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Standard and Non-standard Analysis in Second Order Arithmetic

by

Keita Yokoyama

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Abstract

This research is motivated by the program of Reverse Mathematics. We study real and complex analysis in second order arithmetic. To develop analysis in second order arithmetic, we adopt techniques of non-standard analysis, which enables us to simplify many proofs.

In Chapters 3 and 4, we introduce some suitable notions for differential and integral calculus and develop basic real and complex analysis within RCA_0. Then, we present some results on Reverse Mathematics for real and complex analysis such as the following: the inverse function theorem and Taylor's theorem for holomorphic functions are proved in RCA_0, L^2-convergence of Fourier series and Cauchy's integral theorem is equivalent to WKL_0 over RCA_0.

In Chapter 5, we introduce some model-theoretic arguments of non-standard analysis for WKL_0 and ACA_0 using Tanaka's self-embedding theorem for a model of WKL_0 and Gaifman's conservative extension for a model of PA. Then, applying these techniques, we show that the Jordan curve theorem is equivalent to WKL_0 over RCA_0 and that the Riemann mapping theorem is equivalent to ACA_0 over WKL_0.

In Chapter 6, we introduce systems of non-standard second order arithmetic $\mathsf{ns\text{-}ACA}_0$ and $\mathsf{ns\text{-}WKL}_0$ and formalize the non-standard arguments introduced in Chapter 5. Then, we obtain some effective methods to transform non-standard proofs in $\mathsf{ns\text{-}ACA}_0$ or $\mathsf{ns\text{-}WKL}_0$ into standard proofs in ACA_0 or WKL_0 and show that $\mathsf{ns\text{-}ACA}_0$ and $\mathsf{ns\text{-}WKL}_0$ are conservative extensions of ACA_0 and WKL_0 respectively without using a model-theoretic method.

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I also thank Professor Takeshi Yamazaki. He posed important questions and contributed essential ideas and comments during my development. He has tirelessly helped me professionally and privately throughout my period of study. I remain certain that I could not have completed this thesis without his thoughtful assistance.

Furthermore, I am grateful to Professor Sakae Fuchino for useful discussions and comments. He posed the important question of whether non-standard proofs in second order arithmetic can be transformed into standard proofs directly. This question inspired me to undertake the investigation described in Chapter 6.

I also thank Professor Mitsutaka Murayama, who guided me through my study of basic mathematics when I studied at the Tokyo Institute of Technology. I also extend my gratitude to Professor Ryo Kashima, who encouraged me to study at Tohoku University, and to Professor Theodore A. Slaman for useful comments.

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Contents

1 Introduction

This thesis is a contribution to the foundations of mathematics. The main objectives of this thesis are mathematics in second order arithmetic, especially arguments of non-standard analysis for second order arithmetic and Reverse Mathematics.

Most or all ordinary mathematics can be developed within a system of axiomatic set theory such as ZFC. On the other hand, some parts of basic analysis or linear algebra can be developed using elementary methods such as computable methods. For example, it is difficult to find a fixed point of a continuous function from the unit disk to itself (Brouwer's fixed point theorem). However, if a function is a contraction mapping, one can find a fixed point computably (actually, $0, f(0), f(f(0)), \dots$ converges to a fixed point of a contraction mapping f). Then, *which parts of ordinary mathematics can be formalized in weak systems?* (In this setting, we are especially concerned with the core of ordinary mathematics such as calculus, real and complex analysis, abstract algebra, and geometry, which are learned by undergraduates.) Motivated by this question, we develop some parts of ordinary mathematics (mainly analysis) within some sufficiently weak subsystems of second order arithmetic (introduced in the following section), which consist of axioms to treat computable sets and other plain sets.[1] This development is the first subject of this thesis. The systems we adopt have few axioms to treat infinite sets. For that reason, we encounter many difficulties in developing analysis. Consequently, we require some suitable methods to treat limit, differentiation, integration, and so on. We adopt techniques of non-standard analysis to develop mathematics more richly in these systems.

When a theory of ordinary mathematics is provable by weak axioms, then *are these axioms exactly necessary to prove it?* For example, the fixed point theorem for contraction mappings requires less axioms than Brouwer's theorem.[2] The second subject of this thesis is to answer the previous question. Friedman[11] revealed the following theme: very often, if a theorem τ of ordinary mathematics is proved from the "right" axioms, then τ is equivalent to those axioms over some weaker system in which itself is not provable. This theme is known as *Reverse Mathematics*. Following this theme, we determine which axioms are necessary to prove some theorems of real and complex analysis in second order arithmetic.[3]

[1] There are many other studies to formalize ordinary mathematics in weak systems from various standpoints such as recursive mathematics[23], constructive mathematics[4] and so on.

[2] See Shioji/Tanaka[27].

[3] Aside from the viewpoint of second order arithmetic, Constructive Reverse Mathematics (see,

1.1 Second order arithmetic and Reverse Mathematics

The formal system Z_2 of second order arithmetic is a system in which one can deal with natural numbers and sets of natural numbers. Its language is a two-sorted language, *i.e.*, there are two distinct sorts of variables which are intended to range over two different kinds of objects, natural numbers and sets of natural numbers. Its axioms consist of basic axioms of arithmetic such as ordered semiring, an induction axiom and a comprehension axiom which expresses that, for any second order formula $\varphi(n)$, there exists a set of all n such that $\varphi(n)$ holds. Second order arithmetic is adequate to develop the core of ordinary mathematics. Actually, most or all concepts of 'classical' mathematics can be developed within second order arithmetic.

Friedman pointed out that the study of subsystems of second order arithmetic is necessary and important to answer the theme of Reverse Mathematics. Actually, we can determine the right axioms for many theorems of ordinary mathematics and classify theorems by the strength of axioms they require in second order arithmetic. For this reason, we study ordinary mathematics in subsystems of second order arithmetic. Reverse Mathematics in second order arithmetic is carried forward by Friedman, Simpson, Tanaka, and others. Many theorems of ordinary mathematics are provable within a subsystem of second order arithmetic RCA_0, or equivalent over RCA_0 to one of the following subsystems: WKL_0, ACA_0, ATR_0, and Π^1_1-CA_0. We usually consider these five subsystems of second order arithmetic when we study Reverse Mathematics.

RCA_0 is a system of *recursive comprehension* that guarantees the existence of recursively definable sets. This system is the weakest system that we will consider; it is the basis of Reverse Mathematics. However, it is sufficiently strong to prove some basic theorems of continuous functions, algebra, and so on. For example, the mean value theorem and fixed point theorem for contraction mapping is provable in RCA_0 [14, 27].

WKL_0 consists of RCA_0 and a particular set existence axiom called *weak König's lemma*, which asserts that every infinite tree of sequences of 0's and 1's has an infinite path. Although the first order part of WKL_0 is the same as that of RCA_0, many important theorems, such as the Heine-Borel theorem, Brouwer's fixed point theorem and the uniform continuity of continuous functions on the closed unit interval cannot

e.g. [18]) is also known.

be proved in RCA_0 but can be proved in WKL_0. Since the consistency strength of WKL_0 is equivalent to that of PRA (*Primitive Recursive Arithmetic*), mathematics in WKL_0 present important implications for the foundations of mathematics, especially in relation to Hilbert's program [29, page 382].

The system ACA_0 consists of RCA_0 and *arithmetical comprehension axiom*, which guarantees the existence of arithmetically definable sets. The first order part of ACA_0 is just PA (*Peano Arithmetic*). ACA_0 is stronger than WKL_0 in the sense of consistency, and it proves many theorems related to convergence, e.g., the convergence of a bounded monotone real sequence.

ATR_0 is a system of *arithmetical transfinite recursion*, which says that arithmetical comprehension can be iterated along any countable well ordering. Π^1_1-CA_0 is a system of Π^1_1 *comprehension*, which guarantees the existence of Π^1_1 definable sets. Both ATR_0 and Π^1_1-CA_0 present numerous mathematical consequences in the realms of algebra, analysis, classical descriptive set theory, and countable combinatorics.

For this thesis, we mainly consider RCA_0, WKL_0, ACA_0 and another system, WWKL_0, which is introduced by Simpson/Yu[39]. For mathematics in ATR_0 and Π^1_1-CA_0, see [29, Chapters V, VI].

1.2 Non-standard analysis and second order arithmetic

Non-standard analysis is a noteworthy application of model theory to mathematics. It is a method to handle infinitely large and small numbers and develop analysis as follows. First, fix a model V including $\mathbb{N}$ and $\mathbb{R}$ to carry out mathematics, which is called the standard model. Then, construct an elementary extension ${}^*V \succ V$.[4] In ${}^*V$, one can find infinitely large natural numbers and infinitely large or small real numbers from the standpoint of V. Then, one can carry out mathematics with infinitely small and large numbers. By elementarity, consequences in ${}^*V$ return to V.

Non-standard analysis was initiated by Robinson[24] and carried forward by many people. It provides a suitable framework for the development of differential and integral calculus using intuitive definitions of limits, derivatives and so on using infinitely small and large numbers. Using non-standard analysis, one can substantially simplify many proofs of analysis.

As described earlier, we encounter some difficulties in dealing with various 'in-

[4]Sometimes, an elementary extension with some saturation is needed.

finities' such as limit, continuity and integration when we study analysis in weak systems of second order arithmetic. Consequently, within these systems, it would be extremely convenient to handle 'infinities' if techniques of non-standard analysis were completely available and 'infinities' were expressed by 'hyper finite concepts', *i.e.*, as expressed by finite concepts with infinitely small or large numbers. Unfortunately, second order arithmetic is not sufficiently strong to use full techniques of non-standard analysis. Actually, Henson, Kaufmann and Keisler[15] and Henson and Keisler[16] showed that some systems of non-standard analysis deduce some properties which cannot be proved in Z_2. Therefore, some restriction exists in using arguments of non-standard analysis in subsystems of second order arithmetic. Non-standard analysis is based on constructions of a non-standard model. For that reason, some model theoretic consideration is needed to use techniques of non-standard analysis in second order arithmetic.

In [34], Tanaka introduced some arguments of restricted non-standard analysis for a system WKL_0 using self-embedding theorem for countable non-standard models of WKL_0. On the other hand, many constructions of non-standard models of arithmetic are known. In this thesis, we introduce non-standard arguments for ACA_0 using Gaifman's model constructions. Then, we apply these arguments to some theorems of standard analysis.

According to the model theoretic arguments presented above, we can show that some theorems are provable in second order arithmetic. However, we cannot find the steps of proofs in which some essential axioms are needed. In fact, we need a precise formal proof for Reverse Mathematics. We seek to reconstruct a formal proof of second order arithmetic from a model theoretic non-standard proof. For that reason, we formalize non-standard arguments and transform non-standard proofs into formal proofs of second order arithmetic.

1.3 Outline of this thesis

A main topic of this thesis is non-standard arguments for second order arithmetic and its applications to Reverse Mathematics, which we argue in Chapters 5 and 6. The other is Reverse Mathematics for real and complex analysis related to differentiability and integrability, as described in Chapters 3 and 4. Arguments presented in Chapters 3 and 4 are used to prove the Riemann mapping theorem and the Jordan curve theorem as applications of non-standard arguments in Chapter 5, but the other parts of Chapters 5 and 6 are independent from Chapters 3 and 4.

Chapter 2 is devoted to define the systems RCA_0, WWKL_0, WKL_0, and ACA_0. We introduce the real number system in RCA_0. In addition, we study the strength of compactness and convergence for Euclidean space within RCA_0.

In Chapters 3 and 4, we develop basic real and complex analysis within RCA_0 and show some results for Reverse Mathematics. Analysis in second order arithmetic has been developed well, e.g. in [29]. However, some difficulty remains in dealing with derivatives within RCA_0 because we might not construct the derivative f' in RCA_0 even if f is continuously differentiable (see Theorem 3.8). We introduce a suitable definition and a useful expression of derivatives for RCA_0. Then, we prove some basic theorems such as the inverse function theorem in RCA_0 and present some results on Reverse Mathematics for Fourier expansions. For complex analysis, we show some results for Reverse Mathematics, which are mainly related to Cauchy's integral theorem presented in Chapter 4.

In Chapter 5, we introduce some non-standard arguments for WKL_0 and ACA_0. We also prove the Riemann mapping theorem in ACA_0 and the Jordan curve theorem in WKL_0. Using Tanaka's self-embedding theorem for a model of WKL_0, some proofs of non-standard analysis are available within WKL_0 [33, 32]. Applying this, we show that the Jordan curve theorem is provable in WKL_0. We also introduce non-standard arguments for ACA_0 by Gaifman's conservative extension, by which non-standard analysis for sequential compactness is available in ACA_0. Subsequently, we apply them for the Riemann mapping theorem within ACA_0. We eventually show that the Riemann mapping theorem is provable in ACA_0; moreover, we show that the Riemann mapping theorem is equivalent to ACA_0 over WKL_0.

In Chapter 6, we seek some effective methods to convert non-standard proofs introduced in Chapter 5 into proofs in ACA_0 or WKL_0. Professor Sakae Fuchino inspired this research. We introduce systems $\mathsf{ns\text{-}ACA}_0$ and $\mathsf{ns\text{-}WKL}_0$, corresponding to ACA_0 and WKL_0, following the non-standard arithmetic introduced by Keisler [21]. In these systems, we can formalize non-standard arguments introduced in Chapter 5 such as proofs of the Jordan curve theorem and the Riemann mapping theorem. As stated earlier, although we can find that $\mathsf{ns\text{-}ACA}_0$ and $\mathsf{ns\text{-}WKL}_0$ are conservative extensions of ACA_0 and WKL_0 respectively by model theoretic considerations, we seek to obtain sharper and more effective conservation results without model theoretic considerations. For this reason, we interpret $\mathsf{ns\text{-}ACA}_0$ and $\mathsf{ns\text{-}WKL}_0$ within ACA_0 and WKL_0 and show the conservativities as corollaries, as with the formalization of

Harrington's conservation theorem by Avigad[2]. Thereby, we can find methods to transform non-standard proofs into proofs in WKL_0 or ACA_0.

The works presented in Chapters 4 and 5 appeared respectively in [37] and [38, 25]. Other works in this thesis have been presented at several workshops and as preprints.

2 Preliminaries

In this chapter, we first define four subsystems of second order arithmetic RCA_0, WWKL_0, WKL_0 and ACA_0. Then we introduce the real number system and Euclidean space in RCA_0 and study the strength of compactness and convergence in Euclidean space within RCA_0.

2.1 Subsystems of second order arithmetic

The language $\mathcal{L}_2$ of second order arithmetic is a two-sorted language with number variables $x, y, z, \ldots$ and set variables $X, Y, Z, \ldots$. Numerical terms are built up from numerical variables and constant symbols $0, 1$ by means of binary operations $+$ and $\cdot$. Atomic formulas are $s = t$, $s < t$ and $s \in X$, where s and t are numerical terms. *Bounded* (Σ^0_0 or Π^0_0) formulas are constructed from atomic formulas by propositional connectives and bounded numerical quantifiers $(\forall x < t)$ and $(\exists x < t)$, where t does not contain x. A Σ^0_n formula is of the form $\exists x_1 \forall x_2 \ldots x_n \theta$ with θ bounded, and a Π^0_n formula is of the form $\forall x_1 \exists x_2 \ldots x_n \theta$ with θ bounded. All the Σ^0_n and Π^0_n formulas are the *arithmetical* (Σ^1_0 or Π^1_0) formulas. A Σ^1_n formula is of the form $\exists X_1 \forall X_2 \ldots X_n \varphi$ with φ arithmetical, and a Π^1_n formula is of the form $\forall X_1 \exists X_2 \ldots X_n \varphi$ with φ arithmetical.

The semantics of $\mathcal{L}_2$ are given by the following definition.

Definition 2.1. An $\mathcal{L}_2$-*structure* is an ordered 7-tuple

$$(M, S, +_M, \cdot_M, 0_M, 1_M, <_M),$$

where M is a set which serves as the range of the number variables, S is a set of subsets of M serving as the range of set variables, $+_M$ and $\cdot_M$ are binary operations on M, 0_M and 1_M are distinguished elements of M, and $<_M$ is a binary relation on M. We always assume that the sets M and S are disjoint and nonempty. The structure $(M, S, +_M, \cdot_M, 0_M, 1_M, <_M)$ is simply denoted by (M, S). Formulas of $\mathcal{L}_2$ are interpreted in (M, S) in the obvious way.

We also write M for an $\mathcal{L}_1$-structure $(M, +_M, \cdot_M, 0_M, 1_M, <_M)$. If M is the set (or structure) of standard natural numbers ω, an $\mathcal{L}_2$-structure (M, S) is called an ω-*structure* or an ω-*model*.

We first define RCA_0.

Definition 2.2. The system of RCA_0 consists of

(1) the discrete ordered semiring axioms for $(\omega, +, \cdot, 0, 1, <)$,

(2) Δ^0_1-CA (RCA):

$$\forall x(\varphi(x) \leftrightarrow \psi(x)) \to \exists X \forall x(x \in X \leftrightarrow \varphi(x)),$$

where $\varphi(x)$ is Σ^0_1, $\psi(x)$ is Π^0_1, and X does not occur freely in $\varphi(x)$,

(3) Σ^0_1 induction scheme:

$$\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x \varphi(x),$$

where $\varphi(x)$ is a Σ^0_1 formula.

The acronym RCA stands for recursive comprehension axiom. Roughly speaking, the set existence axioms of RCA_0 are strong enough to prove the existence of recursive sets.

If X and Y are set variables, we use $X \subseteq Y$ and $X = Y$ as abbreviations for the formulas $\forall n(n \in X \to n \in Y)$ and $\forall n(n \in X \leftrightarrow n \in Y)$. We define $\mathbb{N}$ to be the unique set X such that $\forall n(n \in X)$.

Within RCA_0, we define a *pairing map* $(m, n) = (m+n)^2 + m$. We can prove within RCA_0 that for any m, n, i, j in $\mathbb{N}$, $(m, n) = (i, j)$ if and only if $m = i$ and $n = j$. Moreover, using Δ^0_1-CA, we can prove that for any X and Y, there exists a set $X \times Y \subseteq \mathbb{N}$ such that

$$\forall n(n \in X \times Y \leftrightarrow \exists x \leq n \exists y \leq n(x \in X \wedge y \in Y \wedge (x, y) = n)).$$

We can encode a finite sequence of natural numbers in RCA_0 using the method by Shoenfield [28, page 115]. We define $\mathbb{N}^{<\mathbb{N}}$ to be the set of (codes for) finite sequences of elements of $\mathbb{N}$. A *sequence of sets of natural numbers* is defined to be a set $X \subseteq \mathbb{N} \times \mathbb{N}$. By Δ^0_1 comprehension, we define X_k as $m \in X_k \leftrightarrow (k, m) \in X$ and write $X = \{X_k\}_{k \in \mathbb{N}}$ or $X = \langle X_k \mid k \in \mathbb{N} \rangle$. Let $\{A_k\}_{k<n}$ be a sequence of sets. Then, by Δ^0_1 comprehension, we define a direct product $\prod_{k<n} A_k$ as

$$\prod_{k<n} A_k = \{\sigma \in \mathbb{N}^{<\mathbb{N}} \mid \mathrm{lh}(\sigma) = n \wedge \forall i < n \ \sigma(i) \in A_i\}$$

where $\mathrm{lh}(\sigma)$ denotes the length of σ. We write A^n for $\prod_{k<n} A$. A set X is said to be finite if there exists a monotone increasing finite sequence σ such that $\forall x(x \in X \leftrightarrow \exists i \ \sigma(i) = x)$. If a finite set X is denoted by σ, define $|X|$ (the *cardinality*

of X) as $|X| = \mathrm{lh}(\sigma)$. For X and Y, a *function* $f : X \to Y$ is defined to be a set $f \subseteq X \times Y$ such that $\forall x \forall y_0 \forall y_1((x, y_0) \in f \land (x, y_1) \in f \to y_0 = y_1)$ and $\forall x \in X \exists y \in Y (x, y) \in f$. We write $f(x) = y$ for $(x, y) \in f$.

Within $\mathsf{RCA_0}$, the universe of functions is closed under composition, primitive recursion (i.e., given $f : X \to Y$ and $g : \mathbb{N} \times X \times Y \to Y$, there exists a unique $h : \mathbb{N} \times X \to Y$ defined by $h(0, m) = f(m)$, $h(n+1, m) = g(n, m, h(n, m))$ and the least number operator (i.e., given $f : \mathbb{N} \times X \to \mathbb{N}$ such that for any $m \in X$ there exists $n \in \mathbb{N}$ such that $f(n, m) = 1$, there exists a unique $g : X \to \mathbb{N}$ defined by $g(m) =$the least n such that $f(n, m) = 1$). Especially, if (M, S) is an ω-model of $\mathsf{RCA_0}$, then (M, S) contains all recursive functions on ω.

Theorem 2.1. *The following is provable in* $\mathsf{RCA_0}$. *If* $\varphi(x, y)$ *is* Σ^0_1 *and* $\forall n \exists m \varphi(n, m)$ *holds, then there exists a function from* $\mathbb{N}$ *to* $\mathbb{N}$ *such that* $\forall n \varphi(n, f(n))$ *holds.*

Proof. We reason within $\mathsf{RCA_0}$. Write

$$\varphi(x, y) \equiv \exists z \theta(x, y, z)$$

where θ is Σ^0_0. By Δ^0_1 comprehension, we define *projection functions* p_1 and p_2 as follows: $p_i((n_1, n_2)) = n_i$ for all $n_1, n_2 \in \mathbb{N}$. Again using Δ^0_1 comprehension, there exists a function g from $\mathbb{N}^2$ to $\mathbb{N}$ such that

$$\theta(n, p_1(m), p_2(m)) \leftrightarrow g(n, m) = 1.$$

Then $\forall n \exists m g(n, m) = 1$, hence by the least number operator there exists a function h from $\mathbb{N}$ to $\mathbb{N}$ such that $g(n, h(n)) = 1$. Define a function f as $f(n) = p_1(g(n, h(n)))$, then $\forall n \varphi(n, f(n))$ holds. This completes the proof. □

Next, we define $\mathsf{WKL_0}$. Within $\mathsf{RCA_0}$, we define $2^{<\mathbb{N}}$ to be the set of (codes for) finite sequences of 0's and 1's. A set $T \subseteq 2^{<\mathbb{N}}$ is said to be a *tree* (or precisely 0-1 *tree*) if any initial segment of a sequence in T is also in T. A *path* through T is a function $f : \mathbb{N} \to \{0, 1\}$ such that for each n, the sequence $f[n] = \langle f(0), f(1), \dots, f(n-1) \rangle$ belongs to T.

Definition 2.3. $\mathsf{WKL_0}$ is the system which consists of $\mathsf{RCA_0}$ plus *weak König's lemma*: every infinite 0-1 tree T has a path.

In particular, ω-models of $\mathsf{WKL_0}$ are known as Scott systems and extensively studied by e.g. Kaye [20]. The first-order part of $\mathsf{WKL_0}$ is the same as that of $\mathsf{RCA_0}$.

Furthermore, WKL_0 is conservative over Primitive Recursive Arithmetic (PRA) with respect to Π^0_2 sentences. On the other hand, WKL_0 is strong enough to prove many theorems of ordinary mathematics, for example, Heine-Borel covering theorem, maximum principle for continuous functions on [0,1], Brouwer's fixed point theorem and so on.

We next introduce a weaker version of weak König's lemma called *weak weak König's lemma*: if a 0-1 tree T has no path, then

$$\lim_{n\to\infty} \frac{|\{\sigma \in T \mid \mathrm{lh}(\sigma) = n\}|}{2^n} = 0.$$

Definition 2.4. WWKL_0 is the system which consists of RCA_0 plus weak weak König's lemma.

WWKL_0 is introduced by Simpson/Yu[39]. WWKL_0 is properly weaker than WKL_0 and properly stronger than RCA_0. WWKL_0 permits a theory of Lebesgue measurability.

Finally, we define ACA_0.

Definition 2.5. ACA_0 is the system which consists of RCA_0 plus ACA (arithmetical comprehension axioms) :

$$\exists X \forall n (n \in X \leftrightarrow \varphi(n)),$$

where $\varphi(x)$ is arithmetical and X does not occur freely in $\varphi(x)$.

ACA_0 permits a smooth theory of sequential convergence. For any sentence σ of the language of Peano Arithmetic (PA), σ is a theorem of ACA_0 if and only if σ is a theorem of PA. ACA_0 is finitely axiomatizable although PA is not. The following theorem will be useful in showing that ACA is needed in order to prove various theorems of ordinary mathematics.

Theorem 2.2 ([29] Theorem III.1.3)**.** *The following assertions are pairwise equivalent over* RCA_0.

1. *For any one-to-one function f from $\mathbb{N}$ to $\mathbb{N}$, there exists a set $X \subseteq \mathbb{N}$ such that X is the range of f.*

2. Σ^0_1-CA: $\exists X \forall n (n \in X \leftrightarrow \varphi(n))$ *restricted to Σ^0_1 formulas $\varphi(x)$ in which X does not occur freely in $\varphi(x)$.*

3. ACA_0.

For details of the definitions of these four subsystems, see [29].

2.2 Real number system and Euclidean space

Next, we construct the real number system. We first define $\mathbb{Z}$ and $\mathbb{Q}$. Define an equivalence relation $=_{\mathbb{Z}}$ on $\mathbb{N}^2$ as $(m,n) =_{\mathbb{Z}} (p,q) \leftrightarrow m+q = n+p$, and by Δ^0_1 comprehension, define $\mathbb{Z}$, a set of integers, as $(m,n) \in \mathbb{Z} \leftrightarrow \forall k < (m,n)\ (p_1(k), p_2(k)) \neq_{\mathbb{Z}} (m,n)$, *i.e.*, $\mathbb{Z}$ is a set of least number elements of equivalence classes of $=_{\mathbb{Z}}$. We define $+_{\mathbb{Z}} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ as $(l_1, l_2) +_{\mathbb{Z}} (m_1, m_2) = (n_1, n_2)$ where (n_1, n_2) is a unique element in $\mathbb{Z}$ which satisfies $(l_1+m_1, l_2+m_2) =_{\mathbb{Z}} (n_1, n_2)$. We define $\cdot_{\mathbb{Z}}$ similarly. We can also define $|\cdot|_{\mathbb{Z}}$ and $\leq_{\mathbb{Z}}$ (norm and order in $\mathbb{Z}$) naturally. Similarly, we define a relation $=_{\mathbb{Q}}$ on $\mathbb{Z}\times\mathbb{Z}^+$ ($\mathbb{Z}^+$ is a set of positive integers) as $(m,n) =_{\mathbb{Q}} (p,q) \leftrightarrow m \cdot_{\mathbb{Z}} q =_{\mathbb{Z}} n \cdot_{\mathbb{Z}} p$, and define $\mathbb{Q}$ as $(m,n) \in \mathbb{Q} \leftrightarrow \forall k < (m,n)\ (p_1(k), p_2(k)) \neq_{\mathbb{Q}} (m,n)$. We also define $+_{\mathbb{Q}}$, $\cdot_{\mathbb{Q}}$, $|\cdot|_{\mathbb{Q}}$ and $\leq_{\mathbb{Q}}$ as in $\mathbb{Z}$, and then the system $(\mathbb{Q}; 0, 1, +_{\mathbb{Q}}, \cdot_{\mathbb{Q}}; \leq_{\mathbb{Q}})$ is an ordered field.

Definition 2.6 (Real number system)**.** The following definitions are made in RCA_0. A *real number* is an infinite sequence of rational numbers $\alpha = \{q_n\}_{n\in\mathbb{N}}$ (*i.e.* a function from $\mathbb{N}$ to $\mathbb{Q}$) which satisfies $|q_k - q_l|_{\mathbb{Q}} \leq_{\mathbb{Q}} 2^{-k}$ for all $l \geq k$. Here, each q_n is said to be n-th approximation of α. Define $\{p_n\}_{n\in\mathbb{N}} =_{\mathbb{R}} \{q_n\}_{n\in\mathbb{N}}$ as $\forall k\ |p_k - q_k|_{\mathbb{Q}} \leq_{\mathbb{Q}} 2^{-k+1}$. We can also define $+_{\mathbb{R}}$, $\cdot_{\mathbb{R}}$, $|\cdot|_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$ naturally. We usually write $\alpha \in \mathbb{R}$ if α is a real number. For details of the definition of the real number system, see [29, Chapter II].

We usually omit the subscript $_{\mathbb{Z}}$, $_{\mathbb{Q}}$ or $_{\mathbb{R}}$.

Imitating the definition of $\mathbb{R}$, we define Euclidean space $\mathbb{R}^n$. We define the addition and the scalar multiplication naturally, and see $\mathbb{Q}^n$ as a (countable) vector space. We also define $\|\cdot\|_{\mathbb{Q}^n}$ as

$$\|\mathbf{q}\|_{\mathbb{Q}^n} = \sqrt{{q_1}^2 + \cdots + {q_n}^2}$$

where $\mathbf{q} = (q_1, \ldots, q_n)$.

Definition 2.7 (Euclidean space)**.** The following definitions are made in RCA_0. An element of $\mathbb{R}^n$ is an infinite sequence of elements of $\mathbb{Q}^n$ $\mathbf{a} = \{\mathbf{q}_k\}_{k\in\mathbb{N}}$ which satisfies $\|\mathbf{q}_k - \mathbf{q}_l\| \leq 2^{-k}$ for all $l \geq k$. Then, each $a_i = \{q_{ki}\}_{k\in\mathbb{N}}$ is a real number. (Here, $\mathbf{q}_k = (q_{k1}, \ldots, q_{kn})$.) We define $\|\cdot\|_{\mathbb{R}^n}$, the norm of $\mathbb{R}^n$ as the following:

$$\|\mathbf{a}\|_{\mathbb{R}^n} = \sqrt{{a_1}^2 + \cdots + {a_n}^2}.$$

Here, of course the real number field $\mathbb{R}$ is the 1-dimensional Euclidean space $\mathbb{R}^1$.

Remark 2.3. In this thesis, to avoid too many subscripts, we use the intuitive expression such as $\mathbf{q} = (q_1, \ldots, q_n)$ even if the dimension of Euclidean space n may be nonstandard.

Let $X = \{X_k\}_{k\in\mathbb{N}}$ be a sequence of sets. If $X_k = \mathbf{a}_k \in \mathbb{R}^n$, *i.e.*, each X_k is formed an element of $\mathbb{R}^n$, then $X = \{\mathbf{a}_k\}_{k\in\mathbb{N}}$ is said to be a *sequence of points of* $\mathbb{R}^n$. We say that a sequence $\{\mathbf{a}_k\}_{k\in\mathbb{N}}$ converges to $\mathbf{b}$, written $\mathbf{b} = \lim_{k\to\infty} \mathbf{a}_k$, if

$$\forall m\ \exists k\ \forall i\ \|\mathbf{b} - \mathbf{a}_{k+i}\| < 2^{-m}.$$

Note that $\mathbf{b} = \lim_{k\to\infty} \mathbf{a}_k$ is expressed by a Π^0_3 formula. The next theorem shows that $\mathbb{R}^n$ is 'weakly' complete.

Theorem 2.4. *The following is provable in* RCA_0*. Let* $\{\mathbf{a}_k\}_{k\in\mathbb{N}}$ *be a sequence of points of* $\mathbb{R}^n$*. If there exists a sequence of real numbers* $\{r_k\}_{k\in\mathbb{N}}$ *such that* $\lim_{k\to\infty} r_k = 0$ *and* $\forall k \forall i\ \|\mathbf{a}_k - \mathbf{a}_{k+i}\| < r_k$*, then* $\{\mathbf{a}_k\}_{k\in\mathbb{N}}$ *is convergent, i.e., there exists* $\mathbf{b}$ *such that* $\mathbf{b} = \lim_{k\to\infty} \mathbf{a}_k$*.*

Proof. This theorem is a generalization of nested interval completeness [29, Theorem II.4.8], and modifying its proof, we can easily prove this theorem. □

Note that we can prove Theorem 2.4 effectively, *i.e.*, we can effectively find the limit $\lim_{k\to\infty} \mathbf{a}_k$ in Theorem 2.4.[5] Thus, a sequential version of Theorem 2.4 holds.

The next theorem shows that the 'strong' completeness of $\mathbb{R}^n$ is not provable in RCA_0.

Theorem 2.5. *The following assertions are pairwise equivalent over* RCA_0*.*

1. ACA_0*.*
2. *Every Cauchy sequence in* $\mathbb{R}^n$ *is convergent. (A sequence* $\{\mathbf{a}_k\}_{k\in\mathbb{N}}$ *in* $\mathbb{R}^n$ *is said to be Cauchy if* $\forall \varepsilon > 0\ \exists m\ \forall n(n > m \to \|\mathbf{a}_n - \mathbf{a}_m\| < \varepsilon$*.)*

Proof. This theorem is an easy generalization of [29, Theorem III.2.2]. □

[5]In this thesis, 'we can effectively find a set X (using a parameter Y)' means 'a set X is directly constructed by RCA (Δ^0_1 comprehension) with parameter Y'. In this situation, given a sequence of sets $\{Y_n\}_{n\in\mathbb{N}}$, we can find a sequence $\{X_n\}_{n\in\mathbb{N}}$ such that each X_n is constructed from Y_n. For example, 'Theorem 2.4 is effectively provable' means that there exist a Σ^0_1 formula $\varphi(n, X, Y)$ and a Π^0_1 formula $\psi(n, X, Y)$ such that 'for any $\{\mathbf{a}_k\}_{k\in\mathbb{N}}$ and $\{r_k\}_{k\in\mathbb{N}}$ as in Theorem 2.4, $Z = \{n \mid \varphi(n, \{\mathbf{a}_k\}_{k\in\mathbb{N}}, \{r_k\}_{k\in\mathbb{N}})\} = \{n \mid \psi(n, \{\mathbf{a}_k\}_{k\in\mathbb{N}}, \{r_k\}_{k\in\mathbb{N}})\}$ is the limit of $\{\mathbf{a}_k\}_{k\in\mathbb{N}}$' is provable in RCA_0.

Next, we define an open or closed set. It is coded by the countable open basis of $\mathbb{R}^n$.

Definition 2.8 (open and closed sets)**.** The following definitions are made in RCA_0.

1. A (code for an) *open set* U in $\mathbb{R}^n$ is a set $U \subseteq \mathbb{N} \times \mathbb{Q}^n \times \mathbb{Q}$. A point $\mathbf{x} \in \mathbb{R}^n$ is said to *belong to* U (abbreviated $\mathbf{x} \in U$) if

$$\exists n\ \exists \mathbf{a}\ \exists r\ (\|\mathbf{x} - \mathbf{a}\| < r \wedge (n, \mathbf{a}, r) \in U).$$

2. A (code for a) *closed set* C in $\mathbb{R}^n$ is a set $C \subseteq \mathbb{N} \times \mathbb{Q}^n \times \mathbb{Q}$. A point $\mathbf{x} \in \mathbb{R}^n$ is said to *belong to* C (abbreviated $\mathbf{x} \in C$) if

$$\forall n\ \forall \mathbf{a}\ \forall r\ ((n, \mathbf{a}, r) \in C \to \|\mathbf{x} - \mathbf{a}\| \geq r).$$

The following lemma is very useful to construct open or closed sets.

Lemma 2.6 ([29] Lemma II.5.7)**.** *For any Σ^0_1 (or Π^0_1) formula $\varphi(X)$, the following is provable in RCA_0. Assume that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} = \mathbf{y}$ and $\varphi(\mathbf{x})$ imply $\varphi(\mathbf{y})$. Then there exists an open (or closed) set $U \subseteq \mathbb{R}^n$ such that for all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \in U$ if and only if $\varphi(\mathbf{x})$.*

Finally, we consider some versions of Heine-Borel theorem.

Theorem 2.7. *The following assertions are pairwise equivalent over RCA_0.*

1. WKL_0.

2. *Heine-Borel theorem for bounded closed rectangles: if $C \subseteq \mathbb{R}^n$ is a bounded closed rectangle, i.e., C is a products of bounded closed intervals, and $\{U_k\}_{k\in\mathbb{N}}$ be a sequence of open subsets of $\mathbb{R}^n$ which covers C, then there exists m such that $\{U_k\}_{k<m}$ covers C.*

3. *A sequential version of 2: if $\{C_l\}_{l\in\mathbb{N}}$ is a sequence of bounded closed rectangles in $\mathbb{R}^n$ and $\langle \{U_{lk}\}_{k\in\mathbb{N}} \mid l \in \mathbb{N}\rangle$ be a sequence of sequences of open subsets of $\mathbb{R}^n$ such that each $\{U_{lk}\}_{k\in\mathbb{N}}$ covers C_l, then there exists a sequence $\{m_l\}_{l\in\mathbb{N}}$ such that $\{U_{lk}\}_{k<m_l}$ covers C_l.*

Proof. This theorem is a generalization of [29, Theorem IV.1.2], and we can imitate its proof. □

Theorem 2.8. *The following assertions are pairwise equivalent over* RCA_0.

1. WWKL_0.

2. *If* $C \subseteq \mathbb{R}^n$ *is a bounded closed rectangle and* $\{U_k\}_{k\in\mathbb{N}}$ *be a sequence of open subsets of* $\mathbb{R}^n$ *which covers* C*, then there exists a sequence of finite sequences of rectangles* $\langle \{V_{ij}\}_{j<l_i} \rangle \mid i \in \mathbb{N}\rangle$ *such that* $\{U_k\}_{k<i} \cup \{V_{ij}\}_{j<l_i}$ *covers* C *for all* $i \in \mathbb{N}$ *and*
$$\lim_{i\to\infty} \sum_{j<l_i} \lambda(V_{ij}) = 0.$$
Here, $\lambda(V)$ *denotes the volume of* V*, i.e.,* $\lambda(V) = (b_1 - a_1)(b_2 - a_2)\cdots(b_n - a_n)$ *where* $V = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$.

3. *A sequential version of 2 as in Theorem 2.7.*

Proof. This theorem is a generalization of a weak form of Heine-Borel compactness appeared in [39], and we can imitate arguments in [39]. □

A bounded closed rectangle is called a 'totally bounded (or compact)' and 'separably closed' closed set in the theory of complete separable metric spaces[6] in RCA_0, and the above Heine-Borel theorem can be generalized for complete separable metric spaces.

[6] For the theory of complete separable metric spaces, see e.g. [7, 6, 12].

3 Basic analysis in second order arithmetic

In this chapter, we introduce a means to deal with differentiability and integrability within RCA_0, and then, develop basic real analysis. Subsequently, we present some results on Reverse Mathematics for fundamental real analysis.

3.1 Differentiability and integrability

In this section, we define continuous functions, C^1-functions and Riemann integrability. To deal with C^1-functions within RCA_0, we introduce a differentiable condition function for a C^1-function. To consider Riemann integrability of continuous functions, we introduce a modulus of integrability for a continuous function. These are powerful tools to develop differential and integral calculus within RCA_0.

3.1.1 Continuous functions

In this subsection, we define a continuous function and show some basic results for continuous functions. We define continuous functions as a certain code given by the countable open basis of $\mathbb{R}^n$.

Definition 3.1 (continuous functions)**.** The following definition is made in RCA_0. A (code for a) *continuous partial function* f from $\mathbb{R}^n$ to $\mathbb{R}$ is a set of quintuples $F \subseteq \mathbb{N} \times \mathbb{Q}^n \times \mathbb{Q}^+ \times \mathbb{Q} \times \mathbb{Q}^+$ which satisfies the following properties. We write $(\mathbf{a}, r)F(b, s)$ as an abbreviation for $\exists m((m, \mathbf{a}, r, b, s) \in F)$. The properties which we require are:

1. if $(\mathbf{a}, r)F(b, s)$ and $(\mathbf{a}, r)F(b', s')$, then $|b - b'| \leq s + s'$;
2. if $(\mathbf{a}, r)F(b, s)$ and $\|\mathbf{a}' - \mathbf{a}\| + r' < r$, then $(\mathbf{a}', r')F(b, s)$;
3. if $(\mathbf{a}, r)F(b, s)$ and $|b - b'| + s < s'$, then $(\mathbf{a}, r)F(b', s')$.

A point $\mathbf{x} \in \mathbb{R}^n$ is said to belong to the *domain* of f, abbreviated $\mathbf{x} \in \mathrm{dom}(f)$, if and only if for any $\varepsilon > 0$ there exists $(\mathbf{a}, r)F(b, s)$ such that $\|\mathbf{x} - \mathbf{a}\| < r$ and $s < \varepsilon$. If $\mathbf{x} \in \mathrm{dom}(f)$, we define the *value* $f(\mathbf{x})$ to be the unique $y \in \mathbb{R}$ such that $|y - b| < s$ for all $(\mathbf{a}, r)F(b, s)$ with $\|\mathbf{x} - \mathbf{a}\| < r$. The existence of $f(\mathbf{x})$ is provable in RCA_0.

Let U, V be an open or closed subset of $\mathbb{R}^n$, $\mathbb{R}$, respectively. Then f is said to be a continuous function from U to V if and only if for any $\mathbf{x} \in U$, $\mathbf{x} \in \mathrm{dom}(f)$ and $f(\mathbf{x}) \in V$.

Definition 3.2. The following definitions are made in RCA_0. A *continuous partial function from* $\mathbb{R}^n$ *to* $\mathbb{R}^m$ is a (code for a) finite sequence of continuous partial functions $\mathbf{f} = (f_1, \ldots, f_m)$ such that $f_1, \ldots, f_m$ are continuous partial functions from $\mathbb{R}^n$ to $\mathbb{R}$.

Let U, V be an open or closed subset of $\mathbb{R}^n$, $\mathbb{R}^m$, respectively. Then $\mathbf{f}$ is said to be a continuous function from U to V if and only if for any $\mathbf{x} \in U$ and for any $1 \leq i \leq m$, $\mathbf{x} \in \mathrm{dom}(f_i)$ and $\mathbf{y} = (f_1(\mathbf{x}) \ldots f_m(\mathbf{x})) \in V$.

Remark 3.1. Imitating definition 3.1, we can define another code for a continuous partial function from $\mathbb{R}^n$ to $\mathbb{R}^m$. A (code for a) continuous partial function $\mathbf{f}$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a set of quintuples $\hat{F} \subseteq \mathbb{N} \times \mathbb{Q}^n \times \mathbb{Q}^+ \times \mathbb{Q}^m \times \mathbb{Q}^+$ which is required to satisfy:

1. if $(\mathbf{a}, r)\hat{F}(\mathbf{b}, s)$ and $(\mathbf{a}, r)\hat{F}(\mathbf{b}', s')$, then $\|\mathbf{b} - \mathbf{b}'\| \leq s + s'$;
2. if $(\mathbf{a}, r)\hat{F}(\mathbf{b}, s)$ and $\|\mathbf{a}' - \mathbf{a}\| + r' < r$, then $(\mathbf{a}', r')\hat{F}(\mathbf{b}, s)$;
3. if $(\mathbf{a}, r)\hat{F}(\mathbf{b}, s)$ and $\|\mathbf{b} - \mathbf{b}'\| + s < s'$, then $(\mathbf{a}, r)\hat{F}(\mathbf{b}', s')$.

We can easily and effectively find a code for $\mathbf{f}$ from codes for $f_1, \ldots, f_m$. Conversely we can easily and effectively find codes for $f_1, \ldots, f_m$ from a code for $\mathbf{f}$.

First, there exist a code for an identity function, a constant function, a norm function, and so on. We can construct other elementary continuous functions by next theorem.

Theorem 3.2 (Theorems [29] II.6.3 and II.6.4.)**.** *The following is provable in* RCA_0*. There exists a (code for a) continuous function of a sum, a product and quotient of two* $\mathbb{R}$*-valued continuous functions. Also there exists a (code for a) continuous function of a composite of two continuous functions.*[7]

The next two theorems show the basic properties of continuous functions.

Theorem 3.3. *The following assertions are provable in* RCA_0*.*

1. *Let* U *be an open subset of* $\mathbb{R}^n$*,* V *be an open subset of* $\mathbb{R}^m$ *and* f *be a continuous function from* U *to* $\mathbb{R}^m$*. Then there exists an open set* $W = f^{-1}(V) \cap U$*, the inverse image of* V*.*

[7]'There exists a composite of n continuous functions for all $n \in \mathbb{N}$' cannot be proved in RCA_0. See [10].

2. *Let C be a closed subset of $\mathbb{R}^n$, V be an open subset of $\mathbb{R}^m$ and f be a continuous function from C to $\mathbb{R}^m$. Then there exists an open set $W \subseteq \mathbb{R}^n$ such that $W \cap C = f^{-1}(V) \cap C$.*

We write such W as $W = \tilde{f}^{-1}(V)$.

Proof. Immediate from Lemma 2.6. □

We can prove Theorems 3.2 and 3.3 effectively, and thus, sequential versions of these theorems hold.

Theorem 3.4 (intermediate value theorem: [29] Theorem II.6.6)**.** *The following is provable in RCA_0. If f is a continuous function from $[0,1]$ to $\mathbb{R}$ such that $f(0) < 0 < f(1)$, then there exists c such that $0 < c < 1$ and $f(c) = 0$.*

Note that a sequential version of the intermediate value theorem is not provable in RCA_0. Actually, it is equivalent to WKL_0 over RCA_0 (see [29, Section IV]).

Next, we consider some behavior of continuous functions on a bounded closed set, such as uniform continuity and boundedness.

Definition 3.3 (modulus of uniform continuity)**.** The following definition is made in RCA_0. Let U be an open or closed subset of $\mathbb{R}^n$, and let $\mathbf{f}$ be a continuous function from U to $\mathbb{R}^m$. A *modulus of uniform continuity* on U for $\mathbf{f}$ is a function h from $\mathbb{N}$ to $\mathbb{N}$ such that for any $n \in \mathbb{N}$ and for any $\mathbf{x}, \mathbf{y} \in U$, if $\|\mathbf{x} - \mathbf{y}\| < 2^{-h(n)}$, then $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| < 2^{-n}$.

A modulus of uniform continuity for f guarantees stronger uniform continuity of f than that in the usual sense. In fact, in RCA_0, f may not have a modulus of uniform continuity even if f is uniformly continuous. See [29, Exercise IV.2.9].

Theorem 3.5. *The following assertions are pairwise equivalent over RCA_0.*

1. WKL_0.

2. *Every continuous function on a bounded closed set is bounded.*

3. *Every continuous function on a bounded closed set has a modulus of uniform continuity.*

4. *A sequential version of 2: if $\{C_k\}_{k\in\mathbb{N}}$ is a sequence of bounded closed sets in $\mathbb{R}^m$ and $\{\mathbf{f}_k\}_{k\in\mathbb{N}}$ is a sequence of continuous functions such that each $\mathbf{f}_k$ is from C_k to $\mathbb{R}^n$, then, there exists a sequence of rational numbers $\{M_k\}_{k\in\mathbb{N}}$ such that each $\mathbf{f}_k$ is from C_k to $B(\mathbf{0}; M_k) = \{\mathbf{x} \mid \|\mathbf{x}\| < M_k\}$.*

5. *A sequential version of 3: if* $\{C_k\}_{k\in\mathbb{N}}$ *is a sequence of bounded closed sets in* $\mathbb{R}^m$ *and* $\{\mathbf{f}_k\}_{k\in\mathbb{N}}$ *is a sequence of continuous functions on* C_k*, then, there exists a sequence of functions* $\{h_k\}_{k\in\mathbb{N}}$ *such that each* h_k *is a modulus of uniform continuity for* $\mathbf{f}_k$ *on* C_k*.*

Here, a bounded closed set is a closed set which is included in some bounded closed rectangle.

Proof. Easy generalization of [29, Theorem VI.2.2 and VI.2.3]. □

Remark 3.6. Boundedness cannot always provide the maximum value principle. In fact, the property 'every continuous function on some bounded closed rectangle attains a maximum value' is equivalent to WKL_0 over RCA_0, but 'every continuous function on some bounded closed set attains a maximum value' is equivalent to ACA_0 over RCA_0. For details, see [29, IV].

The next theorem is very useful to show that constructing some continuous functions requires ACA_0.

Theorem 3.7. *The following assertions are pairwise equivalent over* RCA_0*.*

1. ACA_0*.*
2. *If* f *is a continuous function from* $(0,1)$ *to* $\mathbb{R}$ *such that* $\lim_{x\to+0} f(x) = 0$*, then there exists a (code for a) continuous function* $\bar{f}$ *from* $[0,1)$ *to* $\mathbb{R}$ *such that*

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in (0,1), \\ 0 & \text{if } x = 0. \end{cases}$$

The following proof of $2 \to 1$ is due to Tanaka.

Proof. We reason within RCA_0. We first prove $1 \to 2$. By arithmetical comprehension, define $\bar{F}$ as

$$\begin{aligned}(n,a,r,b,s) \in \bar{F} \quad \leftrightarrow \quad & a \in \mathbb{Q} \cap [0,1) \wedge b \in \mathbb{Q} \wedge s, r \in \mathbb{Q}^+ \wedge n = (a,r,b,s) \\ & \wedge \forall p \in \mathbb{Q} \cap (0,1)\ |a-p| < r \to |b - f(p)| \leq s.\end{aligned}$$

Let $\bar{f}$ be a continuous function coded by $\bar{F}$. Then clearly $(0,1) \subseteq \mathrm{dom}(\bar{f})$ and $\bar{f}|_{(0,1)} = f$. $\lim_{x\to+0} f(x) = 0$ implies for any $\varepsilon > 0$ there exist $r, s \in \mathbb{Q}^+$ such that $\forall p \in \mathbb{Q} \cap (0,1)\ |0-p| < r \to |0 - f(p)| \leq s$ and $s < \varepsilon$. Hence $0 \in \mathrm{dom}(\bar{f})$ and $\bar{f}(0) = 0$.

Next we show $2 \to 1$. By Theorem 2.2, we show that for any one-to-one function h from $\mathbb{N}$ to $\mathbb{N}$, there exists a set X such that X is the range of h. Let h be a one-to-one function from $\mathbb{N}$ to $\mathbb{N}$. Then $\lim_{n\to\infty} h(n) = \infty$. Define $\{a_n\}_{n\in\mathbb{N}}$ as

$$a_n \quad := \quad \frac{\frac{1}{h(n)+1} - \frac{1}{h(n+1)+1}}{\frac{1}{n+1} - \frac{1}{n+2}}.$$

Then we define a continuous function f from $(0,1)$ to $\mathbb{R}$ such that

$$f(x) = a_n\left(x - \frac{1}{n+1}\right) + \frac{1}{(h(n)+1)}$$

for each n and $x \in \left[\frac{1}{n+2}, \frac{1}{n+1}\right]$. Then, $f(1/(n+1)) = 1/(h(n)+1)$ for all $n \in \mathbb{N}$, and $\lim_{x\to 0} f(x) = 0$. Hence by 2, we can expand f into $\bar{f}$ such that

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in (0,1), \\ 0 & \text{if } x = 0. \end{cases}$$

Now we construct the range of h. Let $\bar{F}$ be a code for $\bar{f}$, and let $\varphi(k,l)$ be a Σ^0_1 formula which expresses that there exists (a,r,b,s) such that $(a,r)\bar{F}(b,s)$, $|a| + 1/(l+1) < r$ and $|b| + s < 1/(k+1)$. Then by conditions of a code for a continuous function, $\forall k \exists l \varphi(k,l)$ holds. Hence, there exists a function h_0 from $\mathbb{N}$ to $\mathbb{N}$ such that $\forall k \varphi(k, h_0(k))$ holds. This implies

$$\forall m \in \mathbb{N}\ m \geq h_0(n) \to n < h(m).$$

By Δ^0_1 comprehension, define a set $X \subseteq \mathbb{N}$ as $n \in X \leftrightarrow \exists m < h_0(n)\ n = h(m)$. Then clearly, X is the range of h. This completes the proof of $2 \to 1$. □

3.1.2 C^1-functions

We first define differentiability and continuous differentiability.

Definition 3.4 (differentiablity)**.** The following definition is made in RCA_0. Let U be an open subset of $\mathbb{R}$, and let f be a continuous function from U to $\mathbb{R}$. Then f is said to be *differentiable* if

$$\forall x \in U\ \exists \alpha_x \in \mathbb{R}\ \alpha_x = \lim_{u\to x} \frac{f(u) - f(x)}{u - x}.$$

A differentiable function f is said to be *continuously differentiable* if

$$\forall x \in U\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall y \in U\ |x - y| < \delta \to |\alpha_x - \alpha_y| < \varepsilon.$$

Though we can deal with continuous functions within RCA_0, the above definition of continuous differentiability cannot work to construct derivatives in RCA_0. Actually, the next theorem shows that the existence of derivatives of continuously differentiable functions requires ACA_0.

Theorem 3.8. *The following assertions are pairwise equivalent over* RCA_0.

1. ACA_0.
2. *If f is a continuously differentiable function from $(-1,1)$ to $\mathbb{R}$, then there exists a (code for a) continuous function f' which is the derivative of f.*

Proof. We reason within RCA_0. We can prove $1 \to 2$ by arithmetical comprehension as in the proof of Theorem 3.7. For the converse, we assume 2. By Theorem 2.2, we show that for any one-to-one function h from $\mathbb{N}$ to $\mathbb{N}$, there exists a set X such that X is the range of h. Let h be a one-to-one function from $\mathbb{N}$ to $\mathbb{N}$. Then $\lim_{n\to\infty} h(n) = \infty$. Define $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ such that

$$\begin{aligned} a_n &:= \frac{\frac{1}{h(n)+1} - \frac{1}{h(n+1)+1}}{\frac{1}{n+1} - \frac{1}{n+2}}; \\ b_n &:= \frac{1}{2}\left(\frac{1}{h(n)+1} + \frac{1}{h(n+1)+1}\right)\left(\frac{1}{n+1} - \frac{1}{n+2}\right). \end{aligned}$$

Then $b_n \leq 1/(n+1) - 1/(n+2)$, hence by Theorem 3.26.1, $\sum_{k=n}^{\infty} b_k$ is convergent for all $n \in \mathbb{N}$. Using these, we define a continuously differentiable function from $(-1,1)$ to $\mathbb{R}$. Define a continuous function f_0 from $(-1,0)\cup(0,1)$ such that

$$f_0(x) = \begin{cases} -\frac{a_n}{2}\left(x - \frac{1}{n+1}\right)^2 + \frac{x(n+1)+1}{(n+1)(h(n)+1)} - \sum_{k=n}^{\infty} b_k & \text{if } x \in \left[\frac{-1}{n+1}, \frac{-1}{n+2}\right], \\ \frac{a_n}{2}\left(x - \frac{1}{n+1}\right)^2 + \frac{x(n+1)-1}{(n+1)(h(n)+1)} + \sum_{k=n}^{\infty} b_k & \text{if } x \in \left[\frac{1}{n+2}, \frac{1}{n+1}\right] \end{cases}$$

for each n. Here, if $|x| < 1/(n+1)$, then $|f_0(x)| < 1/(n+1)$. Hence, we can extend f_0 into f from $(-1,1)$ to $\mathbb{R}$ such that

$$f(x) = \begin{cases} f_0(x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

To extend f_0 into f, we need to construct a code for f. Let F_0 be a code for f_0 and let $\varphi(a,r,b,s)$ be a Σ^0_1 formula which expresses $(a,r)F_0(b,s) \vee \exists m \in \mathbb{N}\ |a| + r < 1/(m+1) < s - |b|$. Write

$$\varphi(a,r,b,s) \equiv \exists m \theta(m,a,r,b,s)$$

where θ is Σ^0_0. By Δ^0_1 comprehension, define F as $(m, a, r, b, s) \in F \leftrightarrow \theta(m, a, r, b, s)$. Then clearly f is coded by F.

Next, we show that f is continuously differentiable. Define α_x as above, then

$$\alpha_x = \begin{cases} -a_n \left(x - \frac{1}{n+1}\right) + \frac{1}{(h(n)+1)} & \text{if } x \in \left[\frac{-1}{n+1}, \frac{-1}{n+2}\right], \\ 0 & \text{if } x = 0, \\ a_n \left(x - \frac{1}{n+1}\right) + \frac{1}{(h(n)+1)} & \text{if } x \in \left[\frac{1}{n+2}, \frac{1}{n+1}\right]. \end{cases}$$

We can easily check the condition of continuously differentiability for f. By 2, there exists a continuous function g from $(-1, 1)$ to $\mathbb{R}$ such that $g(x) = \alpha_x$. Note that this continuous function g is similar to the continuous function we constructed in the proof of Theorem 3.7. Hence, we can construct the range of h as in the proof of Theorem 3.7. This completes the proof of $2 \to 1$. □

Theorem 3.8 pointed out the difficulty of constructing the derivative. To avoid this difficulty, we mainly consider the following C^1-functions to develop differential calculus.

Remark 3.9. There is another difficulty in dealing with differentiable functions within RCA_0. Actually, the following assertions are pairwise equivalent over RCA_0.

1. ACA_0.
2. If f is a differentiable function from $(-1, 1)$ to $\mathbb{R}$ and $\{x_n\}_{n\in\mathbb{N}}$ is a real sequence in $(-1, 1)$, then there exists a real sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ such that
$$\forall n \in \mathbb{N}\ \alpha_n = \lim_{u \to x_n} \frac{f(u) - f(x_n)}{u - x_n}.$$
3. If f is a continuously differentiable function from $(-1, 1)$ to $\mathbb{R}$ and $\{x_n\}_{n\in\mathbb{N}}$ is a real sequence in $(-1, 1)$, then there exists a real sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ such that
$$\forall n \in \mathbb{N}\ \alpha_n = \lim_{u \to x_n} \frac{f(u) - f(x_n)}{u - x_n}.$$

We first define C^1-functions in $\mathbb{R}$, and similarly we define C^r and C^∞-functions in $\mathbb{R}$.

Definition 3.5 (C^1-, C^r- and C^∞-functions)**.** The following definitions are made in RCA_0.

1. Let U be an open subset of $\mathbb{R}$, and let f , f' be continuous functions from U to $\mathbb{R}$. Then a pair (f, f') is said to be of C^1 if and only if

$$\forall x \in U \ \lim_{u \to x} \frac{f(u) - f(x)}{u - x} = f'(x).$$

2. Let U be an open subset of $\mathbb{R}$, and let $\{f^{(n)}\}_{n \leq r}$ be a finite sequence of continuous functions from U to $\mathbb{R}$. Then $\{f^{(n)}\}_{n \leq r}$ is said to be of C^r if and only if for any n less than r, $(f^{(n)}, f^{(n+1)})$ is of C^1.

3. Let U be an open subset of $\mathbb{R}$, and let $\{f^{(n)}\}_{n \in \mathbb{N}}$ be an infinite sequence of continuous functions from U to $\mathbb{R}$. Then $\{f^{(n)}\}_{n \in \mathbb{N}}$ is said to be of C^∞ if and only if for any $r \in \mathbb{N}$, $\{f^{(n)}\}_{n \leq r}$ is of $\mathbb{C}^r$.

We usually write f for $f^{(0)}$ when $\{f^{(n)}\}_{n \leq r}$ is of C^r or $\{f^{(n)}\}_{n \in \mathbb{N}}$ is of C^∞. If (f, f') is of C^1, $\{f^{(n)}\}_{n \leq r}$ is of C^r or $\{f^{(n)}\}_{n \in \mathbb{N}}$ is of C^∞, f is said to be of C^1, C^r or C^∞, respectively.

The next lemma shows that the uniqueness of the derivative is provable in RCA_0.

Lemma 3.10. *The following is provable in* RCA_0. *Let U be an open subset of $\mathbb{R}$, and let f, g be C^r- or C^∞-functions from U to $\mathbb{R}$. If $\forall x \in U \ f(x) = g(x)$, then for any $k \leq r$ or $k \in \mathbb{N}$ $\forall x \in U \ f^{(k)}(x) = g^{(k)}(x)$, respectively.*

Proof. We reason within RCA_0. It is sufficient to prove only the C^∞ case. By definition of continuous functions, the following equivalence is easily derived:

$$\begin{aligned} & \forall x \in \mathbb{R} \ (x \in U \to f^{(k)}(x) = g^{(k)}(x)) \\ \leftrightarrow \quad & \forall q \in \mathbb{Q} \ (q \in U \to f^{(k)}(q) = g^{(k)}(q)). \end{aligned}$$

Write

$$\varphi(k) \equiv \forall q \in \mathbb{Q} \ (q \in U \to f^{(k)}(q) = g^{(k)}(q)).$$

Then $\varphi(k)$ is Π^0_1 and $\varphi(0)$ holds. If $\varphi(k)$ holds, then for any $x \in U$,

$$\begin{aligned} f^{(k+1)}(x) &= \lim_{u \to x} \frac{f^{(k)}(u) - f^{(k)}(x)}{u - x} \\ &= \lim_{u \to x} \frac{g^{(k)}(u) - g^{(k)}(x)}{u - x} \\ &= g^{(k+1)}(x). \end{aligned}$$

Hence $\varphi(k+1)$ holds. Then by Π^0_1-induction, $\forall k \varphi(k)$ holds and this completes the proof. □

To develop differential calculus, we have to begin with the mean value theorem. Fortunately, the mean value theorem for C^1-functions is easily provable in RCA_0 using the intermediate value theorem.

Lemma 3.11 ([29] Exercise II.6.10)**.** *The following is provable in* RCA_0*. Let U be an open subset of $\mathbb{R}$, and let f be a C^1-function from U to $\mathbb{R}$. Let K be a positive real number. If $[a,b] \subseteq U$ and for all $x \in [a,b]$ $|f'(x)| \leq K$, then*

$$\left|\frac{f(b)-f(a)}{b-a}\right| \leq K.$$

Theorem 3.12 (mean value theorem)**.** *The following is provable in* RCA_0*. Let $[a,b]$ be an interval of $\mathbb{R}$ and let f be a continuous function from $[a,b]$ to $\mathbb{R}$. If f is of C^1 on (a,b), i.e. there exists a continuous function f' from (a,b) to $\mathbb{R}$ such that (f,f') is of C^1, then there exists $c \in (a,b)$ such that*

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

Proof. The proof is easy from Lemma 3.11 and Theorem 3.4. □

Note that a sequential version of Theorem 3.12 is not provable in RCA_0. Actually, it is equivalent to WKL_0 (due to Yamazaki).

Remark 3.13. We can prove a stronger version of Theorem 3.12. In fact, the mean value theorem for a differentiable function can be proved in RCA_0. See Hardin/Velleman[14]. However, we do not know whether a sequential version of the mean value theorem for a differentiable function is provable in WKL_0.

Next, we define C^r- or C^∞-function in $\mathbb{R}^n$.

Definition 3.6 (C^r- and C^∞-functions from $U \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$)**.** The following definitions are made in RCA_0. Let U be an open subset of $\mathbb{R}^n$. The notation $\alpha = (a_1, \ldots, a_n) \in \mathbb{N}^n$ is a multi-index and $|\alpha| = a_1 + \cdots + a_n$.

1. A C^r*-function from U to* $\mathbb{R}$ is a finite sequence of continuous functions $\{f_\alpha\}_{|\alpha| \leq r}$ from U to $\mathbb{R}$ which satisfies the following: for any $\alpha = (a_1, \ldots, a_n)$ such that $|\alpha| \leq r-1$, $(f_{(a_1,\ldots,a_i,\ldots,a_n)}, f_{(a_1,\ldots,a_i+1,\ldots,a_n)})$ is of C^1 as a function of x_i, *i.e.*,

$$\forall \mathbf{x} \in U \ f_{(a_1,\ldots,a_i+1,\ldots,a_n)}(\mathbf{x}) = \lim_{t \to 0} \frac{f_{(a_1,\ldots,a_i,\ldots,a_n)}(\mathbf{x}+t\mathbf{e}_i) - f_{(a_1,\ldots,a_i,\ldots,a_n)}(\mathbf{x})}{t}$$

where $\mathbf{e}_i$ is the unit vector along x_i.

2. A C^∞*-function from U to $\mathbb{R}$* is an infinite sequence of continuous functions $\{f_\alpha\}_{\alpha\in\mathbb{N}^n}$ from U to $\mathbb{R}$ such that for any $r \in \mathbb{N}$, $\{f_\alpha\}_{|\alpha|\le r}$ is a C^r-function.

3. A C^r- or C^∞*-function from U to $\mathbb{R}^m$* is a finite sequence of C^r- or C^∞-functions $\mathbf{f} = (f_1, \ldots, f_m)$ from U to $\mathbb{R}$, respectively.

If $\{f_\alpha\}_{|\alpha|\le r}$ is of C^r or $\{f_\alpha\}_{\alpha\in\mathbb{N}^n}$ is of C^∞, then f is said to be of C^r or C^∞. As usual, we write

$$f_{(a_1,\ldots,a_n)} = \frac{\partial^{a_1+\cdots+a_n} f}{\partial^{a_1} x_1 \ldots \partial^{a_n} x_n}.$$

Theorem 3.14. *The following assertions are provable in* RCA_0.

1. *Let U be an open subset of $\mathbb{R}^n$, and let f be a C^1-function from U to $\mathbb{R}$. If its derivatives $f_{x_i} = \partial f/\partial x_i$ and $f_{x_j} = \partial f/\partial x_j$ are also of C^1, i.e., there exist finite sequences $\{(f_{x_i})_\alpha\}_{|\alpha|\le 1}$ and $\{(f_{x_j})_\alpha\}_{|\alpha|\le 1}$ which satisfy the condition for C^1, then*
$$\frac{\partial f_{x_i}}{\partial x_j} = \frac{\partial f_{x_j}}{\partial x_i}.$$

2. *Let U be an open subset of $\mathbb{R}^n$, and let f be a C^1-function from U to $\mathbb{R}$. If each derivative f_{x_i} is also of C^1, then we can expand f into a C^2-function, i.e., we can construct a finite sequences $\{f_\alpha\}_{|\alpha|\le 2}$ which satisfies the condition for C^2.*

Proof. We can prove 1 imitating the usual proof, and 2 is immediate from 1. □

Remark 3.15. Theorem 3.14.1 can be strengthened for a continuously differentiable version, *i.e.*, the derivative along x_j of f_{x_i} is equal to the derivative along x_i of f_{x_j} at each point if f_{x_i} and f_{x_j} are continuously differentiable. To prove this, we use the mean value theorem for differentiable functions (Remark 3.13).

To prove basic properties of C^1-functions in RCA_0, we construct differentiable condition functions. A differentiable condition function for a C^1-function f expresses a condition of differentiability at each point of $\mathrm{dom}(f)$. It also expresses a continuity of the derivative f'. Hence using a differentiable condition function, we can easily prove basic properties of C^1-functions in RCA_0.

Theorem 3.16. *The following is provable in* RCA_0.

1. *Let U be an open subset of $\mathbb{R}^n$, and let f be a C^1-function from U to $\mathbb{R}$. Then, there exists a continuous function e_f from $U \times U$ to $\mathbb{R}$ such that*

$$\forall \mathbf{x} \in U \ e_f(\mathbf{x}, \mathbf{x}) = 0; \tag{1}$$

$$\forall \mathbf{x}, \mathbf{y} \in U \ f(\mathbf{y}) - f(\mathbf{x}) = \sum_{i=1}^{n} f_{x_i}(\mathbf{x})(y_i - x_i) + e_f(\mathbf{x}, \mathbf{y})\|\mathbf{y} - \mathbf{x}\|. \tag{2}$$

(Here, $f_{x_i} = \partial f / \partial x_i$.)

2. *Let U be an open subset of $\mathbb{R}$, and let f be a C^1-function from U to $\mathbb{R}$. Then, there exists a continuous function e_f from $U \times U$ to $\mathbb{R}$ such that*

$$\forall x \in U \ e_f(x, x) = 0; \tag{3}$$

$$\forall x, y \in U \ f(y) - f(x) = (y - x)(f'(x) + e_f(x, y)). \tag{4}$$

We call this e_f a differentiable condition function for f.

Note that we can effectively find a differentiable condition function for a C^1-function. Thus, a sequential version of Theorem 3.16 holds.

Remark 3.17. Theorem 3.16 is not trivial. Actually, for 3.16.2, we want to define e_f as

$$e_f(x, y) = \begin{cases} \frac{f(y)-f(x)}{y-x} - f'(x) & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases} \tag{5}$$

and of course this e_f is a continuous function in the usual sense. However, Theorem 3.7 points out that RCA_0 cannot guarantee the existence of a code for a continuous function which is defined like as above, hence it is not easy to construct (a code for) e_f.

Proof of Theorem 3.16. We reason within RCA_0. To prove 1, define a (code for a) closed set $\Delta \subseteq \mathbb{R}^{2n}$ as $\Delta = \{(\mathbf{x}, \mathbf{x}) \mid \mathbf{x} \in U\}$. By Theorem 3.2, we can construct a continuous function g from U to $\mathbb{R}$ and a continuous function e_f^0 from $U \times U \setminus \Delta$ to $\mathbb{R}$ such that

$$\begin{aligned} g(\mathbf{x}) &= \sum_{i=1}^{n} |f_{x_i}(\mathbf{x})|; \\ e_f^0(\mathbf{x}, \mathbf{y}) &= \frac{f(\mathbf{y}) - f(\mathbf{x}) - \sum_{i=1}^{n} f_{x_i}(\mathbf{x})(y_i - x_i)}{\|\mathbf{y} - \mathbf{x}\|}. \end{aligned}$$

Let E_f^0 be a code for e_f^0, and let G be a code for g. Let $\varphi(\mathbf{a}, r, b, s)$ be a Σ_1^0 formula which expresses the following (i) or (ii) holds:

(i) $(\mathbf{a}, r)E^0_f(b, s)$;

(ii) $b = 0$ and there exists $(m_0, \mathbf{a_0}, r_0, b_0, s_0) \in G$ such that $\|\mathbf{a} - (\mathbf{a_0}, \mathbf{a_0})\| + r < r_0$ and $s > 2ns_0$.

Write

$$\varphi(\mathbf{a}, r, b, s) \equiv \exists m \theta(m, \mathbf{a}, r, b, s)$$

where θ is Σ^0_0. By Δ^0_1 comprehension, define E_f as $(m, \mathbf{a}, r, b, s) \in E_f \leftrightarrow \theta(m, \mathbf{a}, r, b, s)$, *i.e.*, $(\mathbf{a}, r)E_f(b, s)$ holds if and only if (i) or (ii) holds. Then E_f is a code for a continuous (partial) function. To show this, we have to check the conditions of a code for a continuous function. It is clear that E_f satisfies conditions 2 and 3 of definition 3.1. We must check condition 1. Assume $(\mathbf{a}, r)E_f(b, s)$ and $(\mathbf{a}, r)E_f(b', s')$. If $(\mathbf{a}, r, b, s)$ and $(\mathbf{a}, r, b', s')$ satisfy (i), then clearly condition 1 holds. If $(\mathbf{a}, r, b, s)$ and $(\mathbf{a}, r, b', s')$ satisfy (ii), then we can show condition 1 holds easily by G satisfying condition 1. Now we consider the case $(\mathbf{a}, r, b, s)$ satisfies (i) and $(\mathbf{a}, r, b', s')$ satisfies (ii). By condition 2, it is sufficient that we only check the case $\{(\mathbf{x}', \mathbf{y}') \mid \|(\mathbf{x}', \mathbf{y}') - \mathbf{a}\| < r\} \subseteq U \times U \setminus \Delta$ holds. Let $(m_0, \mathbf{a_0}, r_0, b_0, s_0)$ be an element of G such that $\|\mathbf{a} - (\mathbf{a_0}, \mathbf{a_0})\| + r < r_0$ and $s' > 2ns_0$. Here, $(m_0, \mathbf{a_0}, r_0, b_0, s_0) \in G$ implies

$$\forall \mathbf{z} \in U \ \|\mathbf{z} - \mathbf{a_0}\| < r_0 \rightarrow \left| \sum_{i=1}^{n} |f_{x_i}(\mathbf{x})| - b_0 \right| \leq s_0. \tag{6}$$

Write

$$\begin{aligned} \mathbf{a} &= (\mathbf{a}^x, \mathbf{a}^y)(\in \mathbb{R}^n \times \mathbb{R}^n); \\ \mathbf{a}^x &= (a^x_1, \ldots, a^x_n); \\ \mathbf{a}^y &= (a^y_1, \ldots, a^y_n); \\ \mathbf{z}_i &= (a^y_1, \ldots, a^y_i, a^x_{i+1}, \ldots, a^x_n). \end{aligned}$$

Here $\mathbf{a}^x \neq \mathbf{a}^y$, $\mathbf{z}_0 = \mathbf{a}^x$, $\mathbf{z}_n = \mathbf{a}^y$ and each $\mathbf{z}_i$ satisfies $\|\mathbf{z}_i - \mathbf{a_0}\| < r_0$. Then,

$$|e^0_f(\mathbf{a}) - b| \leq s; \tag{7}$$

$$\begin{aligned} |e^0_f(\mathbf{a}) - b'| &= |e^0_f((\mathbf{a}^x, \mathbf{a}^y))| \\ &\leq \sum_{i=1}^{n} \frac{|f(\mathbf{z}_i) - f(\mathbf{z}_{i-1}) - f_{x_i}(\mathbf{a}^x)(a^y_i - a^x_i)|}{\|\mathbf{a}^y - \mathbf{a}^x\|}. \end{aligned} \tag{8}$$

On the other hand, using Theorem 3.12, for any $1 \le i \le n$, if $a_i^x \ne a_i^y$, there exists $0 < \theta < 1$ such that

$$\frac{f(\mathbf{z}_i) - f(\mathbf{z}_{i-1})}{a_i^y - a_i^x} = f_{x_i}(\mathbf{z}_{i-1} + \theta(\mathbf{z}_i - \mathbf{z}_{i-1})).$$

(Here, $\|(\mathbf{z}_{i-1} + \theta(\mathbf{z}_i - \mathbf{z}_{i-1})) - \mathbf{a_0}\| < r_0$.) Then,

$$\begin{aligned} (9) \qquad & \frac{|f(\mathbf{z}_i) - f(\mathbf{z}_{i-1}) - f_{x_i}(\mathbf{a}^x)(a_i^y - a_i^x)|}{\|\mathbf{a}^y - \mathbf{a}^x\|} \\ \le & \ \left|\frac{f(\mathbf{z}_i) - f(\mathbf{z}_{i-1})}{a_i^y - a_i^x} - f_{x_i}(\mathbf{a}^x)\right| \\ = & \ |f_{x_i}(\mathbf{z}_{i-1} + \theta(\mathbf{z}_i - \mathbf{z}_{i-1})) - f_{x_i}(\mathbf{a}^x)| \\ \le & \ |f_{x_i}(\mathbf{z}_{i-1} + \theta(\mathbf{z}_i - \mathbf{z}_{i-1})) - b_0| + |f_{x_i}(\mathbf{a}^x) - b_0|. \end{aligned}$$

Hence by (6) and (9), for all $1 \le i \le n$,

$$(10) \qquad \frac{|f(\mathbf{z}_i) - f(\mathbf{z}_{i-1}) - f_{x_i}(\mathbf{a}^x)(a_i^y - a_i^x)|}{\|\mathbf{a}^y - \mathbf{a}^x\|} \le 2s_0.$$

(If $a_i^x = a_i^y$, then clearly (47) holds.) From (46) and (47),

$$\begin{aligned} (11) \qquad |e_f^0(\mathbf{a}) - b'| & \le \sum_{i=1}^{n} \frac{|f(\mathbf{z}_i) - f(\mathbf{z}_{i-1}) - f_{x_i}(\mathbf{a}^x)(a_i^y - a_i^x)|}{\|\mathbf{a}^y - \mathbf{a}^x\|} \\ & \le \sum_{i=1}^{n} 2s_0 \\ & \le s'. \end{aligned}$$

By (45) and (11), $|b - b'| \le s + s'$ holds. This means E_f satisfies condition 1.

Let e_f be a continuous function which is coded by E_f. Then, (i) provides $U \times U \setminus \Delta \subseteq \mathrm{dom}(e_f)$ and (ii) provides $\Delta \subseteq \mathrm{dom}(e_f)$, hence $U \times U \subseteq \mathrm{dom}(e_f)$. Clearly e_f satisfies (1) and (2), and this completes the proof of 1.

We can prove 2 similarly. □

Remark 3.18. If U is an open subset of $\mathbb{R}^n$ and $\mathbf{f} = (f_1, \dots, f_m)$ is a C^1-function from U to $\mathbb{R}^m$, then we define the differentiable condition function for f as $\mathbf{e_f} = (e_{f_1}, \dots, e_{f_m})$. Then

$$\forall \mathbf{x} \in U \ \mathbf{e_f}(\mathbf{x}, \mathbf{x}) = \mathbf{0};$$

$$\forall \mathbf{x}, \mathbf{y} \in U \ \mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) = \sum_{i=1}^{n} \mathbf{f}_{x_i}(\mathbf{x})(y_i - x_i) + \mathbf{e_f}(\mathbf{x}, \mathbf{y})\|\mathbf{y} - \mathbf{x}\|.$$

(Here, $\mathbf{f}_{x_i} = (f_{1\,x_i}, \dots f_{m\,x_i})$.)

Remark 3.19. Conversely, let U be an open subset of $\mathbb{R}^n$, f, f' be continuous function from U to $\mathbb{R}$ and e_f be a continuous function from $U \times U$ to $\mathbb{R}$. If f, f', e_f satisfy (1) and (2), then clearly (f, f') is of C^1.

Corollary 3.20. *The following assertions are provable in* RCA_0.

1. *Let U be an open subset of $\mathbb{R}$ and let k be a real number. If f and g are C^r- or C^∞-functions from U to $\mathbb{R}$, then $kf, f+g, fg$ are all C^r- or C^∞-functions from U to $\mathbb{R}$. Moreover, $(kf)' = kf'$, $(f+g)' = f' + g'$ and $(fg)' = f'g + fg'$ hold.*

2. *Let U be an open subset of $\mathbb{R}$, and let f be a C^r- or C^∞-functions from U to $\mathbb{R}$. If $f \neq 0$ in U, then $1/f$ is a C^r- or C^∞-function from U to $\mathbb{R}$, and $(1/f)' = -f'/(f^2)$ holds.*

3. *(chain rule) Let U be an open subset of $\mathbb{R}^n$ and let V be an open subset of $\mathbb{R}^m$. If $\mathbf{f} = (f_1, \ldots, f_m)$ is a continuous function from U to V, g is a continuous function from V to $\mathbb{R}$ and both f and g are of C^r or C^∞, then $g \circ f$ is a C^r- or C^∞-function from U to $\mathbb{R}$ and satisfies*

$$\frac{\partial(g \circ \mathbf{f})}{\partial x_i}(\mathbf{x}) = \sum_{j=1}^{m} \frac{\partial g}{\partial y_j}(\mathbf{f}(\mathbf{x})) \frac{\partial f_j}{\partial x_i}(\mathbf{x}).$$

Proof. We reason within RCA_0. We only prove 3. (We can prove 1 and 2 easily.) For any $\mathbf{x} \in U$, $1 \leq i \leq n$ and $\Delta x \in \mathbb{R} \setminus \{0\}$, define Δy_j $(1 \leq j \leq m)$ as

$$\begin{aligned} \Delta y_j &:= f_j(\mathbf{x} + \Delta x \mathbf{e_i}) - f_j(\mathbf{x}) \\ &= \Delta x \frac{\partial f_j}{\partial x_i}(\mathbf{x}) + |\Delta x| e_{f_j}(\mathbf{x}, \mathbf{x} + \Delta x \mathbf{e_i}). \end{aligned}$$

where $\mathbf{e_i}$ is the unit vector along x_i and each e_{f_j} is the differentiable condition function for f_j. Then

$$\begin{aligned} \|\mathbf{\Delta y}\| &:= \sqrt{\sum_{j=1}^{m} (\Delta y_j)^2} \\ &= |\Delta x| \sqrt{\sum_{j=1}^{m} \left(\frac{\Delta x}{|\Delta x|} \frac{\partial f_j}{\partial x_i}(\mathbf{x}) + e_{f_j}(\mathbf{x}, \mathbf{x} + \Delta x \mathbf{e_i}) \right)^2}. \end{aligned}$$

Define $e^i_{g\circ\mathbf{f}}$ as

$$\begin{aligned} e^i_{g\circ\mathbf{f}}(\Delta x) \ := \ & \sum_{j=1}^{m} \frac{\partial g}{\partial y_j}(\mathbf{f}(\mathbf{x}))e_{f_j}(\mathbf{x}, \mathbf{x}+\Delta x\mathbf{e_i}) \\ & + e_g(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{x}+\Delta x\mathbf{e_i}))\sqrt{\sum_{j=1}^{m}\left(\frac{\Delta x}{|\Delta x|}\frac{\partial f_j}{\partial x_i}(\mathbf{x}) + e_{f_j}(\mathbf{x}, \mathbf{x}+\Delta x\mathbf{e_i})\right)^2} \end{aligned}$$

where e_g is the differentiable condition function for g. Then

$$\lim_{\Delta x\to 0} e^i_{g\circ\mathbf{f}}(\Delta x) = 0, \tag{12}$$

$$\begin{aligned} (13)\, g\circ\mathbf{f}(\mathbf{x}+\Delta x\mathbf{e_i}) - g\circ\mathbf{f}(\mathbf{x}) \ &= \ \sum_{j=1}^{m} \Delta y_j \frac{\partial g}{\partial y_j}(\mathbf{f}(\mathbf{x})) + \|\mathbf{\Delta y}\| e_g(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{x}+\Delta x\mathbf{e_i})) \\ &= \ \Delta x \sum_{j=1}^{m} \frac{\partial g}{\partial y_j}(\mathbf{f}(\mathbf{x}))\frac{\partial f_j}{\partial x_i}(\mathbf{x}) + |\Delta x| e^i_{g\circ\mathbf{f}}(\Delta x). \end{aligned}$$

(12) and (13) show that $\sum_{j=1}^{m}(\partial g/\partial y_j)(\partial f_j/\partial x_i)$ is the first derivative of $g\circ\mathbf{f}$ along x_i, and this completes the proof. □

3.1.3 Riemann integration

In this subsection, we define Riemann integrability and study some conditions to integrate continuous functions within $\mathsf{RCA_0}$.

Definition 3.7 (Riemann integral: [29] Lemma IV.2.6)**.** The following definition is made in $\mathsf{RCA_0}$. Let f be a continuous function from $[a, b]$ to $\mathbb{R}$. Then, define the *Riemann integral* $\int_a^b f(x)\,dx$ as

$$\int_a^b f(x)\,dx = \lim_{|\Delta|\to 0} S^{\Delta}_{[a,b]}(f)$$

if this limit exists. Here, Δ is a partition of $[a, b]$, *i.e.* $\Delta = \{a = x_0 \le \xi_1 \le x_1 \le \cdots \le \xi_n \le x_n = b\}$, $S^{\Delta}_{[a,b]}(f)$ is defined as

$$S^{\Delta}_{[a,b]}(f) = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1})$$

and $|\Delta| = \max\{x_k - x_{k-1} \,|\, 1 \le k \le n\}$.

To integrate continuous functions effectively, we introduce a modulus of integrability.

Definition 3.8 (modulus of integrability)**.** The following definition is made in RCA_0. Let f be a continuous function from $[a,b]$ to $\mathbb{R}$. A *modulus of integrability* on $[a,b]$ for f is a function h from $\mathbb{N}$ to $\mathbb{N}$ such that for any $n \in \mathbb{N}$ and for any partitions Δ_1, Δ_2 of $[a,b]$,

$$|\Delta_1| < \frac{2^{-h(n)}}{b-a} \wedge |\Delta_2| < \frac{2^{-h(n)}}{b-a} \to |S^{\Delta_1}_{[a,b]}(f) - S^{\Delta_2}_{[a,b]}(f)| < 2^{-n+1}.$$

Within RCA_0, we can easily show that if h is a modulus of integrability for f, then h is a modulus of integrability for $|f|$.

Lemma 3.21. *The following is provable in* RCA_0*. Let f be a continuous function from $[a,b]$ to $\mathbb{R}$, and let h be a modulus of uniform continuity for f. Then, h is a modulus of integrability on $[a,b]$ for f.*

Proof. We reason within RCA_0. Let Δ_1, Δ_2 be partitions of $[a,b]$ such that $|\Delta_1| < 2^{-h(n)}/(b-a)$ and $|\Delta_2| < 2^{-h(n)}/(b-a)$. Take a common refinement $\bar{\Delta}$ of Δ_1 and Δ_2. Then,

$$|S^{\Delta_1}_{[a,b]}(f) - S^{\bar{\Delta}}_{[a,b]}(f)| < 2^{-n}$$

and

$$|S^{\Delta_2}_{[a,b]}(f) - S^{\bar{\Delta}}_{[a,b]}(f)| < 2^{-n}.$$

Thus,

$$|S^{\Delta_1}_{[a,b]}(f) - S^{\Delta_2}_{[a,b]}(f)| < 2^{-n+1}.$$

□

The next lemma show that if f has a modulus of integrability on $[a,b]$, then we can integrate f effectively.

Lemma 3.22. *The following is provable in* RCA_0*. Let f be a continuous function from $[a,b]$ to $\mathbb{R}$, let $c,d \in [a,b]$, and let h be a modulus of integrability for f. Take a natural number $K > b-a$ and define $t(n)$ as $t(n) = \min\{k \mid 1/k < 2^{-h(n)-2}/K\}$. Define S_n as*

$$S_n(f;c,d) = \sum_{i=1}^{t(n)} \frac{d-c}{t(n)} f\left(\frac{i}{t(n)}(d-c)+c\right).$$

Then, $\int_c^d f(x)\,dx$ exists and

$$\int_c^d f(x)\,dx = \lim_{n\to\infty} S_n(f;c,d).$$

Moreover, for all $n \in \mathbb{N}$,

$$\left| \int_c^d f(x)\, dx - S_n(f) \right| < 2^{-n}.$$

Proof. Obvious from the definition of a modulus of integrability. □

We call $S_n(f; c, d)$ in above lemma n-th approximation of $\int_c^d f(x)\, dx$. We often say that a continuous function f is *effectively integrable* on $[a, b]$ if f has a modulus of integrability on $[a, b]$.

Lemma 3.23 (indefinite integral). *The following is provable in* RCA_0. *Let f be a continuous function from $[a, b]$ to $\mathbb{R}$, let $c \in [a, b]$, and let h be a modulus of integrability for f. Then, there exists a continuous function g from $[a, b]$ to $\mathbb{R}$ such that*

$$g(x) = \int_a^x f(x)\, dx.$$

Proof. We reason within RCA_0. We construct a code G for the desired continuous function g. Let $\varphi(p, r, q, s)$ be a Σ^0_1 formula which expresses the following:

$$\exists m_1, m_2 \in \mathbb{N}\ |S_{m_1}(f; c, p) - q| + S_{m_2}(|f|; p_1, p_2) + 2^{-m_1} + 2^{-m_2} < s$$

where S_m is m-th approximation, $p_1 = \max\{a, p - r\}$ and $p_2 = \min\{p + r, b\}$. As in the proof of Theorem 3.16, define $G \subseteq \mathbb{N} \times \mathbb{Q} \times \mathbb{Q}^+ \times \mathbb{Q} \times \mathbb{Q}^+$ as $(p, r)G(q, s) \leftrightarrow \varphi(p, r, q, s)$. Then, $(p, r)G(q, s)$ if and only if

$$\forall x \in [a, b]\ |x - a| < r \rightarrow \left| \int_c^x f(x)dx - q \right| < s.$$

Hence, we can easily show that G satisfies the conditions for a code for a continuous function and g is the desired continuous function. □

The next theorem shows that the integrability of continuous functions requires WKL_0.

Theorem 3.24. *The following assertions are pairwise equivalent over* RCA_0.

1. WKL_0.
2. *Every continuous function on $[a, b]$ is Riemann integrable.*
3. *Every continuous function on $[a, b]$ has a modulus of integrability.*
4. *A sequential version of 3 as in Theorem 3.5.5.*

Proof. By Theorem 3.5, Lemmas 3.21 and 3.22, $1 \to 4 \to 3 \to 2$ holds. See [29, Theorem IV.2.7] for $2 \to 1$. □

To integrate bounded function, we only need $\mathsf{WWKL_0}$.

Theorem 3.25. *The following assertions are pairwise equivalent over* $\mathsf{RCA_0}$.

1. $\mathsf{WWKL_0}$.
2. *Every bounded continuous function on* $[a, b]$ *is Riemann integrable.*
3. *Every bounded continuous function on* $[a, b]$ *has a modulus of integrability.*
4. *A sequential version of 3 as in Theorem 3.5.5.*

Proof. We first show $1 \to 3$. We reason within $\mathsf{WWKL_0}$. Let f be a bounded continuous function on $[a, b]$. Without loss of generality, we can assume $[a, b] = [0, 1]$ and $f(x) \in [-1, 1]$ for all $x \in [0, 1]$. Let $F \subseteq \mathbb{N} \times \mathbb{Q} \times \mathbb{Q}^+ \times \mathbb{Q} \times \mathbb{Q}^+$ be a code for f. Define Σ^0_1 formula $\varphi(n, a, r)$ as

$$\varphi(n, a, r) \leftrightarrow a \in \mathbb{Q} \wedge r \in \mathbb{Q}^+ \wedge \exists b \in \mathbb{Q}\, \exists s \in \mathbb{Q}^+\, (a, 2r)F(b, s) \wedge s < 2^{-n-2}.$$

By Δ^0_1 comprehension, take a sequence $\{(a_{nk}, r_{nk})\}_{k\in\mathbb{N},n\in\mathbb{N}}$ such that

$$\forall n \forall a \forall r(\varphi(n, a, r) \leftrightarrow \exists k(a, r) = (a_{nk}, r_{nk})).$$

Note that $[0, 1] \subseteq \bigcup_{k\in\mathbb{N}} B(a_{nk}, r_{nk})$ for all $n \in \mathbb{N}$. Thus, by Theorem 2.8, there exists a double sequence of finite sequences of open intervals $\langle \{(c_{nij}, d_{nij})\}_{j\le l_{ni}} \mid n \in \mathbb{N}, i \in \mathbb{N}\rangle$ such that for any $n \in \mathbb{N}$,

$$[0, 1] \subseteq \bigcup_{k<i} B(a_{nk}, r_{nk}) \cup \bigcup_{j\le l_{ni}} (c_{nij}, d_{nij}),$$
$$\lim_{i\to\infty} \sum_{j\le l_{ni}} |d_{nij} - c_{nij}| = 0.$$

Take a sequence $\{\hat{i}_n\}_{n\in\mathbb{N}}$ such that $\sum_{j\le l_{n\hat{i}_n}} |d_{n\hat{i}_n j} - c_{n\hat{i}_n j}| < 2^{-n-2}$ and define $\hat{l}_n := l_{n\hat{i}_n}$.

Define a function h as

$$h(n) := \min\{q \in \mathbb{N} \mid 2^{-q} < \min\{r_{nk} \mid k \le \hat{l}_n\}\}.$$

We show that this h is a modulus of integrability for f. Let Δ_1, Δ_2 be partitions of $[0,1]$ such that $|\Delta_1| < 2^{-h(n)}, |\Delta_2| < 2^{-h(n)}$. Let $\Delta = \{0 = x_0 < x_1 < \cdots < x_{N+1} = 1\}$ be a common refinement of Δ_1, Δ_2, and let $\delta_m := [x_m, x_{m+1}]$. To show

$$|S^{\Delta_1}_{[0,1]}(f) - S^{\Delta_2}_{[0,1]}(f)| < 2^{-n+1},$$

we only need to show that for any $\{\xi_m\}_{m\leq N}, \{\xi'_m\}_{m\leq N}$ such that $\xi_m, \xi'_m \in \delta_m$,

$$\sum_{m=0}^{N} |f(\xi_m) - f(\xi'_m)|(x_{m+1} - x_m) < 2^{-n}.$$

Define $I \subseteq \{0, \cdots N\}$ as

$$I =: \{m \leq N \mid \delta_m \cap \bigcup_{k<\hat{i}_n} B(a_{nk}, r_{nk}) = \emptyset\}.$$

Then,

$$\begin{aligned} &[0,1] \subseteq \bigcup_{k<\hat{i}_n} B(a_{nk}, r_{nk}) \bigcup_{m\in I} \delta_m, \\ &\sum_{m\in I}(x_{m+1} - x_m) \leq \sum_{j\leq \hat{l}_n} |d_{n\hat{i}_n j} - c_{n\hat{i}_n j}|. \end{aligned}$$

If $m \in I$, then, by definition of h, $\xi_m, \xi'_m \in \delta_m \subseteq B(a_{nk}, 2r_{nk})$. Thus, $|f(\xi_m) - f(\xi'_m)| < 2^{-n-1}$ for all $m \in I$. Therefore,

$$\begin{aligned} & \sum_{m=0}^{N} |f(\xi_m) - f(\xi'_m)|(x_{m+1} - x_m) \\ \leq & \sum_{m\in I} 2(x_{m+1} - x_m) + \sum_{m\notin I} 2^{-n-1}(x_{m+1} - x_m) \\ \leq & \; 2 \sum_{j\leq l_{n\hat{i}_n}} |d_{n\hat{i}_n j} - c_{n\hat{i}_n j}| + 2^{-n-1} \\ < & \; 2^{-n}. \end{aligned}$$

This completes the proof of $1 \to 3$.

We can show $1 \to 4$ similarly. The implications $4 \to 3$ and $3 \to 2$ are trivial.

To show $2 \to 1$, we define the following notation. For a tree $T \subseteq 2^{<\mathbb{N}}$, define a set $S_T \subseteq 2^{<\mathbb{N}}$ and $\lambda^T_n \in \mathbb{N}$ as

$$\begin{aligned} S_T &:= \{\sigma \in 2^{<\mathbb{N}} \mid \sigma \notin T \wedge \forall \tau \subseteq \sigma(\tau \neq \sigma \to \tau \in T)\}; \\ \lambda^T_n &:= |\{\sigma \in T \mid \mathrm{lh}(\sigma) = n\}|. \end{aligned}$$

For a finite sequence $\sigma \in 2^{<\mathbb{N}}$, define $a_\sigma, b_\sigma \in \mathbb{Q}$ as

$$a_\sigma := \sum_{i<\mathrm{lh}(\sigma)} \frac{\sigma(i)}{2^{i+1}};$$
$$b_\sigma := a_\sigma + \frac{1}{2^{\mathrm{lh}(\sigma)}}.$$

Thus, if $\sigma, \tau \in S_T$, then, $b_\tau \leq a_\sigma$ or $b_\sigma \leq a_\tau$. Note that a tree T has a path if and only if $[0,1] \not\subseteq \bigcup_{\sigma \in S_T} [a_\sigma, b_\sigma]$.

Now, we show $\neg 1 \to \neg 2$. We reason within $\mathsf{RCA_0}$. Assume $\neg$WWKL. Then, there exist $q > 0$ and a tree T which has no path such that

$$\forall n \in \mathbb{N}\ \frac{\lambda_n^T}{2^n} > q.$$

Since $[0,1] \subseteq \bigcup_{\sigma \in S_T} [a_\sigma, b_\sigma]$, we can define a continuous function f from $[0,1]$ to $[0,1]$ as

$$f(x) := \begin{cases} \frac{x-a_\sigma}{c_\sigma - a_\sigma} & x \in [a_\sigma, c_\sigma] \wedge \sigma \in S_T, \\ \frac{b_\sigma - x}{b_\sigma - c_\sigma} & x \in [c_\sigma, b_\sigma] \wedge \sigma \in S_T \end{cases}$$

where $c_\sigma := (b_\sigma + a_\sigma)/2$.

We show that this f is not Riemann integrable. Define partitions Δ_k of $[0,1]$ as

$$\Delta_k := \left\{0 \leq \frac{1}{2^k} \leq \frac{2}{2^k} \leq \cdots \leq \frac{2^k - 1}{2^k} \leq 1\right\} = \{[a_\eta, b_\eta] \mid \eta \in 2^{<\mathbb{N}} \wedge \mathrm{lh}(\eta) = k\}.$$

Note that we can easily take $M_\sigma := \max\{f(x) \mid x \in [a_\sigma, b_\sigma]\}$ and $m_\sigma := \min\{f(x) \mid x \in [a_\sigma, b_\sigma]\}$. We show that for any $k \in \mathbb{N}$,

$$\sum_{\eta \in 2^{<\mathbb{N}} \wedge \mathrm{lh}(\eta) = k} (M_\eta - m_\eta) 2^{-k} > q.$$

If $\eta \in T$, then, there exists $\sigma \in S_T$ such that $\sigma \supseteq \eta$, thus, $[a_\eta, b_\eta] \supseteq [a_\sigma, b_\sigma]$. Therefore, $\eta \in T$ implies $M_\eta - m_\eta = 1$. Hence, for any $k \in \mathbb{N}$,

$$\begin{aligned} & \sum_{\eta \in 2^{<\mathbb{N}} \wedge \mathrm{lh}(\eta) = k} (M_\eta - m_\eta) 2^{-k} \\ \geq & \sum_{\eta \in T \wedge \mathrm{lh}(\eta) = k} 2^{-k} \\ \geq & \ \lambda_n^T 2^{-n} \\ > & \ q. \end{aligned}$$

This completes the proof of $2 \to 1$. $\square$

3.2 Series

In this section, we construct some C^∞-functions by series in RCA_0. Especially, we construct power series, which are elementary examples of analytic functions. We also prove the termwise differentiation and integration theorems. They are very important in this thesis.

The next theorem is the core of this section.

Theorem 3.26. *Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers whose series $\sum_{n=0}^{\infty}\alpha_n$ is convergent. Then the following assertions are provable in RCA_0.*

1. *If a sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ which satisfies $|a_n|\le\alpha_n$ for all $n\in\mathbb{N}$, then the series $\sum_{n=0}^{\infty}a_n$ is convergent.*

2. *([29, Lemma II.6.5]) Let U be an open subset of $\mathbb{R}^l$, and let $\{f_n\}_{n\in\mathbb{N}}$ be a (code for a) sequence of continuous functions from U to $\mathbb{R}$ which satisfies the following:*
$$\forall \mathbf{x}\in U\ \forall n\in\mathbb{N}\ |f_n(\mathbf{x})|\le\alpha_n.$$
Then there exists a (code for a) continuous function f from U to $\mathbb{R}$ such that
$$\forall \mathbf{x}\in U\ f(\mathbf{x})=\sum_{n=0}^{\infty}f_n(\mathbf{x}).$$

Proof. The proof of 2 is in [29, Lemma II.6.5], and 1 is immediate from 2. □

Corollary 3.27. *The following assertions are provable in RCA_0.*

1. *Absolutely convergent series are convergent.*

2. *Let $\sum_{n=0}^{\infty}a_n$ be an absolutely convergent series, and let h is a bijective function from $\mathbb{N}$ to $\mathbb{N}$. Then $\sum_{n=0}^{\infty}a_{h(n)}$ is convergent and*
$$\sum_{n=0}^{\infty}a_{h(n)}=\sum_{n=0}^{\infty}a_n.$$

3. *If $\sum_{n=0}^{\infty}a_n$ and $\sum_{n=0}^{\infty}b_n$ are absolutely convergent series, then their Cauchy product $\sum_{n=0}^{\infty}\sum_{k+l=n}a_kb_l$ is absolutely convergent and*
$$\sum_{n=0}^{\infty}\sum_{k+l=n}a_kb_l=\sum_{n=0}^{\infty}a_n\sum_{n=0}^{\infty}b_n.$$

Proof. Immediate from Theorem 3.26.1. □

Theorem 3.28. *The following is provable in* RCA_0. *Let* $\sum_{n=0}^{\infty} \alpha_n$ *be a nonnegative convergent series, and let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of continuous functions from* $[a, b]$ *to* $\mathbb{R}$ *which satisfies the following:*

$$\forall x \in [a, b]\ \forall n \in \mathbb{N}\ |f_n(x)| \leq \alpha_n.$$

By Theorem 3.26.2, we define $f = \sum_{n=0}^{\infty} f_n$. *Let* $\{h_n\}_{n\in\mathbb{N}}$ *be a sequence of functions such that each* h_n *is a modulus of uniform continuity for* f_n. *Then,* f *has a modulus of uniform continuity.*

Proof. We reason within RCA_0. Let h_n be a modulus of uniform continuity for f_n. Let $\sum_{n=0}^{\infty} \alpha_n = \alpha$. Define $k(n)$ as the following:

$$k(n) = \min\left\{ k \;\middle|\; \left(\alpha - \sum_{i=0}^{k} \alpha_i\right)_k < 2^{-n-2} - 2^{-k+1} \right\}.$$

(Here, $(\alpha)_k$ is the k-th approximation of α.) Then

$$\sum_{i=k(n)+1}^{\infty} \alpha_i < 2^{-n-2}. \tag{14}$$

Now define h as

$$h(n) = \max\{h_i(n + 2 + i) \mid i \leq k(n)\}.$$

Then for any $x, y \in [a, b]$, $|x - y| < 2^{-h(n)}$ implies

$$\forall i \leq k(n)\ |f_i(x) - f_i(y)| < 2^{-n-2-i}. \tag{15}$$

Hence by (14) and (15), for any $n \in \mathbb{N}$, if $|x - y| < 2^{-h(n)}$, then,

$$\begin{aligned}
|f(x) - f(y)| &\leq \sum_{i=k(n)+1}^{\infty} (|f_i(x)| + |f_i(y)|) + \sum_{i=0}^{k(n)} |f_i(x) - f_i(y)| \\
&\leq \sum_{i=k(n)+1}^{\infty} 2a_i + \sum_{i=0}^{k(n)} 2^{-n-2-i} \\
&< 2 \cdot 2^{-n-2} + 2^{-n-1} \\
&= 2^{-n}.
\end{aligned}$$

This means h is a modulus of uniform continuity for f. □

Next, we prove the termwise differentiation theorem, and construct a power series.

Theorem 3.29 (termwise differentiation)**.** *The following is provable in* RCA_0*. Let* U *be an open interval of* $\mathbb{R}$*, and let* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *be nonnegative convergent series. Let* $\{(f_n, f'_n)\}_{n\in\mathbb{N}}$ *be a sequence of* C^1*-functions from* U *to* $\mathbb{R}$ *which satisfies the following conditions:*

$$\forall x \in U \ \forall n \in \mathbb{N} \ |f_n(x)| \leq a_n,$$

$$\forall x \in U \ \forall n \in \mathbb{N} \ |f'_n(x)| \leq b_n.$$

Then there exists a C^1*-function* (f, f') *from* U *to* $\mathbb{R}$ *such that*

$$f = \sum_{n=0}^{\infty} f_n, \ f' = \sum_{n=0}^{\infty} f'_n.$$

Proof. We reason within RCA_0. By Theorem 3.26.2, there exist continuous functions f and f' from U to $\mathbb{R}$ which satisfy the following condition:

$$f = \sum_{n=0}^{\infty} f_n, \ f' = \sum_{n=0}^{\infty} f'_n.$$

Let e_{f_n} be a differentiable condition function for (f_n, f'_n). By Theorem 3.12, for any n and for any $x \neq y$ in U, there exists $z \in U$ such that

$$\frac{f_n(y) - f_n(x)}{y - x} = f'_n(z).$$

Hence, for any $n \in \mathbb{N}$, if $x \neq y$, then there exists z and

$$\begin{aligned} |e_{f_n}(x,y)| &= \left| \frac{f_n(y) - f_n(x)}{y - x} - f'_n(x) \right| \\ &= |f'_n(z) - f'_n(x)|. \end{aligned}$$

Then for any $n \in \mathbb{N}$,

$$|e_{f_n}(x,y)| \leq 2b_n. \tag{16}$$

(Clearly, (16) holds if $x = y$.) Then by Theorem 3.26.2, $e_f = \sum_{n=0}^{\infty} e_{f_n}$ exists and e_f satisfies

$$\begin{aligned} &\forall x \in U \ e_f(x,x) = 0; \\ &\forall x, y \in U \ f(y) - f(x) = (y - x)(f'(x) + e_f(x,y)), \end{aligned}$$

which means (f, f') is of C^1. This completes the proof. □

Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers, and let r be a positive real number. If the series $\sum_{n=0}^{\infty}|a_n|r^n$ is convergent, then for any $a\in\mathbb{R}$ and for any x such that $|x-a|<r$, $\sum_{n=0}^{\infty}a_n(x-a)^n$ is absolutely convergent and $|a_n(x-a)^n|<|a_n|r^n$. Define an open set U and a sequence of continuous functions $\{f_n\}_{n\in\mathbb{N}}$ from U to $\mathbb{R}$ as $U=\{x \mid |x-a|<r\}$ and $f_n(x)=a_n(x-a)^n$. Then by Theorem 3.26.2 there exists a continuous function f from U to $\mathbb{R}$ such that

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} f_n(x) \\ &= \sum_{n=0}^{\infty} a_n(x-a)^n. \end{aligned}$$

Corollary 3.30. *The following is provable in* RCA_0. *Let* $\{a_n\}_{n\in\mathbb{N}}$ *be a sequence of real numbers, and let* r *be a positive real number which satisfies* $\sum_{n=0}^{\infty}|a_n|r^n$ *is convergent. Let* $U=\{x \mid |x-a|<r\}$. *Define a continuous function* f *from* U *to* $\mathbb{R}$ *as* $f(x)=\sum_{n=0}^{\infty}a_n(x-a)^n$. *Then, there exists a sequence of continuous functions* $\{f^{(n)}\}_{n\in\mathbb{N}}$*,i.e.,* f *is a* C^∞*-function.*

Proof. We reason within RCA_0. Define a function p from $\mathbb{N}^2$ to $\mathbb{N}$ as

$$p(n,k)=\begin{cases} \frac{n!}{(n-k)!} & \text{if } k\le n, \\ 0 & \text{if } k>n. \end{cases}$$

(Here $0!=1$.) Define $a_n^{(k)}=p(n,k)a_n$ and $f_n^{(k)}(x)=a_n^{(k)}x^{n-k}$. Then each $f_n^{(k)}$ is a continuous function from U to $\mathbb{R}$, and for all $n,k\in\mathbb{N}$ a pair $(f_n^{(k)},f_n^{(k+1)})$ is of C^1. Moreover, we can easily show that $\sum_{n=0}^{\infty}|a_n^{(k)}|r^n$ is convergent because $\sum_{n=0}^{\infty}|a_n|r^n$ is convergent. Since the construction of functional series in Theorem 3.29 is effective, $f^{(k)}=\sum_{n=0}^{\infty}f_n^{(k)}$ exists and $(f^{(k)},f^{(k+1)})$ is of C^1 for all $k\in\mathbb{N}$. Then $\{f^{(n)}\}_{n\in\mathbb{N}}$ is of C^∞ and this completes the proof. □

The next lemma is very useful to construct continuous, C^r- or C^∞-functions.

Lemma 3.31. *The following is provable in* RCA_0. *Let* $\{U_n\}_{n\in\mathbb{N}}$ *be a (code for a) sequence of open subsets of* $\mathbb{R}^l$*, and let* $\{f_n\}_{n\in\mathbb{N}}$ *be a (code for a) sequence of continuous,* C^r*- or* C^∞*-functions. Here, each* f_n *is from* U_n *to* $\mathbb{R}$. *If* $\{f_n\}_{n\in\mathbb{N}}$ *satisfies*

$$\forall \mathbf{x}\in\mathbb{R}^l\ \forall i,j\in\mathbb{N}\ (\mathbf{x}\in U_i\cap U_j\rightarrow f_i(\mathbf{x})=f_j(\mathbf{x})),$$

then there exists a continuous, C^r*- or* C^∞*-function* f *from* $U=\bigcup_{n=0}^{\infty}U_n$ *to* $\mathbb{R}$ *such that*

$$\forall \mathbf{x}\in U\ \forall n\in\mathbb{N}\ (\mathbf{x}\in U_n\rightarrow f_n(\mathbf{x})=f(\mathbf{x})).$$

(We usually write $f = \bigcup_{n=0}^{\infty} f_n$.)

Proof. We reason within RCA_0. We first treat the case of continuous functions. Let F_n be a code for f_n. Let $\varphi(\mathbf{a}, r, b, s)$ be a Σ^0_1 formula which express there exists n such that $\exists (m', \mathbf{a}', r') \in U_n \ \|\mathbf{a} - \mathbf{a}'\| + r < r'$ and $(\mathbf{a}, r) F_n (b, s)$ holds. Write

$$\varphi(\mathbf{a}, r, b, s) \equiv \exists m \theta(m, \mathbf{a}, r, b, s)$$

where θ is Σ^0_0. By Δ^0_1 comprehension, define F as $(m, \mathbf{a}, r, b, s) \in F \leftrightarrow \theta(m, \mathbf{a}, r, b, s)$. Then clearly F is a code for a continuous (partial) function and f is from U to $\mathbb{R}$ which satisfies

$$\forall \mathbf{x} \in U \ \forall n \in \mathbb{N} \ (\mathbf{x} \in U_n \rightarrow f_n(\mathbf{x}) = f(\mathbf{x})).$$

This completes the proof of the continuous case.

To prove the C^r or C^∞ case, by Lemma 3.10, for any $\alpha = (a_1, \ldots, a_n)$,

$$\forall \mathbf{x} \in \mathbb{R}^l \ \forall i, j \in \mathbb{N} \left(\mathbf{x} \in U_i \cap U_j \rightarrow \frac{\partial^{a_1 + \cdots + a_n} f_i}{\partial^{a_1} x_1 \ldots \partial^{a_n} x_n}(\mathbf{x}) = \frac{\partial^{a_1 + \cdots + a_n} f_j}{\partial^{a_1} x_1 \ldots \partial^{a_n} x_n}(\mathbf{x}) \right).$$

Then we can use the continuousness case to construct

$$\frac{\partial^{a_1 + \cdots + a_n} f}{\partial^{a_1} x_1 \ldots \partial^{a_n} x_n} = \bigcup_{n=0}^{\infty} \frac{\partial^{a_1 + \cdots + a_n} f_n}{\partial^{a_1} x_1 \ldots \partial^{a_n} x_n}.$$

We can easily check the condition for C^r or C^∞.

□

Example 3.9. The following analytic functions can be constructed in RCA_0.

1. Define $s(n)$ as

$$s(n) = \begin{cases} (-1)^{\frac{n}{2}} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

and define $\{a_n\}_{n \in \mathbb{N}}$, $\{b_n\}_{n \in \mathbb{N}}$ and $\{c_n\}_{n \in \mathbb{N}}$ as

$$a_n = \frac{1}{n!}, \ b_n = \frac{s(n+3)}{n!}, \ c_n = \frac{s(n)}{n!}.$$

Then for any $m \in \mathbb{N}$, $\sum_{n=0}^{\infty} |a_n| m^n$, $\sum_{n=0}^{\infty} |b_n| m^n$ and $\sum_{n=0}^{\infty} |c_n| m^n$ are convergent. Define $U_m = \{x \mid |x| < m\}$. On U_m, define $\exp_m(x) = \sum_{n=0}^{\infty} a_n x^n$, $\sin_m(x) = \sum_{n=0}^{\infty} b_n x^n$ and $\cos_m(x) = \sum_{n=0}^{\infty} c_n x^n$. By Lemma 3.31, C^∞-functions $\exp = \bigcup_{m \in \mathbb{N}} \exp_m$, $\sin = \bigcup_{m \in \mathbb{N}} \sin_m$ and $\cos = \bigcup_{m \in \mathbb{N}} \cos_m$ from $\mathbb{R}$ to $\mathbb{R}$ can be constructed.

2. Define $\{d_n\}_{n\in\mathbb{N}}$ as $d_n = n\cdot(-1)^{n+1}$ and define $t(m)$ as $t(m) = 1-1/m$. Then for any $m \in \mathbb{N}$, $\sum_{n=0}^{\infty} |d_n| t(m)^n$ is convergent. Define $U_m = \{x \mid |x-1| < t(m)\}$. On U_m, define $\log_m(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$. By Lemma 3.31, a C^∞-function $\log = \bigcup_{m\in\mathbb{N}} \log_m$ from $(0,2)$ to $\mathbb{R}$ can be constructed.

Next, we show the termwise integration theorem.

Theorem 3.32 (termwise integration)**.** *The following is provable in* RCA_0*. Let* $\sum_{n=0}^{\infty} \alpha_n$ *be nonnegative convergent series, and let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of effectively integrable continuous functions from* $[a,b]$ *to* $\mathbb{R}$ *which satisfies the following:*[8]

$$\forall x \in [a,b]\ \forall n \in \mathbb{N}\ |f_n(x)| \le \alpha_n.$$

By Theorem 3.26.2, we define $f = \sum_{n=0}^{\infty} f_n$*. Then,* f *is effectively integrable and*

$$\int_a^b f(x)\,dx = \sum_{n=0}^{\infty} \int_a^b f_n(x)\,dx. \tag{17}$$

Proof. We reason within RCA_0. Let h_n be a modulus of integrability for f_n. Let $\sum_{n=0}^{\infty} \alpha_n = \alpha$. Define $k(n)$ as

$$k(n) = \min\left\{k \;\middle|\; \left(\alpha - \sum_{i=0}^{k} \alpha_i\right)_k < 2^{-n-2} - 2^{-k+1}\right\}.$$

and define h as

$$h(n) = \max\{h_i(n+2+i) \mid i \le k(n)\}.$$

Then, similarly to the proof of Theorem 3.28, we can check that h is a modulus of integrability for f. To prove (17), for any $n \in \mathbb{N}$,

$$\begin{aligned} & \left|\int_a^b f(x)\,dx - \sum_{i=0}^{k(n)} \int_a^b f_i(x)\,dx\right| \\ \le\ & \int_a^b \sum_{i=k(n)+1}^{\infty} \alpha_i(x)\,dx \\ \le\ & |b-a| 2^{-n-2}, \end{aligned}$$

which implies (17). This completes the proof. □

[8]We loosely say that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of effectively integrable continuous functions if there exists a sequence of functions $\{h_n\}_{n\in\mathbb{N}}$ such that each h_n is a modulus of integrability for f_n.

Note that we can effectively prove Theorems 3.26, 3.28, 3.29, 3.32, Corollary 3.30 and Lemma 3.31, thus, sequential versions of these theorems, corollary and lemma hold.

Finally, we argue about the commutativity of limits and integrals when a sequence of functions does not uniformly converge. The next theorem is a Riemann integral version of the monotone convergence theorem in [40].

Theorem 3.33. *The following assertions are pairwise equivalent over* RCA_0.

1. WWKL_0.
2. *If a uniformly bounded monotone sequence of effectively integrable continuous functions* $\{f_n\}_{n\in\mathbb{N}}$ *on* $[a, b]$ *converges to an effectively integrable continuous function* f *pointwise, then,*
$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$
(a sequence of continuous functions $\{f_n\}_{n\in\mathbb{N}}$ *on* $[a, b]$ *is said to be uniformly bounded if there exists a natural number* K *such that* $|f_n(x)| \le K$ *for all* $n \in \mathbb{N}$ *and for all* $x \in [a, b]$*.)*
3. *If a uniformly bounded monotone sequence of continuous functions* $\{f_n\}_{n\in\mathbb{N}}$ *on* $[a, b]$ *converges to a continuous function* f *pointwise, then, each of* f_n *and* f *is integrable and*
$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Proof. We reason within RCA_0. We show 1 $\to$ 3. Let $K \in \mathbb{Q}$ and let $\{f_n\}_{n\in\mathbb{N}}$ be a monotone sequence of continuous functions on $[a, b]$ such that $|f_n| < K$ and $\{f_n\}_{n\in\mathbb{N}}$ converges to a continuous function f pointwise. By Theorem 3.25, each of f_n and f is effectively integrable. Let $\varepsilon > 0$. We show that $|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx| < \varepsilon$ for some $n \in \mathbb{N}$. By Lemma 2.6, define open sets U_n as $U_n = \{x \mid |f(x) - f_n(x)| < \varepsilon/2(b-a) \vee x \notin [a, b]\}$. Since $\{f_n\}_{n\in\mathbb{N}}$ is monotone and converges to f pointwise, $\{U_n\}_{n\in\mathbb{N}}$ is a monotone open covering of $[a, b]$. By Theorem 2.8, there exist $n \in \mathbb{N}$ and $\{[c_i, d_i]\}_{i<l}$ such that $[a, b] \subseteq U_n \cup \bigcup_{i<l}[c_i, d_i]$ and $\sum_{i<l}(d_i - c_i) < \varepsilon/2K$. Then,
$$\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| < \frac{\varepsilon}{2(b-a)}(b-a) + K\sum_{i<l}(d_i - c_i) < \varepsilon.$$

3 $\to$ 2 is trivial. Modifying the proof of 2 $\to$ 1 of Theorem 3.25, we can prove 2 $\to$ 1 easily. □

3.3 Inverse function theorem and implicit function theorem

In this section, we prove the inverse function theorem and the implicit function theorem in RCA_0. A differentiable condition function again plays a key role.

Theorem 3.34 (inverse function theorem and implicit function theorem)**.** *The following assertions are provable in* RCA_0*.*

1. *Let U be an open subset of $\mathbb{R}^n$, and let $\mathbf{f}$ be a C^r- ($r \geq 1$) or C^∞-function from U to $\mathbb{R}^n$. Let $\mathbf{a}$ be a point of U such that $|\mathbf{f}'(\mathbf{a})| \neq 0$. Then, there exist open subsets of $\mathbb{R}^n$ V, W and a C^r- or C^∞-function $\mathbf{g}$ from W to V such that $\mathbf{a} \in V$, $\mathbf{f}(\mathbf{a}) \in W$ and*

$$\begin{aligned} \forall \mathbf{x} \in V \qquad & \mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{x}, \\ \forall \mathbf{y} \in W \qquad & \mathbf{f}(\mathbf{g}(\mathbf{y})) = \mathbf{y}. \end{aligned}$$

2. *Let U be an open subset of $\mathbb{R}^n \times \mathbb{R}^m$, and let $\mathbf{F}$ be a C^r- ($r \geq 1$) or C^∞-function from U to $\mathbb{R}^m$. Let $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2)$ be a point of U such that $\mathbf{F}(\mathbf{a}) = \mathbf{0}$ and $|\mathbf{F}_{x_{n+1} \cdots x_{n+m}}(\mathbf{a})| \neq 0$. Then there exist open subsets $V \subseteq \mathbb{R}^n$, $W \subseteq \mathbb{R}^m$ and a C^r- or C^∞-function $\mathbf{f}$ from W to V such that $\mathbf{a}_1 \in V$, $\mathbf{a}_2 \in W$ and*

$$\begin{aligned} & \mathbf{f}(\mathbf{a}_1) = \mathbf{a}_2, \\ & \forall \mathbf{v} \in V \ \mathbf{F}(\mathbf{v}, \mathbf{f}(\mathbf{v})) = \mathbf{0}. \end{aligned}$$

Here, $|\mathbf{f}'(\mathbf{a})|$ and $|\mathbf{F}_{x_{n+1} \cdots x_{n+m}}(\mathbf{a})|$ are the Jacobians, i.e.,

$$\begin{aligned} |\mathbf{f}'(\mathbf{a})| &= \det \left(\frac{\partial f_i}{\partial x_j} \right)_{1 \leq i,j \leq n}, \\ |\mathbf{F}_{x_{n+1} \cdots x_{n+m}}(\mathbf{a})| &= \det \left(\frac{\partial F_i}{\partial x_{n+j}} \right)_{1 \leq i,j \leq m}. \end{aligned}$$

Proof. We reason within RCA_0. We first prove 1. By Theorem 3.3 and Corollary 3.20, we may assume the following condition:

$$\begin{aligned} & \mathbf{a} = \mathbf{f}(\mathbf{a}) = \mathbf{0}; \\ & \forall \mathbf{x} \in U \ |\mathbf{f}'(\mathbf{x})| > 0; \\ & \frac{\partial f_i}{\partial x_j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \end{aligned}$$

Define $\mathbf{u}$ from U to $\mathbb{R}^n$ as $\mathbf{u}(\mathbf{x}) = \mathbf{x} - \mathbf{f}(\mathbf{x})$. Then $\mathbf{u}$ is of C^1, hence we can construct the differentiable condition function $\mathbf{e_u}$ for $\mathbf{u}$. Then for any $\mathbf{x}, \mathbf{y} \in U$,

$$\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x}) = \sum_{i=1}^{n} \mathbf{u}_{x_i}(\mathbf{x})(y_i - x_i) + \mathbf{e_u}(\mathbf{x}, \mathbf{y})\|\mathbf{y} - \mathbf{x}\|.$$

Hence

$$\|\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})\| \leq \left(\sum_{i=1}^{n} \|\mathbf{u}_{x_i}(\mathbf{x})\| + \|\mathbf{e_u}(\mathbf{x}, \mathbf{y})\| \right) \|\mathbf{y} - \mathbf{x}\|.$$

Here, $\sum_{i=1}^{n} \|\mathbf{u}_{x_i}(\mathbf{0})\| = 0$ and $\|\mathbf{e_u}(\mathbf{0}, \mathbf{0})\| = 0$. Hence by continuity of $\sum_{i=1}^{n} \|\mathbf{u}_{x_i}\|$ and $\|\mathbf{e_u}\|$, we can get $\varepsilon > 0$ such that

$$\begin{aligned} W_0 := \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{0}\| < \varepsilon\} \subseteq U, \\ \forall \mathbf{x} \in W_0 \ \sum_{i=1}^{n} \|\mathbf{u}_{x_i}(\mathbf{x})\| < \frac{1}{4}, \\ \forall \mathbf{x}, \mathbf{y} \in W_0 \ \|\mathbf{e}_{\mathbf{u}_{x_i}}(\mathbf{x}, \mathbf{y})\| < \frac{1}{4}. \end{aligned}$$

Then for any $\mathbf{x}, \mathbf{x} \in W_0$,

$$\|\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})\| \leq \frac{1}{2}\|\mathbf{y} - \mathbf{x}\|; \tag{18}$$

$$\begin{aligned} \|\mathbf{y} - \mathbf{x}\| &= \|\mathbf{u}(\mathbf{y}) + \mathbf{f}(\mathbf{y}) - \mathbf{u}(\mathbf{x}) - \mathbf{f}(\mathbf{x})\| \\ &\leq \|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})\| + \|\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})\| \\ &\leq \|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})\| + \frac{1}{2}\|\mathbf{y} - \mathbf{x}\|. \end{aligned} \tag{19}$$

Hence

$$\|\mathbf{y} - \mathbf{x}\| \leq 2\|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})\|. \tag{20}$$

Define open sets V and W as

$$\begin{aligned} W &:= \left\{ \mathbf{x} \in \mathbb{R}^n \,\middle|\, \|\mathbf{x} - \mathbf{0}\| < \frac{\varepsilon}{2} \right\}, \\ V &:= \mathbf{f}^{-1}(W) \cap W_0. \end{aligned}$$

Claim 3.34.1. *For any $\mathbf{y} \in W$, there exists a unique $\mathbf{x} \in V$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{y}$.*

To prove this claim, let $\mathbf{y}$ be a point of W. Define $\mathbf{v_y}$ from W_0 to $\mathbb{R}^n$ as $\mathbf{v_y}(\mathbf{x}) = \mathbf{y} + \mathbf{u}(\mathbf{x})$. Then by (18), for any $\mathbf{x}', \mathbf{x}'' \in W_0$,

$$\|\mathbf{v_y}(\mathbf{x}'') - \mathbf{v_y}(\mathbf{x}')\| \leq \frac{1}{2}\|\mathbf{x}'' - \mathbf{x}'\|. \tag{21}$$

Especially,

$$\|\mathbf{v}_{\mathbf{y}}(\mathbf{x}') - \mathbf{y}\| = \|\mathbf{v}_{\mathbf{y}}(\mathbf{x}') - \mathbf{v}_{\mathbf{y}}(\mathbf{0})\| \leq \frac{1}{2}\|\mathbf{x}'\| < \frac{\varepsilon}{2}. \tag{22}$$

On the other hand, $\mathbf{y} \in W$ implies $\|\mathbf{y}\| < \varepsilon/2$. Hence by (22),

$$\forall \mathbf{x}' \in W_0 \ \|\mathbf{v}_{\mathbf{y}}(\mathbf{x}')\| < \varepsilon. \tag{23}$$

(21) and (23) mean that $\mathbf{h}_{\mathbf{y}}$ is a contraction map from W_0 to W_0. Hence by contraction mapping theorem (particular version of [29] Theorem IV.8.3), there exists a unique $\mathbf{x} \in W_0$ such that $\mathbf{h}_{\mathbf{y}}(\mathbf{x}) = \mathbf{x}$. This implies $\mathbf{f}(\mathbf{x}) = \mathbf{y}$ and then $\mathbf{x} \in V$. This completes the proof of the claim.

Next, we construct a code for the local inverse function. Let F be a code for $\mathbf{f}$. Let $\varphi(\mathbf{b}, s, \mathbf{a}, r)$ be a Σ^0_1 formula which expresses that $\|\mathbf{b}\| + s < \varepsilon/2$ and there exists $(m', \mathbf{a}', r', \mathbf{b}', s') \in F$ such that $\|\mathbf{b} - \mathbf{b}'\| + s < s'$ and $\|\mathbf{a} - \mathbf{a}'\| + 4s' < r$. Write

$$\varphi(\mathbf{b}, s, \mathbf{a}, r) \equiv \exists m \theta(m, \mathbf{b}, s, \mathbf{a}, r)$$

where θ is Σ^0_0. By Δ^0_1 comprehension, define G as $(m, \mathbf{b}, s, \mathbf{a}, r) \in G \leftrightarrow \theta(m, \mathbf{b}, s, \mathbf{a}, r)$.

Claim 3.34.2. *G is a code for a continuous (partial) function (in the sense of remark 3.1).*

We can easily check that the condition 2 and 3 holds. We must check the condition 1. Assume $(\mathbf{b}, s)G(\mathbf{a}_1, r_1)$ and $(\mathbf{b}, s)G(\mathbf{a}_2, r_2)$. By the previous claim, we can take a unique $\mathbf{a}_0 \in V$ such that $\mathbf{f}(\mathbf{a}_0) = \mathbf{b}$. By the definition of G, there exist $(\mathbf{a}'_i, r'_i, \mathbf{b}'_i, s'_i)$ $(i = 1, 2)$ such that $(\mathbf{a}'_i, r'_i)F(\mathbf{b}'_i, s'_i)$, $\|\mathbf{b} - \mathbf{b}'_i\| + s < s'_i$ and $\|\mathbf{a}_i - \mathbf{a}'_i\| + 4s'_i < r_i$ $(i = 1, 2)$. Then

$$\|\mathbf{f}(\mathbf{a}_0) - \mathbf{f}(\mathbf{a}'_i)\| = \|\mathbf{b} - \mathbf{f}(\mathbf{a}'_i)\| \leq \|\mathbf{b} - \mathbf{b}'_i\| + |\mathbf{b}'_i - \mathbf{f}(\mathbf{a}'_i)\| < 2s'_i.$$

Hence by (20),

$$\|\mathbf{a}_0 - \mathbf{a}'_i\| < 4s'_i.$$

This implies $\|\mathbf{a}_0 - \mathbf{a}_i\| < r_i$ $(i = 1, 2)$ and then $\|\mathbf{a}_1 - \mathbf{a}_2\| < r_1 + r_2$. This completes the proof of the claim.

Claim 3.34.3. *Let $\mathbf{g}$ be the continuous function coded by G. Then for any $\mathbf{y} \in W$, $\mathbf{y} \in \mathrm{dom}(\mathbf{g})$.*

For any $\mathbf{y} \in W$ and for any $\delta > 0$, we need to show that there exists $(\mathbf{b}, s, \mathbf{a}, r)$ such that $(\mathbf{b}, s)G(\mathbf{a}, r)$, $\|\mathbf{b} - \mathbf{y}\| < s$ and $r < \delta$. Take $\mathbf{x} \in V$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{y}$. Then there exists $(\mathbf{a}', r', \mathbf{b}', s')$ such that $(\mathbf{a}', r')F(\mathbf{b}', s')$, $\|\mathbf{a}' - \mathbf{x}\| < r'$ and $\|\mathbf{b}' - \mathbf{y}\| < s' < \delta/8$. Then, there exists n such that the following conditions hold:

$$\begin{aligned} \|\mathbf{y}_n - \mathbf{b}'\| + 2^{-n+1} &< s'; \\ \|\mathbf{y}_n\| + 2^{-n+1} &< \frac{\varepsilon}{2}. \end{aligned}$$

Here, $\mathbf{y}_n$ is an n-th approximation of $\mathbf{y}$. These conditions can be expressed by a Σ_1^0 formula, hence we can take $n = n_0$ which satisfies them. Define $(\mathbf{b}, s, \mathbf{a}, r)$ as

$$\begin{aligned} \mathbf{b} &:= \mathbf{y}_{n_0}; \\ s &:= 2^{-n_0+1}; \\ \mathbf{a} &:= \mathbf{a}'; \\ r &:= 5s'. \end{aligned}$$

Then $\|\mathbf{a} - \mathbf{a}'\| + 4s' < r$, hence $(\mathbf{b}, s)G(\mathbf{a}, r)$. Also $\|\mathbf{b} - \mathbf{y}\| < s$ and $r < \delta$ hold. This completes the proof of the claim.

Claim 3.34.4. $\mathbf{g}$ *is the local inverse of* $\mathbf{f}$*, i.e.,*

$$\forall \mathbf{x} \in V \qquad \mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{x}, \tag{24}$$

$$\forall \mathbf{y} \in W \qquad \mathbf{f}(\mathbf{g}(\mathbf{y})) = \mathbf{y}. \tag{25}$$

We first show (24). Let $\mathbf{x} \in V$ and $\mathbf{y} = \mathbf{f}(\mathbf{x})$. To prove $\mathbf{x} = \mathbf{g}(\mathbf{y})$, we need to show that $(\mathbf{b}, s)G(\mathbf{a}, r)$ and $\|\mathbf{y} - \mathbf{b}\| < s$ imply $\|\mathbf{x} - \mathbf{a}\| < r$. Assume $(\mathbf{b}, s)G(\mathbf{a}, r)$ and $\|\mathbf{y} - \mathbf{b}\| < s$. Then by the definition of G, there exists $(\mathbf{a}', r', \mathbf{b}', s')$ such that $(\mathbf{a}', r')F(\mathbf{b}', s')$, $\|\mathbf{b} - \mathbf{b}'\| + s < s'$ and $\|\mathbf{a} - \mathbf{a}'\| + 4s' < r$. Then

$$\begin{aligned} \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}')\| &= \|\mathbf{y} - \mathbf{f}(\mathbf{a}')\| \\ &\leq \|\mathbf{y} - \mathbf{b}\| + \|\mathbf{b} - \mathbf{b}'\| + \|\mathbf{b}' - \mathbf{f}(\mathbf{a}')\| \\ &< 2s'. \end{aligned}$$

Hence by (20),

$$\|\mathbf{x} - \mathbf{a}'\| < 4s'.$$

Therefore

$$\|\mathbf{x} - \mathbf{a}\| \leq \|\mathbf{x} - \mathbf{a}'\| + \|\mathbf{a}' - \mathbf{a}\| < r.$$

(25) is immediate from (24) since $\mathbf{f}$ is bijective on V. This completes the proof of the claim.

Now we expand $\mathbf{g}$ into a C^r- or C^∞-function. We can easily define the derivatives of $\mathbf{g}$. For example, define the first derivatives as

$$\left(\frac{\partial g_i}{\partial x_j}\right)_{1\le i,j\le n} = \left(\left(\frac{\partial f_i}{\partial x_j}\right)_{1\le i,j\le n}\right)^{-1}.$$

It remains to prove that $\mathbf{g}$ and their derivatives surely satisfy the conditions for C^r or C^∞. Using the differentiable condition function for $\mathbf{f}$, this can be achieved as usual. This completes the proof of 1.

We can imitate the usual proof to show the implication $1 \to 2$. □

Mathematics in $\mathsf{RCA_0}$ is concerned with constructive mathematics. The constructive proof of implicit function theorem is in Bridges, Calude, Pavlov and Ştefănescu [5]. For details of constructive mathematics, see Bishop and Bridges [4].

3.4 Fourier expansion

In this section, we show some results of Reverse Mathematics for some basic theories of Fourier expansion.

Remark 3.35 (definition of π)**.** $\mathsf{RCA_0}$ proves that a continuous real function $\sin x$ is monotonous on $[2, 4]$ and there exists a unique $a \in [2, 4]$ such that $\sin a = 0$. Then, we can set $\pi := a$ in $\mathsf{RCA_0}$.

Using this definition, we can prove basic properties of π concerning $\sin x$ and $\cos x$ in $\mathsf{RCA_0}$.

Here, $\sin x$ and $\cos x$ are trigonometric functions defined in Example 3.9. Note that $\sin x$, $\cos x$ and their sum and product have a modulus of continuity, thus, they are effectively integrable. Note also that sum and product of effectively integrable functions are effectively integrable.

We write $f \in \mathcal{P}^{2\pi}$ if f is a continuous periodic function with period 2π. Let $\{a_k\}_{k\in\mathbb{N}}, \{b_k\}_{k\in\mathbb{N}}$ be real sequences. Then, define S_n as

$$S_n[\{a_k\}\{b_k\}](x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx).$$

If $f \in \mathcal{P}^{2\pi}$ is effectively integrable, then, define

$$S_n[f](x) = S_n[\{a_k\}\{b_k\}](x)$$

where a_k and b_k are Fourier coefficients, *i.e.*,

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx dx; \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx dx. \end{aligned}$$

The next lemma is an easy modification of Theorems 3.5 and 3.24

Lemma 3.36. *The following assertions are pairwise equivalent over* RCA_0.

1. WKL_0.
2. *Every periodic* C^1*-function has a modulus of uniform continuity on* $\mathbb{R}$.
3. *Every periodic* C^1*-function is Riemann integrable on any closed intervals.*
4. *Every periodic* C^1*-function has a modulus of integrability on any closed intervals.*

We first show some basic lemmas.

Lemma 3.37 (Bessel inequality)**.** *The following is provable in* RCA_0. *Let* $f \in \mathcal{P}^{2\pi}$ *be effectively integrable and let* a_k *and* b_k *be Fourier coefficients of* f. *Then,*

$$2\pi \sum_{i=0}^{n}(|a_i|^2 + |b_i|^2) \leq \int_{-\pi}^{\pi} f(x)^2 dx$$

for all $n \in \mathbb{N}$.

Proof. Straightforward imitation of the usual proof. □

Lemma 3.38. *The following is provable in* RCA_0. *Let* $f \in \mathcal{P}^{2\pi}$ *be effectively integrable. If*

$$\int_{-\pi}^{\pi} f(x)\cos kx dx = 0;$$
$$\int_{-\pi}^{\pi} f(x)\sin kx dx = 0$$

for all $n \in \mathbb{N}$, *then,* $f \equiv 0$.

Proof. Straightforward imitation of the usual proof. □

Lemma 3.39. *The following is provable in* RCA_0. *Let* $f \in \mathcal{P}^{2\pi}$ *be a* C^1*-function and let* f *and* f' *are effectively integrable. Then, the Fourier series* $S_n[f]$ *uniformly converges to* f.

Proof. We reason within RCA_0. Let a_k and b_k be Fourier coefficients of f, let a'_k and b'_k be Fourier coefficients of f' and let

$$K = \int_{-\pi}^{\pi} f'(x)^2 dx.$$

Then,

$$\begin{aligned} \pi a_k &= \int_{-\pi}^{\pi} f(x) \cos kx dx \\ &= \frac{1}{k} \int_{-\pi}^{\pi} f'(x) \sin kx dx = \frac{\pi b'_k}{k}. \\ \pi b_k &= \frac{\pi a'_k}{k}. \end{aligned}$$

By Schwarz's inequality and Lemma 3.37,

$$\begin{aligned} &\sum_{k=n}^{m} (|a_k| + |b_k|) \\ \leq &\sqrt{2 \sum_{k=n}^{m} \frac{1}{k^2}} \sqrt{\sum_{k=n}^{m} (|b'_k|^2 + |a'_k|^2)} \\ \leq &K \sqrt{2 \sum_{k=n}^{m} \frac{1}{k^2}}. \end{aligned}$$

Thus, by Theorem 2.4, $\sum_{k=0}^{\infty}(|a_k| + |b_k|)$ converges. Then, by Theorem 3.26, there exists $g \in \mathcal{P}^{2\pi}$ such that

$$\begin{aligned} g(x) &= \lim_{n \to \infty} S_n[f](x) \\ &= \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx). \end{aligned}$$

Let $\bar{f} = g - f$. Then, by Lemma 3.38, $\bar{f} \equiv 0$. This means $S_n[f]$ uniformly converges to f. □

The first theorem is concerned with the uniform convergence of Fourier series.

Theorem 3.40. *The following assertions are pairwise equivalent over* $\mathsf{RCA_0}$.

1. $\mathsf{WKL_0}$.
2. *If* $f \in \mathcal{P}^{2\pi}$ *is a* C^1*-function, then, there exist real sequences* $\{a_k\}_{k\in\mathbb{N}}$ *and* $\{b_k\}_{k\in\mathbb{N}}$ *such that* $S_n[\{a_k\}\{b_k\}]$ *uniformly converges to* f.

Proof. By Lemmas 3.36 and 3.39, $1 \to 2$ holds. For the converse, we assume 2. By Lemma 3.36, we only need to show that every periodic C^1-function (with period 2π) has a modulus of uniform continuity. Let $f \in \mathcal{P}^{2\pi}$ be a C^1-function. Then, there exist $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ such that $S_n[\{a_k\}\{b_k\}]$ uniformly converges to f. Since $\sin x$ and $\cos x$ has a modulus of uniform continuity, f has a modulus of uniform continuity by Theorem 3.28. This completes the proof. □

Next, we argue about L^2-convergence of Fourier series.

Definition 3.10 (L^2-convergence)**.** The following definition is made in $\mathsf{RCA_0}$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{P}^{2\pi}$. Then, we say that $\{f_n\}_{n\in\mathbb{N}}$ *L^2-converges* to f if for any $i \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that for any $m \geq k$ there exists a continuous function g such that g^2 is effectively continuous and

$$|f_m(x) - f(x)| \leq g(x),$$
$$\int_{-\pi}^{\pi} g(x)^2 dx < 2^{-i}.$$

Lemma 3.41. *The following is provable in* $\mathsf{RCA_0}$. *Let* $f \in \mathcal{P}^{2\pi}$ *be effectively integrable. Then, the Fourier series* $S_n[f]$ L^2*-converges to* f.

Proof. We reason within $\mathsf{RCA_0}$. Let $h : \mathbb{N} \to \mathbb{N}$ be a modulus of integrability for f on $[-\pi, \pi]$. We can construct a sequence of continuous functions on $[-\pi, \pi]$ $\{\tilde{f}_i\}_{i\in\mathbb{N}}$ (by means of piecewise parabolic functions) which satisfies the following:

- $\tilde{f}_i$ is of C^1 and $\tilde{f}_i$ and $\tilde{f}_i{}'$ are effectively integrable;
- $\tilde{f}_i(t_{ij}) = f(t_{ij})$;
- $\tilde{f}_i$ is monotone on $[t_j, t_{j+1}]$

where $t_{ij} = -\pi + 2\pi j/2^{h(i)}$ ($j \leq 2^{h(i)}$). Then, $\{\tilde{f}_i\}_{i\in\mathbb{N}}$ L^2-converges to f. By Lemma 3.37,

$$\begin{aligned} &\int_{-\pi}^{\pi}(f(x) - S_n[f](x))^2dx \\ \leq\ &\int_{-\pi}^{\pi}(f(x) - \tilde{f}_i(x))^2dx + \int_{-\pi}^{\pi}(\tilde{f}_i(x) - S_n[\tilde{f}_i](x))^2dx + \int_{-\pi}^{\pi}(S_n[\tilde{f}_i](x) - S_n[f](x))^2dx \\ \leq\ &2\int_{-\pi}^{\pi}(f(x) - \tilde{f}_i(x))^2dx + \int_{-\pi}^{\pi}(\tilde{f}_i(x) - S_n[\tilde{f}_i](x))^2dx. \end{aligned}$$

Since $\{\tilde{f}_i\}_{i\in\mathbb{N}}$ L^2-converges to f,

$$\lim_{i\to\infty}\int_{-\pi}^{\pi}(f(x) - \tilde{f}_i(x))^2dx = 0.$$

By Lemma 3.39, $\{S_n[\tilde{f}_i]\}_{n\in\mathbb{N}}$ uniformly converges to $\tilde{f}_i$. Thus,

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}(\tilde{f}_i(x) - S_n[\tilde{f}_i](x))^2dx = 0.$$

Hence,

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}(f(x) - S_n[f](x))^2dx = 0,$$

and this completes the proof. □

Theorem 3.42. *The following assertions are pairwise equivalent over* RCA_0.

1. WKL_0.
2. *If $f \in \mathcal{P}^{2\pi}$, then, there exist real sequences $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ such that $S_n[\{a_k\}\{b_k\}]$ L^2-converges to f.*

Proof. We reason within RCA_0. By Theorem 3.24 and Lemma 3.41, $1 \to 2$ holds. For the converse, we show $\neg 1 \to \neg 2$. Let $\neg\mathsf{WKL}_0$. Then, by Theorem 3.5, there exists an unbounded function $f \in \mathcal{P}^{2\pi}$. Thus, for any real sequences $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$, $|S_n[\{a_k\}\{b_k\}] - f|$ is unbounded. Therefore, if $|S_n[\{a_k\}\{b_k\}] - f| \leq g$, then, g^2 is not integrable, which means that $\neg 2$. This completes the proof of $2 \to 1$. □

Theorem 3.43. *The following assertions are pairwise equivalent over* RCA_0.

1. WWKL_0.
2. *If $f \in \mathcal{P}^{2\pi}$ and $|f| \leq K$ for some $K \in \mathbb{Q}$, then, there exist real sequences $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ such that $S_n[\{a_k\}\{b_k\}]$ L^2-converges to f.*

Proof. We reason within $\mathsf{RCA_0}$. By Theorem 3.25 and Lemma 3.41, $1 \to 2$ holds. For the converse, we show $\neg 1 \to \neg 2$. We use the notation S_T and λ_n^T defined in the proof of Theorem 3.25. Let $\neg\mathsf{WWKL_0}$. Then, there exist $q > 0$ and a tree T which has no path such that

$$\forall n \in \mathbb{N}\ \frac{\lambda_n^T}{2^n} > q.$$

For a finite sequence $\sigma \in 2^{<\mathbb{N}}$, define $a_\sigma, b_\sigma \in \mathbb{R}$ as

$$\begin{aligned} a_\sigma &:= -\pi + 2\pi \sum_{i<\mathrm{lh}(\sigma)} \frac{\sigma(i)}{2^{i+1}}; \\ b_\sigma &:= a_\sigma + \frac{2\pi}{2^{\mathrm{lh}(\sigma)}}. \end{aligned}$$

Since T has no path, $[-\pi, \pi] = \bigcup_{\sigma\in S_T} [a_\sigma, b_\sigma]$. Define a function $f \in \mathcal{P}^{2\pi}$ as

$$f(x) := \begin{cases} \frac{8(x-a_\sigma)}{b_\sigma - a_\sigma} & x \in [a_\sigma, c_\sigma] \wedge \sigma \in S_T, \\ \frac{-8\{x-(a_\sigma+b_\sigma)/2\}}{b_\sigma - a_\sigma} & x \in [c_\sigma, d_\sigma] \wedge \sigma \in S_T, \\ \frac{8(x-b_\sigma)}{b_\sigma - a_\sigma} & x \in [d_\sigma, b_\sigma] \wedge \sigma \in S_T \end{cases}$$

where $c_\sigma := (b_\sigma + 3a_\sigma)/4$ and $d_\sigma := (3b_\sigma + a_\sigma)/4$. Then, $f(c_\sigma) = 2$, $f(d_\sigma) = -2$ for any $\sigma \in S_T$ and $|f| \le 2$.

Now, we show that for any real sequences $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ and for any $n \in \mathbb{N}$, if $|S_n[\{a_k\}\{b_k\}] - f| \le g$, then, $\int_{-\pi}^{\pi} g(x)^2 dx > \pi q$. Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ be real sequences, let $n \in \mathbb{N}$ and let g be a continuous function such that g^2 is effectively integrable and $|S_n[\{a_k\}\{b_k\}] - f| \le g$. Take $M_0 \in \mathbb{Q}$ such that $M_0 \ge \max\{|a_0|, \dots, |a_n|, |b_0|, \dots, |b_n|\}$ and define $M := (n+1)^2 M_0$. Then, $|S_n[\{a_k\}\{b_k\}]'(x)| \le M$ for any $x \in [-\pi, \pi]$. Thus, if $\sigma \in S_T$ and $2\pi/2^{\mathrm{lh}(\sigma)} \le 1/M$, then, $|S_n[\{a_k\}\{b_k\}](x)| < 1$ for all $x \in [a_\sigma, b_\sigma]$ or $|S_n[\{a_k\}\{b_k\}](x)| > -1$ for all $x \in [a_\sigma, b_\sigma]$. Therefore, $g(c_\sigma) > 1$ or $g(d_\sigma) > 1$ for any $\sigma \in S_T$ such that $2\pi/2^{\mathrm{lh}(\sigma)} \le 1/M$. Let h be a modulus of integrability for g^2 on $[-\pi, \pi]$. Take $N \in \mathbb{N}$ such that $2\pi/2^N \le 1/M$ and $2^{-N} < 2^{-h(i)}/2\pi$ where $i = \min\{j \in \mathbb{N} \mid 2^{-j+2} < \pi q\}$. As in the proof of Theorem 3.25, if $\eta \in T$ and $\mathrm{lh}(\eta) = N$, then, there exists $x \in [a_\eta, b_\eta]$ such that $g(x)^2 > 1$. Take $\langle \alpha_\eta \in [a_\eta, b_\eta] \mid \eta \in 2^{<\mathbb{N}} \wedge \mathrm{lh}(\eta) = N \rangle$ such that

$g(\alpha_\eta) > 1$ if $\eta \in T$. Then, as in the proof of Theorem 3.25,

$$\begin{aligned}\int_{-\pi}^{\pi} g(x)^2 dx &\geq \sum_{\eta \in 2^{<\mathbb{N}} \wedge \mathrm{lh}(\eta)=N} g(\alpha_\eta)^2 (b_\eta - a_\eta) - 2^{-i+2} \\ &\geq \sum_{\eta \in T \wedge \mathrm{lh}(\eta)=N} (b_\eta - a_\eta) - 2^{-i+2} \\ &\geq \frac{2\pi\lambda_N^T}{2^N} - \pi q \\ &> \pi q,\end{aligned}$$

which means that $\neg 2$. This completes the proof of $2 \to 1$. $\square$

Imitating the usual arguments for Fourier expansions in $\mathsf{RCA_0}$, we can show the following theorems.

Theorem 3.44 (Perseval equality)**.** *The following is provable in* $\mathsf{RCA_0}$*. Let* $f \in \mathcal{P}^{2\pi}$ *be effectively integrable, and let* a_k *and* b_k *be Fourier coefficients of* f*. Then,*

$$2\pi \sum_{i=0}^{\infty} (|a_i|^2 + |b_i|^2) = \int_{-\pi}^{\pi} f(x)^2 dx.$$

Theorem 3.45 (Riemann-Lebesgue lemma)**.** *The following is provable in* $\mathsf{RCA_0}$*. Let* f *be an effectively integrable continuous function on* $\mathbb{R}$*. Then, for any* $a, b \in \mathbb{R}$*,*

$$\lim_{n\to\infty} \int_a^b f(x) \cos nx dx = 0.$$

Theorem 3.46 (pointwise convergence)**.** *The following is provable in* $\mathsf{RCA_0}$*. Let* $f \in \mathcal{P}^{2\pi}$ *be bounded variation on* $[-\pi, \pi]$*, i.e., there exist monotone increasing functions* g_0, g_1 *such that* $f = g_0 - g_1$*. Then,* f *is effectively integrable and* $S_n[f]$ *pointwise converges to* f*.*

Theorem 3.47. *Let* $f_1, f_2 \in \mathcal{P}^{2\pi}$ *be effectively integrable and let* $x_0 \in \mathbb{R}$*. Let* $f_1 \equiv f_2$ *on some neighborhood of* x_0*. Then,* $S_n[f_1](x_0)$ *converges if and only if* $S_n[f_2](x_0)$ *converges. Moreover, if* $S_n[f_1](x_0)$ *converges, then,*

$$\lim_{n\to\infty} S_n[f_1](x_0) = \lim_{n\to\infty} S_n[f_2](x_0).$$

Finally, we argue about local approximation for continuous functions by trigonometric functions.

Theorem 3.48. *The following assertions are pairwise equivalent over* $\mathsf{RCA_0}$*.*

1. $\mathsf{WWKL_0}$.

2. *If f is a continuous function on $\mathbb{R}$ and $x_0 \in \mathbb{R}$, then, there exist real sequences $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ such that $S_n[\{a_k\}\{b_k\}]$ L^2-converges to f on a neighborhood of x_0.*

Proof. $1 \to 2$ is a straightforward direction from Theorems 3.43 and 3.47. For the converse, we show $\neg 1 \to \neg 2$. We reason within $\mathsf{RCA_0}$. By $\neg\mathsf{WWKL_0}$, define a continuous function f on $[-\pi, \pi]$ as in the proof of $2 \to 1$ of Theorem 3.43. Then, define continuous functions f_i on $[0, \pi/2^i]$ as $f_i(x) = 2^{-i} f(2^{i+1}x - \pi)$. Note that $|f_i| \leq 2^{-i+1}$. Thus, we can define a continuous function $\bar{f}$ on $[-\pi, \pi]$ as

$$\bar{f}(x) := \begin{cases} f_{i+1}(x + \frac{\pi}{2^i}) & x \in [\frac{-\pi}{2^i}, \frac{-\pi}{2^{i+1}}], \\ f_{i+1}(x - \frac{\pi}{2^{i+1}}) & x \in [\frac{\pi}{2^{i+1}}, \frac{\pi}{2^i}], \\ 0 & x = 0. \end{cases}$$

Let U be a neighborhood of 0. Then, there exists $i \in \mathbb{N}$ such that $[\frac{\pi}{2^{i+1}}, \frac{\pi}{2^i}] \subseteq U$. As in the proof of Theorem 3.43, there is no real sequences $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ such that $S_n[\{a_k\}\{b_k\}]$ L^2-converges to $\bar{f}$ on $[\frac{\pi}{2^{i+1}}, \frac{\pi}{2^i}]$, which means that $\neg 2$. This completes the proof of $2 \to 1$. $\square$

4 Complex analysis in second order arithmetic

In this chapter, we develop complex analysis related mainly to Cauchy's integral theorem within RCA_0. In RCA_0, Cauchy's integral theorem holds on a good neighborhood of each point, but Cauchy's integral theorem itself is equivalent to WKL_0 over RCA_0. For that reason, some 'local' properties of holomorphic functions are provable in RCA_0, but 'global' properties are not.

4.1 Complex differentiability and integrability

In this section, we prove basic properties of holomorphic functions on the complex plane within RCA_0. Most of the following results are easy modifications of those in Chapter 3.

We first define the complex numbers and holomorphic functions.

Definition 4.1 (the complex number system)**.** The following definitions are made in RCA_0. We identify a *complex number*, an element of $\mathbb{C}$, as an element of $\mathbb{R}^2$, and we define $+_{\mathbb{C}}$, $\cdot_{\mathbb{C}}$ and $|\cdot|_{\mathbb{C}}$ by:

$$\begin{aligned}(x_1, y_1) +_{\mathbb{C}} (x_2, y_2) &= (x_1 + x_2, y_1 + y_2);\\ (x_1, y_1) \cdot_{\mathbb{C}} (x_2, y_2) &= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1);\\ |(x, y)|_{\mathbb{C}} &= \|(x, y)\|_{\mathbb{R}^2} = \sqrt{x^2 + y^2}.\end{aligned}$$

We write $(0, 1) = i$ and $(x, y) = x + iy$ where $x, y \in \mathbb{R}$. We usually leave out the subscript $\mathbb{C}$. A continuous (partial) function from $\mathbb{C}$ to $\mathbb{C}$ is a continuous (partial) function from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Definition 4.2 (holomorphic functions)**.** The following definition is made in RCA_0. Let D be an open subset of $\mathbb{C}$, and let f , f' be continuous functions from D to $\mathbb{C}$. Then a pair (f, f') is said to be *holomorphic* if

$$\forall z \in D \ \lim_{w \to z} \frac{f(w) - f(z)}{w - z} = f'(z).$$

Informally, we write f for a holomorphic function (f, f').

Let f be a continuous function. We are safe to say that f is holomorphic when we can effectively find a (code for a) continuous function f' which is the derivative of f.

Similarly to C^1-functions, we give another expression of holomorphic functions without using limits.

Theorem 4.1. *The following is provable in* RCA_0. *Let* D *be an open subset of* $\mathbb{C}$, *and let* f *be a holomorphic function from* D *to* $\mathbb{C}$. *Then, there exists a (code for a) continuous function* e_f *from* $D \times D$ *to* $\mathbb{C}$ *such that*

$$\forall z \in D \; e_f(z,z) = 0; \tag{26}$$

$$\forall z_1 \in D, \forall z_2 \in D \; f(z_2) - f(z_1) = (z_2 - z_1)(f'(z_1) + e_f(z_1, z_2)). \tag{27}$$

Such an e_f *is called the differentiable condition function for* f.

Proof. Easy modification of Theorem 3.16. □

Clearly, if f, f' and e_f satisfy (26) and (27), then (f, f') is a holomorphic function. Using differentiable condition functions, we can easily prove in RCA_0 that sum, product, quotient and composite of holomorphic functions are holomorphic.

Let a be an element of $\mathbb{C}$ and let r be a positive real number. Then we define

$$\begin{aligned} B(a;r) &:= \{z \mid |z-a| < r\}, \\ \overline{B(a;r)} &:= \{z \mid |z-a| \le r\}. \end{aligned}$$

The next lemma is a complex version of Lemma 3.11.

Lemma 4.2. *The following is provable in* RCA_0. *Let* D *be an open subset of* $\mathbb{C}$, *and let* f *be a holomorphic function from* D *to* $\mathbb{C}$. *Let* $a \in D$ *and* $r, K > 0$ *be such that* $\overline{B(a;r)} \subseteq D$ *and for all* $z \in \overline{B(a;r)}$, $|f'(z)| \le K$. *Then for all* $z, w \in \overline{B(a;r)}$,

$$|f(w) - f(z)| \le 4K|w - z|.$$

Proof. We reason within RCA_0. Suppose $f = f_1 + if_2$, $z = x_0 + iy_0$ and $w = x_1 + iy_1$. Define C^1-functions g_1, g_2 from $\mathbb{R}$ to $\mathbb{R}$ as

$$g_j(t) = f_j(z + (w - z)t) \; (j = 1, 2).$$

Then

$$g_j'(t) = \frac{\partial f_j}{\partial x}(z + (w-z)t)(x_1 - x_0) + \frac{\partial f_j}{\partial y}(z + (w-z)t)(y_1 - y_0) \; (j = 1, 2).$$

Hence

$$|g_j'(t)| \le 2K|w - z| \; (j = 1, 2).$$

By Lemma 3.11,

$$|g_j(1) - g_j(0)| \le 2K|w - z|(1 - 0) \le 2K|w - z| \; (j = 1, 2).$$

Hence

$$
\begin{aligned}
|f(w) - f(z)| &\leq |f_1(w) - f_1(z)| + |f_2(w) - f_2(z)| \\
&= |g_1(1) - g_1(0)| + |g_2(1) - g_2(0)| \\
&\leq 4K|w - z|.
\end{aligned}
$$

This completes the proof. □

Next, we define analytic functions. We will show in Section 4.3 that a holomorphic function is an analytic function in RCA_0.

Definition 4.3 (analytic functions)**.** The following definition is made in RCA_0. Let D be an open subset of $\mathbb{C}$. An *analytic function* on D is defined to be a triple $(f, \{a_n, r_n\}_{n\in\mathbb{N}}, \{\alpha_{nk}\}_{n\in\mathbb{N},k\in\mathbb{N}})$, where f is a continuous function from D to $\mathbb{C}$, $a_n, \alpha_{nk} \in \mathbb{C}$ and $r_n \in \mathbb{R}^+$, satisfying the following conditions:

1. $\bigcup_{n\in\mathbb{N}} B(a_n; r_n) = D$;
2. $\forall z \in B(a_n; r_n)\ f(z) = \sum_{k\in\mathbb{N}} \alpha_{nk} z^k$ for all $n \in \mathbb{N}$.

Informally, we write f for an analytic function $(f, \{a_n, r_n\}_{n\in\mathbb{N}}, \{\alpha_{nk}\}_{n\in\mathbb{N},k\in\mathbb{N}})$.

As holomorphic functions, we are justified in saying that a continuous function f is analytic when we can effectively find $\{a_n, r_n\}_{n\in\mathbb{N}}$ and $\{\alpha_{nk}\}_{n\in\mathbb{N},k\in\mathbb{N}}$ such that $(f, \{a_n, r_n\}_{n\in\mathbb{N}}, \{\alpha_{nk}\}_{n\in\mathbb{N},k\in\mathbb{N}})$ is an analytic function.

Next, we present complex versions of Theorems 3.26.2, 3.29, Corollary3.30 and Lemma 3.31.

Theorem 4.3. *The following is provable in* RCA_0*. Let* $\{\alpha_n\}_{n\in\mathbb{N}}$ *be a (code for a) sequence of nonnegative real numbers whose series* $\sum_{n=0}^{\infty} \alpha_n$ *is convergent. Let* D *be an open subset of* $\mathbb{C}$*, and let* $\{f_n\}_{n\in\mathbb{N}}$ *be a (code for a) sequence of continuous functions from* D *to* $\mathbb{C}$ *such that*

$$\forall n \in \mathbb{N}\ \forall z \in D\ |f_n(z)| \leq \alpha_n.$$

Then there exists a (code for a) continuous function f *from* D *to* $\mathbb{C}$ *such that*

$$\forall z \in D\ f(z) = \sum_{n=0}^{\infty} f_n(z).$$

Theorem 4.4 (termwise differentiation). *The following is provable in* $\mathsf{RCA_0}$. *Let* D *be an open subset of* $\mathbb{C}$, *and let* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *be nonnegative convergent series. Let* $\{(f_n, f'_n)\}_{n\in\mathbb{N}}$ *be a sequence of holomorphic functions from* D *to* $\mathbb{C}$ *which satisfies the following:*

$$\forall n \in \mathbb{N}\ \forall z \in D\ |f_n(z)| \leq a_n,$$
$$\forall n \in \mathbb{N}\ \forall z \in D\ |f'_n(z)| \leq b_n.$$

Then, there exists a holomorphic function (f, f') *from* D *to* $\mathbb{C}$ *such that*

$$f = \sum_{n=0}^{\infty} f_n,\ f' = \sum_{n=0}^{\infty} f'_n.$$

Theorem 4.5. *The following is provable in* $\mathsf{RCA_0}$. *Let* $\{a_n\}_{n\in\mathbb{N}}$ *be a sequence of complex numbers, and let* r *be a positive real number such that* $\sum_{n=0}^{\infty} |a_n|r^n$ *is convergent. Let* a *be a complex number, and define an open set* $D \subseteq \mathbb{C}$ *as* $D = \{z \mid |z-a| < r\}$. *Define a continuous function* f *from* D *to* $\mathbb{C}$ *as a complex power series on* D, *i.e.,*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n.$$

Then, there exists a sequence of continuous functions $\{f^{(n)}\}_{n\in\mathbb{N}}$.

Lemma 4.6. *The following is provable in* $\mathsf{RCA_0}$. *Let* $\{D_n\}_{n\in\mathbb{N}}$ *be a (code for a) sequence of open subsets of* $\mathbb{C}$, *and let* $\{f_n\}_{n\in\mathbb{N}}$ *be a (code for a) sequence of continuous or holomorphic functions where each* f_n *is from* D_n *to* $\mathbb{C}$. *If* $\{f_n\}_{n\in\mathbb{N}}$ *satisfies*

$$\forall z \in \mathbb{C}\ \forall i, j \in \mathbb{N}\ (z \in D_i \cap D_j \to f_i(z) = f_j(z)),$$

then there exists a continuous or holomorphic function f *from* $D = \bigcup_{n=0}^{\infty} D_n$ *to* $\mathbb{C}$ *such that*

$$\forall n \in \mathbb{N}\ \forall z \in D\ (z \in D_n \to f_n(z) = f(z)).$$

Proofs of Theorems 4.3, 4.4, 4.5 and Lemma 4.6 are similar to those of Theorems 3.26.2, 3.29, Corollary 3.30 and Lemma 3.31 respectively.

We can also construct $\exp(z)$, $\sin(z)$ and $\cos(z)$ as analytic functions in the same way as in Example 3.9.1.

Let f be an analytic function. Then by Theorem 4.5 and Lemma 4.6, we can easily construct its n-th derivative $f^{(n)}$ in $\mathsf{RCA_0}$. Clearly, for each n, $f^{(n)}$ is holomorphic and analytic.

Next, we define line integral. Let a, b, c be elements of $\mathbb{C}$ and let r be a positive real number. Then we define the following:

$$\begin{aligned}
[a,b] &:= \{a + (b-a)x \,|\, 0 \le x \le 1\}, \\
\Delta abc &:= \{ax_1 + bx_2 + cx_3 \,|\, x_1 + x_2 + x_3 = 1, 0 \le x_1, x_2, x_3 \le 1\}, \\
S(a;r) &:= \{a + (x+iy) \,|\, -r \le x, y \le r\}, \\
\partial \Delta abc &:= [a,b] \cup [b,c] \cup [c,a], \\
\partial S(a;r) &:= [a + (-r - ir), a + (r - ir)] \cup [a + (r - ir), a + (r + ir)] \\
&\quad\ \cup [a + (r + ir), a + (-r + ir)] \cup [a + (-r + ir), a + (-r - ir)].
\end{aligned}$$

Definition 4.4 (line integral)**.** Let D be an open or closed subset of $\mathbb{C}$, and let f be a continuous function from D to $\mathbb{C}$. Then the following definitions are made in RCA_0.

1. Let γ be a continuous function from $[0,1]$ to D. Then, we define $\int_\gamma f(z)\,dz$, the *line integral* of f along γ, as

$$\int_\gamma f(z)\,dz = \lim_{|\Delta| \to 0} S_\gamma^\Delta(f)$$

if this limit exists. Here, Δ is a partition of $[0,1]$, *i.e.* $\Delta = \{0 = x_0 \le \xi_1 \le x_1 \le \cdots \le \xi_n \le x_n = 1\}$, $S_\gamma^\Delta(f) = \sum_{k=1}^n f(\gamma(\xi_k))(\gamma(x_k) - \gamma(x_{k-1}))$ and $|\Delta| = \max\{x_k - x_{k-1} \,|\, 1 \le k \le n\}$.

2. If $[a,b] \subseteq D$, we define $\gamma(t) = a + (b-a)t$ and define $\int_{[a,b]} f(z)\,dz$ as

$$\int_{[a,b]} f(z)\,dz = \int_\gamma f(z)\,dz.$$

3. If $\partial \Delta abc \subseteq D$ and $\partial S(a;r) \subseteq D$, we define $\int_{\partial \Delta abc} f(z)\,dz$ and $\int_{\partial S(a;r)} f(z)\,dz$, respectively as

$$\begin{aligned}
\int_{\partial \Delta abc} f(z)\,dz &= \int_{[a,b]} f(z)\,dz + \int_{[b,c]} f(z)\,dz + \int_{[c,a]} f(z)\,dz, \\
\int_{\partial S(a;r)} f(z)\,dz &= \int_{[a+(-r-ir),a+(r-ir)]} f(z)\,dz + \int_{[a+(r-ir),a+(r+ir)]} f(z)\,dz \\
&\quad + \int_{[a+(r+ir),a+(-r+ir)]} f(z)\,dz + \int_{[a+(-r+ir),a+(-r-ir)]} f(z)\,dz.
\end{aligned}$$

Let f be a continuous function from $D \subseteq \mathbb{C}$ to $\mathbb{C}$, and let $[a,b] \subseteq D$. A *modulus of integrability* along $[a,b]$ for f is a function $h_{[a,b]}$ from $\mathbb{N}$ to $\mathbb{N}$ such that for any

$n \in \mathbb{N}$ and for any partitions Δ_1, Δ_2 of $[0,1] \subseteq \mathbb{R}$, if $|\Delta_1|, |\Delta_2| < 2^{-h_{[a,b]}(n)}$ then $|S^{\Delta_1}_{[a,b]}(f) - S^{\Delta_2}_{[a,b]}(f)| < 2^{-n+1}$. We say that f is *effectively integrable* on D when for every $[a,b] \subseteq D$, we can find a modulus of integrability along $[a,b]$.

We can show that if f has a modulus of uniform continuity on D, then f is effectively integrable on D as in the proof of Lemma 3.21. Let f be an effectively integrable continuous function on D. Let $h_{[a,b]}$ be a modulus of integrability on for f and let K be a rational number such that $|f(z)| < K$ for all $z \in [a,b]$. As in Lemma 3.22, define n-approximation $S_n(f;[a,b])$ of $\int_{[a,b]} f(z)\,dz$ as

$$S_n(f;[a,b]) = \sum_{i=1}^{s(n)} \frac{b-a}{s(n)} f\left(\frac{i}{s(n)}(b-a)+a\right)$$

where $s(n) = \min\{k \mid 1/k < 2^{-h_{[a,b]}(n)-2}/K\}$. Then,

$$\left|\int_{[a,b]} f(x)\,dx - S_n(f;[a,b])\right| < 2^{-n}.$$

The next theorem is a line integral version of Theorem 3.32.

Theorem 4.7 (termwise integration). *The following is provable in* RCA_0. *Let* $\sum_{n=0}^{\infty} a_n$ *be a convergent series of nonnegative real numbers, and let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of effectively integrable continuous functions from* $D \subseteq \mathbb{C}$ *to* $\mathbb{C}$ *which satisfies the following:*

$$\forall n \in \mathbb{N}\ \forall z \in D\ |f_n(z)| \leq a_n.$$

Then, $f = \sum_{n=0}^{\infty} f_n$ *is effectively integrable and for all* $[a,b] \subseteq D$*,*

$$\int_{[a,b]} f(z)\,dz = \sum_{n=0}^{\infty} \int_{[a,b]} f_n(z)\,dz.$$

Proof. Similar to the proof of Theorem 3.32. □

The next lemma is some basic properties of line integral.

Lemma 4.8. *The following assertions are provable in* RCA_0.

1. *For any* $a,b,c \in \mathbb{C}$, $\int_{\partial\Delta abc} 1\,dz$ *and* $\int_{\partial\Delta abc} z\,dz$ *exist and*

$$\int_{\partial\Delta abc} 1\,dz = 0, \int_{\partial\Delta abc} z\,dz = 0.$$

2. *Let D be an open or closed subset of $\mathbb{C}$, and let f be a continuous function from D to $\mathbb{C}$. If $[a,b] \subseteq D$ and there exists a modulus of uniform continuity on $[a,b]$ for f, then $\int_{[a,b]} f(z)\,dz$ and $\int_{[b,a]} f(z)\,dz$ exist and*

$$\int_{[a,b]} f(z)\,dz + \int_{[b,a]} f(z)\,dz = 0.$$

Proof. Obvious. □

Note that we can effectively prove Theorems 4.1, 4.3, 4.4, 4.5, 4.7 and Lemma 4.6, and thus, sequential versions of these theorems and lemma hold as in Chapter 3.

4.2 Cauchy's integral theorem

To prove Cauchy's integral theorem in $\mathsf{RCA_0}$, we need to integrate holomorphic (hence analytic) functions effectively. Actually, we can prove Cauchy's integral theorem for power series in $\mathsf{RCA_0}$ since power series are always effectively integrable. However, we cannot prove Cauchy's integral theorem in $\mathsf{RCA_0}$ because there might exist a holomorphic (analytic) function which cannot be integrated.

The next theorem is an $\mathsf{RCA_0}$ version of Cauchy's integral theorem.

Theorem 4.9. *The following is provable in $\mathsf{RCA_0}$. Let D be an open subset of $\mathbb{C}$, and let f be a holomorphic function from D to $\mathbb{C}$. If f is effectively integrable on D then, for any $a, b, c \in D$ such that $\Delta abc \subseteq D$,*

$$\int_{\partial\Delta abc} f(z)\,dz = 0.$$

Proof. We reason within $\mathsf{RCA_0}$. We imitate the usual proof of Cauchy's integral theorem (c.f. [1, page 109]) for triangles (or rectangles) in $\mathsf{RCA_0}$. The existence of $\int_{\partial\Delta abc} f(z)\,dz$ is given by the modulus of uniform continuity on D. Define L as the length of $\partial\Delta abc$, *i.e.*, $L = |b-a| + |c-b| + |a-c|$. Assume

$$\left|\int_{\partial\Delta abc} f(z)\,dz\right| > 0,$$

then there exists a positive rational number q such that

$$\left|\int_{\partial\Delta abc} f(z)\,dz\right| > q > 0. \tag{28}$$

Now by primitive recursion, we construct a sequence $\{(a_n, b_n, c_n)\}_{n\in\mathbb{N}}$ such that

$$\begin{aligned}
(a_0, b_0, c_0) &= (a, b, c);\\
\Delta a_{n+1}b_{n+1}c_{n+1} &\subseteq \Delta a_n b_n c_n;\\
|b_{n+1} - a_{n+1}| &= \frac{|b_n - a_n|}{2};\\
|c_{n+1} - b_{n+1}| &= \frac{|c_n - b_n|}{2};\\
|a_{n+1} - c_{n+1}| &= \frac{|a_n - c_n|}{2};\\
\left|\int_{\partial\Delta a_n b_n c_n} f(z)\,dz\right| &> \frac{q}{4^n}.
\end{aligned}$$

Let (a_n, b_n, c_n) be already defined. Define $\{(a^i_{n+1}, b^i_{n+1}, c^i_{n+1})\}_{1\le i\le 4}$ as $a^1_{n+1} = a_n$, $b^1_{n+1} = a^2_{n+1} = c^4_{n+1} = (a_n + b_n)/2$, $b^2_{n+1} = b_n$, $c^2_{n+1} = b^3_{n+1} = a^4_{n+1} = (b_n + c_n)/2$, $c^3_{n+1} = c_n$ and $c^1_{n+1} = a^3_{n+1} = b^4_{n+1} = (c_n + a_n)/2$. Then

$$\begin{aligned}
&\sum_{i=1}^{4}\left|\int_{\partial\Delta a^i_{n+1}b^i_{n+1}c^i_{n+1}} f(z)\,dz\right|\\
\geq\ &\left|\sum_{i=1}^{4}\int_{\partial\Delta a^i_{n+1}b^i_{n+1}c^i_{n+1}} f(z)\,dz\right|\\
=\ &\left|\int_{\partial\Delta a_n b_n c_n} f(z)\,dz\right|\\
>\ &\frac{q}{4^n}.
\end{aligned}$$

Hence,

$$\exists k\in\mathbb{N}\ \left|S_k(f;\partial\Delta a^i_{n+1}b^i_{n+1}c^i_{n+1})\right| > \frac{q}{4^{n+1}} + \frac{3}{2^k} \tag{29}$$

for some $1 \le i \le 4$ where

$$S_k(f;\partial\Delta a^i_{n+1}b^i_{n+1}c^i_{n+1}) = S_k(f;[a^i_{n+1}, b^i_{n+1}]) + S_k(f;[b^i_{n+1}, c^i_{n+1}]) + S_k(f;[c^i_{n+1}, a^i_{n+1}]).$$

Since (29) is expressed by a Σ^0_1 formula, a function H exists such that $H(a_n, b_n, c_n) = (a^i_{n+1}, b^i_{n+1}, c^i_{n+1})$ where $(a^i_{n+1}, b^i_{n+1}, c^i_{n+1})$ satisfies (29). Then, by primitive recursion, define $\{(a_n, b_n, c_n)\}_{n\in\mathbb{N}}$ as $(a_0, b_0, c_0) = (a, b, c)$ and $(a_{n+1}, b_{n+1}, c_{n+1}) = H((a_n, b_n, c_n))$.

Clearly, $|a_{n+1} - a_n| \le L/2^n$ holds for all $n \in \mathbb{N}$. Hence by theorem 2.4, there exists $z_0 = \lim_{n\to\infty} a_n$. Then clearly, for all $n \in \mathbb{N}$, $z_0 \in \Delta a_n b_n c_n$. Let e_f be the differentiable condition function for f and let E_f be a code for e_f. Let $\varphi(k)$ be a Σ^0_1 formula which express there exists $(m', a', r', b', s') \in E_f$ such that $\|(z_0, z_0) -$

$\mathbf{a}'\| + 2^{-k} < r'$ and $|b'| + s' < q/2L^2$. Then $e_f(z_0, z_0) = 0$ implies $\exists p \varphi(p)$. Hence there exists $k_0 \in \mathbb{N}$ such that $\varphi(k_0)$ holds. Take n_0 such that $L2^{-n_0-1} < 2^{-k_0}$, then $\Delta a_{n_0} b_{n_0} c_{n_0} \subseteq B(z_0; 2^{-k_0})$. Define a continuous function g from D to $\mathbb{C}$ as

$$\begin{aligned} g(z) &= f(z_0) + (z - z_0) f'(z_0) \\ &= f(z) - e_f(z_0, z)(z - z_0). \end{aligned}$$

Then for any $z \in \Delta a_{n_0} b_{n_0} c_{n_0} \subseteq B(z_0; 2^{-k_0})$,

$$\begin{aligned} |f(z) - g(z)| &= |e_f(z_0, z)(z - z_0)| \\ &\leq \frac{q}{2L^2} \cdot \frac{L}{2^{n_0}} \\ &= \frac{q}{L2^{n_0+1}}. \end{aligned}$$

Hence

$$\begin{aligned} \left| \int_{\partial \Delta a_{n_0} b_{n_0} c_{n_0}} f(z) - g(z) \, dz \right| &\leq \frac{L}{2^{n_0}} \cdot \frac{q}{L2^{n_0+1}} \\ &= \frac{q}{2 \cdot 4^{n_0}}. \end{aligned} \tag{30}$$

On the other hand, by Lemma 4.8,

$$\int_{\partial \Delta a_{n_0} b_{n_0} c_{n_0}} g(z) \, dz = 0.$$

Hence

$$\begin{aligned} & \left| \int_{\partial \Delta a_{n_0} b_{n_0} c_{n_0}} f(z) - g(z) \, dz \right| \\ = & \left| \int_{\partial \Delta a_{n_0} b_{n_0} c_{n_0}} f(z) \, dz \right| \\ > & \frac{q}{4^{n_0}}, \end{aligned}$$

which contradicts (30). This completes the proof. □

By this theorem, if f is a effectively integrable holomorphic function on $D = B(a; r)$, then, there exists a continuous function F on D such that

$$F(x) = \int_{[a,x]} f(x) \, dx$$

as in Lemma 3.23. Clearly, (F, f) forms a holomorphic function.

We show that Cauchy's integral theorem is equivalent to WKL_0 over RCA_0.

Theorem 4.10. *The following assertions are pairwise equivalent over* RCA_0.

1. *If f is a holomorphic function on an open set $D \subseteq \mathbb{C}$, for any $\Delta abc \subseteq D$, f is bounded on Δabc.*

2. *If f is a holomorphic function on an open set $D \subseteq \mathbb{C}$, for any $\Delta abc \subseteq D$, there exists a modulus of uniform continuity on Δabc for f.*

3. *Cauchy's integral theorem for triangles: if f is a holomorphic function on an open set $D \subseteq \mathbb{C}$, then for any $\Delta abc \subseteq D$, $\int_{\partial\Delta abc} f(z)\,dz$ exists and*
$$\int_{\partial\Delta abc} f(z)\,dz = 0.$$

4. WKL_0.

Proof. We reason within RCA_0. The implications $4 \to 1$ and $4 \to 2$ are immediate from Theorem 3.5. The implication $2 \to 3$ is immediate from Theorem 4.9.

To prove $1 \to 4$ and $3 \to 4$, we show that $\neg\mathsf{WKL}_0$ implies $\neg 1$ and $\neg 3$. Let T be an infinite tree with no path, *i.e.*, T is an infinite subset of $2^{<\mathbb{N}}$ and for any function h from $\mathbb{N}$ to $\{0,1\}$, there exists $n \in \mathbb{N}$ such that $h[n] \notin T$. By Δ^0_1 comprehension, define $\tilde{T}$ as $\sigma \in \tilde{T} \leftrightarrow \sigma \in 2^{<\mathbb{N}} \setminus T \wedge \forall k < \mathrm{lh}(\sigma)\ \sigma[k] \in T$. Let $\{\sigma_n\}_{n\in\mathbb{N}}$ be an enumeration of $\tilde{T}$ which satisfies for any $n \in \mathbb{N}$, $\mathrm{lh}(\sigma_n) \le \mathrm{lh}(\sigma_{n+1})$. Define $c_\sigma, d_\sigma \in [0,1]$ for each $\sigma \in 2^{<\mathbb{N}}$ as $c_{\langle\rangle} = 0$, $d_{\langle\rangle} = 1$, $c_{\sigma\frown\langle 0\rangle} = c_\sigma$, $d_{\sigma\frown\langle 1\rangle} = d_\sigma$ and $c_{\sigma\frown\langle 1\rangle} = d_{\sigma\frown\langle 0\rangle} = (c_\sigma + d_\sigma)/2$. Then define natural numbers s_n, real numbers a_n and open intervals I_n as follows:

$$\begin{aligned} s_n &= \mathrm{lh}(\sigma_n) + 1; \\ a_n &= \frac{c_{\sigma_n} + d_{\sigma_n}}{2}; \\ I_n &= (c_{\sigma_n}, d_{\sigma_n}) = (a_n - 2^{-s_n}, a_n + 2^{-s_n}). \end{aligned}$$

Note that for any $p, q \in \mathbb{N}$ $I_p \cap I_q = \emptyset$.

Next, by Δ^0_1 comprehension, define X as

$$\begin{aligned} X &= \{\,(\sigma,\tau) \in \tilde{T} \times \tilde{T} \mid d_\sigma = c_\tau\} \cup \{\,(\langle 2\rangle, \tau) \mid \tau \in \tilde{T} \wedge c_\tau = 0\} \\ &\quad \cup \{\,(\sigma, \langle 2\rangle) \mid \sigma \in \tilde{T} \wedge d_\sigma = 1\}. \end{aligned}$$

Let $\{(\hat{\sigma}_n, \hat{\tau}_n)\}_{n\in\mathbb{N}}$ be an enumeration of X. Then define natural numbers t_n, real numbers b_n and open intervals J_n as follows:

$$\begin{aligned} t_n &= \max\{\mathrm{lh}(\hat{\sigma}_n), \mathrm{lh}(\hat{\tau}_n)\} + 1; \\ b_n &= d_{\hat{\sigma}_n} = c_{\hat{\tau}_n}; \\ J_n &= (b_n - 2^{-t_n}, b_n + 2^{-t_n}). \end{aligned}$$

(Here, $d_{\langle 2 \rangle} = 0$, $c_{\langle 2 \rangle} = 1$ and $\mathrm{lh}(\langle 2 \rangle) = 1$.) Then for any $n \in \mathbb{N}$, there exist at most two k's such that $J_n \cap I_k \neq \emptyset$, and for any $n, k \in \mathbb{N}$, $a_k \notin J_n$. Moreover, $\{I_n\}_{n \in \mathbb{N}}, \{J_n\}_{n \in \mathbb{N}}$ cover $[0,1]$, *i.e.*,

$$\bigcup_{n=0}^{\infty} I_n \cup \bigcup_{n=0}^{\infty} J_n \supseteq [0,1].$$

Now, define an open cover of $\mathbb{C}$ as follows:

$$\begin{aligned} A_n &= \{x + iy \,|\, x \in I_n, y \in \mathbb{R}\}; \\ B_n &= \{x + iy \,|\, x \in J_n, y \in \mathbb{R}\}; \\ C &= \{x + iy \,|\, x < 0 \vee x > 1, y \in \mathbb{R}\}. \end{aligned}$$

Then

$$\bigcup_{n=0}^{\infty} A_n \cup \bigcup_{n=0}^{\infty} B_n \cup C = \mathbb{C}.$$

Define a sequence of complex numbers $\{\zeta_n\}_{n \in \mathbb{N}}$ as

$$\zeta_n = a_n + i \cdot 2^{-s_n - 2} \in A_n,$$

and consider each a_n as a complex number, *i.e.*, redefine a_n as

$$a_n = a_n + i \cdot 0 \in A_n.$$

Then for any $p, q \in \mathbb{N}$, $A_p \cap A_q = \emptyset$ and for any $n \in \mathbb{N}$, a_n and ζ_n are not in each B_k and C. Moreover, for any $n \in \mathbb{N}$,

$$\forall z \notin A_n \ |z - \zeta_n| \geq 2^{-s_n}. \tag{31}$$

Define a sequence of holomorphic functions $\{f_n\}_{n \in \mathbb{N}}$ as

$$\begin{aligned} f_n(z) &= \left(\frac{2^{-s_n}}{z - \zeta_n} \cdot \frac{1}{2i} \right)^{n + s_n} \cdot 2^{-s_n}; \\ f_n'(z) &= -\left(\frac{2^{-s_n}}{z - \zeta_n} \cdot \frac{1}{2i} \right)^{n + s_n + 1} \cdot 2i(n + s_n + 1). \end{aligned}$$

(Here, each f_n is from $\mathbb{C} \setminus \{\zeta_n\}$ to $\mathbb{C}$.) Then

$$f_n(a_n) = \left(\frac{2^{-s_n}}{-i \cdot 2^{-s_n-2}} \cdot \frac{1}{2i} \right)^{n+s_n} \cdot 2^{-s_n} = 2^n. \tag{32}$$

Define $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ as $\alpha_n = 2^{-n}$ and $\beta_n = (n+1)2^{-n-1}$. Then their series $\sum_{n=0}^{\infty} \alpha_n$ and $\sum_{n=0}^{\infty} \beta_n$ are convergent. By (31), for any $n \in \mathbb{N}$,

$$\begin{aligned}
&(33) \qquad \forall z \notin A_n \ |f_n(z)| &&= \left(\frac{2^{-s_n}}{|z-\zeta_n|} \cdot \frac{1}{2} \right)^{n+s_n} \cdot 2^{-s_n} \\
& &&\leq \left(\frac{1}{2} \right)^{n+s_n} \cdot 2^{-s_n} \\
& &&\leq \alpha_n, \\
&(34) \qquad \forall z \notin A_n \ |f_n'(z)| &&= \left(\frac{2^{-s_n}}{|z-\zeta_n|} \cdot \frac{1}{2} \right)^{n+s_n+1} \cdot 2(n+s_n+1) \\
& &&\leq \left(\frac{1}{2} \right)^{n+s_n+1} \cdot 2(n+s_n+1) \\
& &&\leq \beta_n.
\end{aligned}$$

Define $\{m_n\}_{n\in\mathbb{N}}$ as

$$m_n = \min\{k \,|\, t_n < s_k\}.$$

Then for any $n \in \mathbb{N}$

$$\forall k \geq m_n \qquad A_k \cap B_n = \emptyset, . \tag{35}$$

Let $k > n$. Then by (33) and (34), $|f_k(z)| \leq \alpha_k$ and $|f_k'(z)| \leq \beta_k$ for all $z \in A_n$. Hence by Theorem 4.4, for each $n \in \mathbb{N}$, we can construct holomorphic functions f_{A_n} from $A_n \setminus \{\zeta_n\}$ to $\mathbb{C}$ as

$$f_{A_n}(z) = \sum_{k=0}^{n} f_k(z) + \sum_{k=n+1}^{\infty} f_k(z).$$

Similarly, by (35), we can construct holomorphic functions f_{B_n} from B_n to $\mathbb{C}$ and a holomorphic function f_C from C to $\mathbb{C}$ as

$$\begin{aligned}
f_{B_n}(z) &= \sum_{k=0}^{m_n} f_k(z) + \sum_{k=m_n+1}^{\infty} f_k(z), \\
f_C(z) &= \sum_{k=0}^{\infty} f_k(z).
\end{aligned}$$

Then by Lemma 4.6, we can construct a holomorphic function $f = \bigcup_{n=0}^{\infty} f_{A_n} \cup \bigcup_{n=0}^{\infty} f_{B_n} \cup f_C$ from $\bigcup_{n=0}^{\infty}(A_n \setminus \{\zeta_n\}) \cup \bigcup_{n=0}^{\infty} B_n \cup C = \mathbb{C} \setminus \{\zeta_n\}_{n\in\mathbb{N}}$ to $\mathbb{C}$. Here, for any $n \in \mathbb{N}$,

$$\begin{aligned} f(a_n) &= f_n(a_n) + \sum_{k\neq n} f_k(a_n) \\ &= 2^n + \sum_{k\neq n} f_k(a_n), \\ \left|\sum_{k\neq n} f_k(a_n)\right| &\leq \sum_{k=0}^{\infty} \alpha_k \leq 2. \end{aligned}$$

Hence, the real part of $f(z)$ cannot be bounded in $[0,1]$, and then, $\int_{\partial\Delta(-i)01} f(z)\,dz$ cannot exist although $\Delta(-i)01 \subseteq \mathbb{C}\setminus\{\zeta_n\}_{n\in\mathbb{N}}$, which means that 1 and 3 are denied. This completes the proofs of $1 \to 4$ and $3 \to 4$. □

We can generalize the above version of Cauchy's integral theorem. Let us regard a C^1-function from $[0,1]$ to $\mathbb{R}^2$ as a C^1-function from $[0,1]$ to $\mathbb{C}$. (At end points, we consider one-side derivative.) A *piecewise* C^1*-Jordan curve* γ on an open set $D \subseteq \mathbb{C}$ is a finite sequence of one-to-one C^1-functions $\{\gamma_j\}_{1\leq j\leq n}$ from $[0,1]$ to $\mathbb{C}$ which have no common points except $\gamma_j(1) = \gamma_{j+1}(0)$, $\gamma_n(1) = \gamma_1(0)$. (We just write γ_j for C^1-function (γ_j, γ_j').) Given a continuous function from D to $\mathbb{C}$, we define a line integral along γ as

$$\int_\gamma f(z)\,dz = \sum_{j=1}^{n} \int_{\gamma_j} f(z)\,dz.$$

Remark 4.11. Let γ is a piecewise C^1-Jordan curve on $\mathbb{C}$. If $\gamma([0,1])$ is a closed set and its complement is divided into two arcwise connected open sets U_0 and U_1 such that U_0 is bounded, then U_0 is said to be the *interior* of γ. However, we cannot prove the existence of the interior of a piecewise C^1-Jordan curve. In fact, we show that the Jordan curve theorem is equivalent to WKL_0 over RCA_0 in Chapter 5. See also [25].

In WKL_0, we can find a broken-line approximation of a piecewise C^1-Jordan curve. Thus, in WKL_0, we can prove Cauchy's integral theorem for piecewise C^1-Jordan curves by Theorem 4.10.

Theorem 4.12. *The following assertions are pairwise equivalent over* RCA_0.

1. *Cauchy's integral theorem: if f is a holomorphic function on an open set $D \subseteq \mathbb{C}$, γ is a piecewise* C^1*-Jordan curve on D such that its interior exists and is included in D, then $\int_\gamma f(z)\,dz$ exists and*

$$\int_\gamma f(z)\,dz = 0.$$

2. $\mathsf{WKL_0}$.

4.3 Taylor's theorem

In this section, we show the Taylor theorem for holomorphic functions within $\mathsf{RCA_0}$, *i.e.*, we show that a holomorphic function can be expanded into a power series on some neighborhood of each point. This means that 'holomorphic functions are analytic' in $\mathsf{RCA_0}$. Holomorphic functions are uniformly continuous and effectively integrable on some neighborhood of each point. Thus, to show the Taylor theorem, we only need Cauchy's integral theorem for effectively integrable functions, which is provable in $\mathsf{RCA_0}$.

Let f be a continuous (partial) function from $\mathbb{C}$ to $\mathbb{C}$, and let $B(a;r) \subseteq \mathrm{dom}(f)$. Let h be a modulus of uniform continuity on $B(a;r)$ for f. Then we call a pair $(B(a;r), h)$ (or just $B(a;r)$) *uniformly continuous neighborhood* (*u.c.-neighborhood*) for f.

Lemma 4.13. *The following is provable in* $\mathsf{RCA_0}$*. Let D be an open subset of $\mathbb{C}$, and let f be a holomorphic function from D to $\mathbb{C}$. Given a positive real number K such that $|f'| \leq K$ on $B(a;r) \subseteq D$, then there exists a modulus of continuity for f on $B(a;r)$, i.e., $B(a;r)$ is a u.c.-neighborhood.*

Proof. We reason within $\mathsf{RCA_0}$. Without loss of generality, we may assume K is a positive rational. Define h from $\mathbb{N}$ to $\mathbb{N}$ by:

$$h(n) = \min\left\{ k \;\middle|\; 2^{-k} < \frac{2^{-n}}{4K} \right\}.$$

Then by Lemma 4.2, h is a modulus of uniform continuity for f on $B(a;r)$, which completes the proof. □

Let f be a holomorphic function from an open set D to $\mathbb{C}$. By the continuity of f', for each $z_0 \in D$ there exists a positive rational number r such that f' is bounded on

$B(z_0;r)$. Then $B(z_0;r)$ is a u.c.-neighborhood. By Theorem 4.9, Cauchy's integral theorem holds on $B(z_0;r)$. Roughly speaking, Cauchy's integral theorem holds locally in RCA_0. Using this, we can show that 'holomorphic functions are analytic' in RCA_0.

Lemma 4.14. *The following is provable in* RCA_0. *Let D be an open subset of $\mathbb{C}$, and let f be a holomorphic function from D to $\mathbb{C}$. Let $a_0 \in D$ and $r_0, K > 0$ be such that $B(a_0;r_0) \subseteq D$ and for all $z \in B(a_0;r_0)$, $|f'(z)| \leq K$. Take $z_0 \in B(a_0;r_0)$ and define a continuous function g_{z_0} from D to $\mathbb{C}$ as the following:*

$$\begin{aligned} g_{z_0}(z) &= e_f(z_0,z) + f'(z_0). \\ &= \begin{cases} \frac{f(z)-f(z_0)}{z-z_0} & \text{if } z \neq z_0, \\ f'(z_0) & \text{if } z = z_0. \end{cases} \end{aligned}$$

(Here e_f is the differentiable condition function for f.) Then, the following hold.

1. *If g_{z_0} is holomorphic on $D \setminus \{z_0\}$, then, there exists a modulus of uniform continuity on $B(a_0;r_0)$ for g_{z_0}.*

2. *For any $a, b, c \in B(a_0;r_0)$, $\int_{\partial\Delta abc} g_{z_0}(z)\,dz$ exist and*

$$\int_{\partial\Delta abc} g_{z_0}(z)\,dz = 0.$$

Proof. We reason within RCA_0. We first prove 1. Clearly g_{z_0} is holomorphic on $D \setminus \{z_0\}$. We only need to find a modulus of uniform continuity on $B(a_0;r_0)$ for g_{z_0} effectively. By Lemma 4.13, we can effectively find a modulus of uniform continuity h_0 on $B(a_0;r_0)$ for f. Let G_{z_0} be a code for g_{z_0} and let $\varphi(n,m)$ be a Σ^0_1 formula which expresses that there exists $(\hat{m},\hat{a},\hat{r},\hat{b},\hat{s}) \in G_{z_0}$ such that $|z_0 - \hat{a}| + 2^{-m} < \hat{r}$ and $|g_{z_0}(z_0) - \hat{b}| + \hat{s} < 2^{-n-2}$. Since $z_0 \in \mathrm{dom}(g_{z_0})$ implies $\forall n \exists m \varphi(n,m)$, we can find a function l from $\mathbb{N}$ to $\mathbb{N}$ such that $\forall n \varphi(n,l(n))$ holds. Thus for any $n \in \mathbb{N}$,

$$(36) \qquad \forall w_1, w_2 \in \overline{B(z_0; 2^{-l(n)-1})}$$
$$\begin{aligned} |g_{z_0}(w_1) - g_{z_0}(w_2)| &\leq |g_{z_0}(w_1) - g_{z_0}(z_0)| + |g_{z_0}(w_2) - g_{z_0}(z_0)| \\ &< 2^{-n-1}. \end{aligned}$$

Now, take $m_0 \in \mathbb{N}$ such that $|f(a_0)| + 4Kr_0 < 2^{m_0}$. By Lemma 4.2, for all $w \in B(a_0;r_0)$, we have $|f(w)| < 2^{m_0}$. Define a function h from $\mathbb{N}$ to $\mathbb{N}$ as

$$h(n) = \max\{h_0(l(n)+n+3), 2l(n)+m_0+5\}.$$

The proof will be completed if we show that h is a modulus of uniform continuity on $B(a_0; r_0)$ for g_{z_0}. To show this, we need to check the following three cases:

(i) $w_1, w_2 \in \overline{B(z_0; 2^{-l(n)-1})}$ and $|w_1 - w_2| < h(n)$;

(ii) $w_1, w_2 \in B(a_0; r_0) \setminus B(z_0; 2^{-l(n)-1})$ and $|w_1 - w_2| < h(n)$;

(iii) $w_1 \in \overline{B(z_0; 2^{-l(n)-1})}$, $w_2 \in B(a_0; r_0) \setminus B(z_0; 2^{-l(n)-1})$ and $|w_1 - w_2| < h(n)$.

If (w_1, w_2) satisfies (i), then by (36),

$$|g_{z_0}(w_1) - g_{z_0}(w_2)| < 2^{-n-1}.$$

If (w_1, w_2) satisfies (ii), then

$$\begin{aligned}
|g_{z_0}(w_1) - g_{z_0}(w_2)| &= \left| \frac{f(w_1) - f(z_0)}{w_1 - z_0} - \frac{f(w_2) - f(z_0)}{w_2 - z_0} \right| \\
&= \left| \frac{f(w_1) - f(w_2)}{w_2 - z_0} + \frac{(f(w_2) + f(z_0))(w_2 - w_1)}{(w_2 - z_0)(w_1 - z_0)} \right| \\
&\leq \frac{1}{2^{-l(n)-1}} |f(w_1) - f(w_2)| + \frac{2 \cdot 2^{m_0}}{2^{-l(n)-1} \cdot 2^{-l(n)-1}} |w_2 - w_1| \\
&< 2^{l(n)+1} \cdot 2^{-l(n)-n-3} + 2^{m_0+2l(n)+3} \cdot 2^{-m_0-2l(n)-n-5} \\
&= 2^{-n-1}.
\end{aligned}$$

Assume (w_1, w_2) satisfies (iii). Let w_3 be an intersection point of $[w_1, w_2]$ and $\{w \mid |w - z_0| = 2^{-l(n)-1}\}$. Then (w_1, w_3) satisfies (i) and (w_2, w_3) satisfies (ii). Hence

$$\begin{aligned}
|g_{z_0}(w_1) - g_{z_0}(w_2)| &\leq |g_{z_0}(w_1) - g_{z_0}(w_3)| + |g_{z_0}(w_2) - g_{z_0}(w_3)| \\
&< 2^{-n-1} + 2^{-n-1} \\
&= 2^{-n}.
\end{aligned}$$

Hence for any $n \in \mathbb{N}$, if $w_1, w_2 \in B(a_0; r_0)$ and $|w_1 - w_2| < h(n)$, then $|g_{z_0}(w_1) - g_{z_0}(w_2)| < 2^{-n}$. This completes the proof of 1.

By 1 and Theorem 4.9, we can prove 2 as usual (c.f. [1, page 111]). □

Since we can effectively prove Lemmas 4.13 and 4.14, sequential versions of these lemmas hold.

Now, we are ready to prove the main theorem of this section.

Theorem 4.15 (Taylor's theorem)**.** *The following is provable in* RCA_0. *Let* (f, f') *be a holomorphic function from an open set* $D \subseteq \mathbb{C}$ *to* $\mathbb{C}$, *then, there exist* $\{a_n, r_n\}_{n \in \mathbb{N}}$ *and* $\{\alpha_{nk}\}_{n \in \mathbb{N}, k \in \mathbb{N}}$ *such that* $(f, \{a_n, r_n\}_{n \in \mathbb{N}}, \{\alpha_{nk}\}_{n \in \mathbb{N}, k \in \mathbb{N}})$ *forms an analytic function from* D *to* $\mathbb{C}$.

We can prove this theorem effectively and thus a sequential version of this theorem also holds. This theorem shows that a holomorphic function is an analytic function. Particularly, the derivative of a holomorphic function is a holomorphic function.

Proof. We reason within RCA_0. We first decompose D into u.c.-neighborhoods. Let F' be a code for f', and let $(m_n, a_n, r_n, b_n, s_n)$ be an enumeration of all elements (m, a, r, b, s) of F' which satisfy $\exists(\hat{m}, \hat{a}, \hat{r}) \in D\ |a - \hat{a}| + r < \hat{r}$. Define $\tilde{r}_n = r_n/2$, $K_n = |f'(a_n)| + s_n$ and $M_n = 4K_n s_n |f(a_n)|$. Then by the definition of continuous functions and $D \subseteq \mathrm{dom}(f')$,

$$\forall z \in B(a_n; r_n)\ |f'(z)| \le K_n, \tag{37}$$

$$\bigcup_{n=0}^{\infty} B(a_n; \tilde{r}_n) = D. \tag{38}$$

By Lemma 4.13, we can find a modulus of uniform continuity on $B(a_n; r_n)$ for f for each n.

Next, we expand f into a power series on each $B(a_n; \tilde{r}_n)$. For any $n \in \mathbb{N}$ and for any $z \in B(a_n; \tilde{r}_n)$, define g_z as

$$\begin{aligned} g_z(w) &= e_f(z, w) + f'(z) \\ &= \begin{cases} \frac{f(w)-f(z)}{w-z} & \text{if } w \neq z, \\ f'(z) & \text{if } w = z. \end{cases} \end{aligned}$$

Here, e_f is the differentiable condition function for f. Then by Lemma 4.14,

$$\int_{\partial S(a_n; \frac{2}{3} r_n)} g_z(w)\, dw = 0.$$

This means

$$f(z) \int_{\partial S(a_n; \frac{2}{3} r_n)} \frac{dw}{w - z} = \int_{\partial S(a_n; \frac{2}{3} r_n)} \frac{f(w)}{w - z}\, dw \tag{39}$$

if these two integrals exist.

Claim 4.15.1. *For any $n \in \mathbb{N}$ and for any $z \in B(a_n; \tilde{r}_n)$ the following integrals exist and,*

$$\int_{\partial S(a_n; \frac{2}{3} r_n)} \frac{dw}{w - z} = \int_{\partial S(0;1)} \frac{dw}{w}. \tag{40}$$

Moreover,

$$4 < \left| \int_{\partial S(0;1)} \frac{dw}{w} \right| < 8. \tag{41}$$

We can find a modulus of uniform continuity on $\partial S(a_n; 2r_n/3)$ for $1/(w-z)$ and a modulus of uniform continuity on $\partial S(0;1)$ for $1/w$ easily. Hence the integrals in (40) exist. The equality (40) is obvious. The estimation (41) can be proved by an approximation which is produced by a modulus of integrability. This completes the proof of this claim.

Claim 4.15.2. *For any $n \in \mathbb{N}$ and for any $z \in B(a_n; \tilde{r}_n)$ the following integrals exist and,*

$$\begin{aligned} &\int_{\partial S(a_n;\frac{2}{3}r_n)} \frac{f(w)}{w-z}\,dw \\ &= \sum_{j=0}^{\infty} (z-a_n)^j \int_{\partial S(a_n;\frac{2}{3}r_n)} \frac{f(w)}{(w-a_n)^{j+1}}\,dw. \end{aligned} \tag{42}$$

To prove this claim, define a sequence of continuous functions $\{p_j\}_{j\in\mathbb{N}}$ as

$$p_j(w) = (z-a_n)^j \frac{f(w)}{(w-a_n)^{j+1}}.$$

For all $w \in \partial S(a_n; 2r_n/3)$,

$$|f(w)| \left| \frac{(z-a_n)^j}{(w-a_n)^{j+1}} \right| < M_n \frac{(\tilde{r}_n)^j}{(\frac{2}{3}r_n)^{j+1}} < \left(\frac{3}{4}\right)^j \cdot \frac{M_n}{2r_n}. \tag{43}$$

Hence by Lemma 4.3, $\sum_{j=0}^{\infty} p_j(w)$ is convergent and,

$$\sum_{j=0}^{\infty} p_j(w) = \frac{f(w)}{w-z}.$$

Using a modulus of uniform continuity for f, we can construct a modulus of uniform continuity on $\partial S(a_n; 2r_n/3)$ for each p_j. Then by Theorem 4.7, (42) holds. This completes the proof of this claim.

Define α_{nj} as

$$\alpha_{nj} = \frac{\int_{\partial S(a_n;\frac{2}{3}r_n)} \frac{f(w)}{(w-a_n)^{j+1}}\,dw}{\int_{\partial S(0;1)} \frac{dw}{w}}.$$

Then, by (41) and (43), for any $n \in \mathbb{N}$

$$\begin{aligned} \forall j \in \mathbb{N}\ \forall z \in B(a_n; \tilde{r}_n)\ \ |\alpha_{nj}(z-a_n)^j| &\le \frac{1}{4} \cdot \frac{8}{3} r_n \cdot \left(\frac{3}{4}\right)^j \cdot \frac{M_n}{2r_n} \\ &= \left(\frac{3}{4}\right)^j \cdot \frac{M_n}{3}. \end{aligned}$$

Hence by (39) and Lemma 4.3,

$$\forall z \in B(a_n; \tilde{r}_n)\ f(z) = \sum_{j=0}^{\infty} \alpha_{nj}(z - a_n)^j$$

for all $n \in \mathbb{N}$. This means $(f, \{a_n, \tilde{r}_n\}_{n\in\mathbb{N}}, \{\alpha_{nj}\}_{n\in\mathbb{N} j\in\mathbb{N}})$ is an analytic function, which completes the proof. $\square$

Let f be a continuous function from $[a, b] \subseteq \mathbb{R}$ to $\mathbb{C}$, we define $\int_a^b f(t)\, dt$ as

$$\int_a^b f(t)\, dt = \int_a^b f_1(t)\, dt + i \int_a^b f_2(t)\, dt$$

where, $f = f_1 + if_2$. The next corollaries are straightforward from the Taylor theorem.

Corollary 4.16 (mean value principle on some u.c.-neighborhood)**.** *The following is provable in* RCA_0*. Let D be an open subset of $\mathbb{C}$, f be a holomorphic function from D to $\mathbb{C}$ and $z_0 \in D$. Let $z_0 \in B(a; r)$ be a u.c.-neighborhood for f. For any positive real number $\tilde{r}$, if $\overline{B(z_0; \tilde{r})} \subseteq B(a; r)$, then the following integral exists and*

$$\int_0^{2\pi} f(z_0 + \tilde{r} \exp(i\theta))\, d\theta = 2\pi f(z_0).$$

Corollary 4.17 (maximal value principle on some u.c.-neighborhood)**.** *The following is provable in* RCA_0*. Let D be an open subset of $\mathbb{C}$, f is a holomorphic function from D to $\mathbb{C}$ and $z_0 \in D$. Let $z_0 \in B(a; r)$ be a u.c.-neighborhood for f. If $|f(z)|$ attains a maximal value at z_0, then f is constant on $B(z_0; r)$.*

4.4 Some other results

In this section, we show some other results for Reverse Mathematics for complex analysis.

Using differentiable condition functions, we can prove the basic theorem for the Cauchy-Riemann equation.

Theorem 4.18. *Let D be an open subset of $\mathbb{C}(= \mathbb{R}^2)$ and let $f = f_1 + if_2$ be a continuous function from D to $\mathbb{C}(= \mathbb{R}^2)$, i.e., f_1, f_2 are continuous functions from D to $\mathbb{R}$ such that $f(z) = f_1(z) + if_2(z)$. (If f is a continuous function from $\mathbb{R}^n$ to $\mathbb{R}^m$, then we can effectively find (codes for) continuous functions $f_1, \ldots, f_m$ from $\mathbb{R}^n$ to $\mathbb{R}$ such that $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$.) Then the following assertions are provable in* RCA_0*.*

1. *Given the continuous derivative of f, we can find continuous partial derivatives of f_1 and f_2 which satisfy the following Cauchy-Riemann equation:*

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y}, \ \frac{\partial f_2}{\partial x} = -\frac{\partial f_1}{\partial y}. \tag{44}$$

2. *Given continuous partial derivatives of f_1 and f_2 which satisfy (44), then $\partial f_1/\partial x + i\partial f_1/\partial y$ is the continuous derivative of f.*

Proof. We reason within RCA_0. We can easily prove 1. To prove 2, let $f = (f_1, f_2)$ be a C^1-function which satisfies the Cauchy-Riemann equation. Let e_{f_1} and e_{f_2} be differentiable condition functions for f_1 and f_2, and let $f' = \partial f_1/\partial x + i\partial f_1/\partial y$. Let Δ be a closed set such that $\Delta = \{(w, w) \mid w \in D\}$. Define a continuous function e_f^0 from $D \times D \setminus \Delta$ to $\mathbb{C}$ as

$$e_f^0(z, w) \ = \ \frac{w - z}{|w - z|}(e_{f_1}(z, w) + i(e_{f_2}(z, w)),$$

where e_{f_j} is a differentiable condition function for f_j. Then,

$$\forall z \in D \ \lim_{w \to z} e_f^0(z, w) = 0,$$

$$\forall (z, w) \in D \times D \setminus \Delta \ \ f(w) - f(z) = (w - z)(f'(z) + e_f^0(z, w)).$$

These imply

$$\forall z \in D \ \lim_{w \to z} \frac{f(w) - f(z)}{w - z} = f'(z),$$

which completes the proof.[9] □

Theorem 4.19 (Morera's theorem)**.** *The following is provable in RCA_0. Let D be an open subset of $\mathbb{C}$, and let f be a continuous function from D to $\mathbb{C}$. If f is effectively integrable on D and for all $\Delta abc \subseteq D$, $\int_{\partial \Delta abc} f(z)\, dz = 0$, then, f is a holomorphic function.*

Proof. We reason within RCA_0. By the definition of open set, write $D = \bigcup_{n \in \mathbb{N}} B(a_n; r_n)$. Using moduli of integrability for f, we can easily construct continuous functions F_n from $B(a_n; r_n)$ to $\mathbb{C}$ such that

$$F_n(z) = \int_{[a_n, z]} f(w)\, dw.$$

Then, clearly (F_n, f) is holomorphic on $B(a_n; r_n)$. By theorem 4.15, F_n is analytic. Hence $f|_{B(a_n; r_n)}$ is a holomorphic function on $B(a_n; r_n)$ for all $n \in \mathbb{N}$. Thus, f is holomorphic on D. This completes the proof. □

[9]Actually, we can expand e_f^0 into the differentiable condition function for f.

A continuous function from an open set D to $\mathbb{C}$ is said to be *complex differentiable* if

$$\forall z \in D\ \exists \alpha \in \mathbb{C}\ \lim_{w \to z} \frac{f(w) - f(z)}{w - z} = \alpha.$$

Theorem 4.9 holds for complex differentiable functions in place of holomorphic functions. Now, the question is 'a complex differentiable function is a holomorphic function?' *i.e.*, can we find the derivative of a complex differentiable function? To answer this, we need to use the complex differentiable version of Theorem 4.9.

Theorem 4.20. *The following is provable in* $\mathsf{WWKL_0}$. *Let D be an open subset of $\mathbb{C}$, and let f be a continuous function from D to $\mathbb{C}$. If f is complex differentiable, then f is analytic. Particularly, complex differentiable functions are holomorphic.*

Proof. We reason within $\mathsf{WWKL_0}$. By Theorem 3.25, all bounded continuous functions are effectively integrable. Let F be a code for f, and let $(m_n, a_n, r_n, b_n, s_n)$ be an enumeration of all elements (m, a, r, b, s) of F which satisfy $\exists(\hat{m}, \hat{a}, \hat{r}) \in D\ |a - \hat{a}| + r < \hat{r}$. Then $D = \bigcup_{n \in \mathbb{N}} B(a_n; r_n)$ and f is bounded on each $B(a_n; r_n)$. Hence f is effectively integrable on each $B(a_n; r_n)$. By the complex differentiable version of Theorem 4.9, for all $\Delta abc \subseteq B(a_n; r_n)$, $\int_{\partial \Delta abc} f(z)\, dz = 0$. Then by Theorem 4.19, f is analytic on each $B(a_n; r_n)$. Thus, f is analytic on D. This completes the proof. □

Remark that for real differentiable functions, the situation is quite different. We showed that the existence of the derivative of a real continuously differentiable function requires $\mathsf{ACA_0}$ in Theorem 3.8. We can find the derivative of a complex differentiable function within a weaker system. We do not know whether $\mathsf{RCA_0}$ proves that complex differentiable functions are holomorphic, but we get a complex differentiable version of Cauchy's integral theorem.

Theorem 4.21. *The following assertions are pairwise equivalent over* $\mathsf{RCA_0}$.

1. *If f is a complex differentiable function on an open set $D \subseteq \mathbb{C}$, γ is a piecewise* C^1*-Jordan curve on D such that its interior exists and included in D, then $\int_\gamma f(z)\, dz$ exists and*
$$\int_\gamma f(z)\, dz = 0.$$
2. $\mathsf{WKL_0}$.

5 Non-standard arguments for WKL_0 and ACA_0

In this chapter, we introduce non-standard arguments and prove the Riemann mapping theorem in ACA_0 and the Jordan curve theorem in WKL_0. Arguments of non-standard analysis in second order arithmetic were first introduced by Tanaka as a corollary to his self-embedding theorem for WKL_0 [34]. He showed that some popular arguments of non-standard analysis can be carried out in WKL_0 [33]. Using that method, Tanaka and Yamazaki[32] constructed the Haar measure in WKL_0. However, arguments of non-standard analysis in WKL_0 are insufficiently strong to carry out some popular applications of non-standard analysis. We carry out some popular methods of non-standard analysis for sequential compactness in ACA_0. For non-standard analysis in ACA_0, we use the construction of models of ACA_0 (cf. [9]). Applying non-standard arguments, we study standard analysis in WKL_0 and ACA_0.

5.1 Model construction for non-standard analysis

In this section, we introduce some model construction to carry out non-standard analysis in WKL_0 and ACA_0.

To carry out arguments of non-standard analysis in WKL_0, we use an extension of a non-standard model of WKL_0 provided by Tanaka's self-embedding theorem.

Theorem 5.1. *Let $V = (M, S)$ be a countable non-standard model of WKL_0. Then, there exists a countable model of WKL_0 ${}^*V = ({}^*M, {}^*S)$ which satisfies the following:*

1. *${}^*M$ is a proper end extension of M;*

2. *$S = \{X \cap M \mid X \in {}^*S\}$;*

3. *$\exists * : V \longrightarrow {}^*V$ s.t. $*|_M = \mathrm{id}_M$ and $*$ is a Σ^0_0 elementary embedding.*

Proof. Easy from the self embedding theorem for WKL_0 [34, Main Theorem 2.7]. $\square$

A careful examination of arguments of non-standard analysis in WKL_0 shows that the above three conditions are essential. These conditions correspond to the techniques of ordinary non-standard analysis. By the first condition, we can find a 'non-standard' element, *i.e.*, we can take an infinite element from the viewpoint of V. Since ${}^*V$ is again a model of WKL_0, we can use the 'overspill principle'. The second condition allows us to construct 'standard parts'. By the third one, we can use the 'transfer principle' for Σ^0_0-sentences. Using these, we can apply methods of non-standard analysis for some theorems such as the maximal value principle for continuous functions.

Although the conditions described above provide us with some useful popular arguments of non-standard analysis, they are not sufficiently strong to prove some other popular theorems. Actually, the transfer principle for Σ^0_0-sentences are not sufficiently strong to carry out methods of non-standard analysis for sequential compactness or others. Therefore, we strengthen the third condition. The next theorem asserts that we can strengthen the third condition for a model of ACA_0. Using this theorem, we can carry out some more popular arguments of non-standard analysis in ACA_0.

Theorem 5.2. *Let $V = (M, S)$ be a countable model of ACA_0. Thereby, a countable model of ACA_0 ${}^*V = ({}^*M, {}^*S)$ exists which satisfies the following:*

1. *${}^*M$ is a proper end extension of M;*
2. *$S = \{X \cap M \mid X \in {}^*S\}$;*
3. *$\exists * : V \longrightarrow {}^*V$ s.t. $*|_M = \mathrm{id}_M$ and $*$ is a Σ^1_1 elementary embedding.*

The third condition corresponds to the 'transfer principle' for Σ^1_1 sentences; we call the third condition Σ^1_1 *transfer principle*.

This theorem is an easy consequence of the following Gaifman's theorem [20, Theorem 8.8]. Let R be a countable set of unary relation symbols, and let $\bar{\mathcal{L}} = \mathcal{L}_{\mathsf{PA}} \cup R$. Define $\mathsf{PA}(\bar{\mathcal{L}})$ as PA^- plus induction axioms for $\bar{\mathcal{L}}$-formulas.

Lemma 5.3 (Gaifmann)**.** *Every model of $\mathsf{PA}(\bar{\mathcal{L}})$ has a proper conservative extension: if M is a model of $\mathsf{PA}(\bar{\mathcal{L}})$, there exists a proper elementary extension ${}^*M$ which satisfies the following:*
for any $\bar{\mathcal{L}}$-formula $\varphi(x, \vec{y})$ and $\vec{d} \in {}^*M$, there exists a $\bar{\mathcal{L}}$-formula $\psi(x, \vec{z})$ and $\vec{c} \in M$ such that

$$\{a \in {}^*M \mid {}^*M \models \varphi(a, \vec{d})\} \cap M = \{a \in M \mid M \models \psi(a, \vec{c})\}.$$

Enayat mentioned in [9] that they can construct the extension of a model of ACA_0 which has the transfer principle for arithmetical sentences using the Gaifman theorem. By a little consideration, we can strengthen the transfer principle. The transfer principle for Σ^1_1 sentences is due to Tanaka.

Proof of Theorem 5.2. Let (M, S) be a countable model of ACA_0. We identify each element of S as a unary relation on M and regard M as an $\bar{\mathcal{L}} = \mathcal{L}_{\mathsf{PA}} \cup R_S$-structure. Here, R_S is a countable set of unary relation symbols that correspond to S. Since (M, S) satisfies induction axioms for arithmetical formulas, M is a model of $\mathsf{PA}(\bar{\mathcal{L}})$. Hence, by the previous lemma, there exists an $\bar{\mathcal{L}}$-structure ${}^*M$ which is a proper conservative extension of M.

Now we construct a second order part for ${}^*M$. Define ${}^*S$ as

$${}^*S = \{\ \{a \in {}^*M \mid {}^*M \models \varphi(a)\} \mid \varphi(x) \in \bar{\mathcal{L}}_{{}^*M}(x)\}.$$

Here, $\bar{\mathcal{L}}_{{}^*M}(x)$ is the set of all $\bar{\mathcal{L}} \cup {}^*M$-formulas with only one free variable x. Clearly, $({}^*M, {}^*S)$ is a model of ACA_0.

For each $X \in S$, define ${}^*X \in {}^*S$ as ${}^*X = \{a \in {}^*M \mid {}^*M \models X(a)\}$. Because ${}^*M$ is an end extension of M as an $\bar{\mathcal{L}}$-structure, we can define a map $* : (M, S) \to ({}^*M, {}^*S)$ as $*(a) := a$ for each $a \in M$ and $*(X) := {}^*X$ for each $X \in S$. Also, because ${}^*M$ is an elementary extension of M, a map $*$ is a Σ^1_0 elementary embedding.

Next, we show that ${}^*V = ({}^*M, {}^*S)$ satisfies the second condition. By the definition of ${}^*S$, a subset Z of ${}^*M$ is definable in ${}^*M$ if and only if $Z \in {}^*S$. For that reason, if $X \in {}^*S$, then $X \cap M$ is definable in M because ${}^*M$ is a conservative extension of M. Because of that fact, $X \cap M$ is arithmetically definable in (M, S). Then, $X \cap M \in S$ since $(M, S) \models \mathsf{ACA}_0$. Therefore, $S = \{X \cap M \mid X \in {}^*S\}$.

To show the third condition, let $\psi(X, \vec{Y}, \vec{x})$ be an arithmetical $\mathcal{L}_2$-formula with no free variables other than $X, \vec{Y}, \vec{x}$ and let $\vec{A} \in S$ and $\vec{a} \in M$. Clearly, $V \models \exists X \psi(X, \vec{A}, \vec{a})$ implies ${}^*V \models \exists X \psi(X, {}^*\vec{A}, \vec{a})$. We show the converse. Let ${}^*V \models \exists X \psi(X, {}^*\vec{A}, \vec{a})$. Then, there exists $X_0 \in {}^*S$ such that ${}^*V \models \psi(X_0, {}^*\vec{A}, \vec{a})$. By the definition of ${}^*S$, there exist $\vec{b} \in {}^*M$ and an arithmetical $\bar{\mathcal{L}}$-formula $\theta(z, \vec{y})$ such that ${}^*V \models \forall z(z \in X_0 \leftrightarrow \theta(z, \vec{b}))$. Let $\psi(\theta(\vec{y}), \vec{Y}, \vec{x})$ be a formula obtained by replacing all subformulas of the form $z \in X$ that appears in $\psi(X, \vec{Y}, \vec{x})$ with $\theta(z, \vec{y})$. Clearly, ${}^*V \models \psi(\theta(\vec{b}), {}^*\vec{A}, \vec{a})$. Hence, ${}^*V \models \exists \vec{y} \psi(\theta(\vec{y}), {}^*\vec{A}, \vec{a})$. Note that $\exists \vec{y} \psi(\theta(\vec{y}), \vec{A}, \vec{a})$ is an $\bar{\mathcal{L}}$-sentence. Thus, $V \models \exists \vec{y} \psi(\theta(\vec{y}), \vec{A}, \vec{a})$. Then, there exists $\vec{c} \in M$ such that $V \models \psi(\theta(\vec{c}), \vec{A}, \vec{a})$. By arithmetical comprehension in V there exists $X_1 \in S$ such that $V \models \forall z(z \in X_1 \leftrightarrow \theta(z, \vec{c}))$. Then, $V \models \psi(X_1, \vec{A}, \vec{a})$,*i.e.*, $V \models \exists X \psi(X, \vec{A}, \vec{a})$. □

5.2 Non-standard arguments for WKL_0 and ACA_0

In this section, we give some examples of applying non-standard arguments to mathematics in second order arithmetic. We prove Heine-Borel covering theorem using methods of non-standard analysis for $\mathsf{WKL_0}$ and we prove the Bolzano Weierstraß theorem and the Ascoli lemma using methods of non-standard analysis for $\mathsf{ACA_0}$. The original proofs of these three theorems using ordinary non-standard analysis are in [24].

The following lemma is a basic tool for non-standard arguments.

Lemma 5.4 (overspill, underspill)**.** *Let $V = (M, S)$ be a countable model of* $\mathsf{WKL_0}$ *and let ${}^*V = ({}^*M, {}^*S)$ be an extension of V that satisfies the three conditions presented in Theorem 5.1, and let $\vec{a} \in {}^*M$ and $\vec{A} \in {}^*S$. Let ${}^*V \models \mathsf{I}\Sigma^0_n$. Then, for all Σ^0_n formula $\varphi(x, \vec{y}, \vec{X})$, the following hold:*

1. *overspill: if $\forall m \in M \exists n \in M\ n \geq m \wedge {}^*V \models \varphi(n, \vec{a}, \vec{A})$, then, $\exists b \in {}^*M \setminus M\ {}^*V \models \varphi(b, \vec{a}, \vec{A})$.*
2. *underspill: if $\forall b \in {}^*M \setminus M \exists c \in {}^*M\ c \leq b \wedge {}^*V \models \varphi(c, \vec{a}, \vec{A})$, then, $\exists n \in M\ {}^*V \models \varphi(n, \vec{a}, \vec{A})$.*

Proof. To prove 1, let $\psi(x, \vec{y}, \vec{X}) \equiv \exists z \leq x \varphi(z, \vec{y}, \vec{X})$. Assume 1 does not hold, then, ${}^*V \models \psi(0, \vec{a}, \vec{A}) \wedge \forall x(\psi(x, \vec{a}, \vec{A}) \to \psi(x+1, \vec{a}, \vec{A})$ and ${}^*V \not\models \forall x \psi(x, \vec{a}, \vec{A})$, but it contradicts $V \models \mathsf{I}\Sigma^0_n$. We can prove 2 similarly. □

Example 5.1 (Heine-Borel covering theorem)**.** The following is provable in $\mathsf{WKL_0}$. If $\{U_k\}_{k \in \mathbb{N}}$ be a sequence of open subsets of $\mathbb{R}$ which covers $[0, 1]$, then there exists m such that $\{U_k\}_{k \leq m}$ covers $[0, 1]$.

Proof. Let $V = (M, S)$ be a countable model of $\mathsf{WKL_0}$ and let ${}^*V = ({}^*M, {}^*S)$ be an extension of V that satisfies the three conditions presented in Theorem 5.1. Let $\bigcup_{k=0}^{\infty} U_k \supseteq [0, 1]$ in V. Without loss of generality, we can assume $U_n = B(f(n), g(n))$ where f and g are sequences of rational numbers (*i.e.*, $f, g : \mathbb{N} \to \mathbb{Q}$) and $\forall n \in \mathbb{N}\ \exists m > n\ \overline{B(f(n), g(n))} \subset B(f(m), g(m))$, thus, we need to show that there exists $K \in \mathbb{N}$ such that

$$\bigcup_{n=0}^{K} \overline{B(f(n), g(n))} \supseteq [0, 1].$$

By overspill, there exists $a \in {}^*M \setminus M$ such that $*(f)|_{\leq a}$ and $*(g)|_{\leq a}$ are sequences of rational numbers in ${}^*V$.

Claim 5.4.1. *Let $\varphi(r)$ be a Σ^0_1 formula, and let $\omega \in {}^*M \setminus M$. Then, there exists a real number $\alpha \in [0, 1]$ such that $V \models \varphi(\alpha)$ if and only if there exists $c \in {}^*M$ such that ${}^*V \models c \leq \omega \wedge \varphi(c/\omega)$.*

This claim is easily proved by overspill (cf. [33]).

Take $\omega \in {}^*M \setminus M$. By the previous claim, for any $b \in {}^*M \setminus M$ such that $b \leq a$,

$$ {}^*V \models \forall x \leq \omega \ \frac{x}{\omega} \in \bigcup_{n=0}^{b} \overline{B(*(f)(n), *(g)(n))}. $$

Thus, by underspill, there exists $K \in M$ such that

$$ {}^*V \models \forall x \leq \omega \ \frac{x}{\omega} \in \bigcup_{n=0}^{K} \overline{B(*(f)(n), *(g)(n))}. $$

Then, by the claim,

$$ \bigcup_{n=0}^{K} \overline{B(f(n), g(n))} \supseteq [0, 1]. $$

$\square$

Let $V = (M, S)$ be a countable model of ACA_0 and let ${}^*V = ({}^*M, {}^*S)$ be an extension of V which satisfies the three conditions in Theorem 5.2. Then, by Σ^1_1 transfer principle, $*(\mathbb{N}^V) = \mathbb{N}^{*V}$, $*(\mathbb{Z}^V) = \mathbb{Z}^{*V}$ and $*(\mathbb{Q}^V) = \mathbb{Q}^{*V}$. Here, $\mathbb{N}^V$, $\mathbb{Z}^V$, $\mathbb{Q}^V$ are $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ defined in V and $\mathbb{N}^{*V}$, $\mathbb{Z}^{*V}$, $\mathbb{Q}^{*V}$ are those defined in ${}^*V$. We usually write $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ for $\mathbb{N}^V$, $\mathbb{Z}^V$, $\mathbb{Q}^V$ and ${}^*\mathbb{N}$, ${}^*\mathbb{Z}$, ${}^*\mathbb{Q}$ for $\mathbb{N}^{*V}$, $\mathbb{Z}^{*V}$, $\mathbb{Q}^{*V}$. If $X \in S$, we write ${}^*X$ for $*(X)(\in {}^*S)$. Note that if $q \in \mathbb{Q}$, then, $*(q) \in {}^*\mathbb{Q}$ by Σ^1_1 transfer principle. Note also that if $q \in \mathbb{Q}$, then, $q = *(q)$ since $\mathbb{Q} \subseteq M$.

Remark 5.5. Let $X \in S$, and let $X_n = \{a \in M \mid (n, a) \in X\}$. ($X$ is a sequence of sets $\{X_n\}_{n\in\mathbb{N}}$ in V.) Then, for any $n \in \mathbb{N}$ and for any $b \in {}^*M$, $b \in *(X_n) \leftrightarrow (n, b) \in *(X) \leftrightarrow b \in (*(X))_n$, *i.e.*, ${}^*(X_n) = ({}^*X)_n$ in ${}^*V$. Thus, we do not distinct ${}^*(X_n)$ and $({}^*X)_n$ and we write ${}^*X_n$ for $({}^*X)_n$. Then, ${}^*X$ is a sequence of sets $\{{}^*X_n\}_{n\in{}^*\mathbb{N}}$ in ${}^*V$.

By Σ^1_1 transfer principle, if $\alpha = \{q_n\}_{n\in\mathbb{N}}$ is a real number in V, ${}^*\alpha = \{q_n\}_{n\in{}^*\mathbb{N}}$ is a real number in ${}^*V$.

A real number $\alpha = \{q_n\}_{n\in\mathbb{N}}$ is said to be *normally expressed* if each q_n is a form of $i/2^{n+1}$ for some $i \in \mathbb{Z}$. Within RCA_0, if $\alpha = \{q_n\}_{n\in\mathbb{N}}$ is a real number, we can find a normally expressed real number $\alpha' = \{q'_n\}_{n\in\mathbb{N}}$ such that $\alpha' =_{\mathbb{R}} \alpha$. If $\beta = \{q_n\}_{n\in{}^*\mathbb{N}}$ is a normally expressed real number in ${}^*V$ and $|\beta| \leq K$ for some $K \in \mathbb{N}$, then, $\beta \cap M(\in S)$ is a real number in V. We write $\beta \cap M = \beta|_M$ and $\beta|_M$ is said to be the standard part of β.

Example 5.2 (Bolzano-Weierstraß theorem)**.** The following is provable in ACA_0. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ for all $n \in \mathbb{N}$. Then, there exists a convergent subsequence $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$, $n_0 < n_1 < \cdots < n_k < \cdots$.

Proof. Let $V = (M, S)$ be a countable model of ACA_0 and let ${}^*V = ({}^*M, {}^*S)$ be an extension of V that satisfies the three conditions presented in Theorem 5.2. Let $A = \{\alpha_n\}_{n\in\mathbb{N}} \in S$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ for all $n \in \mathbb{N}$. Without loss of generality, we can assume that each α_n is normally expressed. Then, by Σ^1_1 transfer principle, ${}^*A \in {}^*S$ is a sequence of real numbers ${}^*A = \{{}^*\alpha_n\}_{n\in{}^*\mathbb{N}}$, where each ${}^*\alpha_n$ is normally expressed and $0 \leq {}^*\alpha_n \leq 1$ for all $n \in {}^*\mathbb{N}$ in ${}^*V$.

Take $\omega \in {}^*\mathbb{N}\setminus\mathbb{N}$. Because ${}^*\alpha_\omega$ is expressed normally, $\gamma = {}^*\alpha_\omega|_M$ is a real number in V. Then ${}^*\gamma \cap M = {}^*\alpha_\omega \cap M$. Consequently, for all $n, m \in \mathbb{N}$, ${}^*V \models \exists y > m\ |{}^*\alpha_y - {}^*\gamma| < 2^{-n}$ (take $y = \omega$). Then, by Σ^1_1 transfer principle, $V \models \exists y > m\ |\alpha_y - \gamma| < 2^{-n}$ for all $n, m \in \mathbb{N}$. Therefore, we can easily find a subsequence of A which converges to γ in V. According to the completeness theorem, ACA_0 proves the existence of a convergent subsequence of $\{\alpha_n\}_{n\in\mathbb{N}}$. □

Next, we show the Ascoli lemma in ACA_0 using non-standard analysis. For the convenience, we redefine continuous functions. We use this definition only in this section.

Definition 5.3 (normally expressed continuous functions)**.** The following definition is made in ACA_0. A code for a *continuous (partial) function* f is a set of quadruples $F \subseteq \mathbb{Q} \times \mathbb{Q}^+ \times \mathbb{Q} \times \mathbb{Q}^+$ that satisfies the following:

1. if $(a, r, b, s) \in F$ and $(a, r, b', s') \in F$, then $|b - b'| \leq s + s'$;
2. if $(a, r, b, s) \in F$ and $|a' - a| + r' < r$, then $(a', r', b, s) \in F$; and
3. if $(a, r, b, s) \in F$ and $|b - b'| + s < s'$, then $(a, r, b', s') \in F$.

As in the definition in Chapter 3, a point $x \in \mathbb{R}$ is said to belong to the *domain* of f, abbreviated $x \in \mathrm{dom}(f)$ if and only if for any $m \in \mathbb{N}$ there exists $(a, r, b, s) \in F$ such that $|x - a| < r$ and $s \leq 2^{-m}$. If $x \in \mathrm{dom}(f)$, we define the *value* of $f(x)$ to be the unique $y \in \mathbb{R}$ such that $|y - b| < s$ for all $(a, r, b, s) \in F$ with $|x - a| < r$.

A code F for a continuous function f from $[c, d]$ to $[c', d']$ is said to be *normally expressed* if F satisfies the following:

(45) $$F \subseteq \mathbb{Q}_{c,d} \times \mathbb{Q}^+ \times \mathbb{Q}_{c',d'} \times \mathbb{Q}^+,$$

(46) $$\varphi(F, a, r, b, s) \to (a, r, b, s) \in F$$

where, $\mathbb{Q}_{\alpha,\beta} = \{q \in \mathbb{Q} \mid \alpha \leq q \leq \beta\}$ and $\varphi(F, a, r, b, s) \equiv \forall q \in \mathbb{Q}_{c,d}\ |q - a| < r \rightarrow \exists(a', r', b', s') \in F\ (|q - a'| < r' \wedge |b - b'| + s' < s)$.

This definition is equivalent to the definition in Chapter 3 over ACA_0.

Remark 5.6. The formula $\varphi(F, a, r, b, s)$ expresses $f(B(a; r)) \subseteq \overline{B(b, s)}$. Therefore, a normally expressed code F for f contains all (a, r, b, s) with $f(B(a; r)) \subseteq \overline{B(b, s)}$, *i.e.*, F is the maximal code. Moreover, given a code F_0 for a continuous function f from $[c, d]$ to $[c', d']$, we can construct a normally expressed code F for f as $(a, r, b, s) \in F \leftrightarrow \varphi(F_0, a, r, b, s)$.

Remark 5.7. Let $V = (M, S)$ be a countable model of ACA_0 and let ${}^*V = ({}^*M, {}^*S)$ be an extension of V that satisfies the three conditions in Theorem 5.2. Let F be a normally expressed code for a continuous function from $[c, d]$ to $[c', d']$. Then, by Σ^1_1 transfer principle, ${}^*F$ satisfies (45), (46) and the three conditions for codes for continuous functions in ${}^*V$.

Example 5.4 (the Ascoli lemma)**.** The following is provable in ACA_0. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous functions from $[0, 1]$ to $[0, 1]$. If $\{f_n\}_{n\in\mathbb{N}}$ is equicontinuous, then, there exists a uniformly convergent subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$, $n_0 < n_1 < \cdots < n_k < \cdots$.

Proof. Let $V = (M, S)$ be a countable model of ACA_0 and let ${}^*V = ({}^*M, {}^*S)$ be an extension of V that satisfies the three conditions in Theorem 5.2. Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}} \in S$ be a sequence of codes for an equicontinuous sequence of continuous functions $\{f_n\}_{n\in\mathbb{N}} \in S$ from $[0, 1]$ to $[0, 1]$. Without loss of generality, we might assume that each F_n is expressed normally. Then, by Σ^1_1 transfer principle, ${}^*\mathcal{F} \in {}^*S$ is a sequence of sets ${}^*\mathcal{F} = \{{}^*F_n\}_{n\in{}^*\mathbb{N}}$. Each ${}^*F_n$ satisfies (45), (46), and the three conditions for a code for a continuous function in Definition 5.3.

We first show the following:

(47) $$\forall m \in \mathbb{N}\ \forall \alpha \in [0, 1](\alpha \in S)\ \exists l \in \mathbb{N}$$
$$ {}^*V \models \forall n \in {}^*\mathbb{N}\ \exists a, b \in {}^*\mathbb{Q}_{0,1}\ |{}^*\alpha - a| < 2^{-l-2} \wedge (a, 2^{-l}, b, 2^{-m}) \in {}^*F_n.$$

By equicontinuity, for any $m \in \mathbb{N}$ and $\alpha \in [0, 1]$, there exists $l \in \mathbb{N}$ such that $\forall n \in \mathbb{N}\ f_n(B(\alpha; 2^{-l})) \subseteq B(f_n(\alpha); 2^{-m})$. Thus, for each $n \in \mathbb{N}$, we can find $a, b \in \mathbb{Q}_{0,1}$ such that $|\alpha - a| < 2^{-l-3}$ and $|f_n(\alpha) - b| < 2^{-m}$. Because F_n is normally expressed, $(a, 2^{-l-1}, b, 2^{-m+1}) \in F_n$. Eventually,

$$\forall m \in \mathbb{N}\ \forall \alpha \in [0, 1](\alpha \in S)\ \exists l \in \mathbb{N}$$
$$V \models \forall n \in \mathbb{N}\ \exists a, b \in \mathbb{Q}_{0,1}\ |\alpha - a| < 2^{-l-2} \wedge (a, 2^{-l}, b, 2^{-m}) \in F_n.$$

According to Σ^1_1 transfer principle, we obtain (47).

Take $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$ and define $G \in S$ as $G = {}^*F_\omega \cap M$. Then, G satisfies the three conditions for a code for a continuous function. Let g be a (partial) continuous function coded by G. We show that $\mathrm{dom}(g) = [0,1]$. Let $\alpha \in [0,1]$, and let $m \in \mathbb{N}$. By (47), there exists $l \in \mathbb{N}$ and $a, b \in {}^*\mathbb{Q}_{0,1}$ such that $|{}^*\alpha - a| < 2^{-l-2}$ and $(a, 2^{-l}, b, 2^{-m-1}) \in {}^*F_\omega$. In ${}^*V$, we can find $i, j \in {}^*\mathbb{N}$ with $|i/2^{l+2} - a| < 2^{-l-2}$ and $|j/2^{m+1} - a| < 2^{-m-1}$. Then $(i/2^{l+2}, 2^{-l-1}, j/2^{m+1}, 2^{-m}) \in {}^*F_\omega$. Because $i \leq 2^{l+2}$ and $j \leq 2^{m+1}$, both i and j are standard, *i.e.*, $i, j \in \mathbb{N}$. Consequently, $(i/2^{l+2}, 2^{-l-1}, j/2^{m+1}, 2^{-m}) \in M \cap {}^*F_\omega = G$. On the other hand, ${}^*V \models |{}^*\alpha - i/2^{l+2}| < 2^{-l-1}$ because $|{}^*\alpha - a| < 2^{-l-2}$ and $|i/2^{l+2} - a| < 2^{-l-2}$ in ${}^*V$. Then, by Σ^1_1 transfer principle, $V \models |\alpha - i/2^{l+2}| < 2^{-l-1}$. Eventually, $\forall \alpha \in [0,1]\ \forall m \in \mathbb{N}\ \exists (a, r, b, s) \in G\ |\alpha - a| < r \wedge s \leq 2^{-m}$, *i.e.*, $\mathrm{dom}(g) = [0,1]$ in V.

Next, we construct a subsequence of $\mathcal{F}$ that converges to g uniformly. Let $\beta \in {}^*S$ be a normally expressed real number in ${}^*V$ with ${}^*V \models \beta \in [0,1]$. Then, $\gamma := \beta|_M$ is a real number in V and $\gamma \in \mathrm{dom}(g)$. Since ${}^*\gamma \cap M = \beta \cap M = \gamma$, ${}^*V \models |{}^*\gamma - \beta| < 2^{-k}$ for all $k \in \mathbb{N}$. Thus, $V \models |\gamma - a| < r$ implies ${}^*V \models |\beta - a| < r$. Therefore, for any $\beta \in [0,1]$ in ${}^*V$ and for any $m \in \mathbb{N}$, there exists $(a, r, b, s) \in G$ such that ${}^*V \models |\beta - a| < r \wedge s \leq 2^{-m}$. Because $G \subseteq {}^*G$ and $G \subseteq {}^*F_\omega$,

$$\forall m \in \mathbb{N}\ {}^*V \models \forall q \in {}^*\mathbb{Q}_{0,1}\ \exists (a, r, b, s) \in {}^*F_\omega \cap {}^*G\ |q - a| < r \wedge s \leq 2^{-m}.$$

Then,

$$\begin{aligned} &\forall m \in \mathbb{N}\ \forall l \in \mathbb{N} \\ &\quad {}^*V \models \exists y > l\ \forall q \in {}^*\mathbb{Q}_{0,1}\ \exists (a, r, b, s) \in {}^*F_y \cap {}^*G\ |q - a| < r \wedge s \leq 2^{-m}. \end{aligned}$$

According to Σ^1_1 transfer principle,

$$\begin{aligned} &\forall m \in \mathbb{N}\ \forall l \in \mathbb{N} \\ &\quad V \models \exists y > l\ \forall q \in \mathbb{Q}_{0,1}\ \exists (a, r, b, s) \in F_y \cap G\ |q - a| < r \wedge s \leq 2^{-m}. \end{aligned}$$

This implies that $V \models \exists y > l\ \|f_y - g\| < 2^{-m}$ for all $m, l \in \mathbb{N}$. Thereby, we can easily find a subsequence of $\mathcal{F}$ that converges to g uniformly in V.

Using the completeness theorem, ACA_0 proves the existence of a uniformly convergent subsequence of $\{f_n\}_{n \in \mathbb{N}}$. □

Remark 5.8. The Bolzano-Weierstraß theorem and the Ascoli lemma are equivalent to ACA_0 over RCA_0. See [29].

5.3 Application 1: the Riemann mapping theorem

In this section, we prove the Riemann mapping theorem in ACA_0 by using arguments of non-standard analysis. It becomes easy to treat a space of conformal maps if we use methods of non-standard analysis. Then, we show that the Riemann mapping theorem is equivalent to ACA_0 over WKL_0.

We first develop some parts of complex analysis within RCA_0 and WKL_0. We define the following notation:

$$\begin{aligned} \bar{\mathbb{Q}} &:= \{q_1 + iq_2 \mid q_1, q_2 \in \mathbb{Q}\} \\ \mathfrak{B} &:= \{(a, r) \mid a \in \bar{\mathbb{Q}}, r \in \mathbb{Q}, r > 0\} \end{aligned}$$

In this section, we denote open sets using $\mathfrak{B}$ as follows: A code for an *open set* U in $\mathbb{C}$ is a sequence of elements of $\mathfrak{B}$ $U = \{(a_n, r_n)\}_{n\in\mathbb{N}}$. A point $z \in \mathbb{C}$ is said to *belong to* U (abbreviated $z \in U$) if

$$\exists n\ |z - a_n| < r_n.$$

Note that the assertion that a closed ball $\overline{B(a;r)}$ is included in an open set U (*i.e.*, $\overline{B(a;r)} \subseteq U$) is expressible by a Σ^0_1 formula in WKL_0. We also redefine holomorphic functions by a suitable code to apply non-standard arguments.

Definition 5.5 (holomorphic functions)**.** The following definition is made in RCA_0. Let D be an open subset of $\mathbb{C}$. A *holomorphic function* on D is defined to be a pair of sequences $f = (\{(a_n, r_n)\}_{n\in\mathbb{N}}, \{\alpha_{nk}\}_{n\in\mathbb{N},k\in\mathbb{N}})$ such that $\alpha_{nk} \in \mathbb{C}$ and $(a_n, r_n) \in \mathfrak{B}$, which satisfies the following conditions:

1. $\bigcup_{n\in\mathbb{N}} B(a_n; r_n) = D$;

2. for all $n \in \mathbb{N}$, $\sum_{k\in\mathbb{N}} |\alpha_{nk}| {r_n}^k$ converges;

3. for all $n, m \in \mathbb{N}$ and for all $z \in B(a_n; r_n) \cap B(a_m; r_m)$,

$$\sum_{k\in\mathbb{N}} \alpha_{nk}(z - a_n)^k = \sum_{k\in\mathbb{N}} \alpha_{mk}(z - a_m)^k.$$

We define $f(z), f'(z), \ldots, f^{(l)}(z), \ldots$ as

$$f^{(l)}(z) = \sum_{k=l}^{\infty} \frac{k!}{(k-l)!} \cdot \alpha_{nk}(z - a_n)^{k-l} \ \text{ if } \ z \in B(a_n; r_n).$$

By Theorem 4.15, the above definition is equivalent to the definition in Chapter 4 in RCA_0. Let $\{D_n\}_{n\in\mathbb{N}}$ be a sequence of open subsets of $\mathbb{C}$; also let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of holomorphic functions where f_n is from D_n to $\mathbb{C}$. If $\{f_n\}_{n\in\mathbb{N}}$ satisfies

$$\forall z \in \mathbb{C}\ \forall p, q \in \mathbb{N}\ (z \in D_p \cap D_q \to f_p(z) = f_q(z)),$$

then, we can construct a holomorphic function f from $D = \bigcup_{n=0}^{\infty} D_n$ to $\mathbb{C}$ such that

$$\forall n \in \mathbb{N}\ \forall z \in D\ (z \in D_n \to f_n(z) = f(z)).$$

We write this f as $f = \bigcup\{f_n\}$.

We first define some concepts on the complex plane in RCA_0.

Definition 5.6. The following definitions are made in RCA_0. Let D be an open subset of $\mathbb{C}$, and let $\alpha, \beta \in D$.

1. A *path* γ from α to β in D is a broken line in D which connects α and β, *i.e.*, γ is a finite sequence $\langle\gamma(0), \dots, \gamma(m)\rangle$ of points in D such that $\gamma(0) = \alpha$, $\gamma(m) = \beta$ and $[\gamma(k), \gamma(k+1)] \subseteq D$.

2. A *circuit* in D is a broken line in D with its two end points are common, *i.e.*, a broken line $\gamma = \langle\gamma(0), \dots, \gamma(m)\rangle$ in D with $\gamma(0) = \gamma(m)$.

Lemma 5.9. *The following is provable in* RCA_0. *Let* γ *be a circuit in* $\mathbb{C}$. *Thereby, there exist two open sets called exterior and interior of* γ *and a closed set called the image of* γ.

Proof. Let $\varphi(z)$ (or $\psi(z)$) be a Σ^0_1 formula that expresses the following:

- $z \notin [\gamma(k), \gamma(k+1)]$ for all $0 \leq k < m$;

- there exist a $0 < \theta < \pi/2, \theta \in \mathbb{Q}$ such that the half-line $l(z, \theta) = \{w \in \mathbb{C} \mid \arg(w - z) = \theta\}$ does not contain each $\gamma(k)$ and the cardinality of $\{0 \leq k < m \mid l(z, \theta) \cap [\gamma(k), \gamma(k+1)] \neq \emptyset\}$ is even (or odd).

Then, by Lemma 2.6, we can find open sets U_1, U_2 such that $z \in U_1 \leftrightarrow \varphi(z)$ and $z \in U_2 \leftrightarrow \psi(z)$. U_1 is said to be the *exterior* of γ and U_2 is said to be the *interior* of γ. The *image* of γ is a closed set $\mathbb{C} \setminus (U_1 \cup U_2)$. □

Definition 5.7. The following definitions are made in RCA_0. Let D be an open subset of $\mathbb{C}$.

1. D is said to be *path connected* if there exists a path from α to β in D for all $\alpha, \beta \in D$.

2. D is said to be *simply connected* if D is path connected and for all circuit γ in D, the interior of γ is included in D.

Next, we prepare some lemmas. As in the usual complex analysis, Cauchy's integral theorem (Theorem 4.10) and Taylor's theorem (Theorem 4.15) play central roles. See [1] for usual proofs of the following lemmas.

Lemma 5.10 (local inverse function)**.** *The following is provable in* $\mathsf{RCA_0}$. *Let D be an open subset of $\mathbb{C}$, and let f be a holomorphic function from D to $\mathbb{C}$. Let $z_0 \in D$ such that $f'(z_0) \neq 0$ and $r \in \mathbb{R}$, $r > 0$. If $|f'(z_0) - f'(z)| \leq |f'(z_0)|/8$ for all $z \in B(z_0; r)$, then $\overline{B(f(z_0); |f'(z_0)|r/2)} \subseteq f(\overline{B(z_0; r)})$. Moreover, a local inverse holomorphic function f^{-1} exists from $B(f(z_0); |f'(z_0)|r/2)$ to $B(z_0; r)$.*

Proof. Let $w \in \overline{B(f(z_0); |f'(z_0)|r/2)}$. Define a holomorphic function h as

$$\begin{aligned} h(\zeta) &:= \frac{w - f(z_0)}{f'(z_0)} + \frac{f(z_0) - f(\zeta) + f'(z_0)(\zeta - z_0)}{f'(z_0)} + z_0 \\ &= \frac{w - f(\zeta)}{f'(z_0)} + \zeta. \end{aligned}$$

Then, $|h'(\zeta)| \leq 1/8$ for all $\zeta \in \overline{B(z_0; r)}$ and $|h(z_0) - z_0| \leq r/2$. Hence, by Lemma 4.2, $h(\overline{B(z_0; r)}) \subseteq \overline{B(z_0; r)}$ and $|h(\zeta_1) - h(\zeta_2)| \leq |\zeta_1 - \zeta_2|/2$ for all $\zeta_1, \zeta_2 \in \overline{B(z_0; r)}$. Therefore, by the contraction mapping theorem, there exists $z \in \overline{B(z_0; r)}$ such that $h(z) = z$. Consequently, $f(z) = w$. (Note that the contraction mapping theorem is provable in $\mathsf{RCA_0}$.)

For construction of the local inverse function, we can imitate the proof of the inverse function theorem for C^1-functions. See Theorem 3.34. □

Lemma 5.11 (maximal value principle)**.** *The following is provable in* $\mathsf{WKL_0}$. *Let f be a holomorphic function on an open subset $D \subseteq \mathbb{C}$, and let $\overline{B(a; r)} \subseteq D$. Then,* $\sup\{|f(z)| \mid z \in \overline{B(a; r)}\} = \sup\{|f(z)| \mid |z - a| = r\}$.

Proof. We can imitate the usual proof using Theorem 4.10. □

Lemma 5.12. *The following is provable in* $\mathsf{WKL_0}$. *Let D be an open subset of $\mathbb{C}$, and let f be a holomorphic function on D. Let $B(a; r) \subseteq D$, and let $M \in \mathbb{R}$ such that $\forall z \in B(a; r)\ |f(z)| < M$. Then,*

$$\forall z \in B(a; r)\ |f^{(n)}(z)| < \frac{M}{r^n}.$$

Particularly, if f and g are holomorphic functions on $B(a;r)$ with $\forall z \in B(a,r)\ |f(z)-g(z)| < \varepsilon$, then $\forall z \in B(a,r/2)\ |f'(z)-g'(z)| < 2\varepsilon/r$.

Proof. We can imitate the usual proof using Lemma 5.11. □

Lemma 5.13 (Schwarz's lemma)**.** *The following is provable in $\mathsf{WKL_0}$. Let f be a holomorphic function from $B(0;1)$ to $B(0;1)$ such that $f(0)=0$. Then, f satisfies either of the following:*

1. *$|f(z)| < |z|$ and $|f'(0)| < 1$.*
2. *There exists $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and $f(z) = \lambda z$.*

Proof. We can imitate the usual proof using Theorem 4.15 and Lemma 5.11. □

Let D be an open subset of $\mathbb{C}$, and let f be a holomorphic function from D to $\mathbb{C}$. Let $\gamma = \langle \gamma(0), \dots, \gamma(m) \rangle$ be a path in D. Then, we define $\int_\gamma f(z)\,dz$, the line integral of f along γ, as

$$\int_\gamma f(z)\,dz = \sum_{k=0}^{m-1} \int_{[\gamma(k),\gamma(k+1)]} f(z)\,dz.$$

In $\mathsf{WKL_0}$, if f is a holomorphic function on $B(a;r)$, then, by Cauchy's integral theorem, we can easily construct a holomorphic function F on $B(a;r)$ such that

$$F(z) = \int_{[a,z]} f(\zeta) d\zeta$$

since f has a modulus of integrability.

Lemma 5.14. *The following is provable in $\mathsf{WKL_0}$. Let D be a simply connected open subset of $\mathbb{C}$, and let f be an holomorphic function on $\mathbb{C}$. If $\forall z \in D\ f(z) \neq 0$, then a holomorphic function g exists such that $f(z) = g(z)^2$.*

Proof. Write $D = \bigcup_{k\in\mathbb{N}} B(a_k;r_k)$ and let γ_k be a path from a_0 to a_k. Define holomorphic functions $F_k : B(a_k;r_k) \to \mathbb{C}$ as

$$F_k(z) := \int_{\gamma_k \frown \langle z \rangle} \frac{f'(\zeta)}{f(\zeta)} d\zeta = \int_{\gamma_k} \frac{f'(\zeta)}{f(\zeta)} d\zeta + \int_{[a_k,z]} \frac{f'(\zeta)}{f(\zeta)} d\zeta.$$

By Theorem 4.10, if $B(a_k;r_k) \cap \mathcal{B}(a_l;r_l) \neq \emptyset$, then $F_k(z) = F_l(z)$ on $B(a_k;r_k) \cap \mathcal{B}(a_l;r_l)$. Hence we can construct the holomorphic function $F = \bigcup_{k\in\mathbb{N}} F_k$ on D. Then,

$$\frac{d}{dz}(f(z)\cdot \exp(-F(z))) = 0.$$

Thus, we write f as

$$f(z) = f(a_0)\exp(F(z)).$$

Take α such that $\alpha^2 = f(a_0)$, and define a holomorphic function g as

$$g(z) := \alpha \exp\left(\frac{F(z)}{2}\right).$$

Then $g(z)^2 = f(z)$. □

Lemma 5.15. *The following is provable in* $\mathsf{WKL_0}$*. Let* f *be a non-constant holomorphic function from* $B(a;r)$ *to* $\mathbb{C}$ *such that* $f'(a) = 0$*. Then,* $z_1, z_2 \in B(a;r)$ *exist such that* $z_1 \neq z_2$*,* $f'(z_1) \neq 0$*,* $f'(z_2) \neq 0$ *and* $f(z_1) = f(z_2)$*.*

Proof. We can imitate the usual proof using Lemmas 5.10 and 5.11. □

Note that we can easily show that sequential versions of Lemmas 5.9, 5.10 and 5.14 also hold.

Definition 5.8 (conformal map)**.** The following definition is made in $\mathsf{RCA_0}$. Let D_1 and D_2 be open subsets of $\mathbb{C}$. A *conformal map* from D_1 to D_2 is a pair of holomorphic functions (h, h^{-1}) such that $h : D_1 \to D_2$, $h^{-1} : D_2 \to D_1$, and $h^{-1} \circ h = \mathrm{id}_{D_1} \wedge h \circ h^{-1} = \mathrm{id}_{D_2}$.

Now, we are prepared to prove the Riemann mapping theorem.

Theorem 5.16 (the Riemann mapping theorem)**.** *The following is provable in* $\mathsf{ACA_0}$*. Let* D_0 *be a simply connected open subset of* $\mathbb{C}$ *such that* $D_0 \neq \mathbb{C}$*, and let* $z_0 \in D_0$*. Then, there exists a conformal map* (f, f^{-1}) *from* D_0 *to* $B(0;1)$ *such that* $f(z_0) = 0$*. Moreover, if* (f, f^{-1}) *and* (g, g^{-1}) *are conformal maps from* D_0 *to* $B(0;1)$ *such that* $f(z_0) = g(z_0) = 0$*, then there exists* $\lambda \in \mathbb{C}$ *such that* $|\lambda| = 1$ *and* $f = \lambda g$*.*

Proof. To prove this theorem, we show the following four sublemmas.

Sublemma 1. *If* D_0 *is a simply connected open subset of* $\mathbb{C}$ *such that* $D_0 \neq \mathbb{C}$ *and* $z_0 \in D_0$*, then there exists an open subset* $0 \in D \subseteq B(0;1)$ *and a conformal map* (h_0, h_0^{-1}) *from* D_0 *to* D *such that* $h_0(z_0) = 0$*.*

Sublemma 2. *If* D *is a simply connected open set such that* $0 \in D \subseteq B(0;1)$*, then there exists a conformal map* (h, h^{-1}) *from* D *to an open set* $0 \in E \subseteq B(0;1)$ *such that* $h(0) = 0$ *and its derivative at the origin is maximal, i.e., if* $(\tilde{h}, \tilde{h}^{-1})$ *is a conformal map from* D *to an open set* $0 \in \tilde{E} \subseteq B(0;1)$ *such that* $\tilde{h}(0) = 0$*, then* $|\tilde{h}'(0)| \leq |h'(0)|$*.*

Sublemma 3. *If* $0 \in D \subseteq B(0;1)$ *is a simply connected open set and* (h, h^{-1}) *is a conformal map from* D *to an open set* $0 \in E \subseteq B(0;1)$ *such that* $h(0) = 0$ *and its derivative at the origin is maximal, then* $E = B(0;1)$.

Sublemma 4. *If* (f, f^{-1}) *and* (g, g^{-1}) *are conformal maps from an open set* $z_0 \in D_0 \neq \mathbb{C}$ *to* $B(0;1)$ *such that* $f(z_0) = g(z_0) = 0$*, then there are* $\lambda \in \mathbb{C}$ *such that* $|\lambda| = 1$ *and* $f = \lambda g$.

Imitating the usual proof, we can readily prove Sublemmas 1 and 3 in $\mathsf{WKL_0}$ using Lemma 5.14. Sublemma 4 is a straightforward direction from Lemma 5.13. We show Sublemma 2 using non-standard arguments.

Let U be a open subset of $B(0;1)$, and let Seq_l be the set of all complex rational sequences with length $l \in \mathbb{N}$. We define the following notation:

$$\begin{aligned}
\mathbb{Q}(m) &:= \{q_0 2^{-m} \in \mathbb{Q} \mid q_0 \in \mathbb{Z}\};\\
\bar{\mathbb{Q}}(m) &:= \{q_1 + iq_2 \in \mathbb{C} \mid q_1, q_2 \in \mathbb{Q}(m)\};\\
\mathbb{Q}(U;m) &:= \{q \in \bar{\mathbb{Q}}(m) \mid \overline{B(q;2^{-m+2})} \subseteq U\};\\
\mathfrak{N}(U;m) &:= \bigcup_{k=0}^{m} \{(q, 2^{-k+2}) \mid q \in \mathbb{Q}(U;k)\};\\
\mathrm{Seq}(l;m) &:= \{\sigma \in \mathrm{Seq}_l \mid \forall k < l\ \sigma(k) \in \bar{\mathbb{Q}}(m+k+3)\}.
\end{aligned}$$

Note that $\mathfrak{N}(U;m)$ is a finite set and it can be coded by a natural number. In addition, note that $\bigcup\{B(a, r/2) \mid \exists m \in \mathbb{N}\ (a,r) \in \mathfrak{N}(U;m)\} = U$.

We first construct an approximation of a conformal map from $D \subseteq B(0;1)$ to $E \subseteq B(0;1)$ coded by a finite set. Let D be an open subset of $B(0;1)$. We define an *m-approximation polynomial on D* as follows. An *m-approximation polynomial* $\mathcal{P}$ on D is a pair of finite sets $\mathcal{P} = (P, Q)$. Here, P is a function from $\mathfrak{N}(D;m)$ to $\mathrm{Seq}(l_{\mathcal{P}};m)$ ($l_{\mathcal{P}} \in \mathbb{N}$) and Q is a function from $\mathfrak{E}_{\mathcal{P}} \subseteq \mathfrak{N}(B(0;1);m)$ to $\mathrm{Seq}(l_{\mathcal{P}};m)$. If $(a,r) \in \mathfrak{N}(D;m)$ and $P(a,r) = \sigma$, we define a polynomial $P_{a,r}$ as $P_{a,r}(z) = \sum_{j=0}^{l_{\mathcal{P}}} \sigma(j) z^j$. We define $Q_{b,s}$ ($(b,s) \in \mathfrak{E}_{\mathcal{P}}$) similarly. P, Q and $\mathfrak{E}_{\mathcal{P}}$ satisfy the following conditions.

1. $\forall z \in B(a_1;r_1) \cap B(a_2;r_2)\ |P_{a_1,r_1}(z) - P_{a_2,r_2}(z)| < 2^{-m}$ for all $(a_1,r_1), (a_2,r_2) \in \mathfrak{N}(D;m)$.

2. $\forall w \in B(b_1;s_1) \cap B(b_2;s_2)\ |Q_{b_1,s_1}(w) - Q_{b_2,s_2}(w)| < 2^{-m}$ for all $(b_1,s_1), (b_2,s_2) \in \mathfrak{E}_{\mathcal{P}}$.

3. For all $(b,s) \in \mathfrak{N}(B(0;1);m)$, then $(b,s) \in \mathfrak{E}_\mathcal{P}$ if $(a_0,r_0) \in \mathfrak{N}(D;m)$ and $(a,\delta) \in \mathfrak{B}$ exist which satisfy the following:

   (a) $\overline{B(a;\delta)} \subseteq B(a_0, r_0/2)$;

   (b) $\forall z \in B(a;\delta) \cap \bar{\mathbb{Q}}$ $(|P'_{a_0,r_0}(a) - P'_{a_0,r_0}(z)| \le |P'_{a_0,r_0}(a)|/8 - 3 \cdot 2^{-m}/r_0)$; and

   (c) $\overline{B(b;s)} \subseteq B(P_{a_0,r_0}(a); |P'_{a_0,r_0}(a)|\delta/2 - 2^{-m})$.

4. For all $(a,r) \in \mathfrak{N}(B(0;1);m)$, then $(a,r) \in \mathfrak{N}(D;m)$ if there exist $(b_0,s_0) \in \mathfrak{E}_\mathcal{P}$ and $(b,\delta) \in \mathfrak{B}$ which satisfy the following:

   (a) $\overline{B(b;\delta)} \subseteq B(b_0, s_0/2)$;

   (b) $\forall w \in B(b;\delta) \cap \bar{\mathbb{Q}}$ $(|Q'_{b_0,s_0}(b) - Q'_{b_0,s_0}(w)| \le |P'_{b_0,s_0}(b)|/8 - 3 \cdot 2^{-m}/s_0)$; and

   (c) $\overline{B(a;r)} \subseteq B(Q_{b_0,s_0}(b); |Q'_{b_0,s_0}(b)|\delta/2 - 2^{-m})$.

5. For all $(a,r) \in \mathfrak{N}(D;m)$, there exist $(b,s) \in \mathfrak{E}_\mathcal{P}$ and $w \in B(b,s) \cap \bar{\mathbb{Q}}$ such that $|P_{a,r}(a) - w| < 2^{-m+1}$ and $|a - Q_{b,s}(w)| < 2^{-m+1}$.

6. For all $(b,s) \in \mathfrak{E}_\mathcal{P}$, there exist $(a,r) \in \mathfrak{N}(D;m)$ and $z \in B(a,r) \cap \bar{\mathbb{Q}}$ such that $|Q_{b,s}(b) - z| < 2^{-m+1}$ and $|b - P_{a,r}(z)| < 2^{-m+1}$.

Intuitively, P is an approximation of a holomorphic function from D to $\mathbb{C}$, and Q is an approximation of a holomorphic function from $E_\mathcal{P} = \bigcup_{(b,s)\in\mathfrak{E}_\mathcal{P}} B(b;s)$ to $\mathbb{C}$. Conditions 1 and 2 mean that P and Q are well-defined. Condition 3 means that if $P'(z) \neq 0$, then $P(z) \in \mathrm{dom}(Q)$. More precisely, we can find a (sufficiently large) neighborhood $U \subseteq E_\mathcal{P}$ such that $P(z) \in U$ and Q can be the local inverse holomorphic function of P on U based on Lemma 5.10. Similarly, condition 4 means that $Q'(w) \neq 0$ implies $Q(w) \in \mathrm{dom}(P)$. Conditions 5 and 6 mean that Q is the inverse function of P if $P(D) \subseteq E_\mathcal{P}$ and $Q(E_\mathcal{P}) \subseteq D$.

We write $\mathcal{P}(z) \sim \alpha$ if $|P_{a,r}(z) - \alpha| < 2^{-m}$ for all $(a,r) \in \mathfrak{N}(D;m)$ with $z \in B(a;r)$. We write $|\mathcal{P}'(z)| \gtrsim K$ $(K \in \mathbb{R})$ if $|P'_{a,r}(z)| > K - 2^{-m}/r$ for all $(a,r) \in \mathfrak{N}(D;m)$ with $z \in B(a;r/2)$.

Let (h, h^{-1}) be a conformal map from D to an open set $E \subseteq B(0;1)$. An m-approximation polynomial $\mathcal{P} = (P,Q)$ is said to be an m-approximation of (h,h^{-1}) if $|h(z) - P_{a,r}(z)| < 2^{-m-1}$ for all $z \in D$ and for all $(a,r) \in \mathfrak{N}(D;m)$.

Claim 5.16.1. *If (h,h^{-1}) is a conformal map from D to an open set $E \subseteq B(0;1)$, then an m-approximation of (h,h^{-1}) exists for all $m \in \mathbb{N}$. Moreover, if $\mathcal{P}$ is an*

m-approximation of (h, h^{-1}), then $h(0) = 0$ implies $\mathcal{P}(0) \sim 0$; also, $|h'(0)| \geq K$ ($K \in \mathbb{Q}$) implies $|\mathcal{P}'(0)| \gtrsim K$.

To show this claim, we construct an m-approximation polynomial using (h, h^{-1}). We define $\mathfrak{E}_{\mathcal{P}}$ as $\mathfrak{E}_{\mathcal{P}} = \mathfrak{N}(E; m)$. Then, by Theorem 4.15, we define $l_{\mathcal{P}}$ as the least l which satisfies the following:

$$\forall (a,r) \in \mathfrak{N}(D;m)\ \forall z \in B(a,r)\ \left|\sum_{j=0}^{l} \frac{h^{(j)}(a)}{j!}(z-a)^j - h(z)\right| < 2^{-m-2}; \textit{and}$$

$$\forall (b,s) \in \mathfrak{N}(E;m)\ \forall w \in B(b,s)\ \left|\sum_{j=0}^{l} \frac{h^{-1(j)}(a)}{j!}(w-b)^j - h^{-1}(w)\right| < 2^{-m-2}.$$

For each $(a, r) \in \mathfrak{N}(d; m)$ and $(b, s) \in \mathfrak{N}(E; m)$, take $\sigma_{a,r}$ and $\tau_{b,s}$ such that $|h^{(k)}(a)/k! - \sigma_{a,r}(k)| < 2^{-m-k-3}$, $|h^{-1(k)}(b)/k! - \tau_{b,s}(k)| < 2^{-m-k-3}$ and $\sigma_{a,r}(k), \tau_{b,s}(k) \in \bar{\mathbb{Q}}(m + k + 3)$. Define $P(a, r) := \sigma_{a,r}$, $Q(b, s) := \tau_{b,s}$ and $\mathcal{P} = (P, Q)$. Then, clearly, $|h(z) - P_{a,r}(z)| < 2^{-m-1}$ for all $(a, r) \in \mathfrak{N}(D; m), z \in B(a, r)$, and $|h^{-1}(w) - Q_{b,s}(w)| < 2^{-m-1}$ for all $(b, s) \in \mathfrak{E}_{\mathcal{P}}, w \in B(b, s)$. Consequently, we can readily verify that conditions 1, 2, 5 and 6 hold. We show that condition 3 also holds. By Lemma 5.12, $|h(z) - P_{a_0,r_0}(z)| < 2^{-m-1}$ for all $z \in B(a_0, r_0)$ and $\overline{B(a;\delta)} \subseteq B(a_0, r_0/2)$ implies $|h'(z) - P'_{a_0,r_0}(z)| < 2^{-m}/r_0$ for all $z \in B(a_0, r_0)$. Therefore, (b) implies $\forall z \in B(a; \delta)$ $|h'(a) - h'(z)| \leq |h'(a)|/8$, and (c) implies $\overline{B(b; s)} \subseteq B(h(a); |h'(a)|\delta/2)$. Therefore, by Lemma 5.10, $\overline{B(b; s)} \subseteq h(\overline{B(a; \delta)}) \subseteq E$, which means that $(b, s) \in \mathfrak{N}(E; m) = \mathfrak{E}_{\mathcal{P}}$. We can show that condition 4 holds similarly.

It is readily apparent that $h(0) = 0$ implies $\mathcal{P}(0) \sim 0$. By Lemma 5.12, $|h'(0)| \geq K$ ($K \in \mathbb{Q}$) implies $|\mathcal{P}'(0)| \gtrsim K$. This completes the proof of the claim.

From now on, we use arguments of non-standard analysis. Let $V = (M, S)$ be a countable model of ACA_0 and let ${}^*V = ({}^*M, {}^*S)$ be an extension of V which satisfies the three conditions in Theorem 5.2. Let $a, b \in {}^*\mathbb{C}$; then we use the notation $a \approx b$ if $\forall p \in \mathbb{N}\ {}^*V \models |a - b| < 2^{-p}$.

Let D be an open subset of $B(0; 1)$ in V. Take $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$. Let $\mathcal{P} = (P, Q)$ be an ω-approximation polynomial on ${}^*D$ in ${}^*V$. (Actually, we should write '*approximation polynomial' for approximation polynomial in ${}^*V$, but we usually omit $*$.) We define the standard part of P as follows. Let $\langle (a_n, r_n) \rangle_{n \leq |\mathfrak{N}({}^*D;\omega)|}$ be an enumeration (in ${}^*V$) of all elements of $\mathfrak{N}({}^*D; \omega)$ such that $r_p > r_q \rightarrow p < q$. Because $|\mathfrak{N}({}^*D; m)| < 4^{m+3}$ and $\mathfrak{N}(D; m) = \mathfrak{N}({}^*D; m)$ for all $m \in \mathbb{N}$, if $n \in \mathbb{N}$, then $(a_n, r_n) \in \mathfrak{N}(D; m)$ for some $m \in \mathbb{N}$. Let $P(a_n, r_n) = \sigma_n$. Take $q_{n,k,p} \in \bar{\mathbb{Q}}(p+1)$ such that $|q_{n,k,p} - \sigma_n(k)|$ (in ${}^*V$). By Lemma 5.12, $q_{n,k,p} < 1 + 1/{r_n}^k$. Consequently, if $n, k, p \in \mathbb{N}$, then $q_{n,k,p} \in M$. Hence, if $n, k \in \mathbb{N}$, $\{q_{n,k,p}\}_{p\in\mathbb{N}} \in S$ is a complex number in V. (Actually, $\{q_{n,k,p}\}_{p\in\mathbb{N}}$ is a standard part of $\sigma_n(k)$, *i.e.*, $\sigma_n(k)|_M = \{q_{n,k,p}\}_{p\in\mathbb{N}}$.) Let $\hat{r}_n := r_n/2$ and $\alpha_{n,k} := \{q_{n,k,p}\}_{p\in\mathbb{N}}$. We define $P|_M \in S$, the standard part of P, as $P|_M = (\{(a_n, \hat{r}_n)\}_{n\in\mathbb{N}}, \{\alpha_{nk}\}_{n\in\mathbb{N},k\in\mathbb{N}})$. (If necessary, we can redefine a code for an approximation polynomial so that a code of the standard part of P is exactly the set $P \cap M$.) Similarly, we define $Q|_M = (\{(b_n, \hat{s}_n)\}_{n\in\mathbb{N}}, \{\beta_{nk}\}_{n\in\mathbb{N},k\in\mathbb{N}})$, the standard part of Q, using an enumeration $\langle (b_n, s_n)\rangle_{n\le|\mathfrak{E}_\mathcal{P}|}$ of all elements of $\mathfrak{E}_\mathcal{P}$. We define the standard part of $E_\mathcal{P} = \bigcup_{(b,s)\in\mathfrak{E}_\mathcal{P}} B(b;s)$ as $E_\mathcal{P}|_M = \bigcup_{n\in\mathbb{N}} B(b_n, \hat{s}_n) \in S$. Next, we show that $P|_M$ and $Q|_M$ are holomorphic functions.

Claim 5.16.2. *Let $\omega \in {}^*\mathbb{N}\backslash\mathbb{N}$, and let $\mathcal{P} = (P, Q)$ be an ω-approximation polynomial on ${}^*D$ with $|\mathcal{P}'(0)| \gtrsim 1$. Define the standard parts of P, Q and $E_\mathcal{P}$ as above. Let $f := P|_M$, $g := Q|_M$ and $E_f := E_\mathcal{P}|_M$. Then, f and g are holomorphic functions and (f, g) is a conformal map from D to E_f in V. Moreover, $f(0) = 0$ if $\mathcal{P}(0) \sim 0$; also, $|f'(0)| \ge K|_M$ if $|\mathcal{P}'(0)| \gtrsim K$ ($K \in {}^*\mathbb{R}$ and $K \le K_0$ for some $K_0 \in \mathbb{N}$).*

We first show that f is a holomorphic function on D. Clearly, $\bigcup_{n\in\mathbb{N}} B(a_n; \hat{r}_n) = D$. Let $n \in \mathbb{N}$. By condition 5, $|P_{a_n,r_n}(z)| \le 1+2^{-\omega+1}$ in ${}^*V$. Hence, by Lemma 5.12, $|P^{(k)}_{a_n,r_n}(a_n)| \le (1 + 2^{-\omega+1})/{r_n}^k$ for all $k \in \mathbb{N}$. Then, $|{}^*\alpha_{n,k}|{\hat{r}_n}^k \approx |P^{(k)}_{a_n,r_n}(a_n)|{\hat{r}_n}^k \le (1+2^{-\omega+1})2^{-k} < 2^{-k+1}$. Consequently, $|\alpha_{n,k}|{\hat{r}_n}^k < 2^{-k+1}$ for all $k \in \mathbb{N}$ and the series $\sum_{k\in\mathbb{N}} |\alpha_{n,k}|{\hat{r}_n}^k$ converges. If $z \in B(a_p; \hat{r}_p) \cap B(a_q; \hat{r}_q)$ ($p, q \in \mathbb{N}$) in V, then, in ${}^*V$,

$$\begin{aligned}
\sum_{k\in{}^*\mathbb{N}} {}^*\alpha_{p,k}({}^*z - a_p)^k &\approx \sum_{k=0}^{l_\mathcal{P}} {}^*\alpha_{p,k}({}^*z - a_p)^k \\
&\approx P_{a_p,r_p}({}^*z) \\
&\approx P_{a_q,r_q}({}^*z) \\
&\approx \sum_{k=0}^{l_\mathcal{P}} {}^*\alpha_{q,k}({}^*z - a_q)^k \\
&\approx \sum_{k\in{}^*\mathbb{N}} {}^*\alpha_{q,k}({}^*z - a_q)^k.
\end{aligned}$$

Hence, in V,

$$\sum_{k\in\mathbb{N}} \alpha_{p,k}(z - a_p)^k = \sum_{k\in\mathbb{N}} \alpha_{q,k}(z - a_q)^k.$$

Thus, f is a holomorphic function on D. We can similarly show that g is a holomorphic function on E_f. It is also apparent that $f(0) = 0$ if $\mathcal{P}(0) \sim 0$. Because $|{}^*f(z) - P_{a_n,r_n}(z)| \approx 0$ for all $z \in B(a_n; r_n)$ in ${}^*V$, we can show readily that

$|f'(0)| \geq K|_M$ if $|\mathcal{P}'(0)| \gtrsim K$ ($K \in {}^*\mathbb{R}$ and $K \leq K_0$ for some $K_0 \in \mathbb{N}$) using Lemma 5.12.

Next, we show that $f(z) \in E_f$ and $g(f(z)) = z$ for all $z \in D$ with $f'(z) \neq 0$. Note that $|\mathcal{P}'(0)| \gtrsim 1$ implies that f is not a constant. Let $z_1 \in D$ with $f'(z_1) \neq 0$. Then, in V, there exists $(a, \delta) \in \mathfrak{B}$ with $z_1 \in B(a; \delta/2)$ which satisfies the following:

- $|f'(a)| > 0$ and $|f'(z) - f'(a)| \leq |f'(a)|/16$ for all $z \in B(a; \delta)$;
- there exists $n \in \mathbb{N}$ such that $\overline{B(a;\delta)} \subseteq B(a_n; \hat{r}_n)$.

In ${}^*V$, by Lemma 5.12, $|{}^*f'(z) - P'_{a_n,r_n}(z)| \approx 0$ for all $z \in B(a_n; \hat{r}_n)$ because $|{}^*f(z) - P_{a_n,r_n}(z)| \approx 0$ for all $z \in B(a_n; r_n)$. Then, in ${}^*V$, for all $z \in B(a; \delta)$,

$$\begin{aligned}
|P'_{a_n,r_n}(z) - P'_{a_n,r_n}(a)| &\leq \frac{|{}^*f'(a)|}{16} + |{}^*f(z) - P_{a_n,r_n}(z)| \\
&\quad + |{}^*f(a) - P_{a_n,r_n}(a)| \\
&\leq \frac{|P'_{a_n,r_n}(a)|}{8} - \frac{3 \cdot 2^{-\omega}}{r_n} - \Big(\frac{|{}^*f'(a)|}{16} - \frac{3 \cdot 2^{-\omega}}{r_n} \\
&\quad - \frac{9}{8}|{}^*f(a) - P_{a_n,r_n}(a)| - |{}^*f(z) - P_{a_n,r_n}(z)|\Big) \\
&\leq \frac{|P'_{a_n,r_n}(a)|}{8} - \frac{3 \cdot 2^{-\omega}}{r_n}.
\end{aligned}$$

On the other hand, ${}^*f({}^*z_1) \in B({}^*f(a); |{}^*f'(a)|\delta/8)$ and $B({}^*f(a); |{}^*f'(a)|\delta/4) \subseteq B(P_{a_n,r_n}(a); |P'_{a_n,r_n}(a)|\delta/2 - 2^{-\omega})$ in ${}^*V$. Then, because $\delta/8 \not\approx 0$, there exists $p \in \mathbb{N}$ and there exists $(b, s) \in \mathfrak{N}(B(0; 1); p)$ such that ${}^*f({}^*z_1) \in B(b, s/4)$ and $B(b, s) \subseteq B(P_{a_n,r_n}(a); |P'_{a_n,r_n}(a)|\delta/2 - 2^{-\omega})$ in ${}^*V$. By condition 3, $(b, s) \in \mathfrak{E}_{\mathcal{P}}$. For that reason, there exists $p \in \mathbb{N}$ such that $(b, s) = (b_p, s_p)$ and $f(z_1) \in B(b_p; \hat{s}_p)$ in V, *i.e.*, $f(z_1) \in E_f$. Moreover, by condition 5, $\check{w}_1 \approx {}^*f({}^*z_1)$ exists such that $\check{w}_1 \in B(b; s)$ and $Q_{b,s}(\check{w}_1) \approx {}^*z_1$ in ${}^*V$. Hence, in V, $f(z_1) = \check{w}_1|_M$ and $g(\check{w}_1|_M) = z_1$, *i.e.*, $g(f(z_1)) = z_1$.

Next, we show that $f'(z) \neq 0$ for all $z \in D$. Then, $f(z) \in E_f$ and $g(f(z)) = z$ for all $z \in D$. Assume that there exists $z_1 \in D$ such that $f'(z_1) = 0$. By Lemma 5.15, there exist $z_2, z_3 \in D$ such that $z_2 \neq z_3$, $f'(z_2) \neq 0$, $f'(z_3) \neq 0$ and $f(z_2) = f(z_3)$. Thereby, $z_2 = g(f(z_2)) = g(f(z_3)) = z_3$, which contradicts $z_2 \neq z_3$.

Similarly, we can show that $g(w) \in D$ and $f(g(w)) = w$ for all $w \in E_f$. This completes the proof of this claim.

Now, we are ready to prove Sublemma 2. Take $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$. In ${}^*V$, we define the set Ω as $\Omega := \{q \in Q(\omega) \mid 1 \leq q$ and there exists an ω-approximation polynomial

$\mathcal{P}$ on ${}^*D$ such that $E_\mathcal{P} \subseteq B(0;1)$, $\mathcal{P}(0) \sim 0$ and $|\mathcal{P}'(0)| \gtrsim q.\}$. (The condition $q \in \Omega$ is expressible by an arithmetical formula.) Then, Ω is not empty because there exists an ω-approximation polynomial $\mathcal{P}_I = (P_I.Q_I)$ of identity map (I, I^{-1}) from D to D. By Lemma 5.12, Ω is finite; therefore, $\max\Omega$ exists. Take an ω-approximation polynomial $\mathcal{P}$ such that $\mathcal{P}(0) \sim 0$ and $|\mathcal{P}'(0)| \gtrsim \max\Omega$, and define $f := P|_M$, $g := Q|_M$, $E_f := E_\mathcal{P}|_M$ and $K := (\max\Omega)|_M$. Then, by Claim 2, $(f, g) \in S$ is a conformal map from D to E_f with $f(0) = 0$ and $|f'(0)| \geq K$. We show that (f, g) meets the requirements of Sublemma 2 in V. If not, there exists a conformal map $(\tilde{f}, \tilde{g}) \in S$ from D to an open set $\tilde{E} \subseteq B(0;1)$ such that $\tilde{f}(0) = 0$ and $\tilde{f}'(0) > f'(0) \geq K$. Then, there exists $m \in \mathbb{N}$ such that $|\tilde{f}'(0)| > K + 2^{-m}$. In ${}^*V$, because $|{}^*\tilde{f}'(0)| > {}^*K + 2^{-m}$, an ω-approximation polynomial $\tilde{\mathcal{P}}$ exists on ${}^*D$ such that $\tilde{\mathcal{P}}(0) \sim 0$ and $|\tilde{\mathcal{P}}'(0)| \gtrsim {}^*K + 2^{-m}$ by Claim 1. Therefore, $|\tilde{\mathcal{P}}'(0)| \gtrsim 2^{-\omega} + \max\Omega$ because ${}^*K \approx \max\Omega \approx 2^{-\omega} + \max\Omega$. However, that would mean that $2^{-\omega} + \max\Omega \in \Omega$ (contradiction). Therefore, Sublemma 2 holds in V. By the completeness theorem, Sublemma 2 can be proven in $\mathsf{ACA_0}$. This completes the proof of this theorem. □

Theorem 5.17 (reversal)**.** *The following assertions are pairwise equivalent over* $\mathsf{WKL_0}$.

1. $\mathsf{ACA_0}$.
2. *If $D \subseteq \mathbb{C}$ is a simply connected open set and $D \neq \mathbb{C}$, then there exists a conformal map $f : D \to B(0;1)$.*

Proof. The implication $1 \to 2$ is already proven in Theorem 5.16.

To prove $2 \to 1$, we show the convergence of bounded increasing real positive Cauchy sequences because this convergence is equivalent to $\mathsf{ACA_0}$ over $\mathsf{RCA_0}$ (see [29]). Let $\{a_n\}_{n\in\mathbb{N}}$ be an increasing real positive Cauchy sequence such that $0 < a_n < 1$. Define an open set $U \subseteq \mathbb{C}$ as

$$U := \bigcup_{n\in\mathbb{N}} B(0; a_n).$$

Clearly, $0 \in U$, $U \neq \mathbb{C}$ and U is simply connected. Thus, there exists a conformal map $h : U \to B(0;1)$ such that $h(0) = 0$.

By $h(0) = 0$ and Taylor expansion, there exists a holomorphic function $g_1 : U \to \mathbb{C}$ such that $h(z) = zg_1(z)$. Then,

$$\forall n \in \mathbb{N}\ \forall z \in \partial B(0; a_n)\ |g_1(z)| = \frac{|h(z)|}{|z|} \leq \frac{1}{a_n}.$$

By the maximal value principle,

$$\forall n \in \mathbb{N}\ \forall z \in B(0; a_n)\ |g_1(z)| \le \frac{1}{a_n}.$$

Thus, by the increasingness of $\{a_n\}_{n\in\mathbb{N}}$,

$$\forall n \in \mathbb{N}\ \forall z \in U\ |g_1(z)| \le \frac{1}{a_n}. \tag{48}$$

Similarly, by $h^{-1}(0) = 0$ and Taylor expansion, there exists a holomorphic function $g_2 : B(0; 1) \to \mathbb{C}$ such that $h^{-1}(w) = zg_2(w)$. If K, r are positive real numbers such that $\forall n \in \mathbb{N}\ a_n \le K$ and $r < 1$,

$$\forall n \in \mathbb{N}\ \forall w \in \partial B(0; r)\ |g_2(w)| = \frac{|h^{-1}(w)|}{|w|} \le \frac{K}{r}.$$

Hence, by the maximal value principle,

$$\forall n \in \mathbb{N}\ a_n \le K \to \forall z \in B(0; 1)\ |g_2(w)| \le K. \tag{49}$$

Let $z_0 \in U \setminus \{0\}$ and $w_0 = h(z_0)$. Let ε be an arbitrary positive real number. Then, by Cauchyness of $\{a_n\}_{n\in\mathbb{N}}$, there exists $N \in \mathbb{N}$ such that $\forall n \ge N\ a_N \le a_n \le a_N + \varepsilon$. By (48),

$$a_N \le \frac{|z_0|}{|h(z_0)|}.$$

By (49),

$$\frac{|z_0|}{|h(z_0)|} = \frac{|h^{-1}(w_0)|}{|w_0|} \le a_N + \varepsilon.$$

Hence,

$$\forall n \ge N\ \left| \frac{|z_0|}{|h(z_0)|} - a_n \right| \le \varepsilon.$$

Consequently, $\{a_n\}_{n\in\mathbb{N}}$ converges to $|z_0|/|h(z_0)|$. □

5.4 Application 2: the Jordan curve theorem

In this section, we prove Jordan curve theorem within $\mathsf{WKL_0}$ using non-standard arguments. This is a joint work with Nobuyuki Sakamoto. To prove the Jordan curve theorem, we need to show that the interior of a given Jordan curve is non-empty. To show this, we usually use assertions which require $\mathsf{ACA_0}$ such as *the leftmost maximal value principle* (appears in [22]) or the Bolzano/Weierstraß theorem. Our main tool to prove the Jordan curve theorem within $\mathsf{WKL_0}$ is an argument of non-standard analysis. By using non-standard arguments appropriately, Jordan curves can be treated as polygons.

5.4.1 Preparations

In this subsection, we prepare some notions and technical lemmas for the Jordan curve theorem.

In this section, we write $d^i_{\mathbb{R}}$ (or just d for simplicity) for the metric in $\mathbb{R}^i$, *i.e.*, $d^i_{\mathbb{R}}(x,x') = \|x - x'\|$ where $x, x' \in \mathbb{R}^i$. For (a,r), $(a',r') \in \mathbb{Q}^i \times \mathbb{Q}^+$, we write $(a,r) < (a',r')$ as an abbreviation for $d(a,a') + r < r'$.

Theorem 5.18 (Simpson[29] Theorem IV.2.3)**.** *The following assertions are pairwise equivalent over* RCA_0*.*

1. WKL_0.
2. *Every continuous function from* $[0,1]$ *to* $\mathbb{R}$ *has a modulus of uniform continuity and a supremum, and attains the supremum.*
3. *Every continuous function from* $[0,1]$ *to* $\mathbb{R}$ *which has a supremum, attains it.*

We can strengthen the equivalence of 1 and 3 as follows:

Theorem 5.19. *The following is provable in* WKL_0*. Let* $\langle \phi_i : i \in \mathbb{N}\rangle$ *be a sequence of continuous functions* $\phi_i : [0,1] \to \mathbb{R}$*. Then, there exists a sequence* $\langle x_i : i \in \mathbb{N}\rangle$ *such that* $\phi_i(x_i) = \max_{y\in[0,1]} \phi_i(y)$*.*

Proof. We reason within WKL_0. Let $\langle \phi_i : i \in \mathbb{N}\rangle$ be a sequence of continuous functions $\phi_i : [0,1] \to \mathbb{R}$. Let $\varphi(i,x)$ be a Π^0_1-formula which expresses $\forall q \in \mathbb{Q} \cap [0,1](\phi_i(x) \geq \phi_i(q))$. By Theorem 5.18, there exists $x \in [0,1]$ such that $\phi_i(x) = \max_{y\in[0,1]} \phi_i(y)$ for each i. Then, we have $\forall i \exists x \in [0,1]\varphi(i,x)$. By Π^0_1-AC_0, which is provable within WKL_0 [29, Lemma VIII.2.5], there exists a sequence $\{x_i : i \in \mathbb{R}\}$ such that $\phi(x_i) \geq \phi_i(q)$ for all $q \in \mathbb{Q} \cap [0,1]$ and all $i \in \mathbb{N}$. This means that $\phi_i(x_i) = \max_{y\in[0,1]} \phi_i(y)$ for each i. □

Let $U \subseteq \mathbb{R}^2$ be an open set. An *arc* A in U is a continuous function from $[0,1]$ to U. By Theorem 5.18, within WKL_0, we can define the metric $d(A,a) = \min_{x\in[0,1]} d(A(x),a)$ between an arc A and a point $a \in \mathbb{R}^2$. Let $\sigma = \langle a_j \in \mathbb{R}^2 : j < n\rangle$ be a finite real sequence. A *broken-line* $B[\sigma]$ is an arc defined by putting $B[\sigma](x + i/n) = nx\sigma(i+1) + (1 - nx)\sigma(i)$ for each $i = 0, \ldots, n-1$ and a *polygon* $P[\sigma]$ is an arc defined as $P[\sigma] = B[\sigma^\frown\langle\sigma(0)\rangle]$. We sometimes put $\sigma(\mathrm{lh}(\sigma)) := \sigma(0)$ so that edges of $P[\sigma]$ can be written as $\{B[\langle\sigma(j), \sigma(j+1)\rangle] : j < \mathrm{lh}(\sigma)\}$. Note that we can define the metric $d(B,a) = \min_{x\in[0,1]} d(B(x),a)$ between a broken-line B and a point $a \in \mathbb{R}^2$ within $\mathsf{RCA_0}$. A *Jordan curve* J in U is an arc in U with $J(0) = J(1)$ and $\forall x, y \in [0,1](J(x) = J(y) \to (x = y \vee |x - y| = 1))$. A polygon is said to be *simple* if it is also a Jordan curve.

Lemma 5.20. *The following is provable in* $\mathsf{WKL_0}$. *Let* U *be an open set in* $\mathbb{R}^2$ *and* A *be an arc in* U. *Let* $k \in \mathbb{N}$. *Then there exists a finite sequence* $\langle (m_i, b_i, s_i) : i < c \rangle$ *such that*

1. A *is an arc in the open set* $\bigcup_{i<c} B(b_i; s_i)$;
2. $d(b_0, A(0)) < s_0$ *and* $d(b_{c-1}, A(1)) < s_{c-1}$;
3. $d(b_i, b_{i+1}) < s_i + s_{i+1}$ *for all* $i < c - 1$;
4. $s_i < 2^{-k}$ *for all* $i < c$;
5. $\forall i < c \; \exists (n, b, s) \in U \; (b_i, s_i) < (b, s)$.

Here, $B(a; r) := \{x \in \mathbb{R}^2 : d(a, x) < r\}$.

Proof. We reason within $\mathsf{WKL_0}$. Let Φ_A be a code for A. Fix $k \in \mathbb{N}$. Define $\Psi \subseteq \mathbb{N} \times \mathbb{Q} \times \mathbb{Q}^+ \times \mathbb{Q}^2 \times \mathbb{Q}^+$ as $\exists m'(m', a', r', b', s') \in \Psi \leftrightarrow \exists (\bar{n}, \bar{b}, \bar{s}) \in U \; \exists (m, a, r, b, s) \in \Phi_A \; (a', r') < (a, r) \wedge (b, s) < (b', s') \wedge (b', s') < (\bar{b}, \bar{s})$.

Because A is defined entirely on $[0,1]$, by Theorem 2.7, we can find a finite sequence $\langle (m_i, a_i, r_i, b_i, s_i) : i < c \rangle$ from Ψ such that $d(a_0, 0) < r_0, d(a_{c-1}, 1) < r_{c-1}$, $\forall i < c - 1(d(a_i, a_{i+1}) < r_i + r_{i+1})$ and $\forall i < c(s_i < 2^{-k})$. Then the sequence $\langle (m_i, b_i, s_i) : i < c \rangle$ satisfies the desired property. □

The following theorem establishes the equivalence of arcwise connectedness and broken-line-wise connectedness within $\mathsf{WKL_0}$.

Theorem 5.21. *The following is provable in* $\mathsf{WKL_0}$. *Let* U *be an open set in* $\mathbb{R}^2$ *and* A *be an arc in* U. *Let* $k \in \mathbb{N}$. *Then there exists a broken-line* B *in* $\mathbb{R}^2$ *such that* $B(0) = A(0)$, $B(1) = A(1)$, $\forall x \in [0,1] \exists y \in [0,1] \; d(B(x), A(y)) < 2^{-k}$ *and* $\forall y \in [0,1] \exists x \in [0,1] \; d(B(x), A(y)) < 2^{-k}$.

Proof. We reason within $\mathsf{WKL_0}$. By Lemma 5.20, let $\langle (m_i, b_i, s_i) : i < c \rangle$ be a finite sequence from U that satisfies Clauses $1, 2, 3, 4, 5$ in Lemma 5.20. Then the broken-line $B[\langle A(0), b_0, \ldots, b_{c-1}, A(1) \rangle]$ satisfies the desired property. □

We define a *locus* L_A of an arc A as a closed set such that $\forall u \in \mathbb{R}^2(u \in L_A \leftrightarrow \exists x \in [0,1]\ A(x) = u)$. We can show the existence of the locus of any given arc within $\mathsf{WKL_0}$

Lemma 5.22. $\mathsf{WKL_0}$ *proves that every arc in* $\mathbb{R}^2$ *has a locus.*

Proof. We reason in $\mathsf{WKL_0}$. Let A be an arc in $\mathbb{R}^2$. By Theorem 5.19, let $\langle r_q : q \in \mathbb{Q}^2 \rangle$ be a sequence of reals such that $r_q = \min_{x\in[0,1]} d(q, A(x))$. By Σ^0_0 comprehension, we can take a (code for an) open set U such that $\exists n\ (n, b, s) \in U$ if and only if $r_b > s$. Then $L_A := U^c$ is the desired locus. □

We define a new notion for the Jordan curve theorem. A pair of open sets (V, W) is said to be a *partition* of an open set U if $V \cup W = U$, $V \cap W = \emptyset$.

The Jordan curve theorem for simple polygons are provable within $\mathsf{RCA_0}$.

Theorem 5.23 (The Jordan curve theorem for simple polygons)**.** *The following is provable in* $\mathsf{RCA_0}$. *Let* P *be a simple polygon. Then, we can find the locus* L_P *and a partition* (U_0, U_1) *of* ${L_P}^c$ *such that*

1. U_0 *is bounded, i.e., there exists* $r \in \mathbb{Q}^+$ *such that* $d(u, (0, 0)) < r$ *for all* $u \in U_0$*;*

2. U_0 *and* U_1 *are broken-line connected, i.e., for any* $u, v \in U_i$*, there exists a broken-lines* B_i *in* U_i *such that* $B_i(0) = u, B_i(1) = v$ *for* $i = 0, 1$*;*

3. *For each* $w \in L_P$ *and each* $r \in \mathbb{Q}^+$*, there exist points* $u \in U_0, v \in U_1$ *such that* $d(w, u) < r$ *and* $d(w, v) < r$*;*

4. *Every arc connecting a point in* U_0 *and a point in* U_1 *meets* P*;*

hold.

In this situation, U_0 and U_1 are said to be the *interior* and *exterior* of P respectively, and ${U_1}^c = U_0 \cup L_P$ is said to be a *Jordan region.*

Proof. Within $\mathsf{RCA_0}$, we can find the locus of a given polygon P as in the proof of Lemma 5.9. Thus, we can imitate a usual proof, e.g., lemma 1 in [35] in $\mathsf{RCA_0}$. □

Lemma 5.24. *The following assertion is provable in* $\mathsf{WKL_0}$. *Let* J *be a Jordan curve. Then there exists a function* $H : \mathbb{N} \to \mathbb{N}$ *such that*

1. $\forall n \exists m \forall l > m\ H(l) > n$,

2. $\forall x, y \in [0,1]\ \forall n(\min(|x-y|, 1-|x-y|) > 2^{-H(n)} \to d(J(x), J(y)) > 2^{-n})$.

Proof. We reason within $\mathsf{WKL_0}$. Let J be a Jordan curve. By Theorem 5.18, let h be a modulus of uniform continuity for J. We can assume that $h(n+1) > h(n)$ for all n. Put $h_n = 2^{h(n+2)}$. Define H by putting

$$\begin{aligned} H(n) = \ & \text{the greatest } l \leq n \text{ such that} \\ & \quad \forall i, j \leq h_n(\min(|i/h_n - j/h_n|, 1 - |i/h_n - j/h_n|) \\ & \qquad > 2^{-l-1} \to d(J(i/h_n), J(j/h_n))_{n+1} > 2^{-n+2}), \end{aligned}$$

where $d(J(i/h_n), J(j/h_n))_{n+1}$ is the $n+1$-st approximation of $d(J(i/h_n), J(j/h_n))$.

We shall prove that H satisfies Clause 2 in the lemma. Let $x, y \in [0,1]$ and $n \in \mathbb{N}$. Assume $\min(|x-y|, 1-|x-y|) > 2^{-H(n)}$. Let i be the least i' such that $i'/h_n \geq x$ and j be the least j' such that $j'/h_n \geq y$. Then

$$\begin{aligned} & |i/h_n - j/h_n| \\ > \ & |x-y| - |x - i/h_n| - |y - j/h_n| \\ \geq \ & |x-y| - 2^{-n-1} \\ \geq \ & 2^{-H(n)} - 2^{-H(n)-1} \\ = \ & 2^{-H(n)-1}. \end{aligned}$$

Similarly, $1-|i/h_n - j/h_n| > 2^{-H(n)-1}$. Thus by the definition of H, $d(J(i/h_n), J(j/h_n)) > 2^{-n+1}$. Hence

$$\begin{aligned} & d(J(x), J(y)) \\ \geq \ & d(J(i/h_n), J(j/h_n)) - d(J(x), J(i/h_n)) - d(J(y), J(j/h_n)) \\ > \ & 2^{-n+1} - 2^{-n-1} \\ > \ & 2^{-n}. \end{aligned}$$

This implies that H satisfies Clause 2 in the lemma.

It remains to be shown that H satisfies Clause 1 in the lemma. Assume that $\forall m\, \exists l > m\ H(l) \leq N$ for some $N \in \mathbb{N}$. Let $\langle (n, (a_n, b_n), r_n) \in \mathbb{N} \times (\mathbb{Q} \cap [0,1])^2 \times \mathbb{Q}^+ : n \in \mathbb{N} \rangle$ be an enumeration of all $((a,b), r) \in (\mathbb{Q} \cap [0,1])^2 \times \mathbb{Q}^+$ such that

$$\begin{aligned} & \min(|a-b|, 1-|a-b|) \leq 2^{-N-2} - r \\ & \qquad \vee \exists m(d(J(a), J(b)) > 2^{-m+3} \wedge r < h(m+1)). \end{aligned}$$

Fix $s \in \mathbb{N}$. We shall show that the open set $\{(n,(a_n,b_n),r_n) : n \leq s\}$ does not cover $[0,1] \times [0,1]$. Take a sequence $m_0, \ldots, m_s$ such that

$$\min(|a_n - b_n|, 1 - |a_n - b_n|) \leq 2^{-N-2} - r \\ \vee ((d(J(a_n), J(b_n)) > 2^{-m_n+3} \wedge r < h(m_n+1))$$

for all $n \leq s$. Choose t so that $t > m_0, \ldots, m_s, N$ and $H(t) \leq N$. By the definition of H, take $i, j \leq h_t$ such that $\min(|i/h_t - j/h_t|, 1 - |i/h_t - j/h_t|) > 2^{-H(t)-2}$, and $d(J(i/h_t), J(j/h_t))_{t+1} \leq 2^{-t+2}$. Then, $\min(|i/h_t - j/h_t|, 1 - |i/h_t - j/h_t|) > 2^{-N-2}$, since $H(t) \leq N$. Moreover, for any $n \leq s$, if $|a_n - i/h_t|, |b_n - j/h_t| < 2^{-h(m_n+1)}$, then

$$\begin{aligned} & d(J(a_n), J(b_n)) \\ \leq\ & d(J(a_n), J(i/h_t)) + d(J(i/h_t), J(j/h_t)) + d(J(j/h_t), J(b_n)) \\ <\ & 2^{-m_n-1} + 2^{-t+2} + 2^{-m_n-1} \leq 2^{-m_n+3}. \end{aligned}$$

This means that $d((i/h_t, j/h_t), (a_n, b_n)) < h(m_n+1) \to d(J(a_n), J(b_n)) < 2^{-m_n+3}$. Therefore, the point $(i/h_t, j/h_t)$ does not belong to the open set $\{(n,(a_n,b_n),r_n) : n \leq s\}$.

By Theorem 2.7, we can find $(x,y) \in [0,1]^2$ such that (x,y) does not belong to the open set $\{(n,(a_n,b_n),r_n) : n \in \mathbb{N}\}$. Then, by the construction of this open set, $\min(|x-y|, 1-|x-y|) > 2^{-N-2}$ and $\forall m \in \mathbb{N} \forall a, b \in \mathbb{Q} \cap [0,1](|a-x| + |b-y| < h(m+1) \to d(J(a), J(b)) \leq 2^{-m+3}$. Thus, $|x-y| \neq 0, 1$ and $J(x) = J(y)$, which contradicts the assumption that J is a Jordan curve. This completes the proof for Clause 1 in the lemma. □

5.4.2 Non-standard proof for the Jordan curve theorem

In this subsection, we prove the Jordan curve theorem using arguments of non-standard analysis within WKL_0.

Theorem 5.25 (The Jordan curve theorem). *The following is provable in* WKL_0. *For each Jordan curve J, we can find the locus L_J of J and a partition (U_0, U_1) of $(L_J)^c$ such that*

1. *U_0 is bounded, i.e., there exists $r \in \mathbb{Q}^+$ such that $d(u,(0,0)) < r$ for all $u \in U_0$;*
2. *For each $w \in L_J$ and each $r \in \mathbb{Q}^+$, there exists a point $a \in U_0 \cap \mathbb{Q}^2$ such that $d(w,a) < r$;*

3. *Let* $N_0 \in \mathbb{N}$ *and let* $u, v \in U_0 \cap \mathbb{Q}^2$. *Let* $x, y \in [0,1]$ *with* $d(u, J) = d(u, J(x))$ *and* $d(v, J) = d(v, J(y))$ *respectively. Without loss of generality, we assume* $x \leq y$. *Then, there exist broken lines* B_0, B_1 *such that* $B_0(0) = B_1(0) = u, B_0(1) = B_1(1) = v$ *and* $d(J_i, B_i(z)) \leq \max(d(J_i, u), d(J_i, v)) + 2^{-N_0}$ *for* $i = 0, 1$ *and for all* $0 \leq z \leq 1$. *Here,* J_0 *is the arc defined by* $J_0(z) = J(zy + (1-z)x)$, *and* J_1 *is the arc defined by*

$$J_1(z) = \begin{cases} J((1-2z)x) & \text{if } 0 \leq z \leq 1/2 \\ J((2z-1)y + 2 - 2z) & \text{if } 1/2 \leq z \leq 1; \end{cases}$$

hold.

In this situation, U_0 and U_1 are said to denote the *interior* and *exterior* of J, respectively. Our proof is a reformulation of a non-standard proof of the Jordan curve theorem [19].

Proof. Fix a countable non-standard model $V = (M, S)$ of $\mathsf{WKL_0}$ and fix a Jordan curve J (in V). By Theorem 5.1, V has an extension ${}^*V = ({}^*M, {}^*S)$ such that $S = \{X \cap M : X \in {}^*S\}$. Using both V and ${}^*V$, we will show that the Jordan curve theorem holds within V. Then, by the completeness theorem, the Jordan curve theorem is provable in $\mathsf{WKL_0}$.

Let $\Phi_J \in S$ be a code for J. Then, by the definition of continuous functions and Theorem 2.7, $\Phi_J \in V$ satisfies the following: for any $k \in M$, there exists $K \in M$ such that

(Ci) there exists a finite sequence $\sigma_k := \langle (n_i, a_i, r_i, b_i, s_i) \in \Phi_J|_{\leq K} : i < l_k \rangle$ such that $s_i < 2^{-k}$ and $[0,1] \subseteq \bigcup_{i \leq l_k} B(a_i; r_i)$;

(Cii) if $(n, a, r, b, s) \in \Phi_J|_{\leq K}$ and $(n, a, r, b', s') \in \Phi_J|_{\leq K}$, then $d(b, b') \leq s + s'$;

(Ciii) if $(n, a, r, b, s) \in \Phi_J|_{\leq K}$, $(n', a', r', b, s) \leq K$ and $(a', r') < (a, r)$, then $(n', a', r', b, s) \in \Phi_J|_{\leq K}$;

(Civ) if $(n, a, r, b, s) \in \Phi_J|_{\leq K}$, $(n', a, r, b', s') \leq K$ and $(b, s) < (b', s')$, then $(n', a, r, b', s') \in \Phi_J|_{\leq K}$;

hold in V. (Here, $\Phi_j|_{\leq K} := \{m \in \Phi_j : m \leq K\}$.)
Note that the above four conditions are expressed by Σ^0_0 formulas. Take a set ${}^*\Phi_J \in {}^*S$ such that ${}^*\Phi_J \cap M = \Phi_J$. Then, by overspill, there exists $\omega \in {}^*M \setminus M$ which satisfies the following: for any $k \leq \omega$, there exists $K \in {}^*M$ such that

(*Ci) there exists a finite sequence $\sigma_k := \langle (n_i, a_i, r_i, b_i, s_i) \in {}^*\Phi_J|_{\le K} : i < l_k \rangle$ such that $s_i < 2^{-k}$ and $[0,1] \subseteq \bigcup_{i<l_k} B(a_i; r_i)$;

(*Cii) if $(n, a, r, b, s) \in {}^*\Phi_J|_{\le K}$ and $(n, a, r, b', s') \in {}^*\Phi_J|_{\le K}$, then $d(b, b') \le s + s'$;

(*Ciii) if $(n, a, r, b, s) \in {}^*\Phi_J|_{\le K}$, $(n', a', r', b, s) \le K$ and $(a', r') < (a, r)$, then $(n', a', r', b, s) \in {}^*\Phi_J|_{\le K}$;

(*Civ) if $(n, a, r, b, s) \in {}^*\Phi_J|_{\le K}$, $(n', a, r, b', s') \le K$ and $(b, s) < (b', s')$, then $(n', a, r, b', s') \in {}^*\Phi_J|_{\le K}$;

hold in ${}^*V$. Let σ_k be a finite sequence taken in (*Ci). Let $\tau_k := \langle b_0, \ldots, b_{l_k-1} \rangle$ if $\sigma_k = \langle (n_i, a_i, r_i, b_i, s_i) : i < l_k \rangle$. Hereafter, we define $\tau(\mathrm{lh}(\tau)) := \tau(0)$. Note that if $k \in M$, we can find σ_k in V. Then, $\tau_k \in M$ and $d(J, P[\tau_k]) < 2^{-k}$ in V. Hence we call $P[\tau_k]$ *a k-th approximation of J*.

Put $P = P[\tau_\omega]$. In ${}^*V$, let $\varphi_0(x)$ (or $\psi_0(x)$) be a Σ^0_1 formula which expresses that $x \in \mathbb{R}^2$, $x \notin L_P$ and there exists $q \in \mathbb{Q}$ such that the half-line $l(x, q) = \{(x_1 + t, x_2 + qt) \in \mathbb{R}^2 : t \in \mathbb{R}, t \ge 0\}$ dose not contain each $\tau_\omega(i)$ and the cardinality of $l(x, q) \cap L_P$ is an odd number (or even number). Then, by Lemma 2.6, we can effectively find the open sets W_0, W_1 such that $x \in W_0 \leftrightarrow \varphi_0(x)$ and $x \in W_1 \leftrightarrow \psi_0(x)$. We can easily show the following:

(Wi) $W_0 \cap W_1 = \emptyset$ and $W_0 \cup W_1 = {L_P}^c$;

(Wii) W_0 is bounded;

(Wiii) if $x_1 \in W_j$ (j=0,1), $x_2 \in {L_P}^c$ and the segment $B[\langle x_1 x_2 \rangle]$ intersects L_P odd number of times, then $x_2 \in W_{1-j}$;

(Wiv) if $x_1 \in W_j$ (j=0,1), $x_2 \in {L_P}^c$ and the segment $B[\langle x_1 x_2 \rangle]$ intersects L_P even number of times, then $x_2 \in W_j$.

Define formulas $\varphi(a, r)$, $\tilde{\varphi}(a, r)$, $\psi(a, r)$ and $\tilde{\psi}(a, r)$ as

$$\varphi(a, r) \equiv a \in \mathbb{Q}^2 \wedge a \in W_0 \wedge d(a, P) > r;$$
$$\tilde{\varphi}(a, r) \equiv a \in \mathbb{Q}^2 \wedge a \notin W_1 \wedge d(a, P) > r;$$
$$\psi(a, r) \equiv a \in \mathbb{Q}^2 \wedge a \in W_1 \wedge d(a, P) > r;$$
$$\tilde{\psi}(a, r) \equiv a \in \mathbb{Q}^2 \wedge a \notin W_0 \wedge d(a, P) > r.$$

Note that $d(a, P) > 0$ can be expressed by Σ^0_0 formula. Hence, φ and ψ are Σ^0_1 formulas and $\tilde{\varphi}$ and $\tilde{\psi}$ are Π^0_1 formulas. Clearly, $\varphi(a, r) \leftrightarrow \tilde{\varphi}(a, r)$ and $\psi(a, r) \leftrightarrow \tilde{\psi}(a, r)$. Then, by Δ^0_1 comprehension, define ${}^*U_0$ and ${}^*U_1$ as:

$$(n, a, r) \in {}^*U_0 \leftrightarrow \varphi(a, r) \wedge n = (a, r),$$
$$(n, a, r) \in {}^*U_1 \leftrightarrow \psi(a, r) \wedge n = (a, r).$$

Put ${}^*U_0 \cap M = U_0 \in S$ and ${}^*U_1 \cap M = U_1 \in S$. Then, in V, it is readily apparent that $U_0 \cup U_1$ is the complement of the locus of J and U_0 is bounded, *i.e.*, Clause 1 in the theorem holds.

Hereafter, we prove Clause 2 in the theorem. Assume $r \in \mathbb{Q}^+$ and $w \in L_J$ in V. Without loss of generality, we let $w = J(1/2)$. Put $z = J(0) = J(1)$. Take $\varepsilon_w, \varepsilon, \varepsilon_z$ such that $0 < \varepsilon_w < \varepsilon < r/2$, $0 < \varepsilon_z$ and $d(w, z) > \sqrt{2}(\varepsilon + \varepsilon_z)$. Then, there exist $p_0, p_1, q_0, q_1 \in \mathbb{Q}$ such that $0 < p_0 < q_0 < 1/2 < q_1 < p_1 < 1$, $J([0, p_0]) \cup J([p_1, 1]) \subseteq \mathrm{Box}(z; \varepsilon_z)$ and $J([q_0, q_1]) \subseteq \mathrm{Box}(w; \varepsilon_w)$. Here, $\mathrm{Box}(a; r) := \{(x_1, x_2) \in \mathbb{R}^2 : a_1 - r < x_1 < a_1 + r, a_2 - r < x_2 < a_2 + r\}$. Put $J_1 = J|_{[p_0,q_0]}$ and $J_2 = J|_{[q_1,p_1]}$. Since J is injective, we can take $\delta \in \mathbb{Q}$ such that $d(J_1, J_2) > \delta > 0$, $\delta < \varepsilon - \varepsilon_w$ and $\delta < r - \sqrt{2}\varepsilon$.

In ${}^*V$, put $P_1 = P|_{[p_0,q_0]}$ and $P_2 = P|_{[q_1,p_1]}$. Let $C : [0, 1] \to \mathbb{R}^2$ be a simple polygon such that L_C is the boundary of $\mathrm{Box}(z; \varepsilon)$, *i.e.*, C draws the square which is a boundary of $\mathrm{Box}(z; \varepsilon)$. Without loss of generality, we may assume $L_P \cap L_C$ is finite. Since $P(p_0), P(p_1) \notin \mathrm{Box}(z; \varepsilon)$ and $P(q_0), P(q_1) \in \mathrm{Box}(z; \varepsilon)$, C intersects L_{P_j} odd number of times ($j = 1, 2$). Let $b_0, \dots, b_{m+1} \in [0, 1]$ such that $0 \le b_0 < b_1 < \dots < b_m \le 1$, $b_{m+1} = b_0$ and $C(\{b_0, \dots, b_m\}) = L_P \cap L_C$. Then, there exist at least two k's such that

(†) $(C(b_k) \in L_{P_1} \wedge C(b_{k+1}) \in L_{P_2}) \vee (C(b_k) \in L_{P_2} \wedge C(b_{k+1}) \in L_{P_1})$

holds. Moreover, there exist k, l such that k and l satisfy (†) and $|k - l|$ is an odd number. Hence, $C((b_k, b_{k+1})) \subseteq W_0$ or $C((b_l, b_{l+1})) \subseteq W_0$. Without loss of generality, we may assume $C(b_k) \in L_{P_1}$, $C(b_{k+1}) \in L_{P_2}$ and $C((b_k, b_{k+1})) \subseteq W_0$. Define $M : [b_k, b_{k+1}] \to \mathbb{R}$ as $M(t) = d(P_1, C(t)) - d(P_2, C(t))$. Then, $M(b_k) < -\delta < 0$ and $M(b_{k+1}) > \delta > 0$. By the intermediate value theorem (provable in RCA_0), there exists $b_k < t_0 < b_{k+1}$ such that $M(t_0) = 0$. Thus, $d(P_1, C(t_0)) = d(P_2, C(t_0)) > \delta/2$. Then, $B(C(t_o); \delta/2) \subseteq W_0$. Hence, there exist $n, a, r \in M$ such that $n = (a, r)$, $a \in \mathbb{Q}^2$, $r \in \mathbb{Q}^+$ and $B(a; r) \subseteq B(C(t_o); \delta/2)$ in ${}^*V$. Then, $(n, a, r) \in U_0$ and $d(a, w) < \sqrt{2}\varepsilon + \delta < r$ in V. This means $a \in U_0$ and $d(w, a) < r$ in V and Clause 2 in the theorem holds.

Clause 3 in the theorem remains to be proven. Let $H_0 : \mathbb{N} \to \mathbb{N}$ be a modulus of uniform continuity for J, *i.e.*, for all $x, y \in [0,1]$, $\min(|x-y|, 1-|x-y|) < 2^{-H(n)}$ implies $d(J(x), J(y)) < 2^{-n}$, and let $H_1 : \mathbb{N} \to \mathbb{N}$ be a function which satisfies the two conditions in Lemma 5.24. Note that the existence of a modulus of uniform continuity is provable in $\mathsf{WKL_0}$ (see Theorem 3.5). Define a function $H : \mathbb{N} \to \mathbb{N}$ as $H(n) := \min\{m : H_0(n) < H_1(m)\}$. Then, for all $x, y \in [0,1]$ with $x \leq y$, $d(J(x), J(y)) < 2^{-H(n)}$ implies $\forall z \in [x,y]\ d(J(x), J(z)) < 2^{-n}$ or $\forall z \in [0,x] \cup [y,1]\ d(J(x), J(z)) < 2^{-n}$. Thus, if $\sigma_k := \langle (n_i, a_i, r_i, b_i, s_i) : i < l_k \rangle$ taken in (*Ci) satisfies $d(b_i, b_j) + s_i + s_j < 2^{-H(n)}$ and $i \leq j$, then, $d(b_i, B[\tau_k|_{\leq i,j\geq}]) < 2^{-n} + 2^{-k+1}$ or $d(b_i, B[\tau_k|_{i\leq j}]) < 2^{-n} + 2^{-k+1}$. Here, $\tau_k|_{\leq i,j\geq} := \langle b_t : j \leq t < l_k \rangle^\frown \langle b_t : 0 \leq t \leq i \rangle$ and $\tau_k|_{i\leq j} := \langle b_t : j \leq i \leq t \leq j \rangle$.

Take n_0 such that $2^{-n_0} < \min(d(u,J), d(v,J))/2$ and take N such that $2^{-N} < \min\{2^{-n_0-2}, 2^{-N_0-1}, 2^{-H(n_0+1)-2}\}$ in V.

Next, we define a sequence ρ. For each side $B[\langle \tau_\omega(i), \tau_\omega(i+1) \rangle]$ of $P[\tau_\omega]$, we draw a rectangle of size $(d(\tau_\omega(i), \tau_\omega(i+1)) + 2 \cdot 2^{-N-1}) \times 2 \cdot 2^{-N-1}$ such that the side $B[\langle \tau_\omega(i), \tau_\omega(i+1) \rangle]$ lies in the rectangle at equal distance 2^{-N-1} from each of the four sides of the rectangle. The parts of the rectangles that lie within W_0 (since W_0 is the 'interior' of P) decompose W_0 into some polygonal domains. Let R be the domain constructed above containing u. Next take the sequence ρ such that $P[\rho]$ is the boundary of R. Then, $L_{P[\tau_\omega]} \cap L_{P[\rho]} = \emptyset$ and $d(P[\tau_\omega], q) < 2^{-N} < 2^{-N_0-1}$ for all $q \in L_{P[\rho]}$. (If $L_{P[\tau_\omega]} \cap L_{P[\rho]} \neq \emptyset$, then $L_{P[\rho]}$ is contained in one rectangle, but it contradicts $d(u, L_{P[\tau_\omega]}) > 2^{-N} - 2^{-\omega}$.) Then, by underspill, we can easily find a finite sequence $\langle a_n, r_n : n < l \rangle \in M$ which satisfies the following:

(Bi) $r_n < d(a_n, J) < 2^{-N_0-1}$ and $B(a_n; r_n) \cap B(a_{n+1}, r_{n+1}) \neq \emptyset$ for all $n < l$ in V;

(Bii) $L_{P[\rho]} \subseteq \bigcup_{n<l} B(a_n; r_n)$ in $^*V$.

If both u and v are contained in R, segments $B[\langle u, x \rangle]$ and $B[\langle v, y \rangle]$ intersect with $\bigcup_{n<l} B(a_n; r_n)$. Therefore, we can easily find the desired broken-lines B_0, B_1 within U_0. So, we will show $v \in R$.

$P[\hat{\tau}]$ is said to be a refinement of $P[\tau]$ if $L_{P[\tau]} = L_{P[\hat{\tau}]}$ and $\{\tau(i) : i < \mathrm{lh}(\tau)\} \subseteq \{\hat{\tau}(i) : i < \mathrm{lh}(\hat{\tau})\}$. Take refinements $P[\hat{\tau}]$ and $P[\hat{\rho}]$ of $P[\tau_\omega]$ and $P[\rho]$ which satisfy the following:

(Ri) $\forall i < \mathrm{lh}(\hat{\rho})\ d(\hat{\rho}(i), \hat{\rho}(i+1)) < 2^{-N}$;

(Rii) there exists a map $\nu : \mathrm{lh}(\hat{\rho}+1) \to \mathrm{lh}(\hat{\tau})+1$ such that $\nu(0)=0$, $\nu(\mathrm{lh}(\hat{\rho})) = \mathrm{lh}(\hat{\tau})$, $\nu(i) \leq \nu(i+1)$ for all $i < \mathrm{lh}(\hat{\rho})$ and $d(\hat{\rho}(i), \hat{\tau}(\nu(i))) < 2^{-N}$ for all $i \leq \mathrm{lh}(\hat{\rho})$.

To show $v \in R$, we show $d(\hat{\rho}(i), B[\hat{\tau}|_{\nu(i)\leq\nu(i+1)}]) < 2^{-n_0}$ for all $i < \mathrm{lh}(\hat{\rho})$. By the above conditions, $d(\hat{\tau}(\nu(i)), \hat{\tau}(\nu(i+1)) < 3 \cdot 2^{-N} < 2^{-H(n_0+1)}$. Then,

$$d(\hat{\tau}(\nu(i)), B[\hat{\tau}|_{\leq\nu(i),\nu(i+1)\geq}]) < 2^{-n_0-1} + 2^{-\omega+1}$$

or

$$d(\hat{\tau}(\nu(i)), B[\hat{\tau}|_{\nu(i)\leq\nu(i+1)}]) < 2^{-n_0-1} + 2^{-\omega+1}.$$

If $d(\hat{\tau}(\nu(i)), B[\hat{\tau}|_{\leq\nu(i),\nu(i+1)\geq}]) < 2^{-n_0-1} + 2^{-\omega+1}$, then

$$d(\hat{\rho}(i), P[\hat{\rho}]) < d(\hat{\tau}(\nu(i)), B[\hat{\tau}|_{\leq\nu(i),\nu(i+1)\geq}]) + 2^{-N} < 2^{-n_0}$$

but it contradicts $u \in R$. Hence $d(\hat{\tau}(\nu(i)), B[\hat{\tau}|_{\nu(i)\leq\nu(i+1)}]) < 2^{-n_0-1} + 2^{-\omega+1}$. Thus,

$$d(\hat{\rho}(i), B[\hat{\tau}|_{\nu(i)\leq\nu(i+1)}]) < d(\hat{\tau}(\nu(i)), B[\hat{\tau}|_{\nu(i)\leq\nu(i+1)}]) + 2^{-N} < 2^{-n_0}.$$

Therefore,

$$\forall q \in L_{P[\hat{\tau}]}\ d(q, P[\hat{\rho}]) < 2^{-n_0}.$$

If $v \in {}^*U_0 \setminus R$, then $d(v, P) = d(v, P[\hat{\tau}]) < 2^{-n_0}$, which contradicts $2^{-n_0} < d(v, J)/2$. Hence $v \in R$. This completes the proof for Theorem 5.25. □

5.4.3 Some more results

In this subsection, we summarize some more results. For the proofs of theorems in this subsection, see [25].

Let D, E be open or closed sets in $\mathbb{R}$ or $\mathbb{R}^2$. Let $\phi : D \to E$ be a continuous function. A *continuous inverse function* of ϕ is a continuous function $\psi : E \to D$ such that $\psi(\phi(u)) = u$ for all $u \in D$ and $\phi(\psi(v)) = v$ for all $v \in E$. The continuous inverse function of ϕ (if it exists) is written as ϕ^{-1}. A pair of continuous functions (ϕ, ψ) is said to be a *homeomorphism* if ψ is the continuous inverse function of ϕ. For the simplicity, we write ϕ for a homeomorphism (ϕ, ϕ^{-1}). By the Jordan curve theorem 5.25, we can show the following Schönflies theorem.

Theorem 5.26 (The Schönflies theorem, first form)**.** *The following is provable in* WKL_0. *Let J be a Jordan curve and let K be the Jordan region of J. Put $Q = P[\langle(1,1),(-1,1),(-1,-1),(1,-1)\rangle]$. Let L be the Jordan region of Q. Then, there exists a homeomorphism $\phi : K \to L$ such that $\phi(J(x)) = Q(x)$. Moreover, ϕ and ϕ^{-1} have moduli of uniform continuities.*

Theorem 5.27 (The Schönflies theorem, second form)**.** *The following is provable in* $\mathsf{WKL_0}$*. Let* J *be a Jordan curve and let* Q *be interpreted as in Theorem* 5.26*. Consequently, there exists a homeomorphism* $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ *such that* $\phi(J(x)) = Q(x)$ *for all* $x \in [0, 1]$.

Then, a strong version of the Jordan curve theorem immediately follows from above theorems.

Theorem 5.28 (The Jordan curve theorem)**.** *The following is provable in* $\mathsf{WKL_0}$*. For each Jordan curve* J*, we can find the locus* L_J *of* J *and a partition* (U_0, U_1) *of* $(L_J)^c$ *such that all of the following hold:*

1. U_0 *is bounded, i.e., there exists* $q \in \mathbb{Q}^+$ *such that* $d(u, (0, 0)) < q$ *for all* $u \in U_0$*;*
2. U_0 *and* U_1 *are broken-line connected, i.e., for all* $u, v \in U_i$*, there exists a broken-lines* B_i *in* U_i *such that* $B_i(0) = u, B_i(1) = v$ *for* $i = 0, 1$*;*
3. *For each* $w \in L_J$ *and each* $q \in \mathbb{Q}^+$*, there exist points* $u \in U_0, v \in U_1$ *such that* $d(w, u) < q$ *and* $d(w, v) < q$*;*
4. *Every arc connecting a point in* U_0 *and a point in* U_1 *meets* J*.*

Theorem 5.27 extends the result in Shioji/Tanaka [27].

Corollary 5.29 (The Brouwer fixed point theorem for Jordan regions)**.** *The following is provable in* $\mathsf{WKL_0}$*. Let* J *be a Jordan curve and let* U_1 *be the exterior of* J*. Let* ψ *be a continuous function from* U_1^c *to itself. Then,* ψ *has a fixed point, i.e., there exists* $u \in U_1^c$ *with* $\psi(u) = u$*.*

Proof. Immediate from Theorem 5.27 and the fact that the Brouwer fixed point theorem for convex hulls in $\mathbb{R}^n$ is provable within $\mathsf{WKL_0}$ ([27]). □

Finally, we show that the Jordan curve theorem and The Schönflies theorem are equivalent to $\mathsf{WKL_0}$ over $\mathsf{RCA_0}$.

Theorem 5.30 (reversal)**.** *The following assertions are pairwise equivalent over* $\mathsf{RCA_0}$*.*

1. $\mathsf{WKL_0}$*.*
2. *Every Jordan curve has a locus.*

3. *The Jordan curve theorem.*

4. *For every Jordan curve* J *and non-empty open set* U, U_0, U_1*, if* U^c *is a locus of* J *and* (U_0, U_1) *is a partition of* U*, then* U_0 *or* U_1 *is bounded.*

5. *For every Jordan curve* J *and non-empty open set* U, U_0, U_1*, if* U^c *is a locus of* J*,* (U_0, U_1) *is a partition of* U *and* U_0 *is bounded, then,* U_0 *and* U_1 *are broken line connected.*

6. *The Schönflies theorem, second form.*

7. *The Schönflies theorem, first form.*

6 Formalizing non-standard arguments

In the previous chapter, we introduced model theoretic arguments to do non-standard analysis for ACA_0 or WKL_0 and applied them to demonstrate that some theorems are provable in ACA_0 or WKL_0. Then, *can we canonically reconstruct formal proofs within* ACA_0 *or* WKL_0 *from such non-standard arguments?* Professor Sakae Fuchino posed this question. In this chapter, we introduce systems of non-standard second order arithmetic $\mathsf{ns\text{-}ACA}_0$ and $\mathsf{ns\text{-}WKL}_0$ corresponding to ACA_0 and WKL_0. In these systems, we can formalize the above non-standard arguments. By model constructions appearing in the previous chapter, we can show that $\mathsf{ns\text{-}ACA}_0$ is a conservative extension of ACA_0 and $\mathsf{ns\text{-}WKL}_0$ is a conservative extension of WKL_0. However, we need some canonical transformations that do not depend on semantics because we want to analyze non-standard techniques in second order arithmetic. To transform non-standard proofs directly into standard proofs, we interpret $\mathsf{ns\text{-}ACA}_0$ in ACA_0 and interpret $\mathsf{ns\text{-}WKL}_0$ in WKL_0, as for the formalization of Harrington's conservation theorem by Avigad[2]. The technical ideas of these interpretations are attributed to [9] and [34]. In addition, systems of non-standard second order arithmetic we introduce are the expansions of non-standard arithmetic introduced in [21].

We first introduce the language of non-standard second order arithmetic.

Definition 6.1. The language of non-standard second order arithmetic $\mathcal{L}_2^*$ is defined by the following:

- standard number variables: $x^{\mathrm{s}}, y^{\mathrm{s}}, \ldots,$
- non-standard number variables: $x^*, y^*, \ldots,$
- standard set variables: $X^{\mathrm{s}}, Y^{\mathrm{s}}, \ldots,$
- non-standard set variables: $X^*, Y^*, \ldots,$
- function and relation symbols: $0^{\mathrm{s}}, 1^{\mathrm{s}}, =^{\mathrm{s}}, +^{\mathrm{s}}, \cdot^{\mathrm{s}}, <^{\mathrm{s}}, \in^{\mathrm{s}}, 0^*, 1^*, =^*, +^*, \cdot^*, <^*, \in^*$ $, \surd\cdot$

Here, $0^{\mathrm{s}}, 1^{\mathrm{s}}, =^{\mathrm{s}}, +^{\mathrm{s}}, \cdot^{\mathrm{s}}, <^{\mathrm{s}}, \in^{\mathrm{s}}$ denote "the standard structure" of second order arithmetic, $0^*, 1^*, =^*, +^*, \cdot^*, <^*, \in^*$ denote "the non-standard structure" of second order arithmetic and $\surd$ denote an embedding from the standard structure to the non-standard structure.

In this chapter, we consider $+, \cdot \in \mathcal{L}_2$ as ternary relations. Similarly, we consider $+^{\mathrm{s}}, \cdot^{\mathrm{s}}, +^*, \cdot^* \in \mathcal{L}_2^*$ as ternary relations.

The terms and formulas of the language of non-standard second order arithmetic are as follows. *Standard numerical terms* are standard number variables and the constant symbols 0^{s} and 1^{s} and *non-standard numerical terms* are non-standard number variables, the constant symbols 0^* and 1^* and $\surd(t^{\mathrm{s}})$ whenever t^{s} is a numerical term. *Standard set terms* are standard set variables and *non-standard set terms* are non-standard set variables and $\surd(X^{\mathrm{s}})$ whenever X^{s} is a standard set term. *Atomic formulas* are $t_1^{\mathrm{s}} =^{\mathrm{s}} t_2^{\mathrm{s}}$, $+^{\mathrm{s}}(t_1^{\mathrm{s}}, t_2^{\mathrm{s}}, t_3^{\mathrm{s}})$, $\cdot^{\mathrm{s}}(t_1^{\mathrm{s}}, t_2^{\mathrm{s}}, t_3^{\mathrm{s}})$, $t_1^{\mathrm{s}} <^{\mathrm{s}} t_2^{\mathrm{s}}$, $t_1^{\mathrm{s}} \in^{\mathrm{s}} X^{\mathrm{s}}$, $t_1^* =^* t_2^*$, $+^*(t_1^*, t_2^*, t_3^*)$, $\cdot^*(t_1^*, t_2^*, t_3^*)$, $t_1^* <^* t_2^*$ and $t_1^* \in^* X^*$ where $t_1^{\mathrm{s}}, t_2^{\mathrm{s}}, t_3^{\mathrm{s}}$ are standard numerical terms, t_1^*, t_2^*, t_3^* are non-standard numerical terms, X^{s} is a standard set term and X^* is a non-standard set term. *Formulas* are built up from atomic formulas by means of propositional connectives $\wedge$, $\neg$ and quantifiers $\exists x^{\mathrm{s}}$, $\exists x^*$, $\exists X^{\mathrm{s}}$, $\exists X^*$. Other connectives $\vee$, $\rightarrow$, $\leftrightarrow$ and quantifiers $\forall x^{\mathrm{s}}$, $\forall x^*$, $\forall X^{\mathrm{s}}$, $\forall X^*$ are introduced by a combination of some of $\wedge$, $\neg$, $\exists x^{\mathrm{s}}$, $\exists x^*$, $\exists X^{\mathrm{s}}$, $\exists X^*$ as usual. A *sentence* is a formula without free variables.

Let φ be an $\mathcal{L}_2$-formula. We write φ^{s} for the $\mathcal{L}_2^*$ formula constructed by adding $^{\mathrm{s}}$ to all occurrences of bounded variables and relations of φ. Similarly, we write φ^* for the $\mathcal{L}_2^*$ formula constructed by adding *. We usually omit $^{\mathrm{s}}$ and * of relations. We write $\check{t}^{\mathrm{s}}$ for $\surd(t^{\mathrm{s}})$ and $\check{X}^{\mathrm{s}}$ for $\surd(X^{\mathrm{s}})$.

In this chapter, we use M to indicate the range of number variables and S to indicate the range of set variables in the system of second order arithmetic. We are not going to describe the semantics of the system by these M and S but these symbols are introduced just to make the argument more accessible. Similarly, in the system of non-standard second order arithmetic, we use M^{s} to indicate the range of standard number variables, M^* to indicate the range of non-standard number variables, S^{s} to indicate the range of standard set variables and S^* to indicate the range of non-standard set variables. Moreover, we use $V^{\mathrm{s}} = (M^{\mathrm{s}}, S^{\mathrm{s}})$ to indicate the range of standard variables and $V^* = (M^*, S^*)$ to indicate the range of non-standard variables.

6.1 The system $\mathsf{ns\text{-}ACA_0}$

In this section, we introduce the system $\mathsf{ns\text{-}ACA_0}$. Then, we interpret $\mathsf{ns\text{-}ACA_0}$ in $\mathsf{ACA_0}$ by forcing notion. For this, we use the forcing relation for generic ultra filters.

Definition 6.2 (the system $\mathsf{ns\text{-}ACA_0}$)**.** The axioms of $\mathsf{ns\text{-}ACA_0}$ are the following:

1. $(\mathsf{ACA_0})^{\mathrm{s}}$, $(\mathsf{ACA_0})^{*}$.
2. $\surd : V^{\mathrm{s}} \to V^{*}$ is an injective homomorphism.
3. end extension:
$$\forall x^{*}\forall y^{\mathrm{s}}(x^{*} < \check{y}^{\mathrm{s}} \to \exists z^{\mathrm{s}}(x^{*} = \check{z}^{\mathrm{s}})).$$
4. standard part:
$$\forall X^{*}\exists Y^{\mathrm{s}}\forall x^{\mathrm{s}}(x^{\mathrm{s}} \in Y^{\mathrm{s}} \leftrightarrow \check{x}^{\mathrm{s}} \in X^{*}).$$
5. Σ^1_1 transfer principle:
$$\forall x^{\mathrm{s}}\forall X^{\mathrm{s}}(\varphi(x^{\mathrm{s}}, X^{\mathrm{s}})^{\mathrm{s}} \leftrightarrow \varphi(\check{x}^{\mathrm{s}}, \check{X}^{\mathrm{s}})^{*})$$
for any Σ^1_1 formula φ.
6. Σ^1_0 overspill:
$$\forall x^{*}\forall X^{*}(\forall y^{\mathrm{s}}\exists z^{\mathrm{s}}(z^{\mathrm{s}} \geq y^{\mathrm{s}} \wedge \varphi(\check{z}^{\mathrm{s}}, x^{*}, X^{*})^{*}) \to \exists y^{*}(\forall w^{\mathrm{s}}(y^{*} > \check{w}^{\mathrm{s}}) \wedge \varphi(y^{*}, x^{*}, X^{*})^{*}))$$
for any Σ^1_0 formula φ.

In fact, the axiom "end extension" is deduced from other axioms. Note that we can check that $\mathsf{ns\text{-}ACA_0}$ is an expansion of $\mathsf{ACA_0^*}$ which is introduced by Keisler[21].

Now, we interpret $\mathsf{ns\text{-}ACA_0}$ in $\mathsf{ACA_0}$ and show that $\mathsf{ns\text{-}ACA_0}$ is a conservative extension of $\mathsf{ACA_0}$. We argue in $\mathsf{ACA_0}$. A set X is said to be *bounded* if $\exists x\ \forall y \in X\ y \leq x$. We write $X \subseteq_{\mathrm{al}} Y$ if $X \setminus Y$ is bounded.

Definition 6.3 (in $\mathsf{ACA_0}$)**.** Let $\bar{M}^{\mathrm{s}} = M$, $\bar{M}^{*} = \{f \mid f : M \to M\}$, $\bar{S}^{\mathrm{s}} = S$ and $\bar{S}^{*} = \{X \mid X \subseteq M \times M\}$. Here, $\bar{M}^{\mathrm{s}}$, $\bar{M}^{*}$, $\bar{S}^{\mathrm{s}}$, $\bar{S}^{*}$ denote the sets of (names of) standard numbers, non-standard numbers, standard sets and non-standard sets in non-standard second order arithmetic. Define $0^{\mathrm{s}}, 1^{\mathrm{s}} \in \bar{M}^{\mathrm{s}}$ as $0^{\mathrm{s}} = 0$, $1^{\mathrm{s}} = 1$. Define $0^{*}, 1^{*} \in \bar{M}^{*}$ as $0^{*}(i) = 0$ and $1^{*}(i) = 1$. Let $P = \{X \mid X \text{ is unbounded}\}$ and $X \leq Y \Leftrightarrow X \subseteq Y$. For $A^{*} \in \bar{S}^{*}$, we write $A^{*}(i)$ for $\{x \mid (x, i) \in A^{*}\}$.

We inductively define $X \Vdash \psi$ for any $\mathcal{L}_2^{*} \cup \bar{M}^{\mathrm{s}} \cup \bar{M}^{*} \cup \bar{S}^{\mathrm{s}} \cup \bar{S}^{*}$ sentence ψ as follows. For $a^{\mathrm{s}} \in \bar{M}^{\mathrm{s}}$ and $A^{\mathrm{s}} \in \bar{S}^{\mathrm{s}}$, we define $(a^{\mathrm{s}})^{\surd} \in \bar{M}^{*}$ and $(A^{\mathrm{s}})^{\surd} \in \bar{S}^{*}$ as $(a^{\mathrm{s}})^{\surd}(i) = a^{\mathrm{s}}$ and $(A^{\mathrm{s}})^{\surd} = \{(x, i) \mid x \in A^{\mathrm{s}} \wedge i \in M\}$. Let $X \in P$, $a^{\mathrm{s}}, b^{\mathrm{s}}, c^{\mathrm{s}} \in M$, $a^{*}, b^{*}, c^{*} \in \bar{M}^{*}$, $A^{\mathrm{s}} \in S$, $A^{*} \in \bar{S}^{*}$.

- $X \Vdash a^{\mathrm{s}} =^{\mathrm{s}} b^{\mathrm{s}} \Leftrightarrow a^{\mathrm{s}} = b^{\mathrm{s}}$.
- $X \Vdash +^{\mathrm{s}}(a^{\mathrm{s}}, b^{\mathrm{s}}, c^{\mathrm{s}}) \Leftrightarrow +(a^{\mathrm{s}}, b^{\mathrm{s}}, c^{\mathrm{s}})$.
- $X \Vdash \cdot^{\mathrm{s}}(a^{\mathrm{s}}, b^{\mathrm{s}}, c^{\mathrm{s}}) \Leftrightarrow \cdot(a^{\mathrm{s}}, b^{\mathrm{s}}, c^{\mathrm{s}})$.
- $X \Vdash a^{\mathrm{s}} <^{\mathrm{s}} b^{\mathrm{s}} \Leftrightarrow a^{\mathrm{s}} < b^{\mathrm{s}}$.
- $X \Vdash a^{\mathrm{s}} \in^{\mathrm{s}} A^{\mathrm{s}} \Leftrightarrow a^{\mathrm{s}} \in A^{\mathrm{s}}$.
- $X \Vdash a^* =^* b^* \Leftrightarrow \{i \mid a^*(i) = b^*(i)\} \supseteq_{\mathrm{al}} X$.
- $X \Vdash +^*(a^*, b^*, c^*) \Leftrightarrow \{i \mid +(a^*(i), b^*(i), c^*(i))\} \supseteq_{\mathrm{al}} X$.
- $X \Vdash \cdot^*(a^*, b^*, c^*) \Leftrightarrow \{i \mid \cdot(a^*(i), b^*(i), c^*(i))\} \supseteq_{\mathrm{al}} X$.
- $X \Vdash a^* <^* b^* \Leftrightarrow \{i \mid a^*(i) < b^*(i)\} \supseteq_{\mathrm{al}} X$.
- $X \Vdash a^* \in^* A^* \Leftrightarrow \{i \mid a^*(i) \in A^*(i)\} \supseteq_{\mathrm{al}} X$.
- $X \Vdash \psi(\check{a}^{\mathrm{s}}, b^*, \check{A}^{\mathrm{s}}, B^*) \Leftrightarrow X \Vdash \psi((a^{\mathrm{s}})^{\checkmark}, b^*, (A^{\mathrm{s}})^{\checkmark}, B^*)$ where ψ is a non-standard atomic formula.
- $X \Vdash \varphi \wedge \psi \Leftrightarrow X \Vdash \varphi \wedge X \Vdash \psi$.
- $X \Vdash \neg\psi \Leftrightarrow \forall Y \leq X \; \neg Y \Vdash \psi$.
- $X \Vdash \exists x^{\mathrm{s}}\psi(x^{\mathrm{s}}) \Leftrightarrow \{Y \mid \exists a^{\mathrm{s}} \in \bar{M}^{\mathrm{s}} \; Y \Vdash \psi(a^{\mathrm{s}})\}$ is dense below X.
- $X \Vdash \exists x^*\psi(x^*) \Leftrightarrow \{Y \mid \exists a^* \in \bar{M}^* \; Y \Vdash \psi(a^*)\}$ is dense below X.
- $X \Vdash \exists X^{\mathrm{s}}\psi(X^{\mathrm{s}}) \Leftrightarrow \{Y \mid \exists A^{\mathrm{s}} \in \bar{S}^{\mathrm{s}} \; Y \Vdash \psi(A)\}$ is dense below X.
- $X \Vdash \exists X^*\psi(X^*) \Leftrightarrow \{Y \mid \exists A^* \in \bar{S}^* \; Y \Vdash \psi(A^*)\}$ is dense below X.

Define $\Vdash \psi$ as $X \Vdash \psi$ for any $X \in P$.

The next two lemmas are easily proved in usual ways.

Lemma 6.1 (in ACA_0)**.** *The following are equivalent.*

1. $X \Vdash \varphi$
2. $\{Y \mid Y \Vdash \varphi\}$ *is dense below* X.
3. $\forall Y \leq X \; Y \Vdash \varphi$.

Lemma 6.2 (in $\mathsf{ACA_0}$). *Let ψ be an $\mathcal{L}_2^*$ sentence. Then, the following are equivalent.*

1. $X \Vdash \psi$ *for some* $X \in P$.

2. $X \Vdash \psi$ *for all* $X \in P$.

Lemma 6.3 (in $\mathsf{ACA_0}$). *Let X be an unbounded set, and let A be an arbitrary set. Let $A_a = \{i \mid (a,i) \in A\}$. Then, there exist an unbounded set $Y \subseteq X$ such that $(Y \subseteq_{\mathrm{al}} A_a) \vee (Y \subseteq_{\mathrm{al}} A_a^c)$ for all $a \in M$.*

Proof. This lemma is an easy consequence of [20, Lemma 8.7]. This lemma is also showed in [9, Theorem 3.2]. □

Lemma 6.4 (in $\mathsf{ACA_0}$).

$$X \Vdash \varphi(\check{\vec{a^{\mathrm{s}}}}, \vec{a^*}, \check{\vec{A^{\mathrm{s}}}}, \vec{A^*})^* \leftrightarrow \{i \mid \varphi(\vec{a^{\mathrm{s}}}, \vec{a^*}(i), \vec{A^{\mathrm{s}}}, \vec{A^*}(i))\} \supseteq_{\mathrm{al}} X \tag{50}$$

for any Σ^1_0 formula φ and for any $\vec{a^{\mathrm{s}}} \in \bar{M}^{\mathrm{s}}$, $\vec{a^*} \in \bar{M}^*$, $\vec{A^{\mathrm{s}}} \in \bar{S}^{\mathrm{s}}$, $\vec{A^*} \in \bar{S}^*$.

Proof. We show this by induction on the complexity of formulas. Atomic formulas satisfy (50) by the definition of $\Vdash$. We can easily check that $\varphi \wedge \psi$ and $\neg\varphi$ satisfy (50) if both φ and ψ satisfy (50). Let $\psi \equiv \exists x \varphi(x)$ (ψ may have parameters from $\bar{M}^{\mathrm{s}} \cup \bar{S}^{\mathrm{s}} \cup \bar{M}^* \cup \bar{S}^*$). If $\{i \mid \psi\} \not\supseteq_{\mathrm{al}} X$, then $\{i \mid \psi\} \cap X \leq X$ and $\{i \mid \psi\} \cap X \Vdash \neg\varphi(a^*)$ for all $a^* \in \bar{M}^*$. Thus, $X \Vdash \psi \to \{i \mid \psi\} \supseteq_{\mathrm{al}} X$. For the converse, we define $a^* \in \bar{M}^*$ as

$$a^*(i) = \begin{cases} \min\{a \mid \varphi(a)\} & \text{if } \exists x \varphi(x), \\ 0 & \text{if } \neg\exists x \varphi(x). \end{cases}$$

Then, $\{i \mid \psi\} \supseteq_{\mathrm{al}} X$ implies $\{i \mid \varphi(a^*(i))\} \supseteq_{\mathrm{al}} X$. Hence, $\{i \mid \psi\} \supseteq_{\mathrm{al}} X \to X \Vdash \psi$. □

Lemma 6.5 (in $\mathsf{ACA_0}$). *Let φ be an $\mathcal{L}_2$ formula and let $\vec{a^{\mathrm{s}}} \in \bar{M}^{\mathrm{s}}$, $\vec{A^{\mathrm{s}}} \in \bar{S}^{\mathrm{s}}$. Then, the following are equivalent.*

1. $\varphi(\vec{a^{\mathrm{s}}}, \vec{A^{\mathrm{s}}})$.

2. $X \Vdash \varphi(\vec{a^{\mathrm{s}}}, \vec{A^{\mathrm{s}}})^{\mathrm{s}}$ *for some* $X \in P$.

3. $X \Vdash \varphi(\vec{a^{\mathrm{s}}}, \vec{A^{\mathrm{s}}})^{\mathrm{s}}$ *for any* $X \in P$.

Proof. We can easily check this by induction on the complexity of φ. □

Lemma 6.6 (in $\mathsf{ACA_0}$)**.**

$$(X \Vdash \varphi(\vec{a^{\mathrm{s}}}, \vec{A}^{\mathrm{s}})^{\mathrm{s}}) \leftrightarrow \varphi(\vec{a^{\mathrm{s}}}, \vec{A}^{\mathrm{s}})$$

for any $\mathcal{L}_2$ formula φ and for any $X \in P$, $\vec{a^{\mathrm{s}}} \in \bar{M}^{\mathrm{s}}$, $\vec{A}^{\mathrm{s}} \in \bar{S}^{\mathrm{s}}$.

Proof. Obvious from Lemma 6.5. □

Lemma 6.7.

$$\mathsf{ACA_0} \vdash\Vdash \mathsf{ns\text{-}ACA_0}.$$

Proof. We show $\Vdash \mathsf{ns\text{-}ACA_0}$ in $\mathsf{ACA_0}$. We can easily check that $\Vdash$ "$\surd$ is an injective homomorphism" (Definition 6.2.2). We first show that $\Vdash$ "Σ^1_1 transfer principle" (Definition 6.2.5). Let ψ be Σ^1_0 and $\Vdash (\exists X \psi(X, \check{a}^{\mathrm{s}}, \check{A}^{\mathrm{s}}))^*$. Then, for any $X \in P$, there exist $Y \le X$ and $A^* \in \bar{S}^*$ such that $Y \Vdash \psi(A^*, \check{a}^{\mathrm{s}}, \check{A}^{\mathrm{s}})^*$. By Lemma 6.4, there exist $j \in Y \cap \{i \mid \psi(A^*(i), a^{\mathrm{s}}, A^{\mathrm{s}})\}$. Then, $\psi(A^*(j), a^{\mathrm{s}}, A^{\mathrm{s}})$ and $Y \Vdash \psi(A^*(j), a^{\mathrm{s}}, A^{\mathrm{s}})^{\mathrm{s}}$. Hence $\Vdash (\exists X \psi(X, a^{\mathrm{s}}, A^{\mathrm{s}}))^{\mathrm{s}}$. Conversely, let $\Vdash (\exists X \psi(X, a^{\mathrm{s}}, A^{\mathrm{s}}))^{\mathrm{s}}$. By Lemma 6.6, $\exists X \psi(X, a^{\mathrm{s}}, A^{\mathrm{s}})$. Hence, there exist $B^{\mathrm{s}} \in \bar{S}^{\mathrm{s}}$ such that $\psi(B^{\mathrm{s}}, a^{\mathrm{s}}, A^{\mathrm{s}})$. By Lemma 6.4, $\Vdash \psi(\check{B}^{\mathrm{s}}, \check{a}^{\mathrm{s}}, \check{A}^{\mathrm{s}})$. Thus, $\Vdash (\exists X \psi(X, \check{a}^{\mathrm{s}}, \check{A}^{\mathrm{s}}))^*$.

Next, we show $\Vdash (\mathsf{ACA_0})^{\mathrm{s}} \wedge (\mathsf{ACA_0})^*$ (Definition 6.2.1). By Lemma 6.6 and $\Vdash$ "Σ^1_1 transfer principle", we get $\Vdash (\mathsf{ACA_0})^{\mathrm{s}}$ and $\Vdash (\mathsf{I}\Sigma^0_1)^*$. We show $\Vdash (\mathrm{ACA})^*$. Let ψ be Σ^1_0 and let $a^* \in \bar{M}^*$, $A^* \in \bar{S}^*$. We need to show $\Vdash (\exists X \forall x (x \in X \leftrightarrow \psi(x, a^*, A^*)))^*$. Define $B^{\mathrm{s}} \in \bar{S}^{\mathrm{s}}$ as $(x, i) \in B^{\mathrm{s}} \leftrightarrow \psi(x, a^*(i), A^*(i))$. Then, $\Vdash (x \in B^{\mathrm{s}*} \leftrightarrow \psi(x, a^*, A^*))^*$. Thus $\Vdash (\mathrm{ACA})^*$.

To show $\Vdash$ "standard part" (Definition 6.2.4), we use Lemma 6.3. For any $A^* \in \bar{S}^*$ and for any $X \in P$, there exist $Y \le X$ such that $(Y \subseteq_{\mathrm{al}} A^*_b) \vee (Y \subseteq_{\mathrm{al}} (A^*_b)^c)$ for all $b \in M$. Define $B^{\mathrm{s}} \in \bar{S}^{\mathrm{s}}$ as $b \in B^{\mathrm{s}} \leftrightarrow Y \subseteq_{\mathrm{al}} A^*_b$. Then, $Y \Vdash \forall x^{\mathrm{s}} (x^{\mathrm{s}} \in B^{\mathrm{s}} \leftrightarrow \check{x}^{\mathrm{s}} \in A^*)$. Thus, $\Vdash \exists X^{\mathrm{s}} \forall x^{\mathrm{s}} (x^{\mathrm{s}} \in X^{\mathrm{s}} \leftrightarrow \check{x}^{\mathrm{s}} \in A^*)$.

Finally, we show $\Vdash$ "Σ^1_0 overspill" (Definition 6.2.6). Let φ be Σ^1_0 formula and $a^* \in \bar{M}^*$, $A^* \in \bar{S}^*$. Let $\Vdash \forall y^{\mathrm{s}} \exists z^{\mathrm{s}} (z^{\mathrm{s}} \ge y^{\mathrm{s}} \wedge \varphi(\check{z}^{\mathrm{s}}, a^*, A^*)^*)$. This implies that there exist cofinitely many $i \in M$ such that $\exists x (x > j \wedge \varphi(x, a^*(i), A^*(i)))$ for all $j \in M$. Define $b^* \in \bar{M}^*$ as

$$b^*(i) = \begin{cases} \min\{b \mid b > i \wedge \varphi(b, a^*(i), A^*(i))\} & \text{if } \exists x (x > i \wedge \varphi(x, a^*(i), A^*(i))), \\ 0 & \text{if } \neg \exists x (x > i \wedge \varphi(x, a^*(i), A^*(i))). \end{cases}$$

Then, $\Vdash (b^* > \check{c}^{\mathrm{s}}) \wedge \varphi(b^*, a^*, A^*)^*$ for any $c^{\mathrm{s}} \in \bar{M}^{\mathrm{s}}$. Thus, $\Vdash \exists y^* (\forall w^{\mathrm{s}} (y^* > \check{w}^{\mathrm{s}}) \wedge \varphi(y^*, a^*, A^*)^*)$. □

Theorem 6.8.

$$\mathsf{ns\text{-}ACA_0} \vdash \psi \Rightarrow \mathsf{ACA_0} \vdash\!\Vdash \psi$$

for any $\mathcal{L}_2^*$ sentence ψ. Moreover, we can transform a proof of $\mathsf{ns\text{-}ACA_0} \vdash \psi$ into a proof of $\mathsf{ACA_0} \vdash\!\Vdash \psi$.

Proof. We can easily check that $\Vdash$ is closed under inference rules. Thus, by Lemma 6.7, we can transform a proof of $\mathsf{ns\text{-}ACA_0} \vdash \psi$ into a proof of $\mathsf{ACA_0} \vdash\!\Vdash \psi$ effectively. □

Corollary 6.9.

$$\mathsf{ns\text{-}ACA_0} \vdash \varphi^{\mathrm{s}} \Rightarrow \mathsf{ACA_0} \vdash \varphi$$

for any $\mathcal{L}_2$ sentence φ. Moreover, we can transform a proof of $\mathsf{ns\text{-}ACA_0} \vdash \varphi^{\mathrm{s}}$ into a proof of $\mathsf{ACA_0} \vdash \varphi$.

Proof. By Theorem 6.8, we can transform a proof of $\mathsf{ns\text{-}ACA_0} \vdash \varphi^{\mathrm{s}}$ into a proof of $\mathsf{ACA_0} \vdash\!\Vdash \varphi^{\mathrm{s}}$. Then, as in the proof of Lemma 6.6, we can get a proof of $\mathsf{ACA_0} \vdash \varphi$ effectively. □

6.2 The system $\mathsf{ns\text{-}WKL_0}$

In this section, we introduce the system $\mathsf{ns\text{-}WKL_0}$ and interpret $\mathsf{ns\text{-}WKL_0}$ in (a conservative expansion of) $\mathsf{WKL_0}$. For this, we introduce another relation $\Vdash_w$. We treat the universe $V = (M, S)$ of second order arithmetic as the non-standard universe of non-standard second order arithmetic. Then, we construct the standard universe within V by forcing.

Definition 6.4 (the system $\mathsf{ns\text{-}WKL_0}$)**.** The axioms of $\mathsf{ns\text{-}WKL_0}$ are the following:

1. $(\mathsf{WKL_0})^{\mathrm{s}}$, $(\mathsf{WKL_0})^*$.
2. $\surd : V^{\mathrm{s}} \to V^*$ is an injective homomorphism.
3. end extension:
$$\forall x^* \forall y^{\mathrm{s}} (x^* < \check{y}^{\mathrm{s}} \to \exists z^{\mathrm{s}} (x^* = \check{z}^{\mathrm{s}})).$$
4. standard part:
$$\forall X^* \exists Y^{\mathrm{s}} \forall x^{\mathrm{s}} (x^{\mathrm{s}} \in Y^{\mathrm{s}} \leftrightarrow \check{x}^{\mathrm{s}} \in X^*).$$

5. Σ^0_0 transfer principle:

$$\forall x^{\mathrm{s}} \forall X^{\mathrm{s}} (\varphi(x^{\mathrm{s}}, X^{\mathrm{s}})^{\mathrm{s}} \leftrightarrow \varphi(\check{x^{\mathrm{s}}}, \check{X^{\mathrm{s}}})^*)$$

for any Σ^0_0 formula φ.

6. Σ^0_1 overspill:

$$\forall x^* \forall X^* (\forall y^{\mathrm{s}} \exists z^{\mathrm{s}} (z^{\mathrm{s}} \geq y^{\mathrm{s}} \wedge \varphi(\check{z^{\mathrm{s}}}, x^*, X^*)^*) \to \exists y^* (\forall w^{\mathrm{s}} (y^* > \check{w^{\mathrm{s}}}) \wedge \varphi(y^*, x^*, X^*)^*))$$

for any Σ^0_1 formula φ.

In fact, the axiom "Σ^0_0 transfer principle" is deduced from other axioms. Note that we can check that $\mathsf{ns\text{-}WKL_0}$ is an expansion of WKL_0^* which is introduced by Keisler[21].

To interpret $\mathsf{ns\text{-}WKL_0}$, we expand $\mathsf{WKL_0}$.

Definition 6.5. 1. Let c be a new constant symbol. Define the system $\mathsf{WKL_0}'$ as

$$\mathsf{WKL_0}' := \mathsf{WKL_0} + \{c > n \mid n \in \omega\}.$$

2. Let $I(\cdot)$ be a new unary relation symbol. Define the system $\mathsf{WKL_0}''$ as

$$\mathsf{WKL_0}'' := \mathsf{WKL_0}' + \{I(0) \wedge \forall x (I(x) \to I(x+1)) \wedge \forall x \forall y < x (I(x) \to I(y)) \wedge \neg I(c)\}.$$

Note that for any $\mathcal{L}_2$ sentence φ, if we get a proof of $\mathsf{WKL_0}'' \vdash \varphi$, we can easily get a proof of $\mathsf{WKL_0}' \vdash \varphi$ and a proof of $\mathsf{WKL_0} \vdash \varphi$.

Next, we prepare generalized Σ^0_1 satisfaction relation and partial embeddings to interpret $\mathsf{ns\text{-}WKL_0}$.

Definition 6.6 (in $\mathsf{WKL_0}$:Definition 2.1 of [34])**.** We define the set of (Gödel numbers of) $\mathcal{L}_2$ formulas G_e as the following:

- $G_0 = \{\varphi \mid \varphi$ is an atomic formula or the negation of an atomic formula$\}$,
- $G^1_e = \{\exists x \varphi \mid \varphi$ is a finite conjunction of G_e formulas$\}$,
- $G^2_e = \{\forall x < y \varphi \mid \varphi$ is a finite disjunction of G_e formulas$\}$,
- $G^3_e = \{\forall X \varphi \mid \varphi$ is a finite disjunction of G_e formulas$\}$,
- $G_{e+1} = G_e \cup G^1_e \cup G^2_e \cup G^3_e$,

where x and y denote arbitrary distinct number variables, and X an arbitrary set variable. Then, we define $G = \bigcup_{e \in M} G_e$.

Within $\mathsf{WKL_0}$, we can show that each formula in G is equivalent to a Σ^0_1 formula. We next define a satisfaction relation for G as in [34]. For a Σ^0_1 satisfaction relation, see also [13]. For $p \in M$, let $M_p = \{a \mid a < p\}$, $S_p = \{X \cap M_p \mid X \in S\}$ and $V_p = (M_p, S_p)$. We can define the full satisfaction predicate $\mathrm{Tr}^p(z, \xi)$ for V_p as a Δ^0_1 relation, where $z \in G$ and ξ is a finite mapping to evaluate the free variables in z by elements of $M_p \cup S_p$. If z is a Gödel number of $\varphi(\vec{x}, \vec{X})$ (φ has only free variables denoted) and if $\xi(\vec{x}) = \vec{a} \in M_p$ and $\xi(\vec{X}) = \vec{A} \in S_p$, then $\mathrm{Tr}^p(z, \xi) \Leftrightarrow V_P \models \varphi(\vec{a}, \vec{A})$. Moreover, we may assume that Tr^p satisfies the Tarski clauses for all formulas (Note that there exists a non-standard formula in $\mathsf{WKL_0}'$).

Definition 6.7 (in $\mathsf{WKL_0}$:Definition 2.2 of [34])**.** Define the satisfaction relation Tr for G as

$$\mathrm{Tr}(z, \xi) \leftrightarrow \exists p (p > \xi \wedge \mathrm{Tr}^p(z, \xi \upharpoonright V_p)).$$

Here, $p > \xi$ means that p is greater than any $\xi(x)$ and $\xi \upharpoonright V_p$ is defined as $\xi \upharpoonright V_p(x) = \xi(x)$ and $\xi \upharpoonright V_p(X) = \xi(X) \cap M_p$.

Lemma 6.10 (in $\mathsf{WKL_0}$:Lemmas 2.3 and 2.4 of [34])**.**

1. Tr *satisfies the Tarski clauses for G.*

2. *For any $e \in M$ and for any evaluation ξ, there exists a natural number $p \in M$ such that* $\mathrm{Tr}(z, \xi) \leftrightarrow \mathrm{Tr}^p(z, \xi)$ *for any formula $z \in G_e$ with only the free variables associated with ξ.*

From now on, we argue in $\mathsf{WKL_0}''$. We fix p as $p = \min\{q \mid \mathrm{Tr}(z, \emptyset) \leftrightarrow \mathrm{Tr}^q(z, \emptyset)$ for any sentence $z \in G_c\}$. Note that for any (standard) $\mathcal{L}_2$ formula φ such that $[\varphi] \in G$, $[\varphi] \in G_{c-a}$ if $I(a)$. Here, $[\varphi]$ denotes the Gödel number of φ.

Definition 6.8 (in $\mathsf{WKL_0}'$)**.** Let ξ and ξ' be two evaluation mappings with the same domain. Then the pair $\eta = (\xi, \xi')$ is said to be a *partial embedding* (p.e., for short) if $p > \xi'$, $|\eta| \leq c$ and $\mathrm{Tr}(z, \xi) \to \mathrm{Tr}^p(z, \xi' \upharpoonright V_p)$ for any $z \in G_{p-|\eta|}$ with only the free variables associated with ξ. Here, $|\eta| = |\operatorname{dom}(\xi)|$.

We write $a \in \operatorname{dom}(\eta)$ ($A \in \operatorname{dom}(\eta)$) if $a = \xi(x)$ for some number variable x ($A = \xi(X)$ for some set variable X), and $\eta(a) = b$ ($\eta(A) = B$) if $\xi(x) = a \wedge (\xi(x') = a \to [x] \leq [x']) \wedge \xi'(x) = b$ for some x ($\xi(X) = A \wedge (\xi(X') = A \to [X] \leq [X']) \wedge \xi'(X) = B$ for some X). (Here, $[x] \leq [x']$ means that the Gödel number of x' is grater than that of x.)

We now define the forcing notion. Let $P = \{\eta \mid \eta \text{ is a p.e. and } I(|\eta|)\}$. Let $\eta = (\xi, \xi')$ and $\iota = (\zeta, \zeta')$ be p.e.'s. Then, we define $\iota \leq \eta$ as $\xi \subseteq \zeta \wedge \xi' \subseteq \zeta'$ and we define $\iota \leq_1 \eta$ as $\iota \leq \eta \wedge |\iota| \leq |\eta| + 1$. Note that $\iota \in P$ if $\eta \in P \wedge \iota \leq_1 \eta$.

The following lemma is an easy modification of Lemma 2.6 of [34].

Lemma 6.11 (in $\mathsf{WKL_0}''$)**.** *Let $\eta \in P$. Then the following hold.*

1. $\forall a\, \exists a' < p\, \exists \iota \leq_1 \eta\; a \in \mathrm{dom}(\iota) \wedge \iota(a) = a'$.

2. $\forall a' < \eta(a_0)\, \exists a < a_0\, \exists \iota \leq_1 \eta\; a \in \mathrm{dom}(\iota) \wedge \iota(a) = a'$ *for any* $a_0 \in \mathrm{dom}(\eta)$.

3. $\forall A\, \exists A'\, \exists \iota \leq_1 \eta\; A \in \mathrm{dom}(\iota) \wedge \iota(A) = A'$.

4. $\forall A'\, \exists A\, \exists \iota \leq_1 \eta\; A \in \mathrm{dom}(\iota) \wedge \forall \iota' \leq \iota\, \forall a \in \mathrm{dom}(\iota')\; (a \in A \leftrightarrow \iota'(a) \in A')$.

Proof. 1, 2 and 3 are straight forward directions from Lemma 2.6 of [34]. We show 4. Let A' be an arbitary set and $\eta = (\xi, \xi') \in P$. Then, there exists a set A such that $\mathrm{Tr}(z, \xi \cup \{(Y, A)\}) \to \mathrm{Tr}^p(z, \xi' \cup \{(Y, A')\})$ for any $G_{c-|\eta|-1}$ formula z with only the free variables associated with $\xi \cup \{(Y, A)\}$, where Y is a set variable not in the domain of ξ. Define a p.e. $\iota \leq_1 \eta$ as $\iota = (\xi \cup \{(Y, A)\}, \xi' \cup \{(Y, A')\})$. We show $\forall \iota' \leq \iota\, \forall a \in \mathrm{dom}(\iota')\; (a \in A \leftrightarrow \iota'(a) \in A')$. Let $\iota' = (\zeta, \zeta') \leq \iota$. For any $x \in \mathrm{dom}(\zeta)$, $[x \in Y] \in G_0$ and $[\neg x \in Y] \in G_0$, thus $\mathrm{Tr}([x \in Y], \zeta) \to \mathrm{Tr}^p([x \in Y], \zeta')$ and $\mathrm{Tr}([\neg x \in Y], \zeta) \to \mathrm{Tr}^p([\neg x \in Y], \zeta')$. Hence, for any $a \in \mathrm{dom}(\iota')$, $a \in A \to \iota'(a) \in A'$ and $\neg a \in A \to \neg \iota'(a) \in A'$. □

Now, we interpret $\mathsf{ns\text{-}WKL_0}$ in $\mathsf{WKL_0}''$ and show that $\mathsf{ns\text{-}WKL_0}$ is a conservative extension of $\mathsf{WKL_0}$.

Definition 6.9 (in $\mathsf{WKL_0}''$)**.** Let $\bar{M}^{\mathrm{s}} = \bar{M}^* = M$, $\bar{S}^{\mathrm{s}} = \bar{S}^* = S$. Here, $\bar{M}^{\mathrm{s}}$, $\bar{M}^*$, $\bar{S}^{\mathrm{s}}$, $\bar{S}^*$ denote the sets of standard numbers, non-standard numbers, standard sets and non-standard sets in non-standard second order arithmetic. Define $0^{\mathrm{s}}, 1^{\mathrm{s}} \in \bar{M}^{\mathrm{s}}$ and $0^*, 1^* \in \bar{M}^*$ as $0^{\mathrm{s}} = 0^* = 0$ and $1^{\mathrm{s}} = 1^* = 1$. Using the forcing notion $(P, \leq)$, we inductively define $\eta \Vdash_w \psi$ for any $\mathcal{L}_2^* \cup M \cup \bar{M}^* \cup S \cup \bar{S}^*$ sentence ψ as follows. Let $\eta \in P$, $a, b, c \in M$, $a^*, b^*, c^* \in \bar{M}^*$, $A \in S$, $A^* \in \bar{S}^*$. Note that we consider $\eta(a^{\mathrm{s}}) \in \bar{M}^*$ for $a^{\mathrm{s}} \in \mathrm{dom}(\eta)$.

- $\eta \Vdash_w a^{\mathrm{s}} =^{\mathrm{s}} b^{\mathrm{s}} \Leftrightarrow \{\iota \mid a^{\mathrm{s}}, b^{\mathrm{s}} \in \mathrm{dom}(\iota) \wedge a^{\mathrm{s}} = b^{\mathrm{s}}\}$ is dense below η.

- $\eta \Vdash_w +^{\mathrm{s}}(a^{\mathrm{s}}, b^{\mathrm{s}}, c^{\mathrm{s}}) \Leftrightarrow \{\iota \mid a^{\mathrm{s}}, b^{\mathrm{s}}, c^{\mathrm{s}} \in \mathrm{dom}(\iota) \wedge +(a^{\mathrm{s}}, b^{\mathrm{s}}, c^{\mathrm{s}})\}$ is dense below η.

- $\eta \Vdash_w \cdot^{\mathrm{s}}(a^{\mathrm{s}}, b^{\mathrm{s}}, c^{\mathrm{s}}) \Leftrightarrow \{\iota \mid a^{\mathrm{s}}, b^{\mathrm{s}}, c^{\mathrm{s}} \in \mathrm{dom}(\iota) \wedge \cdot(a^{\mathrm{s}}, b^{\mathrm{s}}, c^{\mathrm{s}})\}$ is dense below η.

- $\eta \Vdash_w a^{\mathrm{s}} <^{\mathrm{s}} b^{\mathrm{s}} \Leftrightarrow \{\iota \mid a^{\mathrm{s}}, b^{\mathrm{s}} \in \mathrm{dom}(\iota) \wedge a^{\mathrm{s}} < b^{\mathrm{s}}\}$ is dense below η.
- $\eta \Vdash_w a^{\mathrm{s}} \in^{\mathrm{s}} A^{\mathrm{s}} \Leftrightarrow \{\iota \mid a^{\mathrm{s}}, A^{\mathrm{s}} \in \mathrm{dom}(\iota) \wedge a^{\mathrm{s}} \in A^{\mathrm{s}}\}$ is dense below η.
- $\eta \Vdash_w a^* =^* b^* \Leftrightarrow a^* = b^*$.
- $\eta \Vdash_w +^*(a^*, b^*, c^*) \Leftrightarrow +(a^*, b^*, c^*)$.
- $\eta \Vdash_w \cdot^*(a^*, b^*, c^*) \Leftrightarrow \cdot(a^*, b^*, c^*)$.
- $\eta \Vdash_w a^* <^* b^* \Leftrightarrow a^* < b^*$.
- $\eta \Vdash_w a^* \in^* A^* \Leftrightarrow a^* \in A^*$.
- $\eta \Vdash_w \psi(\check{a^{\mathrm{s}}}, b^*, \check{A^{\mathrm{s}}}, B^*) \Leftrightarrow \{\iota \mid a^{\mathrm{s}}, A^{\mathrm{s}} \in \mathrm{dom}(\iota) \wedge \iota \Vdash_w \psi(\eta(a^{\mathrm{s}}), b^*, \eta(A^{\mathrm{s}}), B^*)\}$ is dense below η where ψ is a non-standard atomic formula.
- $\eta \Vdash_w \varphi \wedge \psi \Leftrightarrow \eta \Vdash_w \varphi \wedge \eta \Vdash_w \psi$.
- $\eta \Vdash_w \neg\psi \Leftrightarrow \forall \iota \leq \eta\ \iota \nVdash_w \psi$.
- $\eta \Vdash_w \exists x^{\mathrm{s}} \psi(x^{\mathrm{s}}) \Leftrightarrow \{\iota \mid \exists a^{\mathrm{s}} \in \mathrm{dom}(\iota)\ \iota \Vdash_w \psi(a^{\mathrm{s}})\}$ is dense below η.
- $\eta \Vdash_w \exists X^{\mathrm{s}} \psi(X^{\mathrm{s}}) \Leftrightarrow \{\iota \mid \exists A^{\mathrm{s}} \in \mathrm{dom}(\iota)\ \iota \Vdash_w \psi(A^{\mathrm{s}})\}$ is dense below η.
- $\eta \Vdash_w \exists x^* \psi(x^*) \Leftrightarrow \exists a^* \in \bar{M}^*\ \eta \Vdash_w \psi(a^*)$.
- $\eta \Vdash_w \exists X^* \psi(X^*) \Leftrightarrow \exists A^* \in \bar{M}^*\ \eta \Vdash_w \psi(A^*)$.

Define $\Vdash_w \psi$ as $\emptyset \Vdash_w \psi$.

Note that we can easily prove that the following are equivalent within $\mathsf{WKL_0}''$:

1. $\eta \Vdash \psi$
2. $\{\iota \mid \iota \Vdash \varphi\}$ is dense below η.
3. $\forall \iota \leq \eta\ \iota \Vdash \varphi$.

We next prepare some lemmas.

Lemma 6.12 (in $\mathsf{WKL_0}''$)**.** *Let $a^{\mathrm{s}}, b^{\mathrm{s}}, c^{\mathrm{s}} \in \bar{M}^{\mathrm{s}}$, $A^{\mathrm{s}} \in \bar{S}^{\mathrm{s}}$, and let $\eta \in P$ such that $a^{\mathrm{s}}, b^{\mathrm{s}}, c^{\mathrm{s}}, A^{\mathrm{s}} \in \mathrm{dom}(\eta)$. Let a pair of $\mathcal{L}_2^*$ formula (ψ, ψ') is one of the following: $\{(a^{\mathrm{s}} =^{\mathrm{s}} b^{\mathrm{s}}, \check{a^{\mathrm{s}}} =^* \check{b^{\mathrm{s}}})$, $(+^{\mathrm{s}}(a^{\mathrm{s}}, b^{\mathrm{s}}, c^{\mathrm{s}}), +^*(\check{a^{\mathrm{s}}}, \check{b^{\mathrm{s}}}, \check{c^{\mathrm{s}}}))$, $(\cdot^{\mathrm{s}}(a^{\mathrm{s}}, b^{\mathrm{s}}, c^{\mathrm{s}}), \cdot^*(\check{a^{\mathrm{s}}}, \check{b^{\mathrm{s}}}, \check{c^{\mathrm{s}}}))$, $(a^{\mathrm{s}} <^{\mathrm{s}} b^{\mathrm{s}}, \check{a^{\mathrm{s}}} <^* \check{b^{\mathrm{s}}})$, $(a^{\mathrm{s}} \in^{\mathrm{s}} A^{\mathrm{s}}, \check{a^{\mathrm{s}}} \in^* \check{A^{\mathrm{s}}})$. Then,*

$$\eta \Vdash_w \psi \leftrightarrow \psi'.$$

Proof. Let $\eta = (\xi, \xi')$. By the definition of partial embedding, $\mathrm{Tr}(z, \xi) \to \mathrm{Tr}^p(z, \xi' \upharpoonright V_p)$ for any atomic formula z with only the free variables associated with ξ. Thus, we can easily check that $\eta \Vdash_w \psi \leftrightarrow \psi'$ by the definition of $\Vdash_w$. □

Lemma 6.13 (in $\mathsf{WKL_0}''$)**.**

$$(\eta \Vdash_w \varphi(\vec{a^{\mathrm{s}}}, \vec{A}^{\mathrm{s}})^{\mathrm{s}}) \leftrightarrow \varphi(\vec{a^{\mathrm{s}}}, \vec{A}^{\mathrm{s}}) \tag{51}$$

for any $\mathcal{L}_2$ formula φ and for any $\eta \in P$ such that $\vec{a^{\mathrm{s}}}, \vec{A}^{\mathrm{s}} \in \mathrm{dom}(\eta)$.

Proof. We show this by induction on the complexity of formulas. Atomic formulas satisfy (51) by the definition of $\Vdash_w$. We can easily check that $\varphi \wedge \psi$ and $\neg\varphi$ satisfy (51) if φ and ψ satisfy (51). Let $\vec{a^{\mathrm{s}}} \in \bar{M}^{\mathrm{s}}$, $\vec{A}^{\mathrm{s}} \in \bar{S}^{\mathrm{s}}$, $\psi(\vec{a^{\mathrm{s}}}, \vec{A}^{\mathrm{s}}) \equiv \exists x \varphi(x, \vec{a^{\mathrm{s}}}, \vec{A}^{\mathrm{s}})$ and $\iota \in P$ such that $\vec{a^{\mathrm{s}}}, \vec{A}^{\mathrm{s}} \in \mathrm{dom}(\eta)$. By the induction hypothesis, $(\iota \Vdash_w \varphi(b^{\mathrm{s}}, \vec{a^{\mathrm{s}}}, \vec{A}^{\mathrm{s}})^{\mathrm{s}}) \leftrightarrow \varphi(b^{\mathrm{s}}, \vec{a^{\mathrm{s}}}, \vec{A}^{\mathrm{s}})$ for any $b^{\mathrm{s}} \in \bar{M}^{\mathrm{s}}$ and $\iota \in P$ such that $b^{\mathrm{s}}, \vec{a^{\mathrm{s}}}, \vec{A}^{\mathrm{s}} \in \mathrm{dom}(\iota)$. If $\exists x \varphi(x, \vec{a^{\mathrm{s}}}, \vec{A}^{\mathrm{s}})$, then, by Lemma 6.11.1, for any $\eta' \in P$ such that $\eta' \leq \eta$, there exist $b^{\mathrm{s}} \in \bar{M}^{\mathrm{s}}$ and $\iota \leq_1 \eta'$ such that $\iota \Vdash_w \varphi(b^{\mathrm{s}}, \vec{a^{\mathrm{s}}}, \vec{A}^{\mathrm{s}})^{\mathrm{s}}$ and $b^{\mathrm{s}} \in \mathrm{dom}(\iota)$. Thus, $\exists x \varphi(x, \vec{a^{\mathrm{s}}}, \vec{A}^{\mathrm{s}}) \to (\eta \Vdash_w (\exists x \varphi(x, \vec{a^{\mathrm{s}}}, \vec{A}^{\mathrm{s}}))^{\mathrm{s}})$. Conversely, $(\eta \Vdash_w (\exists x \varphi(x, \vec{a^{\mathrm{s}}}, \vec{A}^{\mathrm{s}}))^{\mathrm{s}}) \to \exists x \varphi(x, \vec{a^{\mathrm{s}}}, \vec{A}^{\mathrm{s}})$ is obvious. Similarly, we can show the case that $\psi(\vec{a^{\mathrm{s}}}, \vec{A}^{\mathrm{s}}) \equiv \exists X \varphi(X, \vec{a^{\mathrm{s}}}, \vec{A}^{\mathrm{s}})$ by Lemma 6.11.3. □

Lemma 6.14 (in $\mathsf{WKL_0}''$)**.**

$$(\eta \Vdash_w \varphi(\vec{a^*}, \vec{A^*})^*) \leftrightarrow \varphi(\vec{a^*}, \vec{A^*})$$

for any $\mathcal{L}_2$ formula φ and for any $\eta \in P$, $\vec{a^*} \in \bar{M}^*$, $\vec{A^*} \in \bar{M}^*$.

Proof. Obvious from the definition of $\Vdash_w$. □

Lemma 6.15.

$$\mathsf{WKL_0}'' \vdash \Vdash_w \mathsf{ns\text{-}WKL_0}.$$

Proof. We show $\Vdash_w$ $\mathsf{ns\text{-}WKL_0}$ in $\mathsf{WKL_0}''$. By Lemma 6.12, $\Vdash_w$ "$\surd$ is an injective homomorphism" (Definition 6.4.2). By Lemmas 6.13 and 6.14, $\Vdash_w (\mathsf{WKL_0})^{\mathrm{s}} \wedge (\mathsf{WKL_0})^*$ (Definition 6.4.1).

We first show that $\Vdash_w$ "end extension" (Definition 6.4.3). We only need to show that $(\eta \Vdash_w b^* <^* \check{a}^{\mathrm{s}}) \to (\eta \Vdash_w \exists x^{\mathrm{s}}\ \check{x}^{\mathrm{s}} =^* b^*)$ for any $a^{\mathrm{s}} \in \bar{M}^{\mathrm{s}}$, $b^* \in \bar{M}^*$ and for any $\eta \in P$ such that $a^{\mathrm{s}} \in \mathrm{dom}(\eta)$. Let $a^{\mathrm{s}} \in \bar{M}^{\mathrm{s}}$, $b^* \in \bar{M}^*$, $\eta \in P$ such that $a^{\mathrm{s}} \in \mathrm{dom}(\eta)$ and $\eta \Vdash_w b^* <^* \check{a}^{\mathrm{s}}$. Then, $b^* < \eta(a^{\mathrm{s}})$. Thus, by Lemma 6.11.2, for any $\eta' \in P$ such that $\eta' \le \eta$, there exist $\iota \le_1 \eta'$ and $c^{\mathrm{s}} \in \bar{M}^{\mathrm{s}}$ such that $c^{\mathrm{s}} \in \mathrm{dom}(\iota)$ and $\iota(c^{\mathrm{s}}) = b^*$. Hence, $\eta \Vdash_w \exists x^{\mathrm{s}}\ \check{x}^{\mathrm{s}} =^* b^*$.

Next, we show $\Vdash_w$ "standard part" (Definition 6.4.4). We only need to show that $\exists \iota \le \eta\ \exists B^{\mathrm{s}} \in \mathrm{dom}(\iota)\ \iota \Vdash_w \forall x^{\mathrm{s}}(x^{\mathrm{s}} \in^{\mathrm{s}} B^{\mathrm{s}} \leftrightarrow \check{x}^{\mathrm{s}} \in^* A^*)$ for any $A^* \in \bar{S}^*$ and for any $\eta \in P$. Let $A^* \in \bar{S}^*$ and $\eta \in P$. Then, by Lemma 6.11.4, there exist $\iota \le_1 \eta$ and $B^{\mathrm{s}} \in \bar{S}^{\mathrm{s}}$ such that $B^{\mathrm{s}} \in \mathrm{dom}(\iota)$ and $\forall \iota' \le \iota\ \forall a^{\mathrm{s}} \in \mathrm{dom}(\iota')\ (a^{\mathrm{s}} \in B^{\mathrm{s}} \leftrightarrow \iota'(a^{\mathrm{s}}) \in A^*)$. Thus, $\iota \Vdash_w \forall x^{\mathrm{s}}(x^{\mathrm{s}} \in^{\mathrm{s}} B^{\mathrm{s}} \leftrightarrow \check{x}^{\mathrm{s}} \in^* A^*)$. Hence, $\emptyset \Vdash_w \forall X^* \exists Y^{\mathrm{s}} \forall x^{\mathrm{s}}(x^{\mathrm{s}} \in^{\mathrm{s}} Y^{\mathrm{s}} \leftrightarrow \check{x}^{\mathrm{s}} \in^* X^*)$.

Finally, we show $\Vdash_w$ "Σ^0_1 overspill" (Definition 6.4.6). Let φ be Σ^0_1 formula and $a^* \in \bar{M}^*$, $A^* \in \bar{S}^*$. Let $\emptyset \Vdash_w \forall y^{\mathrm{s}} \exists z^{\mathrm{s}}(z^{\mathrm{s}} \ge y^{\mathrm{s}} \wedge \varphi(\check{z}^{\mathrm{s}}, a^*, A^*)^*)$. Then, there exist $\eta_0 \in P$ and $a^{\mathrm{s}} \in \eta_0$ such that $\eta' \Vdash_w \varphi(\check{a}^{\mathrm{s}}, a^*, A^*)^*$. Thus, by Lemma 6.14, $\exists x \le p\ \varphi(x, a^*, A^*)$. Define $b^* \in \bar{M}^*$ as $b^* = \max\{x \le p \mid \varphi(x, a^*, A^*)\}$ by Σ^0_1 induction. We show that there is no $\iota \in P$ such that $\iota(b^{\mathrm{s}}) = b^*$ for some $b^{\mathrm{s}} \in \mathrm{dom}(\iota)$. Assume there exist $\iota \in P$ and $c^{\mathrm{s}} \in \bar{M}^{\mathrm{s}}$ such that $c^{\mathrm{s}} \in \mathrm{dom}(\iota)$ and $\iota(c^{\mathrm{s}}) = b^*$. Since $\iota \Vdash_w \forall y^{\mathrm{s}} \exists z^{\mathrm{s}}(z^{\mathrm{s}} \ge y^{\mathrm{s}} \wedge \varphi(\check{z}^{\mathrm{s}}, a^*, A^*)^*)$, there exist $\bar{\iota} \in P$ and $d^{\mathrm{s}} \in \bar{M}^{\mathrm{s}}$ such that $d^{\mathrm{s}} > c^{\mathrm{s}}$, $d^{\mathrm{s}} \in \mathrm{dom}(\bar{\iota})$ and $\bar{\iota} \Vdash_w \varphi(\check{d}^{\mathrm{s}}, a^*, A^*)^*$. Then, $p \ge \bar{\iota}(d^{\mathrm{s}}) > b^*$ and $\varphi(\bar{\iota}(d^{\mathrm{s}}), a^*, A^*)$, but it contradicts the definition of b^*. Thus, $\emptyset \Vdash_w \forall x^{\mathrm{s}}(b^* > \check{x}^{\mathrm{s}})$ and $\emptyset \Vdash_w \varphi(b^*, a^*, A^*)^*$. Hence, $\emptyset \Vdash_w \exists y^*(\forall x^{\mathrm{s}}(y^* > \check{x}^{\mathrm{s}}) \wedge \varphi(y^*, a^*, A^*)^*)$. □

Theorem 6.16.

$$\mathsf{ns\text{-}WKL_0} \vdash \psi \Rightarrow \mathsf{WKL_0}'' \vdash\Vdash_w \psi$$

for any $\mathcal{L}_2^*$ sentence ψ. Moreover, we can transform a proof of $\mathsf{ns\text{-}WKL_0} \vdash \psi$ *into a proof of* $\mathsf{WKL_0}' \vdash\Vdash_w \psi$.

Proof. We can easily check that $\Vdash_w$ is closed under inference rules. Thus, by Lemma 6.15, we can transform a proof of $\mathsf{ns\text{-}WKL_0} \vdash \psi$ into a proof of $\mathsf{WKL_0}'' \vdash\Vdash_w \psi$ effectively. □

Corollary 6.17.

$$\mathsf{ns\text{-}WKL_0} \vdash \varphi^{\mathrm{s}} \Rightarrow \mathsf{WKL_0} \vdash \varphi$$

for any $\mathcal{L}_2$ sentence φ. Moreover, we can transform a proof of $\mathsf{ns\text{-}WKL_0} \vdash \varphi^{\mathrm{s}}$ *into a proof of* $\mathsf{WKL_0} \vdash \varphi$.

Proof. By Theorem 6.16, we can transform a proof of $\mathsf{ns\text{-}WKL_0} \vdash \varphi^{\mathrm{s}}$ into a proof of $\mathsf{WKL_0}'' \vdash\Vdash_w \varphi^{\mathrm{s}}$. Thus, as in the proof of Lemma 6.13, we can get a proof of $\mathsf{WKL_0}'' \vdash \varphi$. Therefore, we can get a proof of $\mathsf{WKL_0} \vdash \varphi$ easily.

□

7 Appendices

We outline some ongoing studies and we present some open questions.

7.1 Some more studies on Reverse Mathematics for complex analysis

We summarize some more results on Reverse Mathematics for complex analysis without proofs[10]. This is a joint work with Yoshihiro Horihata.

We first study Laurent expansions.

Theorem 7.1 (Laurent expansion)**.** *The following is provable in* $\mathsf{RCA_0}$*. Let f be an effectively integrable holomorphic function on $D = \{z \mid 0 \leq R_1 < |z-a| < R_2\}$. Then, for all $z \in D$,*

$$f(z) = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n} + \sum_{n=0}^{\infty} a_n(z-a)^n$$

where $R_1 < r < R_2$ and

$$a_{-n} = \frac{1}{2\pi i} \int_{|\zeta-a|=r} f(\zeta)(\zeta-a)^{n-1}\, d\zeta,$$
$$a_n = \frac{1}{2\pi i} \int_{|\zeta-a|=r} \frac{f(\zeta)}{(\zeta-a)^{n+1}}\, d\zeta.$$

Theorem 7.2. *The following assertions are pairwise equivalent over* $\mathsf{RCA_0}$*.*

1. $\mathsf{WKL_0}$.
2. *If f is a holomorphic function on $D = \{z \mid 0 \leq R_1 < |z-a| < R_2\}$, then, there exists $\{a_n\}_{n\in\mathbb{Z}}$ such that $f(z) = \sum_{n\in\mathbb{Z}} a_n(z-a)^n$ for all $z \in D$.*

Definition 7.1 (isolated essential singularity)**.** The following definition is made in $\mathsf{RCA_0}$. Let f be a holomorphic function on $D = \{z \mid 0 < |z-a| < R\}$. Then, a is said to be an *isolated essential singularity* if there exists $\{a_n\}_{n\in\mathbb{Z}}$ such that $f(z) = \sum_{n\in\mathbb{Z}} a_n(z-a)^n$ for all $z \in D$ and $\forall m \in \mathbb{N}\ \exists k \geq m\ a_{-k} \neq 0$.

By Theorem 3.25, we can integrate bounded continuous functions within $\mathsf{WWKL_0}$. Thus, many theorems are provable in $\mathsf{WWKL_0}$.

[10]Proofs will appear in Horihata's Master's thesis[17].

Theorem 7.3 (Liouville's theorem)**.** *The following is provable in* $\mathsf{WWKL_0}$*. Let* f *be a holomorphic function from* $\mathbb{C}$ *to* $\mathbb{C}$*. If* f *is bounded, then* f *is a constant function.*

Theorem 7.4 (Riemann's theorem on removable singularities)**.** *The following is provable in* $\mathsf{WWKL_0}$*. Let* f *be a holomorphic function on* $D = \{z \mid 0 < |z-a| < r\}$*. If there exists* $r' > 0$ *such that* $r' < r$ *and* f *is bounded on* $\{z \mid 0 < |z-a| < r'\}$*, then, there exists a holomorphic function* $\tilde{f}$ *on* $D \cup \{a\}$ *such that* $\tilde{f}(z) = f(z)$ *for all* $z \in D$*.*

Theorem 7.5 (Casorati/Weierstraß theorem)**.** *The following is provable in* $\mathsf{WWKL_0}$*. Let* f *be a holomorphic function on* $D = \{z \mid 0 < |z-a| < r\}$ *and* a *is an isolated essential singularity. Then,* $f(D)$ *is dense in* $\mathbb{C}$*.*

Theorem 7.6 (Schwarz's reflection principle)**.** *The following is provable in* $\mathsf{WWKL_0}$*. Let* $D \subseteq \mathbb{C}^+\{x+iy \mid y > 0$ *be an open set and let* $L = (a,b) \subseteq \mathbb{R}$ *be an open interval such that* $L[a,b] = \partial D \cap \mathbb{R}$*. Let* f *be a continuous function on* $D \cup L$ *such that* f *is holomorphic on* D *and* $f(z) \in \mathbb{R}$ *for all* $z \in L$*. Then, there exists a holomorphic function* $\tilde{f}$ *on* $\tilde{D} = D \cup \{z \in \mathbb{C} \mid \bar{z} \in D \cup L\}$ *such that* $\tilde{f}(z) = f(z)$ *for all* $z \in D \cup L$*.*

To study singularities, we need covering spaces.

Definition 7.2 (covering space)**.** Let $X, D \subseteq \mathbb{C}$ be open sets, and let π be a continuous surjective function from X to D. Let $\{U_{ij}\}_{i\in I, j\in J}$ and $\{V_i\}_{i\in I}$ be sequences of open sets, and let π_{ij} be homeomorphic functions from U_{ij} to V_j. Then, π is said to be a *covering map* and a sextuple $(X, D, \pi, U_{ij}, V_i, \pi_{ij})$ is said to be a *covering space* of D if they satisfy the following:

$$\begin{aligned}
&\text{each of } U_{ij} \text{ and } V_i \text{ is simply connected;}\\
&D = \bigcup_{i\in I} V_i;\\
&\forall i \in I\ \pi^{-1}(V_i) = \bigcup_{j\in J} U_{ij},\\
&\forall i \in I\ \forall j \in J\ \pi|_{U_{ij}} = \pi_{ij}.
\end{aligned}$$

Lemma 7.7 (lifting)**.** *The following assertions are pairwise equivalent over* $\mathsf{RCA_0}$*.*

1. $\mathsf{WKL_0}$.

2. *Let* $D_0, D \subseteq \mathbb{C}$ *be open sets, and let* $(X, D, \pi, U_{ij}, V_i, \pi_{ij})$ *be a covering space of* D*. If* D_0 *is simply connected and* f *is a continuous function from* D_0 *to* D*, then, there exists a continuous function* $\hat{f}$ *from* D_0 *to* X *such that* $\pi \circ \hat{f} = f$*.*

By Riemann mapping theorem and Schwarz reflection principle, we can construct a holomorphic covering map from $B(0;1)$ to $\mathbb{C} \setminus \{0,1\}$. Then, by Lemma 7.7, we can show the following Picard's theorem.

Theorem 7.8 (Picard's theorem)**.** *The following is provable in* ACA_0*. Let f be a holomorphic function from $\mathbb{C}$ to $\mathbb{C}$. If the range of f omits two points, then, f is a constant function.*

7.2 Reverse Mathematics for groups and rings

We summarize some results on Reverse Mathematics for groups and rings without proofs[11]. This is a joint work with Takashi Sato and Takeshi Yamazaki. Each of the following theorems is a generalization or a refinement of a known result which appeared, e.g., in [31].

Theorem 7.9. *The following assertions are pairwise equivalent over* RCA_0*.*

1. ACA_0.
2. *For any group G and a subset $S \subseteq G$, there exists a subgroup generated by S.*
3. *For any group G and $a \in G$, there exists a subgroup generated by a.*
4. *For any group G, the center of G exists.*
5. *For any group G and a subgroup H, the normalizer of H exists.*
6. *For any group G, the commutator subgroup of G exists.*
7. *If a group G acts on a set X, then the orbit of each point $x \in X$ exists.*
8. *If S is an integral domain and R is a subring of S, then there exists the integral closure of R in S.*

Theorem 7.10. *The following assertions are pairwise equivalent over* RCA_0*.*

1. WKL_0.
2. *For any group G and $g, h \in G$ such that $\forall n \in \mathbb{N}\ h \neq g^n$, there exists a subgroup H such that $g \in H \land h \notin H$.*
3. *Let I be an ideal of a ring R. Then, there is no ideal J such that $J \supset I$ and $J \neq I$ if and only if I is maximal, i.e., R/I is a field.*

[11] Proofs are in Sato's Master's thesis[26].

7.3 Π^1_1-conservativity and Π^1_2-theories

In this section, we study Π^1_1-conservativity for Π^1_2-theories. This study is inspired by the following questions.

Question 7.3 (Cholak, Jockusch, Slaman[8])**.** Let T_0, T_1, T_2 be Π^1_2-theories.

1. [8, Question 13.3] Let T_1 be a Π^1_1-conservative extension of T_0. Then, is every countable model of T_0 an ω-submodel of some model of T_1?
2. [8, Question 13.4] Let T_1, T_2 be Π^1_1-conservative extensions of T_0. Then, is $T_1 + T_2$ Π^1_1-conservative over T_0?

We first answer these questions. We can easily show that a positive answer to 1 implies a positive answer to 2. However, the answer to 1 is no. In [3], Avigad constructed a counter example for 1. Here, we show another counter example. We construct true Π^1_2-theories which denies 1.

Let $T_0 = \Pi^1_1(\mathsf{ACA_0}^+) + \mathsf{ACA_0}$ and let $T_1 = \mathsf{ACA_0}^+$ where $\Pi^1_1(\mathsf{ACA_0}^+) = \{\varphi \mid \varphi$ is a Π^1_1 sentence and $\mathsf{ACA_0}^+ \vdash \varphi\}$. Then, T_1 is a Π^1_1-conservative extension of T_0. Since $T_1 \vdash \exists X$ 'X is a full satisfaction class for $\mathbb{N}$ (as an $\mathcal{L}_{\mathsf{PA}}$-structure)', the first order part of a countable model of T_1 must be recursively saturated (as an $\mathcal{L}_{\mathsf{PA}}$-structure). Note that a recursively saturated model is not a short model. On the other hand, we can construct a countable model of T_0 whose first order part is a short model (as an $\mathcal{L}_{\mathsf{PA}}$-structure). To show this, let T' be the first order part of T_0, *i.e.*, T' be the set of all $\mathcal{L}_{\mathsf{PA}}$ sentences which are proved in T_0. Then, there exists a countable model M of T' which is short. Thus $(M, \mathrm{ARITH}(M))$ is the required countable model of T_0.

Now, we give a positive answer to Question 7.3.2.

Lemma 7.11. *Let T_0 and T_1 be Π^1_2-theories. Then, the following are equivalent.*

1. *T_1 is a Π^1_1-conservative extension of T_0.*
2. *If (M, S) is a countable model of T_0, then there exist $M' \supseteq M$ and $S' \supseteq S$ such that $(M', S') \models T_1$ and $(M, S) \prec (M', S')$ with respect to arithmetical formulas.*

Proof. To show $2 \to 1$, assume $T_0 \not\vdash \forall X \varphi(X)$ where φ is arithmetical. Then, there exists a countable model $(M, S) \models T_0 + \exists X \neg\varphi(X)$. Then, there exists $(M', S') \models T_1$

such that $(M,S) \prec (M',S')$ with respect to arithmetical formulas. Since $(M,S) \models \exists X \neg\varphi(X)$, $(M',S') \models \exists X \neg\varphi(X)$. Hence, $T_1 \not\vdash \forall X \varphi(X)$.

For the converse, let (M,S) be a countable model of T_0. Let Θ be all arithmetical $\mathcal{L}^2 \cup M \cup S$ sentences which are true in (M,S). We show that $\Theta + T_1$ is consistent. Presume $\Theta + T_1$ is inconsistent, there exist an $\mathcal{L}^2 \cup M$ formula $\psi(\vec{X})$ and $\vec{Z} \in S$ such that $(M,S) \models \psi(\vec{Z})$ and $T_1 \vdash \neg\psi(\vec{Z})$. Then, $T_1 \vdash \forall \vec{X} \neg\psi(\vec{X})$. By Π^1_1-conservativity, $T_0 \vdash \forall \vec{X} \neg\psi(\vec{X})$, but it contradicts $(M,S) \models T_0 + \psi(\vec{Z})$. □

Theorem 7.12. *Let T_0, T_1, T_2 be Π^1_2-theories and let T_1 and T_2 be Π^1_1-conservative extensions of T_0. Then, $T_1 + T_2$ is a Π^1_1-conservative extension of T_0.*

Proof. By the previous lemma, we can construct an arithmetical elementary chain $\{(M_i, S_i)\}_{i\in\omega}$ such that $(M_{2i}, S_{2i}) \models T_1$ and $(M_{2i+1}, S_{2i+1}) \models T_2$. Let $M := \bigcup_{i\in\omega} M_i$ and let $S := \bigcup_{i\in\omega} S_i$. Then, $(M,S) \models T_1 + T_2$. Thus, by the previous lemma, $T_1 + T_2$ is a Π^1_1-conservative extension of T_0. □

Professor Tsuboi and Dr. Ikeda pointed out that the previous lemma and theorem can be generalized to a similar theorem for general first order theories. We give another generalization. Let Γ_0 be the set of all sentences of the form $\forall X \exists! Y \varphi(X,Y)$ where φ is arithmetical. In [36], Yamazaki showed that RCA_0^+ is a Γ_0-conservative extension of RCA_0. In [30], Simpson, Tanaka and Yamazaki showed that WKL_0 is a Γ_0-conservative extension of RCA_0, and then, they showed that $\mathsf{WKL}_0^+ = \mathsf{WKL}_0 + \mathsf{RCA}_0^+$ is also a Γ_0-conservative extension of RCA_0. Then, is $T_1 + T_2$ a Γ_0-conservative extension of T_0 if T_1 and T_2 are Γ_0-conservative extensions of T_0? The answer is yes. We can generalize Theorem 7.12 as follows.

Theorem 7.13. *Let Γ be a class of Π^1_3 sentences which satisfies the following:*

(∗) *for all Σ^1_2 formula $\psi(X)$ and for all Σ^1_0 formula $\theta(X)$, if $\forall X \psi(X) \in \Gamma$, then, there exists $\varphi \in \Gamma$ such that*

$$T_0 \vdash \forall X(\theta(X) \to \psi(X)) \leftrightarrow \varphi.$$

If T_0, T_1 and T_2 are Π^1_2-theories and both T_1 and T_2 are Γ-conservative extensions of T_0, then, $T_1 + T_2$ is a Γ-conservative extension of T_0.

Proof. Easy modification of the proof of Theorem 7.12. □

Corollary 7.14. *Let T_0, T_1 and T_2 be Π^1_2-theories such that T_1 and T_2 are Γ_0-conservative extensions of T_0. Then, $T_1 + T_2$ is a Γ_0-conservative extension of T_0*

We can easily generalize this theorem for Π^1_n-theories.

Theorem 7.15. *Let Γ be a class of Π^1_{n+3} sentences which satisfies the following:*

(∗) *for all Σ^1_{n+2} formula $\psi(X)$ and for all $\Pi^1_n \cup \Sigma^1_n$ formula $\theta(X)$, if $\forall X \psi(X) \in \Gamma$, then, there exists $\varphi \in \Gamma$ such that*

$$T_0 \vdash \forall X(\theta(X) \to \psi(X)) \leftrightarrow \varphi.$$

If T_0, T_1 and T_2 are Π^1_{n+2}-theories and both T_1 and T_2 are Γ-conservative extensions of T_0, then, $T_1 + T_2$ is a Γ-conservative extension of T_0.

Remark 7.16. Avigad showed that there exist Π^1_2-theories T_0, T_1 and T_2 which satisfy the assumption of Theorem 7.12 but $T_1 + T_2 + \Pi^1_1$-AC is inconsistent. This shows that we cannot weaken the assumption of Theorem 7.12. Similarly, there exist Π^1_{n+2}-theories T_0, T_1 and T_2 which satisfy the assumption of Theorem 7.15 but $T_1 + T_2 + \Pi^1_{n+1}$-AC is inconsistent.

7.4 Open questions

We finally present some open questions.

We first consider complex analysis. Theorem 4.20 shows that we can construct the derivative of a complex differentiable function within $\mathsf{WWKL_0}$. As we stated in Section 7.1, we can prove many theorems within $\mathsf{WWKL_0}$ because integrability for bounded functions plays a key role in complex analysis. However, we do not know whether WWKL is exactly needed. For example,

Open question 1. can we prove that a complex differentiable function is a holomorphic function in $\mathsf{RCA_0}$?

When we study complex analysis, an entire function is a important and basic object. We aim to deal with entire functions within $\mathsf{RCA_0}$. An entire function must be a power series, but is this provable in $\mathsf{RCA_0}$? To prove this, we need Cauchy's integral theorem for entire functions.

Open question 2. Can we prove Cauchy's integral theorem for entire functions in $\mathsf{RCA_0}$?

We aim to sharpen the result on Reverse Mathematics for the Riemann mapping theorem (Theorem 5.17).

Open question 3. Is the Riemann mapping theorem equivalent to ACA_0 over RCA_0?

By Theorems 5.26 and 5.25, the interior of a Jordan curve exists and is homeomorphic to the open unit disk in WKL_0. Then, we consider another version of the Riemann mapping theorem:

(JRMT) if $D \subseteq \mathbb{C}$ is the interior of a Jordan curve, then there exists a conformal map $f : D \to B(0; 1)$.

Open question 4. Is JRMT provable in WKL_0?

Note that Picard's theorem is provable in $\mathsf{WKL}_0 +$ JRMT.

Next, we consider arguments of non-standard analysis in second order arithmetic. We developed non-standard analysis only in WKL_0 and ACA_0. Then,

Open question 5. develop suitable parts of non-standard analysis in RCA_0.

We aim to find some more good axioms of non-standard second order arithmetic. Define two new axioms as:

(EEQ) $\varphi^{\mathrm{s}} \leftrightarrow \varphi^{*}$ where φ is an $\mathcal{L}_2$-sentence;

(SB) $\forall x^{\mathrm{s}} \exists y^{*} \psi(x^{\mathrm{s}}, y^{*}) \to \exists z^{*} \forall x^{\mathrm{s}} \exists y^{*} \leq z^{*} \psi(x^{\mathrm{s}}, y^{*})$ where ψ is an $\mathcal{L}_2^{*}$-formula.

Open question 6. Is $\mathsf{ns}\text{-}\mathsf{ACA}_0 +$ EEQ is a conservative extension of ACA_0? More precisely, does ACA_0 prove $\Vdash \mathsf{ns}\text{-}\mathsf{ACA}_0 +$ EEQ?

We can show that $\mathsf{ns}\text{-}\mathsf{ACA}_0 +$ SB is a conservative extension of ACA_0 by ω_1 iterations of Theorem 5.2. Then,

Open question 7. does ACA_0 prove $\Vdash \mathsf{ns}\text{-}\mathsf{ACA}_0 +$ SB? Otherwise, is there a good interpretation of $\mathsf{ns}\text{-}\mathsf{ACA}_0 +$ SB in ACA_0?

We seek some more axioms to develop non-standard analysis richly in second order arithmetic.

Open question 8. Find some more good axioms of non-standard second order arithmetic.

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