By the definition of $r^{p-1}_{FG,H}$, the diagram

$$\begin{tikzcd} R^{p-1}(FG)(H(I)/H(A)) \arrow[r] \arrow[d] & R^{p-1}(FG)(H(M)) \arrow[r, "r^{p-1}_{FG,H(M)}"] & R^{p-1}(FGH)(M) \arrow[d] \\ R^p(FG)(H(A)) \arrow[rr, "r^p_{FG,H}(A)"'] & & R^p(FGH)(A) \end{tikzcd}$$

is commutative. Moreover, since $r^{p-1}_{F,G}$ is functorial, the diagram

$$\begin{tikzcd}
R^{p-1}F(G(H(I)/H(A))) \arrow[r, "r^{p-1}_{F,G}(N)"] \arrow[d] & R^{p-1}(FG)(H(I)/H(A)) \arrow[d] \\
R^{p-1}F(GH(M)) \arrow[r, "r^{p-1}_{F,G}(H(M))"'] & R^{p-1}(FG)(H(M))
\end{tikzcd}$$

is also commutative. Therefore the diagram

$$\begin{tikzcd}
R^{p-1}F(GH(I)/GH(A)) \arrow[r] \arrow[d] & R^{p-1}F(GH(M)) \arrow[r, "r^{p-1}_{FG,H}(M)\circ r^{p-1}_{F,G}(H(M))"] & R^{p-1}(FGH)(M) \arrow[d] \\
R^pF(GH(A)) \arrow[rr, "r^p_{FG,H}(A)\circ r^p_{F,G}(H(A))"'] & & R^p(FGH)(A)
\end{tikzcd}$$

is commutative. By induction, $r^{p-1}_{FG,H}(M) \circ r^{p-1}_{F,G}(H(M)) = r^{p-1}_{F,GH}(M)$. So by (b) by the induction. $r^p_{FG,H}(A) \circ r^p_{F,G}(H(A)) = r^p_{F,GH}(A)$. This proves (b). ☐

**Proposition 6.6.** Suppose that $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $F$: $\mathbb{C} \to \mathbb{B}$ and $G$: $\mathbb{A} \to \mathbb{B}$ satisfy Assumption 6.1. Let $a_{F,G}$: id $\to GH$ be the canonical adjoint functor. Then $n^p_{F,G}(A)$:

$$R^pF(A) \to R^pF(GH(A)) \overset{r^p_{F,G}(H(A))}{\to} R^pFG(H(A))$$

is a homomorphism of $\delta$-functors, i.e, when $0 \to A \to B \to C \to 0$ is exact, the diagram

$$\begin{tikzcd}
R^{p-1}F(C) \arrow[r] \arrow[d] & R^{p-1}FG(H(C)) \arrow[d] \\
R^pF(A) \arrow[r] & R^pFG(H(A))
\end{tikzcd}$$

is commutative.

*Proof.* If $0 \to A \to B \to C \to 0$ is exact, the following diagram is commutative.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "a_{G,H}(A)"'] & B \arrow[r] \arrow[d, "a_{G,H}(B)"'] & C \arrow[r] \arrow[d, "\bar{a}_{G,H}"'] & 0 \\
0 \arrow[r] & GH(A) \arrow[r] & GH(B) \arrow[r] & GH(B)/GH(A) \arrow[r] & 0
\end{tikzcd}$$

where the right vertical map is induced by $a_{F,G}(C)$, i.e, $\bar{a}_{G,H}$ satisfies the following commutative diagram

(6.7)
$$\begin{tikzcd}
C \arrow[r] \arrow[dr] & GH(B)/GH(A) \arrow[d] \\
& GH(C).
\end{tikzcd}$$

The diagram

$$\begin{tikzcd} R^{p-1}F(C) \arrow[r] \arrow[d] & R^{p-1}F(GH(B)/GH(A)) \arrow[d] \\ R^pF(A) \arrow[r] & R^pF(GH(A)) \end{tikzcd} \tag{6.8}$$

is commutative. Moreover the diagram

$$\begin{tikzcd} R^{p-1}F(C) \arrow[r] \arrow[dr] & R^{p-1}F(GH(B)/GH(A)) \arrow[d] \\ & R^{p-1}F(GH(C)). \end{tikzcd} \tag{6.9}$$

is commutative. So the statement follows. □

**Proposition 6.10.** Let $K$ be a field, $K'/K$ a field extension, $X = \mathrm{Spec}\,(K)$ and $X' = \mathrm{Spec}\,(K')$. Let $i : X' \to X$ be the morphism of schemes which corresponds to the inclusion map $K \to K'$. Suppose that $F$ is the functor which associates to an étale sheaf $\mathcal{F}$ on $X$ the module of its sections $\Gamma(X, \mathcal{F})$ and that $G$ is the functor $i_* : \mathbf{S}_{X'_{et}} \to \mathbf{S}_{X_{et}}$ (the direct image). Then $r^p_{F,G}(\mathcal{F}) : \mathrm{H}^p(X_{et}, i_*(\mathcal{F})) \to \mathrm{H}^p(X'_{et}, \mathcal{F})$ coincides with

$$\mathrm{H}^p(G_K, M_{G_K}^{\psi(G_{K'})}(N^{\mathrm{Ker}(\psi)})) \xrightarrow{\simeq} \mathrm{H}^p(\psi(G_{K'}), N^{\mathrm{Ker}(\psi)}) \xrightarrow{\mathrm{Inf}} \mathrm{H}^p(G_{K'}, N) \tag{6.11}$$

where the first map is the isomorphism by Shapiro's Lemma. See [S1, p.13, I,§2 Proposition 10] for Shapiro's Lemma and [S2, p.116, VII, §5] for the definition of the inflation Inf.

*Proof.* The category $\mathbf{S}_{X_{et}}$ is equivalent to the category $G_K$-mod
(see [Me, p.53, II.Theorem.1.9]). If $\mathcal{F} \in \mathbf{S}_{X_{et}}$, corresponds to a $G_K$-module $N$, $i_*(\mathcal{F})$ corresponds to the induced module $M_{G_K}^{\psi(G_{K'})}(N^{\mathrm{Ker}(\psi)})$ (see [Me, p.69, II.Remarks 3.1 (e)] and also [S1, p.13, I, 2.5] for the notation). Then $r^p_{F,G}(\mathcal{F})$ and the homomorphism (6.11) satisfy Property 6.2. So the proof is complete. □

Let $X, Y$ be quasi-compact schemes such that every finite subset is contained in an affine open set. Let $f : Y \to X$ be a morphism of schemes and $\mathcal{F}$ a sheaf on $Y_{et}$. If $\mathcal{U} = (U_i \to X)_{i \in I}$ is a covering of $X$, $\mathcal{U}' = (U_i \times_X Y \to Y)_{i \in I}$ is a covering of $Y$. So, the natural map from the Čech cohomology of $f_*(\mathcal{F})$ with respect to $\mathcal{U}$ into the Čech cohomology of $\mathcal{F}$ with respect to $\mathcal{U}'$

$$\mathrm{H}^p(\mathcal{U}/X, f_*(\mathcal{F})) \to \mathrm{H}^p(\mathcal{U}'/Y, \mathcal{F})$$

is given. This map induces a map $\check{\mathrm{H}}^p(X, f_*(\mathcal{F})) \to \check{\mathrm{H}}^p(Y, \mathcal{F})$ from the Čech cohomology of $f_*(\mathcal{F})$ over $X$ into the Čech cohomology of $\mathcal{F}$ over $X'$. Čech étale cohomology groups agree with derived functor étale cohomology groups (see [Me, p.104, III, Theorem 2.17]). So the above map induces the map

$$\mathrm{H}^p(X, f_*(\mathcal{F})) \to \mathrm{H}^p(Y, \mathcal{F}). \tag{6.12}$$

Then we have the following fact.

**Proposition 6.13.** Let the notations $X, Y, f : Y \to X$ and $\mathcal{F}$ be the same as above. Then the homomorphism (6.12) coincides with $r^p_{F,G}$ where $F$ is the functor which associates to an étale sheaf $\mathcal{F}$ the module of its sections $\Gamma(X, \mathcal{F})$ and $G$ is the direct image $f_*$ .

*Proof.* This proposition also follows from Proposition 6.2. □

**Proposition 6.14.** Let the notations $X$, $Y$, $f$: $Y \to X$, $\mathcal{F}$, $F$ and $G$ be the same as above. Moreover, let $k$ be a field and $X = \operatorname{Spec}(k)$. Then $r^p_{F,G}$ coincides with $r^p_{F',G'}$ where $G'$ is $\Gamma(X \otimes_k k_s, \mathcal{F})$ and $F'$ is the action of $G(k_s/k)$.

**Corollary 6.15.** Let $k$ be a field. Then

$$\mathrm{Br}(k) = \mathrm{Br}(\mathbb{P}^1_k).$$

*Proof.* It is easy to show that the natural map $\mathrm{Br}(k) \to \mathrm{Br}(\mathbb{P}^1_k)$ is injective. So it is sufficient to show that this map is surjective. By the Hochschild-Serre spectral sequence $\mathrm{H}^p(G_k, \mathrm{H}^q(\mathbb{P}^1_{k_s}, \mathbb{G}_m)) \Rightarrow \mathrm{H}^n(\mathbb{P}^1_k, \mathbb{G}_m)$ (cf, [Me, p.105, III, Theorem 2.20]), we have the following exact sequence

$$\mathrm{Br}(k) \to \operatorname{Ker}\left(\mathrm{Br}(\mathbb{P}^1_k) \to \mathrm{Br}(\mathbb{P}^1_{k_s})\right) \to \mathrm{H}^1(k, \operatorname{Pic}(\mathbb{P}^1_{k_s})) \tag{6.16}$$

(cf, [Me, p.309, Appendix B]). Then the homomorphism

$$\mathrm{Br}(k) \to \operatorname{Ker}\left(\mathrm{Br}(\mathbb{P}^1_k) \to \mathrm{Br}(\mathbb{P}^1_{k_s})\right)$$

is the canonical map by Proposition 6.14 and Lemma 6.26. On the other, $\mathrm{H}^1(k, \operatorname{Pic}(\mathbb{P}^1_{k_s})) = \mathrm{H}^1(k, \mathbb{Z}) = 0$ because $\mathbb{Z}$ is torsion free and $\mathrm{Br}(\mathbb{P}^1_{k_s}) = 0$ by [G, III, Corollary 5.8]. So the statement follows. □

Suppose that Assumption 6.1 is satisfied. Let

$$0 \to A \to I^0 \xrightarrow{\mathrm{d}^0} I^1 \xrightarrow{\mathrm{d}^1} \cdots \xrightarrow{\mathrm{d}^{p-1}} I^p \xrightarrow{\mathrm{d}^p} I^{p+1} \xrightarrow{\mathrm{d}^{p+1}} \cdots$$

be an injective resolution of $A$. Then, by definition

$$R^p(FG)(A) = \operatorname{Ker}(FG(\mathrm{d}^p)) / \operatorname{Im}(FG(\mathrm{d}^{p-1})),$$
$$R^pG(A) = \operatorname{Ker}(G(\mathrm{d}^p)) / \operatorname{Im}(G(\mathrm{d}^{p-1})).$$

So

$$\operatorname{Im}(FG(\mathrm{d}^{p-1})) \subset F(\operatorname{Im}(G(\mathrm{d}^{p-1}))). \tag{6.17}$$

Since $F$ is left exact functor, the natural map $F(\operatorname{Ker}(G(\mathrm{d}^p))) \to \operatorname{Ker}(FG(\mathrm{d}^p))$ is an isomorphism. Therefore we can define a homomorphism of functors

$$l^p_{F,G}(A) : R^p(FG)(A) \to F(R^pG(A))$$

so that the following diagram is commutative.

$$
\begin{tikzcd}
\mathrm{Ker}(FG(\mathrm{d}^p)) \arrow[d] & F(\mathrm{Ker}(G(\mathrm{d}^p))) \arrow[l, "\simeq"'] \arrow[d] \\
R^p(FG)(A) \arrow[r, "l^p_{F,G}(A)"] & F(R^pG(A))
\end{tikzcd}
\tag{6.18}
$$

Note that $l^p_{F,G}(A)$ is well-defined by (6.17). Also since $\mathrm{Ker}(FG(\mathrm{d}^p)) \to R^p(FG)(A)$ is surjective, this property characterizes $l^p_{F,G}(A)$.

**Lemma 6.19.** (a) If $0 \to A \to B \to C \to 0$ is an exact sequence, then the following diagram

$$
\begin{tikzcd}
R^p(FG)(C) \arrow[r, "\delta"] \arrow[d, "l^p_{F,G}(C)"'] & R^{p+1}(FG)(A) \arrow[d, "l^{p+1}_{F,G}(A)"] \\
F(R^pG(C)) \arrow[r, "F(\delta)"'] & F(R^{p+1}(G)(A))
\end{tikzcd}
$$

is commutative.

(b) Suppose that $H : \mathcal{D} \to \mathcal{C}$ takes injective objects to $G$-acyclic and $FG$-acyclic objects and that $GH$ takes injective objects to $F$-acyclic objects. Then we have

$$l^p_{FG,H}(A) = F(l^p_{G,H}(A)) \circ l^p_{F,GH}(A).$$

*Proof.* We first prove the statement $(a)$. Let

$$0 \to A \to I^0_A \overset{\mathrm{d}^0_A}{\to} I^1_A \overset{\mathrm{d}^1_A}{\to} \cdots \overset{\mathrm{d}^{p-1}_A}{\to} I^p_A \overset{\mathrm{d}^p_A}{\to} I^{p+1}_A \overset{\mathrm{d}^{p+1}_A}{\to} \cdots$$
$$0 \to C \to I^0_C \overset{\mathrm{d}^0_C}{\to} I^1_C \overset{\mathrm{d}^1_C}{\to} \cdots \overset{\mathrm{d}^{p-1}_C}{\to} I^p_C \overset{\mathrm{d}^p_C}{\to} I^{p+1}_C \overset{\mathrm{d}^{p+1}_C}{\to} \cdots$$

be injective resolutions of $A$ and $C$. Let $I^p_A \to I^p_A \oplus I^p_C$, $I^p_A \oplus I^p_C \to I^p_C$ be the natural homomorphisms. Then there exists an injective resolution of $B$ in the form

$$0 \to B \to I^0_A \oplus I^0_C \overset{\mathrm{d}^0_B}{\to} I^1_A \oplus I^1_C \overset{\mathrm{d}^1_B}{\to} I^1_A \oplus I^2_C \overset{\mathrm{d}^2_B}{\to} \cdots$$

which makes the following diagram

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & I^0_A \arrow[r, "\mathrm{d}^0_A"] \arrow[d] & I^1_A \arrow[r, "\mathrm{d}^1_A"] \arrow[d] & I^2_A \arrow[r, "\mathrm{d}^2_A"] \arrow[d] & \cdots \\
0 \arrow[r] & B \arrow[r] \arrow[d] & I^0_A \oplus I^0_C \arrow[r, "\mathrm{d}^0_B"] \arrow[d] & I^1_A \oplus I^1_C \arrow[r, "\mathrm{d}^1_B"] \arrow[d] & I^1_A \oplus I^2_C \arrow[r, "\mathrm{d}^2_B"] \arrow[d] & \cdots \\
0 \arrow[r] & C \arrow[r] & I^0_C \arrow[r, "\mathrm{d}^0_C"] & I^1_C \arrow[r, "\mathrm{d}^1_C"] & I^2_C \arrow[r, "\mathrm{d}^2_C"] & \cdots
\end{tikzcd}
$$

commutative (see [W, p.37, 2.2.8]).

Let $\mathrm{s}^p : I_C^p \to I_A^p \oplus I_C^p$ be the natural homomorphism. Then there exists a homomorphism $\bar{\delta} : \mathrm{Ker}(\mathrm{d}_C^p) \to \mathrm{Ker}(\mathrm{d}_A^{p+1})$ which makes the following diagram

$$\begin{tikzcd}
\mathrm{Ker}(\mathrm{d}_C^p) \arrow[r, "\bar{\delta}"] \arrow[d] & \mathrm{Ker}(\mathrm{d}_A^{p+1}) \arrow[r] & I_A^{p+1} \arrow[d] \\
I_C^p \arrow[r, "\mathrm{s}^p"] & I_A^p \oplus I_C^p \arrow[r, "\mathrm{d}_B^p"] & I_A^{p+1} \oplus I_C^{p+1}.
\end{tikzcd}$$

commutative. Since $G$ is left exact,

$$G(\mathrm{Ker}(\mathrm{d}_C^p)) \simeq \mathrm{Ker}(G(\mathrm{d}_C^p)), \qquad G(\mathrm{Ker}(\mathrm{d}_A^{p+1})) \simeq \mathrm{Ker}(G(\mathrm{d}_A^{p+1})).$$

Moreover $G(s^p)$ is also the natural homomorphism. So

(6.20)
$$\begin{tikzcd}
\mathrm{Ker}(G(\mathrm{d}_C^p)) \arrow[r, "G(\bar{\delta})"] \arrow[d] & \mathrm{Ker}(G(\mathrm{d}_A^{p+1})) \arrow[d] \\
R^p(G)(C) \arrow[r, "\delta"] & R^{p+1}(G)(A)
\end{tikzcd}$$

is a commutative diagram by the construction of $\delta$. Since $F$ is left exact also, the diagram

$$\begin{tikzcd}
\mathrm{Ker}(FG(\mathrm{d}_C^p)) \arrow[r, phantom, "\cong"] & F(\mathrm{Ker}(G(\mathrm{d}_C^p))) \arrow[r, "FG(\bar{\delta})"] \arrow[d] & F(\mathrm{Ker}(G(\mathrm{d}_A^{p+1}))) \arrow[r, phantom, "\cong"] \arrow[d] & \mathrm{Ker}(FG(\mathrm{d}_A^{p+1})) \\
& F(R^p(G)(C)) \arrow[r, "F(\delta)"] & F(R^{p+1}(G)(A)) &
\end{tikzcd}$$

is commutative. On the other hand, the diagram

$$\begin{tikzcd}
\mathrm{Ker}(FG(\mathrm{d}_C^p)) \arrow[r, "FG(\bar{\delta})"] \arrow[d] & \mathrm{Ker}(FG(\mathrm{d}_A^{p+1})) \arrow[d] \\
R^p(FG)(C) \arrow[r, "\delta"] & R^{p+1}(FG)(A)
\end{tikzcd}$$

is commutative as above. So the statement $(a)$ follows from the above consideration.

We next prove the statement $(b)$. Since the functor $H\colon \mathcal{D} \to \mathcal{C}$ takes injective objects to $G$-acyclic objects, the diagram

$$\begin{tikzcd}
\mathrm{Ker}(GH(\mathrm{d}^p)) \arrow[d] & G(\mathrm{Ker}(H(\mathrm{d}^p))) \arrow[l, "\simeq"'] \arrow[d] \\
R^p(GH)(A) \arrow[r, "l^p_{G,H}(A)"] & G(R^pH(A))
\end{tikzcd}$$

is commutative. By applying the functor $F\colon \mathcal{B} \to \mathcal{A}$,

$$\begin{tikzcd}
F(\mathrm{Ker}(GH(\mathrm{d}^p))) \arrow[d] & FG(\mathrm{Ker}(H(\mathrm{d}^p))) \arrow[l, "\simeq"'] \arrow[d] \\
F(R^p(GH)(A)) \arrow[r, "F(l^p_{G,H}(A))"] & FG(R^pH(A))
\end{tikzcd}$$

is commutative. Moreover, since $GH\colon \mathcal{D} \to \mathcal{B}$ takes injective objects to $F$-acyclic objects, the diagram

$$\begin{tikzcd} \mathrm{Ker}(FGH(\mathrm{d}^p)) \arrow[d] & F(\mathrm{Ker}(GH(\mathrm{d}^p))) \arrow[l, "\simeq"'] \arrow[d] \\ R^p(FGH)(A) \arrow[r, "l^p_{F,GH}(A)"] & F(R^pGH(A)) \end{tikzcd}$$

is commutative. So the diagram

$$\begin{tikzcd} \mathrm{Ker}(FGH(\mathrm{d}^p)) \arrow[d] & F(\mathrm{Ker}(GH(\mathrm{d}^p))) \arrow[l, "\simeq"'] & FG(\mathrm{Ker}(H(\mathrm{d}^p))) \arrow[l, "\simeq"'] \arrow[d] \\ R^p(FGH)(A) \arrow[rr, "F(l^p_{G,H}(A))\circ l^p_{F,GH}(A)"] & & FG(R^pH(A)) \end{tikzcd} \tag{6.21}$$

is commutative.

On the other hand, $H\colon \mathcal{D} \to \mathcal{C}$ takes injective objects to $FG$-acyclic objects, the diagram

$$\begin{tikzcd} \mathrm{Ker}(FGH(\mathrm{d}^p)) \arrow[d] & FG(\mathrm{Ker}(H(\mathrm{d}^p))) \arrow[l, "\simeq"'] \arrow[d] \\ R^p(FGH)(A) \arrow[r, "l^p_{FG,H}(A)"] & FG(R^pH(A)) \end{tikzcd} \tag{6.22}$$

is commutative. The statement (b) follows from the commutative diagrams (6.21) and (6.22). ☐

*Remark* 6.23. The functor $l^p_{F,G}(A)\colon\ R^p(FG)(A) \to F(R^pG(A))$ is characterized by the following properties.

(1) $l^0_{F,G}(A) = \mathrm{id}_{FG(A)}$.

(2) $l^p_{F,G}(A)$ satisfies the property of Lemma 6.19 (a).

**Proposition 6.24.** Let $X$ be a connected regular scheme and $g : \mathrm{Spec}\,(K) \to X$ the generic point. Let $F$ be the functor $\mathcal{F} \to \Gamma(X, \mathcal{F})$ and $G$ the direct image $g_*$. For $x \in X$, let $\bar{x}$ be the spectrum of the separable closure $k(x)_s$ of $k(x)$. Then we have

$$\mathrm{Ker}(l^p_{F,G}(\mathcal{F}')) = \mathrm{Ker}\left(\mathrm{H}^p(\mathrm{Spec}\,(K), \mathcal{F}') \overset{\mathrm{Res}}{\to} \prod_{x\in X} \mathrm{H}^p(\mathrm{Spec}\,(K_{\bar{x}}), \widetilde{\mathcal{F}'}_x)\right)$$

for any $p > 0$.

*Proof.* Let $u_x : \bar{x} \to x \hookrightarrow X$ be the composition of the canonical morphisms $\bar{x} \to x$ and $x \hookrightarrow X$, $\mathcal{F}' \in \mathbf{S}_{\mathrm{Spec}\,(K)_{et}}$, $i_x : \mathrm{Spec}\,(K_{\bar{x}}) \to \mathrm{Spec}\,(K)$ the canonical morphism and $\widetilde{\mathcal{F}}'_x = i_x^*\mathcal{F}'$. Then the composition of the homomorphisms

$$\begin{aligned} \mathrm{H}^p(\mathrm{Spec}\,(K), \mathcal{F}') \xrightarrow{l^p_{F,G}(\mathcal{F}')} \Gamma(X, R^pg_*(\mathcal{F}')) &\to \Gamma(X, u_{x*}{u_x}^*(R^pg_*(\mathcal{F}'))) \\ &= R^pg_*(\mathcal{F}')_{\bar{x}} \\ &\simeq \mathrm{H}^p(\mathrm{Spec}\,(K_{\bar{x}}), \widetilde{\mathcal{F}'}_x) \end{aligned} \tag{6.25}$$

and the restriction map Res : $\mathrm{H}^p(\mathrm{Spec}\,(K), \mathcal{F}') \to \mathrm{H}^p(\mathrm{Spec}\,(K_{\bar{x}}), \widetilde{\mathcal{F}'}_x)$ satisfy the following property which characterizes the homomorphism (see Remark 6.23).

(1) $\mathrm{H}^0(\mathrm{Spec}\,(K), \mathcal{F}') \to \mathrm{H}^0(\mathrm{Spec}\,(K_{\bar{x}}), \widetilde{\mathcal{F}'}_x)$ is the natural map.

(2) If $0 \to \mathcal{F}' \to \mathcal{F}'' \to \mathcal{F}''' \to 0$ is an exact sequence of $\mathbf{S}_{\mathrm{Spec}\,(K)_{et}}$, the diagram

$$\begin{tikzcd}
\mathrm{H}^p(\mathrm{Spec}\,(K), \mathcal{F}''') \arrow[r, "\delta"] \arrow[d] & \mathrm{H}^{p+1}(\mathrm{Spec}\,(K), \mathcal{F}') \arrow[d] \\
\mathrm{H}^p(\mathrm{Spec}\,(K_{\bar{x}}), \widetilde{\mathcal{F}'''}_x) \arrow[r, "\delta"'] & \mathrm{H}^{p+1}(\mathrm{Spec}\,(K_{\bar{x}}), \widetilde{\mathcal{F}'}_x)
\end{tikzcd}$$

is commutative.

So the homomorphism (6.25) coincides with the restriction map Res. Moreover $\mathcal{F} \to \prod_{x \in X} u_{x*}u_x{}^*(\mathcal{F})$ is injective for $\mathcal{F} \in \mathbf{S}_{X_{et}}$ (see [Me, p.90, Remark 1.20 (c)]). Therefore the proof is complete. ☐

**Lemma 6.26.** (see [W, p.150, The Grothendieck spectral sequence 5.8.3]) Under Assumption 6.1, $r^p_{F,G}(\mathcal{F}), l^p_{F,G}(\mathcal{F})$ coincide with edge maps which are induced by the Grothendieck spectral sequence $\mathrm{R}^p\, F(\mathrm{R}^q\, G(\mathcal{F})) \Rightarrow \mathrm{R}^{p+q}(FG)(\mathcal{F})$.

*Proof.* The proof of Lemma 6.26 relates to the construction of the Grothendieck spectral sequence. So we review the proof of the Grothendieck spectral sequence ([W, p.150, 5.8.3]). Let $C^{*,*}$ be a double complex in an abelian category, i.e., $C^{*,*}$ has homomorphisms $d_h^{p,q} : C^{p,q} \to C^{p+1,q}$ and $d_v^{p,q} : C^{p,q} \to C^{p,q+1}$ satisfying

$$d_h^{p+1,q} d_h^{p,q} = 0, \qquad d_v^{p,q+1} d_v^{p,q} = 0, \qquad d_h^{p,q+1} d_v^{p,q} + d_v^{p+1,q} d_v^{p,q} = 0.$$

Then we can define the differential $d_t^p$ of the total complex $(\mathrm{tot}(C^{*,*}))_p = \bigoplus_{i+j=p} C^{i,j}$ as the homomorphism satisfying the following commutative diagram

$$\begin{tikzcd}
C^{i,j} \arrow[r] \arrow[d, "d_h^{p,q} \oplus d_v^{p,q}"'] & (\mathrm{tot}(C^{*,*}))_p \arrow[d, "d_t^p"] \\
C^{i+1,j} \oplus C^{i,j+1} \arrow[r] & (\mathrm{tot}(C^{*,*}))_{p+1}.
\end{tikzcd}$$

Let $\mathrm{H}_v^q(C^{p,*}) = \mathrm{Ker}(d_v^{p,q}) / \mathrm{Im}(d_v^{p,q-1})$ and $\mathrm{H}_h^p(C^{*,q}) = \mathrm{Ker}(d_h^{p,q}) / \mathrm{Im}(d_h^{p-1,q})$.

Then $d_h^{p,q}, d_v^{p,q}$ induce homomorphisms

$$\bar{d}_h^{p,q} : \mathrm{H}_v^q(C^{p,*}) \to \mathrm{H}_v^q(C^{p+1,*}), \qquad \bar{d}_v^{p,q} : \mathrm{H}_h^p(C^{*,q}) \to \mathrm{H}_h^p(C^{*,q+1}).$$

Let

$$\mathrm{H}_h^p\,\mathrm{H}_v^q(C^{*,*}) = \mathrm{Ker}(\bar{d}_h^{p,q}) / \mathrm{Im}(\bar{d}_h^{p-1,q})$$
$$\mathrm{H}_v^q\,\mathrm{H}_h^p(C^{*,*}) = \mathrm{Ker}(\bar{d}_v^{p,q}) / \mathrm{Im}(\bar{d}_v^{p,q-1}).$$

Then there exist spectral sequences ([W, p.135, 5.5.1])

$$
\begin{aligned}
{}_{\mathrm{I}}\mathrm{E}_2^{p,q} &= \mathrm{H}_h^p\,\mathrm{H}_v^q(C^{*,*}) \Rightarrow \mathrm{H}^{p+q}(\mathrm{tot}(C^{*,*})),\\
{}_{\mathrm{II}}\mathrm{E}_2^{p,q} &= \mathrm{H}_v^p\,\mathrm{H}_h^q(C^{*,*}) \Rightarrow \mathrm{H}^{p+q}(\mathrm{tot}(C^{*,*})).
\end{aligned} \tag{6.27}
$$

Note that $\mathrm{H}_v^p\,\mathrm{H}_h^q(C^{*,*}) = \mathrm{Ker}(\bar{d}_v^{q,p})/\,\mathrm{Im}(\bar{d}_v^{q,p-1})$. Also note that ${}_{\mathrm{I}}\mathrm{E}_2^{p,q}$ is a subquotient of $C^{p,q}$ whereas ${}_{\mathrm{II}}\mathrm{E}_2^{p,q}$ is a subquotient of $C^{q,p}$. Here the edge map $\mathrm{H}^q(\mathrm{tot}(C^{*,*})) \to \mathrm{H}_h^0\,\mathrm{H}_v^q(C^{*,*})$ of ${}_{\mathrm{I}}\mathrm{E}_2^{p,q} = \mathrm{H}_h^p\,\mathrm{H}_v^q(C^{*,*}) \Rightarrow \mathrm{H}^{p+q}(\mathrm{tot}(C^{*,*}))$ is induced by the natural homomorphism

$$
\bigoplus_{i+j=q} C^{i,j} \to C^{0,q} \to C^{0,q}/\,\mathrm{Im}(d_v^{0,q-1})
$$

and the edge map $\mathrm{H}_h^p\,\mathrm{H}_v^0(C^{*,*}) \to \mathrm{H}^p(\mathrm{tot}(C^{*,*}))$ of ${}_{\mathrm{I}}\mathrm{E}_2^{p,q} = \mathrm{H}_h^p\,\mathrm{H}_v^q(C^{*,*}) \Rightarrow \mathrm{H}^{p+q}(\mathrm{tot}(C^{*,*}))$ is induced by the natural homomorphism

$$
\mathrm{Ker}(d_v^{p,0}) \to C^{p,0} \to \bigoplus_{i+j=p} C^{i,j}. \tag{6.28}
$$

The edge map $\mathrm{H}^p(\mathrm{tot}(C^{*,*})) \to \mathrm{H}_v^0\,\mathrm{H}_h^p(C^{*,*})$ of ${}_{\mathrm{II}}\mathrm{E}_2^{p,q} = \mathrm{H}_v^p\,\mathrm{H}_h^q(C^{*,*}) \Rightarrow \mathrm{H}^{p+q}(\mathrm{tot}(C^{*,*}))$ is induced by the natural homomorphism

$$
\bigoplus_{i+j=p} C^{i,j} \to C^{p,0} \to C^{p,0}/\,\mathrm{Im}(d_h^{p-1,0}) \tag{6.29}
$$

and the edge map $\mathrm{H}_v^p\,\mathrm{H}_h^0(C^{*,*}) \to \mathrm{H}^p(\mathrm{tot}(C^{*,*}))$ of ${}_{\mathrm{II}}\mathrm{E}_2^{p,q} = \mathrm{H}_v^p\,\mathrm{H}_h^q(C^{*,*}) \Rightarrow \mathrm{H}^{p+q}(\mathrm{tot}(C^{*,*}))$ is induced by the natural homomorphism

$$
\mathrm{Ker}(d_h^{0,p}) \to C^{0,p} \to \bigoplus_{i+j=p} C^{i,j}. \tag{6.30}
$$

Let $\mathcal{F}$ be an object of $\mathcal{B}$,

$$
0 \to \mathcal{F} \to I_{\mathcal{F}}^0 \xrightarrow{\mathrm{d}_{\mathcal{F}}^0} I_{\mathcal{F}}^1 \xrightarrow{\mathrm{d}_{\mathcal{F}}^1} I_{\mathcal{F}}^2 \xrightarrow{\mathrm{d}_{\mathcal{F}}^2} \cdots
$$

an injective resolution of $\mathcal{F}$ and $J_{\mathcal{F}}^{*,*}$ a Cartan-Eilenberg resolution of the complex $G(I_{\mathcal{F}}^*)$ (where

$$
0 \to G(I_{\mathcal{F}}^p) \to J_{\mathcal{F}}^{p,0} \to J_{\mathcal{F}}^{p,1} \to \cdots
$$

is an injective resolution of $G(I_{\mathcal{F}}^p)$), $d_{\mathcal{F},h}^{p,q}\colon J_{\mathcal{F}}^{p,q} \to J_{\mathcal{F}}^{p+1,q}$ the horizontal differential, $d_{\mathcal{F},v}^{p,q}\colon J_{\mathcal{F}}^{p,q} \to J_{\mathcal{F}}^{p,q+1}$ the vertical differential and $C_{\mathcal{F}}^{*,*} = F(J_{\mathcal{F}}^{*,*})$. Note that the vertical (resp. the horizontal) differential $d_{\mathcal{F},v}^{p,q}$ (resp. $d_{\mathcal{F},h}^{p,q}$) of $C_{\mathcal{F}}^{*,*}$ corresponds to $d_v^{p,q}$ (resp. $d_h^{p,q}$) of the above $C^{p,q}$.

We have two spectral sequences

$$
\begin{aligned}
{}_{\mathrm{I}}\mathrm{E}_2^{p,q} &= \mathrm{H}_h^p\,\mathrm{H}_v^q(C_{\mathcal{F}}^{*,*}) \Rightarrow \mathrm{H}^{p+q}(\mathrm{tot}(C_{\mathcal{F}}^{*,*})),\\
{}_{\mathrm{II}}\mathrm{E}_2^{p,q} &= \mathrm{H}_v^p\,\mathrm{H}_h^q(C_{\mathcal{F}}^{*,*}) \Rightarrow \mathrm{H}^{p+q}(\mathrm{tot}(C_{\mathcal{F}}^{*,*})).
\end{aligned}
$$

Since $J_{\mathcal{F}}^{*,*}$ is a Cartan-Eilenberg resolution, $\operatorname{Im}(d_{\mathcal{F},h}^{p-1,q})$, $\operatorname{Ker}(d_{\mathcal{F},h}^{p,q})$ and $\operatorname{H}(d_{\mathcal{F},h}^{p,q}) = \operatorname{Ker}(d_{\mathcal{F},h}^{p,q})/\operatorname{Im}(d_{\mathcal{F},h}^{p-1,q})$ are injective objects. So exact sequences

$$
\begin{aligned}
&0 \to \operatorname{Ker}(d_{\mathcal{F},h}^{p,q}) \to C_{\mathcal{F}}^{p,q} \to \operatorname{Im}(d_{\mathcal{F},h}^{p-1,q}) \to 0 \\
&0 \to \operatorname{Im}(d_{\mathcal{F},h}^{p-1,q}) \to \operatorname{Ker}(d_{\mathcal{F},h}^{p,q}) \to \operatorname{H}(d_{\mathcal{F},h}^{p,q}) \to 0
\end{aligned}
\tag{6.31}
$$

split and

$$
\operatorname{Im}(F(d_{\mathcal{F},h}^{p,q})) \simeq F(\operatorname{Im}(d_{\mathcal{F},h}^{p,q})) \qquad F(\operatorname{Ker}(d_{\mathcal{F},h}^{p,q}))/F(\operatorname{Im}(d_{\mathcal{F},h}^{p-1,q})) \simeq F(\operatorname{H}(d_{\mathcal{F},h}^{p,q})).
$$

Therefore

$$
\operatorname{Ker}(F(d_{\mathcal{F},h}^{p,q}))/\operatorname{Im}(F(d_{\mathcal{F},h}^{p-1,q})) \simeq F(\operatorname{H}(d_{\mathcal{F},h}^{p,q})).
\tag{6.32}
$$

It follows from the above isomorphism that the complex $\operatorname{H}_h^q(C_{\mathcal{F}}^{*,*}) = \operatorname{H}(F(d_{\mathcal{F},h}^{q,*}))$ is given by applying $F$ to an injective resolution of $R^qG(\mathcal{F})$. So ${}_{\mathrm{II}}\mathrm{E}_2^{p,q} \cong \mathrm{R}^p\, F(\mathrm{R}^q\, G(\mathcal{F}))$.

On the other hand, since $G(I_{\mathcal{F}}^*)$ is $F$-acyclic objects, the complex $C_{\mathcal{F}}^{p,*}$ is exact and $\operatorname{H}_v^q(C_{\mathcal{F}}^{*,*}) = 0$ if $q > 0$. So ${}_{\mathrm{I}}\mathrm{E}_2^{p,0} \cong R^p(FG)(\mathcal{F})$ and $R^p(FG)(\mathcal{F}) \simeq \operatorname{H}^p(\operatorname{tot}(C_{\mathcal{F}}^{*,*}))$ through the edge map $\operatorname{H}_h^p \operatorname{H}_v^0(C_{\mathcal{F}}^{*,*}) \to \operatorname{H}^p(\operatorname{tot}(C_{\mathcal{F}}^{*,*}))$ of the first spectral sequence of (6.27). So we obtain a spectral sequence $\mathrm{R}^p\, F(\mathrm{R}^q\, G(\mathcal{F})) \Rightarrow \mathrm{R}^{p+q}(FG)(\mathcal{F})$, which is called the Grothendieck spectral sequence. By using the natural homomorphisms (6.28) and (6.29) we see that the edge map $R^n(FG)(\mathcal{F}) \to F(R^nG(\mathcal{F}))$ is induced by the natural homomorphism

$$
\operatorname{Ker}(F(d_{\mathcal{F},v}^{n,0})) \to C_{\mathcal{F}}^{n,0}/\operatorname{Im}(F(d_{\mathcal{F},h}^{n-1,0})).
\tag{6.33}
$$

It satisfies the following commutative diagram
(6.34)

$$
\begin{tikzcd}
\operatorname{Ker}(F(d_{\mathcal{F},v}^{n,0})) \arrow[r, "(6.33)"] & C_{\mathcal{F}}^{n,0}/\operatorname{Im}(F(d_{\mathcal{F},h}^{n-1,0})) \\
\operatorname{Ker}(F(\bar{d}_{\mathcal{F},h}^{n,0})) = \operatorname{Ker}(F(d_{\mathcal{F},v}^{n,0})) \cap \operatorname{Ker}(F(d_{\mathcal{F},h}^{n,0})) \arrow[u] \arrow[d] \arrow[r] & \operatorname{Ker}(F(d_{\mathcal{F},h}^{n,0}))/\operatorname{Im}(F(d_{\mathcal{F},h}^{n-1,0})) \arrow[u, "\text{the inclusion map}"'] \\
\operatorname{Ker}(F(\bar{d}_{\mathcal{F},h}^{n,0}))/\operatorname{Im}(F(\bar{d}_{\mathcal{F},h}^{n-1,0})) \arrow[r, "\text{the edge map}"] & \operatorname{Ker}(F(\bar{d}_{\mathcal{F},v}^{n,0})) \arrow[u, "\text{the inclusion map}"'] \\
R^n(FG)(\mathcal{F}) \arrow[u, "\simeq"] \arrow[d, equal] & F(R^nG(\mathcal{F})) \arrow[u, "\simeq"] \arrow[d, equal] \\
\operatorname{Ker}(FG(\mathrm{d}_{\mathcal{F}}^n))/\operatorname{Im}(FG(\mathrm{d}_{\mathcal{F}}^{n-1})) & F(\operatorname{Ker}(G(\mathrm{d}_{\mathcal{F}}^n))/\operatorname{Im}(G(\mathrm{d}_{\mathcal{F}}^{n-1}))).
\end{tikzcd}
$$

Moreover, the following diagrams

$$
\begin{tikzcd}
\operatorname{Ker}(FG(\mathrm{d}_{\mathcal{F}}^n)) \arrow[r, "\simeq"] \arrow[d] & \operatorname{Ker}(F(d_{\mathcal{F},v}^{n,0})) \cap \operatorname{Ker}(F(d_{\mathcal{F},h}^{n,0})) \arrow[d] \\
\operatorname{Ker}(FG(\mathrm{d}_{\mathcal{F}}^n))/\operatorname{Im}(FG(\mathrm{d}_{\mathcal{F}}^{n-1})) \arrow[r] & \operatorname{Ker}(F(\bar{d}_{\mathcal{F},h}^{n,0}))/\operatorname{Im}(F(\bar{d}_{\mathcal{F},h}^{n-1,0})),
\end{tikzcd}
\tag{6.35}
$$

$$(6.36)\qquad \begin{tikzcd} \operatorname{Ker}(F(d^{n,0}_{\mathcal{F},v}))/\operatorname{Im}(F(d^{n-1,0}_{\mathcal{F},h})) \arrow[dr, "(6.32)", "\simeq"'] & \\ F(\operatorname{Ker}(G(\mathrm{d}^n_{\mathcal{F}}))/\operatorname{Im}(G(\mathrm{d}^{n-1}_{\mathcal{F}}))) \arrow[u] \arrow[r] & F(\mathrm{H}(d^{n,0}_{\mathcal{F},h})) \end{tikzcd}$$

are commutative.

Since the homomorphism $F(R^nG(\mathcal{F})) \to F(\mathrm{H}(d^{n,0}_{\mathcal{F},h}))$ is injective, to show that the edge map $R^n(FG)(\mathcal{F}) \to F(R^nG(\mathcal{F}))$ coincides with $l^n_{F,G}(\mathcal{F})$, it suffices to show that the composition

$$F(\operatorname{Ker}(G(\mathrm{d}^n_{\mathcal{F}}))) \to \operatorname{Ker}(FG(\mathrm{d}^n_{\mathcal{F}})) \to R^n(FG)(\mathcal{F}) \xrightarrow{\text{the edge map}} F(R^nG(\mathcal{F})) \to F(\mathrm{H}(d^{n,0}_{\mathcal{F},h}))$$

coincides with

$$F(\operatorname{Ker}(G(\mathrm{d}^n_{\mathcal{F}}))) \to \operatorname{Ker}(FG(\mathrm{d}^n_{\mathcal{F}})) \to R^n(FG)(\mathcal{F}) \xrightarrow{l^n_{F,G}(\mathcal{F})} F(R^nG(\mathcal{F})) \to F(\mathrm{H}(d^{n,0}_{\mathcal{F},h}))$$
$$\overset{\text{c.f,(6.18)}}{=} F(\operatorname{Ker}(G(\mathrm{d}^n_{\mathcal{F}}))) \xrightarrow{\text{the natural homomorphism}} F(\mathrm{H}(d^{n,0}_{\mathcal{F},h})).$$

This follows from the above diagrams (6.34), (6.35) and (6.36).

Therefore this edge map agrees with $l^n_{F,G}(\mathcal{F})$.

We next prove that the edge map $R^pF(G(\mathcal{F})) \to R^pFG(\mathcal{F})$ coincides with $r^p_{F,G}(\mathcal{F})$. For that purpose it is sufficient to prove that the edge map satisfies (2) of Property 6.2. We prove Lemma 6.5 (a) which does not assume that $\mathcal{F}_2$ is an injective object and implies Property 6.2 (2) (which assumes that $\mathcal{F}_2$ is an injective object). Suppose that

$$0 \to \mathcal{F}_1 \xrightarrow{f} \mathcal{F}_2 \xrightarrow{g} \mathcal{F}_3 \to 0$$

is an exact sequence ($\mathcal{F}_2$ is not always an injective object) and that

$$0 \to \mathcal{F}_1 \to I^0_{\mathcal{F}_1} \to I^1_{\mathcal{F}_1} \to \cdots, \qquad 0 \to \mathcal{F}_3 \to I^0_{\mathcal{F}_3} \to I^1_{\mathcal{F}_3} \to \cdots$$

are injective resolutions of $\mathcal{F}_1$ and $\mathcal{F}_3$. Then there exist an injective resolution of the form

$$0 \to \mathcal{F}_2 \to I^0_{\mathcal{F}_1} \oplus I^0_{\mathcal{F}_3} \to I^1_{\mathcal{F}_1} \oplus I^1_{\mathcal{F}_3} \to \cdots$$

and chain maps $f^*_1\colon I^*_{\mathcal{F}_1} \to I^*_{\mathcal{F}_1} \oplus I^*_{\mathcal{F}_3}$, $g^*_1\colon I^*_{\mathcal{F}_1} \oplus I^*_{\mathcal{F}_3} \to I^*_{\mathcal{F}_3}$ lifting $f$ and $g$. Then

$$0 \to I^*_{\mathcal{F}_1} \xrightarrow{f^*_1} I^*_{\mathcal{F}_1} \oplus I^*_{\mathcal{F}_3} \xrightarrow{g^*_1} I^*_{\mathcal{F}_3} \to 0$$

is an exact sequence by the Horseshoe Lemma [W, p.37, 2.2.8].

Moreover suppose that $G(I^p_{\mathcal{F}_1}) \to J^{p,*}_{\mathcal{F}_1}$, $G(I^p_{\mathcal{F}_3}) \to J^{p,*}_{\mathcal{F}_3}$ are Cartan-Eilenberg resolutions of $G(I^p_{\mathcal{F}_1})$ and $G(I^p_{\mathcal{F}_3})$. Then there exist a Cartan-Eilenberg resolution $G(I^p_{\mathcal{F}_1}) \oplus G(I^p_{\mathcal{F}_3}) \to J^{p,*}_{\mathcal{F}_2}$ and homomorphisms of double complexes $f^{p,q}_2\colon J^{p,q}_{\mathcal{F}_1} \to J^{p,q}_{\mathcal{F}_2}$, $g^{p,q}_2\colon J^{p,q}_{\mathcal{F}_2} \to J^{p,q}_{\mathcal{F}_3}$ lifting $f^p_1$ and $g^p_1$ for all $p, q$ such that

$$(6.37)\qquad 0 \to J^{p,q}_{\mathcal{F}_1} \xrightarrow{f^{p,q}_2} J^{p,q}_{\mathcal{F}_2} \xrightarrow{g^{p,q}_2} J^{p,q}_{\mathcal{F}_3} \to 0$$

is exact. Since $J^{p,q}_{\mathcal{F}_1}, J^{p,q}_{\mathcal{F}_2}$ are injective objects, it splits and there exists a section $s^{p,q}_g$: $J^{p,q}_{\mathcal{F}_3} \to J^{p,q}_{\mathcal{F}_2}$ of $g^{p,q}_2$.

Let $C^{p,q}_{\mathcal{F}_2} = F(J^{p,q}_{\mathcal{F}_2})$ and $C^{p,q}_{\mathcal{F}_3} = F(J^{p,q}_{\mathcal{F}_3})$ as above. Then the homomorphism $\bar{\delta}^{p,q}_t$

$$C^{p,q}_{\mathcal{F}_3} \xrightarrow{F(s^{p,q}_g)} C^{p,q}_{\mathcal{F}_2} \xrightarrow{F(d^{p,q}_{\mathcal{F}_2,h}) \oplus F(d^{p,q}_{\mathcal{F}_2,v})} C^{p+1,q}_{\mathcal{F}_2} \oplus C^{p,q+1}_{\mathcal{F}_2}$$

induces the connecting homomorphism $\delta : \mathrm{H}^n(\mathrm{tot}(\mathcal{F}_3)) \to \mathrm{H}^{n+1}(\mathrm{tot}(\mathcal{F}_1))$.

On the other hand, since $G(I^p_{\mathcal{F}_1}) \to J^{p,*}_{\mathcal{F}_1}$, $G(I^p_{\mathcal{F}_3}) \to J^{p,*}_{\mathcal{F}_3}$ are Cartan-Eilenberg resolutions of $G(I^*_{\mathcal{F}_1})$ and $G(I^*_{\mathcal{F}_3})$ and the exact sequence (6.37) splits, rows and columns of the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & G(\mathcal{F}_1) \arrow[r] \arrow[d] & \mathrm{Ker}(d^{0,0}_{\mathcal{F}_1,h}) \arrow[r] \arrow[d] & \mathrm{Ker}(d^{0,1}_{\mathcal{F}_1,h}) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & G(\mathcal{F}_2) \arrow[r] & \mathrm{Ker}(d^{0,0}_{\mathcal{F}_2,h}) \arrow[r] & \mathrm{Ker}(d^{0,1}_{\mathcal{F}_2,h}) \arrow[r] & \cdots
\end{tikzcd} \tag{6.38}$$

are exact. Let $f^p_3$: $\mathrm{Ker}(d^{0,p}_{\mathcal{F}_1,h}) \to \mathrm{Ker}(d^{0,p}_{\mathcal{F}_2,h})$ be the homomorphism which is induced by the homomorphism $f^{0,p}_2$: $I^{0,p}_{\mathcal{F}_1} \to I^{0,p}_{\mathcal{F}_2}$. Since $\mathrm{Ker}(d^{0,p}_{\mathcal{F}_1,h}), \mathrm{Ker}(d^{0,p}_{\mathcal{F}_2,h})$ are injective objects, the exact sequence

$$0 \to \mathrm{Ker}(d^{0,p}_{\mathcal{F}_{1,h}}) \xrightarrow{f^p_3} \mathrm{Ker}(d^{0,p}_{\mathcal{F}_{2,h}}) \to \mathrm{Coker}(f^p_3) \to 0 \tag{6.39}$$

splits and $\mathrm{Coker}(f^p_3)$ is an injective object. Then

$$0 \to G(\mathcal{F}_2)/G(\mathcal{F}_1) \to \mathrm{Coker}(f^0_3) \to \mathrm{Coker}(f^1_3) \to \cdots \tag{6.40}$$

which is induced by the commutative diagram (6.38) is an injective resolution of $G(\mathcal{F}_2)/G(\mathcal{F}_1)$. The reason is the following. Let

$$\begin{aligned}
A^0 &= \mathrm{Im}(\mathrm{Ker}(d^{0,0}_{\mathcal{F}_1,h}) \to \mathrm{Ker}(d^{0,1}_{\mathcal{F}_1,h})) = \mathrm{Coker}(G(\mathcal{F}_1) \to \mathrm{Ker}(d^{0,0}_{\mathcal{F}_1,h})), \\
B^0 &= \mathrm{Im}(\mathrm{Ker}(d^{0,0}_{\mathcal{F}_2,h}) \to \mathrm{Ker}(d^{0,1}_{\mathcal{F}_2,h})) = \mathrm{Coker}(G(\mathcal{F}_2) \to \mathrm{Ker}(d^{0,0}_{\mathcal{F}_2,h})), \\
C^0 &= \mathrm{Im}(\mathrm{Coker}(f^0_3) \to \mathrm{Coker}(f^1_3)).
\end{aligned}$$

Then the homomorphism $B^0 \to C^0$ is surjective. Moreover, by chasing the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & A^0 \arrow[r] \arrow[d] & \mathrm{Ker}(d^{0,1}_{\mathcal{F}_1,h}) \arrow[r] \arrow[d] & \mathrm{Ker}(d^{0,2}_{\mathcal{F}_1,h}) \arrow[d] \\
0 \arrow[r] & B^0 \arrow[r] \arrow[d] & \mathrm{Ker}(d^{0,1}_{\mathcal{F}_2,h}) \arrow[r] \arrow[d] & \mathrm{Ker}(d^{0,2}_{\mathcal{F}_2,h}) \arrow[d] \\
0 \arrow[r] & C^0 \arrow[r] & \mathrm{Coker}(f^1_3) \arrow[r] & \mathrm{Coker}(f^2_3)
\end{tikzcd} \tag{6.41}$$

whose rows and columns are exact except for the far left column and the bottom row, we see that the sequence

$$0 \to A^0 \to B^0 \to C^0$$

is exact. Therefore, it follows from the 3-3 lemma (see [W], p.11, Exercise 1.3.2) that the sequence

$$0 \to G(\mathcal{F}_2)/G(\mathcal{F}_1) \to \mathrm{Coker}(f_3^0) \to C^0 \to 0$$

is exact. So the sequence (6.40) is an injective resolution of $G(\mathcal{F}_2)/G(\mathcal{F}_1)$ by induction.

Since the exact sequence (6.39) splits, there exists a section $s_h^p\colon \mathrm{Coker}(f_3^p) \to \mathrm{Ker}(d_{\mathcal{F}_2,h}^{0,p})$ of the natural map $\mathrm{Ker}(d_{\mathcal{F}_2,h}^{0,p}) \to \mathrm{Coker}(f_3^p)$. Then we have the homomorphism $\bar{\delta}^p$ which satisfies the following diagram

$$\begin{tikzcd} F(\mathrm{Coker}(f_3^p)) \arrow[r, "\bar{\delta}^p"] \arrow[d, "F(s_h^p)"'] & F(\mathrm{Ker}(d_{\mathcal{F}_1,h}^{0,p+1})) \arrow[d] \\ F(\mathrm{Ker}(d_{\mathcal{F}_2,h}^{0,p})) \arrow[r, "F(\tilde{d}_{\mathcal{F}_2,v}^{0,p})"'] & F(\mathrm{Ker}(d_{\mathcal{F}_2,h}^{0,p+1})) \end{tikzcd}$$

where $\tilde{d}_{\mathcal{F}_2,v}^{0,p}$ is induced from $d_{\mathcal{F}_2,v}^{0,p}$. Moreover $\bar{\delta}^p$ induces the connecting homomorphism

$$R^pF(G(\mathcal{F}_2)/G(\mathcal{F}_1)) \to R^{p+1}F(G(\mathcal{F}_1)).$$

For all $\mathcal{F} \in \mathcal{C}$, we denote by $i_{\mathcal{F}}^p$ the natural inclusion map $F(\mathrm{Ker}(d_{\mathcal{F},h}^{0,p})) \hookrightarrow C_{\mathcal{F}}^{0,p}$. Let $j^p$ be the natural inclusion map $F(\mathrm{Coker}(f_3^p)) \hookrightarrow F(\mathrm{Ker}(d_{\mathcal{F}_3,h}^{0,p}))$. Then, since the diagram

$$\begin{tikzcd} F(\mathrm{Ker}(d_{\mathcal{F}_2,h}^{0,p})) \arrow[r] \arrow[d] & C_{\mathcal{F}_2}^{0,p} \arrow[d] \\ F(\mathrm{Coker}(f_3^p)) \arrow[r] & C_{\mathcal{F}_3}^{0,p} \end{tikzcd}$$

is commutative, the homomorphism

$$F(\mathrm{Coker}(f_3^p)) \xrightarrow{F(s_h^p)} F(\mathrm{Ker}(d_{\mathcal{F}_2,h}^{0,p})) \overset{i_{\mathcal{F}_2}^p}{\hookrightarrow} C_{\mathcal{F}_2}^{0,p} \xrightarrow{F(g_2^{0,p})} C_{\mathcal{F}_3}^{0,p} \tag{6.42}$$

coincides with the homomorphism

$$F(\mathrm{Coker}(f_3^p)) \xrightarrow{F(s_h^p)} F(\mathrm{Ker}(d_{\mathcal{F}_2,h}^{0,p})) \to F(\mathrm{Coker}(f_3^p)) \overset{j^p}{\hookrightarrow} F(\mathrm{Ker}(d_{\mathcal{F}_3,h}^{0,p})) \overset{i_{\mathcal{F}_3}^p}{\hookrightarrow} C_{\mathcal{F}_3}^{0,p}. \tag{6.43}$$

Also, since $F(s_h^p)$ is a section of the natural homomorphism

$$F(\mathrm{Ker}(d_{\mathcal{F}_2,h}^{0,p})) \to F(\mathrm{Coker}(f_3^p)),$$

the homomorphism (6.43) coincides with the homomorphism

$$F(\mathrm{Coker}(f_3^p)) \overset{j^p}{\hookrightarrow} F(\mathrm{Ker}(d_{\mathcal{F}_3,h}^{0,p})) \overset{i_{\mathcal{F}_3}^p}{\hookrightarrow} C_{\mathcal{F}_3}^{0,p}. \tag{6.44}$$

Moreover, since $F(s_g^{0,p})$ is a section of the homomorphism $F(g_2^{0,p})$, the homomorphism (6.44) coincides with the homomorphism

$$F(\mathrm{Coker}(f_3^p)) \xhookrightarrow{j^p} F(\mathrm{Ker}(d_{\mathcal{F}_3,h}^{0,p})) \xhookrightarrow{i_{\mathcal{F}_3}^p} C_{\mathcal{F}_3}^{0,p} \xrightarrow{F(s_g^{0,p})} C_{\mathcal{F}_2}^{0,p} \xrightarrow{F(g_2^{0,p})} C_{\mathcal{F}_3}^{0,p}. \tag{6.45}$$

So the homomorphism (6.42) coincides with (6.45). Then

$$\mathrm{Im}(F(s_g^{0,p}) \circ i_{\mathcal{F}_3}^p \circ j^p - i_{\mathcal{F}_2}^p \circ F(s_h^p)) \subset C_{\mathcal{F}_1}^{0,p}$$

by the above relation. Then

$$\begin{aligned}
&(F(d_{\mathcal{F}_2,v}^{0,p}) \circ F(s_g^{0,p})) \circ i_{\mathcal{F}_3}^p \circ j^p - i_{\mathcal{F}_1}^{p+1} \circ \bar{\delta}^p \\
=&F(d_{\mathcal{F}_2,v}^{0,p}) \circ (F(s_g^{0,p}) \circ i_{\mathcal{F}_3}^p \circ j^p - i_{\mathcal{F}_2}^p \circ F(s_h^p)) \\
=&F(d_{\mathcal{F}_1,v}^{0,p}) \circ (F(s_g^{0,p}) \circ i_{\mathcal{F}_3}^p \circ j^p - i_{\mathcal{F}_2}^p \circ F(s_h^p)).
\end{aligned}$$

Moreover since $\mathrm{Im}(i_{\mathcal{F}_2}^p \circ F(s_h^p)) \subset \mathrm{Ker}(F(d_{\mathcal{F}_2,h}^{0,p}))$,

$$\begin{aligned}
&(F(d_{\mathcal{F}_2,h}^{0,p}) \circ F(s_g^{0,p})) \circ i_{\mathcal{F}_3}^p \circ j^p \\
=&F(d_{\mathcal{F}_2,h}^{0,p}) \circ (F(s_g^{0,p}) \circ i_{\mathcal{F}_3}^p \circ j^p - i_{\mathcal{F}_2}^p \circ F(s_h^p)) \\
=&F(d_{\mathcal{F}_1,h}^{0,p}) \circ (F(s_g^{0,p}) \circ i_{\mathcal{F}_3}^p \circ j^p - i_{\mathcal{F}_2}^p \circ F(s_h^p)).
\end{aligned}$$

Let $i_{\mathcal{F}}^{p,q}\colon C_{\mathcal{F}}^{p,q} \hookrightarrow \bigoplus_{i+j=p+q} C_{\mathcal{F}}^{i,j}$ be the natural inclusion map. Then

$$\begin{aligned}
&\mathrm{Im}(\bar{\delta}_t^{0,p} \circ i_{\mathcal{F}_3}^p \circ j^p - i_{\mathcal{F}_1}^{0,p+1} \circ i_{\mathcal{F}_1}^{p+1} \circ \bar{\delta}^p) \\
\subset&\,\mathrm{Im}\left((F(d_{\mathcal{F}_1,h}^{0,p}) \oplus F(d_{\mathcal{F}_1,v}^{0,p}))\right) \\
=&\,\mathrm{Im}(F(d_{\mathcal{F}_1,t}^p) \circ i_{\mathcal{F}_1}^{0,p})
\end{aligned}$$

Moreover, the homomorphism

$$F(\mathrm{Ker}(d_{\mathcal{F}_1,h}^{0,p})) \xhookrightarrow{i_{\mathcal{F}_1}^p} C_{\mathcal{F}_1}^{0,p} \xhookrightarrow{i_{\mathcal{F}_1}^{0,p}} \bigoplus_{i+j=p} C_{\mathcal{F}_1}^{i,j}$$

which corresponds to the natural homomorphism (6.30) induces the edge map

$$\mathrm{H}_v^p\,\mathrm{H}_h^0(C_{\mathcal{F}_1}^{*,*}) \to \mathrm{H}^p(\mathrm{tot}(C_{\mathcal{F}_1}^{*,*})).$$

$\bar{\delta}_t^{0,p} \circ i_{\mathcal{F}_3}^p \circ j^p$ corresponds to the homomorphism

$$R^pF(G(\mathcal{F}_2/G(\mathcal{F}_1))) \to R^pF(G(\mathcal{F}_3)) \to R^p(FG)(\mathcal{F}_3) \to R^{p+1}(FG)(\mathcal{F}_1)$$

and $i_{\mathcal{F}_1}^{0,p+1} \circ i_{\mathcal{F}_1}^{p+1} \circ \bar{\delta}^p$ corresponds to the homomorphism

$$R^pF(G(\mathcal{F}_2/G(\mathcal{F}_1))) \to R^{p+1}F(G(\mathcal{F}_1)) \to R^{p+1}(FG)(\mathcal{F}_1).$$

This means that the edge map satisfies (2) of Property 6.2. $\square$

# 7 The proof of the main result

This section make reference to [Me, pp.106-107, III, Example 2.22]. Let $X$ be a regular integral quasi-compact scheme and $g : \mathrm{Spec}(K) \to X$ the generic point. For $v \in X^{(1)}$, let $i_v : v \hookrightarrow X$ be the canonical map and $\mathrm{D}_X = \bigoplus_{v \in X^{(1)}} i_{v*}(\mathbb{Z})$. Then, we have the exact sequence

$$0 \to \mathbb{G}_m \to g_*(\mathbb{G}_{m,K}) \to \mathrm{D}_X \to 0 \tag{7.1}$$

so that we have the long exact sequence of étale cohomology

$$\cdots \to \mathrm{H}^n(X, \mathbb{G}_m) \to \mathrm{H}^n(X, g_*(\mathbb{G}_{m,K})) \to \mathrm{H}^n(X, \mathrm{D}_X) \to \cdots . \tag{7.2}$$

Using the Leray spectral sequence

$$\mathrm{H}^p(X, \mathrm{R}^q i_{v*}(\mathbb{Z})) \Rightarrow \mathrm{H}^{p+q}(\mathrm{Spec}(\kappa(v)), \mathbb{Z})$$

for $i_v : v \hookrightarrow X$, we have the injective homomorphism $\mathrm{H}^1(X, i_{v*}(\mathbb{Z})) \to \mathrm{H}^1(\mathrm{Spec}(\kappa(v)), \mathbb{Z})$. On the other hand, since $\mathbb{Z}$ has no finite subgroups,

$$\mathrm{H}^1(\mathrm{Spec}(\kappa(v)), \mathbb{Z}) = \mathrm{Hom}_{\mathrm{conts}}(G_{\kappa(v)}, \mathbb{Z}) = 0$$

where $\mathrm{Hom}_{\mathrm{conts}}$ is the set of continuous homomorphisms. So $\mathrm{H}^1(X, \mathrm{D}_X) = 0$ and $R^1 i_{v*}(\mathbb{Z}) = 0$. Also, since $\mathbb{Q}$ is uniquely divisible, $\mathrm{H}^r(\kappa(v), \mathbb{Q}) = 0$ for any $r > 0$. So $\mathrm{H}^r(\kappa(v), \mathbb{Q}/\mathbb{Z}) = \mathrm{H}^{r-1}(\kappa(v), \mathbb{Z})$. Therefore, we have the exact sequence

$$0 \to \mathrm{H}^2(X, \mathbb{G}_m) \to \mathrm{H}^2(X, g_*(\mathbb{G}_{m,K})) \to \mathrm{H}^2(X, \mathrm{D}_X) \tag{7.3}$$

and

$$\mathrm{H}^2(X, \mathrm{D}_X) \hookrightarrow \mathrm{H}^2(\kappa(v), \mathbb{Z}) = \mathrm{H}^1(\kappa(v), \mathbb{Q}/\mathbb{Z}) = X(G(\kappa(v)_s/\kappa(v))).$$

Moreover, $\mathrm{H}^p(X, g_*(\mathbb{G}_{m,K}))$ can be expressed by a group cohomology in the following Lemmas 7.4 and 7.8.

**Lemma 7.4.** Let $X$ be a regular integral quasi-compact scheme, $K = R(X)$ and $g$: $\mathrm{Spec}\, K \to X$ the generic point of $X$. Then

$$\mathrm{H}^2(X, g_*(\mathbb{G}_{m,K})) = \mathrm{Ker}\left(\mathrm{Br}(K) \overset{\mathrm{Res}}{\to} \prod_{x \in X_{(0)}} \mathrm{Br}(K_{\bar{x}})\right).$$

*Proof.* In general, for a spectral sequence $E_2^{p,q} \Rightarrow H^{p+q}$, we have the exact sequence

$$E_2^{0,1} \to E_2^{2,0} \xrightarrow{\text{the edge map}} E_1^2 \to E_2^{1,1}$$

where $E_1^2 = \mathrm{Ker}(H^2 \to E_2^{0,2})$. Since

$$\mathrm{R}^1 g_*(\mathbb{G}_m) = 0 \tag{7.5}$$

by [Me, p.89, III, Remark 1.17 (a)] and Hilbert Theorem 90, considering (7.5) for the Leray spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(X, \mathrm{R}^q g_*(\mathbb{G}_m)) \Rightarrow \mathrm{H}^{p+q}(\mathrm{Spec}(K), \mathbb{G}_m),$$

we see that

$$\mathrm{H}^2(X, g_*(\mathbb{G}_{m,K})) = \mathrm{Ker}\left(\mathrm{H}^2(\mathrm{Spec}(K), \mathbb{G}_m) \to \mathrm{H}^0(X, \mathrm{R}^2 g_*(\mathbb{G}_m))\right).$$

Therefore, we have

$$\mathrm{H}^2(X, g_*(\mathbb{G}_{m,K})) = \mathrm{Ker}\left(\mathrm{Br}(K) \overset{\mathrm{Res}}{\to} \prod_{x\in X} \mathrm{Br}(K_{\bar{x}})\right) \tag{7.6}$$

by Proposition 6.24 and Lemma 6.26.

Moreover, it follows from the definition of the étale neighborhood of $\bar{x}$ ([Me, p.38, I, Remark 4.11]) and the fact that a flat morphism satisfies the going-down theorem [M, Theorem 9.5] that $K_{\bar{x}} \subset K_{\bar{y}}$ for $x \in \overline{\{y\}}$. Therefore, we can replace $x \in X$ by $x \in X_{(0)}$ in (7.6). So the proof is complete. □

*Remark* 7.7. Suppose that $A$ is a regular integral domain with $\dim(A) = 1$ which contains a field $k$ and $m$ is a positive integer with $(\mathrm{ch}(k), m) = 1$. For $x \in \mathrm{Spec}\,(A)$, $\mathrm{Br}(K_{\bar{x}})_m = 0$ (cf, [S1, p.111, Appendix, §2]). Therefore

$$\mathrm{Br}(K)_m \subset \mathrm{H}^2(X, g_*(\mathbb{G}_{m,K})).$$

Moreover, suppose that $k$ is a perfect field. Then

$$\mathrm{Br}(K) = \mathrm{H}^2(X, g_*(\mathbb{G}_{m,K}))$$

by Lang's theorem [S2, p.162, X, §7].

**Lemma 7.8.** Let $A$ be a Henselian discrete valuation ring, $K$ its quotient field , $k$ its residue field and $K_{nr}$ its maximal unramified extension. Then

$$\mathrm{H}^p(\mathrm{Spec}\,(A), g_*(\mathbb{G}_m)) = \mathrm{H}^p(K_{nr}/K, (K_{nr})^*)$$

for any $p > 0$.

*Proof.* Let $i$: $\mathrm{Spec}\,(k) \to \mathrm{Spec}\,(A)$ be the natural map. Then, $i_*$ is exact. Let $(set)$ be the class of all separated étale morphisms and $f$: $X_{et} \to X_{set}$ the continuous morphism which is induced by identity map on $X$. Then $f_*$ is exact by [Me, p.112, (b) of Examples 3.4]. Let $(fet)$ be the class of all finite étale morphisms and $f'$: $X_{set} \to X_{fet}$ the continuous morphism which is induced by identity map on $X$.

Let $Y \to X$ be a separated étale morphism with $Y$ connected, $R(Y)$ the ring of rational functions of $Y$, $A \to B$ the normalization of $A$ in $R(Y)$ and $X' = \mathrm{Spec}\,(B)$. Then $R(Y)/K$ is a finite separable extension and $Y$ is an open subscheme of $X'$ by [Me, p.29, I, Theorem 3.20]. Moreover $X' \to X$ is finite by [Me, p.4, I, Proposition 1.1]. Then, since $A$ is a Henselian discrete valuation ring, $B$ is a Henselian discrete valuation ring by [Me, p.33, I, (b) of Theorem 4.2] and [Me, p.34, I, Corollary 4.3]. Also $R(X')/R(X)$ is an unramified extension. Therefore $f'_*$ is exact by [Me, p.111, III, Proposition 3.3]. So $f'_* \circ f_*$ is exact and

$$\mathrm{H}^p_{fet}(X, (f' \circ f)_*(\mathcal{F})) \simeq \mathrm{H}^p_{et}(X, \mathcal{F})$$

for any $\mathcal{F} \in \mathbb{S}_{X_{et}}$.

We have the isomorphism $G_K$-**mod** $\simeq \mathbb{S}_{\mathrm{Spec}\,(K)_{et}}$ by [Me, p.53, II.§1,Theorem1.9]. Let the functor $N$ be defined as

$$(G_K\text{-}\mathbf{mod}) \ni M \longmapsto M^{\mathrm{Gal}(K_s/K_{nr})} \in (G_k\text{-}\mathbf{mod})$$

and $N' : \mathbb{S}_{\mathrm{Spec}\,(K)_{et}} \to \mathbb{S}_{\mathrm{Spec}\,(k)_{et}}$ the functor which corresponds to $N$. Let $Y'' \in X_{fet}$ be connected. Moreover, let $K'' = R(Y'')$ and $k''$ the finite extension field of $k$ which corresponds to the closed point of $Y''$. Then

$$N'(F)(\mathrm{Spec}\,(k'')) = F(\mathrm{Spec}\,(K''))$$

for $F \in \mathbb{S}_{\mathrm{Spec}\,(K)_{et}}$ because

$$G(K_{nr}/K'') \simeq G_{k''}, \quad G(K_{nr}/K'') \simeq G_{K''}/G_{K_{nr}}.$$

Therefore the diagram

$$\begin{tikzcd}
G_K\text{-}\mathbf{mod} \arrow[d, "N"'] & \simeq & \mathbb{S}_{\mathrm{Spec}\,(K)_{et}} \arrow[r, "f'_*\circ f_*\circ g_*"] \arrow[d, "N'"] & \mathbb{S}_{X_{fet}} \\
G_k\text{-}\mathbf{mod} & \simeq & \mathbb{S}_{\mathrm{Spec}\,(k)_{et}} \arrow[ur, "f'_*\circ f_*\circ i_*"'] &
\end{tikzcd}.$$

is commutative. So

$$\begin{aligned}
\mathrm{H}^p_{et}(X, g_*(\mathbb{G}_m)) &= \mathrm{H}^p_{fet}(X, f^{'} \circ f \circ g_*(\mathbb{G}_m)) \\
&= \mathrm{H}^p_{fet}(X, f^{'} \circ f \circ i_*(N'(\mathbb{G}_m))) \\
&= \mathrm{H}^p_{et}(X, i_*(N'(\mathbb{G}_m))) \\
&= \mathrm{H}^p_{et}(\mathrm{Spec}\,(k), N'(\mathbb{G}_m)) \\
&= \mathrm{H}^p(k, (K_{nr})^*) = \mathrm{H}^p(K_{nr}/K, (K_{nr})^*).
\end{aligned}$$

Hence the proof is complete. □

*Remark* 7.9. Let $A$ be a Henselian discrete valuation ring, $K$ its quotient field and $K_{nr}$ the maximal unramified extension of $K$ and $X = \mathrm{Spec}\,(A)$.

For any $p > 0$, the following diagram

$$\begin{tikzcd}
\mathrm{H}^p(X, g_*(\mathbb{G}_m)) \arrow[r] \arrow[d, "\text{the edge map}"'] & \mathrm{H}^p(X, i_*(\mathbb{Z})) \arrow[d, "\text{the edge map}"] \\
\mathrm{H}^p(k, (K_{nr})^*) \arrow[r] & \mathrm{H}^p(k, \mathbb{Z})
\end{tikzcd} \tag{7.10}$$

is commutative where two downward morphisms are the edge maps of spectral sequences. So it follows from Lemma 7.8 that the exact sequence (7.3) corresponds to the exact sequence

$$0 \to \mathrm{Br}(A) \overset{(\alpha)}{\to} \mathrm{Ker}\left(\mathrm{Br}(K) \overset{\mathrm{Res}}{\to} \mathrm{Br}(K_{nr})\right) \overset{(\beta)}{\to} \mathrm{H}^2(k, \mathbb{Z})$$

where $(\alpha)$ is the map which is induced by the natural map $\mathrm{Br}(A) \to \mathrm{Br}(K)$ and $(\beta)$ is the composition of the isomorphism $\mathrm{Ker}\left(\mathrm{Br}(K) \overset{\mathrm{Res}}{\to} \mathrm{Br}(K_{nr})\right) = \mathrm{H}^2(k, (K_{nr})^*)$ [S2, p.156, X, Corollary of Proposition 6] and the map $\mathrm{H}^p(k, (K_{nr})^*) \to \mathrm{H}^p(k, \mathbb{Z})$ which is induced by the discrete valuation of $A$.

**Corollary 7.11.** We consider the situation of Lemma 7.8. Let $\widehat{A}, \widehat{K}$ be the completions of $A, K$ and $k$ the residue field of $A$ and $\widehat{A}$. Then

$$\begin{aligned}\mathrm{H}^p(K_{nr}/K, (K_{nr})^*) &= \mathrm{H}^p(\widehat{K}_{nr}/\widehat{K}, (\widehat{K}_{nr})^*) \\ &= \mathrm{H}^p(k, (k_s)^*) \oplus \mathrm{H}^{p-1}(k, \mathbb{Q}/\mathbb{Z})\end{aligned}$$

for any $p > 0$.

*Proof.* We have

$$\mathrm{H}^p(\mathrm{Spec}\,(A), \mathbb{G}_m) = \mathrm{H}^p(\mathrm{Spec}\,(k), \mathbb{G}_m) = \mathrm{H}^p(\mathrm{Spec}\,(\widehat{A}), \mathbb{G}_m)$$

by [Me, p.116, III, Remark 3.11 (a)]. Corollary 7.11 follows from this fact and the fact that the exact sequence (7.2) splits ( because there is a section of the homomorphism

$$\mathrm{H}^p(k, (K_{nr})^*) \to \mathrm{H}^p(k, \mathbb{Z}) = \mathrm{H}^{p-1}(k, \mathbb{Q}/\mathbb{Z}),$$

cf, [S2, p.186, XII, §3, Proposition 4]). ☐

For a regular ring $A$ and $x \in \mathrm{Spec}\, A^{(1)}$, let

$$\mathrm{Br}(A) \to \mathrm{Ker}\left(\mathrm{Br}(K) \overset{\mathrm{Res}}{\to} \prod_{x \in X^{(1)}} \mathrm{Br}(K_{\bar{x}})\right) \tag{7.12}$$

be the homomorphism which is induced by the natural map and

$$\mathrm{Ker}\left(\mathrm{Br}(K) \overset{\mathrm{Res}}{\to} \prod_{x \in \mathrm{Spec}\,(A)^{(1)}} \mathrm{Br}(K_{\bar{x}})\right) \to \bigoplus_{x \in \mathrm{Spec}\,(A)^{(1)}} \mathrm{X}(G_{\kappa(x)}) \tag{7.13}$$

the homomorphism which satisfies the following commutative diagram

$$\begin{tikzcd}
\mathrm{Ker}\left(\mathrm{Br}(K) \overset{\mathrm{Res}}{\to} \prod_{x \in \mathrm{Spec}\,(A)^{(1)}} \mathrm{Br}(K_{\bar{x}})\right) \arrow[r, "(7.13)"] \arrow[d, "(*)"'] & \bigoplus_{x \in \mathrm{Spec}\,(A)^{(1)}} \mathrm{X}(G_{\kappa(x)}) \arrow[d] \\
\mathrm{Ker}\left(\mathrm{Br}(\widehat{K}_x) \overset{\mathrm{Res}}{\to} \mathrm{Br}(\widehat{K}_{\bar{x}})\right) \arrow[r, "(**)"'] & \mathrm{X}(G_{\kappa(x)})
\end{tikzcd}$$

where the homomorphism $(*)$ is induced by the natural map $\mathrm{Br}(K) \to \mathrm{Br}(\widehat{K}_x)$ and the homomorphism $(**)$ is induced by the valuation (see [S2, p.186, XII, Proposition 4]). Then, the following result follows from Lemmas 7.4, 7.8.

**Proposition 7.14.** Let $A$ be an integral domain of finite type over a field $k$ such that $A \otimes \bar{k}$ is regular, i.e., $\mathrm{Spec}\,(A)$ is smooth over $k$, $K$ the quotient field of $A$ and $X = \mathrm{Spec}\,(A)$. Then the sequence

$$0 \to \mathrm{Br}(A) \overset{(7.12)}{\to} \mathrm{Ker}\left( \mathrm{Br}(K) \overset{\mathrm{Res}}{\to} \prod_{x \in X^{(1)}} \mathrm{Br}(K_{\bar{x}}) \right) \overset{(7.13)}{\to} \prod_{x \in X^{(1)}} \mathrm{X}(G_{\kappa(x)}). \tag{7.15}$$

is exact. Moreover, if an integer $m$ is respectively prime to $\mathrm{ch}(k)$, there is an exact sequence

$$0 \to \mathrm{Br}(A)_m \to \mathrm{Br}(K)_m \to \prod_{x \in X^{(1)}} \mathrm{X}(G_{\kappa(x)})_m.$$

*Proof.* Suppose that $A$ is a 1 dimensional regular ring which contains a field $k$ ( $A$ does not have to be smooth). Then the exact sequence (7.15) is given by the exact sequence (7.3) as follows. Proposition 6.13 shows that the first map of (7.3) coincides with the homomorphism (7.12).

Now, for $x \in X^{(1)}$, $\widehat{X}_x$ denotes $\mathrm{Spec}\,\widehat{A}_x$. Then, let

$$i_x : \mathrm{Spec}\,(\kappa(x)) \to X, \qquad \widehat{i}_x : \mathrm{Spec}\,(\kappa(x)) \to \widehat{X}_x$$

be the canonical maps,

$$g : \mathrm{Spec}\,(K) \to X, \qquad \widehat{g}_x : \mathrm{Spec}\,(\widehat{K}_x) \to \widehat{X}_x$$

the generic point, and $j_x^K : \mathrm{Spec}\,(\widehat{K}_x) \to \mathrm{Spec}\,(K)$ ( resp. $j_x^A : \mathrm{Spec}\,(\widehat{A}_x) \to \mathrm{Spec}\,(A)$) the morphisms of schemes which correspond to the extension of field $\widehat{K}_x/K$ ( resp. the extension of ring $\widehat{A}_x/A$).

We have the following commutative diagram

(7.16)

$$\begin{tikzcd}
\mathrm{H}^2(X, g_*(\mathbb{G}_{m,K})) \arrow[r] \arrow[d, "\text{the edge map}"'] \arrow[dr, phantom, "(*)"] & \mathrm{H}^2(X, g_*((j_x^K)_*(\mathbb{G}_{m,\widehat{K}}))) \arrow[r, equal] \arrow[d, "\text{the edge map}"] & \mathrm{H}^2(X, (j_x^A)_*((\widehat{g}_x)_*(\mathbb{G}_{m,\widehat{K}}))) \arrow[d, "\text{the edge map}"] \\
\mathrm{H}^2(K, \mathbb{G}_{m,K}) \arrow[r] \arrow[dr, "(\gamma)"'] & \mathrm{H}^2(K, (j_x^K)_*(\mathbb{G}_{m,\widehat{K}})) \arrow[d, "\text{the edge map}"] \arrow[r, phantom, "(***)"] & \mathrm{H}^2(\widehat{X}_x, (\widehat{g}_x)_*(\mathbb{G}_{m,\widehat{K}})) \arrow[dl, "\text{the edge map}"] \\
 & \mathrm{H}^2(\widehat{K}, \mathbb{G}_{m,\widehat{K}}) &
\end{tikzcd}$$

$(**)$

where the vertical arrows of $(*)$ are induced by the natural map $\mathbb{G}_{m,K} \to (j_x^K)_*(\mathbb{G}_{m,\widehat{K}})$ and $(\gamma)$ is the natural map.

It follows from Proposition 6.10 that the diagram $(**)$ is commutative. The reason why the diagram $(***)$ is commutative is that the left side of $(***)$ is the edge map $\mathrm{H}^2(X, g_*((j_x^K)_*(\mathbb{G}_{m,\widehat{K}}))) \to \mathrm{H}^2(\widehat{K}, \mathbb{G}_{m,\widehat{K}})$ and the right side of $(***)$ is the edge map $\mathrm{H}^2(X, (j_x^A)_*((\widehat{g}_x)_*(\mathbb{G}_{m,\widehat{K}}))) \to \mathrm{H}^2(\widehat{K}, \mathbb{G}_{m,\widehat{K}})$ by Lemma 6.5 (b). Therefore the composition $\mathrm{H}^2(X, g_*(\mathbb{G}_{m,K})) \to \mathrm{H}^2(\widehat{X}_x, (\widehat{g}_x)_*(\mathbb{G}_{m,\widehat{K}}))$ in (7.16) corresponds to the homomorphism $(\gamma)$.

Also, the diagram

(7.17)
$$\begin{tikzcd}
\mathrm{H}^2(X, g_*(\mathbb{G}_{m,K})) \arrow[r, "\text{cf, (7.1)}"] \arrow[d] & \mathrm{H}^2(X, (i_x)_*(\mathbb{Z})) \arrow[dd, equal] \arrow[dddr, "\text{the edge map}"] & \\
\mathrm{H}^2(X, g_*((j_x^K)_*(\mathbb{G}_{m,\widehat{K}}))) \arrow[d, equal] & & \\
\mathrm{H}^2(X, (j_x^A)_*((\widehat{g}_x)_*(\mathbb{G}_{m,\widehat{K}}))) \arrow[r, "\text{cf, (7.1)}"'] \arrow[d, "\text{the edge map}"'] & \mathrm{H}^2(X, (j_x^A)_*((\widehat{i}_x)_*(\mathbb{Z}))) \arrow[d, "\text{the edge map}"] & \\
\mathrm{H}^2(\widehat{X}_x, (\widehat{g}_x)_*(\mathbb{G}_{m,\widehat{K}})) \arrow[r, "\text{cf, (7.1)}"'] & \mathrm{H}^2(\hat{X}_x, (\widehat{i}_x)_*(\mathbb{Z})) \arrow[r, "\text{the edge map}"'] & \mathrm{H}^2(\kappa(x), \mathbb{Z})
\end{tikzcd}$$

is commutative. So the homomorphism (7.13) coincides with the homomorphism $\mathrm{H}^2(X, g_*(\mathbb{G}_m)) \to \mathrm{H}^2(X, (i_x)_*(\mathbb{Z}))$ in the exact sequence (7.2) by commutative diagrams (7.10), (7.16), (7.17). In general, if $\operatorname{Spec}(A)$ is smooth over $k$ ($\operatorname{Spec} A$ is not necessarily of dimension 1), $\mathrm{Br}(A) = \bigcap_{\mathfrak{p} \in \operatorname{Spec}(A)^{(1)}} \mathrm{Br}(A_{\mathfrak{p}})$ by [H, Corollary.2]. Therefore (7.15) is an exact sequence. $\square$

*Remark* 7.18. Let $X$ be a regular integral quasi-compact scheme. Then the sequence

$$0 \to \mathrm{Br}(X) \to \operatorname{Ker}\left( \mathrm{Br}(R(X)) \overset{\mathrm{Res}}{\to} \prod_{x \in X_{(0)}} \mathrm{Br}(R(\mathcal{O}_{X,\bar{x}})) \right) \to \bigoplus_{x \in X^{(1)}} \mathrm{X}(G_{\kappa(x)}).$$

is exact by the proof of Proposition 7.14.

**Corollary 7.19.** Let $X$ be a 1-dimensional connected regular scheme, $K$ its quotient field. Then

(7.20)
$$0 \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(K) \longrightarrow \prod_{\mathfrak{p} \in X^{(1)}} \mathrm{Br}(\widetilde{R(X)_{\mathfrak{p}}}) / \mathrm{Br}(\widetilde{\mathcal{O}}_{X,\mathfrak{p}})$$

is exact.

*Proof.* Let $\mathcal{O}$ be a discrete valuation ring, $K = R(\mathcal{O})$, $X = \operatorname{Spec}(\mathcal{O})$ and $x$ the closed point of $X$. Then we have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{Br}(\mathcal{O}) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Ker}(\mathrm{Br}(K) \to \mathrm{Br}(K_{\bar{x}})) \arrow[r] \arrow[d] & \mathrm{Ker}(\mathrm{Br}(\widetilde{K}) \to \mathrm{Br}(K_{\bar{x}}))/\,\mathrm{Br}(\widetilde{\mathcal{O}}) \arrow[d] \\
0 \arrow[r] & \mathrm{Br}(\mathcal{O}) \arrow[r] \arrow[d] & \mathrm{Br}(K) \arrow[r] \arrow[d] & \mathrm{Br}(\widetilde{K})/\,\mathrm{Br}(\widetilde{\mathcal{O}}) \arrow[d] \\
 & 0 \arrow[r] & \mathrm{Br}(K_{\bar{x}}) \arrow[r] & \mathrm{Br}(K_{\bar{x}})
\end{tikzcd}$$

where the vertical sequences are exact and the horizontal ones are exact except for the middle horizontal sequence. So the middle horizontal sequence is exact. □

*An alternative proof of Corollary 7.19.* Suppose that $B$ is a discrete valuation ring, $L$ is its quotient field, $Y = \operatorname{Spec} B$ and $Z = Y \setminus \operatorname{Spec} L = \{\mathfrak{p}\}$. Then we have the exact sequence

$$\mathrm{H}^p(Y, \mathbb{G}_m) \to \mathrm{H}^p(\operatorname{Spec} L, \mathbb{G}_m) \to \mathrm{H}^{p+1}_Z(Y, \mathbb{G}_m) \tag{7.21}$$

by [Me, p.92, III, Proposition 1.25] and $\mathrm{H}^2(Y, \mathbb{G}_m) \to \mathrm{H}^2(\operatorname{Spec} L, \mathbb{G}_m)$ is injective by [Me, p.145, IV, §2]. Moreover we have

$$\mathrm{H}^p_Z(Y, \mathbb{G}_m) \simeq \mathrm{H}^p_{\{\mathfrak{p}\}}(\operatorname{Spec}(\widetilde{\mathcal{O}}_{Y,\mathfrak{p}}), \mathbb{G}_m) \tag{7.22}$$

by [Me, p.93, III, Corollary 1.28]. Also, the diagram

$$\begin{tikzcd}
\mathrm{Br}(K)/\,\mathrm{Br}(\mathcal{O}_{X,\mathfrak{p}}) \arrow[r] \arrow[d] & \mathrm{Br}(\widetilde{R(X)_{\mathfrak{p}}})/\,\mathrm{Br}(\widetilde{\mathcal{O}}_{\mathfrak{p}}) \arrow[d] \\
\mathrm{H}^3_{\{\mathfrak{p}\}}(\operatorname{Spec}(\mathcal{O}_{X,\mathfrak{p}}), \mathbb{G}_m) \arrow[r, "\simeq", "{cf, (7.22)}"'] & \mathrm{H}^3_{\{\mathfrak{p}\}}(\operatorname{Spec}(\widetilde{\mathcal{O}}_{X,\mathfrak{p}}), \mathbb{G}_m)
\end{tikzcd}$$

is commutative. Therefore

$$\mathrm{Br}(K)/\,\mathrm{Br}(\mathcal{O}_{X,\mathfrak{p}}) \to \mathrm{Br}(\widetilde{R(X)_{\mathfrak{p}}})/\,\mathrm{Br}(\widetilde{\mathcal{O}}_{X,\mathfrak{p}})$$

is injective. So the statement follows from [G, p.77, II, Proposition 2.3]. □

Moreover, we obtain the following result.

**Corollary 7.23.** Let $X$ be an algebraic curve over a separably closed field such that regular and proper. Then, the local-global map

$$\mathrm{Br}(R(X)) \to \prod_{\mathfrak{p} \in X^{(1)}} \mathrm{Br}(\widehat{R(X)_{\mathfrak{p}}})$$

is injective.

*Proof.* The statement follows from Corollary 7.19 and [G, III, Corollary 5.8]. □

We use the following lemmas in the proof of the main result.

**Lemma 7.24.** Let $A, B, C$ be commutative groups and $c$ a group homomorphism from $C$ into $A \oplus B$. Let $a, b$ be the projections from $A \oplus B$ to $A, B$. Moreover, let $i$ be an inclusion map from $\mathrm{Ker}(a \circ c)$ into $C$. Then

$$\mathrm{Ker}(c) = \mathrm{Ker}(b \circ c \circ i).$$

**Lemma 7.25.** Let $A, B, C$ be commutative groups and $f : A \to B$ be a group homomorphism. We assume that $\mathrm{Ker}(f) \subset C \subset A$. Let $i : C \to A$ be the inclusion map. Then

$$\mathrm{Ker}(f) = \mathrm{Ker}(f \circ i).$$

We also need the following propositions.

**Proposition 7.26.** (See [YS] or [Me, pp.153-154, IV, Exercise 2.20 (d)]) Let $\widetilde{K}$ be the quotient field of the Henselization of $k[t]_{(t)}$. Then, there is an exact sequence

$$0 \to \mathrm{Br}(k[t]) \to \mathrm{Br}(\widetilde{K}) \to \mathrm{X}(G_k) \to 0$$

where the first map is induced by $t \mapsto t^{-1}$ .

We now prove the main result.

**Theorem 7.27.** For any field $k$, let $k(t)$ be the purely transcendental extension field in one variable $t$ over $k$. Then, the local-global map

$$\mathrm{Br}(k(t)) \to \prod_{\mathfrak{p} \in \mathbf{P}_k^{1(1)}} \mathrm{Br}(\widehat{k(t)}_{\mathfrak{p}}) \tag{7.28}$$

is injective.

*Proof.* It is known that a finitely generated ring over an excellent ring and its localization are excellent ([EGA, §7.8]). Also, a discrete valuation ring is excellent if and only if its Henselization is excellent (see [Me, I, Remark 1.2]). So, it follows from Proposition 3.9 that the kernel of the local-global map

$$\mathrm{Br}(k(t)) \to \prod_{\mathfrak{p} \in \mathbf{P}_k^{1(1)}} \mathrm{Br}(\widetilde{k(t)}_{\mathfrak{p}}) \tag{7.29}$$

is equal to the kernel of the local-global map (7.28). Since we have the restriction map $\mathrm{Br}(\widetilde{k(t)}_{\mathfrak{p}}) \to \mathrm{Br}(k(t)_{\bar{\mathfrak{p}}})$, it follows from Lemma 7.25 that the kernel of the local-global map (7.29) is equal to the kernel of the local-global map (7.29) restricted to

$$\mathrm{Ker}\left(\mathrm{Br}(k(t)) \overset{\mathrm{Res}}{\to} \prod_{\mathfrak{p} \in \mathrm{Spec}\,(k[t])^{(1)}} \mathrm{Br}(k(t)_{\bar{\mathfrak{p}}})\right).$$

For the rest of this section, we denote the point which corresponds to $(\frac{1}{t}) \in \mathrm{Spec}\,(k[\frac{1}{t}]) \subset \mathbf{P}^1_k$ by $\infty$. Note that

$$\left(\prod_{\mathfrak{p}\in \mathrm{Spec}\,(k[t])^{(1)}} \mathrm{Ker}\left(\mathrm{Br}(\widetilde{k(t)}_{\mathfrak{p}}) \stackrel{\mathrm{Res}}{\to} \mathrm{Br}(k(t)_{\bar{\mathfrak{p}}})\right)\right) \oplus \mathrm{Br}(\widetilde{k(t)}_{\infty})$$

contains the image of the composition of the inclusion map

$$\mathrm{Ker}\left(\mathrm{Br}(k(t)) \stackrel{\mathrm{Res}}{\to} \prod_{\mathfrak{p}\in \mathrm{Spec}\,(k[t])^{(1)}} \mathrm{Br}(k(t)_{\bar{\mathfrak{p}}})\right) \to \mathrm{Br}(k(t))$$

and the local-global map (7.29).

In Lemma 7.24, let

$$C = \mathrm{Ker}\left(\mathrm{Br}(k(t)) \stackrel{\mathrm{Res}}{\to} \prod_{\mathfrak{p}\in \mathrm{Spec}\,(k[t])^{(1)}} \mathrm{Br}(k(t)_{\bar{\mathfrak{p}}})\right) \qquad B = \mathrm{Br}(\widetilde{k(t)}_{\infty})$$

$$A = \prod_{\mathfrak{p}\in \mathrm{Spec}\,(k[t])^{(1)}} \mathrm{Ker}\left(\mathrm{Br}(\widetilde{k(t)}_{\mathfrak{p}}) \stackrel{\mathrm{Res}}{\to} \mathrm{Br}(k(t)_{\bar{\mathfrak{p}}})\right)$$

and $c$ the map which is induced by the local-global map (7.29) restricted to

$$\mathrm{Ker}\left(\mathrm{Br}(k(t)) \stackrel{\mathrm{Res}}{\to} \prod_{\mathfrak{p}\in \mathrm{Spec}\,(k[t])^{(1)}} \mathrm{Br}(k(t)_{\bar{\mathfrak{p}}})\right).$$

Then, by Corollary 7.11 and [S2, X, Corollary of Proposition 6] (which follows from the inflation restriction sequence),

$$\begin{aligned}
&\prod_{\mathfrak{p}\in \mathrm{Spec}\,(k[t])^{(1)}} \mathrm{Ker}\left(\mathrm{Br}(\widetilde{k(t)}_{\mathfrak{p}}) \stackrel{\mathrm{Res}}{\to} \mathrm{Br}(k(t)_{\bar{\mathfrak{p}}})\right)\\
&= \prod_{\mathfrak{p}\in \mathrm{Spec}\,(k[t])^{(1)}} \mathrm{H}^2(k(t)_{\bar{\mathfrak{p}}}/\widetilde{k(t)}_{\mathfrak{p}}, (k(t)_{\bar{\mathfrak{p}}})^*)\\
&= \prod_{\mathfrak{p}\in \mathrm{Spec}\,(k[t])^{(1)}} \mathrm{Br}(\kappa(\mathfrak{p})) \oplus \prod_{\mathfrak{p}\in \mathrm{Spec}\,(k[t])^{(1)}} X(G_{\kappa(\mathfrak{p})}).
\end{aligned}$$

Note that $k(t)_{\bar{\mathfrak{p}}}$ is the maximal unramified extension of $\widetilde{k(t)}_{\mathfrak{p}}$. Therefore by Proposition 7.31, $\mathrm{Ker}(a \circ c) \subset \mathrm{Br}(k[t])$. Moreover, by Proposition 7.26, $b \circ c \circ i$ is injective. Therefore, Theorem 7.27 follows from Lemma 7.24. □

*Remark* 7.30. If $k$ is perfect, it is well-known fact that the sequence

(7.31) $$0 \to \mathrm{Br}(\mathbb{P}^1_k) \to \mathrm{Br}(k(x)) \to \bigoplus_{\mathfrak{p}\in \mathbb{P}^{1(1)}_k} X(G_{\kappa(\mathfrak{p})})$$

is exact. But it is unknown fact whether (7.31) is exact. The sequence (7.20) is exact in Corollary 7.19, but the sequence (7.31) is not exact in the case where $k$ is not perfect as follows.

It is known that $k$ is perfect if and only if $\mathrm{Br}(k) = \mathrm{Br}(k[x])$ (cf, [A-G, p.389, Theorem 7.5]). So $\mathrm{Br}(k[x]) \neq 0$ in the case where $k$ is the separable closure of an imperfect field and $\mathrm{Br}(k(x)) \neq 0$ because $\mathrm{Br}(k[x]) \subset \mathrm{Br}(k(x))$. On the other hand, $X(G_{\kappa(\mathfrak{p})}) = \{1\}$ and $\mathrm{Br}(\mathbb{P}^1_k) = \mathrm{Br}(k) = \{0\}$. So the sequence (7.31) is not exact.

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