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Study of rigidity problems for $C_{2\pi}$-manifolds

by

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A thesis presented

by

Mitsuko Onodera

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Contents

1 Introduction

A Riemannian manifold (M, g) is called a *C_l-manifold* if all geodesics are closed and have the same length l. We call g a C_l-metric. The standard sphere (S^n, g_0) is clearly an example of C_l-manifolds. Other examples are the standard projective spaces $(P^n(\mathbf{K}), g_0)$ (here $\mathbf{K}$ stands for either the field $\mathbf{R}$ of real numbers, or the field $\mathbf{C}$ of complex numbers, or the non-commutative field $\mathbf{H}$ of quaternions), and the Cayley projective plane $(P^2(\mathbf{Ca}), g_0)$. Here $\mathbf{Ca}$ denotes the Cayley algebra. They are often called CROSSes (i.e., compact rank one symmetric space). It has been an open problem in differential geometry to find all C_l-metrics and classify them.

For the standard sphere (S^2, g_0), every geodesic issuing from any point p in S^2 passes the antipodal point p', and the length from p to p' along each geodesic is always π. Let g be a Riemannian metric on S^2. Suppose every geodesic in g from any point p passes a point p' different from p, and the length from p to p' along each geodesic is always π. Is then (S^2, g) necessarily isometric to (S^2, g_0)? This problem was proposed by W. Blaschke in the first edition (1921) of [Bl]. He called these (S^2, g) *Wiedersehensflächen.* The Blaschke conjecture is that every Wiedersehensfläche is isometric to (S^2, g_0).

A Blaschke manifold, which is defined below, is an extension of the notion of the *Wiedersehensflächen.* Let (M, g) be a compact, connected Riemannian manifold and let p, q be points in M. Set

$$\begin{aligned}
\mathrm{Seg}(p, q) &= \{\text{minimizing geodesics from } p \text{ to } q \text{ (parametrized by arc length)}\}, \\
\Lambda(p, q) &= \{\dot\gamma(q) \in U_qM \mid \gamma \in \mathrm{Seg}(p, q)\},
\end{aligned}$$

where U_qM is the unit tangent space at $q \in M$. $\Lambda(p, q)$ is called the *link* from p to q.

Let $d(p, q)$ be the distance from p to q, and let $\gamma(t)$ be a geodesic in (M, g) issuing from $p = \gamma(0)$ which is parametrized by arc length. $q \in M$ is called a cut point of p along γ if there exists a positive real number t_1 such that $\gamma(t_1) = q$ satisfying $d(p, \gamma(t)) = t$ for any t in $[0, t_1]$ and $d(p, \gamma(t)) < t$ for any $t > t_1$. The cut locus of p, denoted by $\mathrm{Cut}(p)$, is the set of all cut points of p along geodesics issuing from p.

Definition *A compact Riemannian manifold (M, g) is called a Blaschke manifold at a point m in M if the link $\Lambda(m, c)$ is a great sphere of U_cM for every c in* $\mathrm{Cut}(m)$.

Definition *A compact Riemannian manifold (M, g) is called a Blaschke manifold if it is a Blaschke manifold at every point in M.*

The geometrical property of geodesics of a Blaschke manifold is similar to that of a CROSS. For example, if (M, g) is a Blaschke manifold, then it is a C_l-manifold (Proposition

2.1). The Blaschke conjecture, which is an extension of the conjecture by Blaschke for *Wiedersehensflächen*, claims that a Blaschke manifold is one of the CROSSes. The Blaschke conjecture has not completely been solved yet, but for a Blaschke manifold whose cut locus of any point is only one point, it was affirmatively solved by M. Berger (Theorem 2.1).

Are there examples of C_l-manifolds other than CROSSes? This question was affirmatively solved by the solution of the existence problem of infinitesimal $C_{2\pi}$-deformations of the standard sphere. In 1976, V. Guillemin obtained the following result.

Theorem 2.4 ([G]) *For every odd function $\dot{\rho}$ on S^2, there exists a smooth one-parameter family of C^∞-functions ρ_t such that $\rho_0 = 0$, $\dot{\rho} = \left.\dfrac{d\rho_t}{dt}\right|_{t=0}$, and $\exp(\rho_t)g_0$ is a $C_{2\pi}$-metric for small t.*

In 1903, O. Zoll constructed $C_{2\pi}$-metrics of revolution on a sphere which are not isometric to the standard one. (A surface of revolution means a surface whose metric has an S^1-action of isometries.) Let M be a manifold which is diffeomorphic to S^2. Then Zoll metrics are described as follows:

Theorem 3.6 ([Be]) *A Riemannian manifold (M, g) is a Zoll surface if and only if g is a metric of revolution which can be described in a parametrization $(U, (r, \theta))$ as*

$$g = \{1 + h(\cos r)\}^2 dr^2 + \sin^2 r\, d\theta^2,$$

where $h : [-1, 1] \to [-1, 1]$ is an odd function centered at zero which satisfies $h(1) = h(-1) = 0$, and $1 + h > 0$.

Note that if we take $h \equiv 0$, the corresponding Zoll metric is the standard one.

Although Zoll only mentioned a metric on a two-dimensional sphere S^2, it can be extended to a metric on an n-dimensional sphere S^n (see section 3.5). Therefore, the C_l-metrics of revolution on a sphere are completely classified. We do not have any examples of $C_{2\pi}$-manifolds other than CROSSes, infinitesimal $C_{2\pi}$-deformations of the standard sphere, and the Zoll surfaces.

According to Theorems 2.4 and 3.6, we know that Zoll surfaces and $C_{2\pi}$-deformations of the standard sphere are characterized by odd functions on a sphere. Hence we wish to characterize general $C_{2\pi}$-metrics on a sphere by giving conditions on a hemisphere.

First, we make some observations on a Zoll surface, since it is a fundamental model of a $C_{2\pi}$-surface. A Zoll surface can be uniquely divided into the Northen and the Southern Hemispheres, since it has a unique parallel which is called the equator (Lemma 3.4). Since a Zoll metric is characterized by an odd function centered at the equator, it holds that if the metric on the Northern Hemisphere of a Zoll surface is given, then the metric on the Southern Hemisphere is automatically determined. In particular, the following holds.

Theorem 5.1 *Let (M, g) be a Zoll surface whose Northern Hemisphere is the standard hemisphere. Then, (M, g) is the standard sphere.*

Motivated by this theorem, we would like to consider the following problem.

Problem 1 *Let (M, g) be a $C_{2\pi}$-surface. Let D be a closed domain in M such that its boundary ∂D is a geodesic in (M, g). Suppose (D, g) is the standard hemisphere. Then, is (M, g) the standard sphere?*

We first approach this problem from the viewpoint of billiard theory.

A billiard is the motion of a point mass (or the light ray) inside a compact convex closed domain D with smooth boundary in a Riemannian manifold. The orbits of such motion consist of geodesic segments inside D joined at boundary points according to the rule that the angle of incidence equals the angle of reflection. Let ζ be a billiard in D issuing from a point x in D with an initial angle θ. We say that ζ is periodic if it returns to x in finite time with the same angle θ. A positive integer p is called the rotation number of ζ if ζ rounds along ∂D p times while it returns to the initial point. For a positive integer q, a periodic billiard ζ is called a q-link periodic billiard if it contains q points of ∂D. The pair of positive integers (q, p) is called a period of the periodic billiard.

The properties of billiards on the Northern Hemisphere of the standard sphere are as follows:

1) all segments of the billiards have the same length π,

2) each billiard is $(2, 1)$-periodic.

Billiards on a Zoll surface do not always satisfy either of the two properties 1) and 2) above (section 5.1). If billiards on the Northern Hemisphere of a Zoll surface satisfy at least one of those two properties, the metric should be the standard one, as is stated in the following two theorems.

Theorem 5.2 *Let (M, g) be a Zoll surface such that all segments of the billiards on the Northern Hemisphere have the same length π. Then, (M, g) is the standard sphere.*

Theorem 5.3 *Let (M, g) be a Zoll surface such that all billiards on the Northern Hemisphere are periodic. Then its period should be $(2, 1)$, and (M, g) is the standard sphere.*

The following definition is useful for describing properties of the billiards on the standard hemisphere. Let (D, g) be a compact convex Riemannian manifold with smooth boundary.

Definition *We say the billiards on (D, g) have the spherical property if they satisfy:*

1) *all segments of the billiards have the same length π,*

2) *each billiard is $(2, 1)$-periodic.*

From a viewpoint of billiards, a rigidity problem for $C_{2\pi}$-surfaces can be stated as follows.

Problem 2 *Let (M, g) be a $C_{2\pi}$-surface. Let D be a closed domain in M such that its boundary ∂D is a geodesic in (M, g). Suppose the billiards on (D, g) have the spherical property. Then, is (M, g) the standard sphere?*

We solve this problem under a certain conformality condition. Let (M, g_0) be the two-dimensional standard sphere, and f be a smooth positive function on M.

Theorem 6.1 *Let (M, g) be a $C_{2\pi}$-surface with metric $g = f^2 g_0$. Let D be a closed domain in M such that its boundary ∂D is a geodesic in (M, g). Suppose (D, g) satisfies the following:*

i) the billiards on (D, g) have the spherical property,

ii) ∂D is a unit speed geodesic in g, and also in g_0.

Then, (M, g) is the standard sphere.

We now sketch the proof of Theorem 6.1. Billiard condition on a hemisphere makes it possible to use the method of boundary rigidity problems for Riemannian manifolds ([C]). If the billiards on (D, g) have the spherical property, its volume $\mathrm{vol}(D, g)$ is described by the geometry of the boundary of D. Hence, the assumption ii) gives us $\mathrm{vol}(D, g) = \mathrm{vol}(D, g_0)$. On the other hand, according to the Cauchy-Schwarz inequality, we have $\mathrm{vol}(D, g) \geq \mathrm{vol}(D, g_0)$, where the equality holds if and only if $g = g_0$. Hence we conclude that $g = g_0$ on D. Since (M, g) is a $C_{2\pi}$-surface, if the billiards on (D, g) have the spherical property, then the billiards on the complement of the interior of D in M, denoted by D', also have the spherical property. Then the same argument applies to D', and we obtain the conclusion.

If a hemisphere of a $C_{2\pi}$-surface, that is, a closed domain enclosed by a geodesic, is the standard hemisphere, the billiards on the remaining half have the spherical property. If we fully use the condition that a half of a $C_{2\pi}$-surface is the standard hemisphere, we can prove that the remaining half is also the standard hemisphere. Hence our Problem 1 is completely solved, and we get the following result. Here M denotes a manifold which is diffeomorphic to S^2.

Theorem 7.1 *Let (M, g) be a $C_{2\pi}$-surface. Let D be a closed domain in M such that its boundary ∂D is a geodesic in (M, g). Suppose (D, g) is the standard hemisphere. Then, (M, g) is the standard sphere.*

The sketch of the proof is as follows. By assumption, $M = (M, g)$ is divided into two manifolds with boundary, that is, $M = U \cup D$, where D is the standard hemisphere and U is the complement of the interior of D in M.

Let $M' = U' \cup D'$ be a copy of M. Since D and D' are the standard hemispheres, we can smoothly attach U to U' by identifying a point in ∂U with its antipodal point in $\partial U'$. We denote $\tilde{M} = U \cup U'$.

Let $\tilde{\tau}$ be a diffeomorphism from U to U' such that $\tilde{\tau}$ maps a point p in U to the same point p' in U'. Let ϕ be a natural diffeomorphism from M to M', and we identify M with M' by this ϕ. Then $\tilde{\tau}$ is defined as $\tilde{\tau}(p) = \phi(p)$, for any $p \in U$.

Let $\tilde{g}$ be the metric on $\tilde{M}$ defined by

$$\tilde{g} = \begin{cases} g & \text{on} \quad U \\ \\ g' & \text{on} \quad U', \end{cases}$$

where the metric g' is a metric on U' which satisfies $\tilde{\tau}^* g' = g$.

Let γ be a geodesic in M, and γ_U be the restriction of γ to U. Then the union of γ_U and $\tilde{\tau}(\gamma_U)$ is a geodesic in $(\tilde{M}, \tilde{g})$. Hence $(\tilde{M}, \tilde{g})$ is a $C_{2\pi}$-surface.

Then we can show that for any point p in $(\tilde{M}, \tilde{g})$, there exists a point p' in $(\tilde{M}, \tilde{g})$ such that every geodesic issuing from p passes p', and the length from p to p' along each geodesic is always π. Then $(\tilde{M}, \tilde{g})$ is a Blaschke manifold whose cut locus of any point is only one point. Hence, by the solution of the Blaschke conjecture for spheres (Theorem 2.1), $(\tilde{M}, \tilde{g})$ is the standard sphere. Therefore, (U, g) is the standard hemisphere, showing that (M, g) is the standard sphere.

The thesis is organized as follows: In section 2, some important properties of $C_{2\pi}$-manifolds are collected. First, we define the Blaschke manifolds, and state the Blaschke conjecture, and give the sketch of the proof by Green of the solution of the Blaschke conjecture for two-dimensional spheres. Green's result plays an important role in the proof of Theorem 7.1. Second, we show that every infinitesimal $C_{2\pi}$-deformation of the standard sphere is characterized by an odd function. Third, if g_t denotes a family of $C_{2\pi}$-metrics on spheres, then geodesic flows on the cotangent bundle of (S^n, g_t) is symplectically isomorphic to one another. In section 3, we investigate Zoll surfaces and their geodesics, and show the existence of the unique parallel which is called the equator. Then the Northern and the Southern Hemispheres are defined, which will be viewed later as manifolds with boundary. In section 4, notation is given from billiard theory, which will be necessary in later sections. In section 5, billiards on the Northern (or the Southern) Hemisphere of a Zoll surface are investigated. We regard geodesic segments on the Northern (or the Southern) Hemisphere as segments of billiards. Then we obtain some results on rigidity problems for Zoll surfaces. Then we will show how to extend our results for Zoll surfaces to general $C_{2\pi}$-surfaces. Section 6 is devoted to the proof of Theorem 6.1. In section 7, we prove Theorem 7.1, which is the extension of Theorem 5.1 to general $C_{2\pi}$-surfaces. Section 8 is an appendix which contains an elementary lemma necessary to prove Lemmas 2.3, 3.6, and 5.1.

2 $C_{2\pi}$-manifolds

2.1 Blaschke manifolds and the Blaschke conjecture

A Riemannian manifold (M, g) is a C_l-manifold if all geodesics are closed and have the same length l. We call that g is a C_l-metric. CROSSes are examples of C_l-manifolds as we mentioned above.

In order to study C_l-manifolds, it is very important to know what characterizes CROSSes. For the standard sphere (S^2, g_0), every geodesic issuing from any point p in S^2 passes the antipodal point p', and the length from p to p' along each geodesic is always π. Let g be a Riemannian metric on S^2. Suppose every geodesic in g from any point p passes a point p' different from p, and the length from p to p' along each geodesic is always π. Is then (S^2, g) necessarily isometric to (S^2, g_0)? This problem was proposed by W. Blaschke in the first edition (1921) of [Bl]. He called these (S^2, g) *Wiedersehensflächen*. The Blaschke conjecture is that every Wiedersehensfläche is isometric to (S^2, g_0). The conjecture was affirmatively solved by L. Green in 1961 ([Gr]). The sketch of his proof is given later.

Blaschke manifold is an extension of the notion of the *Wiedersehensflächen*. Let (M, g) be a compact, connected Riemannian manifold and let p, q be points in M. Set

$$\begin{aligned} \mathrm{Seg}(p,q) &= \{\text{minimizing geodesics from } p \text{ to } q \text{ (parametrized by arc length)}\}, \\ \Lambda(p,q) &= \{\dot\gamma(q) \in U_qM \mid \gamma \in \mathrm{Seg}(p,q)\}, \end{aligned}$$

where U_qM is the unit tangent space at $q \in M$. $\Lambda(p,q)$ is called the *link* from p to q.

Let $d(p,q)$ be the distance from p to q, and let $\gamma(t)$ be a geodesic in (M, g) issuing from $p = \gamma(0)$ which is parametrized by arc length. $q \in M$ is called a cut point of p along γ if there exists a positive real number t_1 such that $\gamma(t_1) = q$ satisfying $d(p, \gamma(t)) = t$ for any t in $[0, t_1]$ and $d(p, \gamma(t)) < t$ for any $t > t_1$. The cut locus of p, denoted by $\mathrm{Cut}(p)$, is the set of all cut points of p along geodesics issuing from p.

Definition *A compact Riemannian manifold* (M, g) *is called a Blaschke manifold at a point* m *in* M *if the link* $\Lambda(m, c)$ *is a great sphere of* U_cM *for every* c *in* $\mathrm{Cut}(m)$.

Definition *A compact Riemannian manifold* (M, g) *is called a Blaschke manifold if it is a Blaschke manifold at every point in* M.

The geometrical property of geodesics of a Blaschke manifold is similar to that of a CROSS. In particular, the following property is important.

Proposition 2.1 *If* (M, g) *is a Blaschke manifold, then* (M, g) *is a* C_l*-manifold.*

Proof. Let m be any point in M. Take any point c in $\mathrm{Cut}(m)$. Let $\gamma(t)$ be a minimizing geodesic from m to c parametrized by arc length whose length is $l/2$. Since the link from m to c is a great sphere, both $\dot\gamma(l/2)$ and $-\dot\gamma(l/2)$ belong to $\Lambda(m, c)$. Then a geodesic issuing from c with velocity $\dot\gamma(l/2)$ is a minimizing geodesic from c to m, and its length is also $l/2$. Hence γ is smooth on the interval $(0, l)$, and satisfies $\gamma(0) = \gamma(l) = m$. Namely, geodesics always come back to the initial point.

First, we have to show that γ is at least C^1-differentiable at m. Assume that $\dot\gamma(l) \neq \dot\gamma(0)$. For $\varepsilon > 0$ small, let m_ε be a point in γ defined by $m_\varepsilon = \gamma(\varepsilon)$. Let c_ε be a cut point of m_ε along γ. As we mentioned above, γ issues m_ε and comes back to m_ε again. Hence $m_\varepsilon = \gamma(t_\varepsilon)$ holds for some $t_\varepsilon > \varepsilon$. Since we assume that γ is not C^1-differentiable at $m = \gamma(l)$, there exists a constant $\delta_0 > 0$ such that $t_\varepsilon > l + \delta_0$. However, since $m_\varepsilon \to m$ and $c_\varepsilon \to c$ as $\varepsilon \to 0$, we have $t_\varepsilon \to l$ as $\varepsilon \to 0$. This is a contradiction. Then γ is at least C^1-differentiable at $m = \gamma(l)$, and hence γ is a closed geodesic.

Next, we show that all geodesics have the same length. Let m be a point in M. Let c and c', $c \neq c'$, be cut points of m. By continuity of cut locus, there exists a smooth family of geodesics from a geodesic passing through m and c to a geodesic passing through m and c'. Since geodesics in the Blaschke manifolds are all closed, the lengths of geodesics issuing from m are the same, say l, by the first variation formula. Take m' in M, $m' \neq m$. By the same argument, geodesics issuing from m' are all closed and have the same length, say l'. Since M is compact, there exists a geodesic from m to m'. This geodesic is closed again, hence $l = l'$. Therefore, we can conclude that a Blaschke manifold is a C_l-manifold.

Q.E.D.

Let (M, g) be a Blaschke manifold at $m \in M$ with real dimension $d \geq 2$. Then the dimension of $\Lambda(m, c)$ is a constant (say $k - 1$) when c runs through $\mathrm{Cut}(m)$ ([Be]). This cut locus $\mathrm{Cut}(m)$ is a $(d - k)$-dimensional submanifold of M. Since all geodesics issuing from m come back to m at the same length, k can take the values $1, 2, 4, 8, d$, and we have only the following possibilities ([Bo]):

· if $k = 1$ and any d, M is diffeomorphic to $P^d(\mathbf{R})$,

· if $k = 2$ and $d = 2n$, M has the homotopy type of $P^n(\mathbf{C})$,

· if $k = 4$ and $d = 4n$, M has the integral cohomology ring of $P^n(\mathbf{H})$,

· if $k = 8$ and $d = 16$, M has the integral cohomology ring of $P^2(\mathbf{Ca})$,

· if $k = d$ and any d, M has the homotopy type of S^d.

Namely, a Blaschke manifold necessarily has the same topology as one of the CROSSes, and the topology is determined by its dimension of the link, or dimension of the cut locus. The notations below are useful.

Definition (M, g) *is called a Blaschke manifold modeled on* S^n *(resp.* $P^n(\mathbf{K})$, *where* $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}, \mathbf{Ca}$) *if it satisfies:*

(i) (M, g) *is a Blaschke manifold with the same dimension as the model space,*

(ii) *the cut locus of any point is one point (resp. the cut locus of any point is a smooth submanifold of* M *with the same dimension as that of the cut locus of the model* $P^n(\mathbf{K})$).

For a Blaschke manifold with the same diameter as the model space, the following is called the generalized Blaschke conjecture.

The Blaschke conjecture
Every Blaschke manifold is isometric to one of the CROSSes.

The Blaschke conjecture is not completely solved yet, but Berger obtained the following results.

Theorem 2.1 ([Be] p.236) *Let* (M, g) *be a Blaschke manifold with diameter* π *which is modeled on* S^n*. Then,* (M, g) *is isometric to the standard sphere* (S^n, g_0).

Corollary 2.1 ([Be] p.236) *Let* (M, g) *be a Blaschke manifold with diameter* $\pi/2$ *which is modeled on* $P^n(\mathbf{R})$*. Then,* (M, g) *is isometric to the standard real projective space* $(P^n(\mathbf{R}), g_0)$.

Theorem 2.1 was first proved by L. Green in the case where $n = 2$. This result plays an important role in the proof of Theorem 7.1. We will sketch his proof.

Lemma 2.1 ([W2]) *Let (M,g) be a n-dimensional C_l-manifold, and (S^n,g_0) be the n-dimensional standard sphere. Then the ratio*

$$\frac{\mathrm{vol}(M,g)}{\mathrm{vol}(S^n,g_0)}\frac{2\pi}{l}=\mathrm{i}(M,g)$$

is an integer.

The integer $\mathrm{i}(M,g)$ is called Weinstein's integer. Weinstein's integers for CROSSes are:

$$\mathrm{i}(S^n,g_0)=1,\quad \mathrm{i}(P^n(\mathbf{R}),g_0)=2^{n-1},\quad \mathrm{i}(P^2(\mathbf{Ca}),g_0)=39,$$

$$\mathrm{i}(P^n(\mathbf{C}),g_0)=\binom{2n-1}{n-1},\quad \mathrm{i}(P^n(\mathbf{H}),g_0)=\frac{1}{2n+1}\binom{4n-1}{2n-1}.$$

In addition, Weinstein [W2] and C.T. Yang [Y] showed that for any C_l-metric g_l on S^n, $\mathrm{i}(S^n,g_l)=1$.

Let (M,g) be a Blaschke manifold modeled on S^2 with diameter π. By Lemma 2.1 and $\mathrm{i}(S^n,g_l)=1$, we have

$$\mathrm{vol}(M,g)=4\pi, \tag{2.1}$$

since Blaschke manifold with diameter π is a $C_{2\pi}$-manifold. For any point p in M, let p' be a conjugate point of p along any geodesic γ. Since the distance from p to p' is less than π, the index form

$$I(X,X)=\int_0^\pi\{g(\nabla X(t),\nabla X(t))-g(R(\dot\gamma(t),X(t))\dot\gamma(t),X(t))\}\,dt$$

is positive or zero for any vector field X along γ with $X(0)=0$ and $X(\pi)=0$. Then we have

$$\mathrm{vol}(M,g)\geq\frac{1}{2}\int_M \mathrm{Scal}\,d\mu_g,$$

where Scal is the scalar curvature of M, and $d\mu_g$ is the volume form. The equality holds if and only if (M,g) is the standard sphere. By the Gauss-Bonnet formula, we have

$$\chi(S^2)=\frac{1}{4\pi}\int_M \mathrm{Scal}\,d\mu_g,$$

where $\chi(S^2)$ is the Euler number of a two-sphere S^2. Then we obtain

$$\mathrm{vol}(M,g)\geq 4\pi, \tag{2.2}$$

and the equality holds if and only if (M,g) is the standard sphere. Then our conclusion follows from (2.1) and (2.2).

2.2 Infinitesimal $C_{2\pi}$-deformations on spheres

From now on to the last, M denotes a manifold which is diffeomorphic to S^2. Let g_0 be the standard metric on M, and $M_0 = (M, g_0)$ denotes the standard sphere of dimension two.

The Blaschke conjecture claims that Blaschke manifold is one of the CROSSes. Are there examples of C_l-manifolds except for CROSSes? This question was affirmatively solved by the solution of the existence problem of infinitesimal $C_{2\pi}$-deformations of the standard sphere.

A Riemannian metric g on M induces a bundle isomorphism $\sharp_g$ from the cotangent bundle TM^* to the tangent bundle TM such that

$$g(\sharp_g(\lambda), v) = \lambda(v) \tag{2.3}$$

for any $\lambda \in T_pM^*$, $v \in T_pM$, $p \in M$. For each symmetric two-form h, let $h^{\sharp_g}$ be the function on TM^* defined by

$$h^{\sharp_g}(\lambda) = h(\sharp_g(\lambda), \sharp_g(\lambda))$$

for any $\lambda \in TM^*$. Then the energy function on TM^* is defined by

$$E = \frac{1}{2} g^{\sharp_g},$$

and X_E denotes the geodesic vector field on TM^* associated with g. For the sake of simplicity, we shall write $\sharp_t$ instead of $\sharp_{g_t}$. Put

$$E_t = \frac{1}{2} g_t^{\sharp_t}.$$

The following theorem is a particular case of A. Weinstein's result.

Theorem 2.2 ([W1], [Be] or [K]) *Let $g_t = \exp(\rho_t) g_0$ be a smooth family of $C_{2\pi}$-metrics on M (i.e. if ζ_t^s denotes the geodesic flow for the metric g_t, then $\zeta_t^{2\pi}$ equals to identity map) such that $\rho_0 = 0$. We set $\dot{\rho} = \left.\frac{d\rho_t}{dt}\right|_{t=0}$. Then the following holds:*

i) for any closed geodesic γ_0 in M_0 parametrized by arc length, $\int_0^{2\pi} \dot{\rho}(\gamma_0(s))ds = 0$,

ii) there exists a smooth family of homogeneous symplectic diffeomorphisms
$$\phi_t : \mathring{T} M^* \to \mathring{T} M^* \text{ such that } \phi_t^* E_t = E_0, \text{ where } \mathring{T} M^* = TM^* \backslash \{0\}.$$

Proof of i). We set $h_t = (d/dt)g_t$. Let $\gamma_t(s)$ be a geodesic in (M, g_t) parametrized by arc length. Since g_t is a $C_{2\pi}$-metric, we have

$$\int_0^{2\pi} g_t(\dot{\gamma}_t(s), \dot{\gamma}_t(s))\, ds = 2\pi, \tag{2.4}$$

where $\dot{\gamma}_t(s)$ means $(d/ds)\gamma_t(s)$. By the first variation formula, we have

$$\int_0^{2\pi} g_t\Big(\frac{d\dot{\gamma}_t(s)}{dt}, \dot{\gamma}_t(s)\Big)\, ds = 0. \tag{2.5}$$

By (2.4), we have

$$\begin{aligned} 0 &= \frac{d}{dt}\Big\{\int_0^{2\pi} g_t(\dot{\gamma}_t(s), \dot{\gamma}_t(s))\, ds\Big\} \\ &= \int_0^{2\pi}\Big\{\frac{dg_t}{dt}(\dot{\gamma}_t(s), \dot{\gamma}_t(s)) + 2g_t\Big(\frac{d\dot{\gamma}_t(s)}{dt}, \dot{\gamma}_t(s)\Big)\Big\}\, ds \\ &\overset{(2.5)}{=} \int_0^{2\pi} h_t(\dot{\gamma}_t(s), \dot{\gamma}_t(s)). \end{aligned} \tag{2.6}$$

Since $h_t = (d/dt)g_t = \dot{\rho}_t \exp(\rho_t) g_0$, if we substitute $t = 0$ in (2.6), we have

$$0 = \int_0^{2\pi} \dot{\rho}_0(\gamma_0(s)) \exp(\rho_0(\gamma_0(s))) g_0(\dot{\gamma}_0(s), \dot{\gamma}_0(s))\, ds = \int_0^{2\pi} \dot{\rho}_0(\gamma_0(s))\, ds.$$

Hence i) is proved.

In order to prove ii), we need the following lemma.

Lemma 2.2 ([K]) *There is a one-parameter family of homogeneous symplectic vector fields $\{Y_t\}$ on TM^* such that*

$$Y_t E_t = \dot{E}_t,$$

where dot denotes the derivative in the parameter t.

Proof. First, differentiate both sides of (2.3), that is, defining equation of the bundle isomorphism $\sharp$. Then we have

$$h_t(\sharp_t(\lambda), v) + g_t\Big(\frac{d\sharp_t(\lambda)}{dt}, v\Big) = 0,$$

for any $v \in T_pM$, $p \in M$. In particular, if we take $v = \sharp_t(\lambda)$, we have

$$\begin{aligned} g_t\Big(\frac{d\sharp_t(\lambda)}{dt}, \sharp_t(\lambda)\Big) &= -h_t(\sharp_t(\lambda), \sharp_t(\lambda)) \\ &= -h_t^{\sharp_t}(\lambda). \end{aligned} \tag{2.7}$$

Then we obtain

$$\begin{aligned}\dot{E}_t(\lambda) &= \frac{1}{2}\frac{d}{dt}\{g_t(\sharp_t(\lambda), \sharp_t(\lambda))\} \\ &= \frac{1}{2}h_t(\sharp_t(\lambda), \sharp_t(\lambda)) + g_t\Big(\frac{d\sharp_t(\lambda)}{dt}, \sharp_t(\lambda)\Big) \overset{(2.7)}{=} -\frac{1}{2}h_t^{\sharp_t}(\lambda). \end{aligned} \tag{2.8}$$

Let γ_t be a geodesic in (M, g_t) parametrized by arc length, and let $\{\xi_s^t\}_{s\in\mathbf{R}}$ be the geodesic flow on TM^* associated with the metric g_t which satisfies $\sharp_t(\xi_s^t(\lambda)) = \dot{\gamma}_t(s)$ for $\xi_0^t(\lambda) = \lambda \in UM^*$. Then $\{\xi_s^t\}_{s\in\mathbf{R}}$ induces a free S^1-action of period 2π on the unit cotangent bundle $UM^* = E_t^{-1}(1/2)$.

For any $\lambda \in UM^*$, we have

$$\begin{aligned}\int_0^{2\pi} \dot{E}_t(\xi_s^t(\lambda))\, ds &\overset{(2.8)}{=} -\frac{1}{2}\int_0^{2\pi} h_t^{\sharp_t}(\xi_s^t(\lambda))\, ds \\ &= -\frac{1}{2}\int_0^{2\pi} h_t(\sharp_t(\xi_s^t(\lambda)), \sharp_t(\xi_s^t(\lambda)))\, ds \\ &= -\frac{1}{2}\int_0^{2\pi} h_t(\dot{\gamma}_t(s), \dot{\gamma}_t(s)) \overset{(2.6)}{=} 0.\end{aligned}$$

Namely, we obtain

$$\int_0^{2\pi} \dot{E}_t(\xi_s^t(\lambda))\, ds = 0. \tag{2.9}$$

Define the function H_t on $\overset{\circ}{T} M^*$ by the following two conditions

(1) $H_t(\lambda) = \dfrac{-1}{2\pi}\displaystyle\int_0^{2\pi}\int_0^s \dot{E}_t(\xi_r^t(\lambda))\, dr\, ds, \qquad \lambda \in UM^*,$

(2) H_t is positively homogeneous of degree one (i.e. $H_t(a\lambda) = aH_t(\lambda)$, $a \in \mathbf{R}_{>0}$, $\lambda \in \overset{\circ}{T} M^*$).

Recall that X_{E_t} denotes the geodesic vector field, that is, the symplectic vector field associated with the Hamiltonian E_t. Then we have

$$\begin{aligned}X_{E_t}H_t(\lambda) &= \frac{d\xi_u^t}{du}\Big|_{u=0} H_t(\lambda) = \frac{dH_t(\xi_u^t(\lambda))}{du}\Big|_{u=0} \\ &= \frac{d}{du}\Big|_{u=0}\Big\{\frac{-1}{2\pi}\int_0^{2\pi}\int_0^s \dot{E}_t(\xi_r^t(\xi_u^t\lambda))\, dr\, ds\Big\} \\ &= \frac{-1}{2\pi}\int_0^{2\pi}\Big\{\frac{d}{du}\Big|_{u=0}\int_0^s \dot{E}_t(\xi_{r+u}^t(\lambda))\, dr\Big\}\, ds,\end{aligned} \tag{2.10}$$

for $\lambda \in UM^*$. Set

$$F(s) = \frac{d}{du}\Big|_{u=0} \int_0^s \dot{E}_t(\xi^t_{r+u}(\lambda))\, dr.$$

Then we have

$$F(s) \overset{(2.8)}{=} \frac{d}{du}\Big|_{u=0} \int_0^s -\frac{1}{2} h_t^{\sharp_t}(\xi^t_{r+u}(\lambda))\, dr = -\frac{1}{2}\frac{d}{du}\Big|_{u=0} \int_u^{u+s} h_t^{\sharp_t}(\xi^t_w(\lambda))\, dw,$$

where $w = r + u$. Take a real number $a \in (u, u+s)$ and decompose the interval $(u, u+s)$ into $(u, a\,]$ and $[\,a, u+s)$. Then we have

$$\begin{aligned} F(s) &= -\frac{1}{2}\frac{d}{du}\Big|_{u=0} \Big\{ \int_u^a h_t^{\sharp_t}(\xi^t_w(\lambda))\, dw + \int_a^{u+s} h_t^{\sharp_t}(\xi^t_w(\lambda))\, dw \Big\} \\ &= -\frac{1}{2}\Big\{ -h_t^{\sharp_t}(\xi^t_0(\lambda)) + h_t^{\sharp_t}(\xi^t_s(\lambda)) \Big\} \\ &\overset{(2.8)}{=} -\dot{E}_t(\lambda) + \dot{E}_t(\xi^t_s(\lambda)). \end{aligned} \tag{2.11}$$

Substituting (2.11) in (2.10), we obtain

$$\begin{aligned} X_{E_t} H_t(\lambda) &= \frac{-1}{2\pi} \int_0^{2\pi} F(s)\, ds = \frac{-1}{2\pi}\int_0^{2\pi} \Big\{ -\dot{E}_t(\lambda) + \dot{E}_t(\xi^t_s(\lambda)) \Big\}\, ds \\ &\overset{(2.9)}{=} \frac{1}{2\pi}\int_0^{2\pi} \dot{E}_t(\lambda)\, ds = \dot{E}_t(\lambda), \end{aligned}$$

for any $\lambda \in UM^*$. Since $\dot{E}_t = (-1/2)\, h_t^{\sharp_t}$ is positively homogeneous of degree two,

$$X_{E_t} H_t(\lambda) = \dot{E}_t(\lambda) \tag{2.12}$$

holds for any $\lambda \in TM^*$, not only for $\lambda \in UM^*$. By the anti-commutativity of Poisson bracket, we have $X_{E_t} H_t = -X_{H_t} E_t$. Set $X_{H_t} = Y_t$. According to (2.12), we can conclude that

$$Y_t E_t(\lambda) = -\dot{E}_t(\lambda)$$

for any $\lambda \in TM^*$.

Q.E.D.

Proof of ii) in Theorem 2.2. We define a smooth family of homogeneous symplectic diffeomorphisms $\phi_t : \overset{\circ}{T} M^* \to \overset{\circ}{T} M^*$ as a one-parameter transformation of Y_t, which is constructed in Lemma 2.2. Namely,

$$\frac{d}{dt}\phi_t(\omega) = (Y_t)_{\phi_t(\omega)}, \qquad \phi_0 = \mathrm{id},$$

for any $\omega \in \overset{\circ}{T} M^*$. Then we have

$$\begin{aligned}
\frac{d}{dt}(\phi_t^* E_t)(\omega) &= \frac{d\{E_t(\phi_t(\omega))\}}{dt} \\
&= \dot{E}_t(\phi_t(\omega)) + (Y_t)_{\phi_t(\omega)} E_t \\
&= -(Y_t E_t)(\phi_t(\omega)) + (Y_t)_{\phi_t(\omega)} E_t \\
&= -(Y_t E_t)(\phi_t(\omega)) + (Y_t E_t)(\phi_t(\omega)) = 0,
\end{aligned}$$

for any $\omega \in \overset{\circ}{T} M^*$. Therefore, we can conclude that

$$\phi_t^* E_t = \phi_0^* E_0 = E_0.$$

Q.E.D.

By ii) of Theorem 2.2, we can say that geodesic flows on the cotangent bundle of (M, g_t) are symplectically isomorphic to those on the cotangent bundle of M_0.

Proposition 2.2 ([Be]) *Let f be a continuous function on M_0. Then, the following properties are equivalent.*

- *i) the function f is odd, that is, $f(\tau m) = -f(m)$ for every m in M_0, where τ denotes the antipodal map of M_0.*
- *ii) for any closed geodesic γ in M_0, $\int_\gamma f = 0$.*

Proof. Any function f on M_0 is described as the sum of an odd function and an even function. Set $f = f^o + f^e$, where f^o is an odd part of f, and f^e is an even part of f.
i) $\Rightarrow$ ii)
Since $\tau(\gamma) = \gamma$, we have

$$\int_\gamma f^o(\gamma(s))ds = \int_{\tau(\gamma)} f^o(\tau(\gamma(s)))ds = \int_\gamma -f^o(\gamma(s))ds.$$

Hence

$$\int_{\gamma} f^{o}(\gamma(s))ds = 0 \tag{2.13}$$

holds for any γ.
ii) $\Rightarrow$ i)
By (2.13), we have

$$0 = \int_{\gamma} f\, ds = \int_{\gamma} (f^{o} + f^{e})\, ds \overset{(2.13)}{=} \int_{\gamma} f^{e}\, ds. \tag{2.14}$$

To prove $f^{e} \equiv 0$, we need the following lemma.

Lemma 2.3 *For an even function h on M_0 invariant under some subgroup Γ of $SO(3)$ isomorphic to S^1, $\int_{\gamma} h\, ds = 0$ implies $h = 0$.*

Suppose Lemma 2.3 is proved. Fix any point p in M_0. Let o_p be an axis which contains p and the origin. $k_{p,t} \in \Gamma \subset SO(3)$ denotes a rotation around the o_p axis. A composite function $f^{e} \circ k_{p,t}$ is an even function on M_0. Then the function $F_p : M_0 \to \mathbf{R}$ defined by $F_p(q) = \int_{S^1} (f^{e} \circ k_{p,t}(q))\, dt$, $q \in M$ is also an even function on M_0 which is invariant under S^1-action $k_{p,t}$. Then we have

$$\begin{aligned} \int_{\gamma} F_p ds &= \int_{\gamma} \left\{ \int_{S^1} (f^{e} \circ k_{p,t})\, dt \right\} ds = \int_{S^1} \left\{ \int_{\gamma} (f^{e} \circ k_{p,t})\, ds \right\} dt \\ &= \int_{S^1} \left\{ \int_{k_{p,t}(\gamma)} f^{e}\, ds \right\} dt \overset{(2.14)}{=} 0. \end{aligned}$$

Namely, we have $\int_{\gamma} F_p ds = 0$. Then we have

$$F_p = 0, \tag{2.15}$$

by Lemma 2.3. Since the North Pole p is a fixed point of $k_{p,t}$, we have

$$F_p(p) = \int_{S^1} (f^{e} \circ k_{p,t}(p))\, dt = \int_{S^1} f^{e}(p)\, dt = 2\pi f^{e}(p).$$

Then we have $f^{e}(p) = 0$ by (2.15). Since p is any point in M_0, we obtain $f^{e} \equiv 0$ on M_0. Hence f is an odd function.

Q.E.D.

The proof of Lemma 2.3 remains.
Proof of Lemma 2.3 Let (x, y, z) be Euclidian coordinates in $\mathbf{R}^3$. Without loss of generality, we may assume h is invariant under rotation around z-axis. Then h depends only on z, and written as $h(z)$. For each fixed $\theta \in (0, \pi\,]$, let γ_θ be a geodesic in M_0 which is described as follows.

$$\begin{aligned} \gamma_\theta &= (x(s), y(s), z(s)) \\ &= (\sin s, \sin\theta\cos s, \cos\theta\cos s), \qquad\qquad s \in (0, 2\pi\,]. \end{aligned}$$

Then we have

$$0 = \int_{\gamma_\theta} h(z(s))\,ds = \int_0^{2\pi} h(\cos\theta\cos s)\,ds = 4\int_0^{\frac{\pi}{2}} h(\cos\theta\cos s)\,ds, \tag{2.16}$$

for any $\theta \in (0, \pi/2]$. Set $\cos r = \cos\theta\cos s$. Since

$$\frac{dr}{ds} = \frac{\cos\theta\sin s}{\sin r} = \frac{\sqrt{\cos^2\theta - \cos^2 r}}{\sin r},$$

(2.16) is written as

$$\int_\theta^{\frac{\pi}{2}} \frac{h(\cos r)\sin r}{\sqrt{\cos^2\theta - \cos^2 r}}\,dr = 0, \tag{2.17}$$

for any $\theta \in (0, \pi/2]$.
Set $t = \cos r$, and $x = \cos\theta$. Then we have

$$0 = \int_x^0 \frac{h(t)\sin r}{\sqrt{x^2 - t^2}} \cdot \frac{dt}{-\sin r} = \int_0^x \frac{h(t)}{\sqrt{x^2 - t^2}}\,dt,$$

for any $x \in (0, 1]$. Then we have $h(t) = 0$ for any $t \in (0, 1]$ by Lemma 8.1 in the Appendix. Since h is an even function centered at zero, we conclude that $h = 0$ for any $t \in [-1, 1]$.

Q.E.D.

As a corollary of Theorem 2.2 and Proposition 2.2, we get the following result.

Theorem 2.3 (Funk) *If $g_t = \exp(\rho_t)g_0$ is a smooth family of $C_{2\pi}$-metrics on M, with $\rho_0 = 0$ and $\dot\rho = \left.\frac{d\rho_t}{dt}\right|_{t=0}$, then $\dot\rho$ is an odd function on M_0.*

The converse of Theorem 2.3 is true. Namely, for any odd function on M_0, one can construct $C_{2\pi}$-deformations of g_0.

Theorem 2.4 ([G]) *For every odd function $\dot{\rho}$ on M_0, there exists a smooth one-parameter family of C^∞-functions ρ_t such that $\rho_0 = 0$, $\dot{\rho} = \left.\dfrac{d\rho_t}{dt}\right|_{t=0}$, and $\exp(\rho_t)g_0$ is a $C_{2\pi}$-metric for small t.*

Theorems 2.3 and 2.4 show that every infinitesimal $C_{2\pi}$-deformation of the standard sphere is characterized by an odd function on M_0.

3 Zoll surfaces

3.1 Metrics on spheres of revolution

Before introducing Zoll surfaces, we will investigate spheres of revolution because a Zoll surface is a sphere of revolution with some conditions.

Let (M, g) be a sphere of revolution. Namely, (M, g) has the S^1-action as an isometry group. The Euler number $\chi(M)$ is the sum of the indices at the zeroes of any vector field with isolated singularities. Each of the zeroes of an infinitesimal isometry on S^2 is isolated, and has the index equal to one. Then (M, g) has exactly two fixed points. We call them the North Pole and the South Pole, and denote by N and S. We assume the distance from N to S is $L \in \mathbf{R}_{>0}$. Here $\mathbf{R}_{>0}$ denotes the set of positive real numbers.

Let θ be the S^1-action, and u be the distance from the North Pole. Set $U = M \backslash \{N, S\}$. We also need two other charts U_N and U_S. They are geodesic balls centered at N or S with radius L. Namely,

$$\begin{aligned} U_N &= \{N\} \cup \{(u, \theta) \in U \mid u < L\}, \\ U_S &= \{S\} \cup \{(u, \theta) \in U \mid u > L\}. \end{aligned}$$

Now $\{U, U_N, U_S\}$ together with associated coordinate functions (u, θ) is a parametrization of M. Then the metric g may be written on U as

$$g = du^2 + a(u)^2 d\theta^2 \tag{3.1}$$

for some smooth function $a : (0, L) \to \mathbf{R}_{>0}$, $a(0) = a(L) = 0$.

3.2 Geodesics on spheres of revolution

In order to investigate geodesics on a sphere of revolution, we compute the Christoffel coefficients. They are

$$\Gamma^u_{\theta\theta} = -a'(u)a(u), \qquad \Gamma^\theta_{u\theta} = \frac{a'(u)}{a(u)}. \tag{3.2}$$

The other Christoffel coefficients are zero. Then a geodesic $\gamma(t) = (u(t), \theta(t))$ in U satisfies the following differential equations.

$$\begin{cases} \dfrac{d^2u}{dt^2} - a'(u)a(u)\Big(\dfrac{d\theta}{dt}\Big)^2 = 0, & u \in (0, L),\ t \in \mathbf{R}, \\ \dfrac{d^2\theta}{dt^2} + 2\dfrac{a'(u)}{a(u)}\Big(\dfrac{du}{dt}\Big)\Big(\dfrac{d\theta}{dt}\Big) = 0, & \theta \in [\,0, 2\pi),\ t \in \mathbf{R}. \end{cases} \tag{3.3}$$

The second equation of (3.3) gives Clairout's first integral

$$a(u)^2\frac{d\theta}{dt} = c, \tag{3.4}$$

where c is a constant. Let γ be a geodesic parametrized by arc length. Then

$$1 = g(\dot\gamma, \dot\gamma) = \Big(\frac{du}{dt}\Big)^2 + a(u)^2\Big(\frac{d\theta}{dt}\Big)^2.$$

Hence

$$1 - \Big(\frac{du}{dt}\Big)^2 = a(u)^2\Big(\frac{d\theta}{dt}\Big)^2 \geq 0.$$

Thus we obtain

$$a(u)\Big|\frac{d\theta}{dt}\Big| \leq 1.$$

Multiplying both sides of the inequality by $a(u) > 0$, we have

$$a(u)^2\Big|\frac{d\theta}{dt}\Big| \leq a(u).$$

According to (3.4), we have

$$|c| \leq a(u). \tag{3.5}$$

Each geodesic corresponds to the value of $|c|$. If $|c| = 0$, then $d\theta/dt = 0$. It implies that γ is a meridian curve $\theta \equiv \theta_0$ (const.). The parallel curve $u \equiv u_0$ (const.) is a geodesic if and only if it satisfies $a'(u_0) = 0$.

If γ is not a meridian nor the equator, there exists a maximal interval $[u_1, u_2] \subset (0, \pi)$ such that $a(u_1) = a(u_2) = |c|$ and $|c| \leq a(u)$ for any $u \in [u_1, u_2]$. Then along the geodesic γ, we have

$$u_1 \leq u \leq u_2.$$

Hence γ is entirely contained between the parallels $u = u_1$ and $u = u_2$. We can see that γ is tangent to the parallels $u = u_1$ and $u = u_2$. Indeed, let $\gamma(0)$ be the intersection point with the parallel $u = u_1$. Then (3.4) and the first equation of (3.3) gives $(du/dt)|_{t=0} = 0$. This means that γ is tangent to the parallel $u = u_1$. We can prove γ is also tangent to the parallel $u = u_2$ in a similar way.

3.3 Zoll metrics

Now, let us define Zoll surfaces which are known for non-trivial examples of $C_{2\pi}$-manifolds.

Definition *A Riemannian manifold (M, g) is called a Zoll surface if g is a $C_{2\pi}$-metric of revolution on a sphere.*

It is known that all geodesics are simple in Zoll surfaces. We can verify this fact later. Throughout this paper, assume our Zoll metric is smooth on M (see section 3.4).

Let (M, g) be a Zoll surface. Since (M, g) is a surface of revolution, the metric g is written as in (3.1). The parallel $u = u_0$ is a geodesic if and only if it satisfies $a'(u_0) = 0$, as we mentioned above. We should notice that such u_0 is unique in the case of a Zoll surface. We show the fact below.

Lemma 3.4 *Let (M, g) be a Zoll surface. There exists a unique u (say u_0) with $a'(u_0) = 0$.*

Proof. We prove the uniqueness of $u = u_0$ such that $a'(u_0) = 0$. Let R be a curvature tensor, and ∇ be a covariant derivative of (M, g). The orthonormal basis of the tangent space at (u, θ) is $\left\{\frac{\partial}{\partial u}, \frac{1}{a(u)}\frac{\partial}{\partial \theta}\right\} = \{X, Y\}$. Since we already know the Christoffel coefficients (3.2), we have

$$\begin{aligned}
R(X,Y)X &= \nabla_Y \nabla_X X - \nabla_X \nabla_Y X + \nabla_{[X,Y]} X \\
&= \nabla_Y \Big(\Gamma^u_{uu}\frac{\partial}{\partial u} + \Gamma^\theta_{uu}\frac{\partial}{\partial \theta}\Big) - \nabla_X \Big\{\frac{1}{a}\Big(\Gamma^u_{\theta u}\frac{\partial}{\partial u} + \Gamma^\theta_{\theta u}\frac{\partial}{\partial \theta}\Big)\Big\} + \nabla_{-\frac{a'}{a^2}\frac{\partial}{\partial \theta}}\frac{\partial}{\partial u} \\
&= -\nabla_X \Big(\frac{a'}{a^2}\frac{\partial}{\partial \theta}\Big) - \frac{a'}{a^2}\Big\{\Gamma^u_{\theta u}\frac{\partial}{\partial u} + \Gamma^\theta_{\theta u}\frac{\partial}{\partial \theta}\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{a'' a - 2(a')^2}{a^3}\frac{\partial}{\partial\theta} - \frac{(a')^2}{a^3}\frac{\partial}{\partial\theta} - \frac{(a')^2}{a^3}\frac{\partial}{\partial\theta} \\
&= -\frac{a''}{a^2}\frac{\partial}{\partial\theta}.
\end{aligned}
$$

Since (M, g) is a surface of revolution, the sectional curvature $\sigma(u, \theta)$ at any point $(u, \theta) \in U$ depends only on u. Thus we can write $\sigma(u, \theta) = \sigma(u)$. Then we have

$$
\begin{aligned}
\sigma(u) &= g(R(X, Y)X, Y) \\
&= g\Big(-\frac{a''}{a^2}\frac{\partial}{\partial\theta}\,,\ \frac{1}{a}\frac{\partial}{\partial\theta}\Big) = -\frac{a''}{a^2}\cdot\frac{1}{a}\cdot a^2 \\
&= -\frac{a''}{a}.
\end{aligned}
$$

Since all geodesics have the length 2π, we have $a(u_0) = 1$. Then the sectional curvature at $u = u_0$ is

$$
\sigma(u_0) = -\frac{a''(u_0)}{a(u_0)} = -a''(u_0). \tag{3.6}
$$

Let $\gamma(t)$ be the geodesic $u = u_0$, and let $J(t)$ be a non-zero normal Jacobi field along $\gamma(t)$ which satisfies $J(0) = 0$. Since (M, g) has the constant sectional curvature in the direction of θ, $J(t)$ is written as $J'' + \sigma(u_0)J = 0$. By (3.6), we have

$$
J'' - a''(u_0)J = 0. \tag{3.7}
$$

Let $w(t)$ be the parallel vector field along γ which satisfies $g(w(t), \dot\gamma(t)) = 0$ and $g(w(t), w(t)) = 1$. The solutions of (3.7) with initial conditions $J(0) = 0$ and $J'(0) = w(0)$ are:

$$
J(t) = \begin{cases}
\dfrac{\sin t\sqrt{\sigma(u_0)}}{\sqrt{\sigma(u_0)}}w(t) & \text{if } \quad \sigma(u_0) > 0, \\[2ex]
tw(t) & \text{if } \quad \sigma(u_0) = 0, \\[2ex]
\dfrac{\sinh t\sqrt{-\sigma(u_0)}}{\sqrt{-\sigma(u_0)}}w(t) & \text{if } \quad \sigma(u_0) < 0.
\end{cases}
$$

Notice that if $J(0) = 0$, then $J(2\pi) = 0$, since all geodesics are closed and have the same length 2π. However, $J(t) = \{\sinh t\sqrt{-\sigma(u_0)}\sqrt{-\sigma(u_0)}\}w(t)$ or $J(t) = tw(t)$ never satisfies $J(0) = J(2\pi) = 0$. Then we can conclude that $J(t) = \{\sin t\sqrt{\sigma(u_0)}/\sqrt{\sigma(u_0)}\}w(t)$ and $\sigma(u_0) > 0$. then we obtain $a''(u_0) < 0$ from (3.6). This implies that any point u which satisfies $a'(u) = 0$ is a local maximum of the function a, hence such u is unique.

Q.E.D.

This unique parallel is called the equator of a Zoll surface (M, g). Then we can define the Northern Hemisphere and the Southern Hemisphere of M. Fix the direction of the equator. Then the closed domain on the left side of the equator is called the Northern Hemisphere of M. The opposite is called the Southern Hemisphere. The equator is supposed to belong to both hemispheres.

We would like to have a representation of Zoll metrics. Let (M, g) be a Zoll surface equipped with a parametrization (U, U_N, U_S) and the coordinate functions (u, θ) on U, so that g may be written on U as (3.1). By Lemma 3.4, there exists a unique u, say u_0, with $a'(u_0) = 0$. Define a new system of coordinates (r, θ) on U by setting $a(u) = \sin r$. $u = u_0$ corresponds to $r = \pi/2$.

We define piecewise differentiable curves $b : [0, \pi] \to [0, \pi]$ and $c : [-1, 1] \to [0, \pi]$ by $b(u) = r$ and $c(\cos r) = a^{-1}(\sin r) = u$. Namely,

$$b(u) = \begin{cases} \arcsin a(u) & u \in [0, u_0], \\ \pi - \arcsin a(u), & u \in [u_0, \pi], \end{cases} \tag{3.8}$$

$$c(v) = \begin{cases} a^{-1}|_{[0,u_0]}(\sqrt{1 - v^2}), & v \in [0, 1], \\ a^{-1}|_{[u_0,\pi]}(\sqrt{1 - v^2}), & v \in [-1, 0], \end{cases} \tag{3.9}$$

where $a^{-1}|_{[0,u_0]}$ and $a^{-1}|_{[u_0,\pi]}$ are the restriction on $[0, u_0]$ and $[u_0, \pi]$ of the inverse function of a. Clearly, $b(u)$ and $c(v)$ have the same order of differentiability as a. We define the smooth function f from $(-1, 1)$ to $\mathbf{R}_{>0}$ as

$$f(v) = \begin{cases} \dfrac{v}{a'(c(v))}, & v \neq 0, \\ \dfrac{1}{\sqrt{-a''(u_0)}}, & v = 0. \end{cases} \tag{3.10}$$

Lemma 3.5 *$f(v)$ is continuous at $v = 0$.*

Proof. Set $v = \cos r$. First we compute the right-side limit of f as v tends to zero. Suppose v is in $[0, 1]$. By differentiating $a(u) = \sin r$ in u, we have

$$a'(u) = \frac{dr}{du}\cos r = b'(u)\cos b(u), \tag{3.11}$$

since $b(u) = r$. Then we have

$$\begin{aligned} \frac{v}{a'(u)} &\overset{(3.11)}{=} \frac{v}{b'(u)\cos b(u)} = \frac{\cos r}{b'(u)\cos b(u)} = \frac{\cos b(u)}{b'(u)\cos b(u)} \\ &= \frac{1}{b'(u)}. \end{aligned} \tag{3.12}$$

Since

$$c(\cos r) = a^{-1}(\sin r) = u, \tag{3.13}$$

we obtain

$$\frac{v}{a'(c(v))} = \frac{v}{a'(u)} \overset{(3.12)}{=} \frac{1}{b'(u)}. \tag{3.14}$$

On the other hand, by differentiating (3.11) in u, we have

$$a^{''}(u) = b^{''}(u)\cos b(u) - (b'(u))^2 \sin b(u). \tag{3.15}$$

Substituting $u = u_0$ in (3.15), we have

$$-a^{''}(u_0) = (b'(u_0))^2,$$

since $b(u_0) = \arcsin a(u_0) = \pi/2$. We know $a^{''}(u_0) < 0$ by the proof of Lemma 3.4. Then we obtain

$$\sqrt{-a''(u_0)} = b'(u_0). \tag{3.16}$$

Recall that if $v \in [0, 1]$ approaches zero from one, then $r \in [0, \pi/2]$ approaches $\pi/2$ from zero, and $u \in [0, u_0]$ approaches u_0 from zero. Then we have

$$\lim_{v \to +0} f(v) = \lim_{v \to +0} \frac{v}{a'(c(v))} \overset{(3.14)}{=} \frac{1}{b'(u_0)} \overset{(3.16)}{=} \frac{1}{\sqrt{-a''(u_0)}}.$$

We can compute the left-side limit of f as v tends to zero in the same way, and obtain

$$\lim_{v \to -0} f(v) = \frac{1}{\sqrt{-a''(u_0)}}.$$

Therefore, $f(v)$ is continuous at $v = 0$.

Q.E.D.

Rewrite the metric g written as in (3.1) with this $f(v)$. By (3.13), we have

$$a(c(\cos r)) = a(u) = \sin r. \tag{3.17}$$

By differentiating (3.17) in u, we have

$$a'(c(\cos r))) = \frac{dr}{du} \cos r.$$

Then we obtain

$$\frac{du}{dr} = \frac{\cos r}{a'(c(\cos r))} = f(\cos r).$$

Hence g may be written on U as

$$\begin{aligned} g &= du^2 + a^2(u)\, d\theta^2 = \Big(\frac{du}{dr}\Big)^2 dr^2 + \sin^2 r\, d\theta^2 \\ &= \{f(\cos r)\}^2 dr^2 + \sin^2 r\, d\theta^2. \end{aligned} \tag{3.18}$$

We will give a necessary and sufficient condition in order that all geodesics in (M, g) are closed, where g is written on U as in (3.18). We suppose the length of the equator is 2π.

Theorem 3.5 ([D]) *Let (M, g) be a Riemannian manifold which is diffeomorphic to S^2 and g is a metric of revolution which can be described in a parametrization (U, U_N, U_S) and coordinate functions (r, θ) on U as in* (3.18). *We assume the distance from N to S is $L \in \mathbf{R}_{>0}$.*

Then, all geodesics in g are closed if and only if

$$\int_i^{\pi - i} \frac{\sin i \cdot f(\cos r)}{\sin r \sqrt{\sin^2 r - \sin^2 i}} dr = \frac{m}{n}\pi,$$

for every i in $(0, \pi/2)$. Here m and n are coprime positive integers.

Every geodesic in U, except for the equator, consists of $2n$ geodesic segments between two consecutive points of contact with the parallels $r = i$ and $r = \pi - i$. Its length is $2nL$, and it turns m times.

Proof. First, we must give some observations about geodesics in Zoll surfaces in order to prove the theorem. The meridians are geodesics through the North Pole N and the South Pole S. Other geodesics are entirely contained in $U = M \backslash \{N, S\}$. We can compute the equations of geodesics in U, and obtain Clairaut's first integral

$$\sin^2 r \frac{d\theta}{dt} = c, \tag{3.19}$$

for some constant c. We may set $c = \varepsilon_1 \sin i$, where the sign ε_1 determines the orientation of the geodesic, and i is in $(0, \pi/2)$. Then the geodesic, denoted by γ_i, is contained between two parallels $r = i$ and $r = \pi - i$. Minimum or maximum value of $r(t)$ is $r = i$ or $r = \pi - i$, respectively.

By (3.19), we have

$$\frac{d\theta}{dt} = \varepsilon_1 \frac{\sin i}{\sin^2 r}. \tag{3.20}$$

Recall that γ_i is parametrized by arc length. Then, since the metric g is written as in (3.18), we have

$$\{f(\cos r)\}^2 \left(\frac{dr}{dt}\right)^2 + \sin^2 r \left(\frac{d\theta}{dt}\right)^2 = 1. \tag{3.21}$$

Substituting (3.20) in (3.21), we have

$$\left(\frac{dr}{dt}\right)^2 = \frac{\sin^2 r - \sin^2 i}{\sin^2 r \{f(\cos r)\}^2}.$$

Hence, $\dot{\gamma_i} = (dr/dt, d\theta/dt)$ is described as follows:

$$\begin{cases} \dfrac{dr}{dt} = \varepsilon \dfrac{\sqrt{\sin^2 r - \sin^2 i}}{\sin r \cdot f(\cos r)}, \\ \dfrac{d\theta}{dt} = \varepsilon_1 \dfrac{\sin i}{\sin^2 r}. \end{cases} \tag{3.22}$$

Here the sign $\varepsilon = \pm 1$ does not change on a segment of a geodesic going from $r = i$ to $r = \pi - i$, and it changes whenever r equals to i or $\pi - i$.

The angle $\theta(i, \pi - i)$ between two consecutive points of contact with the extreme parallel is

$$\theta(i, \pi - i) = \int_i^{\pi - i} \frac{d\theta}{dr} dr = \int_i^{\pi - i} \frac{d\theta}{dt}\frac{dt}{dr} dr = \varepsilon_1 \varepsilon \int_i^{\pi - i} \frac{\sin i \cdot f(\cos r)}{\sin r \sqrt{\sin^2 r - \sin^2 i}} dr.$$

A geodesic γ_i is closed if and only if

$$\varepsilon_1 \varepsilon \theta(i, \pi - i) = \frac{m(i)}{n(i)}\,\pi, \tag{3.23}$$

for some coprime positive integers $n(i)$ and $m(i)$. If all geodesics are closed, then $\varepsilon_1\varepsilon\theta(i,\pi-i)$ is a continuous function from $\mathbf{R}/2\pi\mathbf{Z}$ to $\mathbf{R}$ for any i in $(0,\pi/2)$ though the range of $\theta(i,\pi-i)$ is contained in $\mathbf{Q}\pi$. Here $\mathbf{Q}$ denotes the set of all rational numbers. Then $\theta(i,\pi-i)$ has to be constant, say $m(i)/n(i) = m/n \in \mathbf{Q}$. We suppose $\varepsilon_1\varepsilon = 1$ for the sake of simplicity. Then, γ_i is closed if and only if

$$\int_i^{\pi-i} \frac{\sin i \cdot f(\cos r)}{\sin r\sqrt{\sin^2 r - \sin^2 i}}dr = \frac{m}{n}\,\pi, \tag{3.24}$$

for every i in $(0,\pi/2)$, and for some positive integers m and n. (3.24) means that γ_i consists of $2n$ geodesic segments between two parallels $r = i$ and $r = \pi - i$, and is closed when it turns m times. Then the length of γ_i is

$$\begin{aligned} 2n\int_i^{\pi-i} \frac{dt}{dr}dr &= \varepsilon\, 2n\int_i^{\pi-i} \frac{\sin r f(\cos r)}{\sqrt{\sin^2 r - \sin^2 i}}dr \\ &= 2nL_g(i,\pi-i), \end{aligned} \tag{3.25}$$

where $L_g(i,\pi-i)$ is the length of γ_i from $r = i$ to $r = \pi - i$.

Since the length from N to S is L, we have

$$\lim_{i\to 0} L_g(i,\pi-i) = L.$$

In fact, $L_g(i,\pi-i)$ is independent on i, since $\theta(i,\pi-i) = m/n$ is a constant. Hence, the length of γ_i is $2nL$.

Q.E.D.

Remark

If g is smooth, we have $m = n = 1$. Indeed, suppose g is smooth. If i tends to zero, γ_i tends to n-meridians. Then, if $n \geq 2$, it yields a contradiction to the differentiability of g at N and S. Hence $n = 1$. Then γ_i has length 2π for any i by Theorem 3.5. If i tends to $\pi/2$, then γ_i tends to the equator, which is a simply closed geodesic with length 2π. Then we conclude that γ_i turns M just one time, that is, $m = 1$.

Let $h(\cos r)$ be a smooth function from $[-1, 1]$ to $\mathbf{R}$ which satisfies $f(\cos r) = 1+h(\cos r)$. Using this function h, a metric of revolution (3.18) is written as

$$g = \{1 + h(\cos r)\}^2 dr^2 + \sin^2 r\, d\theta^2. \tag{3.26}$$

We show a necessary and sufficient condition in order that the metric written as in (3.26) is a Zoll metric.

Theorem 3.6 ([Be]) *A Riemannian manifold (M, g) is a Zoll surface if and only if g is a metric of revolution which can be written in a parametrization $(U, (r, \theta))$ as in (3.26), where $h : [-1, 1] \to [-1, 1]$ is an odd function in $\cos r$ centered at zero which satisfies $h(1) = h(-1) = 0$, and $1 + h > 0$.*

Proof. Recall that f is a positive function by definition. Then we have $1 + h(\cos r) > 0$. This, together with the fact that h is an odd function, implies that the range of h is contained in the interval $[-1, 1]$. The condition $h(1) = h(-1) = 0$ guarantees the smoothness of the metric at the North Pole and the South Pole, as one can see in section 3.4. Then we just have to show the metric written as in (3.26) is a Zoll metric if and only if h is an odd function. By Theorem 3.5, g is a Zoll metric if and only if

$$\int_i^{\pi-i} \frac{\sin i\{1 + h(\cos r)\}}{\sin r\sqrt{\sin^2 r - \sin^2 i}} dr = \pi \tag{3.27}$$

holds for any i in $(0, \pi/2)$. Take the odd function $h \equiv 0$ in (3.26). Then g is the standard metric on a two-sphere (see Example 3.1 below). Then (3.27) implies

$$\int_i^{\pi-i} \frac{\sin i}{\sin r\sqrt{\sin^2 r - \sin^2 i}} dr = \pi, \tag{3.28}$$

for any i in $(0, \pi/2)$. Together with (3.27) and (3.28), we have

$$\int_i^{\pi-i} \frac{\sin i \cdot h(\cos r)}{\sin r\sqrt{\sin^2 r - \sin^2 i}} dr = 0, \tag{3.29}$$

for any i in $(0, \pi/2)$.

Then, the following lemma is equivalent to the theorem.

Lemma 3.6 *$h(\cos r)$ is an odd function centered at zero, if and only if (3.29) holds for any i in $(0, \pi/2)$.*

Proof. If $h(\cos r)$ is an odd function centered at zero, clearly (3.29) holds for any i in $(0, \pi/2)$. We will show the converse. Suppose (3.29) holds for any i in $(0, \pi/2)$. Since any function is described as the sum of an even function and an odd function, $h(\cos r)$ is written as $h(\cos r) = h^e(\cos r) + h^o(\cos r)$, where $h^e(\cos r)$ is an even part of h, and $h^o(\cos r)$ is an odd part of h. Indeed, we may set

$$\begin{aligned} h^e(\cos r) &= \frac{h(\cos r) + h(-\cos r)}{2}, \\ h^o(\cos r) &= \frac{h(\cos r) - h(-\cos r)}{2}. \end{aligned}$$

$h(\cos r)$ is an odd function if and only if $h^e(\cos r) = 0$ for any $r \in (0, \pi)$. Hence we have to show that (3.29) implies that $h^e(\cos r) = 0$.

$$\begin{aligned} 0 &\overset{(3.29)}{=} \int_i^{\pi - i} \frac{\sin i \cdot h(\cos r)}{\sin r \sqrt{\sin^2 r - \sin^2 i}} dr \\ &= \sin i \Big\{ \int_i^{\frac{\pi}{2}} \frac{h(\cos r)}{\sin r \sqrt{\sin^2 r - \sin^2 i}} dr + \int_{\frac{\pi}{2}}^{\pi - i} \frac{h(\cos r)}{\sin r \sqrt{\sin^2 r - \sin^2 i}} dr \Big\} \\ &= \sin i \Big\{ \int_i^{\frac{\pi}{2}} \frac{h(\cos r)}{\sin r \sqrt{\sin^2 r - \sin^2 i}} dr + \int_{\frac{\pi}{2}}^{i} \frac{h(-\cos r)}{\sin r \sqrt{\sin^2 r - \sin^2 i}} (-dr) \Big\} \\ &= \sin i \int_i^{\frac{\pi}{2}} \frac{h(\cos r) + h(-\cos r)}{\sin r \sqrt{\sin^2 r - \sin^2 i}} dr \\ &= \sin i \int_i^{\frac{\pi}{2}} \frac{2h^e(\cos r)}{\sin r \sqrt{\sin^2 r - \sin^2 i}} dr. \end{aligned}$$

Namely, we have

$$\int_i^{\frac{\pi}{2}} \frac{h^e(\cos r)}{\sin r \sqrt{\sin^2 r - \sin^2 i}} dr = 0$$

for any $i \in (0, \pi/2)$. Set $t = \cos r$, $x = \cos i$. Then

$$\int_0^x \frac{h^e(t)}{(1 - t^2)\sqrt{x^2 - t^2}} dt = 0$$

holds for any $x \in (0, 1)$. Then we obtain $h^e(t) = 0$ for any $t \in (0, 1)$ by Lemma 8.1 in the Appendix. Since h^e is an even function on $(-1, 1)$ centered at $t = 0$, we have $h^e(t) = 0$ for any $t \in (-1, 1)$. Hence we conclude that h is an odd function.

Q.E.D.

3.4 Differentiability of Zoll metrics

A Zoll metric (3.26) is a metric defined on $U = M \backslash \{N, S\}$. However, if the odd function $h : [-1, 1] \to [-1, 1]$ is smooth, then the Zoll metric g is smooth at the North or the South Pole. Namely, g can be extended from a smooth metric on U to a smooth metric on M. In order to prove this fact, we need the following two lemmas.

Lemma 3.7 *Let $k(r)$ be an even function which is smooth on $\mathbf{R}$. Then, there exists a function $\tilde{k}$ with $\tilde{k}(r^2) = k(r)$ which is smooth on $\mathbf{R}_{\geq 0}$.*

Proof. Fix any natural number n. Expanding k in a Taylor series in r around $r = 0$, we have

$$k(r) = \sum_{i=0}^{2n-1} \frac{1}{i!} r^i k^{(i)}(0) + R_{2n}(r), \tag{3.30}$$

where $R_{2n}(r)$ is a remainder term. We set $k_{2n-1}(r) = \sum_{i=0}^{2n-1} (1/i!) r^i k^{(i)}(0)$.
case 1)

If k is a polynomial, that is, $R_{2n}(r) = 0$ for any r in $\mathbf{R}$, $k(r)$ is equal to $k_{2n-1}(r)$. Since k is an even function, $k^{(i)}(0) = 0$ holds for any odd number $i < 2n$. Then $k_{2n-1}(r)$ is a polynomial in r^2, say $\tilde{k}_1(r^2) = k_{2n-1}(r)$. By smoothness of a polynomial, this $\tilde{k}_1$ is smooth.
case 2)

Assume $k^{(i)}(0) = 0$ holds for any $i < 2n$, that is, $k(r)$ is equal to $R_{2n}(r)$. Define the function $\tilde{k}_2$ as $\tilde{k}_2(r) = k(\sqrt{r})$ for $r \geq 0$. We discuss the differentiability of $\tilde{k}_2$ at $r = 0$. Set $\tilde{k}_2(t) = k(\sqrt{t})$ for any $t > 0$. For any $i \leq n$, the i-th derived function of $\tilde{k}_2$ is

$$\tilde{k}_2^{(i)}(t) = \sum_{l=i}^{2i-1} \mathrm{c}_{i,l} \, \frac{k^{(2i-l)}(\sqrt{t})}{(\sqrt{t})^l}.$$

Here $\mathrm{c}_{i,l}$ is some positive integer. Then we have

$$\lim_{t \to 0} \tilde{k}_2^{(i)}(t) = \lim_{r \to 0} \sum_{l=i}^{2i-1} \mathrm{c}_{i,l} \, \frac{k^{(2i-l)}(r)}{r^l} = \mathrm{c} \, k^{(2i)}(0),$$

where c is some positive integer. The last equality follows from L'Hospital's theorem. The assumption $k^{(m)}(0) = 0$ for any $m < l < 2i \leq 2n$ makes it possible to use it.

Hence, if k is $2n$-times differentiable, then $\tilde{k}_2$ is n-times differentiable.

general case)

By (3.30), $k(r)$ is written as

$$\begin{aligned} k(r) &= k_{2n-1}(r) + R_{2n}(r) \\ &= \tilde{k}_1(r^2) + \tilde{k}_2(r^2). \end{aligned}$$

$\tilde{k}_1$ is smooth by the case 1), and $\tilde{k}_2$ is n-times differentiable by the case 2). Then $\tilde{k}_1(r^2) + \tilde{k}_2(r^2)$ is n-times differentiable, and hence, it is smooth on $\mathbf{R}_{\geq 0}$, since n is any natural number.

Q.E.D.

Lemma 3.8 *Let $k(r)$ be an even function with $k(0) = 0$ which is smooth on $\mathbf{R}$. Then the metric which is described in a parametrization $(U, (r, \theta))$ as*

$$g = \{1 + k(r)\}^2 dr^2 + r^2 d\theta^2$$

can be smoothly extended to $r = 0$. Hence, g is a smooth metric on M.

Proof. Define two metrics g_1, g_2 as $g_1 = dr^2 + r^2 d\theta^2$ and $g_2 = \{2k(r) + k^2(r)\}dr^2$. Then g is written as $g = g_1 + g_2$. Let $(x, y) = (r\cos\theta, r\sin\theta)$ be Euclidian coordinates. Then g_1 is smooth at $(x, y) = (0, 0)$, since

$$g_1 = dr^2 + r^2 d\theta^2 = dx^2 + dy^2$$

is exactly the Euclidian metric.

Since $k(r)$ is even, $2k(r) + k^2(r)$ is also an even function. Then, according to Lemma 3.7, there exists a smooth function $\tilde{k}$ such that $\tilde{k}(r^2) = 2k(r) + k^2(r)$. Set $j(r^2) = \tilde{k}(r^2)/r^2$. Then we have

$$\begin{aligned} g_2 &= \tilde{k}(r^2)dr^2 = r^2 j(r^2)dr^2 \\ &= j(r^2)\{rdr\}^2 \\ &= j(x^2 + y^2)\{xdx + ydy\}^2. \end{aligned}$$

Hence g_2 is smooth at $(x, y) = (0, 0)$. Therefore, g can be smoothly extended to $r = 0$.

Q.E.D.

Let $g = \{1 + h(\cos r)\}^2 dr^2 + \sin^2 r\, d\theta^2$ be a Zoll metric showed in Theorem 3.6. h is a smooth odd function in $\cos r$ and satisfies $h(1) = h(-1) = 0$, and $1 + h > 0$. Set $\sin r = t$. Let $\tilde{h}(t)$ be the function defined by

$$\tilde{h}(t) = \frac{1 - \cos r + h(\cos r)}{\cos r}.$$

Then $\tilde{h}(t)$ is an even function since $\cos r$ is even in t, and satisfies $\tilde{h}(0) = 0$. Then we have

$$\begin{aligned} g &= \{1 + h(\cos r)\}^2 \frac{dt^2}{\cos^2 r} + t^2 d\theta^2 \\ &= \left\{\frac{1 + h(\cos r)}{\cos r}\right\}^2 dt^2 + t^2 d\theta^2 \\ &= \{1 + \tilde{h}(t)\}^2 dt^2 + t^2 d\theta^2. \end{aligned}$$

By Lemma 3.8, we can conclude that g is a smooth metric on the whole M.

3.5 Examples of Zoll surfaces

There is a wide choice of functions h. In particular, notice that there is no condition on the derivatives of h, that is, on the sectional curvature constructed from h. The sectional curvature at r is written as

$$\sigma(r) = \frac{1}{\{1 + h(\cos r)\}^3}\{1 + h(\cos r) - \cos r \cdot h'(\cos r)\}. \tag{3.31}$$

Here we give some examples of Zoll surfaces.

Example 3.1

Take the constant function $h \equiv 0$. Then the Zoll metric is $g = dr^2 + \sin^2 r d\theta^2$, and we can verify that $\sigma(r) = 1$ for any $r \in [0, \pi]$ from (3.31). This is the standard metric on a two-sphere.

Example 3.2

Take the function $h(\cos r) = \cos r \sin(2k+1)r$, for $k \in \mathbf{Z}$. Then the associated sectional curvature $\sigma_k(r)$ is

$$\sigma_k(r) = \frac{1}{\{1+\cos r \sin(2k+1)r\}^3}\Big\{1 + \frac{(2k+1)\cos^2 r \cos(2k+1)r}{\sin r}\Big\}.$$

If $k = 0$, the associated Zoll surface has non-negative sectional curvature for any $r \in [0, \pi]$. On the other hand, by a suitable choice of $k \in \mathbf{Z}$ and $r \in [0, \pi]$, the sectional curvature at r can be made less than any negative number.

Remark

Zoll metric (3.26) can be extended to a metric on a n-dimensional sphere S^n. Its representation is similar to that of the standard metric on S^n. Spherical coordinates in $(n+1)$-dimensional Euclidian space are given below:

$$\begin{cases} x^1 = \cos\theta^1 \\ x^2 = \sin\theta^1 \cos\theta^2 \\ \vdots \\ x^i = \Big(\prod_{j=1}^{i-1} \sin\theta^j\Big)\cos\theta^i \\ \vdots \\ x^{n+1} = \prod_{i=1}^{n} \sin\theta^i, \end{cases}$$

for $\theta^i \in [0, \pi]$, $(i = 1, 2, \ldots, n-1)$ and $\theta^n \in [-\pi, \pi]$. Then the standard metric on S^n, $(n > 2)$, has the following inductive expression:

$$g_0^n = (d\theta^1)^2 + \sin^2\theta^1\, g_0^{n-1}.$$

Here g_0^{n-1} is the standard metric on S^{n-1}, whose coordinates are $(\theta^2, \theta^3, \ldots, \theta^n)$.

One can check n-dimensional Zoll metric is written as

$$g = \{1 + h(\cos\theta^1)\}^2 (d\theta^1)^2 + \sin^2\theta^1\, g_0^{n-1}.$$

4 Billiards on Riemannian manifolds

4.1 Notation about billiards on Riemannian manifolds

Billiards on Riemannian manifolds are the extension of those on the Euclidian plane. Let (D, g) be a compact convex Riemannian manifold with smooth boundary ∂D. Here convex means for any two points p, q in D, there exists a geodesic from p to q which is entirely contained in D. We assume any geodesic issuing from a point in the interior of D intersects ∂D, and it has two endpoints.

Let TM be a tangent bundle over M, and let UM be a unit tangent bundle over M. For a boundary point of D, the set of unit tangent vector toward D is defined by

$$U^+(\partial D) = \{v \in U_p D \mid p \in \partial D\ ,\ g(v, N_p) > 0\},$$

where N_p is the inward normal to ∂D.

By taking $p_0 \in \partial D$ and $v_0 \in U^+_{p_0}(\partial D)$, we obtain a geodesic

$$\zeta_0^1(t) = \exp_{p_0}(tv_0)$$

for $t \in [t_0, t_1]$, where $t_0 = 0$, and t_1 denotes the first value of $t > 0$ such that $\zeta_0^1(t)$ hits ∂D. Then a unit vector $v_1 \in U^+_{\zeta_0^1(t_1)}(\partial D)$ is determined according to the rule that the angle of incidence equals the angle of reflection. Namely, the angle between $T_{\zeta_0^1(t_1)}(\partial D)$ and v_1 is equal to the angle between $T_{\zeta_0^1(t_1)}(\partial D)$ and $\frac{d}{dt}\zeta_0^1(t)\mid_{t=t_1}$. Hence we obtain a geodesic

$$\zeta_1^2(t) = \exp_{\zeta_0^1(t_1)}(t - t_1)v_1$$

for $t \in [t_1, t_2]$, where t_2 denotes the first value of $t > 0$ such that $\zeta_1^2(t)$ hits ∂D. Note that $\zeta_0^1(t_1) = \zeta_1^2(t_1)$. Analogously, geodesic segments are defined by

$$\zeta_i^{i+1}(t) = \exp_{\zeta_{i-1}^i(t_i)}(t - t_i)v_i$$

for $t \in [t_i, t_{i+1}]$ and some non-negative integer $i \in \mathbf{Z}_{\geq 0}$. Broken geodesic segments

$$\zeta(t) = \bigcup_{i=0} \zeta_i^{i+1}([t_i, t_{i+1}])$$

are called a billiard in D.

<u>Remark</u>

∂D is not a billiard, even if it is a geodesic. It can only be considered as a limit of billiards.

4.2 Periodic billiards

Now, we will introduce a periodic billiard and its period. The period of a periodic billiard is constructed of two positive integers, called the link number and the rotation number.

Definition *A billiard $\zeta(t)$ is periodic if there exists a positive integer $q \in \mathbf{Z}_{>0}$ such that*

$$\zeta_i^{i+1}(t - t_i) = \zeta_{i+q}^{i+q+1}(t - t_{i+q}),$$

for any $i \in \mathbf{Z}_{\geq 0}$, and any $t \geq 0$. It is often called a q-link periodic billiard.

A rotation number of a periodic billiard is defined as follows. Let $\zeta(t)$ be a q-link periodic billiard in D, and fix the direction of ∂D. Let $c : \mathbf{R} \longrightarrow D$ be a representation of ∂D parametrized by arc length, where $\mathbf{R}$ is the set of all real numbers. We choose the direction of c in the same direction as ∂D. $(\zeta_i^{i+1}(t_i))_{i \in \mathbf{Z}_{\geq 0}}$ is a sequence of points in ∂D, and is represented by a monotone increasing sequence of real numbers $s = (s_i)_{i \in \mathbf{Z}_{\geq 0}}$, which is defined by the minimum of those which satisfies $\zeta_i^{i+1}(t_i) = c(s_i)$, $s_0 = 0$. Let L be the length of ∂D. Since $\zeta(t)$ is a q-link periodic billiard, there exists a positive integer r such that

$$s_q = rL. \tag{4.1}$$

Take the opposite orientation of ∂D. Let $c^{-1} : \mathbf{R} \longrightarrow D$ be its representation of ∂D. $(\zeta_i^{i+1}(t_i))_{i \in \mathbf{Z}_{\geq 0}}$ is represented by a monotone increasing sequence of real numbers $x = (x_i)_{i \in \mathbf{Z}_{\geq 0}}$, which is defined by the minimum of those which satisfies $\zeta_i^{i+1}(t_i) = c^{-1}(x_i)$, $x_0 = 0$. Since the direction of c and c^{-1} is opposite and $c(s_i) = c^{-1}(x_i)$ for any $i \geq 0$, we have the following equations.

$$\begin{gathered}
s_1 + x_1 = L, \\
s_2 - s_1 + x_2 - x_1 = L, \\
\vdots \\
s_q - s_{q-1} + x_q - x_{q-1} = L.
\end{gathered}$$

Adding both sides of these equations, we have

$$\sum_{i=0}^{q-1} (s_{i+1} - s_i) + \sum_{i=0}^{q-1} (x_{i+1} - x_i) = qL.$$

Since $s_q = s_q - s_0 = \sum_{i=0}^{q-1}(s_{i+1} - s_i)$ and $x_q = x_q - x_0 = \sum_{i=0}^{q-1}(x_{i+1} - x_i)$, we obtain

$$s_q + x_q = qL.$$

By (4.1), this implies

$$x_q = (q - r)L.$$

We can verify $r < q$, since $s_{i+1} - s_i < L$ holds for any i, then we have $rL = s_q = s_q - s_0 = \sum_{i=0}^{q-1}(s_{i+1} - s_i) < qL$.

A positive integer p defined by $p = \min\{r\,, q - r\}$ is called the rotation number of a q-link periodic billiard $\zeta(t)$. Clearly, the rotation number is independent of the choice of the direction of ∂D.

The pair of such positive integers (q, p) is called a period of the periodic billiard. A periodic billiard which has a period (q, p) is often called a (q, p)-periodic billiard.

5 Rigidity problems for Zoll surfaces

5.1 Billiards on Zoll surfaces

Let (M, g) be a Zoll surface. If we give the Northern Hemisphere of a Zoll surface, its Southern Hemisphere is automatically determined since a Zoll metric is characterized by an odd function $h(\cos r)$ centered at $r = \pi/2$. In particular, the following holds.

Theorem 5.1 *Let (M, g) be a Zoll surface whose Northern Hemisphere is the standard sphere. Then, (M, g) is the standard sphere.*

Note: A non-trivial Zoll surface (i.e. a Zoll surface which is not the standard sphere) is always not symmetrical with respect to the equator $r = \pi/2$. We can verify this fact, for example, by computing the meridian curve of a non-trivial Zoll surface which is isometrically embedded in the Euclidian space.

In order to weaken the assumption of Theorem 5.1, we regard geodesic segments on the Northern Hemisphere of a Zoll surface as billiards.

First, we will begin with a simple observation of the billiards on the Northern Hemisphere of the standard sphere. All geodesics starting from a given point p on the equator arrive at the same point p', namely the antipodal point of p. The angle between the equator and the geodesic starting from the equator equals to the angle between the equator and the geodesic arriving at the equator. Since the angle of incidence of a segment of billiards equals the angle of reflection, the billiard starting from p always comes back to p via p'. Thus all billiards are $(2, 1)$-periodic. In particular, each billiard segment has the same length π, which is independent of a choice of an initial point or angle. We can indicate the properties of billiards on the Northern Hemisphere of the standard sphere as follows:

1) all segments of the billiards have the same length π,

2) each billiard is $(2,1)$-periodic.

Let us now attempt to extend the observation into the billiards on the Northern Hemisphere of a Zoll surface. Zoll metric is written as in (3.26). The equator of a Zoll surface is $r = \pi/2$. Then two parallels $r = i$ and $r = \pi - i$ face each other across the equator $r = \pi/2$ for any $i \in (0, \pi/2)$. Since every geodesic in $U = M \backslash \{N, S\}$ is contained between $r = i$ and $r = \pi - i$ (see section 3.3), all geodesics intersect the equator.

Geodesics starting from a given point p in the equator are not arriving at the same point in the equator in general. However, similar to the case of the standard sphere, the angle between the equator and the geodesic starting from the equator equals to the angle between the equator and the geodesic arriving at the equator. Then applying the first variation formula, all segments of the billiards starting from the equator at the angle ϕ have the same length $L(\phi)$. Namely, the length of a geodesic segment on the Northern Hemisphere depends only on its angle with the equator. In addition, billiards are not periodic in general.

Billiards on the Northern Hemisphere yield billiards on the Southern Hemisphere, since all geodesics of the Zoll surfaces are closed, and intersect the equator. If the length of a segment of the Northern Hemisphere is l, then the length of the corresponding segment of the Southern Hemisphere is $2\pi - l$.

According to our observation, we can see that billiards on a Zoll surface do not always satisfy either of the two properties 1) and 2) which are mentioned above. However, if billiards on the Northern Hemisphere of a Zoll surface satisfy at least 1) or 2) above, the metric should be the standard one. These facts are given bellow as Theorems 5.2 and 5.3.

5.2 Some results for Zoll surfaces

Let $\gamma(t) = (r(t), \theta(t))$ be a geodesic in (M, g) contained between two parallels $r = i$ and $r = \pi - i$ for $i \in (0, \pi/2)$ whose minimum or maximum value of $r(t)$ is $r = i$ and $r = \pi - i$, respectively. Let us consider the length of $\gamma(t)$ from $r = i$ to $r = j$, denoted by $L_g(i, j)$, for a positive real number j such that $0 < i < j \leq \pi/2$. Let (M, g_0) be the standard sphere. $L_{g_0}(i, j)$ means the length from $r = i$ to $r = j$ of a geodesic in (M, g_0) contained between parallels $r = i$ and $r = \pi - i$.

Now, fix any $j \in (0, \pi/2]$.

Lemma 5.1 *$L_g(i, j) = L_{g_0}(i, j)$ holds for any $i \in (0, j)$ if and only if*

$$\int_{\cos j}^{y} h(t)dt = 0$$

holds for any $y > \cos j$.

Proof. $L_g(i,j)$ is written as

$$L_g(i,j) = \int_i^j \frac{dt}{dr} dr = \varepsilon \int_i^j \frac{\sin r\{1 + h(\cos r)\}}{\sqrt{\sin^2 r - \sin^2 i}} dr.$$

We suppose $\varepsilon = 1$ for the sake of simplicity. Since g_0 is a Zoll metric with $h(\cos r) \equiv 0$, $L_{g_0}(i,j)$ is written as

$$L_{g_0}(i,j) = \int_i^j \frac{\sin r}{\sqrt{\sin^2 r - \sin^2 i}} dr. \tag{5.1}$$

Then, the assumption $L_g(i,j) = L_{g_0}(i,j)$ equals to

$$\int_i^j \frac{(\sin r) h(\cos r)}{\sqrt{\sin^2 r - \sin^2 i}} dr = 0, \tag{5.2}$$

for any $i \in (0, \pi/2)$. Set $t = \cos r$ and $x = \cos i$. Then we have

$$\int_{\cos j}^x \frac{h(t)}{\sqrt{x^2 - t^2}} dt = 0. \tag{5.3}$$

Then for any $y > \cos j$,

$$\frac{\pi}{2} \int_{\cos j}^y h(t) dt = 0 \tag{5.4}$$

holds by the proof of Lemma 8.1 in the Appendix.

If we follow the proof inversely, (5.4) implies that $L_g(i,j) = L_{g_0}(i,j)$ for any $i \in (0,j)$.

Q.E.D.

Theorem 5.2 *Let (M, g) be a Zoll surface such that all segments of the billiards on the Northern Hemisphere have the same length π. Then, (M, g) is the standard sphere.*

Proof. By assumption, $L_g(i, \pi/2) = L_{g_0}(i, \pi/2) = \pi/2$ for all i in $(0, \pi/2)$. Then Lemma 5.1 gives us

$$\frac{\pi}{2} \int_0^y h(t) dt = 0$$

for all $y > 0$. Since $h(t)$ is an odd function centered at zero, we have $h(t) \equiv 0$. Then we have the conclusion.

Q.E.D.

We considered a geodesic contained between two parallels $r = i$ and $r = \pi - i$, and compare the length from $r = i$ to the equator $r = \pi/2$ in the metric g with that in g_0. This $r = \pi/2$ is the best condition. Namely, if the length from $r = i$ to the parallel above (or below) the equator in the metric g equals that in g_0, (M, g) is not necessary the standard sphere. In fact, we have counter-examples.

Example 5.1

Take an odd function $h_1(t)$ which is equal to zero on the interval $[\cos(\pi/2-\varepsilon), 1]$ for some $0<\varepsilon<\pi/2$, but not identically zero on $[-1,1]$. Then, for all i in $(0,\pi/2)$ and $i<j<\pi/2-\varepsilon$, (M,g) satisfies $L_g(i,j)=L_{g_0}(i,j)$.

Example 5.2

$h_1(t)$ in Example 5.1 satisfies (5.4) not only for $j\in(0,\pi/2)$ but also for $j\in(\pi/2,\pi)$, since

$$\int_{\cos(\pi-j)}^{\cos j} h_1(t)dt=0$$

always holds for $j\in(\pi/2,\pi)$. Then, according to Lemma 5.1, the Zoll surface (M,g) with respect to this $h_1(t)$ satisfies $L_g(i,j)=L_{g_0}(i,j)$ for all i in $(0,\pi/2)$ and $j\in(\pi/2,\pi-i)$.

Theorem 5.3 *Let (M,g) be a Zoll surface such that all billiards on the Northern Hemisphere are periodic. Then its period should be $(2,1)$, and (M,g) is the standard sphere.*

Proof. Let $\gamma(t)=(r(t),\theta(t))$ be a geodesic contained between parallels $r=i$ and $r=\pi-i$ for all $i\in(0,\pi/2)$ whose minimum or maximum value of $r(t)$ is $r=i$ or $r=\pi-i$, respectively. We take the initial point $\gamma(0)=(i,0)$ in the parallel $r=i$. Let $\theta(i,\pi/2)$ be the value of θ which $\gamma(t)$ starting from $\gamma(0)$ hits the equator for the first time. The hit point is $(\pi/2,\theta(i,\pi/2))$.

According to the formula (3.22), $\theta(i,\pi/2)$ is written as

$$\theta(i,\frac{\pi}{2})=\int_i^{\frac{\pi}{2}}\frac{d\theta}{dr}dr=\int_i^{\frac{\pi}{2}}\frac{d\theta}{dt}\frac{dt}{dr}dr=\varepsilon_1\varepsilon\int_i^{\frac{\pi}{2}}\frac{\sin i\{1+h(\cos r)\}}{\sin r\sqrt{\sin^2 r-\sin^2 i}}dr.$$

We suppose $\varepsilon_1\varepsilon=1$ for the sake of simplicity. If all billiards are periodic, then $2\theta(i,\pi/2)q(i)=2\pi p(i)$ for some coprime positive integers $q(i)$ and $p(i)$ such that $p(i)<q(i)$. By the continuity of $\theta(i,\pi/2)$, $p(i)$ and $q(i)$ must be constant, say p and q. Namely,

$$\theta(i,\frac{\pi}{2})=\frac{p}{q}\pi$$

holds for any i in $(0,\pi/2)$, and then, all billiards are (q,p)-periodic. If i tends to zero, γ tends to a meridian curve. Hence we have

$$\lim_{i\to 0}\theta(i,\frac{\pi}{2})=\frac{\pi}{2}.$$

Therefore, $\theta(i,\pi/2)=\pi/2$ holds for any $i\in(0,\pi/2)$, and all billiards are $(2,1)$-periodic. Then we have

$$\theta(i,\frac{\pi}{2})=\int_i^{\frac{\pi}{2}}\frac{\sin i\{1+h(\cos r)\}}{\sin r\sqrt{\sin^2 r-\sin^2 i}}dr=\frac{\pi}{2}, \tag{5.5}$$

for any $i\in(0,\pi/2)$. Since (5.5) always holds for $h(\cos r)=0$, we have

$$\int_i^{\frac{\pi}{2}}\frac{\sin i}{\sin r\sqrt{\sin^2 r-\sin^2 i}}dr=\frac{\pi}{2}, \tag{5.6}$$

for any $i\in(0,\pi/2)$. By (5.5) together with (5.6), we have

$$\int_i^{\frac{\pi}{2}}\frac{(\sin i)h(\cos r)}{\sin r\sqrt{\sin^2 r-\sin^2 i}}dr=0, \tag{5.7}$$

for any $i\in(0,\pi/2)$. Consider the function $H(\cos r)=h(\cos r)/\sin^2 r$. Since $H(\cos r)$ is an odd function centered at zero, (5.7) is the particular case of (5.2). Hence the proof of Theorem 5.3 can be reduced to the proof of Theorem 5.2.

Q.E.D.

5.3 How to extend to $C_{2\pi}$-manifolds

We would like to extend our results for Zoll surfaces to general $C_{2\pi}$-surfaces. Let (M,g) be a $C_{2\pi}$-surface. Let D be a closed domain in M such that its boundary ∂D is a geodesic in (M,g). Suppose the billiards on (D,g) have similar properties to those on the standard hemisphere. Then, is (M,g) the standard sphere?

Let us give a notation which is useful to describe properties of the billiards on the standard hemisphere. The equator is regarded as the boundary of a billiard table. Let (D,g) be a compact convex Riemannian manifold with smooth boundary.

Definition *We say the billiards on (D,g) have the spherical property if they satisfy:*

1) *all segments of the billiards have the same length π,*

2) *each billiard is $(2,1)$-periodic.*

The condition 2) implies that ∂D is a geodesic, since it is the limit of the segments of billiards. Then ∂D is closed, and has length 2π.

Let $\gamma_\theta(t)$ be a geodesic issuing from $p_0\in\partial D$ with the initial angle $\theta\in(0,\pi)$, and denote $\gamma_\theta(0)=p_0$. Define the map $s:(0,\pi)\to\partial D$ by $s(\theta)=\gamma_\theta(\pi)$. If $ds/d\theta\neq 0$, $\dot\gamma_\theta(\pi)$ is orthogonal to ∂D by Gauss' lemma. Since the billiards on (D,g) have the spherical property, geodesics always have the same angle at ∂D. Hence $\theta = \pi/2$. If $\theta \neq \pi/2$, we have $ds/d\theta = 0$. Therefore, $ds/d\theta$ is identically zero, and s is a constant map. Namely, all geodesics issuing from $p_0 \in \partial D$ arrive at only one point.

Let q_0 be a point in ∂D such that the length from p_0 to q_0 along ∂D is π. Since all segments of the billiards have the same length π, we have

$$\lim_{\theta \to 0} s(\theta) = q_0.$$

Let $\overset{\circ}{D}$ be the interior of D. There exists a minimizing geodesic from p_0 to q_0, since D is convex. If it is contained in ∂D, its length is π by the definition of q_0. If it is contained in $\overset{\circ}{D}$, its length is also π, since it is a billiard segment. Hence γ_θ is a minimizing geodesic from p_0 to q_0 for each $\theta \in (0, \pi)$. By the first variation formula, a billiard issuing from a point in ∂D with an angle ϕ arrives at ∂D with the same angle ϕ. Since the angle of incidence equals the angle of reflection, the billiard always has the same angle ϕ at ∂D.

Fix any point $p \in \overset{\circ}{D}$, and $p_0 \in \partial D$. There exists a unique and minimizing geodesic from p_0 to p. Indeed, suppose there are two minimizing geodesics from p_0 to p. By extending these geodesics to ∂D, we can show a contradiction to the fact that all billiard segments issuing from p_0 is minimizing. Let ξ_{p_0} be the geodesic parametrized by arc length, and $\xi_{p_0}(0) = p_0$, $\xi_{p_0}(t) = p$. Define the map ϕ_p by

$$\phi_p : S^1 \cong \partial D \ni p_0 \mapsto \dot{\xi}_{p_0}(t) \in U_p D \cong S^1.$$

Since ξ_{p_0} is a unique geodesic from p_0 to p, ϕ_p is continuous and injective. Then it is surjective. Hence all geodesics in $\overset{\circ}{D}$ are part of billiard segments, and they are minimizing. In particular, there are no closed geodesics in $\overset{\circ}{D}$.

Let (M, g) be a $C_{2\pi}$-surface, and γ be a geodesic in M. Since γ is closed, and particularly simple ([G·G]), γ divides M into two domains. Fix the direction of γ. The closed domain on the left side of γ is called the Northern Hemisphere of M. The opposite is called the Southern Hemisphere. γ is supposed to belong to both hemispheres.

Now we assume that the billiards on the Northern Hemisphere of M have the spherical property. Then the billiards on the Southern Hemisphere also have the spherical property since (M, g) is a $C_{2\pi}$-surface. Then a rigidity problem for $C_{2\pi}$-surfaces can be loosely said as follows. If the half of a $C_{2\pi}$-surface is the standard sphere in the sense of billiards, is it the standard sphere? The problem is precisely stated below.

Problem 2 *Let (M, g) be a $C_{2\pi}$-surface. Let D be a closed domain in M such that its boundary ∂D is a geodesic in (M, g). Suppose the billiards on (D, g) have the spherical property. Then, is (M, g) the standard sphere?*

We could solve this problem under stronger condition. It is stated and proved in the following section.

6 Rigidity problems for $C_{2\pi}$-manifolds from a viewpoint of billiard problems

In this section, we solve Problem 2 under a certain conformality condition. We use the method of boundary rigidity problems for Riemannian manifolds by C. Croke ([C]). Croke's method essentially needs the uniqueness of the geodesics between boundary points. However, our billiard condition makes it possible to obtain the conclusion without the uniqueness of the geodesics.

Let (M, g_0) be the two-dimensional standard sphere, and f be a smooth positive function on M.

Theorem 6.1 *Let (M, g) be a $C_{2\pi}$-surface with metric $g = f^2 g_0$. Let D be a closed domain in M such that its boundary ∂D is a geodesic in (M, g). Suppose (D, g) satisfies the following:*

i) the billiards on (D, g) have the spherical property,

ii) ∂D is a unit speed geodesic in g, and also in g_0.

Then, (M, g) is the standard sphere.

In order to prove the theorem above, we should give some notation and show two lemmas. Let γ be a geodesic in (M, g) such that $\gamma = \partial D$. $\overset{\circ}{D}$ denotes the interior of D. The restriction of g to D is denoted g again.

Let $UD = (UD, G)$ be a unit tangent bundle over D with metric G, where G is a Sasaki metric with respect to g. Let $UD_0 = (UD, G_0)$ be a unit tangent bundle over D with metric G_0, where G_0 is a Sasaki metric with respect to the standard metric g_0.

Let $U^+\gamma = (U^+\gamma, G|_{U^+\gamma})$ be a subbundle of UD whose total space is defined as

$$U^+\gamma = \{v \in U_{\gamma(s)}D \mid g(v, N_{\gamma(s)}) \geq 0\},$$

where $N_{\gamma(s)}$ is the inward normal at $\gamma(s)$. Analogously, we can define $(U^+\gamma)_0$ as a subbundle of UD_0. du and dv denote the standard measures of UD and $U^+\gamma$ with respect to g, respectively. $\| \cdot \|$ denotes a norm with respect to g. In the same way, du_0, dv_0 and $\| \cdot \|_0$ denote the standard measures of UD_0, $(U^+\gamma)_0$ and a norm with respect to g_0, respectively.

Let $q : UM \to M$ be a projection on UM. For each $v \in UM$, there exists a unique geodesic $\xi_v : \mathbf{R} \to M$ which satisfies $\xi_v(0) = q(v)$ and $\dot{\xi}_v(0) = v$. Denote $v_t = \dot{\xi}_v(t)$, where $v_0 = v$. Then $q(v_t) = \xi_v(t)$ holds for any $t \in \mathbf{R}$. We define the map $\varphi : UM \times \mathbf{R} \to UM$ by $\varphi(v,t) = v_t$. Then (UM, φ) is a C^∞ flow called the geodesic flow on UM. Let $Z_g(t)$ be a vector field on UM whose associated one-parameter family is φ. $Z_g(t)$ is called the geodesic vector field on UM. Since M is a $C_{2\pi}$-surface, the geodesic flow φ on the unit tangent bundle UM is periodic. Then Z_g generates a free action of S^1. The quotient space UM/S^1 is a two-dimensional manifold (the manifold of oriented geodesics) which we denote by CM. The projection $\pi : UM \to CM$ is a S^1-principal bundle. Let TM^* be a cotangent bundle over M. Let α be the differential one-form on TM which is a pull-back of the canonical one-form on TM^*. We consider the restriction of α to UM, and still denote by α. Since $\alpha(Z_g) = 1$, the one-form α satisfies the definition of a connection form for the S^1-principal bundle $\pi : UM \to CM$. The distribution $Q : UM \to TUM$ defined by

$$Q_u = \{X \in T_uUM \mid \alpha(X) = 0\,,\ u \in UM\}$$

is a connection whose associated connection form is α. Then the direct sum decomposition of T_uUM is

$$T_uUM = \mathbf{R} \cdot Z_g(u) \oplus Q_u.$$

Moreover, according to the structural equation, $d\alpha$ is the curvature form of this connection form α.

The following lemma is the special case of Santaló 's formula([M]).

Lemma 6.1 *Suppose the assumption is all the same as Theorem* 6.1 *above. The restriction of α and $d\alpha$ to UD or $U^+\gamma$ are still denoted by α and $d\alpha$. Then the following holds:*

$$\int_{UD} \eta(u)\, \alpha \wedge (d\alpha) = \int_{U^+\gamma} \Big\{ \int_0^\pi \eta(\dot{\xi}(t)) dt \Big\} d\alpha,$$

where $\eta(u)$ is an integrable function on UM, $u = \dot{\xi}(t) \in UM$, $\xi(t)$ is a geodesic in (M, g).

Proof. Let ω be the closed two-form on CM defined by $\pi^*\omega = d\alpha$. Since ω is non-degenerate, it can be a symplectic form on CM. By Fubini's theorem, we have

$$\int_{UM} \eta(u)\, \alpha \wedge (d\alpha) = \int_{\xi \in CM} \Big\{ \int_\xi \eta(\dot{\xi}(t)) dt \Big\}\, \omega.$$

Since the billiards on D have the spherical property, all geodesics passing through a point in $\mathring{D}$ have two intersection points with ∂D. Then we can consider the map $\phi : U\,\mathring{D} \longrightarrow U^+\gamma$

defined by $\phi((\xi(t),\dot{\xi}(t))) = (\xi(0),\dot{\xi}(0))$, where $\xi(t)$ is a geodesic whose initial condition is $(\xi(0),\dot{\xi}(0))$. Since $d\alpha|_{U\overset{\circ}{D}} = \phi^*(d\alpha|_{U^+\gamma})$, $(U^+\gamma, d\alpha|_{U^+\gamma})$ is symplectically diffeomorphic to (CM,ω). Recall that all billiard segments have length π on D. Then we have

$$\int_{UD} \eta(u)\, \alpha \wedge (d\alpha) = \int_{(\xi(0),\dot{\xi}(0))\in U^+\gamma} \Big\{ \int_0^\pi \eta(\dot{\xi}(t))dt \Big\}\, d\alpha.$$

Q.E.D.

Lemma 6.2 *Suppose the assumption is all the same as Theorem* 6.1 *above. Let* $\xi_0(t)$ *be a geodesic in* (D,g_0)*. Then the following holds.*

$$\int_0^\pi \parallel \dot{\xi}_0(t) \parallel dt \geq \pi.$$

Proof. (D,g_0) is a closed subdomain of the standard sphere (M,g_0). Recall that $\gamma = \partial D$ is a geodesic in g also in g_0, then we know that (D,g_0) is precisely a hemisphere of (M,g_0).

Since $\xi_0(t)$ is a geodesic in g_0, the distance from $\xi_0(0)$ to $\xi_0(\pi)$ in g_0 is π. On the other hand, the distance from $\xi_0(0)$ to $\xi_0(\pi)$ along γ in g is also π. Then $\xi_0(0)$ and $\xi_0(\pi)$ are connected by minimizing geodesics in $(\overset{\circ}{D},g)$, and their length are also π. Hence the length of $\xi_0(t)$ in g is not less than π.

Q.E.D.

Now, we are ready to prove Theorem 6.1.

Proof of Theorem 6.1 For x in $\overset{\circ}{D}$, let u be a tangent vector at x such that $\parallel u \parallel_0 = 1$. Then, since $g = f^2 g_0$, we have $\parallel u \parallel = f(x)$. Integrating this over UD_0 and applying Fubini's theorem, we have

$$\int_{UD_0} \parallel u \parallel du_0 = 2\pi \int_D f(x)dx_0, \tag{6.1}$$

where dx_0 is the standard measure of the standard hemisphere (D,g_0).

Let $\xi_0(t)$ be a geodesic in g_0, and $u = \dot{\xi}_0(t)$. Set $UM_0 = (UM, G_0)$, and $CM_0 = UM_0/S^1$. Let α_0 be a connection form for S^1-principal bundle

$$\pi_0 : UM_0 \to CM_0$$

which is analogously defined as a connection form α. $d\alpha_0$ is the curvature form of α_0.

Since $\alpha_0 \wedge d\alpha_0$ is a volume form of UD_0, we have

$$\begin{aligned}
2\pi \int_D f(x)dx_0 \quad &\overset{(6.1)}{=} \quad \int_{UD_0} \| u \| \, du_0 \\
&= \quad \int_{UD_0} \| u \| \, \alpha_0 \wedge (d\alpha_0) \\
&\overset{\text{Lemma6.1}}{=} \quad \int_{(U^+\gamma)_0} \Big\{ \int_0^\pi \| \dot{\xi}_0(t) \| dt \Big\} \, d\alpha_0 \\
&\overset{\text{Lemma6.2}}{\geq} \quad \int_{(U^+\gamma)_0} \pi \, d\alpha_0 \\
&= \quad \int_{(U^+\gamma)_0} \Big\{ \int_0^\pi dt \Big\} \, d\alpha_0 \\
&= \quad \int_{UD_0} \alpha_0 \wedge (d\alpha_0) \\
&= \quad \int_{UD_0} du_0 \\
&= \quad 2\pi \, \mathrm{vol}(D, g_0).
\end{aligned}$$

Namely, we have

$$\int_D f(x)dx_0 \geq \mathrm{vol}(D, g_0). \tag{6.2}$$

On the other hand, since ∂D is a geodesic in g_0,

$$\mathrm{vol}(D, g) = \mathrm{vol}(D, f^2 g_0) = \int_D f^2(x)dx_0.$$

According to the Cauchy-Schwarz inequality,

$$\begin{aligned}
\mathrm{vol}(D, g)^{1/2}\mathrm{vol}(D, g_0)^{1/2} \quad &= \quad \Big\{ \int_D f^2(x)dx_0 \Big\}^{1/2} \Big\{ \int_D dx_0 \Big\}^{1/2} \\
&\geq \quad \int_D f(x)dx_0 \\
&\overset{(6.2)}{\geq} \quad \mathrm{vol}(D, g_0).
\end{aligned}$$

Hence we have

$$\mathrm{vol}(D, g) \geq \mathrm{vol}(D, g_0), \tag{6.3}$$

where the equality holds if and only if $f(x) = 1$ for any $x \in D$.

Since γ is a unit speed geodesic in g also in g_0, g equals to g_0 on γ. Namely,

$$g(\dot{\gamma}(t), \dot{\gamma}(t)) = g_0(\dot{\gamma}(t), \dot{\gamma}(t)),$$

for any $t \in \mathbf{R}/2\pi\mathbf{Z}$. Then we obtain

$$\begin{aligned}\int_{UD} du &= \int_{UD} \alpha \wedge (d\alpha) = \int_{U^+\gamma} \Big\{ \int_0^\pi dt \Big\}\, d\alpha \\ &= \int_{(U^+\gamma)_0} \Big\{ \int_0^\pi dt \Big\}\, d\alpha_0 = \int_{UD_0} \alpha_0 \wedge (d\alpha_0) \\ &= \int_{UD_0} du_0.\end{aligned}$$

Applying Fubini's theorem, we have

$$\mathrm{vol}(D, g) = \mathrm{vol}(D, g_0). \tag{6.4}$$

Then the equality holds in (6.3). Hence we have a conclusion that $f(x) = 1$ for any $x \in D$.

Let $D' \subset M$ be the complement of $\mathring{D}$, that is, another closed domain enclosed by γ. Since the billiards on D have the spherical property, the billiards on D' also have the spherical property. Hence the same argument applies to D'. Finally we have $f(x) = 1$ for any $x \in M$.

Q.E.D.

<u>*Remark*</u>

Theorem 6.1 can be extended to the result for n-dimensional $C_{2\pi}$-manifolds. In this case, the boundary of D is not a geodesic γ, but a totally geodesic hypersurface of M. Then we apply the Hölder inequality instead of the Cauchy-Schwarz inequality.

7 Rigidity problems for $C_{2\pi}$-manifolds

Theorem 6.1 is the rigidity problem for $C_{2\pi}$-surfaces under some billiard conditions on a hemisphere. One can easily see that if a hemisphere of a $C_{2\pi}$-surface is the standard hemisphere, the billiards on the remaining half have the spherical property. If we fully use the condition that a half of a $C_{2\pi}$-surface is the standard hemisphere, we can prove that the remaining half will also be the standard hemisphere. This result, the following Theorem 7.1, is the solution of Problem 1 which appeared in section 1. It is the extension of Theorem 5.1 to the general $C_{2\pi}$-surfaces. Here M denotes a manifold which is diffeomorphic to S^2.

Theorem 7.1 *Let (M, g) be a $C_{2\pi}$-surface. Let D be a closed domain in M such that its boundary ∂D is a geodesic in (M, g). Suppose (D, g) is the standard hemisphere. Then, (M, g) is the standard sphere.*

Proof. By assumption, $M = (M, g)$ is divided into two manifolds with boundary, written as $M = U \cup D$, where D is the standard hemisphere and U is the complement of the interior of D in M. Let γ_0 be $\partial D = \partial U$.

Let $M' = U' \cup D'$ be a copy of M. Since D and D' are the standard hemispheres, they are smoothly attached by identifying a point in D with its antipodal point in D'. Then the union $D \cup D'$ is the standard sphere. Recall that U and D are smoothly attached, and U' and D' are also smoothly attached. Then we can smoothly attach U to U' by identifying a point in ∂U with its antipodal point in $\partial U'$ (see Figure 1 below). We denote $\tilde{M} = U \cup U'$.

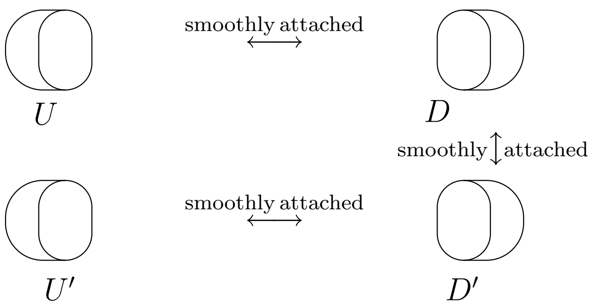

Figure 1: Hemispheres are smoothly attached to each other

Let $\tilde{\tau}$ be a diffeomorphism from U to U' such that $\tilde{\tau}$ maps a point p in U to the same point p' in U'. Namely, let ϕ be a natural diffeomorphism from M to M', and we identify M with M' by this ϕ. Then $\tilde{\tau}$ is defined as $\tilde{\tau}(p) = \phi(p)$, for any $p \in U$.

Let $\tilde{g}$ be the metric on $\tilde{M}$ defined by

$$\tilde{g} = \begin{cases} g & \text{on} \quad U \\ g' & \text{on} \quad U', \end{cases}$$

where the metric g' is a metric on U' which satisfies $\tilde{\tau}^* g' = g$. Note that $\tilde{g}$ is smooth around the equator.

Let $\gamma(t)$ be a geodesic in (M, g) parametrized by arc length. Since M is a $C_{2\pi}$-surface whose hemisphere is the standard one, γ is a closed geodesic which is divided by γ_0 into two geodesic segments with length π. Namely, γ is represented as

$$\gamma(t) = \begin{cases} \gamma_U(t) & t \in [0, \pi] \\ \\ \gamma_D(t) & t \in [\pi, 2\pi]. \end{cases}$$

Here $\gamma_U(t)$ is a geodesic in (U, g) issuing from $\gamma_U(0) \in \gamma_0$, and $\gamma_D(t)$ is a geodesic in (D, g) issuing from $\gamma_D(\pi) \in \gamma_0$, the antipodal point of p_0.

<u>*claim*</u>

$\gamma_U(t)$ is smoothly attached to $\tilde{\tau}(\gamma_U(t))$ in $\tilde{M}$, and hence $\gamma_U(t) \cup \tilde{\tau}(\gamma_U(t))$ is a geodesic in $(\tilde{M}, \tilde{g})$.

Indeed, since D is the standard hemisphere, if γ_D starts γ_0 at some angle, say α, then it arrives γ_0 at the same angle α. Then, γ_U starts γ_0 at the angle α, and it arrives γ_0 at the same angle α again. By the definition of g', (U', g') is isometric to (U, g). Then $\tilde{\tau}(\gamma_U)$ also crosses $\tilde{\tau}(\gamma_0)$ at the angle α.

Therefore, $\tilde{\tau}(\gamma_U)$ and γ_U face each other at the same angle α to the equator, then they are smoothly attached to each other. Since $\tilde{g}$ is smooth around γ_0, $\gamma_U \cup \tilde{\tau}(\gamma_U)$ is a geodesic in $(\tilde{M}, \tilde{g})$.

Then for any point p in $\tilde{M}$, the lengths of geodesics from p to $\tilde{\tau}(p)$ are always π. Indeed, let $\tilde{\gamma}(t)$ be a geodesic in $\tilde{M}$ parametrized by arc length issuing from $\tilde{\gamma}(0) \in \gamma_0$, which satisfies $\tilde{\gamma}(t_1) = p$ and $\tilde{\gamma}(t_2) = \tilde{\tau}(p)$ for some $0 < t_1 < \pi < t_2 < 2\pi$. Since U and U' are isometric, the lengths of geodesics from $\tilde{\gamma}(0)$ to $\tilde{\gamma}(t_1) = p$ are the same as the lengths of geodesics from $\tilde{\gamma}(\pi)$ to $\tilde{\gamma}(t_2) = \tilde{\tau}(p)$. Hence $t_2 = \pi + t_1$.

Recall that D is the standard hemisphere. Then the billiards on U have the spherical property since $M = U \cup D$ is a $C_{2\pi}$-surface. Then for any $p \in \mathring{U}$ and $q \in U$, there exists a unique and minimizing geodesic from p to q (see section 5.3). Hence, every geodesic issuing from p has a first intersection point at $\tilde{\tau}(p)$. Since geodesics are all closed, and the lengths of geodesics from p to $\tilde{\tau}(p)$ are always the same, the cut locus of p is only one point. Hence $(\tilde{M}, \tilde{g})$ is a Blaschke manifold modeled on a sphere, since p is any point in $\tilde{M}$. By the solution of the Blaschke conjecture for spheres (Theorem 2.1), $(\tilde{M}, \tilde{g})$ is the standard sphere. Therefore, (U, g) is the standard hemisphere, showing that (M, g) is the standard sphere.

Q.E.D.

Remark

Theorem 7.1 can be extended to the result for n-dimensional $C_{2\pi}$-manifolds. In this case, we take a totally geodesic hypersurface of M instead of a closed geodesic γ_0.

8 Appendix

Lemma 8.1 *Let f be a continuous function defined on the interval $[a,b]$, $0 \le a < b < \infty$. If*

$$\int_a^x \frac{f(t)}{\sqrt{x^2-t^2}}\,dt = 0 \tag{8.1}$$

holds for any $x \in [a,b]$, then $f \equiv 0$ on $[a,b]$.

Proof. By assumption,

$$\int_a^y \frac{x}{\sqrt{y^2-x^2}}\left\{\int_a^x \frac{f(t)}{\sqrt{x^2-t^2}}\,dt\right\}dx = 0$$

holds for any $y \in [a,b]$. Interchanging the orders of integration, we have

$$\int_a^y f(t)\left\{\int_t^y \frac{x}{\sqrt{y^2-x^2}\sqrt{x^2-t^2}}\,dx\right\}dt = 0. \tag{8.2}$$

Set

$$F(y,t) = \int_t^y \frac{x}{\sqrt{y^2-x^2}\sqrt{x^2-t^2}}\,dx.$$

Setting $x^2 = (y^2-t^2)u + t^2$, we have

$$F(y,t) = \frac{1}{2}\int_0^1 \frac{du}{\sqrt{u(1-u)}} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{2\sin v\cos v}{\sin v\cos v}\,dv = \frac{\pi}{2}.$$

Then we obtain

$$\frac{\pi}{2}\int_a^y f(t)\,dt = 0 \tag{8.3}$$

for any $y \in [a,b]$ by (8.2). Hence $f(t) = 0$ for any $t \in [a,b]$.

Q.E.D.

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