ISSN 1343-9499

TOHOKU
MATHEMATICAL
PUBLICATIONS

Number 32

Global solutions for shape memory alloy systems

by

Shuji Yoshikawa

April 2007

This series aims to publish material related to the activities of the Mathematical Institute of Tohoku University. This may include:
1. Theses submitted to the Institute by grantees of the degree of Doctor of Science.
2. Proceedings of symposia as well as lecture notes of the Institute.
A primary advantage of the series lies in quick and timely publication. Consequently, some of the material published here may very likely appear elsewhere in final form.

Tohoku Mathematical Publications

Mathematical Institute
Tohoku University
Sendai 980-8578, Japan

TOHOKU MATHEMATICAL PUBLICATIONS

Number 32

Global solutions for shape memory alloy systems

by

Shuji YOSHIKAWA

April 2007

Global solutions for shape memory alloy systems

A thesis presented

by

Shuji Yoshikawa

to

The Mathematical Institute

for the degree of

Doctor of Science

Tohoku University

Sendai, Japan

March 2006

Acknowledgments

The author would like to express his sincere thanks to Professor Yoshio Tsutsumi for his dedicated guidance and constant encouragement. Without him, this thesis would never have been completed.

He also wishes to express his special gratitude to Professors Takayoshi Ogawa, Toyohiko Aiki, Irena Pawłow, Wojciech M. Zajączkowski and Takashi Suzuki for their invaluable suggestions. His appreciation also goes to all members of the Applied Mathematics Seminar at Tohoku University and NLPDE Seminar at Kyoto University.

This work was partly supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists.

Contents

Chapter 1

Introduction

Shape memory alloys are the materials which, after being strained, revert back to their original shape at a certain temperature. The most widely used alloys include NiTi (Nickel-Titanium), CuZnAl and CuAlNi. Shape memory alloys are not only well-known materials having a wide variety of applications but also the good examples for the researches of the phase transitions in mathematical field. Shape memory effect is due to first-order phase transitions between different equilibrium configurations of the metallic lattice, called *austenite* and *martensite*.

In this thesis, we consider the unique global existence for the systems which describe the relation between the strain and the heat conduction in shape memory alloys. Although there are many systems representing the phase transition occurring on shape memory alloys, the system we consider seems to be the most popular in these systems.

This thesis consists of two parts. In the rest of this chapter, we introduce the derivation of the system and describe the known results and our main results. In Chapter 2, we consider the unique global existence for the one-dimensional system called the Falk model system. In Chapter 3, two and three dimensional systems are treated. In Appendix A, we give a remark on the existence result for the two-dimensional system with the different type of the nonlinear term from that treated in Chapter 3.

1.1 Derivation of the System

In [21], Falk presented the Ginzburg-Landau type theory using the shear strain $\varepsilon = u_x$ as an order parameter in order to explain the martensitic-austenitic phase transitions occurring in a rod which is made of shape memory alloys with the length l. Here, we denote the displacement by u and the absolute temperature by θ. He chose the free energy density F as follows:

$$\begin{aligned} F &= F(\varepsilon, \varepsilon_x, \theta) \\ &= F_0(\theta) + \widetilde{F}(\varepsilon, \theta) + \frac{\kappa}{2}\varepsilon_x^2, \end{aligned} \tag{1.1.1}$$

where $F_0(\theta)$ is typically taken as the following form:

$$F_0(\theta) = -c_v\theta \log(\theta/\theta_3) + c_v\theta + \tilde{c}, \tag{1.1.2}$$

and $\widetilde{F}(\varepsilon, \theta) = G(\theta)F_1(\varepsilon) + F_2(\varepsilon)$ is given by the Devonshire form:

$$F_1(\varepsilon) = \alpha_1\varepsilon^2, \tag{1.1.3}$$

$$F_2(\varepsilon) = -\alpha_2\varepsilon^4 + \alpha_3\varepsilon^6, \tag{1.1.4}$$

$$G(\theta) = \theta - \theta_c. \tag{1.1.5}$$

Here, $\tilde{c}$, α_1, α_2, α_3 and θ_3 are positive physical constants. Positive constants c_v and θ_c are the caloric specific heat and the critical temperature, respectively. We assume that there is no displacement at the endpoint of the rod, that is,

$$u(0,t) = u(l,t) = 0 \qquad \text{for } t \in [0, \infty).$$

The total free energy and the total kinetic energy at time t are given by

$$\mathcal{F}_{tot}(t) = \int_0^l F(\varepsilon, \varepsilon_x, \theta)(x,t)dx$$

and

$$\mathcal{E}_{kin}(t) = \int_0^l \frac{\rho}{2} u_t^2(x,t)dx,$$

respectively. Here ρ is the mass density of shape memory alloys. Applying *Hamilton's principle* in the usual way to the total *Lagrangian*: $\mathcal{L}(t) = \mathcal{E}_{kin}(t) - \mathcal{F}_{tot}(t)$, one easily deduces the equation of motion:

$$\rho u_{tt} + \kappa u_{xxxx} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \varepsilon}(\varepsilon, \theta)\right) \tag{1.1.6}$$

if the variational boundary condition

$$u_{xx}(0,t) = u_{xx}(l,t) = 0$$

is satisfied for $t \in [0, \infty)$.

We assume that the volume is not changed. Then, from the definition of the free energy density $F = U - \theta S$ for the entropy density S and the first law of thermodynamics $\delta U = p\delta V + \theta \delta S$, we deduce that

$$U = F - \theta \frac{\partial F}{\partial \theta}. \tag{1.1.7}$$

According to Falk [21], the balance law of internal energy is represented by

$$U_t + q_x = \frac{\partial F}{\partial \varepsilon}\varepsilon_t + \frac{\partial F}{\partial \varepsilon_x}\varepsilon_{xt}, \tag{1.1.8}$$

where U is the internal energy and q is the heat flux. The heat flux is assumed to be given by the *Fourier form*

$$q = -k\theta_x, \tag{1.1.9}$$

where the heat conductivity k is assumed to be a positive constant. Differentiating the both sides of (1.1.7) with respect to the time variable t and substituting (1.1.8) and (1.1.9) to the resulting equation, we obtain

$$-\theta \frac{\partial^2 F}{\partial \theta^2}\theta_t - k\theta_{xx} = \theta \frac{\partial^2 F}{\partial \theta \partial \varepsilon}\varepsilon_t + \theta \frac{\partial^2 F}{\partial \theta \partial \varepsilon_x}\varepsilon_{xt}. \tag{1.1.10}$$

If we assume (1.1.1)–(1.1.5), then we have $\frac{\partial^2 F}{\partial \theta \partial \varepsilon_x} = 0$ and

$$\frac{\partial^2 F}{\partial \theta^2} = -c_v \frac{1}{\theta}. \tag{1.1.11}$$

Therefore we can simplify (1.1.10) to

$$c_v \theta_t - k\theta_{xx} = \theta \varepsilon_t \frac{\partial F_1}{\partial \varepsilon}. \tag{1.1.12}$$

Combining (1.1.6) and (1.1.12), we obtain the following system:

$$\begin{cases} \rho u_{tt} + \kappa u_{xxxx} = (f_1(u_x)(\theta - \theta_c) + f_2(u_x))_x, \\ c_v \theta_t - k\theta_{xx} = f_1(u_x)\theta u_{xt}, \qquad (t,x) \in \mathbb{R}^+ \times (0,l), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad \theta(0,x) = \theta_0(x) \geq 0, \\ u(t,0) = u(t,l) = u_{xx}(t,0) = u_{xx}(t,l) = \theta_x(t,0) = \theta_x(t,l) = 0, \end{cases} \tag{1.1.13}$$

where $\mathbb{R}^+ = (0, \infty)$, $f_1(r) = F_1'(r) = 2\alpha_1 r$ and $f_2(r) = F_2'(r) = -4\alpha_2 r^3 + 6\alpha r^5$.

In the three-dimensional case, the problem is somewhat complicated. The model is based on the linearized strain tensor $\epsilon(\mathbf{u}) = (\epsilon_{ij})$ such that

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{1.1.14}$$

Here $\mathbf{u} = (u_1, u_2, u_3)$ is the displacement vector. From an argument similar to the one-dimensional case (1.1.1), Falk and Konopka [22] gave the free energy density as follows:

$$\begin{aligned} F(\epsilon, Qu, \theta) &= F_0(\theta) + \widetilde{F}(\theta, \epsilon) + \frac{\kappa}{2}|Qu|^2 \\ &= F_0(\theta) + G(\theta)F_1(\epsilon) + F_2(\epsilon) + \frac{\kappa}{2}|Qu|^2, \end{aligned} \tag{1.1.15}$$

where

$$F_1(\epsilon) = \sum_{i=1}^{3} \alpha_i^2 J_i^2(\epsilon) + \sum_{i=1}^{5} \alpha_i^2 J_i^4(\epsilon), \tag{1.1.16}$$

$$F_2(\epsilon) = \sum_{i=1}^{2} \alpha_i^2 J_i^6(\epsilon) \tag{1.1.17}$$

and $F_0(\theta)$ and $G(\theta)$ are given by (1.1.2) and (1.1.5), respectively. Here α_i^k and θ_c are constants and J_i^k is given as follows:

$$\begin{aligned} &J_1^2 = \epsilon_{\bar{1}}^2, \qquad J_2^2 = 3\epsilon_{\bar{2}}^2 + \epsilon_{\bar{3}}^2, \qquad J_3^2 = \epsilon_{\bar{4}}^2 + \epsilon_{\bar{5}}^2 + \epsilon_{\bar{6}}^2, \\ &J_1^4 = (J_2^2)^2, \qquad J_2^4 = \epsilon_{\bar{4}}^4 + \epsilon_{\bar{5}}^4 + \epsilon_{\bar{6}}^4, \qquad J_3^4 = (J_3^2)^2, \\ &J_4^4 = J_2^2 J_3^2, \qquad J_5^4 = \epsilon_{\bar{4}}^2(\epsilon_{\bar{2}} - \epsilon_{\bar{3}})^2 + \epsilon_{\bar{5}}^2(\epsilon_{\bar{2}} + \epsilon_{\bar{3}})^2 + 4\epsilon_{\bar{6}}^2\epsilon_{\bar{2}}^2, \\ &J_1^6 = (J_2^2)^3, \qquad J_2^6 = \epsilon_{\bar{2}}^2(\epsilon_{\bar{2}}^2 - \epsilon_{\bar{3}}^2)^2 \end{aligned}$$

with

$$\begin{aligned} &\epsilon_{\bar{1}} = \operatorname{trace}\epsilon/3, \qquad \epsilon_{\bar{2}} = (2\epsilon_{33} - \epsilon_{11} - \epsilon_{22})/6, \\ &\epsilon_{\bar{3}} = (\epsilon_{11} - \epsilon_{22})/2, \qquad \epsilon_{\bar{4}} = \epsilon_{23}, \qquad \epsilon_{\bar{5}} = \epsilon_{13}, \qquad \epsilon_{\bar{6}} = \epsilon_{12}. \end{aligned}$$

We define the linearized elasticity operator Q by the following second order differential operator

$$Qu = \mu\Delta\mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}).$$

As in the one-dimensional case, Pawłow [35] derived the three-dimensional thermoelasticity system of shape memory alloys:

$$\begin{cases} \rho\mathbf{u}_{tt} + \kappa Q^2\mathbf{u} = \nabla \cdot F_{,\epsilon}(\epsilon, \theta), \\ c_v\theta_t - k\Delta\theta = \theta F_{,\theta\epsilon}(\epsilon, \theta) : \epsilon_t & \text{in } \Omega_\infty := \mathbb{R}^+ \times \Omega, \\ \mathbf{u} = Q\mathbf{u} = \nabla\theta \cdot \mathbf{n} = 0 & \text{on } S_\infty := \overline{\mathbb{R}}^+ \times \partial\Omega, \\ (\mathbf{u}(0,\cdot), \mathbf{u}_t(0,\cdot)) = (\mathbf{u}_0, \mathbf{u}_1), \quad \theta(0,\cdot) = \theta_0 \geq 0 & \text{in } \Omega, \end{cases} \tag{1.1.18}$$

where $\overline{\mathbb{R}}^+ = [0,\infty)$, $F_{,\epsilon} = (\frac{\partial F}{\partial \epsilon_{ij}})$, $F_{,\theta} = (\frac{\partial F}{\partial \theta})$ and $\tilde{\epsilon} : \epsilon = \sum_{i,j=1}^{3} \tilde{\epsilon}_{ij}\epsilon_{ij}$. We assume that the Lamé constants λ and μ satisfy

$$\mu > 0 \quad \text{and} \quad n\lambda + 2\mu > 0 \tag{1.1.19}$$

for $n = 3$, which assure the strong ellipticity of Q.

Shape memory alloys have another interesting property called *hysteresis*. There are a lot of models and results from this point of view. For related results to hysteresis, we refer to [3]–[6], [30] and [31] (see also [51]).

In Sections 1.2 and 1.3, after introducing the known results we present our main results for the one-dimensional Falk model system and for the multi-dimensional system, respectively. In Section 1.4, we give the notation which will be often used.

1.2 One-Dimensional Case

In this section, we present a brief review on the results of the one-dimensional system (1.1.13) and its related system.

Sprekels and Zheng [41] proved the unique global existence of smooth solution for (1.1.13). In [11], Bubner and Sprekels established the unique global existence of (1.1.13) for data $(u_0, u_1, \theta_0) \in H^3 \times H^1 \times H^1$ and discussed the optimal control problem in the case

$$f_1(r) = 2\alpha_1 r \quad \text{and} \quad f_2(r) = 6\alpha_3 r^5 - 4\alpha_2 r^3. \tag{1.2.1}$$

Here the spaces W_p^m and H^m are the standard Sobolev spaces, that is, W_p^m is equipped with the norm

$$\|f\|_{W_p^m} = \sum_{0 \le k \le m} \|D_x^k f\|_{L^p},$$

and $H^m = W_2^m$. Aiki [2] proved the unique global existence of solution with $(u_0, u_1, \theta_0) \in H^3 \times H^1 \times H^1$ for more general nonlinearity, that is,

$$f_1, f_2 \in \mathrm{C}^2(\mathbb{R}) \tag{1.2.2}$$

and

$$F_2(r) \ge -C \text{ for } r \in \mathbb{R}, \tag{1.2.3}$$

where $F_2(r) = \int_0^s f_2(s)ds$. We note that the condition (1.2.1) implies the conditions (1.2.2) and (1.2.3).

We show the unique global existence of the solution for (1.1.13) in the energy class. The energy class is the function space which is characterized from the form of the Hamiltonian

$\mathcal{H} = \mathcal{E}_{kin} + \mathcal{F}_{tot}$, and hence this is the most natural class of the solution in which to consider the equation from not only a physical point of view but also a mathematical point of view. Precisely, in our case the energy class E is defined by

$$E = H^2 \times L^2 \times L^1 \ni (u(t), u_t(t), \theta(t)). \tag{1.2.4}$$

The energy norm of solutions does not increase from the energy conservation law. Therefore the energy space is expected to be useful to investigate the temporal behavior of the solution. For these reasons we consider the existence of the solution for (1.1.13) in the energy class. We give the precise formulation of the problem in Chapter 2.

We give some related results. Systems related to (1.1.13) have been studied by many authors for the case of viscous materials which has the shear stress σ containing additional viscous component of the following form:

$$\sigma = \frac{\partial F}{\partial \varepsilon} + \nu \varepsilon_t,$$

where the viscosity coefficient ν is a positive constant. Correspondingly, the equations (1.1.13) are modified as follows:

$$\begin{cases} \rho u_{tt} + \kappa u_{xxxx} - \nu u_{xxt} = (f_1(\varepsilon)\theta + f_2(\varepsilon))_x, \\ c_v \theta_t - k\theta_{xx} = f_1(\varepsilon)\theta\varepsilon_t + \nu|\varepsilon_t|^2. \end{cases} \tag{1.2.5}$$

The viscosity term changes the feature of the system because this term has smoothing property. In fact, K.-H. Hoffmann and Żochowski in [29] established the unique global existence result by decomposing the first equation in (1.2.5) into a system of two parabolic equations. There are also some results for the system without capillarity (i.e. $\kappa = 0$ and $\nu > 0$) called *thermoviscoelasticity* (see e.g. Dafermos and Hsiao [15]). Sprekels, Zheng and Zhu [42] studied the asymptotic behavior of the solution for (1.2.5) as $t \to \infty$. However, it seems to be an open problem to determine the asymptotic behavior of the solution for (1.1.13).

Our result is concerned with the unique global existence for (1.1.13) in the energy class $H^2 \times L^2 \times L^1$. We define $L^p_{loc}(\mathbb{R}^+)$ by the set of all functions u such that $\|u\|_{L^p(I)} < \infty$ for each compact subinterval I of $\mathbb{R}^+$.

Theorem 1.1. *Assume that* (1.2.2)–(1.2.3) *holds. Let any* $p \in [4, \infty]$, $q \in [2, 4]$ *and* $r \in (4/3, 8/5)$ *be fixed such that*

$$\frac{1}{p} = \frac{1}{2} - \frac{1}{q}, \qquad r > p', \qquad \frac{1}{r} + \frac{1}{2q'} > 1. \tag{1.2.6}$$

Then for any $(u_0, u_1, \theta_0) \in H^2 \times L^2 \times L^1$*, there exists a unique solution* (u, θ) *to the problem* (1.1.13) *satisfying*

$$\begin{aligned} &u \in \mathrm{C}(\overline{\mathbb{R}}^+; H^2(0,l)), && u_{xx} \in L^p_{loc}(\mathbb{R}^+; L^q(0,l)),\\ &u_t \in L^\infty(\overline{\mathbb{R}}^+; L^2(0,l)), && u_t \in L^p_{loc}(\mathbb{R}^+; L^q(0,l)),\\ &\theta \in \mathrm{C}(\overline{\mathbb{R}}^+; L^1(0,l)), && \theta_x \in L^r_{loc}(\mathbb{R}^+; L^{q'}(0,l)). \end{aligned}$$

The main tools of the proof of our theorem are the maximal regularity estimate and the Strichartz estimate. The maximal regularity estimate is the classical estimate of parabolic equations, and is concerned with the solvability of linear parabolic equations. This can be proved by using the Mikhlin multiplier theorem (see [32] and [33]). The Strichartz estimate established in [43] is closely related to the restriction theory of the Fourier transform to surfaces and used often in various areas of the study of nonlinear wave and dispersive equations (see [12]).

The Strichartz estimate in the spatially periodic setting was established by J. Bourgain [9] and the more transparent proof was given by Fang and Grillakis in [24]. We consider the following initial value problem with periodic boundary conditions, which is closely related to (1.1.13).

$$\begin{cases} \rho u_{tt} + \kappa u_{xxxx} = (f_1(u_x)\theta + f_2(u_x))_x, & \\ c_v\theta_t - k\theta_{xx} = f_1(u_x)\theta u_{xt} & \text{in } \mathbb{R}^+ \times \Omega, \\ u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1, \quad \theta(0,\cdot) = \theta_0 \geq 0 & \text{in } \Omega \end{cases} \tag{1.2.7}$$

for $\Omega = \mathbb{T} = \mathbb{R}/\mathbb{Z}$. From a physical point of view, the problem (1.2.7) describes the dynamics of the ring made of shape memory alloys. This is also an interesting problem. Besides, Theorem 1.1 follows immediately from the following theorem for (1.2.7), because we can regard the initial boundary value problem (1.1.13) as the problem (1.2.7) with periodic boundary conditions, extending the solutions u and θ of (1.1.13) as odd and even periodic functions, respectively.

We can also obtain the result for (1.2.7) with $\Omega = \mathbb{R}$. This is motivated by the work of Falk, Laedke and Spatschek [23]. They studied the stability and existence of the solitary wave appearing on the shape memory alloy rod in $\mathbb{R}$ without heat conduction. Theorems 1.1 and 1.2 are based on the results in [47] an [50].

Theorem 1.2. (*i*) *Assume that* $\Omega = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ *and* (1.2.2)–(1.2.3) *hold. Let any* $p \in [4, \infty]$, $q \in [2, 4]$ *and* $r \in (4/3, 8/5)$ *be fixed satisfying* (1.2.6). *Then for any* $(u_0, u_1, \theta_0) \in$

$H^2 \times L^2 \times L^1$, *there exists a unique solution* (u, θ) *to the problem* (1.2.7) *satisfying*

$$
\begin{aligned}
&u \in \mathrm{C}(\overline{\mathbb{R}}^+; H^2(\mathbb{T})), && u_{xx} \in L^p_{loc}(\mathbb{R}^+; L^q(\mathbb{T})),\\
&u_t \in L^\infty(\overline{\mathbb{R}}^+; L^2(\mathbb{T})), && u_t \in L^p_{loc}(\mathbb{R}^+; L^q(\mathbb{T})),\\
&\theta \in \mathrm{C}(\overline{\mathbb{R}}^+; L^1(\mathbb{T})), && \theta_x \in L^r_{loc}(\mathbb{R}^+; L^{q'}(\mathbb{T})).
\end{aligned}
$$

(ii) Assume that $\Omega = \mathbb{R}$ *and that* (1.2.1) *hold. Let any* $p \in [4, \infty]$, $q \in [2, \infty]$ *and* $r \in (4/3, 2)$ *be fixed such that*

$$
\frac{2}{p} = \frac{1}{2} - \frac{1}{q}, \qquad r > p', \qquad \frac{1}{r} + \frac{1}{2q'} > 1. \tag{1.2.8}
$$

Then for any $(u_0, u_1, \theta_0) \in H^2 \times L^2 \times L^1$, *there exists a unique solution* (u, θ) *to the problem* (1.2.7) *satisfying*

$$
\begin{aligned}
&u \in \mathrm{C}(\overline{\mathbb{R}}^+; H^2(\mathbb{R})), && u_{xx} \in L^p_{loc}(\mathbb{R}^+; L^q(\mathbb{R})),\\
&u_t \in L^\infty(\overline{\mathbb{R}}^+; L^2(\mathbb{R})), && u_t \in L^p_{loc}(\mathbb{R}^+; L^q(\mathbb{R})),\\
&\theta \in \mathrm{C}(\overline{\mathbb{R}}^+; L^1(\mathbb{R})), && \theta_x \in L^r_{loc}(\mathbb{R}^+; L^{q'}(\mathbb{R})).
\end{aligned}
$$

Remark. We note that the nonlinear terms of the second equation in (1.2.7) and (1.1.13) are rewritten as follows:

$$
f_1(u_x)\theta u_{tx} = (f_1(u_x)\theta u_t)_x - f_1'(u_x)u_{xx}\theta u_t - f(u_x)\theta_x u_t,
$$

which makes sense in the distribution class.

1.3 Multi-Dimensional Case

In this section we describe the results for the multi-dimensional thermoelastic system. At first, we cite the following sentences in [10]:

- "*Falk-Konopka (1990) proposed a three-dimensional Landau theory for the martensitic phase transformations in shape memory alloys. Apparently, this model has not yet been studied mathematically.*" (**Remark 5.2.3**, p. 216).

As is written here, the multi-dimensional problem is generally difficult due to several reasons. Comparing (1.1.16) with (1.1.3), we must take $F_1(\epsilon)$ as the fourth order polynomial in three-dimensional case, where the shear strain tensor ϵ is defined by (1.1.14). This makes it difficult to treat the system (1.1.18). Moreover, the useful embedding $H^1 \hookrightarrow L^\infty$ does not hold in the multi-dimensional case, which causes another difficulty. Indeed, there have been no result on the solvability of (1.1.18) up to now (the monograph [10] cited above was published in 1996).

Recently, Pawłow and Żochowski studied the n-dimensional system ($n = 2$ or 3) with viscosity such as (1.2.5), namely, the system with shear stress tensor σ satisfying that

$$\sigma = F_{,\epsilon}(\epsilon, \theta) - \kappa A\epsilon(\nabla \cdot A\epsilon(\mathbf{u})) + \nu A\epsilon_t.$$

Here, the fourth order tensor A represents linear isotropic Hooke's law, being defined by

$$A_{ijkl} := \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

We note that the tensor has the following symmetry properties

$$A_{ijkl} = A_{klij}, \quad A_{ijkl} = A_{jikl}, \quad A_{ijkl} = A_{ijlk} \tag{1.3.1}$$

and the relation $Qu = \nabla \cdot \varepsilon(\mathbf{u})A$ holds. It follows from (1.1.19) that

$$a_*|\epsilon|^2 \le (A\epsilon) : \epsilon \le a^*|\epsilon|^2 \tag{1.3.2}$$

holds for $a_* = \min\{n\lambda + 2\mu, 2\mu\}$ and $a^* = \max\{n\lambda + 2\mu, 2\mu\}$.

The essence of the choice of the polynomial forms (1.1.3)–(1.1.5) and (1.1.16)–(1.1.17) is to have two different local minima depending on the temperature. Although the polynomial forms give the simplest form satisfying this property, we can also represent this property by other general nonlinearities G, F_1 and F_2. For the general nonlinearity $\widetilde{F}(\epsilon, \theta) = G(\theta)F_1(\epsilon) + F_2(\epsilon)$, we can deduce the following quasilinear system:

$$\begin{cases} \rho\mathbf{u}_{tt} + \kappa Q^2\mathbf{u} - \nu Q\mathbf{u}_t = \nabla \cdot [G(\theta)F_{1,\epsilon}(\epsilon) + F_{2,\epsilon}(\epsilon)], & \\ \{c_v - \theta G''(\theta)F_1(\epsilon)\}\theta_t - k\Delta\theta = \theta G'(\theta)\partial_t F_1(\epsilon) + \nu(A\varepsilon_t) : \varepsilon_t & \text{in } \Omega_T, \\ \mathbf{u} = Q\mathbf{u} = \nabla\theta \cdot \mathbf{n} = 0 & \text{on } S_T, \\ (\mathbf{u}(0,\cdot), \mathbf{u}_t(0,\cdot)) = (\mathbf{u}_0, \mathbf{u}_1), \quad \theta(0,\cdot) = \theta_0 \ge 0 & \text{in } \Omega. \end{cases} \tag{1.3.3}$$

In this case, the equation corresponding to the second equation of (1.1.18) is as above because

$$\frac{\partial^2 F}{\partial\theta^2} = -c_v\frac{1}{\theta} + G''(\theta)F_1(\epsilon)$$

instead of (1.1.11). We note that if $G(\theta) = C(\theta - \theta_c)$ then the quasilinear term $\theta G''(\theta)H(\epsilon)\theta_t$ does not appear. Here, we assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain with a smooth boundary $\partial\Omega$.

We consider the following structure of the nonlinearity: the elastic energy density $\widetilde{F}(\theta, \epsilon) = G(\theta)F_1(\epsilon) + F_2(\epsilon)$ satisfies that

$(N1)$ $G \in \mathrm{C}^4(\mathbb{R}, \mathbb{R})$ is as follows:

$$G(\theta) = \begin{cases} C_1\theta & \text{if } \theta \in [0, \theta_1], \\ \varphi(\theta) & \text{if } \theta \in [\theta_1, \theta_2], \\ C_2\theta^r & \text{if } \theta \in [\theta_2, \infty), \end{cases}$$

where $\varphi \in \mathrm{C}^4(\mathbb{R}, \mathbb{R})$, $\varphi'' \leq 0$ and C_1 and C_2 are positive constants for some fixed θ_1, θ_2 satisfying $0 < \theta_1 < \theta_2 < \infty$. We extend G to an odd function on $\mathbb{R}$.

$(N2)$ $F_1 \in \mathrm{C}^4(\mathrm{Sym}(n, \mathbb{R}), \mathbb{R})$ satisfies that $F_1(\epsilon) \geq 0$, where $\mathrm{Sym}(n, \mathbb{R})$ denotes the set of all symmetric second order tensors in $\mathbb{R}^3$.

$(N3)$ $F_2 \in \mathrm{C}^4(\mathrm{Sym}(n, \mathbb{R}), \mathbb{R})$ satisfies that $F_2(\epsilon) \geq -C_3$, where C_3 is some real constant.

$(N4)$ $F_1(\epsilon)$ and $F_2(\epsilon)$ satisfy the following growth conditions:

$$\begin{aligned} |F_{1,\epsilon}(\epsilon)| &\leq C|\epsilon|^{K_1-1}, & |F_{2,\epsilon}(\epsilon)| &\leq C|\epsilon|^{K_2-1}, \\ |F_{1,\epsilon\epsilon}(\epsilon)| &\leq C|\epsilon|^{K_1-2}, & |F_{2,\epsilon\epsilon}(\epsilon)| &\leq C|\epsilon|^{K_2-2}, \\ |F_{1,\epsilon\epsilon\epsilon}(\epsilon)| &\leq C|\epsilon|^{K_1-3}, & |F_{2,\epsilon\epsilon\epsilon}(\epsilon)| &\leq C|\epsilon|^{K_2-3} \end{aligned}$$

for large $|\epsilon|$.

We first state the results for the thermoelastic system in the three-dimensional case $(n = 3)$. Pawłow and Żochowski [36] studied the following semilinearized system

$$\begin{cases} \rho\mathbf{u}_{tt} + \kappa Q^2\mathbf{u} - \nu Q\mathbf{u}_t = \nabla \cdot F_{,\epsilon}(\epsilon, \theta), \\ c_v\theta_t - k\Delta\theta = \theta F_{,\theta\epsilon}(\epsilon, \theta) : \varepsilon_t + \nu(A\epsilon_t) : \epsilon_t. \end{cases} \tag{1.3.4}$$

The system corresponds to the model (1.3.3) without quasilinear term $\theta G''(\theta)H(\epsilon)\theta_t$. They showed unique global existence of the sufficiently smooth solution for the three-dimensional system (1.3.4) under the assumptions:

$$0 \leq r < \frac{1}{2}, \qquad 0 \leq K_1 \leq \left(\frac{1}{2} - r\right) K_2 + 1 \quad \text{and} \quad 0 \leq K_2 \leq \frac{7}{2}. \tag{1.3.5}$$

In addition, when they apply parabolic decomposition of elasticity system, they need to assume the relation $0 < 2\sqrt{\kappa} < \nu$ between viscosity and capillarity. Such an assumption, however, seems not realistic for shape memory alloys whose viscosity effects are negligibly small. In [48], the author showed the unique global existence of the solution for (1.3.4) in a larger class, by using the contraction mapping principle. In the result we does not need conditions between κ and ν and the upper bound of K_2 is generalized to $K_2 < 6$. The first two assumptions of (1.3.5) appear due to the semilinearization which causes the lack of energy conservation laws.

We recall the result of the system called *thermoviscoelasticity* in the case that $\kappa = 0$ and $\nu > 0$. The thermoviscoelasticity system was treated in one-dimensional case [15] and in three-dimensional case [53]. For the viscoelastic system neglecting heat conduction, we refer to [39].

Recently, Pawłow and Zajączkowski [37] proved the unique global existence theorem for the three-dimensional quasilinear system (1.3.3) under the assumptions:

$$0 \le r < \frac{2}{3}, \qquad 0 < K_1 < \frac{15}{4} \quad \text{and} \quad 0 < K_2 \le \frac{9}{2}, \tag{1.3.6}$$

where r and K_1 are linked by the equality $4K_1 + 15r = 15$. In [52], we showed the unique global existence of solution for (1.3.3) under the following power of nonlinearity:

$$0 \le r < \frac{5}{6}, \qquad 0 \le K_1,\ K_2 < 6 \quad \text{and} \quad 6r + K_1 < 6. \tag{1.3.7}$$

In addition, we admit arbitrary positive coefficients of capillarity $\kappa > 0$ and viscosity $\nu > 0$.

Next, we state several remarks on the two-dimensional case. We can deduce the two-dimensional model (1.3.3) from obvious modifications of the three-dimensional case. In [36], Pawłow and Żochowski also showed the unique global existence of solution for the two-dimensional semilinear system (1.3.4) which is the semilinearized model of (1.3.3). The unique global existence for the quasilinear system (1.3.3) was established in [38] under the assumption:

$$0 \le r < \frac{7}{8} \quad \text{and} \quad 0 \le K_1, K_2 < \infty. \tag{1.3.8}$$

In [52], we showed that the system (1.3.3) has a unique global solution under the assumptions:

$$0 \le r < 1 \quad \text{and} \quad 0 \le K_1, K_2 < \infty. \tag{1.3.9}$$

Before stating our results more precisely, we introduce several function spaces. The Sobolev space $W_p^{2l,l}(\Omega_T)$ is equipped with the norm

$$\|u\|_{W_p^{2l,l}(\Omega_T)} := \sum_{j=0}^{2l} \sum_{2r+|\alpha|=j} \|D_t^r D_x^\alpha u\|_{L^p(\Omega_T)},$$

where $D_t := i\frac{\partial}{\partial t}$, $D_x^\alpha = \prod_{\alpha=\alpha_1+\alpha_2+\alpha_3} D_k^{\alpha_k}$ and $D_k := i\frac{\partial}{\partial x_k}$ for multi index $\alpha = (\alpha_1, \ldots, \alpha_n)$, and $W_{p,loc}^{2l,l}$ is the set of all functions u such that $\|u\|_{W_p^{2l,l}(\Omega_I)} < \infty$ for each compact subinterval

I of $\mathbb{R}^+$. The Besov space $B^s_{p,q} = B^s_{p,q}(\Omega)$ is defined by $B^s_{p,q} := [L^p(\Omega), W^j_p(\Omega)]_{s/j,q}$, where $[X,Y]_{s/j,q}$ is the real interpolation space between Banach spaces X and Y. For more details of the Besov space we refer to [1] and [45].

We now state our results. These results are based on a joint work with Irena Pawłow and Wojciech M. Zajączkowski [52].

Theorem 1.3 (Unique Global Existence for Three-Dimensional System). *Let $n = 3$ and $5 < p \le q < \infty$. Assume that $\nu > 0$ and (1.3.7) hold. Then for any $(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \in B^{4-2/p}_{p,p} \times B^{2-2/p}_{p,p} \times B^{2-2/q}_{q,q}$, there exists a unique solution $(\mathbf{u}, \theta)$ to the three-dimensional system (1.3.3) satisfying*

$$(\mathbf{u}, \theta) \in W^{4,2}_{p,loc} \times W^{2,1}_{q,loc}.$$

Moreover, if we assume that $\min_\Omega \theta_0 = \theta_* > 0$ then there exists a positive constant ω such that

$$\theta \ge \theta_* \exp(-\omega t),$$

where ω depends only on A, θ_* and F.

Theorem 1.4 (Unique Global Existence for Two-Dimensional System). *Let $n = 2$ and $4 < p \le q < \infty$. Suppose that $\nu > 0$ and (1.3.9) hold. Then for the two-dimensional system (1.3.3) the same conclusion as in Theorem 1.3 holds.*

We shall describe the proof of this theorem in Chapter 3. We prove the existence part in Theorem 1.3 by using the Leray-Schauder fixed point principle. The key estimates to the proof are the maximal regularity estimate for the first equation of (1.3.3), the classical energy estimate and the parabolic De Giorgi method for the second equation of (1.3.3). The maximal regularity theory is concerned with the theory of solvability for linear parabolic equations, and the maximal regularity is the subordinate estimate to the maximal regularity theory. In the maximal regularity, a loss of regularity does not occur, such as the Schauder estimate for elliptic equations. The maximal regularity theory was extensively studied by many authors. For more details of the maximal regularity, we refer to [7]. In particular, for more recent developments of the maximal L^p-regularity we refer to [18]. We also give a brief review of the maximal regularity theory in Section 3.2 of this thesis.

Since the maximal regularity theory is limited to linear parabolic equations, we cannot use it directly for the second equation of the problem (1.3.3). To obtain the higher order a priori estimates we also use the classical energy methods and the parabolic De Giorgi method (see [32], [34]). Using these methods we can show the Hölder continuity of θ. From this regularity result, we arrive at the estimate for the higher Sobolev norm $W_p^{4,2}(\Omega_T) \times W_q^{2,1}(\Omega_T)$ for $T < \infty$.

Comparing these assumptions with (1.3.8), we see that the restriction for r is weaker, and we can choose r arbitrarily close to 1.

In [49], the author showed the unique global existence for the two-dimensional system (1.3.3) under $r = 1$, $K_1 \in [0,1]$, $K_2 \in [0,\infty)$ and the smallness for the energy of initial data $\|\mathbf{u}_0\|_{H^2} + \|\mathbf{u}_1\|_{L^2} + \|\theta_0\|_{L^1}$. We give the proof of this theorem in Appendix.

Theorem 1.5 (Small Energy Global Existence). *Let $n = 2$, $\nu > 0$, $p > 4$ and suppose that F satisfies $C_3 = 0$ and*

$$r = 1, \qquad K_1 \in [0,1], \qquad K_2 \in [0,\infty), \qquad F_2(\epsilon) \le C|\epsilon|^{K_2}. \tag{1.3.10}$$

Then there exists $\eta > 0$ such that for any $(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \in B_{p,p}^{4-\frac{2}{p}} \times B_{p,p}^{2-\frac{2}{p}} \times B_{\frac{3p}{4},\frac{3p}{4}}^{2-\frac{8}{3p}}$ satisfying $\|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_E < \eta$ there exists a unique global solution (u, θ) of the two-dimensional system (1.3.3) satisfying that

$$(\mathbf{u}, \theta) \in W_{p,loc}^{4,2} \times W_{\frac{3p}{4},loc}^{2,1}$$

and that there exists the monotone increasing function $K(x) > 0$ such that $K(0) = 0$ and

$$\|(\mathbf{u}(t), \mathbf{u}_t(t), \theta(t))\|_E \le K(\|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_E)$$

for any $t \in [0,\infty)$.

1.4 Notation

We show the notation and collect definitions that we use throughout this treatise.

- By C, M and c, we denote various positive constants which may differ from line to line. In particular, the constant at which we emphasis the dependence of the variable r is denoted by $C(r)$.
- A number q' means the Hölder conjugate of $q \in [1, \infty]$, that is, q' and q satisfy the relation $1/q + 1/q' = 1$.
- We denote a partial derivative with respect to a variable y by $\partial_y := \frac{\partial}{\partial y}$ and a weak derivative by D_y. We also use the notation u_x for $\partial_x u$. In particular, we denote a weak derivative with respective to x_j direction by D_j.
- We denote the norm of f for a normed space X by $\| f\,; X\|$ or $\|f\|_X$.
- L^p is the standard Lebesgue space. We denote by $L^p_{loc}(J)$ the set of the functions u such that $\|u\|_{L^p(I)} < \infty$ for each compact subinterval I of J.
- The spaces W^m_p and H^m are the Sobolev spaces, that is, W^m_p is equipped with the norm
$$\|f\|_{W^m_p} = \sum_{0\le k\le m} \|D^k_x f\|_{L^p},$$
and $H^m = W^m_2$.
- In this thesis, the energy class of the shape memory alloy systems is $H^2 \times L^2 \times L^1 \ni (u, u_t, \theta)$. We denote the energy norm of (u, u_t, θ) by $\|(u, u_t, \theta)\|_E = \|u\|_{H^2} + \|u_t\|_{L^2} + \|\theta\|_{L^1}$.
- We frequently use the following abbreviations: $L^p_I L^q_x$ or $L^p_I L^q(\Omega)$ for $L^p(I; L^q(\Omega))$ and $L^p_{I,x}$ or $L^p(\Omega_I)$ for $L^p(I; L^p(\Omega))$ for a connected interval $I \subset \mathbb{R}$. In particular, $L^p_T L^q_x$ means $L^p(0, T; L^q)$ for $T \in (0, \infty]$. A similar notation is applied to other cases such as $\mathrm{C}_I L^p$.

Next we give the notation used in each of chapters.

Chapter 2.

- We denote by $f^{(j)}$ the j-th derivative of f.
- We denote the one-dimensional torus by $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, the set of positive real numbers by $\mathbb{R}^+ = (0, \infty)$ and the set of nonnegative real numbers by $\overline{\mathbb{R}}^+ = [0, \infty)$.

- We write $\partial_x^s E := H^{s+2} \times H^s \times W_1^s$.
- The heat kernel $\frac{1}{\sqrt{4\pi t}} \exp(\frac{-|x|^2}{2\pi t})$ is denoted by $G_t(x)$.

Chapter 3 and Appendix.

- For $I = (a, b)$ we set $\Omega_I = (a, b) \times \Omega$ and $S_I = [a, b) \times \partial\Omega$, where $0 < a < b \leq \infty$. In particular, $\Omega_t = (0, t) \times \Omega$ and $S_t = [0, t) \times \partial\Omega$ for $t \in (0, \infty]$.
- The Sobolev space $W_p^{2l,l}(\Omega_T)$ is the Banach space equipped with the norm

$$\|u\|_{W_p^{2l,l}(\Omega_T)} := \sum_{j=0}^{2l} \sum_{2r+|\alpha|=j} \|D_t^r D_x^\alpha u\|_{L^p(\Omega_T)},$$

  for multi index $\alpha = (\alpha_i)_{i=1}^n$.
- The Besov space $B_{p,q}^s = B_{p,q}^s(\Omega)$ is defined by $B_{p,q}^s := [L^p(\Omega), W_p^j(\Omega)]_{s/j,q}$, where $[E_0, E_1]_{s/j,q}$ is the real interpolation space of the interpolation couple $[E_0, E_1]$.

  The pair $[E_0, E_1]$ is said to be an interpolation couple if there exists a locally convex space X such that $E_j \hookrightarrow X$ for $j = 0, 1$.

  The real interpolation space $[E_0, E_1]_{s/j,q}$ is the Banach space equipped with the norm

$$\|x\|_{s/j,q} := \|t^{-s/j} K(t, x)\|_{L^q(\mathbb{R}^+, dt/t)}$$

  for $0 < s < j$, $1 \leq q \leq \infty$ and $x \in E_0 + E_1$, and

$$K(t, x) := K(t, x, E_0, E_1) := \inf\{\| x_0 \,;\, E_0\| + t\| x_1 \,;\, E_1\| \mid x = x_0 + x_1\}.$$

- $W_{p,loc}^{2l,l}$ is the set of the functions such that $\|u\|_{W_p^{2l,l}(\Omega_I)} < \infty$ for each compact subinterval I of $\mathbb{R}^+$.
- $\mathrm{C}^{\alpha,\alpha/2}(\Omega_T)$ is the Hölder space: the set of all continuous functions in Ω_T satisfying Hölder condition in x with exponent α and in t with exponent $\alpha/2$.
- BUC(I) consists of all bounded and uniformly continuous functions on a interval I.
- Let $F_{,\epsilon} := (\frac{\partial F}{\partial \epsilon_{ij}})$, $F_{,\theta} := (\frac{\partial F}{\partial \theta})$ and $\tilde{\epsilon} : \epsilon := \sum_{i,j=1}^n \tilde{\epsilon}_{ij}\epsilon_{ij}$.
- We set $U(p, q) := B_{p,p}^{4-2/p} \times B_{p,p}^{2-2/p} \times B_{q,q}^{2-2/q}$ and $V_T^{p,q} := W_p^{4,2}(\Omega_T) \times W_q^{2,1}(\Omega_T)$. In particular, we write $U^p = U(p, 3p/4)$, which we will use in Appendix. In the proof of Lemma 3.4.5, to shorten the notation, we set

$$U_1(m) = B_{10/3,10/3}^{17/5} \times B_{10/3,10/3}^{7/5} \times (L^m \cap H^1),$$
$$U_2 = (B_{p,p}^{3-2/p} \cap B_{10/3,10/3}^{17/5}) \times (B_{p,p}^{1-2/p} \cap B_{10/3,10/3}^{7/5}) \times (L^\infty \cap H^1).$$

- For the linear operator A, we denote the domain of A by $D(A)$.
- We define the linear operators $\mathcal{Q}$ and $\mathfrak{D}$ in $L^p(\Omega)$ for $p \in (1, \infty)$ by

$$\begin{cases} D(\mathcal{Q}) = \{\mathbf{u} \in W_p^2(\Omega) \mid \mathbf{u} = 0 \text{ on } \partial\Omega\}, \\ \mathcal{Q}\mathbf{u} = Q\mathbf{u} \end{cases}$$

  and

$$\begin{cases} D(\mathfrak{D}) = \{\mathbf{u} \in W_p^2(\Omega) \mid \mathbf{u} = 0 \text{ on } \partial\Omega\}, \\ \mathfrak{D}\mathbf{u} = \Delta\mathbf{u}, \end{cases}$$

  respectively.
- We denote a number less than p by $p-$.
- $\mathrm{Sym}(n, \mathbb{R})$ denotes the set of all symmetric second order tensors in $\mathbb{R}^n$

Chapter 2

One-Dimensional Case

This chapter is based on the result of [47] and [50]. Let $u = u(t,x)\colon \overline{\mathbb{R}}^+ \times \Omega \to \mathbb{R}$ be the displacement of shape memory alloys and $\theta = \theta(t,x)\colon \overline{\mathbb{R}}^+ \times \Omega \to \mathbb{R}$ be the temperature, where $\mathbb{R}^+ = (0,\infty)$ and $\overline{\mathbb{R}}^+ = [0,\infty)$. In this chapter, we study the initial boundary value problem of the Boussinesq-heat system:

$$u_{tt} + u_{xxxx} = (f_1(u_x)\theta + f_2(u_x))_x, \tag{2.0.1}$$

$$\theta_t - \theta_{xx} = f_1(u_x)\theta u_{xt} \qquad \text{in } \mathbb{R}^+ \times \Omega, \tag{2.0.2}$$

$$u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1, \quad \theta(0,\cdot) = \theta_0 \qquad \text{on } \Omega, \tag{2.0.3}$$

$$u(t,0) = u(t,1) = u_{xx}(t,0) = u_{xx}(t,1) = \theta_x(t,0) = \theta_x(t,1) = 0 \qquad \text{on } \overline{\mathbb{R}}^+, \tag{2.0.4}$$

where $\Omega = (0,1)$. For simplicity, we normalize all the physical coefficients and the length l to unity. We also consider the initial value problems:

$$u_{tt} + u_{xxxx} = (f_1(u_x)\theta + f_2(u_x))_x, \tag{2.0.5}$$

$$\theta_t - \theta_{xx} = f_1(u_x)\theta u_{xt} \qquad \text{in } \mathbb{R}^+ \times \Omega, \tag{2.0.6}$$

$$u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1, \quad \theta(0,\cdot) = \theta_0 \qquad \text{on } \Omega, \tag{2.0.7}$$

where $\Omega = \mathbb{T}$ or $\mathbb{R}$.

The nonlinearity satisfies

$$f_1, f_2 \in C^2(\mathbb{R}) \tag{2.0.8}$$

and

$$F_2(r) \geq -M \text{ for } r \in \mathbb{R}, \tag{2.0.9}$$

where $F_2(r) = \int_0^s f_2(s)ds$. The typical and realistic example of f_1 and f_2 are given by

$$f_1(r) = r \quad \text{ and } \quad f_2(r) = r^5 - r^3 - r. \tag{2.0.10}$$

In this chapter, we prove the unique global existence for (2.0.1)–(2.0.4) and (2.0.5)–(2.0.7) in the energy class $E = H^2 \times L^2 \times L^1$. Theorems 1.1 and 1.2 follow immediately from the following theorems.

Theorem 2.1 (Unique Local Existence on $\mathbb{T}$). *Assume that $\Omega = \mathbb{T}$ and (2.0.8) hold. Let $p \in [4, \infty]$, $q \in [2, 4]$ and $r \in (4/3, 8/5)$ be arbitrary constants satisfying*

$$\frac{1}{p} = \frac{1}{2} - \frac{1}{q}, \qquad r > p', \qquad \frac{1}{r} + \frac{1}{2q'} > 1. \tag{2.0.11}$$

Then for any $(u_0, u_1, \theta_0) \in E = H^2 \times L^2 \times L^1$, there exists $T = T(\|(u_0, u_1, \theta_0)\|_E) > 0$ such that the initial value problem (2.0.5)–(2.0.7) has a unique solution (u, θ) on the time interval $[0, T]$, satisfying

$$\begin{aligned} &u \in \mathrm{C}_T H^2(\Omega), && u_{xx} \in L^p_T L^q(\Omega), \\ &u_t \in L^\infty_T L^2(\Omega), && u_t \in L^p_T L^q(\Omega), \\ &\theta \in \mathrm{C}_T L^1(\Omega), && \theta_x \in L^r_T L^{q'}(\Omega). \end{aligned} \tag{2.0.12}$$

Theorem 2.2 (Unique Local Existence). *Assume that $\Omega = (0, 1)$ and (2.0.8) hold. Let $p \in [4, \infty]$, $q \in [2, 4]$ and $r \in (4/3, 8/5)$ be arbitrary constants satisfying the relations (2.0.11). Then for any $(u_0, u_1, \theta_0) \in E$, there exists $T = T(\|(u_0, u_1, \theta_0)\|_E) > 0$ such that the initial boundary value problem (2.0.1)–(2.0.4) has a unique solution (u, θ) on the time interval $[0, T]$, satisfying (2.0.12).*

Theorem 2.3 (Unique Local Existence on $\mathbb{R}$). *Assume that $\Omega = \mathbb{R}$ and (2.0.8) hold. Let $p \in [4, \infty]$, $q \in [2, \infty]$ and $r \in (4/3, 2)$ be arbitrary constants satisfying*

$$\frac{2}{p} = \frac{1}{2} - \frac{1}{q}, \qquad r > p', \qquad \frac{1}{r} + \frac{1}{2q'} > 1. \tag{2.0.13}$$

Then for any $(u_0, u_1, \theta_0) \in E$, there exists $T = T(\|(u_0, u_1, \theta_0)\|_E) > 0$ such that the initial value problem (2.0.5)–(2.0.7) has a unique solution (u, θ) on the time interval $[0, T]$, satisfying (2.0.12) with (2.0.13).

Combining these results with the energy conservation law, we obtain the following global result.

Theorem 2.4 (Global Existence). (*i*) *In addition to the assumptions of Theorem 2.2 (Theorem 2.1, resp.), suppose that (2.0.9) and $\theta_0 \geq 0$ hold. Then the solution for (2.0.1)–(2.0.4) ((2.0.5)–(2.0.7), resp.) given by Theorem 2.2 (Theorem 2.1, resp.) can be extended globally in time.*
(*ii*) *In addition to the assumptions of Theorem 2.3, suppose that (2.0.10) and $\theta_0 \geq 0$ hold. Then the solution for (2.0.5)–(2.0.7) given by Theorem 2.3 can be extended globally in time.*

In Section 2.1, we introduce the several preliminary lemmas. In Sections 2.2, 2.3 and 2.4, we prove Theorems 2.1, 2.2 and 2.3, respectively. In Section 2.5, we prove the global existence theorem (Theorem 2.4).

2.1 Preliminary Results

In this section, we summarize several lemmas to be used in the proof of theorems. The key estimates for this result are a space-time estimate for the free solution of the Schrödinger equation (the so-called Strichartz estimate) and the maximal regularity estimate of the heat equation.

Lemma 2.1.1 (Strichartz Estimate). *Let $e^{\pm it\partial_x^2}$ be the Schrödinger group on $\mathbb{R}$ or $\mathbb{T}$.*

(i) Let $p_i \in [4,\infty]$ and $q_i \in [2,\infty]$ satisfy $\frac{2}{p_i} = \frac{1}{2} - \frac{1}{q_i}$ $(i = 1,2)$. Then,

$$\| e^{\pm it\partial_x^2} u_0 \,;\, L_T^{p_1} L_x^{q_1}(\mathbb{R}) \| \leq C \| u_0 \,;\, L_x^2(\mathbb{R}) \| \tag{2.1.1}$$

and

$$\left\| \int_0^t e^{\pm i(t-s)\partial_x^2} f(s) ds \,;\, L_T^{p_1} L_x^{q_1}(\mathbb{R}) \right\| \leq C \| f \,;\, L_T^{p_2'} L_x^{q_2'}(\mathbb{R}) \|. \tag{2.1.2}$$

(ii) Let $p_i \in [4,\infty]$ and $q_i \in [2,\infty]$ satisfy $\frac{1}{p_i} = \frac{1}{2} - \frac{1}{q_i}$ $(i = 1,2)$. Then,

$$\| e^{\pm it\partial_x^2} u_0 \,;\, L_T^{p_1} L_x^{q_1}(\mathbb{T}) \| \leq C \| u_0 \,;\, L_x^2(\mathbb{T}) \| \tag{2.1.3}$$

and

$$\left\| \int_0^t e^{\pm i(t-s)\partial_x^2} f(s) ds \,;\, L_T^{p_1} L_x^{q_1}(\mathbb{T}) \right\| \leq C \| f \,;\, L_T^{p_2'} L_x^{q_2'}(\mathbb{T}) \|. \tag{2.1.4}$$

For the proof of (i), see the literature by Cazenave [12]. For the periodic case (ii), we refer to [9] or [24].

Lemma 2.1.2 (Maximal Regularity). *Let Ω be $\mathbb{R}$ or $\mathbb{T}$. For any $p, q \in (1,\infty)$, we have*

$$\left\| \partial_x^2 \int_0^t e^{(t-s)\partial_x^2} f(s) ds \,;\, L_T^p L_x^q(\Omega) \right\| \leq C \| f \,;\, L_T^p L_x^q(\Omega) \|, \tag{2.1.5}$$

where $e^{t\partial_x^2}$ is the heat semigroup on Ω.

For the proof, we refer to the literature by Lemarié-Rieusset [33].

Proposition 2.1.3 (**L^p-L^q Estimate**)**.** *(i) In the case of* $\mathbb{T}$*, if* $1 \le q \le p \le \infty$ *and* $t > 0$*, then we have*

$$\| e^{t\partial_x^2} g\,;\, L_x^p(\mathbb{T})\| \le C\left(1 + t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\right)\| g\,;\, L_x^q(\mathbb{T})\| \tag{2.1.6}$$

and

$$\| \partial_x e^{t\partial_x^2} g\,;\, L_x^p(\mathbb{T})\| \le Ct^{-\frac{1}{2}}\left(1 + t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\right)\| g\,;\, L_x^q(\mathbb{T})\|. \tag{2.1.7}$$

(ii) In the case of $\mathbb{R}$*, if* $1 \le q \le p \le \infty$ *and* $t > 0$*, then we have*

$$\| e^{t\partial_x^2} g\,;\, L_x^p(\mathbb{R})\| \le Ct^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\| g\,;\, L_x^q(\mathbb{R})\| \tag{2.1.8}$$

and

$$\| \partial_x e^{t\partial_x^2} g\,;\, L_x^p(\mathbb{R})\| \le Ct^{-\frac{1}{2}-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\| g\,;\, L_x^q(\mathbb{R})\|. \tag{2.1.9}$$

Proof. We first prove the case (i). We notice that the following fundamental estimates for the analytic semigroup $e^{t\partial_x^2}$ hold

$$\| \partial_x e^{t\partial_x^2} f\,;\, L_x^p\| \le Ct^{-\frac{1}{2}}\| f\,;\, L_x^p\| \tag{2.1.10}$$

and

$$\| e^{t\partial_x^2} f\,;\, L_x^p\| \le C\| f\,;\, L_x^p\| \tag{2.1.11}$$

for any $p \in [1, \infty]$. If we obtain

$$\| e^{t\partial_x^2} g\,;\, L_x^\infty\| \le C\left(t^{-\frac{1}{2}} + 1\right)\| g\,;\, L_x^1\|, \tag{2.1.12}$$

then we have (2.1.4) by interpolation with (2.1.7).

We can write the heat kernel $\tilde{G}_t$ on $\mathbb{T}$ as the following form

$$\tilde{G}_t = \sum_{n=-\infty}^{\infty} G_t(x+n), \tag{2.1.13}$$

where the $G_t(x)$ is given by

$$G_t(x) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{2\pi t}}.$$

Then we have

$$\begin{aligned}
\sup_{x\in[0,1]} |\tilde{G}_t(x)| &= \frac{C}{t^{1/2}} \sup_{x\in[0,1]} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{|x+n|^2}{t}\right) \\
&\le \frac{C}{t^{1/2}} \sum_{n=0}^{\infty} \sup_{x\in[0,1]} \exp\left(-\frac{|x+n|^2}{t}\right) \\
&\le \frac{C}{t^{1/2}} \sum_{n=0}^{\infty} \exp\left(-\frac{|n|^2}{t}\right) \\
&\le \frac{C}{t^{1/2}} \left(1 + \int_0^\infty \exp\left(-\frac{x^2}{t}\right) dx\right) \\
&\le \frac{C}{t^{1/2}} \left(1 + \sqrt{t} \int_0^\infty \exp(-y^2) dy\right) \\
&\le \frac{C}{t^{1/2}} (1 + C\sqrt{t}).
\end{aligned}$$

Therefore, it holds that

$$\begin{aligned}
\left| \int_{\mathbb{T}} \tilde{G}_t(x-y) g(y) dy \right| &\le \sup_{x\in[-1,1]} |\tilde{G}_t(x)| \, \| g \,;\, L^1_x(\mathbb{T}) \| \\
&\le C \sup_{x\in[0,1]} |\tilde{G}_t(x)| \, \| g \,;\, L^1_x(\mathbb{T}) \| \\
&\le C \left(1 + \frac{1}{\sqrt{t}}\right) \| g \,;\, L^1_x \|.
\end{aligned}$$

Then we have the desired estimate (2.1.12), and hence we obtain (2.1.6).

In an argument similar to above, we prove (2.1.7). By (2.1.11), it is sufficient to prove

$$\| \partial_x e^{t\partial_x^2} g \,;\, L^\infty_x \| \le \frac{C}{t^{1/2}} \left(1 + \frac{1}{t^{1/2}}\right) \| g \,;\, L^1_x \|. \tag{2.1.14}$$

By (2.1.13), we have

$$\partial_x \tilde{G}_t = \sum_{n=-\infty}^{\infty} \frac{C}{t^{3/2}} |x+n| \exp\left(-\frac{|x+n|^2}{t}\right).$$

Then it follows that

$$\begin{aligned}
\sup_{x\in[0,1]} |\partial_x \tilde{G}_t(x)| &\leq \frac{C}{t} \sup_{x\in[0,1]} \sum_{n=-\infty}^{\infty} \left\{ \frac{|x+n|}{t^{1/2}} \exp\left(-\frac{|x+n|^2}{t} \right) \right\} \\
&\leq \frac{C}{t} \sum_{n=0}^{\infty} \sup_{x\in[0,1]} \left\{ \frac{|x+n|}{t^{1/2}} \exp\left(-\frac{|x+n|^2}{t} \right) \right\} \\
&= \frac{C}{t} \Bigg[\sum_{n=0}^{I(\sqrt{t/2})} \sup_{x\in[0,1]} \left\{ \frac{|x+n|}{t^{1/2}} \exp\left(-\frac{|x+n|^2}{t} \right) \right\} \\
&\qquad + \sum_{n=I(\sqrt{t/2})+1}^{\infty} \sup_{x\in[0,1]} \left\{ \frac{|x+n|}{t^{1/2}} \exp\left(-\frac{|x+n|^2}{t} \right) \right\} \Bigg],
\end{aligned}$$

where $I(x)$ denotes the integral part of x, i.e., $I(x)$ is the integer n satisfying $n \leq x < n+1$. Noting that $x/t \cdot \exp(-x^2/t)$ attains the maximum value $\sqrt{1/2e}$ at $x = \sqrt{t/2}$ and is monotone decreasing for $x > \sqrt{t/2}$, we have

$$\begin{aligned}
\sum_{n=0}^{I(\sqrt{t/2})} \sup_{x\in[0,1]} \left\{ \frac{|x+n|}{t^{1/2}} \exp\left(-\frac{|x+n|^2}{t} \right) \right\} &\leq \left(\frac{1}{2e} \right)^{1/2} \left(\sqrt{\frac{t}{2}} + 1 \right) \\
&\leq C(\sqrt{t}+1),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=I(\sqrt{t/2})+1}^{\infty} \sup_{x\in[0,1]} \left\{ \frac{|x+n|}{t^{1/2}} \exp\left(-\frac{|x+n|^2}{t} \right) \right\} &\leq \int_{\sqrt{\frac{t}{2}}}^{\infty} \frac{x}{t^{1/2}} \exp\left(-\frac{x^2}{t} \right) dx \\
&\leq C.
\end{aligned}$$

Consequently, we arrive at

$$\begin{aligned}
\sup_{x\in[0,1]} |\partial_x \tilde{G}_t(x)| &\leq \frac{C}{t} \left((\sqrt{t}+1) + C \right) \\
&\leq \frac{C}{t^{1/2}} \left(1 + \frac{1}{t^{1/2}} \right),
\end{aligned}$$

which implies the desired inequality (2.1.14). For the proof of the case (ii), we refer to [13] □

Remarks. (i) One could place other numbers of derivative in (2.1.7).
(ii) In this thesis, since these estimates in time global setting are not needed, we may regard these estimates as the following well-known inequality:

$$\| e^{t\partial_x^2} g \,;\, L_x^p(\mathbb{T}) \| \leq C t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})} \| g \,;\, L_x^q(\mathbb{T}) \|$$

and

$$\| \partial_x e^{t\partial_x^2} g\,;\, L_x^p(\mathbb{T})\| \le C t^{-\frac{1}{2}-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})} \| g\,;\, L_x^q(\mathbb{T})\|.$$

Next, we formulate the estimates obtained by using the Gagliardo-Nirenberg inequality. We shall make frequent use of the following lemmas in this chapter.

Lemma 2.1.4 (Leibniz's Rule). *If $s \ge 1$ is any integer and $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{r_1} = \frac{1}{q_2} + \frac{1}{r_2}$, then*

$$\| \partial_x^s (fg)\,;\, L_x^p\| \le C(\| \partial_x^s f\,;\, L^{q_1}\|\| g\,;\, L_x^{r_1}\| + \| f\,;\, L_x^{q_2}\|\| \partial_x^s g\,;\, L^{r_2}\|).$$

Proof. This is easy consequence of the Leibniz rule $\partial_x^s(fg) = \sum_{s_1+s_2=s}(\partial_x^{s_1} f \partial_x^{s_2} g)$ and the Gagliardo-Nirenberg inequality. □

Next we introduce a useful lemma.

Lemma 2.1.5. *Let Ω be $\mathbb{R}$ or $\mathbb{T}$. If the assumption*

$$F_{q,r}(a,b) := \left(1 + \frac{1}{q}\right)\frac{1}{a} + \frac{1}{rb} - \frac{1}{r} \ge 0 \tag{2.1.15}$$

holds for $a,\ b,\ q,\ r \in [1,\infty]$, then we have

$$\| f\,;\, L_T^a L_x^b(\Omega)\| \le C T^{\delta} \| f\,;\, L_T^\infty L_x^1(\Omega)\|^{1-\sigma} \| f_x\,;\, L_T^r L_x^{q'}(\Omega)\|^{\sigma}, \tag{2.1.16}$$

where

$$\delta = \delta_{q,r}(a,b) = \frac{b}{b-1} F_{q,r}(a,b),$$
$$\sigma = \sigma_q(b) = \frac{(b-1)q}{(q+1)b}.$$

Proof. By the Gagliardo-Nirenberg inequality we have

$$\| \theta\,;\, L_x^b\| \le C\| \theta\,;\, L_x^1\|^{1-\sigma} \| \theta_x\,;\, L_x^{q'}\|^{\sigma}.$$

Therefore, we have

$$\begin{aligned} \| f\,;\, L_T^a L_x^b\| &\le C \left\| \| f\,;\, L_x^1\|^{1-\sigma} \| f_x\,;\, L_x^{q'}\|^{\sigma}\,;\, L_T^a \right\| \\ &\le C \| f\,;\, L_T^\infty L_x^1\|^{1-\sigma} \left\| \| f_x\,;\, L_x^{q'}\|^{\sigma}\,;\, L_T^a \right\| \\ &\le C \| f\,;\, L_T^\infty L_x^1\|^{1-\sigma} \| f_x\,;\, L_T^{a\sigma} L_x^{q'}\|^{\sigma}. \end{aligned}$$

It follows from the assumption (2.1.15) that

$$\frac{1}{a\sigma} - \frac{1}{r} = \frac{b}{b-1}\left(\frac{q+1}{aq} - \frac{b-1}{rb}\right) = \frac{b}{b-1} F_{q,r}(a,b) \ge 0.$$

Then it follows that

$$\| f_x\,;\, L_T^{a\sigma} L_x^{q'}\| \le C T^{\frac{1}{a\sigma}-\frac{1}{r}} \| f_x\,;\, L_T^r L_x^{q'}\| = C T^{\delta} \| f_x\,;\, L_T^r L_x^{q'}\|,$$

which completes the proof. □

2.2 Local Existence on $\mathbb{T}$

In this section, we prove the local existence for the problem (2.0.5)–(2.0.6) with $\Omega = \mathbb{T}$. We denote by $\widehat{f}(k)$ the Fourier coefficient of the function f with respect to the space variable, i.e.,

$$\widehat{f}(k) = \int_0^1 e^{-2\pi ixk} f(x)dx.$$

We write $F := f_1(\varepsilon)\theta + f_2(\varepsilon)$, where $\varepsilon := u_x$. Since $\widehat{F}(0)$ does not depend on x, the equation (2.0.1) can be rewritten as follows:

$$u_{tt} + u_{xxxx} = \{F - \widehat{F}(0)\}_x. \tag{2.2.1}$$

Differentiating both sides of (2.2.1) for sufficiently smooth solutions u, we can derive the equation of ε as:

$$\varepsilon_{tt} + \varepsilon_{xxxx} = \{F - \hat{F}(0)\}_{xx}. \tag{2.2.2}$$

Here for any f such that $\widehat{f}(0) = 0$, we define ∂_x^{-2} by

$$\partial_x^{-2} f(x) = -\sum_{k\neq 0} \frac{e^{2\pi ikx}}{(2\pi k)^2} \widehat{f}(k).$$

We note that $\widehat{\varepsilon}(0) = 0$ and $\widehat{\varepsilon}_t(0) = 0$ by the definition of ε. Putting

$$\varepsilon^{\pm} := \varepsilon \pm i\partial_x^{-2}\varepsilon_t,$$

we have

$$\begin{aligned}
\partial_t\varepsilon &= \varepsilon_t \pm i\partial_x^{-2}\varepsilon_{tt} \\
&= \varepsilon_t \pm i\partial_x^{-2}\left\{-(\partial_x)^4\varepsilon + \partial_x^2(F - \widehat{F}(0))\right\} \\
&= \varepsilon_t \mp i\partial_x^2\varepsilon \pm i(F - \widehat{F}(0)) \\
&= \mp i\partial_x^2(\varepsilon \pm i\partial_x^{-2}\varepsilon_t) \pm i(F - \widehat{F}(0)).
\end{aligned} \tag{2.2.3}$$

Then (2.2.2) is reduced to the following Schrödinger type equations:

$$\partial_t\varepsilon^{\pm} = \mp i\partial_x^2\varepsilon^{\pm} \pm i\{F - \widehat{F}(0)\}.$$

Noticing that

$$\varepsilon_t = \frac{\partial_x^2}{2i}(\varepsilon^+ - \varepsilon^-), \tag{2.2.4}$$

this transformation is useful for the estimate of ε_t.

We first show the time local existence and uniqueness of solution $(\varepsilon^+, \varepsilon^-, \theta)$ with $\widehat{\varepsilon}^{\pm}(0) = 0$ in the space $H^1 \times H^1 \times L^1$. We set

$$\|\varepsilon^{\pm}\|_S := \| \varepsilon^{\pm} \,;\, L_T^\infty H_x^1\| + \| \partial_x \varepsilon^{\pm} \,;\, L_T^p L_x^q\|,$$
$$\|\theta\|_H := \| \theta \,;\, L_T^\infty L_x^1\| + \| \partial_x \theta \,;\, L_T^r L_x^{q'}\|.$$

Given $L > 0$, we define the space

$$X_T^L = \{(\varepsilon^+, \varepsilon^-, \theta) \mid \|(\varepsilon^+, \varepsilon^-, \theta)\|_{X_T} := \|\varepsilon^+\|_S + \|\varepsilon^-\|_S + \|\theta\|_H \le L\}$$

and the operator $\Lambda : (\varepsilon^+, \varepsilon^-, \theta) \mapsto (\Lambda_+ \varepsilon^+, \Lambda_- \varepsilon^-, \Lambda_H \theta)$ is defined by

$$\Lambda_{\pm} \varepsilon^{\pm} = e^{\pm it\partial_x^2} \varepsilon^{\pm}(0) \pm i \int_0^t e^{\pm i(t-s)\partial_x^2} (F - \widehat{F}(0)) ds, \tag{2.2.5}$$

$$\Lambda_H \theta = e^{t\partial_x^2} \theta_0 + \int_0^t e^{(t-s)\partial_x^2} (f_1(u_x)\theta u_{tx})(s) ds. \tag{2.2.6}$$

We shall prove that the operator Λ is a contraction of X_T^L into itself for an appropriate choice of L and T. Without loss of generality we may assume $T < 1$. We note that $\varepsilon^{\pm}(0, x)$ and $(F - \widehat{F}(0))$ have average zero, therefore, so do $\Lambda_{\pm} \varepsilon^{\pm}$.

For the linear part of (2.2.5), it follows from (2.1.3) that

$$\begin{aligned}
\|e^{\pm it\partial_x^2} \varepsilon^{\pm}(0)\|_S &\le \| e^{\pm it\partial_x^2} \varepsilon^{\pm}(0) \,;\; L_T^\infty H_x^1\| + \| \partial_x e^{\pm it\partial_x^2} \varepsilon^{\pm}(0) \,;\, L_T^p L_x^q\| \\
&\le C(\| u_0 \,;\, H^2\| + \| u_1 \,;\, L^2\|).
\end{aligned}$$

Since by the embedding inequality $\| \varepsilon \,;\, L_x^\infty\| \le CL$, we have

$$\| f_i^{(j)}(\varepsilon) \,;\; L_{T,x}^\infty\| \le \sup_{|r| \in CL} |f_i^{(j)}(r)| \le C(L) \tag{2.2.7}$$

for $i = 1, 2$ and $j = 0, 1, 2$. By (2.0.11) we have

$$\begin{aligned}
F_{q,r}\left(\frac{rp}{p-r}, \frac{q}{q-2}\right) &= \left(1 - \frac{1}{q}\right)\frac{1}{r} - \frac{1}{p}\left(1 + \frac{1}{q}\right) \\
&> \frac{1}{2}\left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{q}\right) - \left(\frac{1}{2} - \frac{1}{q}\right)\left(1 + \frac{1}{q}\right) \\
&= \frac{1}{2q}\left(1 + \frac{1}{q}\right) \\
&> 0.
\end{aligned} \tag{2.2.8}$$

Then it follows from (2.1.16) and (2.2.7) that

$$\begin{aligned}
\| \partial_x (f_1(\varepsilon)\theta) \,;\, L_T^\nu L_x^{q'}\| &\le CT^{\frac{1}{\nu} - \frac{1}{r}} \big(\| \theta_x \,;\, L_T^r L_x^{q'}\| \| f_1(\varepsilon) \,;\, L_{T,x}^\infty\| \\
&\qquad + \| \theta \,;\, L_T^{\frac{rp}{p-r}} L_x^{\frac{q}{q-2}}\| \| \varepsilon_x \,;\, L_T^p L_x^q\| \| f_1'(\varepsilon) \,;\, L_{T,x}^\infty\|\big) \\
&\le C(L) T^{\frac{1}{\nu} - \frac{1}{r}} (\|\theta\|_H \|\varepsilon\|_S + \|\theta\|_H),
\end{aligned} \tag{2.2.9}$$

for $\nu \in [p', r]$, and

$$\begin{aligned}\| \partial_x(f_2(\varepsilon)) \,;\, L_T^\infty L_x^2\| &\le C\| f_2'(\varepsilon) \,;\, L_{T,x}^\infty\| \| \varepsilon_x \,;\, L_T^\infty L_x^2\| \\ &\le C(L)\|\varepsilon\|_S.\end{aligned} \tag{2.2.10}$$

For the nonlinear part, by (2.1.4), (2.1.16), (2.2.9) and (2.2.10), we have

$$\begin{aligned}\left\|\int_0^t e^{\pm i(t-s)\partial_x^2}(F(s)-\widehat{F}(0))ds\right\|_S &\le \left\|\int_0^t e^{\pm i(t-s)\partial_x^2}(F(s)-\widehat{F}(0))ds \,;\, L_T^\infty H_x^1\right\| \\ &\quad + \left\|\partial_x\int_0^t e^{\pm i(t-s)\partial_x^2}(F(s)-\widehat{F}(0))ds \,;\, L_T^p L_x^q\right\| \\ &\le C\| \partial_x(f_1(\varepsilon)\theta) \,;\, L_T^{p'} L_x^{q'}\| + C\| \partial_x f_2(\varepsilon) \,;\, L_T^1 L_x^2\| \\ &\le CT^{\frac{1}{\nu}-\frac{1}{r}}\| \partial_x(f_1(\varepsilon)\theta) \,;\, L_T^{\nu} L_x^{q'}\| + CT\| \partial_x f_2(\varepsilon) \,;\, L_T^\infty L_x^2\| \\ &\le C(L)T^{\frac{1}{\nu}-\frac{1}{r}}(\|\theta\|_H\|\varepsilon\|_S + \|\varepsilon\|_S).\end{aligned}$$

Therefore, noting that $\varepsilon = \varepsilon^+ + \varepsilon^-$ holds, we obtain the following estimate

$$\|\Lambda_\pm\varepsilon^\pm\|_S \le C(\|u_0\|_{H^2} + \|u_1\|_{L^2}) + C(L)T^{\frac{1}{\nu}-\frac{1}{r}}(\|\varepsilon^\pm\|_S\|\theta\|_H + \|\theta\|_H + \|\varepsilon^\pm\|_S). \tag{2.2.11}$$

Next, we estimate the heat equation (2.0.6). It follows from Lemma 2.1.3 that

$$\begin{aligned}\| \partial_x e^{t\partial_x^2}\theta_0 \,;\, L_T^r L_x^{q'}\| &\le \left(\int_0^T \frac{C}{t^{(1-1/2q')r}}\| \theta_0 \,;\, L_x^1\|^r dt\right)^{1/r} \\ &\le CT^{\frac{1}{r}+\frac{1}{2q'}-1}\| \theta_0 \,;\, L_x^1\|\end{aligned}$$

for $\frac{1}{r}+\frac{1}{2q'} > 1$. We can split the nonlinear term into the four parts as follows:

$$\begin{aligned}\left\|\int_0^t e^{(t-s)\partial_x^2}(u_{tx}\theta f_1(\varepsilon))(s)ds\right\|_H &\le \left\|\int_0^t e^{(t-s)\partial_x^2}u_t(s)(\theta f_1(\varepsilon))_x(s)ds \,;\, L_T^\infty L_x^1\right\| \\ &\quad + \left\|\partial_x\int_0^t e^{(t-s)\partial_x^2}u_t(s)(\theta f_1(\varepsilon))_x(s)ds \,;\, L_T^r L_x^{q'}\right\| \\ &\quad + \left\|\int_0^t e^{(t-s)\partial_x^2}(u_t\theta f_1(\varepsilon))_x(s)ds \,;\, L_T^\infty L_x^1\right\| \\ &\quad + \left\|\partial_x\int_0^t e^{(t-s)\partial_x^2}(u_t\theta f_1(\varepsilon))_x(s)ds \,;\, L_T^r L_x^{q'}\right\| \\ &:= I_{1,1} + I_{1,2} + I_{2,1} + I_{2,2}.\end{aligned}$$

Using the Hölder inequality, the Hardy-Littlewood-Sobolev inequality and (2.2.9), we have

$$
\begin{aligned}
I_{1,2} &\leq \left\| \int_0^t \frac{C}{(t-s)^{1-\frac{1}{2q'}}} \| (u_t(\theta f_1(\varepsilon))_x)(s) \,;\, L_x^1 \| ds \,;\, L_T^r \right\| \\
&\leq CT^{\frac{1}{4}} \left\| \int_0^t \frac{1}{(t-s)^{1+\frac{1}{2q'}}} \| (u_t(\theta f_1(\varepsilon))_x)(s) \,;\, L_x^1 \| ds \,;\, L_T^{\frac{4r}{4-r}} \right\| \\
&\leq CT^{\frac{1}{4}} \| u_t(\theta f_1(\varepsilon))_x \,;\, L_T^{\frac{pr}{p+r}} L_x^1 \| \\
&\leq CT^{\frac{1}{4}} \| u_t \,;\, L_T^p L_x^q \| \| (\theta f_1(\varepsilon))_x \,;\, L_T^r L_x^{q'} \| \\
&\leq CT^{\frac{1}{4}} \|\varepsilon^{\pm}\|_S^2 \|\theta\|_H
\end{aligned}
$$

for r, p and q satisfying (2.0.11). Similarly, by (2.2.9) we have

$$
\begin{aligned}
I_{1,1} &\leq \| u_t(\theta f_1(\varepsilon))_x \,;\, L_T^1 L_x^1 \| \\
&\leq \| u_t \,;\, L_T^p L_x^q \| \| (\theta f_1(\varepsilon))_x \,;\, L_T^{p'} L_x^{q'} \| \\
&\leq C(L) T^{\frac{1}{p'} - \frac{1}{r}} (\|\varepsilon^{\pm}\|_S^2 \|\theta\|_H + \|\varepsilon^{\pm}\|_S \|\theta\|_H).
\end{aligned}
$$

Since by (2.0.11) we have

$$
\begin{aligned}
F_{q,r}\left(\frac{8}{3}, 2\right) &= \left(1 + \frac{1}{q}\right)\frac{3}{8} - \frac{1}{2r} \\
&> \frac{1}{8}\left(1 - \frac{1}{q}\right) > 0,
\end{aligned}
$$

it follows from the Hölder inequality and (2.1.16) that

$$
\begin{aligned}
I_{1,2} &\leq \left\| \int_0^t \frac{C}{(t-s)^{1/2}} \| u_t \theta f_1(\varepsilon) \,;\, L_x^1 \| ds \,;\, L_T^{\infty} \right\| \\
&\leq CT^{\frac{1}{8}} \| u_t \theta f_1(\varepsilon) \,;\, L_T^{\frac{8}{3}} L_x^1 \| \\
&\leq CT^{\frac{1}{8}} \| u_t \,;\, L_T^{\infty} L_x^2 \| \| \theta \,;\, L_T^{\frac{8}{3}} L_x^2 \| \| f_1(\varepsilon) \,;\, L_{T,x}^{\infty} \| \\
&\leq C(L) T^{\frac{1}{4}} \|\varepsilon^{\pm}\|_S \|\theta\|_H.
\end{aligned}
$$

By (2.1.5) and (2.2.8) we have

$$
\begin{aligned}
I_{2,2} &\leq C \| u_t \theta f_1(\varepsilon) \,;\, L_T^r L_x^{q'} \| \\
&\leq C \| u_t ; L_T^p L_x^q \| \| \theta \,;\, L_T^{\frac{pr}{p-r}} L_x^{\frac{q}{q-2}} \| \| f_1(\varepsilon) \,;\, L_{T,x}^{\infty} \| \\
&\leq C(L) T^{\frac{1}{4}} \|\varepsilon^{\pm}\|_S \|\theta\|_H.
\end{aligned}
$$

Then, combining these estimates, we obtain the following estimate:

$$
\|\Lambda_H \theta\|_H \leq C \|\theta_0 ; L_x^1\| + CT^{\kappa} (\|\theta\|_H \|\varepsilon^{\pm}\|_S^2 + \|\theta\|_H \|\varepsilon^{\pm}\|_S), \tag{2.2.12}
$$

where $\kappa := \min\left\{\frac{1}{p'} - \frac{1}{r}, \frac{1}{4}\right\}$. Consequently, from (2.2.11) and (2.2.12), we arrive at

$$\begin{aligned}\|\Lambda(\varepsilon^+, \varepsilon^-, \theta)\|_{X_T} \le C(\|\,\varepsilon^{\pm}(0)\,;\,H^1\| + \|\,\theta_0\,;\,L^1\|)\\ + CT^\kappa h(\|(\varepsilon^+, \varepsilon^-, \theta)\|_{X_T})\|(\varepsilon^+, \varepsilon^-, \theta)\|_{X_T},\end{aligned} \tag{2.2.13}$$

where $h(r) := 1 + r + r^2$.

Here, it follows from the mean value theorem and (2.2.7) that

$$\|f_i^{(j)}(\varepsilon) - f_i^{(j)}(\tilde{\varepsilon}); L^p\| \le \left\| |\varepsilon - \tilde{\varepsilon}| \int_0^1 f_i^{(j+1)}(s\varepsilon + (1-s)\tilde{\varepsilon})ds\,;\,L^p \right\| \le C\|\varepsilon - \tilde{\varepsilon}\,;\,L^p\|$$

for $i = 1, 2$ and $j = 0, 1$. By using this, we obtain

$$\begin{aligned}&\|\Lambda(\varepsilon^+,\varepsilon^-, \theta) - \Lambda(\tilde{\varepsilon}^+, \tilde{\varepsilon}^-, \tilde{\theta})\|_{X_T}\\ &\quad\le CT^\kappa \Big(h(\|(\varepsilon^+, \varepsilon^-, \theta)\|_{X_T}) + h(\|(\tilde{\varepsilon}^+, \tilde{\varepsilon}^-, \tilde{\theta})\|_{X_T})\Big)\\ &\qquad\times \|(\varepsilon^+, \varepsilon^-, \theta) - (\tilde{\varepsilon}^+, \tilde{\varepsilon}^-, \tilde{\theta})\|_{X_T}.\end{aligned} \tag{2.2.14}$$

Hence, it is sufficient to choose $L = 2C(\|u_0\|_{H^2}, \|u_1\|_{L^2}, \|\theta\|_{L^1})$ and T such that

$$C(L)T^\kappa\big(h(\|(\varepsilon^+, \varepsilon^-, \theta)\|_{X_T}) + h(\|(\tilde{\varepsilon}^+, \tilde{\varepsilon}^-, \tilde{\theta})\|_{X_T})\big) \le \frac{1}{2} \tag{2.2.15}$$

to obtain from (2.2.13) that Λ maps X_T^L into itself. The inequality (2.2.14) implies that under the same restrictions (2.2.15) on L and T, the mapping Λ is a contraction on X_T^L. The contraction mapping principle shows the existence of a unique solution in the ball $\|(\varepsilon^+, \varepsilon^-, \theta)\|_{X_T} \le L$. To prove the uniqueness in the whole of the space, it is enough to take T sufficiently small. Then the solution $(\varepsilon^+, \varepsilon^-, \theta) \in H^1 \times H^1 \times L^1$ with $\widehat{\varepsilon}^{\pm}(0) = 0$ is obtained and this also means the existence of $(\varepsilon, \theta) \in H^1 \times L^1$ with $\widehat{\varepsilon}(0) = 0$ because $\varepsilon = \varepsilon^+ + \varepsilon^-$.

Finally, we verify that the unique existence of $\varepsilon \in H^1$ leads to that of $u \in H^2$. We can expand ε into the trigonometric series:

$$\varepsilon(x) = \sum_{k \ne 0} \widehat{\varepsilon}(k)e^{2\pi ikx}.$$

Then if $\widehat{u}(0)$ is obtained, u can be written as

$$u = \sum_{k \ne 0} \frac{\widehat{\varepsilon}(k)}{2\pi ik}e^{2\pi ikx} + \widehat{u}(0).$$

Obviously the first term of the right hand side converges. The remaining problem is how $\hat{u}(0)$ should be determined. Since $\widehat{u}_{tt}(0) = 0$ by (2.2.1), we have

$$\widehat{u}(0) = t\widehat{u}_1(0) + \widehat{u}_0(0). \tag{2.2.16}$$

It is also necessary to show $u \in H^2$. It follows from the Poincaré inequality that

$$\left\| u - \widehat{u}(0)\,;\, L_x^2 \right\| \le \|\, \varepsilon\,;\, L_x^2\|.$$

Then it follows from (2.2.13) that $\|\, u\,;\, L_x^2\| \le C$ $(0 \le t \le T)$. This implies $u \in H^2$. We have completed the proof of the local existence for the problem (2.0.5)–(2.0.6) with $\Omega = \mathbb{T}$.

2.3 Initial Boundary Value Problem

In this section, we prove Theorem 2.2 for the initial boundary value problem (2.0.1)–(2.0.4). We first define the operator A such that

$$A = \partial_x^2$$

and

$$D(A) = \{f \in H^2 \,|\, f_x(0) = f_x(1) = 0\}.$$

In this section, we denote $\widehat{f}(k)$ by the coefficient of the Fourier cosine expansion of f, i.e.

$$\widehat{f} = 2\int_0^1 f(x)\cos 2\pi kx dx.$$

As in Section 2.2, we restate the equation (2.0.1). Differentiating both sides of (2.0.1) and putting $\varepsilon := u_x$, the equation can be written as follows:

$$\varepsilon_{tt} + \varepsilon_{xxxx} = (F - \widehat{F}(0))_{xx}. \tag{2.3.1}$$

Notice that we can expand

$$\varepsilon = \sum_{k\ge 1} \widehat{\varepsilon}(k)\cos 2\pi kx.$$

For any f such that $\widehat{f}(0) = 0$, we define the inverse map of A by

$$A^{-1}\varepsilon = -\sum_{k\ge 1} \frac{\cos 2\pi kx}{(2\pi k)^2}\widehat{\varepsilon}(k).$$

By the boundary condition (2.0.4), we have

$$F_x\big|_{x=\{0,1\}} = (f_1'(\varepsilon)\varepsilon_x\theta + f_1(\varepsilon)\theta_x + f_2'(\varepsilon)\varepsilon_x)\big|_{x=\{0,1\}} = 0, \tag{2.3.2}$$
$$u_{tt}\big|_{x=\{0,1\}} = 0.$$

Then the smooth solution of the equation (2.0.1) satisfies

$$u_{xxxx}(t,0) = u_{xxxx}(t,1) = 0.$$

Therefore the boundary condition of (2.3.1) is made into

$$\begin{aligned} \varepsilon_x(t,0) &= \varepsilon_x(t,1) = 0, \\ \varepsilon_{xxx}(t,0) &= \varepsilon_{xxx}(t,1) = 0. \end{aligned} \tag{2.3.3}$$

Next, put $\varepsilon^{\pm} := \varepsilon \pm iA^{-1}\varepsilon_t$. Observing (2.3.2) and (2.3.3),

$$\begin{aligned} \partial_t \varepsilon^{\pm} &= \varepsilon_t \pm iA^{-1}\varepsilon_{tt} \\ &= \varepsilon_t \pm iA^{-1}\left\{-A^2\varepsilon + A(F - \widehat{F}(0))\right\} \\ &= \varepsilon_t \mp iA\varepsilon \pm i(F - \widehat{F}(0)) \\ &= \mp iA(\varepsilon \pm iA^{-1}\varepsilon_t) \pm i(F - \widehat{F}(0)) \\ &= \mp iA\varepsilon^{\pm} \pm i(F - \widehat{F}(0)). \end{aligned}$$

Then, (2.3.1) and (2.3.3) are rewritten as the following form:

$$\begin{cases} \partial_t \varepsilon^{\pm} = \mp i\partial_x^2 \varepsilon^{\pm} \pm i(F - \widehat{F}(0)), \\ \varepsilon_x^{\pm}(t,0) = \varepsilon_x^{\pm}(t,1) = 0. \end{cases}$$

We can prove similar results to Propositions 2.1.1, 2.1.2 and 2.1.3 under the Neumann boundary condition.

Proposition 2.3.1. *Let $p_i \in [4,\infty]$ and $q_i \in [2,4]$ satisfy $\frac{1}{p_i} = \frac{1}{2} - \frac{1}{q_i}$ $(i = 1,2)$. Then, we have*

$$\| e^{\pm itA}u_0\,;\, L_T^{p_1}L_x^{q_1}(0,1)\| \le C\| u_0\,;\, L_x^2(0,1)\| \tag{2.3.4}$$

and

$$\left\| \int_0^t e^{\pm i(t-s)A}f(s)ds\,;\, L_T^{p_1}L_x^{q_1}(0,1)\right\| \le C\| f\,;\, L_T^{p_2'}L_x^{q_2'}(0,1)\|. \tag{2.3.5}$$

Proof. Note that $\varepsilon := e^{\pm itA}\varepsilon_0(x)$ is the solution of the initial boundary value problem:

$$\begin{cases} \varepsilon_t = \pm i\partial_x^2\varepsilon, \\ \varepsilon_x(t,0) = \varepsilon_x(t,1) = 0, \\ \varepsilon(0,x) = \varepsilon_0(x). \end{cases}$$

Let $\tilde{\varepsilon}$ be an extension of ε as an even function on $\mathbb{R}$, i.e.,

$$\tilde{\varepsilon}(2m \pm x) = \varepsilon(x),$$

where $m \in \mathbb{Z}$ and $x \in [0, 1]$. Then we can prove the estimate on a torus of period 2, that is, $(\mathbb{R}/2\mathbb{Z})$. If we restrict the estimate to $[0, 1]$, the desired estimate (2.3.4) is obtained. Similarly, we can obtain (2.3.5). $\square$

By the same idea, we can show the following two propositions.

Proposition 2.3.2. *For any $p, q \in (1, \infty)$, we have*

$$\left\| \partial_x^2 \int_0^t e^{(t-s)A} f(s) ds \,;\, L_T^p L_x^q(0,1) \right\| \leq C \| f \,;\, L_T^p L_x^q(0,1) \|. \tag{2.3.6}$$

Proposition 2.3.3. *If $1 \leq q \leq p \leq \infty$ and $t > 0$, then we have*

$$\| e^{tA} g \,;\, L_x^p \| \leq C \left(1 + t^{-\frac{1}{2}(\frac{1}{q} - \frac{1}{p})} \right) \| g \,;\, L_x^q \| \tag{2.3.7}$$

and

$$\| \partial_x e^{tA} g \,;\, L_x^p \| \leq C t^{-\frac{1}{2}} \left(1 + t^{-\frac{1}{2}(\frac{1}{q} - \frac{1}{p})} \right) \| g \,;\, L_x^q \|. \tag{2.3.8}$$

As in Section 2.2, we can prove the local existence of the solution $(\varepsilon^+, \varepsilon^-, \theta)$ (namely, (ε, θ)). Thus it remains to verify the existence of $u \in H^2$. Obviously u can be determined uniquely because of the boundary condition (2.0.4), and the Poincaré inequality implies

$$\| u \,;\, L^2 \| \leq C \| \varepsilon \,;\, L^2 \|.$$

This assures that $u \in H^2$, which completes the proof.

2.4 Local Existence on $\mathbb{R}$

In this section, we prove the local existence and uniqueness for the problem (2.0.5)–(2.0.7) in the case of $\Omega = \mathbb{R}$ (Theorem 2.3). We give a slightly different proof from the one of Theorem 2.2.

To shorten notation, we write $F := (f_1(u_x)\theta + f_2(u_x))$. Putting

$$u^{\pm} := u \pm i(1 - \partial_x^2)^{-1} u_t, \tag{2.4.1}$$

we restate the equation (2.0.1) as follows:

$$\begin{aligned}
\partial_t u^{\pm} &= u_t \pm i(1 - \partial_x^2)^{-1} u_{tt} \\
&= u_t \pm i(1 - \partial_x^2)^{-1} \left\{ -u_{xxxx} + F_x \right\} \\
&= u_t \pm i(1 - \partial_x^2)^{-1} \left\{ -(1 - \partial_x^2)^2 u + F_x - 2\partial_x^2 u + u \right\} \\
&= u_t \mp i(1 - \partial_x^2) u \pm i(1 - \partial_x^2)^{-1} \left\{ F_x - 2\partial_x^2 u + u \right\} \\
&= \mp i(1 - \partial_x^2) u^{\pm} \pm i(1 - \partial_x^2)^{-1} \left\{ F_x - 2\partial_x^2 u + u \right\}.
\end{aligned}$$

Then the equation (2.0.1) is reduced to the following Schrödinger type equations

$$\partial_t u^{\pm} = \pm i\partial_x^2 u^{\pm} \pm \widetilde{F},$$

where we set

$$\widetilde{F} = i(1-\partial_x^2)^{-1}\left(F_x - 2\partial_x^2 u + u\right) - iu^{\pm}$$

Notice that since

$$u_t = \frac{1-\partial_x^2}{2i}(u^+ - u^-), \tag{2.4.2}$$

the transformation (2.4.1) is useful for the estimate of u_t.

We set

$$\|u^{\pm}\|_{\widetilde{S}} := \|\, u^{\pm}\,;\, L_T^\infty H_x^2\| + \|\,\partial_x^2 u^{\pm}\,;\, L_T^p L_x^q\|,$$
$$\|\theta\|_H := \|\,\theta\,;\, L_T^\infty L_x^1\| + \|\,\partial_x\theta\,;\, L_T^r L_x^{q'}\|.$$

We show the time local existence and uniqueness of solution (u^+, u^-, θ) in the space $H^2 \times H^2 \times L^1$. For $L > 0$ to be determined later, we define the space

$$\widetilde{X}_T^L = \{(u^+, u^-, \theta)| \ ; \|(u^+, u^-, \theta)\|_{\widetilde{X}_T} := \|u^+\|_{\widetilde{S}} + \|u^-\|_{\widetilde{S}} + \|\theta\|_H \le L\},$$

The operator $\Lambda : (u^+, u^-, \theta) \mapsto (\widetilde{\Lambda}_+ u^+, \widetilde{\Lambda}_- u^-, \Lambda_H \theta)$ is defined by

$$\widetilde{\Lambda}_{\pm} u^{\pm} = e^{\pm it\partial_x^2} u^{\pm}(0) \pm i\int_0^t e^{\pm i(t-s)\partial_x^2}\widetilde{F}(s)ds, \tag{2.4.3}$$

$$\Lambda_H\theta = e^{t\partial_x^2}\theta_0 + \int_0^t e^{(t-s)\partial_x^2}(f_1(u_x)\theta u_{tx})(s)ds. \tag{2.4.4}$$

We shall prove that for an appropriate choice of L and T, the operator Λ is a contraction of $\widetilde{X}_T^L$ into itself. Without loss of generality we may assume $T < 1$.

For the linear part, it follows from (2.1.1) that

$$\begin{aligned}\|e^{\pm it\partial_x^2}u^{\pm}(0)\|_{\widetilde{S}} &\le \|\, e^{\pm it\partial_x^2}u^{\pm}(0)\,;\, L_T^\infty H_x^2\| + \|\,\partial_x^2 e^{\pm it\partial_x^2}u^{\pm}(0)\,;\, L_T^p L_x^q\| \\ &\le C\|\, u^{\pm}(0); H^2\|.\end{aligned}$$

By (2.0.13) we have

$$\begin{aligned}F_{q,r}\left(\frac{rp}{r-p}, \frac{q}{q-2}\right) &= \left(1-\frac{1}{q}\right)\frac{1}{r} - \frac{1}{p}\left(1+\frac{1}{q}\right) \\ &> \frac{1}{2}\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right) - \frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)\left(1+\frac{1}{q}\right) \\ &= \frac{1}{4}\left(1+\frac{1}{q}\right) \\ &> 0.\end{aligned} \tag{2.4.5}$$

Then it follows from (2.1.16) that

$$\begin{aligned}\|\,(f_1(u_x)\theta)_x\,;\,L_T^{\nu}L_x^{q'}\| &\le CT^{\frac{1}{\nu}-\frac{1}{r}}\Big(\|\,\theta_x\,;\,L_T^{r}L_x^{q'}\|\|\,f_1(u_x)\,;\,L_{T,x}^{\infty}\| \\ &\qquad + \|\,\theta\,;\,L_T^{\frac{rp}{p-r}}L_x^{\frac{q}{q-2}}\|\|\,u_{xx}\,;\,L_T^{p}L_x^{q}\|\|\,f_1'(u_x)\,;\,L_{T,x}^{\infty}\|\Big) \\ &\le CT^{\frac{1}{\nu}-\frac{1}{r}}\|\theta\|_H\|u\|_{\widetilde{S}},\end{aligned} \tag{2.4.6}$$

for $\nu \in [p', r]$. We remark that $\partial_x^2(1-\partial_x^2)^{-1}$ and $(1-\partial_x^2)^{-1}$ are L^q-bounded operators for any $q \in (1,\infty)$ because ∂_x^2 admits a bounded $\mathcal{H}^\infty$-calculus (see [18]). By (2.1.2), (2.4.6) and the Sobolev inequality, we have

$$\begin{aligned}\left\|\int_0^t e^{\pm i(t-s)\partial_x^2}\widetilde{F}(s)ds\right\|_{\widetilde{S}} &\le C\Big(\|\,\partial_x^2(1-\partial_x^2)^{-1}(f_1(u_x)\theta)_x\,;\,L_T^{p'}L_x^{q'}\| \\ &\qquad + \|\,\partial_x^2(1-\partial_x^2)^{-1}((f_2(u_x))_x - 2u_{xx} + u + \partial_x^2u^{\pm})\,;\,L_T^1L_x^2\|\Big) \\ &\le C\big(\|\,(f_1(u_x)\theta)_x\,;\,L_T^{p'}L_x^{q'}\| + T\|\,u^{\pm}\,;\,L_T^{\infty}H_x^2\|\big) \\ &\le C\big(T^{\frac{1}{p'}-\frac{1}{r}}\|u^{\pm}\|_{\widetilde{S}}\|\theta\|_H + T\|u^{\pm}\|_{\widetilde{S}}\big).\end{aligned}$$

Therefore, we obtain the following estimate,

$$\|\,\widetilde{\Lambda}_{\pm}u^{\pm}\|_{\widetilde{S}} \le C\|\,u^{\pm}(0)\,;\,H^1\| + CT^{\kappa}(\|u^{\pm}\|_{\widetilde{S}}\|\theta\|_H + \|u^{\pm}\|_{\widetilde{S}}), \tag{2.4.7}$$

where $\kappa := \min\left\{\frac{1}{p'}-\frac{1}{r}, 1\right\}$.

Note that (2.4.5) and

$$F_{q,r}\left(\frac{8}{3}, 2\right) = \left(1+\frac{1}{q}\right)\frac{3}{8} - \frac{1}{2r} > \frac{1}{8q} > 0.$$

Then the estimate for the heat equation (2.4.4) follows from the same calculation as that in the proof of Section 2.2. Therefore, we have

$$\|\Lambda_H\theta\|_H \le C\|\,\theta_0\,;\,L_x^1\| + CT^{\kappa}\|\theta\|_H\|u^{\pm}\|_{\widetilde{S}}^2. \tag{2.4.8}$$

Consequently, from (2.4.7) and (2.4.8), we arrive at

$$\begin{aligned}\|\Lambda(u^+,u^-,\theta)\|_{\widetilde{X}_T} &\le C(\|\,u^{\pm}(0)\,;\,H^2\| + \|\,\theta_0\,;\,L^1\|) \\ &\qquad + CT^{\kappa}h(\|(u^+,u^-,\theta)\|_{\widetilde{X}_T})\|(u^+,u^-,\theta)\|_{\widetilde{X}_T},\end{aligned} \tag{2.4.9}$$

where $h(r) = 1 + r + r^2$. Similarly we have

$$\begin{aligned}&\|\Lambda(u^+,u^-,\theta) - \Lambda(\tilde{u}^+,\tilde{u}^-,\tilde{\theta})\|_{\widetilde{X}_T} \\ &\quad \le CT^{\kappa}\Big[h(\|(u^+,u^-,\theta)\|_{\widetilde{X}_T}) + h(\|(\tilde{u}^+,\tilde{u}^-,\tilde{\theta})\|_{\widetilde{X}_T})\Big] \\ &\qquad \times \|(u^+,u^-,\theta) - (\tilde{u}^+,\tilde{u}^-,\tilde{\theta})\|_{\widetilde{X}_T}.\end{aligned} \tag{2.4.10}$$

Hence it is sufficient to take $L = 2C(\| u^{\pm}(0)\,;\, H^2\| + \| \theta_0\,;\, L^1\|)$ and T such that $CT^{\kappa}h(L) \leq \frac{1}{2}$ to obtain from (2.4.9) that Λ maps $\widetilde{X}_T^L$ into itself. The inequality (2.4.10) implies that under the same restrictions on L and T, the mapping Λ is a contraction on $\widetilde{X}_T^L$. The contraction mapping principle shows the existence of a unique solution within the ball $\|(u^+, u^-, \theta)\|_{\widetilde{X}} \leq L$. To prove the uniqueness in the whole of the space, it is enough to take T sufficiently small. Then the solution $(u^+, u^-, \theta) \in H^2 \times H^2 \times L^1$ is obtained, and hence this also means the existence and uniqueness of $(u, \theta) \in H^2 \times L^1$ because $u = u^+ + u^-$. We have completed the proof.

2.5 Global Existence

In this section, we prove Theorem 2.4. We first consider the case of periodic boundary conditions. For the smooth solution (u, θ), multiplying (2.0.5) by u_t, we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\| u_t(t)\,;\, L_x^2\|^2 + \| u_{xx}(t)\,;\, L_x^2\|^2) &= \int_{\mathbb{T}} [f_1(u_x)\theta + f_2(u_x)]_x u_t dx \\
&= -\int_{\mathbb{T}} [f_1(u_x)\theta + f_2(u_x)] u_{tx} dx \\
&= -\int_{\mathbb{T}} f_1(u_x)\theta u_{tx} dx - \frac{d}{dt}\int_{\mathbb{T}} F_2(u_x) dx.
\end{aligned}
\tag{2.5.1}
$$

Integrating (2.0.6) over $\mathbb{T}$, we have

$$
\frac{d}{dt}\int_{\mathbb{T}} \theta(t) dx = \int_{\mathbb{T}} f_1(u_x)\theta u_{tx} dx. \tag{2.5.2}
$$

Adding (2.5.1) to (2.5.2) yields

$$
\frac{d}{dt}\left(\frac{1}{2}\| u_t(t)\,;\, L_x^2\|^2 + \frac{1}{2}\| u_{xx}(t)\,;\, L_x^2\|^2 + \int_{\mathbb{T}} \theta(t) dx + \int_{\mathbb{T}} F_2(u_x)(t) dx\right) = 0. \tag{2.5.3}
$$

Integrating (2.5.3) with respect to the time variable, we have

$$
\begin{aligned}
\frac{1}{2}\| u_t(t)\,;\, L_x^2\|^2 + \frac{1}{2}\|u_{xx}(t)\,;\, L_x^2\|^2 + \int_{\mathbb{T}} \theta(t) dx \leq \frac{1}{2}\|u_1\,;\, L_x^2\|^2 + \frac{1}{2}\|u_0\,;\, H_x^2\|^2 + \int_{\mathbb{T}} \theta_0 dx \\
+ \int_{\mathbb{T}} F_2(\partial_x u_0) dx + M.
\end{aligned}
$$

Next, in the case of $\Omega = \mathbb{R}$, the same calculation as above yields

$$
\begin{aligned}
\frac{d}{dt}\Bigg(\frac{1}{2}\| u_t(t)\,;\, L_x^2\|^2 + \frac{1}{2}\| u_{xx}(t)\,;\, L_x^2\|^2 + \int_{\mathbb{R}} \theta(t) dx \\
+ \int_{\mathbb{R}} (\frac{1}{6}u_x^6 - \frac{1}{4}u_x^4 - \frac{1}{2}u_x^2)(t) dx\Bigg) = 0.
\end{aligned}
\tag{2.5.4}
$$

Here we note that

$$\begin{aligned}\frac{1}{2}\| u_x\,;\, L^2_x\|^2 &\le C\| u\,;\, L^2_x\|\| u_{xx}\,;\, L^2_x\| \\ &\le \frac{1}{8}\| u_{xx}\,;\, L^2_x\|^2 + C\| u\,;\, L^2_x\|^2,\end{aligned}$$

$$\begin{aligned}\frac{1}{4}\| u_x\,;\, L^4_x\|^4 &\le \frac{1}{6}\| u_x\,;\, L^6_x\|^6 + C\| u_x\,;\, L^2_x\|^2 \\ &\le \frac{1}{6}\| u_x\,;\, L^6_x\|^6 + \frac{1}{8}\| u_{xx}\,;\, L^2_x\|^2 + C\| u\,;\, L^2_x\|^2\end{aligned}$$

and

$$\begin{aligned}\| u\,;\, L^2_x\|^2 &= \left\| u_0 + \int_0^t u_s(s)ds\,;\, L^2_x\right\|^2 \\ &\le C\|u_0\,;\, L^2_x\|^2 + C\int_0^t \| u_s(s)\,;\, L^2_x\|^2 ds.\end{aligned}$$

Therefore, integrating (2.5.4) over $[0,t]$, we have

$$\begin{aligned}&\frac{1}{2}\| u_t(t)\,;\, L^2_x\|^2 + \frac{1}{4}\| u_{xx}(t)\,;\, L^2_x\|^2 + \int_{\mathbb{R}} \theta(t)dx \\ &\qquad \le \frac{1}{2}\| u_1\,;\, L^2_x\|^2 + C\| u_0\,;\, H^2\|^2 + \int_{\mathbb{R}} \theta_0 dx + C\int_0^t \| u_s(s)\,;\, L^2_x\|^2 ds.\end{aligned}$$

Hence, if $\theta \ge 0$ holds, then by the Gronwall inequality we obtain

$$\frac{1}{2}\| u_t(t)\,;\, L^2_x\|^2 + \frac{1}{4}\| u_{xx}(t)\,;\, L^2_x\|^2 + \| \theta(t)\,;\, L^1_x\| \le C(T, \|(u_0,u_1,\theta_0)\|_E).$$

These formal calculations can be justified by the following Lemma 2.5.1, which is concerned with the regularized approximation of weak solution. From now on, we only prove the problem of periodic boundary conditions (obviously the proof of other cases follows from a modification similar to this case). We denote $H^{s+2}\times H^s \times W^s_1$ by $\partial^s_x E$.

Lemma 2.5.1. *Let p, $q \in [2,4]$ and r be fixed satisfying* (2.0.11)*. Assume that $f_1, f_2 \in \mathrm{C}^{s+2}$ where $s \ge 1$ is any integer. Then for any $(u_0,u_1,\theta_0) \in \partial^s_x E$, there exists $T = T(\|(u_0,u_1,\theta_0)\|_E) > 0$ such that the problem* (2.0.4)–(2.0.7) *has a unique solution (u,θ) on the time interval $[0,T]$, satisfying*

$$\begin{aligned}&u \in \mathrm{C}_T H^{s+2}(\mathbb{T}) \cap L^p_T W^{s+2}_q(\mathbb{T}), \\ &\partial_t u \in L^\infty_T H^s(\mathbb{T}) \cap\ L^p_T W^s_q(\mathbb{T}), \\ &\theta \in \mathrm{C}_T W^s_1(\mathbb{T}), \quad \partial^{s+1}_x \theta \in L^r_T L^{q'}(\mathbb{T}).\end{aligned} \tag{2.5.5}$$

Moreover, for any T' *with* $0 < T' < T$*, there exists* $\eta = \eta(\|(u_0, u_1, \theta_0)\|_E, \|f_1\|_{C^{s+2}}, \|f_2\|_{C^{s+2}}) > 0$ *such that the map* $\widetilde{u}_0 \mapsto \widetilde{u}(t)$ *from*

$$\big\{(\widetilde{u}_0, \widetilde{u}_1, \widetilde{\theta}_0, \widetilde{f}_1, \widetilde{f}_2) \in \partial_x^s E \times (C^{s+2}_{loc}(\mathbb{R}))^2 \;\big|\; \|(u_0, u_1, \theta_0, f_1, f_2) - (\widetilde{u}_0, \widetilde{u}_1, \widetilde{\theta}_0, \widetilde{f}_1, \widetilde{f}_2)\|_{\partial_x^s E \times (C^{s+2}_{loc}(\mathbb{R}))^2} < \eta\big\}$$

into the class defined by (2.5.5) *with* T' *instead of* T *is Lipschitz, where* T *is the existence time of solution given above for* $(u_0, u_1, \theta_0, f_1, f_2)$ *and* $\widetilde{u}$ *is a solution of* (2.0.5)–(2.0.7) *for* $(\widetilde{u}_0, \widetilde{u}_1, \widetilde{\theta}_0, \widetilde{f}_1, \widetilde{f}_2)$.

Proof. As in the proof of Theorem 2.1, we prove the unique local existence of sufficiently smooth solution. Using Lemma 2.1.4, we can obtain the following estimates

$$\begin{aligned}\|\partial_x^s \Lambda(\varepsilon^+, \varepsilon^-, \theta)\|_{X_T} \le\ & C(\|u_0\|_{H^{s+2}}, \|u_1\|_{H^s}, \|\theta\|_{W^{s,1}}) \\ &+ CT^\kappa h(\|(\varepsilon^+, \varepsilon^-, \theta)\|_{X_T})\|\partial_x^s(\varepsilon^+, \varepsilon^-, \theta)\|_{X_T}\end{aligned}$$

and

$$\begin{aligned}\|\partial_x^s\{\Lambda(\varepsilon^+, \varepsilon^-, \theta) - \Lambda(\tilde{\varepsilon}^+, \tilde{\varepsilon}^-, \tilde{\theta})\}\|_{X_T} \le CT^\kappa &\left(h(\|(\varepsilon^+, \varepsilon^-, \theta)\|_{X_T}) + h(\|(\tilde{\varepsilon}^+, \tilde{\varepsilon}^-, \tilde{\theta})\|_{X_T})\right) \\ &\times \|\partial_x^s\{(\varepsilon^+, \varepsilon^-, \theta) - (\tilde{\varepsilon}^+, \tilde{\varepsilon}^-, \tilde{\theta})\}\|_{X_T}.\end{aligned}$$

Therefore taking the same local time T as in the proof of Theorem 2.1, we can show the local existence result. The continuous dependence of the solution upon the data in the $L^\infty(0, T; \partial_x^s E)$-norm follows from a similar argument. □

In order to regard the third term of the right hand side of (2.5.3) as L^1-norm of θ, we give a claim related to a sign property for the temperature θ.

Proposition 2.5.2 (Maximum Principle). *If* $\theta_0 \ge 0$ *then the solution* θ *of* (2.0.5)–(2.0.7) *satisfies* $\theta \ge 0$ *a.e. on* $[0, T] \times \mathbb{T}$.

Proof. The smooth solution satisfies the maximum principle (e.g. [2]). Therefore, approximating the energy class solution by smooth solutions with the relation:

$$(u_{0n}, u_{1n}, \theta_{0n}, f_{1n}, f_{2n}) \to (u_0, u_1, \theta_0, f_1, f_2) \text{ in } E \times (C^2_{loc}(\mathbb{R}))^2,$$

we obtain the desired result. We observe that the local existence time depends only on the energy norm of the data ($\|(u_0, u_1, \theta_0)\|_E$) by Lemma 2.5.1. This means that the local existence time T does not tend to 0 as $n \to 0$. □

Combining this proposition with the energy conservation law (2.5.3), we obtain

$$\|(u(t), u_t(t), \theta(t))\|_E \leq C(\|(u_0, u_1, \theta_0)\|_E) \qquad \text{for } 0 \leq t \leq T.$$

Then the solution obtained by Theorem 2.1 can be extended globally in time.

Remark. If we take $q = 2$ in Theorems 1.1 and 1.2, the Strichartz estimate is not necessarily needed for the proof. This is because we can take a number p greater than q in the maximal regularity. In other words, we can say that the smoothing effect of the heat equation is sufficient for the unique global existence theorem for the problems (1.1.13) and (1.2.7) with $q = 2$.

Chapter 3

Multi-Dimensional Case

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$. In this chapter, we study the following n-dimensional ($n = 2$ or 3) thermoelastic system with internal viscosity:

$$\mathbf{u}_{tt} + Q^2\mathbf{u} - \nu Q\mathbf{u}_t = \nabla \cdot [G(\theta)F_{1,\epsilon}(\epsilon) + F_{2,\epsilon}(\epsilon)], \tag{3.0.1}$$

$$[1 - \theta G''(\theta)F_1(\epsilon)]\theta_t - \Delta\theta = \theta G'(\theta)\partial_t F_1(\epsilon) + \nu(A\varepsilon_t) : \varepsilon_t \quad \text{in } \Omega_T = (0, T) \times \Omega, \tag{3.0.2}$$

$$\mathbf{u} = Q\mathbf{u} = \nabla\theta \cdot \mathbf{n} = 0 \quad \text{on } S_T = [0, T) \times \partial\Omega, \tag{3.0.3}$$

$$(\mathbf{u}(0, \cdot), \mathbf{u}_t(0, \cdot)) = (\mathbf{u}_0, \mathbf{u}_1), \quad \theta(0, \cdot) = \theta_0 \geq 0 \quad \text{in } \Omega, \tag{3.0.4}$$

where $\mathbf{n}$ is unit outward normal on $\partial\Omega$. The relation $Qu = \nabla \cdot \epsilon(\mathbf{u})A$ holds between the second order differential operator $Q = \mu\Delta\mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u})$ and the fourth order tensor $A = (A_{ijkl})$ such that $A_{ijkl} := \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. We assume that the Lamé constants λ and μ satisfy

$$\mu > 0 \qquad \text{and} \qquad n\lambda + 2\mu > 0. \tag{3.0.5}$$

We normalize the physical coefficients to unity except the viscosity coefficient $\nu > 0$. We let ν lie in order to emphasize that we can take ν sufficiently small. We restate the structure of nonlinearity: $\widetilde{F}(\theta, \epsilon) = G(\theta)F_1(\epsilon) + F_2(\epsilon)$ satisfies that

$(N1)$ $G \in \mathrm{C}^3(\mathbb{R}, \mathbb{R})$ is as follows:

$$G(\theta) = \begin{cases} C_1\theta & \text{if } \theta \in [0, \theta_1], \\ \varphi(\theta) & \text{if } \theta \in [\theta_1, \theta_2], \\ C_2\theta^r & \text{if } \theta \in [\theta_2, \infty), \end{cases}$$

where $\varphi \in \mathrm{C}^3(\mathbb{R}, \mathbb{R})$, $\varphi'' \leq 0$ and C_1, C_2 are positive constants for some fixed θ_1, θ_2 satisfying $0 < \theta_1 < \theta_2 < \infty$. We extend G defined on $\mathbb{R}$ as an odd function.

$(N2)$ $F_1 \in \mathrm{C}^3(\mathrm{Sym}(n,\mathbb{R}),\mathbb{R})$ satisfies that $F_1(\epsilon) \geq 0$, where $\mathrm{Sym}(n,\mathbb{R})$ denotes the set of all symmetric second order tensors in $\mathbb{R}^n$.

$(N3)$ $F_2 \in \mathrm{C}^3(\mathrm{Sym}(n,\mathbb{R}),\mathbb{R})$ satisfies that $F_2(\epsilon) \geq -C_3$, where C_3 is some real constant.

$(N4)$ $F_1(\epsilon)$ and $F_2(\epsilon)$ satisfy the following growth conditions:

$$\begin{aligned} &|F_{1,\epsilon}(\epsilon)| \leq C|\epsilon|^{K_1-1}, && |F_{2,\epsilon}(\epsilon)| \leq C|\epsilon|^{K_2-1}, \\ &|F_{1,\epsilon\epsilon}(\epsilon)| \leq C|\epsilon|^{K_1-2}, && |F_{2,\epsilon\epsilon}(\epsilon)| \leq C|\epsilon|^{K_2-2}, \\ &|F_{1,\epsilon\epsilon\epsilon}(\epsilon)| \leq C|\epsilon|^{K_1-3}, && |F_{2,\epsilon\epsilon\epsilon}(\epsilon)| \leq C|\epsilon|^{K_2-3} \end{aligned}$$

for large $|\epsilon|$.

We also restate the assumptions of nonlinearity:

$$0 \leq r < \frac{5}{6}, \qquad 0 \leq K_1,\ K_2 < 6 \quad \text{and} \quad 6r + K_1 < 6 \tag{3.0.6}$$

in three-dimensional case and

$$0 \leq r < 1 \quad \text{and} \quad 0 \leq K_1, K_2 < \infty \tag{3.0.7}$$

in two-dimensional case.

Theorem 3.1 (Existence for Three-Dimensional System). *Let $n = 3$, $\nu > 0$ and $5 < p \leq q < \infty$. Assume $(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \in U(p,q) := B^{4-2/p}_{p,p} \times B^{2-2/p}_{p,p} \times B^{2-2/q}_{q,q}$ and that* (3.0.6) *holds. Then for any $T > 0$ there exists at least one solution $(\mathbf{u}, \theta)$ to the three-dimensional system* (3.0.1)–(3.0.4) *satisfying*

$$(\mathbf{u}, \theta) \in V_T(p,q) := W^{4,2}_p(\Omega_T) \times W^{2,1}_q(\Omega_T).$$

Moreover, if we assume $\min_\Omega \theta_0 = \theta_* > 0$ then there exists a positive constant ω such that

$$\theta \geq \theta_* \exp(-\omega t) \qquad \text{in } \Omega_T.$$

Theorem 3.2 (Existence for Two-Dimensional System). *Let $n = 2$, $\nu > 0$ and $4 < p \leq q < \infty$, and suppose that* (3.0.7) *holds. Then for the two-dimensional system* (3.0.1)–(3.0.4) *the same conclusion as in Theorem 3.1 holds.*

We can obtain the following uniqueness result.

Theorem 3.3 (Uniqueness). *In addition to the assumptions of Theorems 3.1 and 3.2, suppose that $F(\epsilon,\theta) \in \mathrm{C}^4(\mathrm{Sym}(n,\mathbb{R}) \times \mathbb{R}^+, \mathbb{R})$. Then the solution $(\mathbf{u}, \theta) \in V_T(p,q)$ to* (3.0.1)–(3.0.4) *constructed above is unique.*

Noting the embedding $\mathrm{BUC}([0,T]; B_{p,p}^{2-\frac{2}{p}}) \hookrightarrow W_p^{2,1}(\Omega_T)$, we can immediately obtain Theorems 1.3 and 1.4 from Theorems 3.1, 3.2 and 3.3.

In Section 3.1, we introduce the several lemmas. In Section 3.2, after giving a brief review of the maximal regularity, we prove the maximal regularity estimate used in the proofs of this chapter. In Sections 3.3 and 3.4, we prove the existence theorem for the three-dimensional system (Theorem 3.1). In Section 3.5, we show the uniqueness result (Theorem 3.3). In Section 3.6, we state the proof of the existence theorem for the two-dimensional system (Theorem 3.2).

3.1 Preliminary Results

In this section, we present some auxiliary results which will be used in the subsequent sections. We recall the useful space-time embedding lemma.

Lemma 3.1.1 (Embedding [32, Lemma II.3.3]). *Let $f \in W_p^{2m,m}(\Omega_T)$. Then, for $m \in \mathbb{Z}^+$ and multi index α, it follows that*

$$\| D_t^r D_x^\alpha f \,;\, L^q(\Omega_T)\| \le C\delta^{m-\psi} \| f \,;\, W_p^{2m,m}(\Omega_T)\| + C\delta^{-\psi} \| f \,;\, L^p(\Omega_T)\|, \tag{3.1.1}$$

provided $q \ge p$ and $\psi := r + \frac{|\alpha|}{2} + \frac{n+2}{2}\left(\frac{1}{p} - \frac{1}{q}\right) \le m$. If $\varphi := r + \frac{|\alpha|}{2} + \frac{n+2}{2p} < m$, then

$$\| D_t^r D_x^\alpha f \,;\, L^\infty(\Omega_T)\| \le C\delta^{m-\varphi} \| f \,;\, W_p^{2m,m}(\Omega_T)\| + C\delta^{-\varphi} \| f \,;\, L^p(\Omega_T)\|, \tag{3.1.2}$$

moreover, $D_t^r D_x^\alpha f$ is Hölder continuous. Here, $\delta \in (0, \min(T, \zeta^2)]$ and ζ is the altitude of the cone in the statement of the cone condition satisfied by Ω.

The next lemma is the technical one which we use to assure the nonnegativity of energy.

Lemma 3.1.2. *Let φ be given in $(N1)$. Then the function $\varphi(s)$ satisfies*

$$\varphi(s) - s\varphi'(s) \ge 0 \tag{3.1.3}$$

for any $s \in [\theta_1, \theta_2]$.

Proof. Putting $f(s) = \varphi(s) - s\varphi'(s)$, we have $f'(s) = -s\varphi''(s) \ge 0$ and $f(\theta_1) = 0$. Then $f(s) = \varphi(s) - s\varphi'(s) \ge 0$ in $[\theta_1, \theta_2]$. □

Next, we introduce the Aubin compactness theorem.

Lemma 3.1.3 (Aubin Compactness Theorem). *Let X_0, X_1 and X be Banach spaces, X_0 and X_1 are reflexive, for which the following embeddings hold:*

$$X_0 \hookrightarrow X \hookrightarrow X_1,$$

where the first embedding is compact and the last embedding is continuous. Assuming p_0, $p_1 > 1$, define the space

$$Y = \{u \mid u \in L_I^{p_0} X_0,\ u_t \in L_I^{p_1} X_1\}$$

with an appropriate norm. Then the embedding $Y \hookrightarrow L_I^{p_0} X$ is compact.

To show Theorem 3.1 we apply the Leray-Schauder fixed point principle. We recall it here in one of its equivalent formulations for the reader's convenience .

Theorem 3.1.4 (Leray-Schauder Fixed Point Principle [15]). *Let X be a Banach space. Assume that $\Phi : [0,1] \times X \to X$ is a map with the following properties.*

$(L1)$ *For any fixed $\tau \in [0,1]$ the map $\Phi(\tau, \cdot) : X \to X$ is compact.*

$(L2)$ *For every bounded subset $\mathcal{B}$ of X, the family of maps $\Phi(\cdot, \xi) : [0,1] \to X$, $\xi \in \mathcal{B}$, is uniformly equicontinuous.*

$(L3)$ *$\Phi(0, \cdot)$ has precisely one fixed point in X.*

$(L4)$ *There is a bounded subset $\mathcal{B}$ of X such that any fixed point in X of $\Phi(\tau, \cdot)$ is contained in $\mathcal{B}$ for every $0 \le \tau \le 1$.*

Then $\Phi(1, \cdot)$ has at least one fixed point in X.

3.2 Maximal Regularity

Our purpose of this section is to prove the following lemma called the maximal regularity.

Lemma 3.2.1 (Maximal Regularity). *Let $p \in (1, \infty)$. Denote by $\mathbf{u}$ the solution of the linear problem*

$$\begin{cases} \mathbf{u}_{tt} + Q^2\mathbf{u} - \nu Q\mathbf{u}_t = \nabla \cdot f & \text{in } \Omega_T, \\ \mathbf{u} = Q\mathbf{u} = 0 & \text{on } S_T, \\ \mathbf{u}(0, \cdot) = \mathbf{u}_0, \quad \mathbf{u}_t(0, \cdot) = \mathbf{u}_1 & \text{in } \Omega. \end{cases} \tag{3.2.1}$$

(*i*) *The solution* $\mathbf{u}$ *to* (3.2.1) *satisfies the following estimate:*

$$\|\,\mathbf{u}\,;\ W_p^{4,2}(\Omega_T)\| \le C\big(\|\,\mathbf{u}_0\,;\,B_{p,p}^{4-\frac{2}{p}}\| + \|\,\mathbf{u}_1\,;\,B_{p,p}^{2-\frac{2}{p}}\| + \|\,\nabla\cdot f\,;\,L^p(\Omega_T)\|\big) \qquad (3.2.2)$$

for any $(\mathbf{u}_0,\mathbf{u}_1)\in B_{p,p}^{4-2/p}\times B_{p,p}^{2-2/p}$ *and* $\nabla\cdot f\in L^p(\Omega_T)$.

(*ii*) *The solution* $\mathbf{u}$ *to* (3.2.1) *satisfies the following estimate:*

$$\|\,\nabla\mathbf{u}\,;\,W_p^{2,1}(\Omega_T)\| \le C\big(\|\,\mathbf{u}_0\,;\,B_{p,p}^{3-\frac{2}{p}}\| + \|\,\mathbf{u}_1\,;\,B_{p,p}^{1-\frac{2}{p}}\| + \|\,f\,;\,L^p(\Omega_T)\|\big) \qquad (3.2.3)$$

for any $(\mathbf{u}_0,\mathbf{u}_1)\in B_{p,p}^{3-2/p}\times B_{p,p}^{1-2/p}$ *and* $f\in L^p(\Omega_T)$.

We give a brief review of the maximal regularity theory before we prove Lemma 3.2.1. Let X be a Banach space and A a closed linear unbounded operator in X with dense domain $D(A)$. Consider the abstract Cauchy problem

$$\begin{cases} u_t(t)+Au(t)=f(t), \qquad t>0,\\ u(0)=0, \end{cases} \qquad (3.2.4)$$

where $f:\mathbb{R}^+\to X$ is a given function. We say that this problem has the property of *maximal regularity* if for each $f\in L^p(\mathbb{R}^+;X)$ there exists a unique solution $u\in W_p^1(\mathbb{R}^+;X)\cap L^p(\mathbb{R}^+;X)$ satisfying (3.2.4) in the $L^p(\mathbb{R}^+;X)$-sense. The important estimate

$$\|\,u\,;\,W_p^1(\mathbb{R}^+;X)\| + \|\,Au\,;\,L^p(\mathbb{R}^+;X)\| \le C\|\,f\,;\,L^p(\mathbb{R}^+;X)\| \qquad (3.2.5)$$

follows from the property of maximal regularity and the closed graph theorem, where $C>0$ is independent of f. In this thesis we call this estimate *the maximal regularity* as well. The first abstract result on sufficient conditions for the maximal regularity was obtained by de Simon [16]. He shows that in the case of Hilbert spaces X if $-A$ is the generator of a bounded analytic C_0-semigroup in X with negative exponential type, then the problem (3.2.4) has the property of maximal regularity. In 1987, Dore and Venni [19] obtained that if $A\in\mathcal{BIP}(X)$ (bounded imaginary powers) with power angle $\varphi_A^{BIP}<\pi/2$ provided X in a Banach space of class $\mathcal{HT}$. We give the definitions of these concepts below. The class $\mathcal{HT}$ is known to coincide with the class of UMD Banach space and also with ζ-convex Banach space (see [7]). We note that $L^p(\Omega)$ belongs to $\mathcal{HT}$ for any $p\in(1,\infty)$. There is a more extended concept called a bounded $\mathcal{H}^\infty$-calculus. The class $\mathcal{H}^\infty(X)$ is of operators which admit a functional calculus for a large function class including bounded imaginary powers. We comment on the important result obtained in Weis [46], although we do not use these facts explicitly. He proved that, in the case of $X\in\mathcal{HT}$, the problem (3.2.4) has the property of maximal regularity if and only if $A\in\mathcal{RS}(X)$ with $\mathcal{R}$-angle $\varphi_A^R<\pi/2$. We give the definition of these concepts following the monograph [18].

Definition 3.2.2 ([18]). Let X be a complex Banach space and A a closed linear operator in X. We define the sector Σ_φ in the complex plane by $\Sigma_\varphi := \{z \in \mathbb{C} \mid |\arg z| < \varphi\}$.

(i) A Banach space X belongs to $\mathcal{HT}$ if the Hilbert transform is bounded on $L^p(\mathbb{R}; X)$ for some (and then all) $p \in (1, \infty)$.

(ii) A closed operator A is called *sectorial* if A has the dense domain and range, $(-\infty, 0) \subset \rho(A)$ and

$$\|t(t + A)^{-1}\| \leq M \qquad \text{for all } t > 0 \text{ and some } M < \infty.$$

The class of sectorial operators in X will be denoted by $\mathcal{S}(X)$.

(iii) A sectorial operator A is said to admit *bounded imaginary powers* if A^{is} is bounded on X for each $s \in \mathbb{R}$, and there is a constant $C > 0$ such that $\|A^{is}\| \leq C$ for each $|s| \leq 1$. The class of such operators is denoted by $\mathcal{BIP}(X)$. The *power angle* of A is defined by

$$\varphi_A^{BIP} = \overline{\lim}_{s\to\infty} \frac{1}{|s|} \log \|A^{is}\|.$$

(iv) A sectorial operator A is said to admit a *bounded $\mathcal{H}^\infty(X)$-calculus* if there are $\varphi > \varphi_A$ and a constant $K_\varphi < \infty$ such that

$$|f(A)| \leq K_\varphi |f|_\infty^\varphi \qquad \text{for all} \quad f \in \mathcal{H}_0(\Sigma_\varphi), \tag{3.2.6}$$

where $|f|_\infty^\varphi := \sup\{|f(\lambda)| \mid |\arg \lambda| < \varphi\}$, and

$$\mathcal{H}_0(\Sigma_\varphi) := \cup_{\alpha,\beta<0}\{f \in \mathcal{H}(\Sigma_\varphi) \mid \sup_{|\lambda|<1} |\lambda^\alpha f| + \sup_{|\lambda|\geq 1} |\lambda^{-\beta} f| < \infty\},$$
$$\mathcal{H}(\Sigma_\varphi) := \{f \mid \Sigma_\varphi \to \mathbb{C}, \text{ holomorphic}\}.$$

The class of such sectorial operators A will be denoted by $\mathcal{H}^\infty(X)$. The *$\mathcal{H}^\infty$-angle* of A is defined by $\varphi_A^\infty = \inf\{\varphi > \varphi_A \mid \text{ (3.2.6) is valid }\}$.

(v) A family of operators $\mathcal{T} \subset B(X)$ is called $\mathcal{R}$-bounded if there are a constant $C > 0$ and $p \in [1, \infty)$ such that for each $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ and for all independent, symmetric, $\{1, 1\}$-valued random variables r_j on a probability space $(\Omega, \mathcal{M}, \mu)$ the inequality

$$|\sum_{j=1}^N r_j T_j x_j|_{L^p(\Omega;X)} \leq C|\sum_{j=1}^N r_j x_j|_{L^p(\Omega;X)}$$

is valid. The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$ (we denote it by $\mathcal{R}(\mathcal{T})$).

(vi) A sectorial operator A is called $\mathcal{R}$-*sectorial* if $\mathcal{R}_A(0) < \infty$, where

$$\mathcal{R}_A(\psi) := \mathcal{R}(\{\lambda(\lambda + A)^{-1};\ |\arg \lambda| \leq \psi\}).$$

The class of such operators is denoted by $\mathcal{RS}(X)$. The $\mathcal{R}$-*angle* φ_A^R of A is defined by $\varphi_A^R := \inf\{\varphi \in (0, \pi);\ \mathcal{R}_A(\pi - \varphi) < \infty\}$.

We have the inclusions $\mathcal{H}^\infty(X) \subset \mathcal{BIP}(X) \subset \mathcal{RS}(X)$ and the inequalities $\varphi_A^\infty \geq \varphi_A^{BIP} \geq \varphi_A^R$. Hence, if we obtain $A \in \mathcal{H}^\infty(X)$ with $\varphi_A^\infty < \pi/2$, then the maximal regularity (3.2.5) holds. For more detail of these facts, we refer to the monograph [18].

We turn back to the argument for Lemma 3.2.1. We define the linear operator $\mathcal{Q}$ in $L^p(\Omega)$ for $p \in (1, \infty)$ by

$$\begin{cases} D(\mathcal{Q}) = \{\mathbf{u} \in W_p^2(\Omega) \mid \mathbf{u} = 0 \text{ on } \partial\Omega\}, \\ \mathcal{Q}\mathbf{u} = Q\mathbf{u}, \end{cases} \tag{3.2.7}$$

where we denote the domain of an operator A by $D(A)$.

Proof of Lemma 3.2.1 (*i*). For $\alpha := \frac{\nu}{2} + i\sqrt{1 - \frac{\nu^2}{4}}$, we write $\mathbf{w} := \mathbf{u}_t - \alpha Q\mathbf{u}$. Then the equation (3.2.1) can be decomposed as follows:

$$\begin{cases} \mathbf{u}_t - \bar{\alpha}\mathcal{Q}\mathbf{u} = \mathbf{w} & \text{in } \Omega_T, \\ \mathbf{u}(0, x) = \mathbf{u}_0(x) & \text{in } \Omega \end{cases} \tag{3.2.8}$$

and

$$\begin{cases} \mathbf{w}_t - \alpha\mathcal{Q}\mathbf{w} = \nabla \cdot f & \text{in } \Omega_T, \\ \mathbf{w}(0, x) = \mathbf{u}_1(x) - \alpha Q\mathbf{u}_0(x) & \text{in } \Omega, \end{cases} \tag{3.2.9}$$

where $\mathbf{u}$ and $\mathbf{w}$ are extended as $\mathbb{C}^3$-valued functions. We claim the maximal regularity of both (3.2.8) and (3.2.9):

$$\|\, \mathbf{u}_t\,;\, L^p(\Omega_T)\| + \|\, Q\mathbf{u}\,;\, L^p(\Omega_T)\| \leq C\big(\|\, \mathbf{w}\,;\, L^p(\Omega_T)\| + \|\, \mathbf{u}_0\,;\, B_{p,p}^{2-\frac{2}{p}}\|\big) \tag{3.2.10}$$

and

$$\|\, \mathbf{w}_t\,;\, L^p(\Omega_T)\| + \|\, Q\mathbf{w}\,;\, L^p(\Omega_T)\| \leq C\big(\|\, \nabla \cdot f\,;\, L^p(\Omega_T)\| + \|\, \mathbf{u}_1 - Q\mathbf{u}_0\,;\, B_{p,p}^{2-\frac{2}{p}}\|\big). \tag{3.2.11}$$

We give the proof of the claim as Lemma 3.2.3. Combining these estimates and restricting to $\mathbb{R}^3$-valued functions we obtain the desired estimate. □

Hence, we only show the maximal regularity for (3.2.10) and (3.2.11). This is completely covered by Denk, Hieber and Prüss [18] and by Denk, Dore, Hieber, Prüss and Venni [17]. These papers say that the L^p-realization of the parameter elliptic operator admits $\mathcal{H}^\infty$-calculus and the elliptic angle $\varphi_A^e := \sup_{|\xi|=1} |\arg\sigma(A(\xi))|$ satisfies the inequality $\varphi_A^e \geq \varphi_A^\infty$. Here, the parameter elliptic operator is of order m and has Hölder continuous coefficients and C^m-compact boundary domain under general boundary condition (see [18]). Although our case is included in [18] and [17], it is not so easy to check whether the assumptions are satisfied. We give a simple proof here.

Lemma 3.2.3. *Let* $\operatorname{Re}\alpha > 0$. *Then* $-\alpha\mathcal{Q}$ *and* $-\bar{\alpha}\mathcal{Q}$ *have the property of maximal regularity, i.e., the solutions of the equation* (3.2.8) *and* (3.2.9) *satisfy* (3.2.10) *and* (3.2.11), *respectively, for any* $p \in (1, \infty)$.

Proof. We first consider the problem with the zero initial data. We prove that $-\alpha\mathcal{Q}$ and $-\bar{\alpha}\mathcal{Q} \in \mathcal{H}^\infty(X)$ with $\varphi_{-\alpha Q}^\infty$ and $\varphi_{-\bar{\alpha}Q}^\infty < \pi/2$. Let $\alpha = re^{i\psi}$. Notice that $\psi < \frac{\pi}{2}$ since $\operatorname{Re}\alpha > 0$. Since $\mathcal{Q}$ is a strong elliptic operator, hence, it is the generator of a bounded C_0-semigroup on L^p-spaces as well. Therefore $-\mathcal{Q}$ admits a bounded $\mathcal{H}^\infty$-calculus, (see [20]), so that $-\alpha\mathcal{Q}$ and $-\bar{\alpha}\mathcal{Q}$ also admit a bounded $\mathcal{H}^\infty$-calculus with $\varphi_{-\alpha\mathcal{Q}}^\infty$ and $\varphi_{-\bar{\alpha}\mathcal{Q}}^\infty \leq \varphi_{-\mathcal{Q}}^\infty + \psi$ (see [18, Proposition 2.11]). Here, by the strong ellipticity of $\mathcal{Q}$, we have $\varphi_{-\mathcal{Q}}^\infty = 0$. Indeed, we may write $-Q(\xi) := (-Q(\xi)_{ij}) = (\sum_{k,l=1}^3 A_{ikjl}\xi_k\xi_l)$ since

$$
\begin{aligned}
-Q\mathbf{u} &= -\frac{1}{2}\left(\sum_{j,k,l} A_{ijkl}(\partial_l\partial_j u_k + \partial_j\partial_k u_l)\right) \\
&= \frac{1}{2}\begin{pmatrix} A_{1j1l}(i\partial_j)(i\partial_l) & A_{1j2l}(i\partial_j)(i\partial_l) & A_{1j3l}(i\partial_j)(i\partial_l) \\ A_{2j1l}(i\partial_j)(i\partial_l) & A_{2j2l}(i\partial_j)(i\partial_l) & A_{2j3l}(i\partial_j)(i\partial_l) \\ A_{3j1l}(i\partial_j)(i\partial_l) & A_{3j2l}(i\partial_j)(i\partial_l) & A_{3j3l}(i\partial_j)(i\partial_l) \end{pmatrix}\mathbf{u} \\
&\quad + \frac{1}{2}\begin{pmatrix} A_{1jk1}(i\partial_j)(i\partial_k) & A_{1jk2}(i\partial_j)(i\partial_k) & A_{1jk3}(i\partial_j)(i\partial_k) \\ A_{2jk1}(i\partial_j)(i\partial_k) & A_{2jk2}(i\partial_j)(i\partial_k) & A_{2jk3}(i\partial_j)(i\partial_k) \\ A_{3jk1}(i\partial_j)(i\partial_k) & A_{3jk2}(i\partial_j)(i\partial_k) & A_{3jk3}(i\partial_j)(i\partial_k) \end{pmatrix}\mathbf{u} \\
&= \begin{pmatrix} A_{1j1l}(i\partial_j)(i\partial_l) & A_{1j2l}(i\partial_j)(i\partial_l) & A_{1j3l}(i\partial_j)(i\partial_l) \\ A_{2j1l}(i\partial_j)(i\partial_l) & A_{2j2l}(i\partial_j)(i\partial_l) & A_{2j3l}(i\partial_j)(i\partial_l) \\ A_{3j1l}(i\partial_j)(i\partial_l) & A_{3j2l}(i\partial_j)(i\partial_l) & A_{3j3l}(i\partial_j)(i\partial_l) \end{pmatrix}\mathbf{u}
\end{aligned}
$$

by symmetry property (1.3.1) of (A_{ijkl}). Then for any $x \in \mathbb{C}^3$ and $\xi \in \mathbb{R}^3$ such that

$|\xi| = 1$ we have

$$
\begin{aligned}
{}^t\bar{x} \cdot (-Q(\xi))x &= \sum_{i,j,k,l} A_{ijkl}\xi_j\xi_l\bar{x}_i x_k \\
&= \sum_{i,j,k,l} A_{ijkl}(\xi_j\bar{x}_i)(\xi_l x_k) \\
&\geq a_* \sum_{i,j} \xi_j^2 |x_i|^2 = a_*|x|^2,
\end{aligned}
$$

where $a_* := \min[3\lambda + 2\mu, 2\mu] > 0$. Therefore, since $(-Q(\xi))$ is positive definite for any ξ, these eigenvalues are positive. This implies that $\varphi^\infty_{-\mathcal{Q}} = \varphi^e_{-\mathcal{Q}} = 0$. Consequently, we have $\varphi^\infty_{-\alpha\mathcal{Q}},\ \varphi^\infty_{-\bar{\alpha}\mathcal{Q}} \leq \pi/2$.

Next, we consider the case of nonzero initial data. From the above argument it also yields that $\alpha\mathcal{Q}$ and $\bar{\alpha}\mathcal{Q}$ generate analytic semigroups $T(t)$ and $\overline{T}(t)$ on $L^p(\Omega)$, respectively. It is well-known that $\mathbf{u}_0$ and $(\mathbf{u}_1 - \alpha Q\mathbf{u}_0)$ are in the trace space of $W^{2,1}_p(\Omega_T)$ (i.e., $B^{2-\frac{2}{p}}_{p,p}$) if and only if $\overline{T}(\cdot)\mathbf{u}_0$ and $T(\cdot)(\mathbf{u}_1 - \alpha Q\mathbf{u}_0) \in W^{2,1}_p(\Omega_T)$, respectively (see [45, Theorem 1.14.5]).

Hence, for (3.2.8) and (3.2.9) without inhomogeneous terms, it follows that

$$
\begin{aligned}
\|\, \mathbf{u}\,;\, W^{2,1}_p(\Omega_T)\| &\leq C\|\, \mathbf{u}_0\,;\, B^{2-\frac{2}{p}}_{p,p}\|, \\
\|\, \mathbf{w}\,;\, W^{2,1}_p(\Omega_T)\| &\leq C\|\, \mathbf{u}_1 - \alpha Q\mathbf{u}_0\,;\, B^{2-\frac{2}{p}}_{p,p}\|,
\end{aligned}
$$

from which we have (3.2.10) and (3.2.11). □

Remark. In general, the constants C of the estimates (3.2.10) and (3.2.11) should depend on time T. However, by using the cutoff argument, we obtain that for any given $T_0 > 0$ there exists $\Lambda(T_0) > 0$ independent of $T \in (0, T_0]$ such that

$$
\|\, \mathbf{u}\,;\, W^{2,1}_p(\Omega_T)\| \leq \Lambda(T_0)(\|\, \mathbf{w}\,;\, L^p(\Omega_T)\| + \|\, \mathbf{u}_0\,;\, B^{2-1/p}_{p,p}\|).
$$

Therefore we may write the constant such as independent of time T (see [14]).

Although the estimate (3.2.3) can be obtained by using the Friedman-Nečas method (see [36], [25]), we give a proof by using the another method.

Proof of Lemma 3.2.1 (*ii*). We show that, for the equation (3.2.9),

$$
\|\, \nabla\mathbf{w}\,;\, L^p(\Omega_T)\| \leq C\|\, f\,;\, L^p(\Omega_T)\| + C\|\, \mathbf{u}_1 - \alpha Q\mathbf{u}_0\,;\, B^{1-\frac{2}{p}}_{p,p}\|.
$$

We first consider the equation with zero initial data. Denote the operator Δ with the Dirichlet boundary condition by $\mathfrak{D}$. We know that $I - \mathcal{Q}$ and $I - \mathfrak{D} \in \mathcal{BIP}(X)$ by the permanence property for $\mathcal{BIP}$ (see [18, Proposition 2.6]), and $D(I-\mathcal{Q}) = D(I-\mathfrak{D}) = W^2_{p,0}(\Omega)$, where

$$W^j_{p,0}(\Omega) := \{\mathbf{u} \in W^j_p(\Omega)\,;\ \mathbf{u} = 0 \text{ on } \partial\Omega\}.$$

It follows from [18, Theorem 2.5] that $D((I-\mathcal{Q})^{\frac{1}{2}}) = D((I-\mathfrak{D})^{\frac{1}{2}})$. Hence, using the result by D. Fujiwara [26], we know that $D((I-\mathcal{Q})^{\frac{1}{2}}) = W^1_{p,0}$. Here, if we put $\mathbf{v} := (I-\mathcal{Q})^{-1/2}\mathbf{w}$, then the equation (3.2.9) with zero initial data becomes

$$\begin{cases} \mathbf{v}_t - \alpha\mathcal{Q}\mathbf{v} = (I-\mathcal{Q})^{-1/2}\nabla\cdot f & \text{in } \Omega_T, \\ \mathbf{v}(0) = 0 & \text{in } \Omega. \end{cases} \tag{3.2.12}$$

By the permanence property of $\mathcal{H}^\infty$ ([18, Proposition 2.11]), the maximal regularity with perturbation

$$\|\,\mathbf{v}_t\,;\,L^p(\Omega_T)\| + \|\,(I-\mathcal{Q})\mathbf{v}\,;\,L_p(\Omega_T)\| \le C\|\,(I-\mathcal{Q})^{-1/2}\nabla\cdot f\,;\,L^p(\Omega_T)\|$$

holds, so that

$$\begin{aligned} &\|\,(I-\mathcal{Q})^{-1/2}\mathbf{w}_t\,;\,L^p(\Omega_T)\| + \|\,(I-\mathcal{Q})^{1/2}\mathbf{w}\,;\,L^p(\Omega_T)\| \\ &\qquad\qquad \le C\|\,(1-\mathcal{Q})^{-1/2}\nabla\cdot f\,;\,L^p(\Omega_T)\|. \end{aligned} \tag{3.2.13}$$

Now we claim that for any $q \in (1,\infty)$,

$$\|\,\nabla g\,;\,L^q(\Omega)\| \le C\|\,(I-\mathcal{Q})^{\frac{1}{2}} g\,;\,L^q(\Omega)\|. \tag{3.2.14}$$

Using the claim (3.2.14), the second term of the left hand side of (3.2.13) is estimated as follows:

$$\|\,(I-\mathcal{Q})^{1/2}\mathbf{w}\,;\,L^p(\Omega_T)\| \ge C\|\,\nabla\mathbf{w}\,;\,L^p(\Omega_T)\|.$$

The right hand side of (3.2.13) satisfies

$$\|\,(I-\mathcal{Q})^{-1/2}\nabla\cdot f\,;\,L^p(\Omega_T)\| \le C\|\,f\,;\,L^p(\Omega_T)\|. \tag{3.2.15}$$

Indeed, for smooth function g, we have

$$\begin{aligned} ((I-\mathcal{Q})^{-\frac{1}{2}}(\nabla\cdot f), g) &= (\nabla\cdot f, (I-\mathcal{Q})^{-\frac{1}{2}} g) && (3.2.16)\\ &= (f\,;\,\nabla(I-\mathcal{Q})^{-\frac{1}{2}} g) && (3.2.17)\\ &\le C\|f\|_{L^p}\|g\|_{L^{\frac{p}{p-1}}}, && (3.2.18) \end{aligned}$$

where $(f,g) := \int f\cdot g dx$ and $(f;g) := \int f : g dx$. The first equality (3.2.16) follows from the self-adjointness of $(I-\mathcal{Q})^{-\frac{1}{2}}$, the second one (3.2.17) is obtained by the divergence formula with $(I-\mathcal{Q})^{-\frac{1}{2}}g \in D((I-\mathcal{Q})^{1/2}) = W^{\frac{p}{p-1}}_{p,0}$ and the last inequality (3.2.18) is the consequence of (3.2.14). Hence, the desired results follow if we accept the claim (3.2.14).

The claim follows from the complex interpolation ([26, Theorem 5]) between the following Calderon-Zygmund inequality for the strong elliptic operator $\mathcal{Q}-I$ (see [27]):

$$\| \mathbf{w}\,;\, W^2_q(\Omega)\| \le C\| \,(I-\mathcal{Q})\mathbf{w}\,;\, L^q(\Omega)\|$$

and the trivial equality:

$$\| \mathbf{w}\,;\, L^q(\Omega)\| = \| \mathbf{w}\,;\, L^q(\Omega)\|.$$

Next we consider the case of the homogeneous equation with nonzero data. By (3.2.14) and the real interpolation, we have

$$\| \mathbf{v}(0)\,;\, B^{2-\frac{2}{p}}_{p,p}\| \le C\| \,(I-\mathcal{Q})^{1/2}\mathbf{v}(0)\,;\, B^{1-\frac{2}{p}}_{p,p}\|,$$

which completes the proof. □

We also give the maximal regularity for the heat equation with the Hölder continuous coefficient. The estimate is the particular case of [28, Example 3.2, A), 2)].

Lemma 3.2.4. *Let $q \in (1,\infty)$. Assume that $\rho(x)$ is Hölder continuous in Ω such that* $\inf_\Omega \rho > 0$. *Denote by θ the solution of the linear problem*

$$\begin{cases} \theta_t - \rho\Delta\theta = g & \text{in } \Omega_T, \\ \mathbf{n}\cdot\nabla\theta = 0 & \text{on } S_T, \\ \theta(0,x) = \theta_0(x) & \text{in } \Omega. \end{cases} \tag{3.2.19}$$

Then the following estimate holds

$$\| \theta\,;\, W^{2,1}_q(\Omega_T)\| \le C(\big\| \theta_0\,;\, B^{2-\frac{2}{q}}_{q,q}\big\| + \| g\,;\, L^q(\Omega)\|) \tag{3.2.20}$$

for any $\theta_0 \in B^{2-2/q}_{q,q}$, where C depends on $\inf_\Omega \rho$.

3.3 Truncated Problem

In this section, we consider the three-dimensional case. We define the truncation function Γ_L of level L such as

$$\Gamma_L(x) = \begin{cases} x & \text{if } |x| \le L, \\ L\frac{x}{|x|} & \text{if } |x| \ge L. \end{cases}$$

In order to assure the nonnegativity of temperature θ, we need the sufficiently strong regularity assumption to $\mathbf{u}$. Then we first consider the following truncated problem:

$$\begin{cases} \mathbf{u}_{tt} + Q^2\mathbf{u} - \nu Q\mathbf{u}_t = \Gamma_L\Big(\nabla \cdot [G(\theta)F_{1,\epsilon}(\epsilon) + F_{2,\epsilon}(\epsilon)]\Big), \\ \theta_t - \Delta\theta = \theta G''(\theta)\theta_t F_1(\epsilon) + \theta G'(\theta)\partial_t F_1(\epsilon) + \nu(A\epsilon_t) : \epsilon_t & \text{in } \Omega_T, \\ \mathbf{u} = Q\mathbf{u} = \nabla\theta \cdot \mathbf{n} = 0 & \text{on } S_T, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x), \quad \mathbf{u}_t(0,x) = \mathbf{u}_1(x), \quad \theta(0,x) = \theta_0(x) \geq 0 & \text{in } \Omega. \end{cases} \tag{3.3.1}$$

We prove the unique global existence for the truncated system (3.3.1).

Theorem 3.3.1. *Let $L > 0$ and $5 < p \leq q < \infty$. Assume that $\theta_0 \geq 0$, that r, K_1, K_2 satisfy* (3.0.6)*, and that $F(\epsilon, \theta) \in \mathrm{C}^4(\mathrm{Sym}(n, \mathbb{R}) \times \mathbb{R}^+, \mathbb{R})$ holds. Then for any $T > 0$ and $(u_0, u_1, \theta_0) \in U(p, q)$, there exists a unique solution (u_L, θ_L) to* (3.3.1) *satisfying $(u_L, \theta_L) \in V_T(p, q)$.*

Proof. We apply Theorem 3.1.4 to the map Φ_τ^L from $V_T(p,q)$ into $V_T(p,q)$,

$$\Phi_\tau^L : (\bar{\mathbf{u}}, \bar{\theta}) \mapsto (\mathbf{u}, \theta), \qquad \tau \in [0,1],$$

defined by means of the following initial-boundary value problems:

$$\begin{cases} \mathbf{u}_{tt} + Q^2\mathbf{u} - \nu Q\mathbf{u}_t = \tau\Gamma_L\Big(\nabla \cdot [G(\bar{\theta})F_{1,\epsilon}(\bar{\epsilon}) + F_{2,\epsilon}(\bar{\epsilon})]\Big), \\ \theta_t - \Delta\theta = \tau\left(\bar{\theta} G''(\bar{\theta})\theta_t F_1(\epsilon) + \bar{\theta} G'(\bar{\theta})\partial_t F_1(\epsilon) + \nu(A\epsilon_t) : \epsilon_t\right) & \text{in } \Omega_T, \\ \mathbf{u} = Q\mathbf{u} = \nabla\theta \cdot \mathbf{n} = 0 & \text{on } S_T, \\ \mathbf{u}(0,x) = \tau\mathbf{u}_0(x), \quad \mathbf{u}_t(0,x) = \tau\mathbf{u}_1(x), \quad \theta(0,x) = \tau\theta_0(x) & \text{in } \Omega, \end{cases}$$

where $\bar{\epsilon} = \epsilon(\bar{\mathbf{u}})$. A fixed point of $\Phi_\tau^L(1, \cdot)$ in $V_T(p,q)$ is the desired solution of the system (3.3.1). Therefore, to prove the existence statement, it is sufficient to check that the map Φ_τ^L satisfies assumptions $(L1)$–$(L4)$ of Theorem 3.1.4.

Step 1.

We can check the assumptions $(L1)$, $(L2)$ and $(L3)$ for Φ_τ as the same as in [37, Section 3]. For the sake of completeness, we state the proof of these parts. Here Φ_τ from $V_T(p,q)$ into $V_T(p,q)$ is defined as follows:

$$\Phi_\tau : (\bar{\mathbf{u}}, \bar{\theta}) \mapsto (\mathbf{u}, \theta), \qquad \tau \in [0,1],$$

defined by means of the following initial-boundary value problems:

$$\begin{cases} \mathbf{u}_{tt} + Q^2\mathbf{u} - \nu Q\mathbf{u}_t = \tau\nabla \cdot [G(\bar{\theta})F_{1,\epsilon}(\bar{\epsilon}) + F_{2,\epsilon}(\bar{\epsilon})], \\ \theta_t - \Delta\theta = \tau\left[\bar{\theta} G''(\bar{\theta})\theta_t F_1(\epsilon) + \bar{\theta} G'(\bar{\theta})\partial_t F_1(\epsilon) + \nu(A\epsilon_t) : \epsilon_t\right] & \text{in } \Omega_T, \\ \mathbf{u} = Q\mathbf{u} = \nabla\theta \cdot \mathbf{n} = 0 & \text{on } S_T, \\ \mathbf{u}(0,x) = \tau\mathbf{u}_0(x), \quad \mathbf{u}_t(0,x) = \tau\mathbf{u}_1(x), \quad \theta(0,x) = \tau\theta_0(x) & \text{in } \Omega. \end{cases} \tag{3.3.2}$$

We remark that if the conditions $(L1)$–$(L3)$ for Φ_τ are satisfied then the conditions $(L1)$–$(L3)$ for Φ_τ^L are also satisfied from the Lipschitz continuity of Γ_L.

The property $(L1)$ follows by showing that for any fixed $\tau \in [0,1]$, Φ_τ maps the bounded subsets into precompact subsets in $V_T(p,q)$. Let $(\bar{\mathbf{u}}^n, \bar{\theta}^n)$ be a bounded sequence in $V(p,q)$ such that

$$\begin{aligned} \bar{\mathbf{u}}^n \to \bar{\mathbf{u}} &\quad \text{weakly in } W_p^{4,2}(\Omega_T) \quad \text{for } 5 < p < \infty, \\ \bar{\theta}^n \to \bar{\theta} &\quad \text{weakly in } W_q^{4,2}(\Omega_T) \quad \text{for } 5 < q < \infty, \end{aligned} \tag{3.3.3}$$

as $n \to \infty$. Our aim is to show that for the values of $\Phi_\tau(\cdot)$ given by

$$(\mathbf{u}^n, \theta^n) = \Phi_\tau(\bar{\mathbf{u}}^n, \bar{\theta}^n), \tag{3.3.4}$$

the following convergences hold

$$\mathbf{u}^n \to \mathbf{u} \quad \text{strongly in } W_p^{4,2}(\Omega_T) \quad \text{for } 5 < p < \infty, \tag{3.3.5}$$

$$\theta^n \to \theta \quad \text{strongly in } W_q^{2,1}(\Omega_T) \quad \text{for } 5 < q < \infty, \tag{3.3.6}$$

as $n \to \infty$, where

$$(\mathbf{u}, \theta) = \Phi_\tau(\bar{\mathbf{u}}, \bar{\theta}). \tag{3.3.7}$$

Applying the Aubin compactness theorem (Lemma 3.1.3), we obtain

$$W_p^{4,2}(\Omega_T) \hookrightarrow W_p^{2,1}(\Omega_T) \quad \text{and} \quad W_q^{2,1}(\Omega_T) \hookrightarrow L_T^q W_q^1 \quad \text{are compact.}$$

With the help of the compact embeddings results, it follow from (3.3.3) that

$$\begin{aligned} \epsilon(\bar{\mathbf{u}}^n) \to \epsilon(\bar{\mathbf{u}}) &\quad \text{strongly in } W_p^{2,1}(\Omega_T) \quad \text{for } 5 < p < \infty, \\ \bar{\theta}^n \to \bar{\theta} &\quad \text{strongly in } L_T^q W_q^1 \quad \text{for } 5 < q < \infty \end{aligned} \tag{3.3.8}$$

as $n \to \infty$.

We also obtain that recalling Ω is bounded,

$$W_q^{2,1}(\Omega_T) \hookrightarrow \mathrm{C}^{\alpha,\alpha/2}(\Omega_T) \quad \text{is compact}$$

for $q > 5/2$ and $\alpha < 2 - 5/q$, since $W_q^{2,1}(\Omega_T) \hookrightarrow \mathrm{C}^{\beta,\beta/2}(\Omega_T)$ is continuous from Lemma 3.1.1 for $\beta = 2 - 5/q$ and $\mathrm{C}^{\beta,\beta/2}(\Omega_T) \hookrightarrow \mathrm{C}^{\alpha,\alpha/2}(\Omega_T)$ is compact for $\alpha < \beta$ (see [1]). This, by virtue of this compact embedding, implies that

$$\bar{\epsilon}^n \to \bar{\epsilon}, \quad \nabla\bar{\epsilon}^n \to \nabla\bar{\epsilon}, \quad \bar{\theta}^n \to \bar{\theta}, \tag{3.3.9}$$

strongly in spaces of Hölder continuous functions in Ω_T, where

$$\bar{\epsilon}^n = \epsilon(\bar{\mathbf{u}}^n), \quad \bar{\epsilon} = \epsilon(\bar{\mathbf{u}}).$$

Thanks to the above convergences, it follows that

$$\begin{aligned}\nabla \cdot F_{\epsilon}(\bar{\epsilon}^n, \bar{\theta}^n) = F_{,\epsilon\epsilon}(\bar{\epsilon}^n, \bar{\theta}^n)\nabla\bar{\epsilon}^n + F_{,\epsilon\theta}(\bar{\epsilon}^n, \bar{\theta}^n)\nabla\bar{\theta}^n \\ \to F_{,\epsilon\epsilon}(\bar{\epsilon}, \bar{\theta})\nabla\bar{\epsilon} + F_{,\epsilon\theta}(\bar{\epsilon}, \bar{\theta})\nabla\bar{\theta} = \nabla \cdot F_{,\epsilon}(\bar{\epsilon}, \bar{\theta}) \\ \text{strongly in } L^p(\Omega_T) \quad \text{for } 5 < p \leq q < \infty.\end{aligned} \tag{3.3.10}$$

Consequently, it follows from the maximal regularity that

$$\mathbf{u}^n \to \mathbf{u} \quad \text{strongly in } W_p^{4,2}(\Omega_T).$$

This implies the convergence (3.3.5).

Furthermore, we note that, by (3.3.5) and continuous embeddings (Lemma 3.1.1),

$$\epsilon^n \longrightarrow \epsilon, \quad \epsilon_t^n \longrightarrow \epsilon_t \tag{3.3.11}$$

strongly in spaces of Hölder continuous functions in Ω_T, where

$$\epsilon^n = \epsilon(\mathbf{u}^n), \quad \epsilon_t^n = \epsilon(u_t^n), \quad \epsilon = \epsilon(\mathbf{u}), \quad \epsilon_t = \epsilon(\mathbf{u}_t).$$

In order to prove convergence (3.3.6), we consider the difference

$$\eta^n = \theta^n - \theta.$$

By definition, η^n satisfies the following problem

$$\begin{aligned}& c_0(\epsilon, \bar{\theta}, \tau)\eta_t^n - \Delta\eta^n = \tau R(\epsilon^n, \bar{\theta}^n) - \tau R(\epsilon, \bar{\theta}) \\ & \qquad - (c_0(\epsilon^n, \bar{\theta}^n, \tau) - c_0(\epsilon, \bar{\theta}, \tau))\theta_t^n && \text{in } \Omega_T, \\ & \eta^n(0, \cdot) = 0 && \text{in } \Omega, \\ & \mathbf{n} \cdot \nabla\eta^n = 0 && \text{on } S_T,\end{aligned} \tag{3.3.12}$$

where

$$\begin{aligned}c_0(\epsilon, \bar{\theta}, \tau) &= 1 - \tau\bar{\theta}G''(\bar{\theta})F_1(\epsilon), \\ R(\epsilon^n, \bar{\theta}^n) &= \bar{\theta}^n G'(\bar{\theta}^n)F_{1,\epsilon}(\epsilon^n) : \epsilon_t^n + \nu(A\epsilon_t^n) : \epsilon_t^n, \\ R(\epsilon, \bar{\theta}) &= \bar{\theta}G'(\bar{\theta})F_{1,\epsilon}(\epsilon) : \epsilon_t + \nu(A\epsilon_t) : \epsilon_t.\end{aligned}$$

In view of Hölder continuity of the coefficient $c_0(\epsilon, \bar{\theta}, \tau)$, in order to prove that

$$\eta^n \to 0 \quad \text{strongly in } W_q^{2,1}(\Omega_T) \quad \text{as } n \to \infty,$$

it is sufficient, by virtue of the maximal regularity, to show that the right hand side of (3.3.12) converges to 0 is $L^q(\Omega_T)$-norm. Indeed, we have

$$
\begin{aligned}
&\big\| R(\epsilon^n, \bar\theta^n) - R(\epsilon, \bar\theta)\,;\, L^q(\Omega_T)\big\| \\
&\quad \le C\big\| \,|\bar\theta^n - \bar\theta|\, |F_{,\theta\epsilon}(\epsilon^n, \bar\theta^n)|\, |\epsilon^n_t|\,;\, L^q(\Omega_T)\big\| + C\big\|\,\bar\theta|\epsilon^n_t|(|\epsilon^n - \epsilon| + |\bar\theta^n - \bar\theta|)\,;\, L^q(\Omega_T)\big\| \\
&\qquad + C\big\|\,\bar\theta|F_{,\theta\epsilon}(\epsilon, \bar\theta)|\,|\epsilon^n_t - \epsilon_t|\,;\, L^q(\Omega_T)\big\| + C\big\|\,|\epsilon^n_t - \epsilon_t|(|\epsilon^n_t| + |\epsilon_t|)\,;\, L^q(\Omega_T)\big\| \\
&\quad \to 0 \qquad \text{as } n \to 0,
\end{aligned}
$$

where we have used uniform Hölder bounds on ϵ^n, ϵ^n_t and $\bar\theta^n$ with respect to n, and the convergences (3.3.9) and (3.3.11). Furthermore,

$$
\begin{aligned}
\big\| (c_0(\epsilon^n, \bar\theta^n, \tau) - &c_0(\epsilon, \bar\theta, \tau))\theta^n_t\,;\, L^q(\Omega_T)\big\| \\
&\le \big\| (c_0(\epsilon^n, \bar\theta^n, \tau) - c_0(\epsilon, \bar\theta, \tau))\,;\, L^\infty(\Omega_T)\big\|\, \big\|\,\theta^n_t\,;\, L^q(\Omega_T)\big\| \\
&\to 0 \qquad \text{as } n \to \infty.
\end{aligned}
$$

This shows (3.3.6) and thereby the complete continuity of $\Phi_\tau(\cdot)$.

The uniform equicontinuity property ($L2$) follows by direct comparison of two solutions $(\mathbf{u}, \theta)$ to the problem (3.3.2) with $\tau = \tau_1$ and $(\widetilde{\mathbf{u}}, \widetilde\theta)$ to the problem (3.3.2) with $\tau = \tau_2$, and applying the maximal regularity estimate. The property ($L3$) is obvious by the definition of $\Phi_\tau(\cdot)$.

From the Lipschitz continuity of Γ_L we can immediately check the conditions ($L1$)–($L3$) for Φ^L_τ in the same way as above.

Step 2.

Next, we check the assumption ($L4$), namely, to derive a priori bounds for a fixed point of the solution map Φ^L_τ. Without loss of generality we may set $\tau = 1$. Hence from now on our purpose is to obtain a priori bounds for (3.3.1). To this end we prepare several lemmas. If there is no danger of confusion we write for simplicity $(\mathbf{u}, \theta)$ instead of $(\mathbf{u}_L, \theta_L)$.

Lemma 3.3.2 (Maximum Principle). *Let* $(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \in B^{4-2/p}_{p,p} \times B^{2-2/p}_{p,p} \times L^2$ *for* $p > 5$. *Assume that* $\min_\Omega \theta_0 \ge 0$. *Then the solution* θ *to the truncated problem* (3.3.1) *is non-negative almost everywhere in* Ω_T.

Proof. It follows from the maximal regularity (3.2.2) that

$$
\begin{aligned}
\|\,\mathbf{u}\,;\, W^{4,2}_p(\Omega_T)\| &\le C\Big(\|\,\mathbf{u}_0\,;\, B^{4-2/p}_{p,p}\| + \|\,\mathbf{u}_1\,;\, B^{2-2/p}_{p,p}\| \\
&\qquad + \big\|\,\Gamma_L\left(\nabla\cdot[G(\bar\theta)F_{1,\epsilon}(\bar\epsilon) + F_{2,\epsilon}(\bar\epsilon)]\right)\,;\, L^p(\Omega_T)\big\|\Big) \\
&\le C\left(\|\,\mathbf{u}_0\,;\, B^{4-2/p}_{p,p}\| + \|\,\mathbf{u}_1\,;\, B^{2-2/p}_{p,p}\| + L|\Omega_T|^{\frac{1}{p}}\right) \\
&\le \Lambda(L).
\end{aligned}
\tag{3.3.13}
$$

Then taking $p > 5$, by Lemma 3.1.1 we have

$$\| \epsilon \,;\, L^\infty(\Omega_T)\| + \| \epsilon_t \,;\, L^\infty(\Omega_T)\| \le \Lambda(L) < \infty. \tag{3.3.14}$$

Therefore it holds that

$$\| \partial_t F_1(\epsilon) \,;\, L^\infty(\Omega_T)\| \le \| \epsilon_t \,;\, L^\infty(\Omega_T)\| \| \epsilon \,;\, L^\infty(\Omega_T)\|^{K_1-1} \le \Lambda(L)$$

for $K_1 > 1$. Since $\sup_{\epsilon\in\mathbb{S}} |F_{1,\epsilon}(\epsilon)| \le M$ for $K_1 \le 1$, we conclude that

$$\| \partial_t F_1(\epsilon) \,;\, L^\infty(\Omega_T)\| \le \Lambda(L) \tag{3.3.15}$$

for every $K_1 \ge 0$. From now on throughout this section we shall write $\Lambda = \Lambda(L)$.

Multiplying the second equation of (3.3.1) by $\theta_- := \min\{\theta, 0\}$ and integrating over Ω, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \theta_-^2 dx + \int_\Omega |\nabla\theta_-|^2 dx \\
&\quad = \int_\Omega \left[\theta_-\theta G''(\theta)\theta_t F_1(\epsilon) + \theta_-\theta G'(\theta)\partial_t F_1(\epsilon) + \nu\theta_- A\epsilon_t : \epsilon_t\right] dx \\
&\quad = \frac{d}{dt}\int_\Omega F_1(\epsilon)G_2(\theta_-)dx + \int_\Omega \overline{G}_2(\theta_-)\partial_t F_1(\epsilon)dx + \int_\Omega \nu\theta_- A\epsilon_t : \epsilon_t dx,
\end{aligned}$$

where $G_2(\theta) = \theta^2 G'(\theta) - \overline{G}_2(\theta)$ and $\overline{G}_2(\theta) = 2\int_0^\theta sG'(s)ds$. We have $G_2(0) = 0$ and $G_2'(y) = y^2 G''(y) \ge 0$ for $y \le 0$, because G'' is the odd function such that $G''(y) \le 0$ for $y \ge 0$. Then $G_2(y) \le 0$ for $y \ge 0$. Hence we have

$$-\int_\Omega F_1(\epsilon)G_2(\theta_-)dx \ge 0.$$

It follows from (1.3.2) that

$$\int_\Omega \nu\theta_- A\epsilon_t : \epsilon_t dx \le \nu a_* \int_\Omega \theta_- |\epsilon_t|^2 dx \le 0.$$

Noting that $\overline{G}_2(\theta) = \frac{1}{2}C_1\theta^2$ for $\theta \in [-\theta_1, \theta_1]$, we have $\sup_{s\in\mathbb{R}} \frac{|\overline{G}_2(s)|}{s^2} \le C$. Therefore we conclude that

$$\begin{aligned}
\int_\Omega \overline{G}_2(\theta_-)\partial_t F_1(\epsilon)dx &\le \int_\Omega |\theta_-|^2 \frac{|\overline{G}_2(\theta_-)|}{|\theta_-|^2} |\partial_t F_1(\epsilon)| dx \\
&\le \Lambda \|\theta_-\|_{L^2}^2.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\| \theta_-(t) \,;\, L^2(\Omega)\|^2 - \int_\Omega F_1(\epsilon)G_2(\theta_-)dx\right) \\
&\qquad\qquad \le \Lambda\left(\| \theta_-(t) \,;\, L^2(\Omega)\|^2 - \int_\Omega F_1(\epsilon)G_2(\theta_-)dx\right).
\end{aligned}$$

Using the Gronwall inequality, we obtain

$$\begin{aligned}\|\,\theta_-(t)\,;\,L^2(\Omega)\|^2 &\le \|\,\theta_-(t)\,;\,L^2(\Omega)\|^2 - \int_\Omega F_1(\epsilon)G_2(\theta_-)dx \\ &\le \Lambda e^{\Lambda t}\left(\|\,\theta_-(0)\,;\,L^2(\Omega)\|^2 - \int_\Omega F_1(\epsilon(0))G_2(\theta_-(0))dx\right) \\ &= 0,\end{aligned}$$

which completes the proof. □

Lemma 3.3.3. *Let $m > 2$ be arbitrary integer, and assume that $r \le 1$. Then for any $(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \in B^{4-2/p}_{p,p} \times B^{2-2/p}_{p,p} \times L^m$, the solution $(\mathbf{u}, \theta)$ to the truncated problem* (3.3.1) *satisfies*

$$\|\,\theta\,;\,L^\infty_T L^m\| \le \Lambda,$$

where $\Lambda = \Lambda(T, \|\,(\mathbf{u}_1, \mathbf{u}_2, \theta_0)\,;\,B^{4-2/p}_{p,p} \times B^{2-2/p}_{p,p} \times L^m\|)$. Moreover, if $(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \in B^{4-2/p}_{p,p} \times B^{2-2/p}_{p,p} \times L^\infty$, then we have

$$\|\,\theta\,;\,L^\infty(\Omega_T)\| \le \Lambda,$$

where $\Lambda = \Lambda(T, \|\,(\mathbf{u}_1, \mathbf{u}_2, \theta_0)\,;\,B^{4-2/p}_{p,p} \times B^{2-2/p}_{p,p} \times L^\infty\|)$.

Proof. Multiplying the second equation of (3.3.1) by θ^{m-1} and integrating over Ω, we have

$$\begin{aligned}\frac{1}{m}\frac{d}{dt}\|\,\theta\,;\,L^m(\Omega)\|^m + (m-1)\int_\Omega \theta^{m-2}|\nabla\theta|^2 dx &= \int_\Omega \nu\theta^{m-1}A\epsilon_t : \epsilon_t dx \\ &\quad + \int_\Omega \left(\theta^m G''(\theta)\theta_t F_1(\epsilon) + \theta^m G'(\theta)\partial_t F_1(\epsilon)\right) dx \\ &= \nu\int_\Omega \theta^{m-1}A\epsilon_t : \epsilon_t dx + \frac{d}{dt}\int_\Omega G_m(\theta)F_1(\epsilon)dx \\ &\quad + \int_\Omega \overline{G}_l(\theta)\partial_t F_1(\epsilon)dx,\end{aligned} \tag{3.3.16}$$

where $G_m(\theta) = \theta^m G'(\theta) - \overline{G}_m(\theta)$ and $\overline{G}_m(\theta) = m\int_0^\theta s^{m-1}G'(s)ds$. Since

$$\theta^m G''(\theta) = \begin{cases} C_2 r(r-1)\theta^{m+r-1} \le 0 & \text{for } \theta \ge \theta_2, \\ \theta^m \varphi''(\theta) \le 0 & \text{for } \theta_1 \le \theta \le \theta_2, \\ 0 & \text{for } \theta \le \theta_1, \end{cases} \tag{3.3.17}$$

we have $G'_m(\theta) = \theta^m G''(\theta) \le 0$ for $\theta \ge 0$ and $G'_m(0) = 0$. Thereby, we obtain

$$G_m(\theta) \le 0 \quad \text{for} \quad \theta \ge 0. \tag{3.3.18}$$

We put

$$\hat{\theta} = \theta \left(1 - \frac{mG_m(\theta)F_1(\epsilon)}{\theta^m}\right)^{1/m}.$$

We note that $\hat{\theta} \geq \theta$ due to (3.3.18). Since $\sup_{s\in[0,\infty)} |G'(s)| =: M < \infty$, we have

$$|\overline{G}_m(\theta)| = \left| m \int_0^\theta s^{m-1} G'(s) ds \right| \leq C\theta^m$$

and

$$|G_m(\theta)| \leq M\theta^m + |\overline{G}_m(\theta)| \leq C\theta^m.$$

In view of (3.3.14) and (3.3.15) we obtain

$$\left| \int_\Omega \overline{G}_m(\theta) \partial_t F_1(\epsilon) dx \right| \leq C\| \theta^m ; L^1(\Omega)\| \| \partial_t F_1(\epsilon) ; L^\infty(\Omega)\|^{K_1 - 1} \leq \Lambda \| \theta ; L^m(\Omega)\|^m$$

and

$$\int_\Omega \theta^{m-1} A\epsilon_t : \epsilon_t \leq C\| \epsilon_t ; L^\infty(\Omega)\|^2 \| \theta ; L^{m-1}(\Omega)\|^{m-1} \leq \Lambda \| \theta ; L^m(\Omega)\|^{m-1}.$$

Since $\frac{1}{m}\partial_t \| \hat{\theta} ; L^m\|^m = \| \hat{\theta} ; L^m\|^{m-1} \partial_t \| \hat{\theta} ; L^m\|$, it follows from (3.3.16) that

$$\begin{aligned} \frac{d}{dt} \| \hat{\theta} ; L^m(\Omega)\| &\leq \Lambda \| \theta ; L^m(\Omega)\| + \Lambda \\ &\leq \Lambda \| \hat{\theta} ; L^m(\Omega)\| + \Lambda. \end{aligned}$$

Thus by the Gronwall inequality we have

$$\| \hat{\theta} ; L_T^\infty L^m\| \leq \Lambda \| \hat{\theta}_0 ; L^m(\Omega)\| + \Lambda. \tag{3.3.19}$$

Since

$$\begin{aligned} \hat{\theta}_0 &= \theta_0 \left(1 - \frac{mG_m(\theta_0)F_1(\epsilon_0)}{\theta_0^m}\right)^{1/m} \\ &\leq \theta_0 \left(1 + mM\Lambda\right)^{1/m}, \end{aligned}$$

we can obtain the first assertion. Here we note that the constant Λ in (3.3.19) is independent of m. Therefore taking a limit as $m \to \infty$ we can obtain the second assertion. This completes the proof. □

Lemma 3.3.4. *Let $r \leq 1$. Then for any $(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \in B_{p,p}^{4-2/p} \times B_{p,p}^{2-2/p} \times H^1$, the solution $(\mathbf{u}, \theta)$ to the truncated problem* (3.3.1) *satisfies*

$$\| \theta ; W_2^{2,1}(\Omega_T)\| \leq \Lambda,$$

where Λ depends on T and $\| (\mathbf{u}_0, \mathbf{u}_1, \theta_0) ; B_{p,p}^{4-2/p} \times B_{p,p}^{2-2/p} \times H^1\|$.

Proof. By using Lemma 3.3.3 thanks to $\theta_0 \in H^1 \hookrightarrow L^2$, we have

$$\|\,\theta\,;\,L^\infty_T L^2\| \le \Lambda. \tag{3.3.20}$$

Since $\theta G''(\theta) \le 0$ from (3.3.17) for $m = 1$, the following estimate holds true

$$\iint_{\Omega_T} \theta_t^2 \theta G''(\theta) F_1(\epsilon) dxdt \le 0. \tag{3.3.21}$$

Multiplying the second equation of (3.3.1) by θ_t and integrating over Ω_T, we have

$$\begin{aligned}
\|\,\theta_t\,;\,L^2(\Omega_T)\|^2 + \frac{1}{2}\|\,\nabla\theta\,;L^\infty_T L^2\|^2 &\le \frac{1}{2}\|\,\theta_0\,;\,H^1(\Omega)\|^2 + \iint_{\Omega_T} \nu\theta_t A\epsilon_t : \epsilon_t dxdt \\
&\quad + \iint_{\Omega_T} \theta_t \theta G'(\theta)\partial_t F_1(\epsilon) dxdt + \iint_{\Omega_T} \theta_t^2 \theta G''(\theta) F_1(\epsilon) dxdt \\
&\le \|\theta_0\|^2_{H^1(\Omega)} + \Lambda\|\,\theta_t\,;\,L^\infty_T L^2\|\|\,\epsilon_t\,;\,L^\infty(\Omega_T)\|^2 \\
&\quad + \Lambda\|\,\theta_t\,;\,L^2(\Omega_T)\|\|\,\theta\,;\,L^\infty_T L^2\|^r\|\,\partial_t F_1(\epsilon)\,;\,L^\infty(\Omega_T)\| \\
&\le \|\,\theta_0\,;\,H^1(\Omega)\|^2 + \frac{1}{2}\|\,\theta_t\,;\,L^2(\Omega_T)\|^2 + \Lambda,
\end{aligned}$$

thanks to (3.3.14), (3.3.15), (3.3.20) and (3.3.21). Therefore we arrive at

$$\|\,\theta_t\,;\,L^2(\Omega_T)\| + \|\,\nabla\theta\,;\,L^\infty_T L^2\| \le \Lambda(\|\,(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\,;\,U_2\|).$$

Next multiplying the second equation of (3.3.1) by $\frac{-\Delta\theta}{1-\theta G''(\theta)F_1(\epsilon)}$ and integrating over Ω, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\,\nabla\theta(t)\,;L^2\|^2 + \int_\Omega \frac{(\Delta\theta)^2}{1-\theta G''(\theta)F_1(\epsilon)} dx \\
&\qquad \le \int_\Omega \frac{\Delta\theta}{1-\theta G''(\theta)F_1(\epsilon)} \left(\theta G'(\theta)\partial_t F_1(\epsilon) + \nu A\epsilon_t : \epsilon_t\right) dx.
\end{aligned}$$

Here we remark that

$$1 \le 1 - \theta G''(\theta)F_1(\epsilon) \le 1 + M\Lambda,$$

where $0 \le \sup_{\theta\ge 0}(-\theta G''(\theta)) =: M < \infty$. Then integrating over $[0,t]$ for $t \le T$, we conclude the estimate

$$\begin{aligned}
\|\,\nabla\theta(t)\,;\,L^2(\Omega)\|^2 + \frac{2}{1+\Lambda M}\|\,\Delta\theta\,;\,L^2(\Omega_T)\|^2 &\le \|\,\nabla\theta_0\,;\,L^2(\Omega)\|^2 \\
&\quad + 2\|\,\Delta\theta\,;\,L^2(\Omega_T)\|\,\big\|\,\theta G'(\theta)\partial_t F_1(\epsilon) + \nu A\epsilon_t : \epsilon_t\,;\,L^2(\Omega_T)\big\| \\
&\le \|\,\nabla\theta_0\,;\,L^2\|^2 + \frac{1}{(1+\Lambda M)}\|\,\Delta\theta\,;\,L^2(\Omega_T)\|^2 \\
&\quad + (1+\Lambda M)\Big(\Lambda\|\,\theta\,;\,L^\infty_T L^2\|\|\,\partial_t F_1(\epsilon)\,;\,L^\infty(\Omega_T)\| + \Lambda\|\,\epsilon_t\,;\,L^\infty(\Omega_T)\|\Big)^2 \\
&\le \frac{1}{(1+\Lambda M)}\|\,\Delta\theta\,;\,L^2(\Omega_T)\|^2 + \Lambda.
\end{aligned}$$

Consequently we arrive at the desired result. □

The same procedure as in [37, Section 6] yields that $\theta \in \mathrm{C}^{\alpha,\alpha/2}(\Omega_T)$ for some Hölder exponent $0 < \alpha < 1$ depending on T, $\sup_\Omega \theta_0$ and $\| \theta ; L^\infty(\Omega_T)\|$.

Lemma 3.3.5 ([37, Lemma 6.1]). *Assume that* $\sup_\Omega \theta_0 < \infty$ *and that* $\theta \geq 0$ *in* Ω_T*. Suppose that the solution for the problem* (3.3.1) *satisfies that*

$$\| \epsilon ; L^\infty(\Omega_T)\| + \| \epsilon ; W^{2,1}_s(\Omega_T)\| + \| \theta ; W^{2,1}_2(\Omega_T)\| + \| \theta ; L^\infty(\Omega_T)\| \leq \Lambda \tag{3.3.22}$$

for any $s \in (1,\infty)$*. Then* $\theta \in \mathrm{C}^{\alpha,\alpha/2}(\Omega_T)$ *with the Hölder exponent* $\alpha \in (0,1)$ *depending on* Λ *and* k*.*

For the sake of completeness we give the proof of this lemma. Essentially, the proof of this lemma relies on the classical De Giorgi method for parabolic equations. For more precise information of this method we refer to [32, Chapter II, §7] and [34, Chapter VI, §12]. Here we note that ϵ is Hölder continuous because of Lemma 3.1.1. We first define the parabolic De Giorgi class $\mathcal{B}_2(\Omega_T, M, \gamma, r, \delta, \kappa)$.

Definition 3.3.6 (Parabolic De Giorgi Class). Let M, γ, r, δ, κ be positive numbers. The function u belongs to $\mathcal{B}_2(\Omega_T, M, \gamma, r, \delta, \kappa)$ if the following conditions are satisfied:

$(D1)$ $u \in V^{1,0}_2(\Omega_T) = \mathrm{C}_{[0,T]}L^2 \cap L^2_T H^1$,

$(D2)$ $\| u ; L^\infty(\Omega_T)\| \leq M$,

$(D3)$ the function $w(x,t) = \pm u(x,t)$ satisfies the following inequalities:

$$\max_{t_0 \leq t \leq t_0+\tau} \| (w-k)_+ ; L^2(B_{(1-\sigma_1)\rho}(x_0))\|^2 \leq \|(w-k)_+(\cdot, t_0) ; L^2(B_\rho(x_0))\|^2 + \gamma \left[(\sigma_1\rho)^{-2}\|(w-k)_+ ; L^2(Q(\rho,\tau))\| + \mu^{2(1+\kappa)/r}(k,\rho,\tau)\right],$$

and

$$\| (w-k)_+ ; V_2(Q((1-\sigma_1)\rho, (1-\sigma_2)\tau))\| \leq \gamma \left\{ \left(\frac{1}{(\sigma_1\rho)^2} + \frac{1}{\sigma_2\tau}\right) \| (w-k)_+ ; L^2(Q(\rho,\tau))\|^2 + \mu^{2(1+\kappa)/r}(k,\rho,\tau) \right\}.$$

Here we denote by $(w-k)_+ = \max\{w-k, 0\}$, $B_\rho(x_0) = \{x \in \Omega \mid |x-x_0| < \rho\}$ and $Q(\rho,\tau) = B_\rho(x_0) \times (t_0, t_0+\tau) = \{(x,t) \in \Omega_T \mid |x-x_0| < \rho,\ t_0 < t < t_0+\tau\}$, where ρ and τ are arbitrary positive numbers, σ_1 and σ_2 are arbitrary numbers from the interval $(0,1)$, and k is an arbitrary number satisfying the condition:

$$\| w(x,t) ; L^\infty(Q(\rho,\tau))\| - k \leq \delta.$$

Moreover, we set

$$A_{k,\rho}(t) = \{x \in B_\rho(x_0) \,|\, w(x,t) > k\}\,,$$
$$\mu(k,\sigma,\tau) = \int_{t_0}^{t_0+\tau} |A_{k,\rho}(t)|^{\frac{r}{q}} dt,$$

where positive numbers q and r are linked by the relation

$$\frac{1}{r} + \frac{n}{2q} = \frac{n}{4},$$

with the admissible ranges

$$\begin{aligned} &q \in (2, 2n/(n-2)], && r \in [2,\infty) && \text{for } n \geq 3,\\ &q \in (2,\infty), && r \in (2,\infty) && \text{for } n \geq 2,\\ &q \in (2,\infty], && r \in [4,\infty) && \text{for } n \geq 1. \end{aligned}$$

Besides, we write

$$V_2(\Omega_T) = L^\infty_T L^2 \cap L^2_T H^1.$$

We call $\mathcal{B}_2(\Omega_T, M, \gamma, r, \delta, \kappa)$ the *parabolic De Giorgi class*.

The embedding $\mathcal{B}_2(\Omega_T, M, \gamma, r, \delta, \kappa) \hookrightarrow \mathrm{C}^{\alpha,\alpha/2}(\Omega_T)$ holds (see [32, Theorem II.7.1]). Hence, if we prove $\theta \in \mathcal{B}_2(\Omega_T, M, \gamma, r, \delta, \kappa)$, then we can obtain the desired result.

Proof of Lemma 3.3.5. We shall prove $\theta \in \mathcal{B}_2(\Omega_T, M, \gamma, r, \delta, \kappa)$, where $r = q = 10/3$, $\kappa = 1/3$, $M := \|\,\theta\,;\, L^\infty(\Omega_T)\|$, $\gamma = \Lambda$ and δ is some constant such that $\delta > M - k$ for a positive number satisfying $k > \sup_\Omega \theta_0(x)$. We determine δ later.

It is sufficient to check that θ satisfies conditions $(D1)$–$(D3)$ in the definition of the space $\mathcal{B}_2(\Omega_T, M, \gamma, r, \delta, \kappa)$. Since $\theta \in W^{2,1}_2(\Omega_T)$, by the embedding theorem, it follows that

$$\theta \in \mathrm{C}_{[0,T]} H^1,$$

so that the condition $(D1)$ is clearly satisfied. Furthermore, thanks to the assumption (3.3.22), condition $(D2)$ is also satisfied with the constant $M = \Lambda$.

We proceed now to check that θ satisfies the second inequality in condition $(D3)$. Let $Q(\rho,\tau) = B_\rho(x_0) \times (t_0, t_0+\tau)$ be an arbitrary cylinder in Ω_T, and $\zeta(x,t)$ be a smooth function such that $\operatorname{supp} \zeta(x,t) \subset Q(\rho,\tau)$ and $\zeta(x,t) = 1$ for $(x,t) \in Q((1-\sigma_1)\rho, (1-\sigma_2)\tau)$, where σ_1, $\sigma_2 \in (0,1)$. Moreover, let

$$A_{k,\rho}(t) = \{x \in B_\rho(x_0) \mid \theta(x,t) > k\}\,.$$

Multiplying the equation (3.0.2) by $\zeta^2(\theta-k)_+$ and integrating over Ω, we obtain

$$\begin{aligned}\frac{1}{2}\int_\Omega c_0\zeta^2\frac{\partial}{\partial t}(\theta-k)_+^2dx+\int_\Omega|\nabla(\theta-k)_+|^2\zeta^2dx\\ +2\int_\Omega\zeta(\theta-k)_+\nabla(\theta-k)_+\cdot\nabla\zeta dx=\int_\Omega R\zeta^2(\theta-k)_+dx,\end{aligned}\tag{3.3.23}$$

where for simplicity we have denoted the right hand side of (3.0.2) by f, i.e.,

$$R=R(\epsilon,\theta)=\theta G'(\theta)F_{1,\epsilon}(\epsilon):\epsilon_t+\nu(A\epsilon_t):\epsilon_t,$$

and $c_0(\epsilon,\theta)=1-\theta G''(\theta)F_1(\epsilon)$. The first term on the left hand side of (3.3.23) is rearranged as

$$\begin{aligned}\frac{1}{2}\int_\Omega c_0\zeta^2\frac{\partial}{\partial t}(\theta-k)_+^2dx&=\frac{1}{2}\frac{d}{dt}\int_\Omega c_0(\theta-k)_+^2\zeta^2dx\\ &-\frac{1}{2}\int_{A_{k,\rho}(t)}(c_{0,\epsilon}:\epsilon_t)(\theta-k)_+^2\zeta^2dx-\frac{1}{2}\int_{A_{k,\rho}(t)}c_{0,\theta}\theta_t(\theta-k)_+^2\zeta^2dx\\ &-\int_{A_{k,\rho}(t)}c_0(\theta-k)_+^2\zeta\zeta_tdx.\end{aligned}\tag{3.3.24}$$

The third integral on the right hand side of the above inequality requires special technical treatment because of the presence of θ_t. To this end we first observe that on the set $A_{k,\rho}(t)$ it holds that

$$c_0(\epsilon,\theta)=c_0\big(\epsilon,(\theta-k)_++k\big),$$

and hence we have

$$c_{0,\theta}(\epsilon,\theta)=c_{0,(\theta-k)_+}\big(\epsilon,(\theta-k)_++k\big)\quad\text{ on } A_{k,\rho}(t).$$

Now, restricting considerations to the set $A_{k,\rho}(t)$, we define the function

$$H(\epsilon,(\theta-k)_+)=\int_0^{(\theta-k)_+}c_{0,\xi}(\epsilon,\xi+k)\xi d\xi.\tag{3.3.25}$$

Clearly, it satisfies the conditions

$$\begin{aligned}H(\epsilon,0)=0,\\ H_{,(\theta-k)_+}\big(\epsilon,(\theta-k)_+\big)=c_{0,(\theta-k)_+}\big(\epsilon,(\theta-k)_++k\big)(\theta-k)_+^2.\end{aligned}$$

Then the third mentioned above integral transforms as follows:

$$\begin{aligned}-\frac{1}{2}\int_{A_{k,\rho}(t)}c_{0,\theta}\theta_t(\theta-k)_+^2\zeta^2dx&=-\frac{1}{2}\int_{A_{k,\rho}(t)}c_{0,(\theta-k)_+}(\theta-k)_+^2\zeta^2\partial_t(\theta-k)_+dx\\ &=-\frac{1}{2}\int_{A_{k,\rho}(t)}H_{,(\theta-k)_+}\zeta^2\partial_t(\theta-k)_+dx\\ &=-\frac{1}{2}\int_{A_{k,\rho}(t)}(\partial_tH)\zeta^2dx-\frac{1}{2}\int_{A_{k,\rho}(t)}(H_{,\epsilon}:\epsilon_t)\zeta^2dx.\end{aligned}\tag{3.3.26}$$

Setting

$$H_+ = \begin{cases} H(\epsilon, (\theta - k)_+) & \text{for } \theta > k, \\ 0 & \text{for } \theta \le k, \end{cases}$$

we rewrite the first integral in the last equality as

$$\begin{aligned} -\frac{1}{2}\int_{A_{k,\rho}(t)} (\partial_t H)\zeta^2 dx &= -\frac{1}{2}\int_\Omega (\partial_t H_+)\zeta^2 dx \\ &= -\frac{1}{2}\frac{d}{dt}\int_\Omega H_+\zeta^2 dx + \int_\Omega H_+\zeta\zeta_t dx. \end{aligned} \tag{3.3.27}$$

Summarizing, in view of (3.3.24), (3.3.26) and (3.3.27), the identity (3.3.23) takes the form

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega c_0(\theta - k)_+^2\zeta^2 dx + \int_\Omega |\nabla(\theta - k)_+|^2\zeta^2 dx \\ &\quad = \frac{1}{2}\frac{d}{dt}\int_\Omega H_+\zeta^2 dx - \int_\Omega H_+\zeta\zeta_t dx - \frac{1}{2}\int_{A_{k,\rho}(t)} \zeta^2 H_{,\epsilon} : \epsilon_t dx \\ &\qquad + \int_{A_{k,\rho}(t)} c_0(\theta - k)_+^2\zeta\zeta_t dx + \frac{1}{2}\int_{A_{k,\rho}(t)} \zeta^2(\theta - k)_+^2 c_{0,\epsilon} : \epsilon_t dx \\ &\qquad - 2\int_{A_{k,\rho}(t)} \zeta(\theta - k)_+\nabla(\theta - k)_+ \cdot \nabla\zeta dx + \int_{A_{k,\rho}(t)} R\zeta^2(\theta - k)_+ dx. \end{aligned} \tag{3.3.28}$$

Integrating (3.3.28) with respect to t, and taking into account that $(\theta_0 - k)_+ = 0$ and $H(\epsilon_0, (\theta_0 - k)_+) = 0$, we obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega (\theta - k)_+^2\zeta^2 dx + \int_{\Omega_t} |\nabla(\theta - k)_+|^2\zeta^2 dxds \\ &\quad \le \bigg(\int_\Omega |H_+|\zeta^2 dx + \int_{\Omega_t} |H_+|\,|\zeta_s| dxds + \int_{\Omega_t} |H_{+,\epsilon}|\,|\epsilon_s|\zeta^2 dxds \\ &\qquad + \int_{\Omega_t} |c_0|(\theta - k)_+^2|\zeta_s| dxds + \int_{\Omega_t} |c_{0,\epsilon}|\,|\epsilon_s|(\theta - k)_+^2\zeta^2 dxds \\ &\qquad + \int_{\Omega_t} |R|(\theta - k)_+|\zeta|^2 dxds + \int_{\Omega_t} (\theta - k)_+|\nabla(\theta - k)_+|\,|\zeta|\,|\nabla\zeta| dxds \bigg). \end{aligned} \tag{3.3.29}$$

Now we observe that owing to the boundedness of functions $c_{0,\theta}$ and $c_{0,\theta\epsilon}$, it follows that

$$|H(\epsilon, (\theta - k)_+)| + |H_{,\epsilon}(\epsilon, (\theta - k)_+)| \le c(\theta - k)_+^3. \tag{3.3.30}$$

Moreover, by the assumption on k,

$$|H(\epsilon, (\theta - k)_+)| \le c\delta(\theta - k)_+^2. \tag{3.3.31}$$

Therefore, choosing δ appropriately, the first integral on the right hand side of (3.3.29) can be absorbed by the left hand side. The last integral on the right hand side of (3.3.29) is estimated by use of the Young inequality as follows:

$$\begin{aligned}\int_{\Omega_t}(\theta-k)_+|\nabla(\theta-k)_+|\,|\zeta|\,|\nabla\zeta|dxds\\ \le \frac{1}{2}\int_{\Omega_t}|\nabla(\theta-k)_+|^2\zeta^2dxds+\frac{1}{2}\int_{\Omega_t}(\theta-k)_+^2|\nabla\zeta|^2dxds,\end{aligned}\tag{3.3.32}$$

then the first integral on the right hand side of the above inequality is absorbed by the left hand side of (3.3.29). Combining (3.3.30)–(3.3.32) in (3.3.29), we arrive at

$$\begin{aligned}\int_{\Omega}(\theta-k)_+^2\zeta^2dx+\int_{\Omega_t}|\nabla(\theta-k)_+|^2\zeta^2dxds\\ \le C\int_{\Omega_t}(\theta-k)_+^2\left(\zeta^2+|\nabla\zeta|^2+|\zeta_s|\right)dxds\\ +C\int_{\Omega_t}\left(|\epsilon_s|(\theta-k)_+^2+|f|(\theta-k)_+\right)\zeta^2dxds\\ =: I_1+I_2.\end{aligned}\tag{3.3.33}$$

Clearly, the integral I_1 is estimated by

$$I_1\le C\left(\frac{1}{(\sigma_1\rho)^2}+\frac{1}{\sigma_2\tau}\right)\int_{Q(\rho,\tau)}(\theta-k)_+^2dxds.$$

For the integral I_2, using the boundedness of t and applying the Hölder inequality, we obtain

$$\begin{aligned}I_2&\le C\int_{t_0}^{t_0+\tau}\int_{A_{k,\rho}(s)}(|\epsilon_s|+|f|)\zeta^2dxds\\ &\le C\left(\int_{t_0}^{t_0+\tau}\int_{A_{k,\rho}(s)}(|\epsilon_s|^5+|f|^5)\zeta^2dxds\right)^{\frac{1}{5}}\left(\int_{t_0}^{t_0+\tau}|A_{k,\rho}(s)|ds\right)^{\frac{4}{5}}.\end{aligned}$$

Consequently, we obtain

$$I_2\le\left(\|\,\epsilon_t\,;\,L^5(\Omega_t)\|+\|\,R\,;\,L^5(\Omega_T)\|\right)\mu^{\frac{4}{5}}(k,\rho,\tau).$$

Taking into account that by assumptions

$$\|\,f\,;\,L^5(\Omega_T)\|\le C\left(\|\,\epsilon_t\,;\,L^5(\Omega_T)\|+\|\,\epsilon_t\,;\,L^{10}(\Omega_T)\|^2\right)\le\Lambda,$$

we have

$$I_2\le\Lambda\mu^{2(1+\kappa)/r}(k,\rho,\tau)$$

for $\kappa = 1/3$ and $r = 10/3$. Combining estimates on I_1 and I_2 in (3.3.33) leads to

$$\begin{aligned}
&\| (\theta - k)_+ \,; V_2(Q((1-\sigma_1)\rho, (1-\sigma_2)\tau))\|^2 \\
&\qquad = \left\| \int_\Omega (\theta - k)_+^2 \zeta^2 dx \,;\, L_T^\infty \right\| + \int_{\Omega_T} |\nabla(\theta - k)_+|^2 \zeta^2 dxdt \\
&\qquad \le \Lambda \left[\left(\frac{1}{(\sigma_1 \rho)^2} + \frac{1}{\sigma_2 \tau} \right) \| (\theta - k)_+ \,;\, L^2(Q(\rho, \tau))\|^2 + \mu^{2(1+\kappa)/r}(k, \rho, \tau) \right].
\end{aligned}$$

Since $\theta \ge 0$, this shows that the second inequality in condition $(D3)$ is satisfied with constant $\gamma = \Lambda$.

The first inequality in $(D3)$ can be proved by multiplying (3.0.2) by $\zeta_0^2(\theta - k)_+$, where $\zeta_0(x)$ is a smooth function such that $\operatorname{supp} \zeta_0(x) \subset B_\rho(x_0)$, $\zeta_0(x) = 1$ for $x \in B_{(1-\sigma_1)\rho}(x_0)$, the next integrating over $\Omega \times (t_0, t_0 + \tau)$. In this case, repeating the above arguments, inequality (3.3.33) is replaced by

$$\begin{aligned}
&\int_\Omega (\theta - k)_+^2 \zeta_0^2 dx + \int_{Q(\rho,\tau)} |\nabla(\theta - k)_+|^2 \zeta_0^2 dxdt \\
&\qquad \le C \left[\int_{B_\sigma(x_0)} (\theta(t_0) - k)_+^2 \zeta_0^2 dx + \int_{Q(\rho,\tau)} (\theta - k)_+^2 (\zeta_0^2 + |\nabla \zeta_0|^2) dxdt \right. \\
&\qquad\qquad \left. + \int_{Q(\rho,\tau)} (|\epsilon_t|(\theta - k)_+^2 |f|(\theta - k)_+) \zeta_0^2 dxdt \right].
\end{aligned} \tag{3.3.34}$$

Since the last two integrals on the right hand side of (3.3.34) are estimated as above, this leads to the required inequality. The proof is completed. □

Lemma 3.3.7. *Assume that* (3.3.22) *holds. Then for any* $(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \in U(p, q)$ *and* $5 < p$, $q < \infty$, *we have*

$$\| (\mathbf{u}, \theta) \,; V_T(p, q)\| = \| \mathbf{u} \,; W_p^{4,2}(\Omega_T)\| + \| \theta \,; W_q^{2,1}(\Omega_T)\| \le \Lambda,$$

where Λ *depends on* $\|(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \,; U(p, q)\|$ *and* T.

Proof. By using Lemma 3.3.5, we have θ is Hölder continuous, i.e., $\theta \in \mathrm{C}^{\alpha,\alpha/2}(\Omega_T)$. At the first step we prove the Hölder continuous of θ in $\Omega_{[0,T]}$.

Step 1.

To prove the Hölder continuity of θ in the domain $\Omega_{[0,T)} := [0, T) \times \Omega$ $(\supset \Omega_T)$, we first show the unique local existence to the problem (3.0.1)–(3.0.4) for sufficiently small time interval $[0, \delta]$ for $\delta < T$. We remark that the unique local existence to the truncated problem (3.3.1) follows from the easy modification of this proof, thanks to the Lipschitz continuity of Γ_L. We consider the map $\Psi : (\mathbf{u}, \theta) \mapsto (\widetilde{\mathbf{u}}, \widetilde{\theta})$ such that

$$\begin{cases} \widetilde{\mathbf{u}}_{tt} + QQ\widetilde{\mathbf{u}} - \nu Q\widetilde{\mathbf{u}}_t = \nabla \cdot (G(\theta)F_{1,\epsilon}(\epsilon) + F_{2,\epsilon}(\epsilon)), \\ [1 - \theta G''(\theta)F_1(\epsilon)]\widetilde{\theta}_t - \Delta\widetilde{\theta} = \theta G'(\theta)\partial_t F_1(\epsilon) + \nu(A\epsilon_t) : \epsilon_t & \text{in } \Omega_\delta, \\ \widetilde{\mathbf{u}} = Q\widetilde{\mathbf{u}} = \nabla\widetilde{\theta} \cdot \mathbf{n} = 0 & \text{on } S_\delta, \\ \mathbf{u}(0, \cdot) = \mathbf{u}_0, \quad \mathbf{u}_t(0, \cdot) = \mathbf{u}_1, \quad \theta(0, \cdot) = \theta_0 & \text{in } \Omega. \end{cases}$$

For some $M > 0$, we define the subset $\widetilde{\mathcal{V}}_T^M(p, q)$ of $\widetilde{\mathcal{V}}_T(p, q)$ by

$$\begin{aligned} \widetilde{\mathcal{V}}_T^M(p, q) := \big\{(\mathbf{u}, \theta) \in V_T(p, q) \mid \|(\mathbf{u}, \theta)\|_{\widetilde{\mathcal{V}}_T(p,q)} \le M, \\ (\mathbf{u}(0, \cdot), \mathbf{u}_t(0, \cdot), \theta(0, \cdot)) = (\mathbf{u}_0, \mathbf{u}_1, \theta_0)\big\}. \end{aligned}$$

In this case of $5 < p \le q < \infty$, the norm of $\widetilde{\mathcal{V}}_T(p, q)$ is

$$\begin{aligned} \|(\mathbf{u}, \theta)\|_{\widetilde{\mathcal{V}}_T(p,q)} := \|\, \mathbf{u} \,;\, W_p^{4,2}(\Omega_T)\| + \|\, \epsilon \,;\, W_\infty^{2,1}(\Omega_T)\| \\ + \|\, \theta \,;\, W_q^{2,1}(\Omega_T)\| + \|\, \theta \,;\, L_T^\infty W_\infty^1\|. \end{aligned}$$

We shall show that the map $\Psi(u, \theta)$ is a contraction from $\widetilde{\mathcal{V}}_\delta^M(p, q)$ into $\widetilde{\mathcal{V}}_\delta^M(p, q)$, where positive number M is determined later.

From the Hölder inequality, it follows that

$$\begin{aligned} \|\, \nabla \cdot F_{,\epsilon} \,;\, L^p(\Omega_T)\| &\le CT^{\frac{1}{p}} \|\, \nabla \cdot F_{,\epsilon} \,;\, L^\infty(\Omega_T)\| \\ &\le CT^{\frac{1}{p}} (\|\theta\|_{\widetilde{\mathcal{V}}_T(p,q)} + \|\theta\|^r_{\widetilde{\mathcal{V}}_T(p,q)}) \|\mathbf{u}\|^{K_1-1}_{\widetilde{\mathcal{V}}_T(p,q)} + \|\mathbf{u}\|^{K_2-1}_{\widetilde{\mathcal{V}}_T(p,q)} \\ &\le CT^{\frac{1}{p}} h_1(\|(\mathbf{u}, \theta)\|_{\widetilde{\mathcal{V}}_T(p,q)}) \|(\mathbf{u}, \theta)\|_{\widetilde{\mathcal{V}}_T(p,q)}, \end{aligned}$$

where $h_1(y) = y^{K_1+r-2} + y^{K_1-1} + y^{K_2-2}$. By the maximal regularity (3.2.2) and Proposition A.1.3, we have

$$\begin{aligned} \|(\widetilde{\mathbf{u}}, 0)\|_{\mathcal{V}_T(p,q)} &\le C\|\, (\mathbf{u}_0, \mathbf{u}_1, 0) \,;\, \widetilde{U}(p, q)\| + C\|\, \nabla \cdot F_{,\epsilon} \,;\, L^p(\Omega_T)\| \\ &\le C\|\, (\mathbf{u}_0, \mathbf{u}_1, 0) \,;\, \widetilde{U}(p, q)\| + T^{\frac{1}{q}} h_1(\|(\mathbf{u}, \theta)\|_{\widetilde{\mathcal{V}}_T(p,q)}) \|(\mathbf{u}, \theta)\|_{\widetilde{\mathcal{V}}_T(p,q)}. \end{aligned}$$

On the other hand, we can rewrite the heat equation as follows

$$[1 - \theta_0 G''(\theta_0)F_1(\epsilon_0)]\widetilde{\theta}_t - \Delta\widetilde{\theta} = \mathcal{D}(\epsilon, \theta)\widetilde{\theta}_t + R(\epsilon, \theta),$$

where

$$c_0(\epsilon, \theta) = [1 - \theta G''(\theta)F_1(\epsilon)], \tag{3.3.35}$$

$$R(\epsilon, \theta) = [\theta G'(\theta)\partial_t F_1(\epsilon) + \nu(A\epsilon_t) : \epsilon_t], \tag{3.3.36}$$

$$\mathcal{D}(\epsilon, \theta) = \theta_0 G''(\theta_0)F_1(\epsilon_0) - \theta G''(\theta)F_1(\epsilon). \tag{3.3.37}$$

We note that $c_0(\epsilon_0,\theta_0)$, $\mathcal{D}(\epsilon,\theta)$ and $R(\epsilon,\theta)$ are the given Hölder continuous functions. Then, by the maximal regularity (3.2.20) and Proposition A.1.3, we have

$$
\begin{aligned}
\|(0,\widetilde{\theta})\|_{\widetilde{\mathcal{V}}_T(p,q)} &\le C\|\,(0,0,\theta_0)\,;\,\widetilde{U}(p,q)\| \\
&\qquad + C\|\,\mathcal{D}(\mathbf{u},\theta)\widetilde{\theta}_t\,;\,L^q(\Omega_T)\| + CT^{\frac{1}{q}}\|\,R(\mathbf{u},\theta)\,;\,L^\infty(\Omega_T)\| \\
&\le C\|\,(0,0,\theta_0)\,;\,\widetilde{U}(p,q)\| + \|\mathcal{D}(\mathbf{u},\theta)\,;\,L^\infty(\Omega_T)\|\|\,\widetilde{\theta}_t\,;\,L^q(\Omega_T)\| \\
&\qquad + \Lambda_2 h_2(\|(\mathbf{u},\theta)\|_{\widetilde{\mathcal{V}}_T(p,q)})\|(\mathbf{u},\theta)\|_{\widetilde{\mathcal{V}}_T(p,q)},
\end{aligned}
$$

where $h_2(y):=y^{K_1-2+r}+y$. We set $M=2C\|(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\|_{U(p,q)}$. If we choose $T_1\ll T$ such that $\|\,\mathcal{D}(\mathbf{u},\theta)\,;\,L^\infty(\Omega_{T_1})\|\le 1/2$ and $2CT_1^{\frac{1}{q}}h(M)<M$, then we have

$$
\begin{aligned}
\|(\widetilde{\mathbf{u}},\widetilde{\theta})\|_{\widetilde{\mathcal{V}}_{T_1}(p,q)} &\le C(\|(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\|_{U(p,q)}) + CT_1^{\frac{1}{q}}h(\|(\mathbf{u},\theta)\|_{\widetilde{\mathcal{V}}_{T_1}(p,q)})\|(\mathbf{u},\theta)\|_{\widetilde{\mathcal{V}}_{T_1}(p,q)} \\
&\le M,
\end{aligned}
$$

where $h(y)=h_1(y)+h_2(y)$. This implies that $(\widetilde{\mathbf{u}},\widetilde{\theta})\in\widetilde{\mathcal{V}}_{T_1}^M$.

Given $(\mathbf{u},\theta)\in\widetilde{\mathcal{V}}_T^M(p,q)$ and $(\overline{\mathbf{u}},\overline{\theta})\in\widetilde{\mathcal{V}}_T^M(p,q)$, we set $(\widetilde{\overline{\mathbf{u}}},\widetilde{\overline{\theta}})=\Psi(\overline{\mathbf{u}},\overline{\theta})$, $\mathbf{w}=\mathbf{u}-\overline{\mathbf{u}}$, $\eta=\theta-\overline{\theta}$, $\widetilde{\mathbf{w}}=\widetilde{\mathbf{u}}-\widetilde{\overline{\mathbf{u}}}$ and $\widetilde{\eta}=\widetilde{\theta}-\widetilde{\overline{\theta}}$. Then we have

$$
\begin{aligned}
&\widetilde{\mathbf{w}}_{tt}+Q^2\widetilde{\mathbf{w}}-\nu Q\widetilde{\mathbf{w}}_t=\nabla\cdot(F_{,\epsilon}(\epsilon(\mathbf{u}),\theta)-F_{,\epsilon}(\epsilon(\overline{\mathbf{u}}),\overline{\theta})),\\
&c_0(\epsilon_0,\theta_0)\widetilde{\eta}_t-\Delta\widetilde{\eta}=\mathcal{D}(\epsilon(\mathbf{u}),\theta)\partial_t\widetilde{\eta}\\
&\qquad\qquad +[\mathcal{D}(\epsilon(\mathbf{u}),\theta)-\mathcal{D}(\epsilon(\overline{\mathbf{u}}),\overline{\theta})]\partial_t\widetilde{\overline{\theta}}+(R(\epsilon(\mathbf{u}),\theta)-R(\epsilon(\overline{\mathbf{u}}),\overline{\theta})),\\
&\mathbf{w}=Q\mathbf{w}=\nabla\eta\cdot\mathbf{n}=0,\\
&\mathbf{w}(0,\cdot)=\mathbf{w}_t(0,\cdot)=\eta(0,\cdot)=0.
\end{aligned}
$$

From Lemma 3.2.1 and Proposition A.1.3, we obtain

$$
\|(\widetilde{\mathbf{w}},0)\|_{\widetilde{\mathcal{V}}_T(p,q)}\le CT^{\frac{1}{p}}\big[h_1(\|(\mathbf{u},\theta)\|_{\widetilde{\mathcal{V}}_T(p,q)})+h_1(\|(\overline{\mathbf{u}},\overline{\theta})\|_{\widetilde{\mathcal{V}}_T(p,q)})\big]\|(\mathbf{w},\eta)\|_{\widetilde{\mathcal{V}}_T(p,q)}
$$

and

$$
\begin{aligned}
\|(0,\widetilde{\eta})\|_{\widetilde{\mathcal{V}}_T(p,q)} &\le C\|\,\mathcal{D}(\epsilon(\mathbf{u}),\theta)\,;\,L^\infty(\Omega_T)\|\|\,\partial_t\widetilde{\eta}\,;\,L^q(\Omega_T)\| \\
&\quad + C\|\,\mathcal{D}(\epsilon(\mathbf{u}),\theta)-\mathcal{D}(\epsilon(\overline{\mathbf{u}}),\overline{\theta})\,;\,L^\infty(\Omega_T)\|\|\,\partial_t\widetilde{\overline{\theta}}\,;\,L^q(\Omega_T)\| \\
&\quad + CT^{\frac{1}{q}}\|\,R(\epsilon(\mathbf{u}),\theta)-R(\epsilon(\overline{\mathbf{u}}),\theta)\,;\,L^\infty(\Omega_T)\| \\
&\le C\|\,\mathcal{D}(\epsilon(\mathbf{u}),\theta)\,;\,L^\infty(\Omega_T)\|\|\,\partial_t\widetilde{\eta}\,;\,L^q(\Omega_T)\| \\
&\quad + C\|\,\partial_t\widetilde{\overline{\theta}}\,;\,L^q(\Omega_T)\|\big[h_3(\|(\mathbf{u},\theta)\|_{\widetilde{\mathcal{V}}_T(p,q)})+h_3(\|(\overline{\mathbf{u}},\overline{\theta})\|_{\widetilde{\mathcal{V}}_T(p,q)})\big]\|(\mathbf{w},\eta)\|_{\widetilde{\mathcal{V}}_T(p,q)} \\
&\quad + CT^{\frac{1}{q}}\big[h_2(\|(\mathbf{u},\theta)\|_{\widetilde{\mathcal{V}}_T(p,q)})+h_2(\|(\overline{\mathbf{u}},\overline{\theta})\|_{\widetilde{\mathcal{V}}_T(p,q)})\big]\|(\mathbf{w},\eta)\|_{\widetilde{\mathcal{V}}_T(p,q)},
\end{aligned}
$$

where $h_3(r) = r^{K^1-1} + r^{K^1}$. Since $\widetilde{\overline{\theta}} \in W^{2,1}_q(\Omega_T)$, we can take sufficiently small T_2 such that

$$C\| \partial_t\widetilde{\overline{\theta}}\,;\, L^q(\Omega_{T_2})\| h_3(M) \le \frac{1}{16}.$$

Therefore, if we take $T_3 \le T_2$ such that

$$C\| \mathcal{D}(\epsilon(\mathbf{u}),\theta)\,;\, L^\infty(\Omega_{T_3})\| < 1/2 \quad \text{and} \quad C T_3^{\frac{1}{q}} h_2(M) < 1/16,$$

then we arrive at

$$\begin{aligned}\| \Psi(\mathbf{u},&\theta) - \Psi(\overline{\mathbf{u}},\overline{\theta})\,;\, \mathcal{V}_T(p)\| \\ &\le \frac{1}{2}\| (\mathbf{u},\theta) - (\overline{\mathbf{u}},\overline{\theta})\,;\, \mathcal{V}_T(p)\|.\end{aligned}$$

Choosing the time $\delta = \min\{T_1, T_2, T_3\}$, we obtain the unique existence of the solution for (3.0.1)–(3.0.4) in $\widetilde{\mathcal{V}}^M_\delta(p,q)$. To prove the uniqueness in the whole of the space $\widetilde{\mathcal{V}}_\delta(p,q)$, it is enough to take δ sufficiently small. By the embedding (Lemma 3.1.1) we have $\widetilde{\mathcal{V}}_\delta(p,q) = V_\delta(p,q)$. It follows from the Banach fixed point principle that there exists a unique local solution $(\mathbf{u},\theta)$ for the system on small time interval $[0,\delta]$.

We know the embedding $B^{2-2/q}_{q,q} \hookrightarrow \mathrm{C}(\overline{\Omega})$ for $q > 5/2$, then from the above we have $\theta \in \mathrm{C}(\Omega_{[0,\delta]})$. Therefore, since if $f \in \mathrm{C}([0,\delta]) \cap \mathrm{C}^\alpha((0,T))$ then $f \in \mathrm{C}^\alpha([0,T))$, we have $\theta \in \mathrm{C}^{\alpha,\alpha/2}(\Omega_{[0,T)})$.

Step 2.

Using the definitions (3.3.35)–(3.3.36), the equation (3.0.2) can be rewritten as

$$c_0(\epsilon_0,\theta_0)\theta_t - \Delta\theta = \mathcal{D}(\epsilon,\theta)\theta_t + R(\epsilon,\theta).$$

It follows from the assumptions that

$$\begin{aligned}\| R(\epsilon,\theta)\,;\, L^q(\Omega_T)\| &\le C\| \theta\,;\, L^\infty(\Omega_T)\|^r \| F_{1,\epsilon}(\epsilon)\,;\, L^\infty(\Omega_T)\| \| \epsilon_t\,;\, L^q(\Omega_T)\| \\ &\quad + C\| \epsilon_t\,;\, L^{2q}(\Omega_T)\|^2 \\ &\le \Lambda.\end{aligned}$$

From the Hölder continuity of the solution $(\mathbf{u},\theta)$ on $\Omega_{[0,T]}$, it follows that

$$\| \mathcal{D}(\epsilon,\theta)\,;\, L^\infty(\Omega_{[0,T_1]})\| \le K T_1^{\frac{\alpha}{2}},$$

where K is the Hölder constant independent of T_1. Here, $T_1 \ll T$ will be determined later.

Next we show that for fixed T_2, $1/c_0(\epsilon,\theta)(x,T_2)$ is Hölder continuous with respect to the space variable. Noting that

$$\mathcal{G}(y) := yG''(y) \in [0,M]$$

and $\mathcal{G} \in \mathrm{C}^1$ is Lipschitz continuous, we have

$$\begin{aligned}
\left|\frac{1}{c_0}(x,T_2) - \frac{1}{c_0}(x',T_2)\right| &= \left|\frac{\mathcal{G}(\theta(x',T_2))F_1(\epsilon(x',T_2)) - \mathcal{G}(\theta(x,T_2))F_1(\epsilon(x,T_2))}{\{1-\mathcal{G}(\theta(x,T_2))F_1(\epsilon(x,T_2))\}\{1-\mathcal{G}(\theta(x',T_2))F_1(\epsilon(x',T_2))\}}\right| \\
&\le \Big|\big\{\mathcal{G}(\theta(x',T_2))F_1(\epsilon(x',T_2)) - \mathcal{G}(\theta(x,T_2))F_1(\epsilon(x',T_2))\big\} \\
&\qquad + \big\{\mathcal{G}(\theta(x,T_2))F_1(\epsilon(x',T_2)) - \mathcal{G}(\theta(x,T_2))F_1(\epsilon(x,T_2))\big\}\Big| \\
&\le |F_1(\epsilon(x',T_2))|\,|\mathcal{G}(\theta(x',T_2)) - \mathcal{G}(\theta(x,T_2))| \\
&\qquad + |\mathcal{G}(\theta(x,T_2))|\,|F_1(\epsilon(x',T_2)) - F_1(\epsilon(x,T_2))| \\
&\le \Lambda K|x-x'|^\alpha + CM|x-x'|^\alpha \\
&\le \Lambda|x-x'|^\alpha,
\end{aligned}$$

where Λ is independent of T_2. Therefore $[1/c_0(\epsilon,\theta)](x,T_2)$ is Hölder continuous for any $T_2 \in [0,T]$. Moreover, we have $\sup_{\Omega_T}[1/c_0(\epsilon,\theta)] \ge 1/(1+M\Lambda)$. These assure that $\frac{1}{c_0(\epsilon(T_2),\theta(T_2))}\Delta$ has the maximal regularity property according to Lemma 3.2.4. Hence, taking $T_1 = \left(\frac{1}{2\Lambda(K,M,T)K}\right)^{\frac{1}{\alpha}}$, we have

$$\begin{aligned}
\|\,\theta\,;\,W_q^{2,1}(\Omega_{T_1})\| &\le \Lambda(K,M,T)\big(\|\,\mathcal{D}(\epsilon,\theta)\,;\,L^\infty(\Omega_{T_1})\|\|\,\theta_t\,;\,L^q(\Omega_{T_1})\| \\
&\qquad + \|\,R(\epsilon,\theta)\,;\,L^q(\Omega_{T_1})\| + \|\theta_0\,;\,B_{q,q}^{2-2/q}(\Omega)\|\big) \\
&\le \frac{1}{2}\|\,\theta_t\,;\,L^q(\Omega_{T_1})\| + \Lambda + \Lambda\|\,\theta_0\,;\,B_{q,q}^{2-2/q}(\Omega)\|,
\end{aligned}$$

which yields

$$\|\,\theta\,;\,W_q^{2,1}(\Omega_{T_1})\| \le \Lambda + \Lambda\|\,\theta_0\,;\,B_{q,q}^{2-2/q}(\Omega)\|.$$

Here we remark that

$$\|\,\theta(T_1)\,;\,B_{q,q}^{2-2/q}\| \le C(T_1)\|\,\theta\,;\,W_q^{2,1}(\Omega_{T_1})\| \le C(T_1)(\Lambda + \Lambda\|\,\theta_0\,;\,B_{q,q}^{2-2/q}\|)$$

thanks to the embedding $W_q^{2,1}(\Omega_{T_1}) \hookrightarrow \mathrm{BUC}([0,T_1], B_{q,q}^{2-\frac{2}{q}})$ (see [7], [45]). Then similarly for the interval $[T_1, 2T_1]$ we have

$$\|\,\theta\,;\,W_q^{2,1}(\Omega_{[T_1,2T_1]})\| \le \Lambda + \Lambda\|\,\theta(T_1)\,;\,B_{q,q}^{2-2/q}\| \le \Lambda + \Lambda\|\,\theta_0\,;\,B_{q,q}^{2-2/q}\| \le \Lambda.$$

Repeating the same operation as above, we obtain

$$\|\,\theta\,;\,W_q^{2,1}(\Omega_{[kT_1,(k+1)T_1]})\| \le \Lambda.$$

Summing the inequalities from $k=0$ to $k=m$ satisfying $(m+1)T_1 > T$ and $mT_1 \le T$, we conclude that

$$\|\,\theta\,;W_q^{2,1}(\Omega_T)\| \le \Lambda.$$

Next we estimate the norm $\| \mathbf{u}\,;\, W_p^{4,2}(\Omega_T)\|$. From Lemma 3.1.1 it follows that

$$\| \nabla\theta\,;\, L^\infty(\Omega_T)\| + \| \nabla\epsilon\,;\, L^\infty(\Omega_T)\| \le \Lambda$$

for $q > 5$. Therefore, by virtue of the maximal regularity (3.2.2), we have

$$\begin{aligned}
\| \mathbf{u}\,;\, W_p^{4,2}(\Omega_T)\| &\le C\| (\mathbf{u}_0, \mathbf{u}_1, 0)\,;\, U(p,q)\| + \| \nabla \cdot F_{2,\epsilon}(\epsilon)\,;\, L^p(\Omega_T)\| \\
&\qquad + \| \nabla \cdot (G(\theta)F_{1,\epsilon}(\epsilon))\,;\, L^p(\Omega_T)\| \\
&\le C\| (\mathbf{u}_0, \mathbf{u}_1, 0)\,;\, U(p,q)\| + \Lambda\| \nabla\epsilon\,;\, L^\infty(\Omega_T)\|\| F_{2,\epsilon\epsilon}(\epsilon)\,;\, L^\infty(\Omega_T)\| \\
&\qquad + \Lambda\| \nabla\theta\,;\, L^\infty(\Omega_T)\|\| G'(\theta)\,;\, L^\infty(\Omega_T)\|\| F_{2,\epsilon}(\epsilon)\,;\, L^\infty(\Omega_T)\| \\
&\qquad + \Lambda\| \theta\,;\, L^\infty(\Omega_T)\|^r\| \nabla\epsilon\,;\, L^\infty(\Omega_T)\|\| F_{1,\epsilon\epsilon}(\epsilon)\,;\, L^\infty(\Omega_T)\| \\
&\le \Lambda(\| (\mathbf{u}_0, \mathbf{u}_1, 0)\,;\, U(p,q)\|),
\end{aligned}$$

which completes the proof.

□

Proof of Theorem 3.3.1 (*continuation*). The assumption ($L4$) is satisfied thanks to Lemma 3.3.7. Then the existence of a solution to the problem (3.3.1) results from Theorem 3.1.4. Noting that Γ_L is Lipschitz continuous, we can obtain the uniqueness result by repeating the proof of uniqueness theorem which we shall give in Section 3.6. Thereby the proof of Theorem 3.3.1 is completed. □

3.4 Global Existence

The idea of the proof consists in showing that the solution (u_L, θ_L) to the truncated problem (3.3.1) constructed in Section 3 satisfies also the original system (3.0.1)–(3.0.4) for sufficiently large truncation size L. To this purpose, assuming that there exists a sufficiently smooth solution of problem (3.0.1)–(3.0.4) such that $\theta \ge 0$, we derive for it a sequence of a priori estimates which are independent of L.

Lemma 3.4.1 (Energy Conservation Law). *Assume that $\theta \ge 0$ a.e. in Ω_T and that*

$$0 \le r \le 1, \qquad 0 \le K_1,\, K_2 \le 6, \qquad 6r + K_1 \le 6.$$

Then for any $t \in [0,T]$ a smooth solution of (3.0.1)–(3.0.4) *satisfies*

$$\begin{aligned}
&\| \theta(t)\,;\, L^1(\Omega)\| + \| \mathbf{u}_t(t)\,;\, L^2(\Omega)\| + \| Q\mathbf{u}(t)\,;\, L^2(\Omega)\| \\
&\qquad\qquad \le C(\| (\mathbf{u}_0, \mathbf{u}_1, \theta_0)\,;\, H^2 \times L^2 \times L^1\|).
\end{aligned} \tag{3.4.1}$$

Proof. Multiplying (3.0.1) by u_t and integrating the resulting equation with respect to the space variable, we have

$$\frac{d}{dt}\left(\frac{1}{2}\| \mathbf{u}_t\,;\, L^2(\Omega)\|^2+\frac{1}{2}\| Q\mathbf{u}\,;\, L^2\|^2+\int_\Omega F_2(\epsilon)dx\right)$$
$$= -\nu\int_\Omega (A\epsilon_t):\epsilon_t dx - \int_\Omega G(\theta)\frac{\partial}{\partial t}F_1(\epsilon)dx.$$

Integrating (3.0.2) over Ω, we obtain

$$\frac{d}{dt}\int_\Omega \theta dx = \nu\int_\Omega (A\epsilon_t):\epsilon_t dx + \int_\Omega \theta G'(\theta)\frac{\partial}{\partial t}F_1(\epsilon)dx + \int_\Omega \theta G''(\theta)\theta_t F_1(\epsilon)dx.$$

Combining these equalities, we deduce

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}_t\,;\, L^2(\Omega)\|^2+\frac{1}{2}\|Q\mathbf{u}\,;\, L^2\|^2+\int_\Omega \theta dx+\int_\Omega F_2(\epsilon)dx\right)$$
$$= \int_\Omega\left(\theta G'(\theta)\frac{\partial}{\partial t}F_1(\epsilon)+\theta G''(\theta)\theta_t F_1(\epsilon) - G(\theta)\frac{\partial}{\partial t}F_1(\epsilon)\right)dx$$
$$= -\frac{d}{dt}\int_\Omega \overline{G}(\theta)F_1(\epsilon)dx,$$

where $\overline{G}(\theta) = G(\theta)-\theta G'(\theta)$. Consequently, we have

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}_t\,;\, L^2\|^2+\frac{1}{2}\|Q\mathbf{u}\,;\, L^2\|^2+\int_\Omega \theta dx+\int_\Omega F_2(\epsilon)dx+\int_\Omega \overline{G}(\theta)F_1(\epsilon)dx\right)=0.$$

Here we recall that $\theta \geq 0$ and $F_1(\epsilon)\geq 0$. By the structure of $G(\theta)$ the function $\overline{G}(\theta)$ is as follows:

$$\overline{G}(r) = \begin{cases} 0 & \text{if } \theta\in[0,\theta_1], \\ \varphi(\theta)-\theta\varphi'(\theta) & \text{if } \theta\in[\theta_1,\theta_2], \\ C_2(1-r)\theta^r & \text{if } \theta\in[\theta_2,\infty). \end{cases}$$

Since from Lemma 3.1.2 we have $\overline{G}(\theta)\geq 0$. Consequently, it follows from the structure of the nonlinearity $(N3)$ that

$$\frac{1}{2}\| \mathbf{u}_t(t)\,;\, L^2(\Omega)\|^2+\frac{1}{2}\| \mathbf{u}(t)\,;\, H^2(\Omega)\|^2+\| \theta(t)\,;\, L^1(\Omega)\|$$
$$\leq \frac{1}{2}\| \mathbf{u}_0\,;\, H^2\|^2+\frac{1}{2}\| \mathbf{u}_1\,;\, L^2(\Omega)\|^2+\| \theta_0\,;\, L^1(\Omega)\|+\int_\Omega |F_2(\epsilon_0)|dx + C_3|\Omega|$$
$$+\int_{\{\theta_2\geq\theta_0\geq\theta_1\}\cap\Omega}[\varphi(\theta_0)-\theta_0\varphi'(\theta_0)]F_1(\epsilon_0)dx + C_2(1-r)\int_{\{\theta_0>\theta_2\}\cap\Omega}\theta_0^r F_1(\epsilon_0)dx,$$

where $\epsilon_0 = \epsilon(u_0)$. Since the smooth function $\varphi(s) - s\varphi'(s)$ is bounded for $s \in [\theta_1, \theta_2]$, we have

$$\int_{\{\theta_2 \geq \theta_0 \geq \theta_1\}\cap\Omega} [\varphi(\theta_0) - \theta_0\varphi'(\theta_0)]F_1(\epsilon_0)dx \leq C\int_\Omega |\epsilon_0|^{K_1}dx \leq C\|\mathbf{u}_0\|_{H^2}^{K_1}$$

for $K_1 \leq 6$,

$$\int_{\{\theta_0 > \theta_2\}\cap\Omega} \theta_0^r H(\epsilon_0)dx \leq C\|\,\theta_0\,;\,L^1(\Omega)\|^r\|\,\epsilon_0\,;\,L^{\frac{K_1}{1-r}}(\Omega)\|^{K_1} \leq C\|\,\theta_0\,;\,L^1(\Omega)\|^r\|\,\mathbf{u}_0\,;\,H^2(\Omega)\|^{K_1}$$

for $6r + K_1 \leq 6$ and

$$\int_\Omega |F_2(\epsilon_0)|dx \leq \|\mathbf{u}_0\|_{H^2}^{K_2}$$

for $K_2 \leq 6$. Hence we conclude the assertion. □

Lemma 3.4.2. *Assume that* $\theta \geq 0$ *a.e. in* Ω_T *and that* (3.0.6) *holds. Then for any* $(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \in B^{19/8}_{16/5,16/5} \times B^{3/8}_{16/5,16/5} \times L^2$, *the solution* $(\mathbf{u}, \theta)$ *to* (3.0.1)–(3.0.4) *satisfies*

$$\|\,\epsilon\,;\,W^{2,1}_{16/5}(\Omega_T)\| + \|\,\nabla\theta\,;\,L^2(\Omega_T)\| + \|\,\theta\,;\,L^\infty_T L^2\| \leq \Lambda, \tag{3.4.2}$$

where Λ *depends on* T *and* $\|\,(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\,;\,B^{19/8}_{16/5,16/5} \times B^{3/8}_{16/5,16/5} \times L^2\|$. *Moreover we have*

$$\|\,\epsilon\,;\,L^\infty(\Omega_T)\| + \|\,\theta\,;\,L^{10/3}(\Omega_T)\| \leq \Lambda. \tag{3.4.3}$$

Proof. Remark that from the embedding (see [1]) we have

$$\|\,(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\,;\,H^2 \times L^2 \times L^1\| \leq C\|\,(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\,;\,B^{19/8}_{16/5,16/5} \times B^{3/8}_{16/5,16/5} \times L^2\|.$$

From the Gagliardo-Nirenberg inequality and Lemma 3.4.1 it follows that

$$\begin{aligned}\|\,\epsilon\,;\,L^{5p}(\Omega_T)\| &\leq C\left\|\,\|\,\epsilon\,;\,L^6(\Omega)\|^{\frac{4}{5}}\|\,\epsilon\,;\,W^2_p(\Omega)\|^{\frac{1}{5}}\,;\,L^{5p}_T\right\| \\ &\leq C\|\,\epsilon\,;\,L^\infty_T L^6\|^{\frac{4}{5}}\|\,\epsilon\,;\,W^{2,1}_p(\Omega_T)\|^{\frac{1}{5}} \\ &\leq C\|\,\mathbf{u}\,;\,L^\infty_T H^2\|^{\frac{4}{5}}\|\,\epsilon\,;\,W^{2,1}_p(\Omega_T)\|^{\frac{1}{5}} \\ &\leq C\|\,\epsilon\,;\,W^{2,1}_p(\Omega_T)\|^{\frac{1}{5}}\end{aligned} \tag{3.4.4}$$

and

$$\begin{aligned}\|\,\theta\,;\,L^{8/3}(\Omega_T)\| &\leq C\left\|\,\|\,\theta\,;\,L^1(\Omega)\|^{\frac{1}{4}}\|\,\theta\,;\,H^1(\Omega)\|^{\frac{3}{4}}\,;\,L^\infty_T\right\| \\ &\leq C\|\,\theta\,;\,L^\infty_T L^1\|^{\frac{1}{4}}\|\,\theta\,;\,L^2_T H^1\|^{\frac{3}{4}} \\ &\leq \Lambda(\|\,\nabla\theta\,;\,L^2(\Omega_T)\| + \|\,\theta\,;\,L^\infty_T L^2\|)^{\frac{3}{4}}.\end{aligned} \tag{3.4.5}$$

It follows from (3.4.3) that

$$\begin{aligned}\| F_{2,\epsilon}(\epsilon)\,;\, L^{16/5}(\Omega_T)\| &\le C\|\,\epsilon\,;\, L^{16}(\Omega_T)\|^{K_2-1}\\ &\le \Lambda\|\,\epsilon\,;\, W^{2,1}_{16}(\Omega_T)\|^{\frac{K_2-1}{5}}\\ &\le \frac{1}{4}\|\,\epsilon\,;\, W^{2,1}_{16}(\Omega_T)\| + \Lambda\end{aligned}$$

for $K_2 \in [1,6)$ and

$$\| F_{2,\epsilon}(\epsilon)\,;\, L^{16/5}(\Omega_T)\| \le M|\Omega_T|^{\frac{5}{16}} \le \Lambda$$

for $K_2 \in [0,1)$.

We first consider the case of $K_1 \ge 1$. Applying the growth condition and the Young inequality, we have

$$\begin{aligned}\| G(\theta)F_{1,\epsilon}(\epsilon)\,;\, L^{\frac{16}{5}}(\Omega_T)\| &\le \|\,\theta\,;\, L^{\frac{8}{3}}(\Omega_T)\|^r\|\,\epsilon\,;\, L^{\frac{16(K_1-1)}{5-6r}}(\Omega_T)\|^{K_1-1}\\ &\quad+\sup_{\theta\in[0,\theta_2]}|G(\theta)|\|\,\epsilon\,;\, L^{\frac{16(K_1-1)}{5}}(\Omega_T)\|^{K_1-1}\\ &\le \Lambda\|\,\theta\,;\, L^{\frac{8}{3}}(\Omega_T)\|^r\|\,\epsilon\,;\, L^{16}(\Omega_T)\|^{K_1-1}+\Lambda\|\,\epsilon\,;\, L^{16}(\Omega_T)\|^{K_1-1}\end{aligned}$$

for $6r+K_1 \le 6$ (and $K_1 \le 6$). Then we have

$$\begin{aligned}(1+\|\,\theta\,;\, L^{8/3}(\Omega_T)\|^r)\|\,\epsilon\,;\, L^{16}(\Omega_T)\|^{K_1-1} &\le \Lambda\|\,\epsilon\,;\, W^{2,1}_{16/5}(\Omega_T)\|^{(K_1-1)/5}+\Lambda\big(\|\,\nabla\theta\,;\, L^2(\Omega_T)\|\\ &\quad+\|\,\theta\,;\, L^\infty_T L_2\|\big)^{3r/4}\|\,\epsilon\,;\, W^{2,1}_{16/5}(\Omega_T)\|^{(K_1-1)/5}\\ &\le \frac{1}{4}\|\,\epsilon\,;\, W^{2,1}_{16/5}(\Omega_T)\|+\Lambda\\ &\quad+\Lambda(\|\,\nabla\theta\,;\, L^2(\Omega_T)\|+\|\,\theta\,;\, L^\infty_T L_2)\|^{\frac{15r}{4(6-K_1)}}\end{aligned}$$

for $6r+K_1 < 6$ (and $K_1 < 6$). From the maximal regularity (3.2.3) it follows that

$$\begin{aligned}\|\,\epsilon\,;\, W^{2,1}_{16/5}(\Omega_T)\| &\le C\|\,(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\,;\, B^{19/8}_{16/5,16/5}\times B^{3/8}_{16/5,16/5}\times L^2\|\\ &\quad+\|\,G(\theta)F_{1,\epsilon}(\epsilon)\,;\, L^{16/5}(\Omega_T)\|+\|\,F_{2,\epsilon}(\epsilon)\,;\, L^{16/5}(\Omega_T)\|\\ &\le C\|\,(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\,;\, B^{19/8}_{16/5,16/5}\times B^{3/8}_{16/5,16/5}\times L^2\|+\Lambda\\ &\quad+\Lambda(\|\,\nabla\theta\,;\, L^2(\Omega_T)\|+\|\,\theta\,;\, L^\infty_T L_2\|)^{\frac{15r}{4(6-K_1)}}.\end{aligned}\tag{3.4.6}$$

Next, multiplying (3.0.2) by θ and integrating over Ω, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\| \theta(t)\,;\, L^2(\Omega)\|^2 + \| \nabla\theta\,; L^2(\Omega)\|^2 &= \int_\Omega \theta^2 G''(\theta)\theta_t F_1(\epsilon)dx \\
&\quad + \int_\Omega \theta^2 G'(\theta)\partial_t F_1(\epsilon)dx + \nu\int_\Omega \theta A\epsilon_t : \epsilon_t dx \\
&= \int_\Omega G_2'(\theta)\theta_t F_1(\epsilon)dx + \int_\Omega G_2(\theta)\partial_t F_1(\epsilon)dx \\
&\quad + 2\int_\Omega \overline{G}_2(\theta)\partial_t F_1(\epsilon)dx + \nu\int_\Omega \theta A\epsilon_t : \epsilon_t dx \\
&= \frac{d}{dt}\int_\Omega G_2(\theta)F_1(\epsilon)dx + 2\int_\Omega \overline{G}_2(\theta)\partial_t F_1(\epsilon)dx \\
&\quad + \nu\int_\Omega \theta A\epsilon_t : \epsilon_t dx,
\end{aligned} \tag{3.4.7}$$

where $G_2(\theta)$ and $\overline{G}_2(\theta)$ are given in the proof of Lemma 3.3.2. Recall that

$$G_2(\theta) = \frac{C_2 r(r-1)}{r+1}\theta^{r+1} \le 0 \qquad \text{and} \qquad \overline{G}_2(\theta) = \frac{2C_2 r}{r+1}\theta^{r+1} \qquad \text{for } \theta \ge \theta_2,$$

and

$$\sup_{\theta\in[0,\theta_2]} |G_2(\theta)| + \sup_{\theta\in[0,\theta_2]} |\overline{G}_2(\theta)| =: M < \infty.$$

Then we have

$$\begin{aligned}
-\int_\Omega G_2(\theta)F_1(\epsilon)dx &= -\int_{\Omega\cap\{\theta\ge\theta_2\}} G_2(\theta)F_1(\epsilon)dx - \int_{\Omega\cap\{\theta_1\le\theta\le\theta_2\}} G_2(\theta)F_1(\epsilon)dx \\
&\ge -M\int_\Omega |F_1(\epsilon)|dx.
\end{aligned}$$

Hence, integrating (3.4.7) with respect to time variable, we obtain

$$\begin{aligned}
\frac{1}{2}\| \theta\,; L^\infty_T L^2\|^2 + \| \nabla\theta\,; L^2(\Omega_T)\|^2 &\le \frac{1}{2}\| \theta_0\,; L^2(\Omega)\|^2 + \| \overline{G}_2(\theta)\partial_t F_1(\epsilon)\,; L^1(\Omega_T)\| \\
&\quad + \nu\| \theta A\epsilon_t : \epsilon_t\,; L^1(\Omega_T)\| + M\sup_{t\in[0,T]}\int_\Omega |F_1(\epsilon(t))|dx \\
&\quad + \int_\Omega |G_2(\theta_0)F_1(\epsilon_0)|dx.
\end{aligned}$$

By (3.4.3), (3.4.5) and the assumptions we have

$$\begin{aligned}
\| \theta^{r+1}\partial_t F_1(\epsilon)\,; L^1(\Omega_T)\| &\le \Lambda\| \theta\,; L^{8/3}(\Omega_T)\|^{r+1}\| \mathbf{u}\,; W^{2,1}_{16/5}(\Omega_T)\|\| \epsilon\,; L^{16}(\Omega_T)\|^{K_1-1} \\
&\le \Lambda(\| \nabla\theta\,; L^2(\Omega_T)\| + \| \theta\,; L^\infty_T L_2\|)^{\frac{3(r+1)}{4}}\| \mathbf{u}\,; W^{2,1}_{16/5}(\Omega_T)\|^{1+\frac{K_1-1}{5}},
\end{aligned}$$

$$\|\,\theta A\epsilon_t : \epsilon_t\,;\,L^1(\Omega_T)\| \le C\|\,\theta\,;\,L^{\frac{8}{3}}(\Omega_T)\|\|\,\epsilon_t\,;\,L^{\frac{16}{5}}(\Omega_T)\|^2$$
$$\le \Lambda(\|\,\nabla\theta\,;\,L^2(\Omega_T)\| + \|\,\theta\,;\,L^\infty_T L_2\|)^{\frac{3}{4}}\|\,\epsilon_t\,;\,L^{\frac{16}{5}}(\Omega_T)\|^2,$$

$$\int_\Omega |F_1(\epsilon(t))|dx \le C\|\,\mathbf{u}(t)\,;\,H^2(\Omega)\|^{K_1} \le \Lambda$$

and

$$\|\,\theta_0^{r+1}F_1(\epsilon_0)\,;\,L^1(\Omega)\| \le C\|\,\theta_0\,;\,L^2(\Omega)\|^{r+1}\|\,\epsilon_0\,;\,L^{\frac{2K_1}{1-r}}(\Omega)\|^{K_1}$$
$$\le C\|\,\theta_0\,;\,L^2(\Omega)\|^{r+1}\|\,\mathbf{u}_0;H^2(\Omega)\|^{K_1}.$$

Consequently we arrive at

$$\begin{aligned}\|\,\theta\,;\,L^\infty_T L^2\|^2 + \|\,\nabla\theta\,;\,L^2(\Omega_T)\|^2 &\le \Lambda(\|\,(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\,;\,B^{19/8}_{16/5,16/5}\times B^{3/8}_{16/5,16/5}\times L^2\|)\\ &+\Lambda\big(\|\,\nabla\theta\,;\,L^2(\Omega_T)\| + \|\,\theta\,;\,L^\infty_T L_2\|\big)^{\frac{3(r+1)}{4}}\|\,\epsilon\,;\,W^{2,1}_{16/5}(\Omega_T)\|^{\frac{4}{5}+\frac{K_1}{5}} \qquad (3.4.8)\\ &+\Lambda(\|\,\nabla\theta\,;\,L^2(\Omega_T)\| + \|\,\theta\,;\,L^\infty_T L_2\|)^{\frac{3}{4}}\|\,\epsilon_t\,;\,L^{\frac{16}{5}}(\Omega_T)\|^2.\end{aligned}$$

Substituting (3.4.6) into (3.4.8), we have

$$\begin{aligned}&\|\,\theta\,;\,L^\infty_T L^2\|^2 + \|\,\nabla\theta\,;\,L^2(\Omega_T)\|^2 \le \Lambda(\|\,(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\,;\,B^{19/8}_{16/5,16/5}\times B^{3/8}_{16/5,16/5}\times L^2\|)\\ &\quad+\Lambda\big(\|\,\nabla\theta\,;\,L^2(\Omega_T)\| + \|\,\theta\,;\,L^\infty_T L_2\|\big)^{\frac{3(r+1)}{4}}\\ &\quad\times\Big(\|\,(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\,;\,B^{19/8}_{16/5,16/5}\times B^{3/8}_{16/5,16/5}\times L^2\| + \|\,\nabla\theta\,;\,L^2(\Omega_T)\|^{\frac{15r}{4(6-K_1)}}\Big)^{\frac{4}{5}+\frac{K_1}{5}}\\ &\quad+\Lambda(\|\,\nabla\theta\,;\,L^2(\Omega_T)\| + \|\,\theta\,;\,L^\infty_T L_2\|)^{\frac{3}{4}}\\ &\quad\times\Big(\|\,(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\,;\,B^{19/8}_{16/5,16/5}\times B^{3/8}_{16/5,16/5}\times L^2\| + \|\,\nabla\theta\,;\,L^2(\Omega_T)\|^{\frac{15r}{4(6-K_1)}}\Big)^2.\end{aligned}$$

Here from the assumption $6r + K_1 < 6$ it follows that

$$\frac{3(r+1)}{4} + \frac{15r}{4(6-K_1)}\left(\frac{4}{5}+\frac{q}{5}\right) = \frac{30r + 3(6-K_1)}{4(6-K_1)} < \frac{5(6-K_1)+3(6-K_1)}{4(6-K_1)} = 2,$$
$$\frac{3}{4} + \frac{30r}{4(6-K_1)} < \frac{3}{4}+\frac{5}{4} = 2.$$

Thus we obtain

$$\begin{aligned}&\|\,\theta\,;\,L^\infty_T L^2\| + \|\,\nabla\theta\,;\,L^2(\Omega_T)\|\\ &\quad\le \Lambda(\|\,(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\,;\,B^{19/8}_{16/5,16/5}\times B^{3/8}_{16/5,16/5}\times L^2\|) + \Lambda\|\,\nabla\theta\,;\,L^2(\Omega_T)\|^{1-}.\end{aligned}$$

Here we use $p-$ to denote a number less than p. Hence by the Young inequality we have

$$\|\,\theta\,;\,L^\infty_T L^2\| + \|\,\nabla\theta\,;\,L^2(\Omega_T)\| \le \Lambda(\|\,(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\,;\,B^{19/8}_{16/5,16/5}\times B^{3/8}_{16/5,16/5}\times L^2\|).$$

Substituting the above inequality into (3.4.6), we also obtain the following

$$\|\,\epsilon\,;\,W^{2,1}_{16/5}(\Omega_T)\| \le \Lambda(\|\,(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\,;\,B^{19/8}_{16/5,16/5}\times B^{3/8}_{16/5,16/5}\times L^2\|).$$

Next, we consider the case of $0 \le K_1 \le 1$ and $0 \le r < 5/6$. In this case it follows that

$$|F_{1,\epsilon}(\epsilon)| \le C < \infty.$$

From an argument similar to the above we have

$$\begin{aligned}
\|\,\epsilon\,;\,W^{2,1}_{16/5}(\Omega_T)\| &\le \|\,(\mathbf{u}_0,\mathbf{u}_1,0)\,;\,B^{19/8}_{16/5,16/5}\times B^{3/8}_{16/5,16/5}\times L^2\| \\
&\quad + \|\,G(\theta)F_{1,\epsilon}(\epsilon)\,;\,L^{16/5}(\Omega_T)\| \\
&\le \|\,(\mathbf{u}_0,\mathbf{u}_1,0)\,;\,B^{19/8}_{16/5,16/5}\times B^{3/8}_{16/5,16/5}\times L^2\| \\
&\quad + C\|\,\theta\,;\,L^{\frac{16r}{5}}(\Omega_T)\|^r + C\sup_{\theta\in[0,\theta_2]} G(\theta) \\
&\le \|\,(\mathbf{u}_0,\mathbf{u}_1,0)\,;\,B^{19/8}_{16/5,16/5}\times B^{3/8}_{16/5,16/5}\times L^2\| \\
&\quad + \Lambda\|\,\theta\,;\,L^{\frac{8}{3}}(\Omega_T)\|^r + C.
\end{aligned} \tag{3.4.9}$$

Noting that

$$\|\,\theta^{r+1}\partial_t F_1(\epsilon)\,;\,L^1(\Omega_T)\| \le \Lambda\|\,\theta\,;\,L^{8/3}(\Omega_T)\|^{r+1}\|\,\mathbf{u}\,;\,W^{2,1}_{16/5}(\Omega_T)\|,$$

we obtain

$$\begin{aligned}
\|\,\theta\,;\,L^\infty_T L^2\|^2 + \|\,\nabla\theta\,;L^2(\Omega_T)\|^2 &\le \|\,\theta_0\,;\,L^2\|^2 + \|\,\theta^{r+1}\partial_t F_1(\epsilon)\,;\,L^1(\Omega_T)\| \\
&\quad + \|\,\theta A\epsilon_t : \epsilon_t\,;\,L^1(\Omega_T)\| \\
&\quad + M\sup_{t\in[0,T]}\int_\Omega |F_1(\epsilon(t))|dx + \int_\Omega |G_2(\theta_0)F_1(\epsilon_0)|dx \\
&\le \Lambda(\|\,(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\,;\,B^{19/8}_{16/5,16/5}\times B^{3/8}_{16/5,16/5}\times L^2\|) \\
&\quad + \Lambda\|\,\theta\,;\,L^{8/3}(\Omega_T)\|^{r+1}\|\,\mathbf{u}\,;\,W^{2,1}_{16/5}(\Omega_T)\| \\
&\quad + C\|\,\theta\,;\,L^{8/3}(\Omega_T)\|\|\,\mathbf{u}\,;\,W^{2,1}_{16/5}(\Omega_T)\|^2 \\
&\le \Lambda(\|\,(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\,;\,B^{19/8}_{16/5,16/5}\times B^{3/8}_{16/5,16/5}\times L^2\|) \\
&\quad + \Lambda(\|\,\nabla\theta\,;\,L^2(\Omega_T)\| + \|\,\theta\,;\,L^\infty_T L^2\|)^{3(2r+1)/4}.
\end{aligned}$$

Since $3(2r+1)/4 < 2$, we obtain the desired estimate (3.4.2).

The estimate (3.4.3) follows with the help of the embedding

$$\|\,\epsilon\,;\,L^\infty(\Omega_T)\| \le \Lambda\|\,\epsilon\,;\,W^{2,1}_{16/5}(\Omega_T)\|$$

and of the inequality

$$\|\,\theta\,;\,L^{10/3}(\Omega_T)\| \le C\left\|\,\|\,\theta\,;\,L^2(\Omega)^{2/5}\|\|\,\theta\,;\,H^1(\Omega)^{3/5}\|\,;\,L_T^{10/3}\right\|$$
$$\le C\|\,\theta\,;\,L_T^\infty L^2\|^{2/5}\|\,\theta\,;\,L^2H^1\|^{3/5}.$$

This completes the proof. □

Lemma 3.4.3. *Assume that $\theta \ge 0$ a.e. in Ω_T and that* (3.0.6) *holds. Then for any $(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \in B^{5/2}_{4,4} \times B^{1/2}_{4,4} \times H^1$ the following estimate holds*

$$\|\,\epsilon\,;\,W_4^{2,1}(\Omega_T)\| + \|\,\nabla\theta\,;\,L_T^\infty L^2\| + \|\,\theta\,;\,W_2^{2,1}(\Omega_T)\| \le \Lambda,$$

where constant Λ depends on T and $\|\,(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\,;\,B^{5/2}_{4,4} \times B^{1/2}_{4,4} \times H^1\|$. Moreover, we have

$$\|\,\nabla\theta\,;\,L^{10/3}(\Omega_T)\| + \|\,\theta\,;\,L^{10}(\Omega_T)\| + \|\,\nabla\epsilon\,;\,L^{20}(\Omega_T)\| \le \Lambda.$$

Proof. Remark that $B^{5/2}_{4,4} \times B^{1/2}_{4,4} \times H^1 \hookrightarrow B^{19/8}_{16/5,16/5} \times B^{3/8}_{16/5,16/5} \times L^2$. Set $r \le 5/6$. From (3.4.3), we have

$$\|\,G(\theta)F_{1,\epsilon}(\epsilon)\,;\,L^4(\Omega_T)\| \le \|\,\theta\,;\,L^{10/3}(\Omega_T)\|^r\|\,\epsilon\,;\,L^\infty(\Omega_T)\|^{K_1-1}$$

for $K_1 \ge 1$, and

$$\|\,G(\theta)F_{1,\epsilon}(\epsilon)\,;\,L^4(\Omega_T)\| \le \Lambda \sup|F_{1,\epsilon}|\|\,\theta\,;\,L^{10/3}(\Omega_T)\|^r$$

for $K_1 \le 1$. Then we arrive at

$$\|\,G(\theta)F_{1,\epsilon}(\epsilon)\,;\,L^4(\Omega_T)\| \le \Lambda. \tag{3.4.10}$$

From the maximal regularity (3.2.3) it follows that

$$\|\,\epsilon\,;\,W_4^{2,1}\| \le \|\,(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\,;\,B^{5/2}_{4,4} \times B^{1/2}_{4,4} \times H^1\| + \|\,G(\theta)F_{1,\epsilon}(\epsilon)\,;\,L^4(\Omega)\| \le \Lambda. \tag{3.4.11}$$

Multiplying (3.0.2) by θ_t and integrating over Ω_T, we get

$$\begin{aligned}
\|\,\theta_t\,;\,L^2(\Omega_T)\|^2 + \frac{1}{2}\|\,\nabla\theta\,;\,L_T^\infty L^2\|^2 &\le \frac{1}{2}\|\,\theta_0\,;\,H^1(\Omega)\|^2 + \iint_{\Omega_T} \theta_t^2\theta G''(\theta)F_1(\epsilon)dxdt \\
&\quad + \iint_{\Omega_T} \theta_t\theta G'(\theta)\partial_t F_1(\epsilon)dxdt + \iint_{\Omega_T} \theta_t A\epsilon_t : \epsilon_t dxdt \\
&\le \frac{1}{2}\|\,\theta_0\,;\,H^1(\Omega)\|^2 + C\|\,\theta_t\,;\,L^2(\Omega_T)\|\|\,\theta^r F_{1,\epsilon}(\epsilon)\,;\,L^4(\Omega)\|\|\,\epsilon_t\,;\,L^4(\Omega)\| \\
&\quad + C\|\,\theta_t\,;\,L^2(\Omega)\|\|\,\epsilon_t\,;\,L^4(\Omega)\|^2 \\
&\le \frac{1}{2}\|\,\theta_0\,;\,H^1(\Omega)\|^2 + \Lambda\|\,\theta_t\,;\,L^2(\Omega_T)\| \\
&\le \Lambda(\|\,(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\,;\,B^{5/2}_{4,4} \times B^{1/2}_{4,4} \times H^1\|) + \frac{1}{2}\|\,\theta_t\,;\,L^2(\Omega_T)\|^2,
\end{aligned}$$

where we applied (3.4.3), (3.4.10) and (3.4.11). Therefore we arrive at

$$\begin{aligned} \| \epsilon ; W_4^{2,1}(\Omega_T)\| + \| \theta_t ; L^2(\Omega_T)\| + \| \nabla\theta ; L_T^\infty L^2\| \\ \le \Lambda(\| (\mathbf{u}_0, \mathbf{u}_1, \theta_0) ; B_{4,4}^{5/2} \times B_{4,4}^{1/2} \times H^1\|). \end{aligned} \tag{3.4.12}$$

Next multiplying (3.0.2) by $\frac{-\Delta\theta}{1-\theta G''(\theta)F_1(\epsilon)}$ and integrating over Ω, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\| \nabla\theta(t) ; L^2\|^2 + \int_\Omega \frac{|\Delta\theta|^2}{1-\theta G''(\theta)F_1(\epsilon)} dx \\ \le \int_\Omega \frac{\Delta\theta}{1-\theta G''(\theta)F_1(\epsilon)} \left(\theta G'(\theta)\partial_t F_1(\epsilon) + \nu A\epsilon_t : \epsilon_t\right) dx. \end{aligned}$$

Here we recall that

$$1 \le 1 - \theta G''(\theta)F_1(\epsilon) \le 1 + M\Lambda,$$

where $0 \le \sup_{\theta\ge0}(-\theta G''(\theta)) =: M < \infty$. Then integrating with respect to time variable, we conclude that

$$\begin{aligned} &\| \nabla\theta(t) ; L^2(\Omega)\|^2 + \frac{2}{1+\Lambda M}\| \Delta\theta ; L^2(\Omega_T)\|^2 \\ &\le \| \nabla\theta_0 ; L^2(\Omega)\|^2 + \frac{1}{1+\Lambda M}\| \Delta\theta ; L^2(\Omega_T)\|^2 \\ &\quad + (1+\Lambda M) \left\| \theta G'(\theta)\partial_t F_1(\epsilon) + A\epsilon_t : \epsilon_t ; L^2(\Omega_T)\right\|^2 \\ &\le \Lambda + \frac{1}{1+\Lambda M}\| \Delta\theta ; L^2(\Omega_T)\|^2 + \Lambda(M)\| \epsilon_t ; L^4(\Omega_T)\|^2 \\ &\quad + \Lambda(M)\| \theta^r F_{1,\epsilon}(\epsilon) ; L^4(\Omega_T)\| \|\epsilon_t ; L^4(\Omega_T)\| \\ &\le \Lambda + \frac{1}{2(1+\Lambda M)}\| \Delta\theta ; L^2(\Omega_T)\|^2 \end{aligned}$$

due to (3.4.10) and (3.4.11). Consequently we obtain the first assertion.

With the help of Lemma 3.1.1, we also obtain

$$\begin{aligned} \| \nabla\theta ; L^{10/3}(\Omega_T)\| + \| \theta ; L^{10}(\Omega_T)\| + \| \nabla\epsilon ; L^{20}(\Omega_T)\| \\ \le \Lambda(\| \theta ; W_2^{2,1}(\Omega_T)\| + \| \epsilon ; W_4^{2,1}(\Omega_T)\|) \\ \le \Lambda, \end{aligned}$$

which completes the proof. □

Lemma 3.4.4. *Assume that* $\theta \ge 0$ *a.e. in* Ω_T *and that* (3.0.6) *holds. Then for any* $(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \in B_{p,p}^{4-2/p} \times B_{p,p}^{2-2/p} \times H^1$ *with* $p \in [20/9, 10/3]$, *the solution* $(\mathbf{u}, \theta)$ *to* (3.0.1)–(3.0.4) *satisfies*

$$\| \mathbf{u} ; W_p^{4,2}(\Omega_T)\| \le \Lambda,$$

where Λ *depends on* T *and* $\| (\mathbf{u}_0, \mathbf{u}_1, \theta_0) ; B_{p,p}^{4-2/p} \times B_{p,p}^{2-2/p} \times H^1\|$.

Proof. Since the embedding $B_{p,p}^{4-\frac{2}{p}} \hookrightarrow B_{4,4}^{\frac{5}{2}}$ holds for any $\frac{20}{9} \le p$, by the Lemma 3.4.3 we have

$$\begin{aligned}\|\,\epsilon\,;\,W_4^{2,1}(\Omega_T)\| + \|\,\theta\,;\,W_2^{2,1}(\Omega_T)\| &\le \Lambda(\|\,(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\,;\,B_{4,4}^{5/2}\times B_{4,4}^{1/2}\times H^1\|)\\ &\le \Lambda(\|\,(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\,;\,B_{p,p}^{4-2/p}\times B_{p,p}^{2-2/p}\times H^1\|).\end{aligned}$$

For any $p \le \frac{10}{3}$ we have

$$\begin{aligned}\|\,\nabla\cdot(G(\theta)F_{1,\epsilon}(\epsilon))\,;\,L^p(\Omega_T)\| &\le \Lambda\|\,\nabla\theta\,;\,L^{10/3}(\Omega_T)\|\|\,G'(\theta)\,;\,L^\infty(\Omega_T)\|\|\,F_{1,\epsilon}(\epsilon)\,;\,L^\infty(\Omega_T)\|\\ &\quad+\Lambda\|\,\theta\,;\,L^{10}(\Omega_T)\|^r\|\,\nabla\epsilon\,;\,L^{20}(\Omega_T)\|\|\,F_{1,\epsilon\epsilon}(\epsilon)\,;\,L^\infty(\Omega_T)\|\\ &\le \Lambda\end{aligned}$$

and

$$\|\,\nabla\cdot F_{2,\epsilon}(\epsilon)\,;\,L^p(\Omega_T)\| \le \Lambda\|\,\nabla\epsilon\,;\,L^{20}(\Omega_T)\|\|\,F_{2,\epsilon\epsilon}(\epsilon)\,;\,L^\infty(\Omega_T)\| \le \Lambda,$$

thanks to Lemmas 3.4.2 and 3.4.3. Then by the maximal regularity (3.2.3) we have

$$\begin{aligned}\|\,\mathbf{u}\,;\,W_p^{4,2}(\Omega_T)\| &\le C\|\,(\mathbf{u}_0,\mathbf{u}_1,0)\,;\,B_{p,p}^{4-2/p}\times B_{p,p}^{2-2/p}\times H^1\|\\ &\quad+C(\|\,\nabla\cdot(G(\theta)F_{1,\epsilon}(\epsilon))\,;\,L^p(\Omega_T)\|+\|\,\nabla\cdot F_{2,\epsilon}(\epsilon)\,;\,L^p(\Omega_T)\|)\\ &\le \Lambda(\|\,(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\,;\,B_{p,p}^{4-2/p}\times B_{p,p}^{2-2/p}\times H^1\|).\end{aligned}$$

This completes the proof. □

To shorten the notation we write

$$\begin{aligned}U_1(m) &= B_{10/3,10/3}^{17/5}\times B_{10/3,10/3}^{7/5}\times(L^m\cap H^1),\\ U_2 &= (B_{p,p}^{3-2/p}\cap B_{10/3,10/3}^{17/5})\times(B_{p,p}^{1-2/p}\cap B_{10/3,10/3}^{7/5})\times(L^\infty\cap H^1).\end{aligned}$$

Lemma 3.4.5. *Let $m > 2$ be arbitrary integer and $p \in (1,\infty)$. Assume that $\theta \ge 0$ a.e. in Ω_T and that (3.0.6) holds. Then for any $(\mathbf{u}_0,\mathbf{u}_1,\theta_0) \in U_1(m)$, the solution $(\mathbf{u},\theta)$ to (3.0.1)–(3.0.4) satisfies*

$$\|\,\theta\,;\,L_T^\infty L_x^m\| \le \Lambda,$$

where $\Lambda = \Lambda(T,\|\,(\mathbf{u}_1,\mathbf{u}_2,\theta_0)\,;\,U_1(m)\|)$. Moreover, if $(\mathbf{u}_0,\mathbf{u}_1,\theta_0) \in U_1(\infty)$, then we have

$$\|\,\theta\,;\,L^\infty(\Omega_T)\| \le \Lambda,$$

where $\Lambda = \Lambda(T,\|\,(\mathbf{u}_1,\mathbf{u}_2,\theta_0)\,;\,U_1(\infty)\|)$, and for $(\mathbf{u}_0,\mathbf{u}_1,\theta_0) \in U_2$ it holds that

$$\|\,\epsilon\,;\,W_p^{2,1}(\Omega_T)\| \le \Lambda,$$

where $\Lambda = \Lambda(T,\|\,(\mathbf{u}_1,\mathbf{u}_2,\theta_0)\,;\,U_2\|)$.

Proof. The same operation as in the proof of Lemma 3.3.3 yields

$$\begin{aligned}\frac{1}{m}\frac{d}{dt}\|\,\hat{\theta}\,;\,L^m\|^m + (m-1)\int_\Omega \theta^{m-2}|\nabla\theta|^2 dx \\ = \int_\Omega \overline{G}_m(\theta)\partial_t F_1(\epsilon)dx + \nu\int \theta^{m-1}A\epsilon_t : \epsilon_t dx.\end{aligned} \tag{3.4.13}$$

Here we recall that $G_m(\theta) = \theta^m G'(\theta) - \overline{G}_m(\theta)$, $\overline{G}_m(t) = m\int_0^\theta s^{m-1}G'(s)ds$ and

$$\hat{\theta} = \theta\left(1 - \frac{mG_m(\theta)F_1(\epsilon)}{\theta^m}\right)^{1/m} \geq \theta.$$

Since $\|\,F_{1,\epsilon}(\epsilon)\,;\,L^\infty(\Omega_T)\| = \Lambda < \infty$ from (3.4.3), we have

$$\begin{aligned}\left|\int_\Omega \overline{G}_m(\theta)\partial_t F_1(\epsilon)dx\right| &\leq C\|\,\theta^{m-1}\,;\,L^1(\Omega)\|\|\,\theta\,;\,L^\infty(\Omega)\|\|\,\epsilon_t\,;\,L^\infty(\Omega)\|\|\,F_{1,\epsilon}(\epsilon)\,;\,L^\infty(\Omega)\| \\ &\leq \Lambda\|\,\theta\,;\,L^m(\Omega)\|^{m-1}\|\,\theta\,;\,H^2(\Omega)\|\|\,\epsilon_t\,;\,L^\infty(\Omega)\|.\end{aligned}$$

Therefore, we conclude from (3.4.13) that

$$\begin{aligned}\frac{1}{l}\frac{d}{dt}\|\,\hat{\theta}\,;\,L^m(\Omega)\|^m \leq \Lambda\|\,\epsilon_t\,;\,L^\infty(\Omega)\|\|\,\theta\,;\,H^2(\Omega)\|\|\,\theta\,;\,L^m(\Omega)\|^{m-1} \\ + C\|\,\epsilon_t\,;\,L^\infty(\Omega)\|^2\|\,\theta\,;\,L^m(\Omega)\|^{m-1}.\end{aligned} \tag{3.4.14}$$

Here note that $\frac{d}{dt}\|\,\hat{\theta}\,;\,L^m(\Omega)\|^m = m\|\,\hat{\theta}\,;\,L^m(\Omega)\|^{m-1}\frac{d}{dt}\|\,\hat{\theta}\,;\,L^m(\Omega)\|$ and that from the Sobolev embedding and Lemma 3.4.4

$$\begin{aligned}\|\,\epsilon_t\,;\,L^2_T L^\infty\| \leq \Lambda\|\,\epsilon_t\,;\,L^2_T W^1_{10/3}\| \leq \Lambda\|\,\mathbf{u}\,;\,W^{4,2}_{10/3}(\Omega_T)\| \leq \Lambda, \\ \|\,\theta\,;\,L^2_T H^2\| \leq \|\,\theta\,;\,W^{2,1}_2(\Omega_T)\| \leq \Lambda,\end{aligned}$$

where Λ is independent of m. Thus, integrating (3.4.14) with respect to time variable, we obtain

$$\begin{aligned}\|\,\hat{\theta}\,;\,L^\infty_T L^m\| &\leq \|\,\hat{\theta}_0\,;\,L^m\| + \Lambda\|\,\epsilon_t\,;\,L^2_T L^\infty\|\|\,\theta\,;\,L^2_T H^2\| + \Lambda\|\,\epsilon_t\,;\,L^2_T L^\infty\|^2 \\ &\leq \Lambda + \|\,\hat{\theta}_0\,;\,L^m(\Omega)\|\end{aligned}$$

Since we have $\hat{\theta}_0 \leq \theta_0\,(1 + mM\Lambda)^{1/m}$, the desired result can be obtained. For the $W^{2,1}_p$-norm of ϵ, we have

$$\begin{aligned}\|\,\epsilon\,;\,W^{2,1}_p\| &\leq C\|\,(\mathbf{u}_0,\mathbf{u}_1,0)\,;\,U_2\| \\ &\quad + \Lambda\|\,\theta\,;\,L^\infty(\Omega_T)\|^r\|\,F_{1,\epsilon}(\epsilon)\,;\,L^\infty(\Omega_T)\| + \Lambda\|\,F_{2,\epsilon}(\epsilon)\,;\,L^\infty(\Omega_T)\| \\ &\leq \Lambda(\|\,(\mathbf{u}_0,\mathbf{u}_1,0)\,;\,U_2\|)\end{aligned}$$

for $p \in (1,\infty)$, by virtue of the maximal regularity (3.2.3). This completes the proof. □

Using again Lemma 3.3.4, we can also prove the Hölder continuity of θ. The Hölder continuity of ϵ is assured on account of Lemma 3.1.1. Hence from Lemma 3.3.7 we can obtain the bounds in higher Sobolev norms, i.e., for $5 < p, q < \infty$

$$\| (\mathbf{u}, \theta); V_T(p,q)\| = \|\mathbf{u}\,;\, W_p^{4,2}(\Omega_T)\| + \|\,\theta\,;\, W_q^{2,1}(\Omega_T)\| \le \Lambda =: \widehat{\Lambda}, \tag{3.4.15}$$

where $\widehat{\Lambda}$ is independent of L.

This a priori estimate says that if there exists a solution to the problem (3.0.1)–(3.0.4) such that $\theta \ge 0$ then this solution satisfies estimate (3.4.15). Let us consider now problem (3.3.1) from Section 3.3 assuming that the truncation size L is sufficiently large such that

$$|\nabla \cdot [G(\theta)F_{1,\epsilon}(\epsilon) + F_{2,\epsilon}(\epsilon)]| \le \widehat{\Lambda}^{K_1+r-1} + \widehat{\Lambda}^{K_2-1} \ll L.$$

In this case we may regard Γ_L as the identity operator because the internal part of Γ_L in (3.3.1) is smaller than L. Therefore the unique solution $(\mathbf{u}_L, \theta_L)$ to (3.3.1) satisfies (3.4.15) for large L. In other words, $V_T(p,q)$-norm bound for $(\mathbf{u}_L, \theta_L)$ does not depend on L. Hence $(\mathbf{u}_L, \theta_L)$ satisfies also the original system (3.0.1)–(3.0.4).

A positivity of θ follows by the same argument as the proof of Lemma 3.3.2 in [38]. This completes the proof of Theorem 3.1.

3.5 Uniqueness

For the sake of completeness of this thesis, we give the proof of Theorem 3.3, although this was established by Pawłow and Zajączkowski in [38].

Proof of Theorem 3.3. Let $(\mathbf{u}, \theta)$ and $(\widetilde{\mathbf{u}}, \widetilde{\theta})$ be two solutions of (3.0.1)–(3.0.4) corresponding to the same data. We denote

$$\mathbf{w} = \mathbf{u} - \widetilde{\mathbf{u}}, \qquad \eta = \theta - \widetilde{\theta}.$$

To simplify notation, we set

$$\epsilon = \epsilon(\mathbf{u}), \quad \epsilon_t = \epsilon(\mathbf{u}_t), \quad F_{,\epsilon} = F_{,\epsilon}(\epsilon, \theta), \quad F_{,\theta\epsilon} = F_{,\theta\epsilon}(\epsilon, \theta)$$
$$c_0 = c_0(\epsilon, \theta), \quad \gamma_0 = \gamma_0(\epsilon, \theta)$$

and respectively $\widetilde{\epsilon}$, $\widetilde{\epsilon}_t$, $\widetilde{F}_{,\epsilon}$, $\widetilde{F}_{,\theta\epsilon}$, $\widetilde{c}_0$ and $\widetilde{\gamma}_0$ for the functions of $(\widetilde{\mathbf{u}}, \widetilde{\theta})$.

Further, it is convenient to rewrite the equation (3.0.2) in the form

$$\theta_t - \gamma_0 \Delta\theta = \gamma_0 \theta G'(\theta) F_{1,\epsilon}(\epsilon) : \epsilon_t + \nu\gamma_0 A\epsilon_t : \epsilon_t, \tag{3.5.1}$$

where

$$\gamma_0 = \gamma_0(\epsilon, \theta) = \frac{1}{c_0(\epsilon, \theta)} = \frac{1}{1 + \theta G''(\theta) F_1(\epsilon)}.$$

We note also that

$$\frac{1}{c_*} \le \gamma_0 \le 1, \tag{3.5.2}$$

where $c_* = \sup c_0(\epsilon, \theta)$. Subtracting the corresponding equations, we see that $\mathbf{w}$, η satisfy the following problems:

$$\begin{aligned}
&\mathbf{w}_{tt} + Q^2\mathbf{w} - \nu Q\mathbf{w}_t = \nabla \cdot (F_{,\varepsilon} - \widetilde{F}_{,\varepsilon}), && && (3.5.3)\\
&\eta_t - \gamma_0 \Delta \eta = R_1 + R_2 + R_3 && \text{in } \Omega_T, && (3.5.4)\\
&\mathbf{w} = Q\mathbf{w} = \nabla \eta \cdot \mathbf{n} = 0 && \text{on } S_T, && (3.5.5)\\
&\mathbf{w}(0, \cdot) = \mathbf{w}_t(0, \cdot) = \eta(0, \cdot) = 0 && \text{on } \Omega, && (3.5.6)
\end{aligned}$$

where we put

$$\begin{aligned}
R_1 &= \gamma_0 \theta F_{,\theta\epsilon} : \epsilon_t - \widetilde{\gamma}_0 \widetilde{\theta} \widetilde{F}_{,\theta\epsilon} : \widetilde{\epsilon}_t,\\
R_2 &= \nu(\gamma_0 A\epsilon_t : \epsilon_t - \widetilde{\gamma}_0 A \widetilde{\epsilon}_t : \widetilde{\epsilon}_t),\\
R_3 &= (\gamma_0 - \widetilde{\gamma}_0)\Delta \widetilde{\theta}.
\end{aligned}$$

Multiplying (3.5.3) by $\mathbf{w}_t$ and integrating over Ω_t yields

$$\begin{aligned}
\frac{1}{2}\int_\Omega |\mathbf{w}_t|^2 dx + \frac{1}{2}\int_\Omega |Q\mathbf{w}|^2 dx + \nu \int_{\Omega_t} A\epsilon(\mathbf{w}) : \epsilon(\mathbf{w}) dx ds&\\
= - \int_{\Omega_t} (F_{,\epsilon} - \widetilde{F}_{,\epsilon}) : \epsilon(\mathbf{w}_t) dx ds&
\end{aligned} \tag{3.5.7}$$

Next, adding to (3.5.7) the identity

$$\frac{1}{2}\int_\Omega |\epsilon(\mathbf{w})|^2 dx = \int_{\Omega_t} \epsilon(\mathbf{w}) : \epsilon(\mathbf{w}_t) dx ds$$

valid thanks to the initial condition (3.5.6), and recalling (1.3.2), we obtain

$$\begin{aligned}
\frac{1}{2}\int_\Omega \big(|\mathbf{w}_t|^2 + |\epsilon(\mathbf{w})|^2 + |Q\mathbf{w}|^2\big) dx + \nu a_* \int_{\Omega_t} |\epsilon(\mathbf{w}_t)|^2 dx ds&\\
\le \delta \int_{\Omega_t} |\epsilon(\mathbf{w}_t)|^2 dx ds + C(\delta) \int_{\Omega_t} \Big(|F_{,\epsilon} - \widetilde{F}_{,\epsilon}|^2 + |\epsilon(\mathbf{w})|^2\Big) dx ds&
\end{aligned}$$

Hence, using the estimate

$$|F_{,\epsilon} - \widetilde{F}_{,\epsilon}| \le C(|\epsilon(\mathbf{w})| + |\eta|)$$

which follows due to uniform bounds on ϵ and θ in Ω_T, and choosing δ appropriately, we obtain

$$\int_\Omega \left(|\mathbf{w}_t|^2 + |\varepsilon(\mathbf{w})|^2 + |Q\mathbf{w}|^2\right) dx + \int_{\Omega_t} |\varepsilon(\mathbf{w}_t)|^2 dxds \le C \int_{\Omega_t} \left(|\epsilon(\mathbf{w})|^2 + |\eta|^2\right) dxds.$$

Consequently, with the help of the Gronwall inequality, we arrive at the estimate

$$\|\mathbf{w}_t\|_{L^\infty_T L^2} + \|\epsilon(\mathbf{w})\|_{L^\infty_T L^2} + \|Q\mathbf{w}\|_{L^\infty_T L^2} + \|\epsilon(\mathbf{w}_t)\|_{L^2(\Omega_T)} \le C\|\eta\|_{L^2(\Omega_T)}. \tag{3.5.8}$$

By virtue of the strong ellipticity of Q, it follows from (3.5.8) that

$$\|\mathbf{w}\|_{L^\infty H^2} \le C\|\eta\|_{L^2(\Omega_T)}. \tag{3.5.9}$$

Now we multiply (3.5.4) by η and integrate over Ω_t to get, after integration by parts,

$$\frac{1}{2}\int_\Omega \eta^2 dx + \int_{\Omega_t} \gamma_0 |\nabla\eta|^2 dxds = -\int_{\Omega_t} \eta\nabla\eta\cdot\nabla\gamma_0 dxds + \sum_{i=1}^{3}\int_{\Omega_t} R_i\eta dxds.$$

Hence, by (3.5.2), we have

$$\frac{1}{2}\int_\Omega \eta^2 dx + \frac{1}{c_*}\int_{\Omega_t} |\nabla\eta|^2 dxds \le -\int_{\Omega_t} \eta\nabla\eta\cdot\nabla\gamma_0 dxds + \sum_{i=1}^{3}\int_{\Omega_t} R_i\eta dxds. \tag{3.5.10}$$

We proceed now to estimate the terms on the right hand side of (3.5.10). Note that by virtue of the Hölder estimates on ϵ, θ, $\nabla\epsilon$ and $\nabla\theta$ in Ω_T, we have

$$|\nabla\gamma_0| \le \frac{1}{c_0^2}\left(|c_{0,\epsilon}||\nabla\epsilon| + |c_{0,\theta}||\nabla\theta|\right)$$

in Ω_T. Consequently, the first term on the right hand side of (3.5.10) is, with the help of the Young inequality, estimated by

$$\int_{\Omega_t} |\eta||\nabla\eta||\nabla\gamma_0| dxds \le \delta_1 \int_{\Omega_t} |\nabla\eta|^2 dxds + C(\delta_1)\int_{\Omega_t} \eta^2 dxds. \tag{3.5.11}$$

Further, thanks to the uniform bounds on ϵ, θ, ϵ_t, γ_0, $\widetilde{\epsilon}$, $\widetilde{\theta}$, ϵ_t, $\widetilde{\gamma}_0$ in Ω_T, we have

$$|\gamma_0 - \widetilde{\gamma}_0| \le C(|\epsilon(\mathbf{w})| + |\eta|) \tag{3.5.12}$$

and

$$|R_1| + |R_2| \le C(|\epsilon(\mathbf{w})| + |\eta| + |\epsilon(\mathbf{w}_t)|).$$

Hence, by virtue of (3.5.8), we obtain

$$\sum_{i=1}^{2}\int_{\Omega_t} |R_i||\eta| dxds \le C\int_{\Omega_T} \eta^2 dxdt \tag{3.5.13}$$

The R_3-term is first integrated by part

$$\begin{aligned}\int_{\Omega_t} \eta R_3 dxds &= \int_{\Omega_t} (\gamma_0 - \widetilde{\gamma}_0)\nabla\widetilde{\theta}\cdot\nabla\eta dxds + \int_{\Omega_t} \eta\nabla\widetilde{\theta}\cdot\nabla(\gamma_0-\widetilde{\gamma}_0)dxds \\ &=: R_{41} + R_{42}.\end{aligned} \tag{3.5.14}$$

Utilizing (3.5.11), the uniform bound on $\nabla\widetilde{\theta}$ and (3.5.8) yield

$$R_{41} \leq \delta_2 \int_{\Omega} |\nabla\eta|^2 dxds + c(\delta_2)\int_{\Omega_t} \eta^2 dxds. \tag{3.5.15}$$

Similarly, in view of the bounds

$$|\gamma_{0,\epsilon} - \widetilde{\gamma}_{0,\epsilon}| + |\gamma_{0,\theta} - \widetilde{\gamma}_{0,\theta}| \leq C(|\epsilon(\mathbf{w})| + |\eta|),$$

which follow thanks to the assumption $F \in \mathrm{C}^4$, utilizing the uniform bounds on ϵ, θ, $\nabla\epsilon$, $\nabla\theta$, $\gamma_{0,\epsilon}$ and $\gamma_{0,\theta}$, we see that

$$\begin{aligned}|\nabla(\gamma_0 - \widetilde{\gamma}_0)| &\leq |\nabla\epsilon||\gamma_{0,\epsilon} - \widetilde{\gamma}_{0,\epsilon}| + |\nabla\theta||\gamma_{0,\theta} - \widetilde{\gamma}_{0,\theta}| + |\widetilde{\gamma}_{0,\epsilon}||\nabla\epsilon(\mathbf{w})| + |\widetilde{\gamma}_{0,\theta}||\nabla\eta| \\ &\leq C(|\epsilon(\mathbf{w})| + |\nabla\epsilon(\mathbf{w})| + |\eta| + |\nabla\eta|).\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}R_{42} &\leq \delta_3 \int_{\Omega_t} (|\epsilon(\mathbf{w})|^2 + |\nabla\epsilon(\mathbf{w})|^2 + \eta^2 + |\nabla\eta|^2) dxds + C(\delta_3)\int_{\Omega_t} \eta^2 dxds \\ &\leq \delta_3 \int_{\Omega_t} (\eta^2 + |\nabla\eta|^2) dxds + C(\delta_3)\int_{\Omega_t} \eta^2 dxds,\end{aligned} \tag{3.5.16}$$

where in the last inequality we have applied (3.5.9). Finally, combining estimates (3.5.10), (3.5.12)–(3.5.15) in (3.5.2), and choosing constants δ_i appropriately, we arrive at

$$\int_{\Omega} \eta^2 dx + \int_{\Omega_t} |\nabla\eta|^2 dxds \leq C\int_{\Omega_t} \eta^2 dxds$$

for $t \leq T$. Hence, by the Gronwall inequality, we have $\eta = 0$ in Ω_T. Simultaneously, from the inequality (3.5.9) it follows that $\mathbf{w} = 0$ in Ω_T. This completes the proof.

□

3.6 Two-Dimensional Case

In this section, we consider the solvability of the two-dimensional system (3.0.1)–(3.0.4).

Proof of Theorem 3.2. With the exception of a priori bounds the result follows by the same procedure as in the proof of the three-dimensional case. Thus, it remains to check the bounds corresponding to Lemmas 3.4.1, 3.4.2 and 3.4.3 under (3.0.7).

Lemma 3.6.1 (Energy Conservation Law). *Assume that $\theta \geq 0$ a.e. in Ω_T and that* (3.0.6) *holds. Then for any $t \in [0, T]$ the smooth solution of the two-dimensional system* (3.0.1)–(3.0.4) *satisfies*

$$\|\, \theta(t)\, ;\, L^1(\Omega)\| + \|\, \mathbf{u}_t(t)\, ;\, L^2(\Omega)\| + \|\, Q\mathbf{u}(t)\, ;\, L^2(\Omega)\| \leq C(\|\, (\mathbf{u}_0, \mathbf{u}_1, \theta_0)\, ;\, H^2 \times L^2 \times L^1\|).$$

Proof. The same operation as in the proof of Lemma 3.4.1 yields

$$\frac{d}{dt}\left(\frac{1}{2}\|\, \mathbf{u}_t\, ;\, L^2(\Omega)\|^2 + \frac{1}{2}\|\, Q\mathbf{u}\, ;\, L^2(\Omega)\|^2 + \int_\Omega \theta dx + \int_\Omega F_2(\epsilon) dx + \int_\Omega \overline{G}(\theta) F_1(\epsilon) dx\right) = 0,$$

where $\overline{G}(\theta) = G(\theta) - \theta G'(\theta)$. Here we recall that $\theta \geq 0$, $H(\epsilon) \geq 0$ and $\overline{G}(\theta) \geq 0$. Consequently, it follows from $(N3)$ in the structure of the nonlinearity that

$$\begin{aligned}
\frac{1}{2}\|\, \mathbf{u}_t\, ;\, L^\infty_T L^2\|^2 &+ \frac{1}{2}\|\, \mathbf{u}\, ;\, L^\infty_T H^2\|^2 + \|\, \theta\, ;\, L^\infty_T L^1\| \\
&\leq \frac{1}{2}\|\, \mathbf{u}_0\, ;\, H^2(\Omega)\|^2 + \frac{1}{2}\|\, \mathbf{u}_1\, ;\, L^2(\Omega)\|^2 + \|\, \theta_0\, ;\, L^1(\Omega)\| \\
&\quad + \int_\Omega \left\{|F_2(\epsilon_0)| + |\overline{G}(\theta_0) F_1(\epsilon_0)|\right\} dx + C_3|\Omega|,
\end{aligned}$$

where $\epsilon_0 = \epsilon(\mathbf{u}_0)$. From the Sobolev embedding it holds that

$$\|\, \epsilon_0\, ;\, L^s(\Omega)\| \leq C\|\, u_0\, ;\, H^2(\Omega)\| \tag{3.6.1}$$

for any $s \in [1, \infty)$. Then we have

$$\begin{aligned}
\int_\Omega |\overline{G}(\theta_0) F_1(\epsilon_0)| dx &\leq C\|\, \theta_0\, ;\, L^1(\Omega)\|^r \|\, \epsilon_0\, ;\, L^{\frac{K_1}{1-r}}(\Omega)\|^{K_1} \\
&\leq C\|\, \theta_0\, ;\, L^1(\Omega)\|^r \|\, u_0\, ;\, H^2\|^{K_1}
\end{aligned}$$

for $r < 1$ and $K_1 < \infty$, and

$$\begin{aligned}
\int_\Omega F_2(\epsilon_0) dx &\leq \|\, \epsilon_0\, ;\, L^{K_2}\|^{K_2} \\
&\leq C\|\, \mathbf{u}_0; H^2\|^{K_2}
\end{aligned}$$

for $K_2 < \infty$. This completes the proof. □

Lemma 3.6.2. *Let $p \in [2,4)$. Assume that* (3.0.6) *holds. Then for any* $(u_0, u_1, \theta_0) \in B^{3-2/p}_{p,p} \times B^{1-2/p}_{p,p} \times L^2$, *the solution* (u, θ) *to the two-dimensional system* (3.0.1)–(3.0.4) *satisfies*

$$\| \epsilon\,;\, W^{2,1}_p(\Omega_T)\| + \| \nabla\theta\,;\, L^2(\Omega_T)\| + \| \theta\,;\, L^\infty_T L^2\| \leq \Lambda, \tag{3.6.2}$$

where Λ *depends on* T *and* $\| (\mathbf{u}_0, \mathbf{u}_1, \theta_0)\,;\, B^{3-2/p}_{p,p} \times B^{1-2/p}_{p,p} \times L^2\|$. *Moreover we have*

$$\| \epsilon\,;\, L^\infty(\Omega_T)\| + \| \theta\,;\, L^p(\Omega_T)\| \leq \Lambda. \tag{3.6.3}$$

Proof. We first show (3.6.2) for p such that $p < 3$. From the Sobolev inequality (3.6.1) and Lemma 3.6.1, it follows that

$$\| \epsilon\,;\, L^s(\Omega_T)\| \leq \Lambda \| \mathbf{u}\,;\, L^\infty_T H^2\| \leq \Lambda$$

for every $s < \infty$, and hence we obtain

$$\| F_{1,\epsilon}(\epsilon)\,;\, L^s(\Omega_T)\| + \| F_{2,\epsilon}(\epsilon)\,;\, L^s(\Omega_T)\| \leq \Lambda \tag{3.6.4}$$

for any K_1, $K_2 < \infty$. Moreover, by using the Hölder inequality, we have

$$\begin{aligned} \| \theta\,;\, L^p(\Omega_T)\| &\leq C \left\| \| \theta\,;\, L^1\|^{1-2/p} \| \theta\,;\, L^{2/(3-p)}\|^{2/p}\,;\, \right\|_{L^p_T} \\ &\leq C \| \theta\,;\, L^\infty_T L^1\|^{1-2/p} \| \theta\,;\, L^2_T H^1\|^{2/p} \\ &\leq \Lambda \| \theta\,;\, L^2_T H^1\|^{2/p} \end{aligned} \tag{3.6.5}$$

for $p \in [2,3)$.

We fix $\bar{p}$ such that $r + 2 < \bar{p} < 3$. From (3.6.4), (3.6.5) and the maximal regularity (3.2.3) it follows that

$$\begin{aligned} \| \epsilon\,;\, W^{2,1}_{\bar{p}}(\Omega_T)\| &\leq C \| (\mathbf{u}_0, \mathbf{u}_1, \theta_0)\,;\, U'_3(\bar{p})\| + C \| G(\theta) F_{1,\epsilon}(\epsilon)\,;\, L^{\bar{p}}(\Omega_T)\| \\ &\qquad + C \| F_{2,\epsilon}(\epsilon)\,;\, L^{\bar{p}}(\Omega_T)\| \\ &\leq \Lambda + C \| \theta\,;\, L^{\bar{p}}(\Omega_T)\|^r \| F_{1,\epsilon}(\epsilon)\,;\, L^{\frac{\bar{p}}{(1-r)}}(\Omega_T)\| \\ &\qquad + C \| F_{2,\epsilon}(\epsilon)\,;\, L^{\bar{p}}(\Omega_T)\| \\ &\leq \Lambda + \Lambda \| \theta\,;\, L^2_T H^1\|^{2r/\bar{p}}. \end{aligned} \tag{3.6.6}$$

Next, the same operation as in the proof of Lemma 3.4.2 yields

$$\begin{aligned} \frac{1}{2} \| \theta\,;\, L^\infty_T L^2\|^2 + \| \nabla\theta\,;\, L^2(\Omega_T)\|^2 \leq \frac{1}{2} \| \theta_0\,;\, L^2(\Omega)\|^2 + \| \overline{G}_2(\theta) \partial_t F_1(\epsilon)\,;\, L^1(\Omega_T)\| \\ + \nu \| \theta A \epsilon_t : \epsilon_t\,;\, L^1(\Omega_T)\| + M \sup_{t \in [0,T]} \int_{\Omega_T} |F_1(\epsilon(t))| dx \\ + \int_\Omega |G_2(\theta_0) F_1(\epsilon_0)| dx. \end{aligned}$$

By (3.6.4), (3.6.5) and (3.6.6) we have

$$\| \theta^{r+1}\partial_t F_1(\epsilon) \,;\, L^1(\Omega_T)\| \le \Lambda \| \theta \,;\, L^{\bar p}(\Omega_T)\|^{r+1} \| \epsilon \,;\, W^{2,1}_{\bar p}(\Omega_T)\| \| F_{1,\epsilon}(\epsilon) \,;\, L^{\frac{\bar p}{\bar p-(r+2)}}(\Omega_T)\| \\ \le \Lambda \| \theta \,;\, L^2_T H^1\|^{\frac{2(r+1)}{\bar p}}(\Lambda + \| \theta \,;\, L^2_T H^1\|^{\frac{2r}{\bar p}})$$

for $\bar p > r+2$,

$$\| \theta A\epsilon_t : \epsilon_t \,;\, L^1(\Omega_T)\| \le \Lambda \| \theta \,;\, L^{\bar p}(\Omega_T)\| \| \epsilon_t \,;\, L^{\frac{2\bar p}{\bar p-1}}(\Omega_T)\|^2 \\ \le \Lambda \| \theta \,;\, L^2_T H^1{}^{\frac{2}{\bar p}} \| (\Lambda + \| \theta \,;\, L^2_T H^1\|^{\frac{4r}{\bar p}}),$$

$$\int_\Omega |F_1(\epsilon(t))| dx \le C \| \mathbf{u}(t) \,;\, H^2(\Omega)\|^{K_1} \le \Lambda$$

and

$$\| \theta_0^{r+1} F_1(\epsilon_0) \,;\, L^1(\Omega)\| \le C \| \theta_0 \,;\, L^2(\Omega)\|^{r+1} \| \epsilon_0 \,;\, L^{\frac{2K_1}{1-r}}(\Omega)\|^{K_1} \\ \le C \| \theta_0 \,;\, L^2(\Omega)\|^{r+1} \| \mathbf{u}_0 \,;\, H^2(\Omega)\|^{K_1}.$$

Consequently we arrive at

$$\| \theta \,;\, L^\infty_T L^2\|^2 + \| \nabla\theta \,;\, L^2(\Omega_T)\|^2 \le \Lambda(\| (\mathbf{u}_0, \mathbf{u}_1, \theta_0) \,;\, U_3'(p)\|) + \Lambda \| \theta \,;\, L^2_T H^1\|^{\frac{2(2r+1)}{\bar p}}.$$

Since $2r+1 < r+2 < \bar p$, by using the Young inequality we have

$$\| \theta \,;\, L^\infty_T L^2\| + \| \nabla\theta \,;\, L^2(\Omega_T)\| \le \Lambda(\| (\mathbf{u}_0, \mathbf{u}_1, \theta_0) \,;\, B^{3-2/p}_{p,p} \times B^{1-2/p}_{p,p} \times L^2\|). \tag{3.6.7}$$

Substituting (3.6.7) into (3.6.6), we obtain (3.6.2) for $p < 3$.

We shall show the rest of proof. Taking $p \in [2,4)$, from the same operation as (3.6.5) we have

$$\| \theta \,;\, L^p(\Omega_T)\| \le C \| \theta \,;\, L^\infty_T L^2\|^{1-2/p} \| \theta \,;\, L^2_T H^1\|^{2/p} \le \Lambda$$

for $p < 4$ thanks to (3.6.7). Then from the maximal regularity (3.2.3) we conclude that

$$\| \epsilon \,;\, W^{2,1}_p\| \le \Lambda + \| \theta \,;\, L^p\|^r \| F_{1,\epsilon}(\epsilon) \,;\, L^{\frac{p}{(1-r)}}\| + \| F_{2,\epsilon}(\epsilon) \,;\, L^p\| \\ \le \Lambda.$$

This completes the proof. □

Lemma 3.6.3. *Assume that* (3.0.6) *holds. Then for any* $(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \in B^{5/2}_{4,4} \times B^{1/2}_{4,4} \times H^1$ *the following estimate holds*

$$\| \epsilon \,;\, W^{2,1}_4(\Omega_T)\| + \| \nabla\theta \,;\, L^\infty_T L^2\| + \| \theta \,;\, W^{2,1}_2(\Omega_T)\| \le \Lambda,$$

where constant Λ *depends on* T *and* $\| (\mathbf{u}_0, \mathbf{u}_1, \theta_0) \,;\, B^{5/2}_{4,4} \times B^{1/2}_{4,4} \times H^1\|$. *Moreover, we have*

$$\| \nabla\theta \,;\, L^4(\Omega_T)\| + \| \theta \,;\, L^s(\Omega_T)\| + \| \nabla\epsilon \,;\, L^s(\Omega_T)\| \le \Lambda$$

for any $s < \infty$.

Proof. It follows from Lemma 3.6.2 and (3.2.3) that

$$\begin{aligned}\|\,\epsilon\,;\,W_4^{2,1}(\Omega_T)\| &\le C\|\,(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\,;\,B_{4,4}^{5/2}\times B_{4,4}^{1/2}\times H^1\| \\ &\qquad + C\|\,\theta\,;\,L^{4r}\|^r\|\,F_{1,\epsilon}(\epsilon)\,;\,L^\infty(\Omega_T)\| + C\|\,F_{2,\epsilon}(\epsilon)\,;\,L^\infty(\Omega_T)\| \\ &\le \Lambda(\|\,(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\,;\,B_{4,4}^{5/2}\times B_{4,4}^{1/2}\times H^1\|),\end{aligned} \tag{3.6.8}$$

thanks to $r<1$. The same operation as in the proof of Lemma 3.4.3 yields

$$\begin{aligned}\|\,\theta_t\,;\,L^2(\Omega_T)\|^2 &+ \frac{1}{2}\|\,\nabla\theta\,;\,L_T^\infty L^2\|^2 \\ &\le \frac{1}{2}\|\,\theta_0\,;\,H^1(\Omega)\|^2 + C\|\,\theta_t\,;\,L^2(\Omega_T)\|\|\,\epsilon_t\,;\,L^4(\Omega_T)\|^2 \\ &\qquad + C\|\,\theta_t\,;\,L^2(\Omega_T)\|\|\,\theta^r F_{1,\epsilon}(\epsilon)\,;\,L^4(\Omega_T)\|\|\,\epsilon_t\,;\,L^4(\Omega_T)\| \\ &\le \Lambda + \frac{1}{2}\|\,\theta_t\,;\,L^2(\Omega_T)\|^2\end{aligned}$$

on account of (3.6.3) and (3.6.8). Therefore, we arrive at the estimate

$$\|\,\epsilon\,;\,W_4^{2,1}(\Omega_T)\| + \|\,\theta_t\,;\,L^2(\Omega_T)\| + \|\,\nabla\theta\,;\,L_T^\infty L^2\| \le \Lambda(\|\,(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\,;\,B_{4,4}^{5/2}\times B_{4,4}^{1/2}\times H^1\|).$$

Moreover, applying the same argument as in the proof of Lemma 3.4.3, we get

$$\|\,\Delta\theta\,;\,L^2(\Omega_T)\| \le \Lambda.$$

This completes the proof of the first assertion. With the help of Lemma 3.1.1 we obtain the second assertion. We have thus proved the Lemma 3.6.3. □

From a modification similar to that presented in three-dimensional case we can derive the estimate

$$\|\,(\mathbf{u},\theta)\,;\,V_T(p,q)\| = \|\,\mathbf{u}\,;\,W_p^{4,2}(\Omega_T)\| + \|\,\theta\,;\,W_q^{2,1}(\Omega_T)\| \le \Lambda.$$

Hence the proof of Theorem 3.2 are completed. □

Appendix:
Two-Dimensional Semilinear System

This appendix is concerned with the unique global existence theorem for the two-dimensional thermoelastic system:

$$\mathbf{u}_{tt} + Q^2\mathbf{u} - \nu Q\mathbf{u}_t = \nabla\cdot(\theta F_{1,\epsilon}(\epsilon) + F_{2,\epsilon}(\epsilon)), \tag{A.0.1}$$

$$\theta_t - \Delta\theta = \theta\partial_t F_1(\epsilon) + \nu(A\epsilon_t) : \epsilon_t \qquad \text{in } \Omega_\infty, \tag{A.0.2}$$

$$\mathbf{u} = Q\mathbf{u} = \nabla\theta\cdot\mathbf{n} = 0 \qquad \text{on } S_\infty, \tag{A.0.3}$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0, \quad \mathbf{u}_t(0,\cdot) = \mathbf{u}_1, \quad \theta(0,\cdot) = \theta_0 \geq 0 \qquad \text{in } \Omega, \tag{A.0.4}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$. Let $\mathbf{u} := (u_1, u_2) \in \mathbb{R}^2$ denote the displacement vector of shape memory alloys and θ the temperature.

This system corresponds to the two-dimensional system (3.0.1)–(3.0.4) with $r = 1$. The nonlinear functions F_1 and F_2 take the same structure in Chapter 3, and we restate it.

$(N2)$ $F_1 \in \mathrm{C}^3(\mathrm{Sym}(n,\mathbb{R}),\mathbb{R})$ satisfies that $F_1(\epsilon) \geq 0$.

$(N3)$ $F_2 \in \mathrm{C}^3(\mathrm{Sym}(n,\mathbb{R}),\mathbb{R})$ satisfies that $F_2(\epsilon) \geq -C_3$, where C_3 is some real constant.

$(N4)$ $F_1(\epsilon)$ and $F_2(\epsilon)$ satisfy the following growth conditions:

$$|F_{1,\epsilon}(\epsilon)| \leq C|\epsilon|^{K_1-1}, \qquad |F_{1,\epsilon\epsilon}(\epsilon)| \leq C|\epsilon|^{K_1-2}, \qquad |F_{1,\epsilon\epsilon\epsilon}(\epsilon)| \leq C|\epsilon|^{K_1-3},$$
$$|F_{2,\epsilon}(\epsilon)| \leq C|\epsilon|^{K_2-1}, \qquad |F_{2,\epsilon\epsilon}(\epsilon)| \leq C|\epsilon|^{K_2-2}, \qquad |F_{2,\epsilon\epsilon\epsilon}(\epsilon)| \leq C|\epsilon|^{K_2-3}$$

for large $|\epsilon| > \epsilon_c$. Here ϵ_c is a positive constant.

We show the unique global existence of the solution for two-dimensional system (A.0.1)–(A.0.4) under the assumption

$$K_1 \in [0,1], \qquad K_2 \in [0,\infty), \qquad C_3 = 0, \qquad F_2(\epsilon) \leq C|\epsilon|^{K_2} \tag{A.0.5}$$

and the smallness assumption of the energy norm of the data. Here we recall that

$$\|(\mathbf{u}, \mathbf{u}_t, \theta)\|_E := \|(\mathbf{u}, \mathbf{u}_t, \theta)\|_{H^2 \times L^2 \times L^1}$$

is the energy norm of $(\mathbf{u}, \mathbf{u}_t, \theta)$.

We restate our main result in this appendix. This is based on the result in [49].

Theorem A.1 (Small Energy Global Existence). *Assume that* $p > 4$ *and that* F *satisfies the condition* (A.0.5). *Then there exists* $\eta > 0$ *such that for any* $(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \in U^p := B^{4-\frac{2}{p}}_{p,p} \times B^{2-\frac{2}{p}}_{p,p} \times B^{2-\frac{8}{3p}}_{\frac{3p}{4},\frac{3p}{4}}$ *satisfying* $\|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_E < \eta$ *there exists a unique global solution* $(\mathbf{u}, \theta)$ *of the problem* (A.0.1)–(A.0.4) *satisfying*

$$(\mathbf{u}, \theta) \in W^{4,2}_{p,loc} \times W^{2,1}_{\frac{3p}{4},loc}.$$

Furthermore, there exists the monotone increasing function $K(x) > 0$ *such that* $K(0) = 0$ *and*

$$\|(\mathbf{u}(t), \mathbf{u}_t(t), \theta(t))\|_E \leq K(\|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_E)$$

for any $t \in [0, \infty)$.

A.1 Local Existence

In this section, we show the unique local existence result which can be obtained by using Banach's fixed point principle. The proof of Theorem A.1.1 below is the same as that of [48, Theorem 3.2]. For the sake of completeness, we give the proof of this theorem.

Theorem A.1.1 (Unique Local Existence). *We denote by* $U(p, q)$ *the space* $B^{4-2/p}_{p,p} \times B^{2-2/p}_{p,p} \times B^{2-2/q}_{q,q}$. *Assume that* F *satisfies the condition* $(N2)$–$(N4)$ *with* (A.0.5) *and that* $p > 4/3$ *and* $q > 1$ *are arbitrary numbers satisfying*

$$\frac{2}{p} - \frac{1}{2} \leq \frac{1}{q} < \frac{1}{p} + \frac{1}{4}. \tag{A.1.1}$$

Then for any $(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \in U(p, q)$ *there exists* $T = T(\|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_{U(p,q)}) > 0$ *such that the problem* (A.0.1)–(A.0.4) *has a unique solution* $(\mathbf{u}, \theta)$ *on the time interval* $[0, T]$, *satisfying* $(\mathbf{u}, \theta) \in W^{4,2}_p(\Omega_T) \times W^{2,1}_q(\Omega_T) = V_T(p, q)$.

Remarks. (i) In the proof of Theorem A.1.1, we does not need the assumption $\theta_0 \geq 0$.
(ii) Of course, this result is also true in the case of $r = 1$, $K_1 \in [0, \infty)$ and $K_2 \in [0, \infty)$.

We give several preliminary results which are used in the proof of Theorem A.1.1. Since the operators $\alpha\mathcal{Q}$ and $\bar{\alpha}\mathcal{Q}$ have the maximal regularity property, these operators generate analytic semigroups, where we recall that the linear operator $\mathcal{Q}$ is the differential operator Q with the homogeneous Dirichlet boundary condition, defined by (3.2.7) in Chapter 3. Therefore, we also obtain the following L^p-L^q estimates.

Lemma A.1.2. *For any $j \in \mathbb{Z}^+$, $1 < p < \infty$, and $p \le q \le \infty$, there exists constant $C > 0$ such that*

$$\| D_x^j T(t)\mathbf{v}\,;\, L_x^q\| + \| D_x^j \overline{T}(t)\mathbf{v}\,;\, L_x^q\| \le \frac{C}{t^{\frac{j}{2}}}\left(\frac{1}{t^{\frac{1}{p}-\frac{1}{q}}} + 1\right)\|\mathbf{v}\,;\, L_x^p\|, \tag{A.1.2}$$

where $T(t) = e^{t\alpha\mathcal{Q}}$ and $\overline{T}(t) = e^{t\overline{\alpha}\mathcal{Q}}$ for $\alpha = -\frac{\nu}{2} + i\sqrt{1-\frac{\nu^2}{4}}$.

Proof. By the general theory of analytic semigroups, for any $1 < p < \infty$ and $\beta \ge 0$, we have

$$\| (-\mathcal{Q})^\beta T(t)\mathbf{v}\,;\, L_x^p\| \le \frac{C}{t^\beta}\|\mathbf{v}\,;\, L_x^p\|.$$

From the Sobolev embedding and this inequality, it also follows that

$$\| T(t)\mathbf{v}\,;\, L_x^r\| \le C\left(\| (-\mathcal{Q})^\gamma T(t)\mathbf{v}\,;\, L_x^p\| + \| T(t)\mathbf{v}\,;\, L_x^p\|\right) \le C\left(\frac{1}{t^\gamma} + 1\right)\|\mathbf{v}\,;\, L_x^p\|,$$

where $\frac{1}{r} = \frac{1}{p} - \gamma$. An interpolation ([26, Theorem 5]) between these inequalities yields that for any $\rho \in (0,1)$

$$\| (-\mathcal{Q})^{\beta\rho} T(t)\mathbf{v}\,;\, L_x^q\| \le \frac{C}{t^{\beta\rho}}\left(\frac{1}{t^{\gamma(1-\rho)}} + 1\right)\|\mathbf{v}\,;\, L_x^p\|,$$

where $\frac{1}{q} = \frac{1}{p}\rho + \frac{1}{r}(1-\rho)$. Taking $\beta\rho = j/2$, we obtain the desired result since $\mathcal{Q}^{j/2} \sim D_x^j$. □

Remark. In this paper, since this estimate is used only under time local setting (for example $T < 1$), we may regard this estimate as the following well-known inequality:

$$\| D_x^j T(t)\mathbf{v}\,;\, L_x^q\| + \| D_x^j \overline{T}(t)\mathbf{v}\,;\, L_x^q\| \le \frac{C}{t^{\frac{j}{2}+\frac{1}{p}-\frac{1}{q}}}\|\mathbf{v}\,;\, L_x^p\|.$$

Proposition A.1.3. *Assume that $1 < p, q < 2$, $\nu > 0$ and $T < 1$. Denote the solution of (3.2.1) and (3.2.19) by $\mathbf{u}$ and θ, respectively. Set $\epsilon = (\epsilon_{ij})$ such that $\epsilon_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$. Then the following inequalities hold*

$$\| \epsilon\,;\, L^\infty(\Omega_T)\| \le C\| \nabla\cdot f\,;\, L^p(\Omega_T)\| + C\big\| (\mathbf{u}_0, \mathbf{u}_1)\,;\, B_{p,p}^{3-\frac{2}{p}} \times B_{p,p}^{1-\frac{2}{p}}\big\|, \tag{A.1.3}$$

$$\| \nabla\cdot\epsilon\,;\, L^{\frac{2p}{2-p}}(\Omega_T)\| \le C\| \nabla\cdot f\,;\, L^p(\Omega_T)\| + C\big\| (\mathbf{u}_0, \mathbf{u}_1)\,;\, B_{p,p}^{3-\frac{2}{p}} \times B_{p,p}^{1-\frac{2}{p}}\big\|, \tag{A.1.4}$$

$$\| \theta\,;\, L^{\frac{2q}{2-q}}(\Omega_T)\| \le C\| g\,;\, L^q(\Omega_T)\| + C\big\| \theta_0\,;\, B_{q,q}^{2-\frac{2}{q}}\big\|, \tag{A.1.5}$$

$$\| \nabla\cdot\theta\,;\, L^{\frac{4q}{4-q}}(\Omega_T)\| \le C\| g\,;\, L^q(\Omega_T)\| + C\big\| \theta_0\,;\, B_{q,q}^{2-\frac{2}{q}}\big\|. \tag{A.1.6}$$

Proof. We define the operator $R(t)$ by

$$R(t) = \frac{T(t) - \overline{T}(t)}{\mathcal{Q}}$$

For the linear term, it follows from Lemma 3.2.1 that for any $T_0 > 0$

$$\begin{aligned}
&\| R(\cdot)\epsilon_1 ; L^\infty(\Omega_{T_0})\| + \| \dot{R}(\cdot)\epsilon_0 ; L^\infty(\Omega_{T_0})\| \\
&\qquad \le \| (I-\mathcal{Q})^{\frac{1}{2}} R(\cdot)(I-\mathcal{Q})^{-\frac{1}{2}}\epsilon_1 ; L^\infty(\Omega_T)\| \\
&\qquad\qquad + \| (I-\mathcal{Q})^{\frac{1}{2}} \dot{R}(\cdot)(I-\mathcal{Q})^{-\frac{1}{2}}\epsilon_0 ; L^\infty(\Omega_T)\| \\
&\qquad \le \Lambda(T_0)\Big(\| R(\cdot)(I-\mathcal{Q})^{-\frac{1}{2}}\epsilon_1 ; W_p^{4,2}(\Omega_T)\| \\
&\qquad\qquad + \| \dot{R}(\cdot)(I-\mathcal{Q})^{-\frac{1}{2}}\epsilon_0 ; W_p^{4,2}(\Omega_T)\| \Big) \\
&\qquad \le \Lambda(T_0)(\| (I-\mathcal{Q})^{-\frac{1}{2}}\epsilon_1 ; B_{p,p}^{2-\frac{2}{p}}\| + \| (I-\mathcal{Q})^{-\frac{1}{2}}\epsilon_0 ; B_{p,p}^{4-\frac{2}{p}}\|) \\
&\qquad \le \Lambda(T_0)(\| \mathbf{u}_1 ; B_{p,p}^{2-\frac{2}{p}}\| + \| \mathbf{u}_0 ; B_{p,p}^{4-\frac{2}{p}}\|),
\end{aligned}$$

where $\epsilon_0 := \epsilon(\mathbf{u}_0)$ and $\epsilon_1 := \epsilon(\mathbf{u}_1)$. By a cut off argument we obtain for any $T < T_0$,

$$\begin{aligned}
&\| R(\cdot)\epsilon_0 ; L^\infty(\Omega_T)\| + \| \dot{R}(\cdot)\epsilon_1 ; L^\infty(\Omega_T)\| \\
&\qquad \le C(\| \mathbf{u}_0 ; B_{p,p}^{4-\frac{2}{p}}\| + \| \mathbf{u}_1 ; B_{p,p}^{2-\frac{2}{p}}\|),
\end{aligned}$$

where C depends on T_0. Similarly, we have

$$\begin{aligned}
&\| \nabla\cdot R(\cdot)\epsilon_0 ; L^{\frac{2p}{2-p}}(\Omega_T)\| + \| \nabla\cdot \dot{R}(\cdot)\epsilon_1 ; L^{\frac{2p}{2-p}}(\Omega_T)\| \\
&\qquad \le C(\| \mathbf{u}_0 ; B_{p,p}^{4-\frac{2}{p}}\| + \| \mathbf{u}_1 ; B_{p,p}^{2-\frac{2}{p}}\|),
\end{aligned}$$

We shall give the estimate for the following integral equations:

$$\mathbf{u}(t) = \int_0^t \overline{T}(t-s)\mathbf{w}(s)ds, \tag{A.1.7}$$

$$\mathbf{w}(t) = \int_0^t T(t-s)\nabla\cdot f(s)ds, \tag{A.1.8}$$

which are associated with the equations (3.2.8) and (3.2.9) with zero data, respectively. By using Lemma A.1.2 we have

$$\begin{aligned}
\| \mathbf{w} ; L^{\frac{2p}{2-p}}(\Omega_T)\| &= C\left\| \int_0^t \overline{T}(t-s)\nabla\cdot f(s)ds ; L^{\frac{2p}{2-p}}(\Omega_T)\right\| \\
&\le C\left\| \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}}\| \nabla\cdot f(s) ; L_x^p\| ds ; L_T^{\frac{2p}{2-p}}\right\| .
\end{aligned}$$

From the Hardy-Littlewood-Sobolev inequality it follows that

$$\left\| \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \| \nabla \cdot f(s) \,;\, L_x^p \| ds \,;\, L_T^{\frac{2p}{2-p}} \right\| \leq C \| \nabla \cdot f \,;\, L^p(\Omega_T) \|.$$

By Lemma A.1.2 and the Hölder inequality, we have for any $q \in (1,2)$

$$\begin{aligned} \| \epsilon \,;\, L^\infty(\Omega_T) \| &\leq C \left\| \int_0^t \frac{C}{(t-s)^{\frac{1}{2}}} \| \mathbf{w}(s) \,;\, L_x^{\frac{2p}{2-p}} \| ds \,;\, L_T^\infty \right\| \\ &\leq \left\| \left(\int_0^t \frac{C}{(t-s)^{\frac{p}{3p-2}}} ds \right)^{\frac{3p-2}{2p}} \| \mathbf{w} \,;\, L^{\frac{2p}{2-p}}(\Omega_T) \| \,;\, L_T^\infty \right\| \\ &\leq \Lambda_2 \| \mathbf{w} \,;\, L^{\frac{2p}{2-p}}(\Omega_T) \|. \end{aligned}$$

Combining these inequalities, we have proved the first assertion (A.1.3).

Similarly, from Lemma A.1.2 and the Hardy-Littlewood-Sobolev inequality it follows that

$$\begin{aligned} \| D_x \mathbf{w} \,;\, L^{\frac{4p}{4-p}}(\Omega_T) \| &= C \left\| D_x \int_0^t \overline{T}(t-s) \nabla \cdot f(s) ds \,;\, L^{\frac{4p}{4-p}}(\Omega_T) \right\| \\ &\leq C \left\| \int_0^t \frac{1}{(t-s)^{\frac{3}{4}}} \| \nabla \cdot f(s) \,;\, L_x^p \| ds \,;\, L_T^{\frac{4p}{4-p}} \right\| \\ &\leq C \| \nabla \cdot f(s) \,;\, L^p(\Omega_T) \|. \end{aligned}$$

By the same calculation as above, we have

$$\| D_x \epsilon \,;\, L^{\frac{2p}{2-p}}(\Omega_T) \| \leq \| D_x \mathbf{w} \,;\, L^{\frac{4p}{4-p}}(\Omega_T) \|.$$

We have thus proved the inequality (A.1.4). The third assertion (A.1.5) and the fourth assertion (A.1.6) follow in a similar fashion, which completes the proof. □

Proof of Theorem A.1.1. We denote $W_p^{4,2}(\Omega_T) \times W_q^{2,1}(\Omega_T)$ by $V_T(p,q)$. We introduce the map $\Phi(\mathbf{u}, \theta) := (\widetilde{\mathbf{u}}, \widetilde{\theta})$, where

$$\begin{cases} \widetilde{\mathbf{u}}_{tt} + Q^2 \widetilde{\mathbf{u}} - \nu Q \widetilde{\mathbf{u}}_t = \nabla \cdot (\theta F_{1,\epsilon}(\epsilon) + F_{2,\epsilon}(\epsilon)), & \\ \widetilde{\theta}_t - \Delta \widetilde{\theta} = \theta F_{1,\epsilon}(\epsilon) : \epsilon_t + \nu (A \epsilon_t) : \epsilon_t & \text{in } \Omega_T, \\ \widetilde{\mathbf{u}} = Q \widetilde{\mathbf{u}} = \nabla \widetilde{\theta} \cdot \mathbf{n} = 0 & \text{on } S_T, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0, \quad \mathbf{u}_t(0,\cdot) = \mathbf{u}_1, \quad \theta(0,\cdot) = \theta_0 & \text{in } \Omega. \end{cases}$$

For some $\delta > 0$ and $M > 0$, we define the subset $\mathcal{V}_T^M(p,q)$ of $\mathcal{V}_T(p,q)$ by $\mathcal{V}_T^M(p,q) := \{(\mathbf{u},\theta) \in V_T(p,q);\ \|(\mathbf{u},\theta)\|_{\mathcal{V}_T(p,q)} \le M\}$, where

$$\begin{aligned} &\|(\mathbf{u},\theta)\|_{\mathcal{V}_T(p,q)} \\ &\quad := \|\,\mathbf{u}\,;\, W_p^{4,2}(\Omega_T)\| + \|\,\epsilon\,;\, L^\infty(\Omega_T)\| + \|\,\nabla\epsilon\,;\, L^{m_2(p)}(\Omega_T)\| + \|\,\epsilon_t\,;\, L^{2q}(\Omega_T)\| \\ &\qquad + T^\delta\big(\|\,\theta\,;\, W_q^{2,1}(\Omega_T)\| + \|\,\nabla\theta\,;\, L^{m_1(q)}(\Omega_T)\| + \|\,\theta\,;\, L^{m_2(q)}(\Omega_T)\|\big). \end{aligned} \tag{A.1.9}$$

Here we have denoted

$$m_j(r) = \begin{cases} \frac{4r}{4-jr} & \text{for } 4-jr>0, \\ \infty & \text{for } 4-jr \le 0. \end{cases}$$

We shall show that the map $\Phi(u,\theta)$ is a contraction from $\mathcal{V}_T^M(p,q)$ into $\mathcal{V}_T^M(p,q)$, where positive numbers δ and M are determined later. We only prove the case of $q < 2$ and $p < 2$, hence, $m_2(p) = \frac{2p}{2-p}$, $m_1(q) = \frac{4q}{4-q}$ and $m_2(q) = \frac{2q}{2-q}$. The proofs in the other cases follow from the easy modifications. Without loss of generality, we may assume $T < 1$.

From the Hölder inequality it follows that

$$\begin{aligned} \|\,\nabla\cdot F_{,\epsilon}\,;\, L^p(\Omega_T)\| &\le C\left(\int_{\Omega_T} |\theta F_{1,\epsilon\epsilon}(\epsilon):\nabla\epsilon|^p dxdt\right)^{\frac{1}{p}} + C\left(\int_{\Omega_T} |F_{1,\epsilon}\cdot\nabla\theta|^p dxdt\right)^{\frac{1}{p}} \\ &\quad + C\left(\int_{\Omega_T} |F_{2,\epsilon\epsilon}(\epsilon):\nabla\epsilon|^p dxdt\right)^{\frac{1}{p}} \\ &\le C\left(\int_{\Omega_T} |\theta|^p|\epsilon|^{p(K_1-2)}|\nabla\epsilon|^p dxdt\right)^{\frac{1}{p}} + C\left(\int_{\Omega_T} |\epsilon|^{p(K_1-1)}|\nabla\theta|^p dxdt\right)^{\frac{1}{p}} \\ &\quad + C\left(\int_{\Omega_T} |\epsilon|^{p(K_2-2)}|\nabla\epsilon|^p dxdt\right)^{\frac{1}{p}} \\ &\le CT^{1-\frac{1}{q}}\|\,\epsilon\,;\, L^\infty(\Omega_T)\|^{K_1-2}\|\,\nabla\epsilon\,;\, L^{\frac{2p}{2-p}}(\Omega_T)\|\|\,\theta\,;\, L^{\frac{2q}{2-q}}(\Omega_T)\| \\ &\quad + T^{\frac{1}{p}-\frac{1}{q}+\frac{1}{4}}\|\,\epsilon\,;\, L^\infty(\Omega_T)\|^{K_1-1}\|\,\nabla\cdot\theta\,;\, L^{\frac{4q}{4-q}}(\Omega_T)\| \\ &\quad + CT^{\frac{1}{2}}\|\,\epsilon\,;\, L^\infty(\Omega_T)\|^{K_2-2}\|\,\nabla\epsilon\,;\, L^{\frac{2p}{2-p}}(\Omega_T)\| \end{aligned}$$

and

$$\begin{aligned} \|\,F_{,\epsilon}\,;\, L^{2q}(\Omega_T)\| &\le CT^{1/2q} + C\|\,|\epsilon|^{K_1-1}\theta\,;\, L^{2q}(\Omega_T)\| + C\|\,|\epsilon|^{K_2-1}\,;\, L^{2q}(\Omega_T)\| \\ &\le CT^{1/2q} + CT^{\frac{1}{2}\left(1-\frac{1}{q}\right)}\|\,\epsilon\,;\, L^\infty(\Omega_T)\|^{K_1-1}\|\,\theta\,;\, L^{\frac{2q}{2-q}}(\Omega_T)\| \\ &\quad + CT^{1/2q}\|\,\epsilon\,;\, L^\infty(\Omega_T)\|^{K_2-1}. \end{aligned}$$

Notice that all the exponents of power of T are positive from the assumptions (A.1.1). Here we choose

$$\delta = \min\left\{\frac{1}{4}\left(1-\frac{1}{q}\right),\ \frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}+\frac{1}{4}\right)\right\} \quad (>0).$$

Then we obtain

$$
\begin{aligned}
\| \nabla\cdot F_{,\epsilon}\,;\, L^p(\Omega_T)\| &\le CT^{\left(1-\frac{1}{q}\right)-\delta}\| \nabla\cdot\epsilon\,;\, L^{m_2(p)}\|\|\epsilon\,;\, L^\infty\|^{K_1-2}\left(T^\delta\|\theta\,;\, L^{m_2(q)}\|\right)\\
&\quad + CT^{\left(\frac{1}{p}-\frac{1}{q}+\frac{1}{4}\right)-\delta}\left(T^\delta\|\nabla\theta\,;\, L^{m_1(q)}(\Omega_T)\|\right)\|\epsilon\,;\, L^\infty(\Omega_T)\|^{K_1-1}\\
&\quad + CT^{\frac{1}{2}}\|\epsilon\,;\, L^\infty(\Omega_T)\|^{K_2-1}\|\nabla\cdot\epsilon\,;\, L^{m_2(p)}(\Omega_T)\|\\
&\le \Lambda_2\tilde{h}_1(\|(\mathbf{u},\theta)\,;\,\mathcal{V}_T(p,q)\|)\|(\mathbf{u},\theta)\,;\,\mathcal{V}_T(p,q)\|
\end{aligned}
$$

and

$$
\begin{aligned}
\| F_{,\epsilon}\,;\, L^{2q}(\Omega_T)\| &\le CT^{1/2q} + CT^{\frac{1}{2}\left(1-\frac{1}{q}\right)-\delta}\|\epsilon\,;\, L^\infty(\Omega_T)\|^{K_1-1}\left(T^\delta\|\theta\,;\, L^{m_2(q)}(\Omega_T)\|\right)\\
&\quad + CT^{1/2q}\|\epsilon\,;\, L^\infty(\Omega_T)\|^{K_2-1}\\
&\le CT^{1/2q} + \Lambda_2\tilde{h}_1(\|(\mathbf{u},\theta)\,;\,\mathcal{V}_T(p,q)\|)\|(\mathbf{u},\theta)\,;\,\mathcal{V}_T(p,q)\|,
\end{aligned}
$$

where $\tilde{h}_1(y) := y^{K_1-1} + y^{K_2-2}$ and $\Lambda_2 = CT^C$ for some constant C.

By the maximal regularity (3.2.2) we have

$$
\begin{aligned}
\|\widetilde{\mathbf{u}}\,;\, W_p^{4,2}(\Omega_T)\| &\le C(\|(\mathbf{u}_0,\mathbf{u}_1,0)\,;\, U(p,q)\| + 1)\\
&\quad + \Lambda_2\tilde{h}_1(\|(\mathbf{u},\theta)\,;\,\mathcal{V}_T(p,q)\|)\|(\mathbf{u},\theta)\,;\,\mathcal{V}_T(p,q)\|.
\end{aligned}
$$

By Proposition A.1.3 we also obtain

$$
\begin{aligned}
\|\widetilde{\epsilon}\,;\, L^\infty(\Omega_T)\| &\le C\|(\mathbf{u}_0,\mathbf{u}_1,0)\,;\, U(p,q)\| + C\|\nabla\cdot F_{,\epsilon}\,;\, L^p(\Omega_T)\|\\
&\le C(\|(\mathbf{u}_0,\mathbf{u}_1,0)\,;\, U(p,q)\| + 1)\\
&\quad + \Lambda_2\tilde{h}_1(\|(\mathbf{u},\theta)\,;\,\mathcal{V}_T(p,q)\|)\|(\mathbf{u},\theta)\,;\,\mathcal{V}_T(p,q)\|,\\
\|\nabla\cdot\widetilde{\epsilon}\,;\, L^{m_2(p)}(\Omega_T)\| &\le C\|(\mathbf{u}_0,\mathbf{u}_1,0)\,;\, U(p,q)\| + C\|\nabla\cdot F_{,\epsilon}\,;\, L^p(\Omega_T)\|\\
&\le C(\|(\mathbf{u}_0,\mathbf{u}_1,0)\,;\, U(p,q)\| + 1)\\
&\quad + \Lambda_2\tilde{h}_1(\|(\mathbf{u},\theta)\,;\,\mathcal{V}_T(p,q)\|)\|(\mathbf{u},\theta)\,;\,\mathcal{V}_T(p,q)\|.
\end{aligned}
$$

It follows from the embedding $B_{p,p}^{4-2/p} \hookrightarrow B_{2q,2q}^{3-1/q}$ for $\frac{2}{p}-\frac{1}{q}\le\frac{1}{2}$ and the maximal regularity (3.2.3) that

$$
\begin{aligned}
\|\widetilde{\epsilon}_t\,;\, L^{2q}(\Omega_T)\| &\le C\left\|(\mathbf{u}_0,\mathbf{u}_1)\,;\, B_{2q,2q}^{3-1/q}\times B_{2q,2q}^{1-1/q}\right\| + C\|F_{,\epsilon}\,;\, L^{2q}(\Omega_T)\|\\
&\le C(\|(\mathbf{u}_0,\mathbf{u}_1,0)\,;\, U(p,q)\| + 1)\\
&\quad + \Lambda_2\|\epsilon\,;\, L^\infty(\Omega_T)\|\left(T^\delta\|\theta\,;\, L^{m_2(q)}(\Omega_T)\|\right)\\
&\quad + \Lambda_2\|\epsilon\,;\, L^\infty(\Omega_T)\|^{K_2-1}.
\end{aligned}
$$

On the other hand, by the maximal regularity (3.2.20) for the heat equation, we have

$$
\begin{aligned}
T^\delta \| \widetilde{\theta}\,;\, W^{2,1}_q(\Omega_T)\| \le{}& CT^\delta \| (0,0,\theta_0)\,;\, U(p,q)\| \\
&+ CT^\delta \| \epsilon_t\,;\, L^{2q}(\Omega_T)\| \, \| |\epsilon|^{K_1-1}\theta\,;\, L^{2q}(\Omega_T)\| \\
&+ CT^\delta \| \epsilon_t\,;\, L^{2q}(\Omega_T)\|^2 \\
\le{}& C\| (0,0,\theta_0)\,;\, U(p,q)\| \\
&+ \Lambda_2 \tilde{h}_2 \left(\| (\mathbf{u},\theta)\,;\, \mathcal{V}_T(p,q)\|\right) \| (\mathbf{u},\theta)\,;\, \mathcal{V}_T(p,q)\|,
\end{aligned}
$$

where $\tilde{h}_2(y) := y^{K_1-1} + y$. In the same way as above, it follows from (A.1.5) and (A.1.6) that

$$
\begin{aligned}
T^\delta \| \widetilde{\theta}\,;\, L^{m_2(q)}(\Omega_T)\| \le{}& C(\| (0,0,\theta_0)\,;\, U(p,q)\|) \\
&+ \Lambda_2 \tilde{h}_2(\| (\mathbf{u},\theta)\,;\, \mathcal{V}_T(p,q)\|)\| (\mathbf{u},\theta)\,;\, \mathcal{V}_T(p,q)\|, \\
T^\delta \| \nabla\widetilde{\theta}\,;\, L^{m_1(q)}(\Omega_T)\| \le{}& C(\| (0,0,\theta_0)\,;\, U(p,q)\|) \\
&+ \Lambda_2 \tilde{h}_2(\| (\mathbf{u},\theta)\,;\, \mathcal{V}_T(p,q)\|)\| (\mathbf{u},\theta)\,;\, \mathcal{V}_T(p,q)\|.
\end{aligned}
$$

Consequently, combining these inequalities, we arrive at

$$
\begin{aligned}
\| (\widetilde{\mathbf{u}},\widetilde{\theta})\,;\, \mathcal{V}_T(p,q)\| \le{}& C(\| (\mathbf{u}_0,\mathbf{u}_1,\theta_0)\,;\, U(p,q)\|) \\
&+ \Lambda_2 \tilde{h}(\| (\mathbf{u},\theta)\,;\, \mathcal{V}_T(p,q)\|)\| (\mathbf{u},\theta)\,;\, \mathcal{V}_T(p,q)\|,
\end{aligned}
$$

where $\tilde{h}(y) := \tilde{h}_1(y) + \tilde{h}_2(y)$.

Similarly, for $(\mathbf{u},\theta)$ and $(\overline{\mathbf{u}},\overline{\theta}) \in \mathcal{V}^M_T(p,q)$ we have

$$
\begin{aligned}
&\| \Phi(\mathbf{u},\theta) - \Phi(\overline{\mathbf{u}},\overline{\theta})\,;\, \mathcal{V}_T(p,q)\| \\
&\qquad \le \Lambda_2[h(\| (\mathbf{u},\theta)\,;\, \mathcal{V}_T(p,q)\|) + h(\| (\overline{\mathbf{u}},\overline{\theta})\,;\, \mathcal{V}_T(p,q)\|)] \\
&\qquad\qquad \times \| (\mathbf{u},\theta) - (\overline{\mathbf{u}},\overline{\theta})\,;\, \mathcal{V}_T(p,q)\|.
\end{aligned}
$$

Indeed, it holds that

$$
\begin{aligned}
|F_{,\epsilon}(\epsilon,\theta) - F_{,\epsilon}(\overline{\epsilon},\overline{\theta})| \le{}& (|F_{1,\epsilon}(\epsilon)\theta - F_{1,\epsilon}(\overline{\epsilon})\theta| + |F_{1,\epsilon}(\overline{\epsilon})\theta - F_{1,\epsilon}(\overline{\epsilon})\overline{\theta}|) \\
&+ |F_{2,\epsilon}(\epsilon) - F_{2,\epsilon}(\overline{\epsilon})| \\
\le{}& C|\theta|(|\epsilon|^{K_1-2} + |\overline{\epsilon}|^{K_1-2})|\epsilon - \overline{\epsilon}| + C|\overline{\epsilon}|^{K_1-1}|\theta - \overline{\theta}| \\
&+ C(|\epsilon|^{K_2-2} + |\overline{\epsilon}|^{K_2-2})|\epsilon - \overline{\epsilon}|,
\end{aligned}
$$

where $\overline{\epsilon} = \epsilon(\overline{\mathbf{u}})$. In order to obtain the above inequality, we have used the following inequality:

$$
\begin{aligned}
|F_{1,\epsilon}(\epsilon) - F_{1,\epsilon}(\overline{\epsilon})| \le{}& |\epsilon - \overline{\epsilon}| \int_0^1 |F_{1,\epsilon\epsilon}(s(\epsilon - \overline{\epsilon}) + \overline{\epsilon})| ds \\
\le{}& C|\epsilon - \overline{\epsilon}|(|\overline{\epsilon}|^{K_1-2} + |\epsilon|^{K_1-2}),
\end{aligned}
$$

since $\sup_{s\in[0,1]} F_{1,\epsilon\epsilon}(s(\epsilon-\bar{\epsilon})+\bar{\epsilon}) \le C(|\epsilon|^{K_1-2}+|\bar{\epsilon}|^{K_1-2})$ holds.

We put $2M := C(\|\,(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\,;\,U(p,q)\|+1)$, and choose the time T sufficiently small such that

$$\Lambda_2 h(M) < \frac{1}{2}.$$

We note that $\lim_{T\to 0}\Lambda_2 = 0$. Then we obtain the unique existence of the solution for (A.0.1)–(A.0.4) in $\mathcal{V}_T^M(p,q)$. To prove the uniqueness in the whole space $\mathcal{V}_T(p,q)$, it is enough to take T sufficiently small. By the embedding (Lemma 3.1.1) we have $\mathcal{V}_T(p,q) = V_T(p,q)$. For the other case of p, we can prove the result in the same way. Then the desired result is obtained.

□

A.2 Small Energy Global Existence

Our main purpose of this section is to obtain the global estimate of the above solution class. At the first step we state the energy conservation law.

Lemma A.2.1 (Energy Conservation Law). *Assume that F satisfies the conditions (N2)–(N4) with* (A.0.5). *Then for any $t\in[0,\infty)$ the smooth solution of* (A.0.1)–(A.0.4) *satisfies*

$$\|\theta(t)\|_{L^1(\Omega)} + \|\mathbf{u}_t(t)\|_{L^2(\Omega)} + \|\mathbf{u}(t)\|_{H^2(\Omega)} \le K(\|(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\|_E), \tag{A.2.1}$$

where $K(x)$ is the monotone increasing function such that $K(0)=0$.

Proof. Multiplying (A.0.1) by u_t and integrating the resulting equation with respect to the space variable, we have

$$\frac{d}{dt}\left(\frac{1}{2}\|\,\mathbf{u}_t\,;\,L^2\|^2 + \frac{1}{2}\|\,Q\mathbf{u}\,;\,L^2\|^2 + \int_\Omega F_2(\epsilon)dx\right) + \nu\int_\Omega (A\epsilon_t):\epsilon_t dx = -\int_\Omega \theta F_{1,\epsilon}(\epsilon):\epsilon_t dx.$$

Integrating (A.0.2) with respect to x, we obtain

$$\frac{d}{dt}\int_\Omega \theta dx = \nu\int_\Omega (A\epsilon_t):\epsilon_t dx + \int \theta F_{1,\epsilon}(\epsilon):\epsilon_t dx.$$

Combining these equalities, we deduce

$$\frac{d}{dt}\left(\frac{1}{2}\|\,\mathbf{u}_t\,;\,L^2\|^2 + \frac{1}{2}\|\,Q\mathbf{u}\,;\,L^2\|^2 + \int_\Omega F_2(\epsilon)dx + \int_\Omega \theta dx\right) = 0.$$

Then from $F_2(\epsilon) \geq 0$ and the Sobolev embedding $H^1 \hookrightarrow L^{K_2}$ for any $K_2 < \infty$ it follows that

$$\begin{aligned}\frac{1}{2}\| \mathbf{u}_t(t)\,;L^2\|^2 + \frac{1}{2}\| Q\mathbf{u}(t)\,;\, L^2\|^2 + \int_\Omega \theta(t)dx \\ \leq \frac{1}{2}\| \mathbf{u}_1\,;\, L^2\|^2 + \frac{1}{2}\| Q\mathbf{u}_0\,;\, L^2\|^2 + \int_\Omega \theta_0 dx + C\| \epsilon(0)\,;\, L^{K_2}\|^{K_2} \\ \leq \frac{1}{2}\| \mathbf{u}_1\,;\, L^2\|^2 + \frac{1}{2}\| \mathbf{u}_0\,;\, H^2\|^2 + \| \theta_0\,;\, L^1\| + C\| \mathbf{u}_0\,;\, H^2\|^{K_2}.\end{aligned}$$

By the maximum principle (Lemma 3.3.2), if $\theta_0 \geq 0$, then we obtain $\theta(t) \geq 0$ for sufficiently smooth solution $(\mathbf{u}, \theta)$ (e.g. $(\mathbf{u}, \theta) \in W_p^{4,2}(\Omega_T) \times L_T^\infty L^2$ for $p > 4$). We have completed the proof of Lemma A.2.1. □

Using this energy bound, we can obtain the following global bound.

Theorem A.2.2. *Let $T < \infty$ be arbitrarily fixed. Assume that F satisfies the conditions* (N2)*–*(N4) *with* (A.0.5)*. Let $(\mathbf{u}, \theta)$ be the solution of* (A.0.1)*–*(A.0.4) *for $(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \in U^p$.*

There exists $\eta > 0$ independent of T such that if $(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \in U^p$ satisfies

$$\|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_E \leq \eta,$$

then we have

$$\| \mathbf{u}\,;\, W_p^{4,2}(\Omega_T)\| + \| \theta\,;\, W_{\frac{3p}{4}}^{2,1}(\Omega_T)\| \leq \Lambda,$$

where Λ depends on p, T, η, M, and $\|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_{U^p}$.

Proof. We first note that the following Gagliardo-Nirenberg type estimates hold

$$\begin{aligned}\| \theta\,;\, L^{\frac{3p}{2}}(\Omega_T)\| &\leq C\| \theta\,;\, L_T^\infty L_x^1\|^{\frac{1}{2}}\| \theta\,;\, W_{\frac{3p}{4}}^{2,1}(\Omega_T)\|^{\frac{1}{2}},\\ \| \nabla\theta\,;\, L^p(\Omega_T)\| &\leq C\| \theta\,;\, L_T^\infty L_x^1\|^{\frac{1}{4}}\| \theta\,;\, W_{\frac{3p}{4}}^{2,1}(\Omega_T)\|^{\frac{3}{4}},\\ \| \mathbf{u}_t\,;\, L^{3p}(\Omega_T)\| &\leq C\| \mathbf{u}_t\,;\, L_T^\infty L_x^2\|^{\frac{2}{3}}\| \mathbf{u}\,;\, W_p^{4,2}(\Omega_T)\|^{\frac{1}{3}},\\ \| \nabla\cdot\epsilon\,;\, L^{3p}(\Omega_T)\| &\leq C\| \mathbf{u}\,;\, L_T^\infty H_x^2\|^{\frac{2}{3}}\| \mathbf{u}\,;\, W_p^{4,2}(\Omega_T)\|^{\frac{1}{3}},\\ \| \epsilon_t\,;\, L^{\frac{3p}{2}}(\Omega_T)\| &\leq C\| \mathbf{u}_t\,;\, L_T^\infty L_x^2\|^{\frac{1}{3}}\| \mathbf{u}\,;\, W_p^{4,2}(\Omega_T)\|^{\frac{2}{3}}\end{aligned}$$

and that from the Sobolev inequality it follows that

$$\| \epsilon\,;\, L_T^\infty L_x^a\| \leq C\| \mathbf{u}\,;\, L_T^\infty H_x^2\|$$

for any $a < \infty$, where C is independent of T.

Since $F_1 \in \mathrm{C}^3(\mathrm{Sym}(2,\mathbb{R}),\mathbb{R})$, we have

$$\sup_{|\epsilon|\leq\epsilon_c} (|F_{1,\epsilon\epsilon}(\epsilon)| + |F_{1,\epsilon}(\epsilon)|) \leq C.$$

For $K_1 \in [0,1]$ it holds that

$$\sup_{|\epsilon|>\epsilon_c} \left(|F_{1,\epsilon\epsilon}(\epsilon)| + |F_{1,\epsilon}(\epsilon)|\right) \le C \sup_{|\epsilon|>\epsilon_c} \left(|\epsilon|^{K_1-2} + |\epsilon|^{K_1-1}\right) \le C.$$

Therefore we have

$$|F_{1,\epsilon\epsilon}| + |F_{1,\epsilon}| \le M,$$

where M is a positive constant depending only on F_1.

By the energy conservation law (A.2.1) and the Young inequality, we have

$$\begin{aligned}
\|\nabla\cdot F_{,\epsilon}\,;\,L^p(\Omega_T)\| &\le C\|F_{1,\epsilon\epsilon}\,;\,L^\infty(\Omega_T)\|\|\nabla\cdot\epsilon\,;\,L^{3p}(\Omega_T)\|\|\theta\,;\,L^{\frac{3p}{2}}(\Omega_T)\| \\
&\quad + C\|F_{1,\epsilon}\,;\,L^\infty(\Omega_T)\|\|\nabla\theta\,;\,L^p(\Omega_T)\| \\
&\quad + C\|F_{2,\epsilon\epsilon}\,;\,L^{\frac{3p}{2}}(\Omega_T)\|\|\nabla\cdot\epsilon\,;\,L^{3p}(\Omega_T)\| \\
&\le CMK(\eta)^{\frac{2}{3}+\frac{1}{2}}\|\mathbf{u}\,;\,W_p^{4,2}(\Omega_T)\|^{\frac{1}{3}}\|\theta\,;\,W_{\frac{3p}{4}}^{2,1}(\Omega_T)\|^{\frac{1}{2}} \\
&\quad + CMK(\eta)^{\frac{1}{4}}\|\theta\,;\,W_{\frac{3p}{4}}^{2,1}(\Omega_T)\|^{\frac{3}{4}} \\
&\quad + CT^{\frac{2}{3p}}K(\eta)^{K_2+\frac{2}{3}}\|\mathbf{u}\,;\,W_p^{4,2}(\Omega_T)\|^{\frac{1}{3}} \\
&\le \frac{1}{2}\|\mathbf{u}\,;\,W_p^{4,2}(\Omega_T)\| + \frac{2}{3}\sqrt{\frac{2}{3}}CMK(\eta)^{\frac{7}{6}}\|\theta\,;\,W_{\frac{3p}{4}}^{2,1}(\Omega_T)\|^{\frac{3}{4}} \\
&\quad + CMK(\eta)^{\frac{1}{4}}\|\theta\,;\,W_{\frac{3p}{4}}^{2,1}(\Omega_T)\|^{\frac{3}{4}} + \frac{2}{3}\sqrt{\frac{2}{3}}CT^{\frac{2}{3p}}K(\eta)^{K_2+\frac{2}{3}}.
\end{aligned}$$

From the maximal regularity estimates (3.2.2) and (3.2.20) it follows that

$$\begin{aligned}
\|\mathbf{u}\,;\,W_p^{4,2}(\Omega_T)\| &\le C\|(\mathbf{u}_0,\mathbf{u}_1,0)\|_{U^p} + \frac{4}{3}\sqrt{\frac{2}{3}}CT^{\frac{2}{3p}}K(\eta)^{K_2+\frac{2}{3}} \\
&\quad + \left(\frac{4}{3}\sqrt{\frac{2}{3}}K(\eta)^{\frac{7}{6}} + 2K(\eta)^{\frac{1}{4}}\right)CM\|\theta\,;\,W_{\frac{3p}{4}}^{2,1}(\Omega_T)\|^{\frac{3}{4}}
\end{aligned} \tag{A.2.2}$$

and

$$\begin{aligned}
\|\theta\,;\,W_{\frac{3p}{4}}^{2,1}(\Omega_T)\| &\le C\|(0,0,\theta_0)\|_{U^p} + C\|\epsilon_t\,;\,L^{\frac{3p}{2}}(\Omega_T)\|^2 \\
&\quad + C\|\epsilon_t\,;\,L^{\frac{3p}{2}}(\Omega_T)\|\|\theta\,;\,L^{\frac{3p}{2}}(\Omega_T)\|\|F_{1,\epsilon}\,;\,L^\infty(\Omega_T)\| \\
&\le C\|(0,0,\theta_0)\|_{U^p} + CK(\eta)^{\frac{2}{3}}\|\mathbf{u}\,;\,W_p^{4,2}(\Omega_T)\|^{\frac{4}{3}} \\
&\quad + CMK(\eta)^{\frac{1}{3}+\frac{1}{2}}\|\mathbf{u}\,;\,W_p^{4,2}(\Omega_T)\|^{\frac{2}{3}}\|\theta\,;\,W_p^{2,1}(\Omega_T)\|^{\frac{1}{2}}.
\end{aligned} \tag{A.2.3}$$

Substituting (A.2.2) into (A.2.3), we have

$$\|\theta\,;\,W_{\frac{3p}{4}}^{2,1}(\Omega_T)\| \le C(\|(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\|_{U^p}, T, M, K(\eta)) + \tilde{C}(M,K(\eta))\|\theta\,;\,W_{\frac{3p}{4}}^{2,1}(\Omega_T)\|.$$

Here we note that $\lim_{\eta \to 0} \tilde{C}(M, K(\eta)) = 0$. Therefore, taking η sufficiently small such that

$$\tilde{C}(M, K(\eta)) < \frac{1}{2},$$

we obtain

$$\| \theta \,;\, W^{2,1}_{\frac{3p}{4}}(\Omega_T) \| \le C(\|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_{U^p}, T, M, K(\eta)). \tag{A.2.4}$$

Substituting (A.2.4) into (A.2.2) yields

$$\| \mathbf{u} \,;\, W^{4,2}_p(\Omega_T) \| \le C(\|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_{U^p}, T, M, K(\eta)),$$

which completes the proof. □

Theorem A.1 immediately follows from Theorems A.1.1 and A.2.2.

References

[1] R. A. Adams and J. J. F. Fournier, *Sobolev spaces* (*2nd ed.*), Academic Press, Amsterdam, 2003.

[2] T. Aiki, Weak solutions for Falk's model of shape memory alloys, Math. Methods Appl. Sci. **23** (2000), 299–319.

[3] T. Aiki, A model of 3D shape memory alloy materials, J. Math. Soc. Japan **57** (2005), 903–933.

[4] T. Aiki, One-dimensional shape memory alloy problems including a hysteresis operator, Funkcial. Ekvac. **46** (2003), 441–469.

[5] T. Aiki, A. Kadoya and S. Yoshikawa, Hysteresis model for one-dimensional shape memory alloy with small viscosity, Mathematical Approach to Nonlinear Phenomena: Modelling, Analysis and Simulations, 1–8, GAKUTO Internat. Ser. Math. Sci. Appl. **23**, Gakkotosho, Tokyo, 2005.

[6] T. Aiki and N. Kenmochi, Models for shape memory alloys described by subdifferentials of indicator functions, Elliptic and parabolic problems (Rolduc/Gaeta, 2001), 1–10, World Sci. Publishing, River Edge, NJ, 2002.

[7] H. Amann, *Linear and quasilinear parabolic problems, Vol. I, Abstract linear theory*, Monographs in Mathematics **89**, Birkhäuser, Basel, 1995.

[8] J. P. Aubin, Un théorème de comapcité, C. R. Acad. Sci. Paris Sér. I Math. **256** (1963), 5042–5044.

[9] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I, II. , Geom. Funct. Anal. **3** (1993), 107–156, 209–262.

[10] M. Brokate and J. Sprekels, *Hysteresis and phase transitions*, Appl. Math. Sci. **121**, Springer, Berlin, 1996.

[11] N. Bubner and J. Sprekels, Optimal control of martensitic phase transitions in a deformation-driven experiment on shape memory alloys, Adv. Math. Sci. Appl. **8** (1998), 299–325.

[12] T. Cazenave, *Semilinear Schrödinger equations,* Courant Lect. Notes Math. **10**, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[13] T. Cazenave and A. Haraux, *An introduction to semilinear evolution equations,* Oxford Lecture Ser. Math. Appl. **13**, The Clarendon Press, Oxford University Press, New York, 1998.

[14] P. Clément and S. Li, Abstract parabolic quasilinear equations and application to a groundwater flow problem, Adv. Math. Sci. Appl. **3** (1993/94), 17–32.

[15] C. M. Dafermos and L. Hsiao, Global smooth thermomechanical processes in one-dimensional nonlinear thermoviscoelasticity, Nonlinear Anal. **6** (1982), 435–454.

[16] L. de Simon, Un'applicazione della teoria degli integrali singolri allo studio delle equazioni differenziali astratta del primo ordine, Rend. Sem. Mat. Univ. Padova **34** (1964), 205–223.

[17] R. Denk, G. Dore, M. Hieber, J. Prüss and A. Venni, New thoughts on old results of R. T. Seeley, Math. Ann. **328** (2004), 545–583.

[18] R. Denk, M. Hieber and J. Prüss, *$\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type,* Mem. Amer. Math. Soc. **166** (2003), no. 788.

[19] G. Dore and A. Venni, On the closedness of the sum of two closed operators, Math. Z. **196** (1987), 189–201.

[20] X. T. Duong, $\mathcal{H}_\infty$ functional calculus of second order elliptic partial differential operators on L^p spaces, Miniconference on operators in analysis, Proc. Centre Math. Anal. A.N.U. **24** (1989), 91–102.

[21] F. Falk, Elastic phase transitions and nonconvex energy functions, Free boundary problems: theory and applications, Vol. I (Irsee, 1987), 45–59, Pitman Res. Notes Math. Ser. **185**, Longman Sci. Tech., Harlow, 1990.

[22] F. Falk and P. Konopka, Three-dimensional Landau theory describing the martensitic phase transformation of shape memory alloys, Journal of Physics: Condensed Matter **2** (1990), 61–77.

[23] F. Falk, E. W. Laedke and K. H. Spatschek, Stability of solitary-wave pulses in shape-memory alloys, Physical Rev. B **36** (1987), 3031–3041.

[24] Y. Fang and M.-G. Grillakis, Existence and uniqueness for Boussinesq type equations on a circle, Comm. Partial Differential Equations **21** (1996), 1253–1277.

[25] A. Friedman and J. Nečas, Systems of nonlinear wave equations with nonlinear viscosity, Pacific J. Math. **135** (1988), 29–55.

[26] D. Fujiwara, L^p-theory for characterizing the domain of the fractional powers of $-\Delta$ in the half space, J. Fac. Sci. Univ. Tokyo Sect. I. **15** (1968), 169–177.

[27] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order (2nd ed.),* Grundlehren der Mathematischen Wissenschaften **224**, Springer-Verlag, Berlin, 1983.

[28] M. Hieber and J. Prüss, Heat kernels and maximal L^p-L^q estimates for parabolic evolution equations, Comm. Partial Differential Equations **22** (1997), 1647–1669.

[29] K.-H. Hoffmann and A. Żochowski, Existence of solutions to some non-linear thermoelastic systems with viscosity, Math. Methods Appl. Sci. **15** (1992), 187–204.

[30] P. Krejčí and J. Sprekels, On a system of nonlinear PDEs with temperature-dependent hysteresis in one-dimensional thermoplasticity, J. Math. Anal. Appl. **209** (1997), 25–46.

[31] P. Krejčí and J. Sprekels, Temperature-dependent hysteresis in one-dimensional thermovisco-elastoplasticity, Appl. Math. **43** (1998), 173–205.

[32] O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and quasi-linear equations of parabolic type,* Trans. Math. Monographs **23**, American Mathematical Society, Providence, RI, 1967.

[33] P. G. Lemarié-Rieusset, *Recent development in the Navier-Stokes problem.* Chapman & Hall/CRC Res. Notes Math. **431**, Chapman & Hall/CRC, Boca Raton, FL, 2002.

[34] G. M. Lieberman, *Second order parabolic differential equations,* World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[35] I. Pawłow, Three-dimensional model of thermomechanical evolution of shape memory materials, Control Cybernet. **29** (2000), 341–365.

[36] I. Pawłow and A. Żochowski, Existence and uniqueness of solutions for a three-dimensional thermoelastic system, Dissertationes Math. (Rozprawy Mat.) **406** (2002), 1–46.

[37] I. Pawłow and W. M. Zajączkowski, Global existence to a three-dimensional nonlinear thermoelasticity system arising in shape memory materials, Math. Methods Appl. Sci. **28** (2005), 407–442.

[38] I. Pawłow and W. M. Zajączkowski, Unique global solvability in two-dimensional non-linear thermoelasticity, Math. Methods Appl. Sci. **28** (2005), 551–592.

[39] P. Rybka, Dynamical modelling of phase transitions by means of viscoelasticity in many dimensions, Proc. Roy. Soc. Edinburgh Sect. A **121** (1992), 101–138.

[40] V. A. Solonnikov, On boundary value problems for linear parabolic systems of differential equations of general form, Trudy Mat. Inst. Steklov. **83** (1965), 3–163, (in Russian).

[41] J. Sprekels and S. Zheng, Global solutions to the equations of a Ginzburg-Landau theory for structural phase transitions in shape memory alloys, Phys. D **39** (1989), 59–76.

[42] J. Sprekels, S. Zheng and P. Zhu, Asymptotic behavior of the solutions to a Landau-Ginzburg system with viscosity for martensitic phase transitions in shape memory alloys, SIAM J. Math. Anal. **29** (1998), 69–84.

[43] R. S. Strichartz, Restriction of Fourier transforms to quadratic surface and decay of solutions of wave equations, Duke Math. J. **44** (1977), 705–714.

[44] R. Temam, *Navier-Stokes equations, theory and numerical analysis,* AMS Chelsea Publishing, Providence, RI, 2001

[45] H. Triebel, *Interpolation theory, function spaces, differential operators,* North-Holland Mathematical Library **18**, North-Holland Publishing Co., Amsterdam-New York, 1978.

[46] L. Weis, Operator-valued Fourier multiplier theorems and maximal L_p-regularity, Math. Ann. **319** (2001), 735–758.

[47] S. Yoshikawa, Weak solutions for the Falk model system of shape memory alloys in energy class, Math. Methods Appl. Sci. **28** (2005), 1423–1443.

[48] S. Yoshikawa, Unique global existence for a three-dimensional thermoelastic system of shape memory alloys, Adv. Math. Sci. Appl. **15** (2005), 603–627.

[49] S. Yoshikawa, Small energy global existence for a two-dimensional thermoelastic system of shape memory materials, Mathematical approach to nonlinear phenomena: modelling analysis and simulations, 297–306, GAKUTO Internat. Ser. Math. Sci. Appl. **23**, Gakkotosho, Tokyo, 2005.

[50] S. Yoshikawa, Remarks on the energy class solution for the Falk model system of shape memory alloys, Nonlinear Dispersive Equations, 227–237, GAKUTO Internat. Ser. Math. Sci. Appl. **26**, Gakkotosho, Tokyo, 2007.

[51] S. Yoshikawa, On thermoelastic systems arising in shape memory alloys, MSJ-IRI 2005, Asymptotic Analysis and Singularity, Advanced Studies in Pure Mathematics, Kinokuniya, Tokyo, to appear.

[52] S. Yoshikawa, I. Pawłow and W. M. Zajączkowski, Quasilinear thermoelasticity system arising in shape memory materials, SIAM J. Math. Anal. **38** (2007), 1733–1759.

[53] J. Zimmer, Global existence for a nonlinear system in thermoviscoelasticity with nonconvex energy, J. Math. Anal. Appl. **292** (2004), 589–604.

Tohoku Mathematical Publications

No.1 Hitoshi Furuhata: *Isometric pluriharmonic immersions of Kähler manifolds into semi-Euclidean spaces*, 1995.

No.2 Tomokuni Takahashi: *Certain algebraic surfaces of general type with irregularity one and their canonical mappings*, 1996.

No.3 Takeshi Ikeda: *Coset constructions of conformal blocks*, 1996.

No.4 Masami Fujimori: *Integral and rational points on algebraic curves of certain types and their Jacobian varieties over number fields*, 1997.

No.5 Hisatoshi Ikai: *Some prehomogeneous representations defined by cubic forms*, 1997.

No.6 Setsuro Fujiié: *Solutions ramifiées des problèmes de Cauchy caractéristiques et fonctions hypergéométriques à deux variables*, 1997.

No.7 Miho Tanigaki: *Saturation of the approximation by spectral decompositions associated with the Schrödinger operator*, 1998.

No.8 Y. Nishiura, I. Takagi and E. Yanagida: *Proceedings of the International Conference on Asymptotics in Nonlinear Diffusive Systems — towards the Understanding of Singularities in Dissipative Structures —*, 1998.

No.9 Hideaki Izumi: *Non-commutative L^p-spaces constructed by the complex interpolation method*, 1998.

No.10 Youngho Jang: *Non-Archimedean quantum mechanics*, 1998.

No.11 Kazuhiro Horihata: *The evolution of harmonic maps*, 1999.

No.12 Tatsuya Tate: *Asymptotic behavior of eigenfunctions and eigenvalues for ergodic and periodic systems*, 1999.

No.13 Kazuya Matsumi: *Arithmetic of three-dimensional complete regular local rings of positive characteristics*, 1999.

No.14 Tetsuya Taniguchi: *Non-isotropic harmonic tori in complex projective spaces and configurations of points on Riemann surfaces*, 1999.

No.15 Taishi Shimoda: *Hypoellipticity of second order differential operators with sign-changing principal symbols*, 2000.

No.16 Tatsuo Konno: *On the infinitesimal isometries of fiber bundles*, 2000.

No.17 Takeshi Yamazaki: *Model-theoretic studies on subsystems of second order arithmetic*, 2000.

No.18 Daishi Watabe: *Dirichlet problem at infinity for harmonic maps*, 2000.

No.19 Tetsuya Kikuchi: *Studies on commuting difference systems arising from solvable lattice models*, 2000.

No.20 Seiki Nishikawa: *Proceedings of the Fifth Pacific Rim Geometry Conference*, 2001.

No.21 Mizuho Ishizaka: *Monodromies of hyperelliptic families of genus three curves*, 2001.

No.22 Keisuke Ueno: *Constructions of harmonic maps between Hadamard manifolds*, 2001.

No.23 Hiroshi Sato: *Studies on toric Fano varieties*, 2002.

No.24 Hiroyuki Kamada: *Self-dual Kähler metrics of neutral signature on complex surfaces*, 2002.

No.25 Reika Fukuizumi: *Stability and instability of standing waves for nonlinear Schrödinger equations*, 2003.

No.26 Tôru Nakajima: *Stability and singularities of harmonic maps into spheres*, 2003.

No.27 Yasutsugu Fujita: *Torsion of elliptic curves over number fields*, 2003.

No.28 Shin-ichi Ohta: *Harmonic maps and totally geodesic maps between metric spaces*, 2004.

No.29 Satoshi Ishiwata: *Geometric and analytic properties in the behavior of random walks on nilpotent covering graphs*, 2004.

No.30 Soondug Kim: *Computing in the Jacobian of a C_{34} curve*, 2004.

No.31 Mitsuko Onodera: *Study of rigidity problems for $C_{2\pi}$-manifolds*, 2006.

No.32 Shuji Yoshikawa: *Global solutions for shape memory alloy systems*, 2007.

Tohoku Mathematical Publications

Mathematical Institute
Tohoku University
Sendai 980-8578, Japan