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Harmonic maps and totally geodesic maps between metric spaces

by

Shin-ichi OHTA

April 2004

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A thesis presented

by

Shin-ichi Ohta

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Chapter 1

Introduction

This thesis concerns analysis and differential geometry on metric spaces. We shall generalize several analytic and geometric notions including totally geodesic maps, Sobolev spaces, and harmonic maps. It is noted that the known facts for the finer objects (Riemannian manifolds, functions etc.) do not always continue to hold in our general setting (metric spaces, mappings etc.). Completely different phenomena may appear. In the case where we actually find such a phenomenon, it is probably much more interesting to investigate why such a difference occurs. We will also answer to this further question in several cases. This work was first inspired by Margulis' celebrated superrigidity theorem ([Mar1], see also [Mar2] and [Z]), a somewhat weakened version of which is described as follows:

Let k be a local field. By an *algebraic k-group*, we mean a Zariski k-closed subgroup of $GL(n; K)$ for some $n \geq 1$ and some algebraically closed field K containing k. We denote by G_k a group which consists of the k-points of an algebraic k-group G, and by G_k^0 its identity component. Typical examples are $K = \mathbb{C}$, $k = \mathbb{R}$, $G = SL(n; \mathbb{C})$, and $G_{\mathbb{R}} = G_{\mathbb{R}}^0 = SL(n; \mathbb{R})$.

Theorem 1.0.1 (*Margulis' superrigidity theorem*) *Let k and k' be local fields, G and H be connected algebraic k- and k'-groups, respectively. Let $\Gamma \subset G$ be an irreducible lattice, and $\rho : \Gamma \longrightarrow H_{k'}$ be a homomorphism. Suppose that G is semi-simple and its k-rank is greater than one, and that G_k^0 has no compact factor, H is k'-simple, and $\rho(\Gamma)$ is Zariski dense in H. Then either one of the following is true*:

(i) *The homomorphism ρ extends to a rational homomorphism $\bar{\rho} : G \longrightarrow H$.*

(ii) *The image $\rho(\Gamma)$ is relatively compact in $H_{k'}$.*

A differential geometric approach to proving Margulis' superrigidity theorem, which is established by Corlette [Co], is as follows:

(I) We first show the existence of a ρ-equivariant harmonic map $u : X \longrightarrow Y$.

(II) We next show that u is totally geodesic or constant.

(III) We finally show that u is homothetic or constant.

Here each of X and Y is a symmetric space of noncompact type or a Euclidean building. Indeed, in the case where $X = G/K$ and $Y = H/L$ are symmetric spaces and $\rho : \Gamma \longrightarrow H$ is a homomorphism from a lattice $\Gamma \subset G = \mathrm{Isom}(X)$, (III) above implies that ρ extends to a homomorphism $\bar{\rho} : G \longrightarrow H$ ($g \longmapsto u \circ g \circ u^{-1}$) or the closure of $\rho(\Gamma)$ is compact.

Corlette [Co], Jost and Yau [JY], and Mok, Siu, and Yeung [MSY] each deal with the case where X is a symmetric space of noncompact type and Y is a Riemannian manifold of nonpositive sectional curvature. There they prove (I) by Eells and Sampson's theorem and need more concrete calculations to show (II). On the other hand, (III) is generally obtained from the following well-known fact:

Fact 1.0.2 Any totally geodesic map from an irreducible Riemannian manifold into a Riemannian manifold is homothetic or constant.

Gromov and Schoen [GS] treat the case where X is a symmetric space of noncompact type and Y is a Euclidean building, and Wang [W] deals with the reverse case (under a somewhat stronger assumption than that in Margulis' theorem). The map u is always a constant map in these cases. We emphasize that the results of Corlette and Gromov-Schoen are in the rank-one case which is not covered by Margulis' original proof.

In the case where both X and Y are Euclidean buildings, although u is expected to be homothetic for some X and Y, even (III), i.e., the homothetic property of a totally geodesic map, is unknown. Furthermore, there are only a few investigations on totally geodesic maps for metric spaces. On the other hand, there are many works on the Sobolev spaces for maps between metric spaces and harmonic maps in these spaces. Our purpose in this thesis is to study totally geodesic maps, harmonic maps as well as Sobolev spaces to which they belong, and the relation between them. The thesis is organized as follows:

In Chapter 2, we recall the precise definitions of CAT(K)-spaces and Alexandrov spaces with an upper curvature bound. We also give the proofs, for completeness, of the first variation formula (Theorem 2.2.3) and the 1-Lipschitz continuity of the foot-point map (Proposition 2.3.2).

In Chapter 3, we consider totally geodesic maps between metric spaces. For a smooth map $u : M \longrightarrow N$ between Riemannian manifolds and for two vector fields V and W on M, we define $\nabla(u_*)(V, W) := {}'\nabla_V u_*(W) - u_*(\nabla_V W)$. Here $u_* : TM \longrightarrow TN$ denotes the differential of u, and ∇ and ${}'\nabla$ denote the Levi-Civita connection of M and the connection on $u^{-1}TN$ induced from the Levi-Civita connection of N, respectively. There the totally geodesic property of u is defined by the differential equation $\nabla(u_*) = 0$, or equivalently, by the property that it maps any geodesic of M to a geodesic of N. The latter definition makes sense for a map between metric spaces. Typical examples of totally geodesic maps are an isometry, a homothety, a Riemannian covering and a projection from the product of metric spaces to one of its factors, which are all continuous. An example of discontinuous totally geodesic map is an unbounded linear map from a Hilbert space to a Banach space.

It is intuitively clear that the totally geodesic property is a very strong condition. Indeed, as we mentioned above, any totally geodesic map from an irreducible Riemannian manifold into a Riemannian manifold is homothetic or constant (Fact 1.0.2). However, if we consider maps between arbitrary metric spaces, then we can derive almost nothing from this property. This is because the behavior of geodesics varies depending on an underlying metric space. For instance, in the case where (X, d_X) is a metric space in which any geodesic is constant, e.g., discrete metric spaces or fractals such as the von Koch curve, any map from X is totally geodesic and, conversely, only constant maps are totally geodesic maps from a metric space (which has sufficiently many geodesics) into X. Therefore it is natural and necessary to put some restrictions on the source and the target spaces.

In this chapter, we treat a totally geodesic map $u : M \longrightarrow X$ from a Riemannian manifold M to a metric space X whose convex radii are assumed to be positive. We first prove that M splits locally and isometrically into the vertical and the horizontal parts with respect to u (Theorem 3.2.1). By using these horizontal parts, we next show that the image of a sufficiently small open ball in M has a Finslerian structure (Corollary 3.4.4). This will be globalized in Theorem 3.3.1, which asserts that, if M is geodesically complete, then u can be represented as the composite of a totally geodesic map from M to a Finslerian manifold N and a locally isometric embedding from N to X. Intuitively, N is a space constructed by resolving the singularities of $u(M)$. If X is an Alexandrov space with local curvature upper bound, then N is a Riemannian manifold. As a corollary, we prove the homothetic property of a totally geodesic map from an irreducible Riemannian manifold to an Alexandrov space with local curvature upper bound. This is a generalization of Fact 1.0.2.

In Chapter 4, we treat the Sobolev spaces for maps between metric spaces. The theory of Sobolev spaces for maps from or into metric spaces starts from Ambrosio [A] and is making remarkable progress in these years. There are several definitions of such kind of Sobolev spaces — for example, by Korevaar and Schoen [KS] (see also [J1], [J3], and [Ra]), and by Hajłasz [Ha] (see also [HaK], [He], and [K]). In [Ch], Cheeger also defines the Sobolev spaces for real-valued functions on an arbitrary metric measure space (see also [S] and [HKST]). He not only treats the usual topics of Sobolev spaces including Dirichlet problem, but also proves many independently interesting theorems including generalized Rademacher's theorems. Moreover, such results are used in [CC] to study the structure of the Gromov-Hausdorff limit of Riemannian manifolds whose Ricci curvatures are bounded uniformly from below. For this reason, the author is most interested in Sobolev spaces defined by Cheeger.

When one intends to generalize Sobolev spaces for functions to those for maps into an arbitrary metric space, say X, considerably many difficulties arise. One of the most critical one is that X has no linear structure. One way to bypass this difficulty is to consider CAT(0)-spaces. There we can take the average of maps by using the center of mass argument (cf. [KS] and [J3]). However, we do not have the additive operator for them. The difference between them is not so serious if we treat only L^p spaces, but causes some trouble in the case where we consider Sobolev spaces. Another way is to consider Banach spaces (cf. [HKST]). This is a useful observation because every metric space can be isometrically embedded in some Banach space. Nevertheless, it is not yet sufficient only to consider Banach spaces. Indeed, Cheeger-type Sobolev spaces may change by such an embedding of the target space.

In this chapter, we generalize Cheeger's definition of Sobolev spaces for functions to that for maps into an arbitrary metric space X and, despite the difficulties above, we obtain some interesting results.

In §4.1, we define the energy of a map and the Sobolev spaces. However, for completely general metric spaces, we can prove only a few things.

In §4.2, we treat the case where (X, d_X) is a geodesic length space such that d_X is convex, and show the analogues of the bulk of fundamental results proved in [Ch]. For example, we prove the existence and the uniqueness of the minimal generalized upper gradient (Theorem 4.2.2).

In §4.3, we deal with mappings into a CAT(0)-space and solve the Dirichlet problem in accordance with the strategy in [J3] (Theorems 4.3.4 and 4.3.8).

In §4.4, we prove the minimality of the function $\operatorname{Lip} u$ (defined in §4.1) for any locally Lipschitz continuous map u into a locally compact, locally geodesics extendable, and separable Alexandrov space with local curvature upper bound (Theorem 4.4.8). This is a generalization of [Ch, Theorem 6.1]. However, our proof is based on a new idea different from the original one. The first variation formula (Theorem 2.2.3) is essential in our proof.

Finally, in §4.5, we consider what happens to the Sobolev spaces if we isometrically embed X into a Banach space. By such an embedding, Cheeger-type Sobolev spaces may change in general (Example 4.5.8). We will prove that this change does not occur in some cases (Theorem 4.5.9) by studying the relation between several types of Sobolev spaces.

In Chapter 5, we consider the following problem: Are totally geodesic maps harmonic? The harmonicity of a totally geodesic map is clear for a map between Riemannian manifolds. This is because both the harmonicity and the totally geodesic property of a map are defined by the differential equation and the one for the totally geodesic property ($\nabla(u_*) = 0$) is stronger than that for the harmonicity ($\operatorname{trace}(\nabla(u_*)) = 0$). However, for a map between metric spaces, they are defined in the different categories and it is difficult to compare them generally.

In this chapter, we consider mappings between a Riemannian manifold and a metric space. Our theorems (Theorems 5.1.4 and 5.1.8) assert that any totally geodesic map into an Alexandrov space of curvature ≤ 0 is harmonic with respect to both Korevaar-Schoen-type and Cheeger-type energies. These theorems enable us to construct many examples of harmonic maps whose images are not manifolds (Example 5.2.1). Namely, we can treat a map which maps a non-singular point to a singular point. Examples of harmonic maps of this kind have been scarcely known.

In this connection, we remark that Eells and Fuglede prove the harmonicity of totally geodesic maps from a Riemannian polyhedron into a Riemannian manifold ([EF]). Their proof is more differential geometric than ours. In that situation, it is difficult to construct examples of totally geodesic map except for projections (from the product of a metric space and a Riemannian manifold). In fact, we can easily prove that totally geodesic maps from a tripod or a surface of circular cone into a Riemannian manifold must be constant.

If the source space is not Riemannian, then a totally geodesic map is possibly not harmonic. In the last section, we will give an example of such map between CAT(0)-spaces, or compact, geodesically complete Alexandrov spaces of curvature ≤ 0 (Example 5.2.3 and Claim 5.2.4). We also prove that there exists no continuous harmonic map which is homotopic to that totally geodesic map. The non-uniformity of the dimension of the source space is the essential cause of this phenomenon. We note that any Euclidean building has a uniform dimension.

Chapter 2

CAT(K)-spaces and Alexandrov spaces

In this chapter, we briefly explain main geometric objects in this thesis, CAT(K)-spaces and Alexandrov spaces of curvature bounded from above. These are metric spaces with "curvature bounded from above" in some sense. We shall see some basic facts about them which will be necessary for this thesis. General references of this chapter are [ABN], [Ba], [BBI], [BH], and [OT].

2.1 Definitions

Let (X, d_X) be a metric space. We denote the open (closed respectively) ball with center $x \in X$ and radius $r > 0$ by $B_r(x)$ or $B_r(x; X)$ ($\overline{B}_r(x)$ or $\overline{B}_r(x; X)$ respectively). For $a, b \in \mathbb{R}$, we define $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$. For a continuous path $\gamma : [0, l] \longrightarrow X$, we define the *length* $l(\gamma)$ of γ by

$$l(\gamma) := \sup_{0=t_0<t_1<\cdots<t_N=l} \sum_{i=0}^{N-1} d_X(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is taken over all sequences $\{t_i\}_{i=0}^{N}$ as above and all $N \in \mathbb{N}$. A continuous path $\gamma : [0, l] \longrightarrow X$ is called a *geodesic* if it has a constant speed and is locally minimizing, that is, it satisfies

$$l(\gamma|_{[a,b]}) = (|b - a|/l) \cdot l(\gamma)$$

for every $0 \leq a < b \leq l$ and, for any $a \in [0, l]$, there exists some $\varepsilon > 0$ such that $l(\gamma|_{[a',a'']}) = d_X(\gamma(a'), \gamma(a''))$ holds, where we put $a' := (a - \varepsilon) \vee 0$ and $a'' := (a + \varepsilon) \wedge l$. Unlike those on a Riemannian manifold, geodesics on a metric space (e.g., a tree) may branch.

Fix $K \in \mathbb{R}$ and let M_K^2 be a two-dimensional, complete, and simply connected space form with constant sectional curvature K.

Definition 2.1.1 A metric space (X, d_X) is called a CAT(K)-*space* if it satisfies the following:

(i) For any two points $x, y \in X$ (with $d_X(x, y) \leq \pi/\sqrt{K}$ if $K > 0$), there exists a minimal geodesic joining them, that is, a geodesic $\gamma : [0, 1] \longrightarrow X$ which satisfies $\gamma(0) = x$, $\gamma(1) = y$, and $l(\gamma) = d_X(x, y)$.

(ii) For any three points $x, y, z \in X$ (with $d_X(x, y) + d_X(y, z) + d_X(z, x) < 2\pi/\sqrt{K}$ if $K > 0$), any minimal geodesic $\gamma : [0, 1] \longrightarrow X$ from x to y and $\lambda \in [0, 1]$, we have

$$d_X(z, \gamma(\lambda)) \leq d_{M_K^2}(\tilde{z}, \tilde{\gamma}(\lambda)).$$

Here we denote by $\triangle\tilde{x}\tilde{y}\tilde{z}$ a *comparison triangle* of $\triangle xyz$ in M_K^2 which is defined as a geodesic triangle in M_K^2 satisfying $d_X(x, y) = d_{M_K^2}(\tilde{x}, \tilde{y})$, $d_X(y, z) = d_{M_K^2}(\tilde{y}, \tilde{z})$, and $d_X(z, x) = d_{M_K^2}(\tilde{z}, \tilde{x})$, and we denote by $\tilde{\gamma} : [0, 1] \longrightarrow M_K^2$ the unique minimal geodesic from $\tilde{x}$ to $\tilde{y}$.

Any two points x and y in a CAT(K)-space (X, d_X) (with $d_X(x, y) < \pi/\sqrt{K}$ if $K > 0$) are connected by a unique minimal geodesic $\gamma : [0, 1] \longrightarrow X$ from x to y. Indeed, the existence is guaranteed by (i), and the uniqueness is derived from (ii) in Definition 2.1.1. Then we set $\gamma_{xy} := \gamma$ and we also use the notation $(1 - \lambda)x + \lambda y := \gamma_{xy}(\lambda)$ for $\lambda \in [0, 1]$.

If $K = 0$, then the inequality $d_X(z, \gamma(\lambda)) \leq d_{M_K^2}(\tilde{z}, \tilde{\gamma}(\lambda))$ in Definition 2.1.1(ii) is equivalent to

$$d_X(z, \gamma(\lambda))^2 \leq (1 - \lambda)d_X(z, x)^2 + \lambda d_X(z, y)^2 - (1 - \lambda)\lambda d_X(x, y)^2. \tag{2.1}$$

We know that any CAT(0)-space is contractible. For example, Hadamard manifolds, trees, Euclidean buildings, and Hilbert spaces are CAT(0)-spaces. Note that a Banach space is a CAT(0)-space if and only if it is a Hilbert space, i.e., it satisfies the parallelogram identity.

Definition 2.1.2 A metric space (X, d_X) is called an *Alexandrov space of curvature* $\leq K$ if, for any $x \in X$, there exists an open neighborhood D of x such that

(i) any minimal geodesic joining two points in D is contained in D;

(ii) (D, d_X) is a CAT(K)-space.

We call a set $D \subset X$ satisfying (i) and (ii) above an R_K-*domain*.

In fact, any Riemannian manifold whose sectional curvature is not greater than K is an Alexandrov space of curvature $\leq K$. We observe that an Alexandrov space of curvature ≤ 0 is not necessarily contractible, e.g., a flat torus is an Alexandrov space of curvature ≤ 0. We call a metric space (X, d_X) an *Alexandrov space with local curvature upper bound* if, for any $x \in X$, there exists an open neighborhood D of x which is an R_K-domain for some $K = K(x) \in \mathbb{R}$. Any Riemannian manifold is an Alexandrov space with local curvature upper bound.

In the remainder of this section, let (X, d_X) be an Alexandrov space of curvature $\leq K$. For two nonconstant geodesics $\gamma_i : [0, l_i] \longrightarrow X$ $(i = 1, 2)$ with a common starting point $x := \gamma_1(0) = \gamma_2(0)$, we define

$$\angle_x(\gamma_1, \gamma_2) := \lim_{s,t \to 0} \tilde{\angle}\gamma_1(s)x\gamma_2(t).$$

Here we set $\tilde{\angle}\gamma_1(s)x\gamma_2(t) := \angle\widetilde{\gamma_1(s)}\tilde{x}\widetilde{\gamma_2(t)}$, where $\triangle\widetilde{\gamma_1(s)}\tilde{x}\widetilde{\gamma_2(t)}$ is a comparison triangle of $\triangle\gamma_1(s)x\gamma_2(t)$ in M_K^2. It is not difficult to show that, from Definition 2.1.1(ii), the function $(s, t) \longmapsto \tilde{\angle}\gamma_1(s)x\gamma_2(t)$ is monotone non-decreasing in both s and t if they are sufficiently small. Hence the limit in the definition of $\angle_x(\gamma_1, \gamma_2)$ always exists. In particular, it holds that

$$\angle_x(\gamma_1, \gamma_2) \leq \tilde{\angle}\gamma_1(s)x\gamma_2(t)$$

for small $s, t > 0$. If one of γ_1 and γ_2 is a subarc of the other one, then clearly $\angle_x(\gamma_1, \gamma_2) = 0$ holds. However, the converse is not true, that is, $\angle_x(\gamma_1, \gamma_2) = 0$ does not imply that $\gamma_1(s) = \gamma_2(as)$ for some $a, \varepsilon > 0$ and all $s \in [0, \varepsilon]$. If $x, y, z \in X$ are in an R_K-domain (and $d_X(y, x) \vee d_X(y, z) < \pi/\sqrt{K}$ if $K > 0$), then we put $\angle xyz := \angle_y(\gamma_{yx}, \gamma_{yz})$. We prove some fundamental facts about the angle $\angle$ for the later use.

Proposition 2.1.3 (i) *For three geodesics $\gamma_i : [0, l_i] \longrightarrow X$ $(i = 1, 2, 3)$ emanating from a point $x \in X$, we have the triangle inequality*

$$\angle_x(\gamma_1, \gamma_3) \leq \angle_x(\gamma_1, \gamma_2) + \angle_x(\gamma_2, \gamma_3).$$

(ii) *If x_i, y_i, z_i tend to distinct points $x, y, z \in X$ in an R_K-domain (with $d_X(y, x) \vee d_X(y, z) < \pi/\sqrt{K}$ if $K > 0$) respectively as $i \to \infty$, then we have*

$$\angle xyz \geq \limsup_{i \to \infty} \angle x_i y_i z_i.$$

(iii) *For three distinct points $x, y, z \in X$ in an R_K-domain (with $d_X(y, x) \vee d_X(y, z) < \pi/\sqrt{K}$ if $K > 0$), we have*

$$\angle xyz = \lim_{t \to 0} \tilde{\angle}xy\gamma_{yz}(t).$$

Proof. (i) We may assume $\angle_x(\gamma_1, \gamma_2) + \angle_x(\gamma_2, \gamma_3) < \pi$. Fix a sufficiently small $\varepsilon > 0$ and take $t_1, t_2, t_3 > 0$ which are small enough to satisfy

$$\begin{aligned}
&\tilde{\angle}\gamma_1(t_1)x\gamma_2(t_2) \leq \angle_x(\gamma_1, \gamma_2) + \varepsilon, \quad \tilde{\angle}\gamma_2(t_2)x\gamma_3(t_3) \leq \angle_x(\gamma_2, \gamma_3) + \varepsilon, \\
&\tilde{\angle}\gamma_1(t_1)x\gamma_2(t_2) + \tilde{\angle}\gamma_2(t_2)x\gamma_3(t_3) < \pi.
\end{aligned}$$

Put $y_i := \gamma_i(t_i)$ for $i = 1, 2, 3$. Let $\triangle\tilde{y}_1\tilde{x}\tilde{y}_2$ and $\triangle\tilde{y}_2\tilde{x}\tilde{y}_3$ be comparison triangles of $\triangle y_1xy_2$ and $\triangle y_2xy_3$ in M_K^2 respectively such that they share the edge $\gamma_{\tilde{x}\tilde{y}_2}$ and that $\tilde{y}_1$ and $\tilde{y}_3$ lie on the opposite sides of this edge. Taking t_1 and t_3 much smaller than t_2, we can assume that $\gamma_{\tilde{x}\tilde{y}_2}$ and $\gamma_{\tilde{y}_1\tilde{y}_3}$ intersect, and we denote the intersection point by $p = \gamma_{\tilde{x}\tilde{y}_2}(t)$. Then we have

$$\begin{aligned} d_X(y_1, y_3) &\le d_X(y_1, \gamma_{xy_2}(t)) + d_X(\gamma_{xy_2}(t), y_3) \le d_{M_K^2}(\tilde{y_1}, p) + d_{M_K^2}(p, \tilde{y_3}) \\ &= d_{M_K^2}(\tilde{y_1}, \tilde{y_3}). \end{aligned}$$

The second inequality follows from Definition 2.1.1(ii) for $\triangle y_1 x y_2$ and $\triangle y_2 x y_3$. This implies $\tilde{\angle} y_1 x y_3 \le \angle \tilde{y_1}\tilde{x}\tilde{y_3}$. Moreover, it follows from $\angle \tilde{y_1}\tilde{x}\tilde{y_2} + \angle \tilde{y_2}\tilde{x}\tilde{y_3} < \pi$ that $\angle \tilde{y_1}\tilde{x}\tilde{y_3} = \angle \tilde{y_1}\tilde{x}\tilde{y_2} + \angle \tilde{y_2}\tilde{x}\tilde{y_3}$. Hence we obtain

$$\begin{aligned} \angle_x(\gamma_1, \gamma_3) &\le \tilde{\angle} y_1 x y_3 \le \angle \tilde{y_1}\tilde{x}\tilde{y_3} = \angle \tilde{y_1}\tilde{x}\tilde{y_2} + \angle \tilde{y_2}\tilde{x}\tilde{y_3} \\ &\le \angle_x(\gamma_1, \gamma_2) + \angle_x(\gamma_2, \gamma_3) + 2\varepsilon. \end{aligned}$$

(ii) Given $\varepsilon > 0$, let $x' = \gamma_{yx}(s)$ and $z' = \gamma_{yz}(t)$ $(s, t > 0)$ satisfy $\tilde{\angle} x' y z' \le \angle xyz + \varepsilon$. If we take $x'_i = \gamma_{y_i x_i}(s_i)$ and $z'_i = \gamma_{y_i z_i}(t_i)$ such that $d_X(y_i, x'_i) = d_X(y, x')$ and $d_X(y_i, z'_i) = d_X(y, z')$, then it follows from Definition 2.1.1(ii) that x'_i and z'_i tend to x' and z' respectively as $i \to \infty$. This implies that

$$|\tilde{\angle} x' y z' - \tilde{\angle} x'_i y_i z'_i| \le \varepsilon$$

holds for sufficiently large i. Thus we obtain

$$\angle xyz \ge \tilde{\angle} x' y z' - \varepsilon \ge \tilde{\angle} x'_i y_i z'_i - 2\varepsilon \ge \angle x_i y_i z_i - 2\varepsilon$$

for large i. Letting i tend to infinity, we have the required inequality.

(iii) Fix $\varepsilon > 0$. By (ii), for sufficiently small $t > 0$, it holds that $\angle xyz \ge \angle x \gamma_{yz}(t) z - \varepsilon$. Moreover, we also find

$$|\tilde{\angle} x y \gamma_{yz}(t) + \tilde{\angle} x \gamma_{yz}(t) y - \pi| \le \varepsilon$$

for small $t > 0$. There we have

$$\begin{aligned} \angle xyz &\le \tilde{\angle} x y \gamma_{yz}(t) \le \pi - \tilde{\angle} x \gamma_{yz}(t) y + \varepsilon \\ &\le \angle y \gamma_{yz}(t) z - \angle x \gamma_{yz}(t) y + \varepsilon \\ &\le \angle x \gamma_{yz}(t) z + \varepsilon \\ &\le \angle xyz + 2\varepsilon. \end{aligned}$$

Here we use the equality $\angle y \gamma_{yz}(t) z = \pi$, which holds since γ_{yz} is a geodesic, in the third inequality. Therefore we obtain

$$\angle xyz = \lim_{t \to 0} \tilde{\angle} x y \gamma_{yz}(t).$$

$\square$

For $x \in X$, we set

$$\Sigma_x := \{\text{geodesics emanating from } x\} / \sim,$$

where the equivalence relation $\sim$ is defined such that $\gamma_1 \sim \gamma_2$ holds if $\angle_x(\gamma_1, \gamma_2) = 0$. Then $(\Sigma_x, \angle_x)$ is a metric space by Proposition 2.1.3(i). Define the *space of directions* (Σ_x, ρ_x) at $x \in X$ as the inner metric space induced from $(\Sigma_x, \angle_x)$, that is, for $\gamma_1, \gamma_2 \in \Sigma_x$, we define

$$\rho_x(\gamma_1, \gamma_2) := \inf\{l(c) \,|\, c : [0,1] \longrightarrow (\Sigma_x, \angle_x), \text{ continuous}, c(0) = \gamma_1, c(1) = \gamma_2\},$$

and we set $\rho_x(\gamma_1, \gamma_2) := \infty$ if there exists no such curve. It is not difficult to show $\angle_x(\gamma_1, \gamma_2) = \rho_x(\gamma_1, \gamma_2) \wedge \pi$. We also define the *tangent cone* (K_x, d_x) at x by $K_x := \Sigma_x \times [0, \infty)/\Sigma_x \times \{0\}$ and

$$d_x((s, \gamma_1), (t, \gamma_2)) := \{s^2 + t^2 - 2st \cos \angle_x(\gamma_1, \gamma_2)\}^{1/2}.$$

If we consider a triangle in $\mathbb{R}^2$ such that two sides of it have lengths s and t and that the angle between them is $\angle_x(\gamma_1, \gamma_2)$, then $d_x((s, \gamma_1), (t, \gamma_2))$ is the length of the other side (see Figure 1).

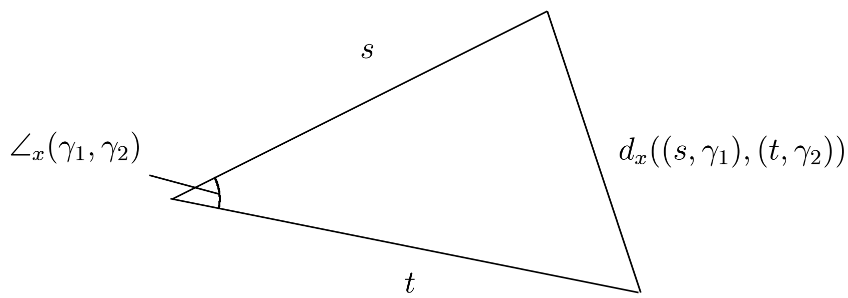

Figure 1

The space of directions and the tangent cone coincide with the unit tangent sphere and the tangent space for a Riemannian manifold. It is known that each connected component of the completion of (Σ_x, ρ_x) is an Alexandrov space of curvature ≤ 1, and the completion of (K_x, d_x) is a CAT(0)-space.

2.2 First variation formula

A metric space (X, d_X) is said to be *locally geodesics extendable* if, for each $x \in X$, there exists $\delta = \delta(x) > 0$ such that any unit speed geodesic $\gamma : [-\varepsilon, 0] \longrightarrow X$ with $\gamma(0) = x$ can be extended as a geodesic $\overline{\gamma} : [-\delta, \delta] \longrightarrow X$ satisfying $\gamma = \overline{\gamma}$ on $[-\varepsilon, 0]$. (We have called this property *locally geodesically complete* in [O2].) In this section, let (X, d_X) be a locally compact, locally geodesics extendable Alexandrov space of curvature $\leq K$.

Lemma 2.2.1 *Let $D \subset X$ be an R_K-domain. Then any geodesic $\gamma : [0, l] \longrightarrow X$ contained in D (with $l(\gamma) \leq \pi/\sqrt{K}$ if $K > 0$) is minimal.*

Proof. Suppose that $\gamma|_{[0,a]}$ and $\gamma|_{[a,a+\varepsilon]}$ $(0 < a < a + \varepsilon \le l)$ are minimal and $\gamma|_{[0,a+\varepsilon]}$ is not minimal. Put $x := \gamma(0)$, $y := \gamma(a)$, and $z := \gamma(a+\varepsilon)$ and let $\triangle\tilde{x}\tilde{y}\tilde{z}$ be a comparison triangle of $\triangle xyz$ in M_K^2. We observe that, by assumption, it holds that

$$d_X(x,y) + d_X(y,z) + d_X(z,x) < 2l(\gamma) \le 2\pi/\sqrt{K}$$

if $K > 0$. Since $d_X(x,y) + d_X(y,z) > d_X(x,z)$, we have $\pi > \tilde{\angle}xyz \ge \angle xyz = \pi$. This is a contradiction, so that γ is minimal on $[0,l]$. □

For a closed ball $\overline{B}_\varepsilon(x)$ (with $0 < \varepsilon < \pi/2\sqrt{K}$ if $K > 0$) contained in an R_K-domain, we set

$$S_\varepsilon(x) := \{y \in X \mid d_X(x,y) = \varepsilon\}, \quad \tilde{\angle}_x(y,z) := \tilde{\angle}yxz \text{ for } y,z \in S_\varepsilon(x).$$

We note that $\tilde{\angle}yxz$ depends only on ε and $d_X(y,z)$. We can show that $(S_\varepsilon(x), \tilde{\angle}_x)$ is a metric space as in the proof of Proposition 2.1.3(i).

Proposition 2.2.2 *For any $x \in X$ and a sufficiently small $\varepsilon > 0$, the metric space $(S_\varepsilon(x), \tilde{\angle}_x)$ is compact. Moreover, $(\Sigma_x, \angle_x)$ is compact.*

Proof. Let D be an R_K-domain containing x, and choose $\varepsilon \in (0, \delta(x) \wedge \pi/2\sqrt{K})$ such that $\overline{B}_\varepsilon(x) \subset D$ and $\overline{B}_\varepsilon(x)$ is compact. Given a sequence $\{y_i\}_{i=1}^\infty \subset S_\varepsilon(x)$, since $\overline{B}_\varepsilon(x)$ is compact, we can take a subsequence $\{y_{i_j}\}$ of $\{y_i\}$ which converges to some point $y \in D$. Then clearly we have $y \in S_\varepsilon(x)$ and $\tilde{\angle}_x(y_{i_j}, y) \to 0$ as $j \to \infty$. Hence $(S_\varepsilon(x), \tilde{\angle}_x)$ is compact. By the geodesics extendable property of $\overline{B}_\varepsilon(x)$, the map

$$S_\varepsilon(x) \ni y \longmapsto \gamma_{xy} \in (\Sigma_x, \angle_x)$$

is surjective and 1-Lipschitz continuous. Therefore $(\Sigma_x, \angle_x)$ is also compact. □

The symbols $o_{\alpha,\beta}(\varepsilon)$ and $\theta_{\alpha,\beta}(\varepsilon)$ denote functions depending only on α and β with $\lim_{\varepsilon\to 0} o_{\alpha,\beta}(\varepsilon)/\varepsilon = 0$ and $\lim_{\varepsilon \to 0} \theta_{\alpha,\beta}(\varepsilon) = 0$. The first variation formula for length below plays a crucial role in §4.4.

Theorem 2.2.3 ([OT], cf. [OS, Theorem 3.5]) *Fix an R_K-domain $D \subset X$ and distinct points $x, y \in D$. Then, for each $z \in D$, it holds that*

$$d_X(y,x) - d_X(z,x) = d_X(y,z)\cos\angle xyz + o_{x,y}(d_X(y,z)).$$

Proof. Fix a sufficiently small $\varepsilon > 0$. Since the metric space $(S_\varepsilon(y), \tilde{\angle}_y)$ is compact, we can take a finite set $\{z_i\}_{i=1}^N \subset S_\varepsilon(y)$ satisfying $B_\varepsilon(\{z_i\}_{i=1}^N; S_\varepsilon(y)) = S_\varepsilon(y)$. For each i, we have $\lim_{t\to 0} \tilde{\angle}xy\gamma_{yz_i}(t) = \angle xyz_i$ by Proposition 2.1.3(iii). Hence we find $t_\varepsilon \in (0,\varepsilon)$ for which

$$\angle xyz_i \le \tilde{\angle}xy\gamma_{yz_i}(t) \le \angle xyz_i + \varepsilon$$

holds for all i and $t \in (0, t_\varepsilon/\varepsilon]$. In particular, for every $w \in B_{t_\varepsilon}(y) \setminus \{y\}$, by taking $w' \in S_\varepsilon(y)$ and i such that $w = \gamma_{yw'}(s)$ for some $s \in (0, t_\varepsilon/\varepsilon)$ and that $\tilde{\angle}_y(w', z_i) \le \varepsilon$, we have $\angle wyz_i = \angle w'yz_i \le \tilde{\angle}_y(w', z_i) \le \varepsilon$ and

$$
\begin{aligned}
\angle xyw \le \tilde{\angle} xyw &\le \tilde{\angle} xy\gamma_{yw'}(t_\varepsilon/\varepsilon) \\
&\le \tilde{\angle} xy\gamma_{yz_i}(t_\varepsilon/\varepsilon) + \tilde{\angle}\gamma_{yz_i}(t_\varepsilon/\varepsilon)y\gamma_{yw'}(t_\varepsilon/\varepsilon) \\
&\le \angle xyz_i + \varepsilon + \tilde{\angle} z_i yw' \\
&\le \angle xyw + 3\varepsilon.
\end{aligned}
$$

We may assume $z \in B_{t_\varepsilon}(y) \setminus \{y\}$. In fact, then we have $\varepsilon = \theta_{x,y}(d_X(y,z))$, more precisely, $\inf\{\varepsilon > 0 \,|\, t_\varepsilon > r\} = \theta_{x,y}(r)$ since $\inf\{\varepsilon > 0 \,|\, t_\varepsilon > t_{\varepsilon_0}/2\} \le \varepsilon_0$. Thus we obtain

$$|\tilde{\angle} xyz - \angle xyz| = \theta_{x,y}(d_X(y,z)).$$

On the other hand, we know

$$\cos \tilde{\angle} xyz = \frac{d_X(x,y) - d_X(x,z)}{d_X(y,z)} + \theta(d_X(y,z)).$$

Consequently, we obtain

$$
\begin{aligned}
d_X(x,y) - d_X(x,z) &= d_X(y,z) \cos \tilde{\angle} xyz + o(d_X(y,z)) \\
&= d_X(y,z) \cos \angle xyz + o_{x,y}(d_X(y,z)).
\end{aligned}
$$

□

2.3 Foot-points

Throughout this section, let (X, d_X) be a CAT(0)-space. A subset $A \subset X$ is said to be *geodesically convex* if any two points in A can be connected by a minimal geodesic contained in A. For a complete, geodesically convex subset $A \subset X$, the *foot-point* (or the *nearest point projection*) of a point $x \in X$ to the set A is defined as a point in A which is closest to x (see [KS, Proposition 2.5.4]). From (2.1), we can easily prove that such a point exists and is unique, and hence we denote it by $F[A](x)$. We shall obtain that this map is distance non-increasing by the sub-embedding property due to Reshetnyak [Re] (see also [KS, Theorem 2.1.1]). Here we only show a much restricted version of it.

Theorem 2.3.1 *For any four points $w, x, y, z \in X$, there exists a convex quadrilateral $\bar{w}\bar{x}\bar{y}\bar{z} \subset \mathbb{R}^2$ (a sub-embedding of $\{w, x, y, z\}$) satisfying*

$$
\begin{aligned}
&d_X(w,x) = |\bar{w} - \bar{x}|, \quad d_X(x,y) = |\bar{x} - \bar{y}|, \quad d_X(y,z) = |\bar{y} - \bar{z}|, \quad d_X(z,x) = |\bar{z} - \bar{x}|, \\
&d_X(w,y) \le |\bar{w} - \bar{y}|, \quad d_X(x,z) \le |\bar{x} - \bar{z}|.
\end{aligned}
$$

Proof. As in the proof of Proposition 2.1.3(i), let $\triangle\tilde{x}\tilde{w}\tilde{y}$ and $\triangle\tilde{y}\tilde{w}\tilde{z}$ be comparison triangles of $\triangle xwy$ and $\triangle ywz$ in $\mathbb{R}^2$ respectively such that they share the edge $\gamma_{\tilde{w}\tilde{y}}$ and that $\tilde{x}$ and $\tilde{z}$ lie on the opposite sides of this edge. If $\tilde{\angle}xyw + \tilde{\angle}wyz \le \pi$, then we can derive $d_X(x,z) \le |\tilde{x} - \tilde{z}|$ as in the proof of Proposition 2.1.3(i), so that $\tilde{w}\tilde{x}\tilde{y}\tilde{z}$ itself is a sub-embedding.

Next we consider the case where $\tilde{\angle}xyw + \tilde{\angle}wyz > \pi$. We move $\tilde{y}$ and $\tilde{z}$ until $\tilde{y}$ lies on $\gamma_{\tilde{x}\tilde{z}}$ while keeping $\tilde{w}, \tilde{x}$, $|\tilde{x} - \tilde{y}|$, $|\tilde{y} - \tilde{z}|$, and $|\tilde{z} - \tilde{w}|$ fixed (see Figure 2). We denote the resulting points by $\bar{w}$ $(= \tilde{w})$, $\bar{x}$ $(= \tilde{x})$, $\bar{y}$, and $\bar{z}$.

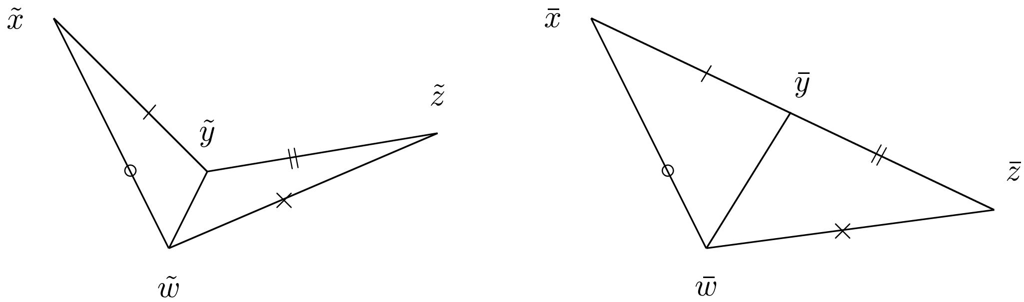

Figure 2

Then we have $d_X(x,z) \le d_X(x,y) + d_X(y,z) = |\bar{x} - \bar{z}|$ and $d_X(w,y) = |\tilde{w} - \tilde{y}| \le |\bar{w} - \bar{y}|$. Therefore $\bar{w}\bar{x}\bar{y}\bar{z}$ is a sub-embedding of $\{w, x, y, z\}$. $\square$

If $\bar{w}\bar{x}\bar{y}\bar{z} \subset \mathbb{R}^2$ is a sub-embedding of $\{w, x, y, z\} \subset X$, then we have

$$\begin{aligned} d_X(w, (1-t)x + ty)^2 &\le (1-t)d_X(w,x)^2 + td_X(w,y)^2 - (1-t)td_X(x,y)^2 \\ &\le (1-t)|\bar{w} - \bar{x}|^2 + t|\bar{w} - \bar{y}|^2 - (1-t)t|\bar{x} - \bar{y}|^2 \\ &= |\bar{w} - \{(1-t)\bar{x} + t\bar{y}\}|^2 \end{aligned}$$

for each $t \in [0,1]$.

Proposition 2.3.2 *For any complete, geodesically convex set $A \subset X$, the map $F[A]$ is 1-Lipschitz continuous.*

Proof. For two points $x, y \in X$, put $x' := F[A](x)$ and $y' := F[A](y)$, and set $z_t := (1-t)x' + ty' \in A$ for $t \in [0,1]$. Let $\bar{x}\bar{x}'\bar{y}'\bar{y} \subset \mathbb{R}^2$ be a sub-embedding of $\{x, x', y', y\}$. Then we have

$$\begin{aligned} &d_X(x, z_t)^2 + d_X(y, z_{1-t})^2 \\ &\le |\bar{x} - \{(1-t)\bar{x}' + t\bar{y}'\}|^2 + |\bar{y} - \{t\bar{x}' + (1-t)\bar{y}'\}|^2 \\ &= |(\bar{x} - \bar{x}') + t(\bar{x}' - \bar{y}')|^2 + |(\bar{y} - \bar{y}') + t(\bar{y}' - \bar{x}')|^2 \\ &= |\bar{x} - \bar{x}'|^2 + |\bar{y} - \bar{y}'|^2 + 2t^2|\bar{x}' - \bar{y}'|^2 + 2t\langle(\bar{x} - \bar{x}') - (\bar{y} - \bar{y}'), \bar{x}' - \bar{y}'\rangle. \end{aligned}$$

Here we denote the canonical inner product on $\mathbb{R}^2$ by $\langle\cdot,\cdot\rangle$. It holds that

$$\begin{aligned} 2t\langle(\bar{x} - \bar{x}') - (\bar{y} - \bar{y}'), \bar{x}' - \bar{y}'\rangle &= 2t\langle(\bar{x} - \bar{y}) - (\bar{x}' - \bar{y}'), \bar{x}' - \bar{y}'\rangle \\ &= t\{|\bar{x} - \bar{y}|^2 - |(\bar{x} - \bar{y}) - (\bar{x}' - \bar{y}')|^2 - |\bar{x}' - \bar{y}'|^2\} \\ &\le t\{|\bar{x} - \bar{y}|^2 - |\bar{x}' - \bar{y}'|^2\}. \end{aligned}$$

Hence we obtain

$$\begin{aligned}&d_X(x,z_t)^2+d_X(y,z_{1-t})^2\\&\le d_X(x,x')^2+d_X(y,y')^2+2t^2d_X(x',y')^2+t\{d_X(x,y)^2-d_X(x',y')^2\}.\end{aligned}$$

On the other hand, the definitions of x' and y' imply $d_X(x,x')\le d_X(x,z_t)$ and $d_X(y,y')\le d_X(y,z_{1-t})$. Therefore we have

$$0\le 2t^2d_X(x',y')^2+t\{d_X(x,y)^2-d_X(x',y')^2\}.$$

Dividing this inequality by t and letting t tend to zero, we obtain $d_X(x',y')\le d_X(x,y)$. $\square$

Chapter 3

Totally geodesic maps

In this chapter, we study totally geodesic maps from a Riemannian manifold to a metric space, and will obtain a kind of rigidity. The main results of this chapter are Theorems 3.2.1 and 3.3.1, and Corollaries 3.4.4 and 3.4.5. This chapter is based on [O1].

3.1 Basic properties

Let (M, g) be an n-dimensional C^∞-Riemannian manifold and (X, d_X) be a metric space. We denote by d_M the distance function on M induced from g. We define $\partial M := \overline{M} \setminus M$, where $\overline{M}$ is the completion of M with respect to d_M. For $p \in M$, the *convexity radius* at p, denoted by $r_M(p)$, is defined by

$$\sup \left\{ \delta \geq 0 \;\middle|\; \begin{array}{l} \text{Any geodesic contained in } B_\delta(p) \text{ is minimal and} \\ \text{any ball } B_\tau(q) \text{ with } d_M(p,q) \leq \delta - \tau \text{ is strongly convex} \end{array} \right\},$$

where a subset $U \subset M$ is said to be *strongly convex* if, for any $q, r \in U$, a minimal geodesic from q to r uniquely exists and is contained in U. The same definition makes sense for $x \in X$ and we denote it by $r_X(x)$. Note that $r_X(x) = 0$ may happen (e.g., at a singular point of an Alexandrov space of curvature bounded from below), while $r_M(p) > 0$ holds for any $p \in M$. For $q, r \in B_{r_M(p)}(p)$, we denote the unique minimal geodesic from q to r by $\gamma_{qr} : [0,1] \longrightarrow M$ as in the previous chapter, and denote the image of γ_{qr} by qr. For $y, z \in B_{r_X(x)}(x)$, we define γ_{yz} and yz in the same manner.

We assume, throughout this chapter, that there exists a *totally geodesic* map $u : M \longrightarrow X$, that is, u maps any geodesic in M to a geodesic in X. For such u, we define a function $|du| : TM \longrightarrow \mathbb{R}$ as, for $v \in T_pM$,

$$\begin{aligned} |du|(v) &:= \text{the speed of the geodesic } u \circ \gamma_v \\ &= \frac{d_X\big(u(p), u(\gamma_v(t))\big)}{t} \quad \text{for sufficiently small } t > 0, \end{aligned}$$

where $\gamma_v(t) := \exp_p tv$.

Proposition 3.1.1 *The map u is continuous on M.*

Proof. Take any $p \in M$ and let $\{e_i\}_{i=1}^n$ be an orthonormal basis of T_pM. Then, for $\varepsilon \in (0, r_M(p) \wedge \mathrm{dist}(p, \partial M))$, we set

$$\begin{aligned}
a_{\pm i}^{\varepsilon} &:= \gamma_{e_i}(\pm\varepsilon) \quad \text{for } i = 1, 2, \dots, n, \\
V_0^{\varepsilon} &:= \{a_i \,|\, i = \pm 1, \pm 2, \dots, \pm n\}, \\
V_1^{\varepsilon} &:= \{r \in a_i a_j \,|\, i < j, i \neq \pm j\}, \\
V_2^{\varepsilon} &:= \{r \in a_i a_j a_k \,|\, i < j < k, i \neq \pm j, j \neq \pm k, k \neq \pm i\}, \\
&\qquad \text{where } a_i a_j a_k := \{a_i q \,|\, q \in a_j a_k\}, \\
&\ \vdots \\
V_{n-1}^{\varepsilon} &:= \left\{ r \in a_{i_0} \cdots a_{i_{n-1}} \,\middle|\, \begin{array}{l} i_0 < \cdots < i_{n-1}, \\ \{\pm i\} \not\subset \{i_0, \dots, i_{n-1}\} \text{ for any } i = 1, \dots, n \end{array} \right\}, \\
&\qquad \text{where we inductively define } a_{i_0} \cdots a_{i_{n-1}} := \{a_{i_0} q \,|\, q \in a_{i_1} \cdots a_{i_{n-1}}\}.
\end{aligned}$$

Then, for any $r \in a_{i_0} q \subset a_{i_0} \cdots a_{i_m} \subset V_m^{\varepsilon}$ with $1 \leq m \leq n-1$, we have

$$\begin{aligned}
d_X(u(p), u(r)) &\leq \frac{1}{2}\{d_X(u(p), u(a_{i_0})) + d_X(u(p), u(q)) + d_X(u(q), u(a_{i_0}))\} \\
&\leq d_X(u(p), u(a_{i_0})) + d_X(u(p), u(q)) \\
&\leq 2 \max\{d_X(u(p), u(a_{i_0})), d_X(u(p), u(q))\}.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
\sup\{d_X(u(p), u(r)) \,|\, r \in V_{n-1}^{\varepsilon}\} &\leq 2 \sup\{d_X(u(p), u(r)) \,|\, r \in V_{n-2}^{\varepsilon}\} \\
&\leq \cdots \\
&\leq 2^{n-1} \max\{d_X(u(p), u(r)) \,|\, r \in V_0^{\varepsilon}\} \\
&< \infty.
\end{aligned}$$

We put

$$a_{\varepsilon} := \inf\{d_M(p, r) \,|\, r \in V_{n-1}^{\varepsilon}\} > 0, \quad b_{\varepsilon} := 2^{n-1} \max\{d_X(u(p), u(r)) \,|\, r \in V_0^{\varepsilon}\}.$$

Then $\lim_{\varepsilon \to 0} a_{\varepsilon}/\varepsilon = 1/\sqrt{n}$ and

$$b_{\varepsilon} = 2^{n-1} \varepsilon \max\{|du|(e_i) \,|\, i = 1, \dots, n\}$$

for sufficiently small $\varepsilon > 0$. Since any geodesic emanating from p intersects V_{n-1}^{ε}, we have

$$\sup_{v \in U_pM} |du|(v) \leq \lim_{\varepsilon \to 0} \frac{b_{\varepsilon}}{a_{\varepsilon}} = 2^{n-1} \sqrt{n} \max\{|du|(e_i) \,|\, i = 1, \dots, n\} < \infty,$$

where we define $U_pM := \{v \in T_pM \,|\, |v| = 1\}$. Hence u is continuous at p. $\square$

If $\sup_{U_pM}|du| = 0$ at some $p \in M$, then u is a constant map. So without loss of generality, we may assume that $\sup_{U_pM}|du| > 0$ for any $p \in M$. Define

$$
\begin{aligned}
a_u(p) &:= \sup_{U_pM}|du| \quad (= \text{the local dilatation of } u \text{ at } p),\\
r_u(p) &:= r_M(p) \wedge \frac{r_X(u(p))}{a_u(p)},\\
\sigma(p) &:= r_u(p) \wedge \operatorname{dist}(p, \partial M).
\end{aligned}
$$

The function r_u has a useful property:

$$
B_{r_u(p)}(p) \subset B_{r_M(p)}(p), \quad u\big(B_{r_u(p)}(p)\big) \subset B_{r_X(u(p))}(u(p)).
$$

Fix $p \in M$ and put $x = u(p)$. We assume $r_X(x) > 0$ in this and the next sections, so that $r_u(p), \sigma(p) > 0$. Then, by the property of $r_u(p)$ we noted above, the continuity of u immediately implies that the function $|du|$ is continuous on $TB_{r_u(p)}(p)$. Hence u is locally Lipschitz continuous, and a_u, r_u, and σ are continuous on $B_{r_u(p)}(p)$.

Definition 3.1.2 We define $V_p := \{v \in T_pM \mid |du|(v) = 0\}$ and call it the *vertical part* at p.

Lemma 3.1.3 *The set V_p is a subspace of T_pM.*

Proof. It suffices to show that $v + w \in V_p$ for any $v, w \in V_p$. We may assume $|v|, |w| \le 1$ and $v + w \neq 0$. For $\varepsilon \in (0, \sigma(p))$, we put

$$
q_\varepsilon := \gamma_{\gamma_v(\varepsilon)\gamma_w(\varepsilon)}(1/2), \quad v_\varepsilon := \exp_p^{-1} q_\varepsilon / |\exp_p^{-1} q_\varepsilon| \in U_pM.
$$

Since $B_\varepsilon(p) \subset B_{r_M(p)}(p)$ and $u(B_\varepsilon(p)) \subset B_{r_X(x)}(x)$, we have $u(q_\varepsilon) = \gamma_{u(\gamma_v(\varepsilon))u(\gamma_w(\varepsilon))}(1/2)$ $= x$. Hence $v_\varepsilon \in V_p$. On the other hand, $\lim_{\varepsilon\to 0} v_\varepsilon = (v+w)/|v+w|$. Therefore $v+w \in V_p$ since V_p is closed. $\square$

Lemma 3.1.4 *Each connected component of $u^{-1}(x)$ is a totally geodesic submanifold of M whose tangent space coincides with the vertical part at each point.*

Proof. For any $p' \in u^{-1}(x)$, we have

$$
B_{r_u(p')}(p') \cap u^{-1}(x) = \exp_{p'}\big(B_{r_u(p')}(0; T_{p'}M) \cap V_{p'}\big).
$$

Here we have $r_u(p') > 0$ since $r_X(u(p')) = r_X(x) > 0$. Hence the function $q \longmapsto \dim V_q$ is constant on any connected component S of $u^{-1}(x)$, and S is a totally geodesic submanifold of M which is coordinated by $\{(U_q, \varphi_q)\}_{q\in S}$, where we set $U_q := \exp_q\big(B_{\sigma(q)}(0; T_qM) \cap V_q\big)$ and $\varphi_q := (\exp_q|_{B_{\sigma(q)}(0;T_qM)\cap V_q})^{-1}$. $\square$

Definition 3.1.5 We define $H_p := V_p^\perp$ and call it the *horizontal part* at p.

For $x \in X$, we set

$$\Sigma'_x := \{\text{unit speed geodesics emanating from } x\}/\sim,$$

where the equivalence relation $\sim$ is defined such that $\gamma_1 \sim \gamma_2$ holds if $\gamma_1 = \gamma_2$ on $[0, \varepsilon]$ for some $\varepsilon > 0$. This set is slightly different from the space of directions Σ_x defined in §2.1. In fact, geodesics γ_1 and γ_2 emanating from the same point x can branch while satisfying $\angle_x(\gamma_1, \gamma_2) = 0$ in the case where (X, d_X) is an Alexandrov space of curvature bounded from above. We define $(du)_p : U_pM \setminus V_p \longrightarrow \Sigma'_x$ as

$$(du)_p(v) := \left[t \longmapsto u \circ \gamma_v\left(\frac{t}{|du|(v)}\right)\right],$$

where $[\cdot]$ indicates the equivalence class.

Lemma 3.1.6 *The map $(du)_p|_{U_pM \cap H_p} : U_pM \cap H_p \longrightarrow (du)_p(U_pM \setminus V_p)$ is bijective.*

Proof. We first show the surjectivity. Take any $v \in U_pM \setminus V_p$. For sufficiently small $\varepsilon \in (0, \sigma(p))$, we can choose two vectors $w \in V_p$ and $w' \in H_p$ such that $|w|, |w'| < \sigma(p)$ and $\gamma_v(\varepsilon) \in \gamma_w(1)\gamma_{w'}(1)$ because the set

$$\{q \in \gamma_w(1)\gamma_{w'}(1) \mid w \in V_p, w' \in H_p, |w| = |w'| = \varepsilon\}$$

is connected and of $(n-1)$-dimensional. Since $u(\gamma_v(0)) = u(\gamma_w(1)) = u(\gamma_{w'}(0)) = x$, we have $u(\gamma_v|_{[0,\varepsilon]}) \subset u(\gamma_w(1)\gamma_{w'}(1)) = u(\gamma_{w'}|_{[0,1]})$. Hence $(du)_p(w'/|w'|) = (du)_p(v)$.

Next, we show the injectivity. If there exist two distinct vectors $v, w \in U_pM \cap H_p$ satisfying $(du)_p(v) = (du)_p(w)$, then $u \circ \gamma_v = u \circ \gamma_w \circ a$ on $[0, \delta]$ for some $a > 0$ and $\delta > 0$ (where a is a multiplication function). We may assume $0 < a \le 1$. For $0 < \varepsilon < \delta \wedge \sigma(p)$, we have $u(\gamma_v(\varepsilon)\gamma_w(a\varepsilon)) = \{u(\gamma_v(\varepsilon))\} = \{u(\gamma_w(a\varepsilon))\}$. Hence

$$|du|\left(\frac{\exp^{-1}_{\gamma_w(a\varepsilon)}(\gamma_v(\varepsilon))}{|\exp^{-1}_{\gamma_w(a\varepsilon)}(\gamma_v(\varepsilon))|}\right) = 0.$$

We notice that $\gamma_v(\varepsilon) \neq \gamma_w(a\varepsilon)$ since $v \neq w$. Letting ε tend to zero, we obtain $|du|((v - aw)/|v - aw|) = 0$. Therefore $v - aw \in V_p \cap H_p$, so that $v - aw = 0$, which is a contradiction. $\square$

Corollary 3.1.7 *Any geodesic contained in $u(B_{r_u(p)}(p))$ does not branch. Namely, if two geodesics $\gamma_1, \gamma_2 : [-\delta, \delta] \longrightarrow u(B_{r_u(p)}(p))$ satisfy $\gamma_1 = \gamma_2$ on $[-\delta, 0]$, then $\gamma_1 = \gamma_2$ holds on $[-\delta, \delta]$.*

Proof. Assume that there exist two unit speed geodesics $\gamma_1, \gamma_2 : [-\delta, \delta] \longrightarrow u(B_{r_u(p)}(p))$ with $\delta > 0$ such that $\gamma_1 = \gamma_2$ on $[-\delta, 0]$ and $\gamma_1(t) \neq \gamma_2(t)$ for some $t \in (0, \delta]$. Then, since $u(B_{r_u(p)}(p)) \subset B_{r_X(x)}(x)$, we have $\gamma_1(s) \neq \gamma_2(s)$ for any $s \in (t, \delta]$. Hence, without loss of generality, we may assume $\gamma_1(t) \neq \gamma_2(t)$ for all $t \in (0, \delta]$.

Choose a point $p' \in u^{-1}(\gamma_1(0)) \cap B_{r_u(p)}(p)$ $(= u^{-1}(\gamma_2(0)) \cap B_{r_u(p)}(p))$. Note that

$$r_X(u(p')) \geq r_X(x) - d_X(x, u(p')) > r_X(x) - a_u(p) r_u(p) \geq 0.$$

By Lemma 3.1.6, there exist three vectors $w_1, w_2, v \in U_{p'}M \cap H_{p'}$ satisfying

$$(du)_{p'}(w_i) = [\gamma_i|_{[0,\delta]}] \quad \text{for } i = 1, 2,$$
$$(du)_{p'}(v) = [(\gamma_1 \circ (-1))|_{[0,\delta]}] \quad (= [(\gamma_2 \circ (-1))|_{[0,\delta]}]).$$

Since $\gamma_1(t) \neq \gamma_2(t)$ for any $t \in (0, \delta]$, we obtain $[\gamma_1|_{[0,\delta]}] \neq [\gamma_2|_{[0,\delta]}]$, so that $w_1 \neq w_2$. Hence we may assume $v \neq -w_1$. However, for sufficiently small $\varepsilon > 0$, we have

$$u(\gamma_{\gamma_v(\varepsilon/|du|(v))\gamma_{w_1}(\varepsilon/|du|(w_1))}(1/2)) = \gamma_{\gamma_1(-\varepsilon)\gamma_1(\varepsilon)}(1/2) = \gamma_1(0) = u(p').$$

Hence $\exp_{p'}^{-1}[\gamma_{\gamma_v(\varepsilon/|du|(v))\gamma_{w_1}(\varepsilon/|du|(w_1))}(1/2)]$ is contained in $V_{p'} \setminus \{0\}$, and we denote it by v_ε. Letting ε tend to zero, we have $v_\varepsilon/\varepsilon \to (v/|du|(v) + w_1/|du|(w_1))/2 \in V_{p'}$, which implies $v/|du|(v) + w_1/|du|(w_1) = 0$. This contradicts the hypothesis $v \neq -w_1$. $\square$

3.2 Local isometric splitting

This section is devoted to proving a local isometric splitting property of M. The precise statement of our result is as follows:

Theorem 3.2.1 *For any $p \in M$ with $r_X(u(p)) > 0$, there exists a natural isometric embedding*

$$B_{r_u(p)}(p) \hookrightarrow M_1 \times M_2.$$

Here we define

$$M_1 := B_{r_u(p)}(p)/\sim, \quad \textit{where } q \sim r \textit{ holds if } \exp_q^{-1} r \in H_q,$$
$$M_2 := B_{r_u(p)}(p)/\sim, \quad \textit{where } q \sim r \textit{ holds if } \exp_q^{-1} r \in V_q.$$

It is sufficient to show that the vertical parts, and hence also the horizontal parts, are invariant under the parallel translation along any geodesic whose initial vector belongs to the vertical or horizontal part. The invariance along a geodesic whose initial vector is contained in the vertical part is easily obtained by Lemma 3.1.4. Thus it suffices to show the invariance along the horizontal parts.

We first show the following:

Lemma 3.2.2 *If w is the projection to H_p of $w' \in T_pM$, then $u(\gamma_{w'}(t)) = u(\gamma_w(t))$ for any $t \in [-\sigma(p)/|w'|, \sigma(p)/|w'|]$.*

Proof. Take any $w' \in B_{\sigma(p)}(0; T_pM)$ and let $v \in V_p$ and $w \in H_p$ be the vertical and horizontal projections of w' respectively. For any $\varepsilon \in (0, 1)$, we put (see Figure 3)

$$q_\varepsilon := \gamma_{\gamma_v(\varepsilon)\gamma_w(\varepsilon)}(1/2), \quad w_\varepsilon := \frac{|w'|}{|\exp_p^{-1} q_\varepsilon|} \exp_p^{-1} q_\varepsilon.$$

By Corollary 3.1.7, we have

$$u(q_\varepsilon) = u\biggl(\gamma_w\Bigl(\frac{\varepsilon}{2}\Bigr)\biggr), \quad u(\exp_p w_\varepsilon) = u\biggl(\gamma_w\Bigl(\frac{|w'|}{|\exp_p^{-1} q_\varepsilon|}\frac{\varepsilon}{2}\Bigr)\biggr).$$

On the other hand, we find that $w_\varepsilon \to w'$ and $|\exp_p^{-1} q_\varepsilon|/\varepsilon \to |w'|/2$ as $\varepsilon \to 0$, so that $u(\gamma_{w'}(1)) = u(\gamma_w(1))$. Moreover, we have $u(\gamma_{w'}(t)) = u(\gamma_w(t))$ for any $t \in [-1, 1]$. This completes the proof. $\square$

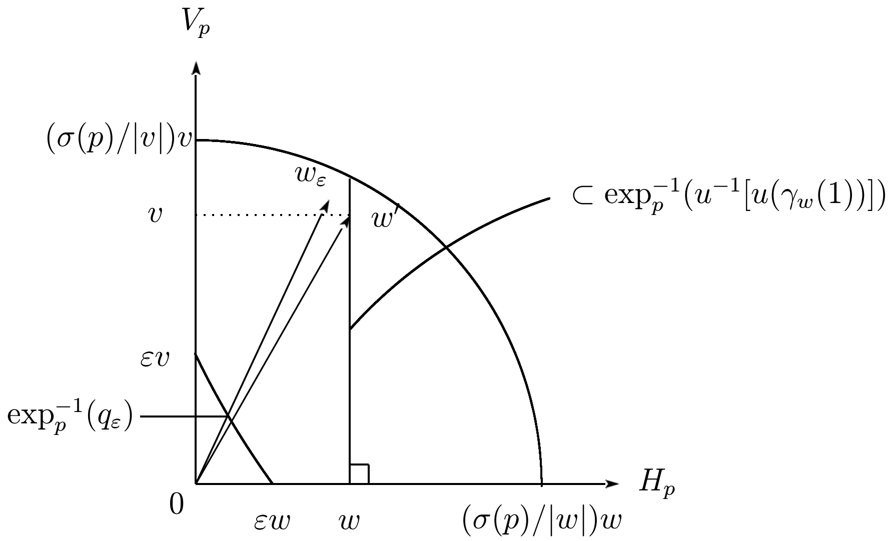

Figure 3

Corollary 3.2.3 *Let w', w be as in Lemma 3.2.2. Then*

(i) $(du)_p(w'/|w'|) = (du)_p(w/|w|)$ *if* $w \neq 0$,

(ii) $|du|(w') = |du|(w) \cos \angle(w', w)$.

In particular, for any $w \in U_pM \cap H_p$, the geodesic γ_w is perpendicular to $u^{-1}[u(\gamma_w(t))]$ for each $t \in (-\sigma(p), \sigma(p))$.

Take any $v \in U_pM \cap V_p$ and $w \in U_pM \cap H_p$. For each $s \in (0, \sigma(p))$, we put $q_s := \gamma_v(s)$. Since $u \circ \gamma_{q_s\gamma_w(\sigma(p)/2)}(t) = u \circ \gamma_w(\sigma(p)t/2)$ for any $t \in [0, 1]$, we have $(du)_p(w) \in (du)_{q_s}(U_{q_s}M \setminus V_{q_s})$. Hence, by Lemma 3.1.6 and its corollary, there exist a unique unit vector $w_s \in U_{q_s}M \cap H_{q_s}$ and a number $c_s > 0$ such that $u(\gamma_w(t)) = u(\gamma_{w_s}(c_s t))$ for any $t \in [0, t_s]$ (see Figure 4), where we put

$$t_s := \sup\{t > 0 \,|\, \gamma_w([0, t]) \cup \gamma_{w_s}([0, c_s t]) \subset B_{\sigma(p)}(p)\}.$$

By Lemma 3.1.4, we have $\gamma_w(t)\gamma_{w_s}(c_s t) \subset u^{-1}[u(\gamma_w(t))]$, so that this geodesic is perpendicular to both γ_w and γ_{w_s}. It follows from the first variation formula that

$$d(\gamma_w(t), \gamma_{w_s}(c_s t)) = s \quad \text{for } 0 \leq t \leq t_s.$$

Similarly, for any $0 < s_1 < s_2 < \sigma(p)$, we have

$$d(\gamma_{w_{s_1}}(c_{s_1}t), \gamma_{w_{s_2}}(c_{s_2}t)) = s_2 - s_1 \quad \text{for } 0 \le t \le t_{s_1} \wedge t_{s_2}.$$

It implies that

$$\begin{aligned} d(\gamma_w(t), \gamma_{w_{s_1}}(c_{s_1}t)) + d(\gamma_{w_{s_1}}(c_{s_1}t), \gamma_{w_{s_2}}(c_{s_2}t)) &= s_1 + (s_2 - s_1) = s_2 \\ &= d(\gamma_w(t), \gamma_{w_{s_2}}(c_{s_2}t)), \end{aligned}$$

so that $\gamma_{w_{s_1}}(c_{s_1}t) \in \gamma_w(t)\gamma_{w_{s_2}}(c_{s_2}t)$. Therefore, by the first variation formula, it holds that $c_s = 1$ for $s \in (0, \sigma(p))$. Consequently, we obtain $t_s \ge \sqrt{\sigma(p)^2 - s^2}$ and the surface $S := \{\gamma_{w_s}(t) \mid \sqrt{s^2 + t^2} < \sigma(p)\}$ is flat with respect to the metric induced from g.

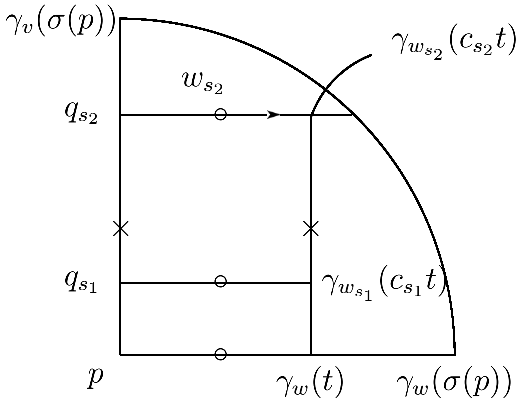

Figure 4

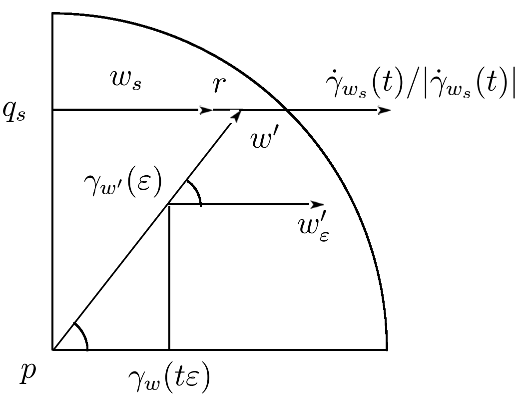

Figure 5

Furthermore, we shall prove that S is totally geodesic in M. Choose $r = \gamma_{w_s}(t) \in S$ and put $w' = \exp_p^{-1} r$ (see Figure 5). Since S is flat with respect to the induced metric, we obtain

$$|w'| \le \sqrt{s^2 + t^2}. \tag{3.1}$$

On the other hand, since $(du)_p(w'/|w'|) = (du)_{q_s}(w_s) = (du)_p(w)$ and $u(\gamma_{w'}(1)) = u(r) = u(\gamma_w(t))$, together with Lemma 3.2.2 and Lemma 3.1.6, we have $w' = v' + tw$ for some $v' \in V_p$. For each $0 < \varepsilon < 1$, let $w'_\varepsilon \in U_{\gamma_{w'}(\varepsilon)}M \cap H_{\gamma_{w'}(\varepsilon)}$ be the unique unit vector satisfying $(du)_{\gamma_{w'}(\varepsilon)}(w'_\varepsilon) = (du)_{\gamma_{w'}(\varepsilon)}(\dot\gamma_{w'}(\varepsilon)/|\dot\gamma_{w'}(\varepsilon)|) = (du)_{\gamma_w(t\varepsilon)}(\dot\gamma_w(t\varepsilon))$. Note that we do not know whether $\gamma_{w'}(\varepsilon) \in S$ or not. By Corollary 3.2.3(ii) and the discussion in the previous paragraph, we have

$$|du|(w'_\varepsilon)\cos\angle(\dot\gamma_{w'}(\varepsilon), w'_\varepsilon) = |du|(w'/|w'|) = |du|(w)\cos\angle(w', w)$$

and $|du|(w'_\varepsilon) = |du|(\dot\gamma_w(t\varepsilon)) = |du|(w)$. Hence we have $\angle(\dot\gamma_{w'}(\varepsilon), w'_\varepsilon) = \angle(w', w)$. Therefore we obtain

$$\angle(\dot\gamma_{w'}(\varepsilon), -\exp^{-1}_{\gamma_{w'}(\varepsilon)}\gamma_w(t\varepsilon)) \ge \frac{\pi}{2} - \angle(\dot\gamma_{w'}(\varepsilon), w_\varepsilon) = \frac{\pi}{2} - \angle(w', w)$$

for any $0 < \varepsilon < 1$. By the first variation formula, it holds that

$$\begin{aligned} s &= d(r, \gamma_w(t)) \\ &\leq \int_0^1 g\left(\dot{\gamma}_{w'}(\varepsilon), -\frac{\exp^{-1}_{\gamma_{w'}(\varepsilon)} \gamma_w(t\varepsilon)}{|\exp^{-1}_{\gamma_{w'}(\varepsilon)} \gamma_w(t\varepsilon)|}\right) d\varepsilon \\ &\leq |w'| \sin \angle(w', w) \\ &= \sqrt{|w'|^2 - t^2}. \end{aligned}$$

The last equality is easily derived from $g(w', w) = g(v' + tw, w) = t$. Thus we obtain $|w'| \geq \sqrt{s^2 + t^2}$. Combining this with (3.1), we have $|w'| = \sqrt{s^2 + t^2}$ and hence S is totally geodesic. Therefore S is flat also with respect to the original metric g. In particular, the vector field $\xi(t) := \exp^{-1}_{\gamma_w(t)}(\gamma_{w_s}(t))$ is parallel along γ_w on $(-\sqrt{\sigma(p)^2 - s^2}, \sqrt{\sigma(p)^2 - s^2})$. Consequently, we obtain that the vertical parts are invariant under the parallel translation along $\gamma_w|_{(-\sigma(p), \sigma(p))}$.

This argument is also true for any ball $B_\delta(p') \subset B_{r_u(p)}(p)$ with $0 < \delta \leq \sigma(p')$. Recall that $a_u(q) = \sup_{U_q M} |du|$, and define $b_u(q) := \inf_{U_q M \cap H_q} |du|$ (> 0) for $q \in M$. As a consequence, we obtain the following lemma and Theorem 3.2.1.

Lemma 3.2.4 (i) *$\dim V_q$ and $\dim H_q$ are independent of the choice of $q \in B_{r_u(p)}(p)$.*

(ii) *The functions a_u and b_u are constant on $u^{-1}(y) \cap B_{r_u(p)}(p)$ for each $y \in u\big(B_{r_u(p)}(p)\big)$.*

3.3 Differentiable structure

In the last two sections of the present chapter, we shall prove the following.

Theorem 3.3.1 *Let (M, g) be a geodesically complete Riemannian manifold and (X, d_X) be a metric space. Suppose that there exists a totally geodesic map $u : M \longrightarrow X$ and that the convexity radius at any point of $u(M)$ is positive. Then we have the following*:

(i) *There exist a C^∞-differentiable and C^0-Finslerian manifold $(N, |\cdot|)$ and maps $u_1 : M \longrightarrow N$ and $u_2 : N \longrightarrow X$ satisfying that*

  (a) $u = u_2 \circ u_1$,

  (b) *u_1 is C^∞ and totally geodesic (as a map between metric spaces),*

  (c) *u_2 is a locally isometric embedding.*

(ii) *If, in addition, (X, d_X) is an Alexandrov space with local curvature upper bound, then N is a C^∞-Riemannian manifold, and then u_1 is totally geodesic as a map between Riemannian manifolds.*

Namely, u_1 has all data of the "metric part" of u with no singularity, and u_2 has all data of the "singular part" (or "branching part") of u. The assumption, the positivity of the convexity radius at any point of $u(M)$, is not a strong one. In fact, any Alexandrov space with local curvature upper bound satisfies this condition.

If (M, g) is not geodesically complete, then, in general, we can not construct N satisfying the condition above. In fact, we see a counter-example in the following:

Example 3.3.2 Let $M := \mathbb{R}^2 \setminus \{(0, q) \,|\, q \geq 0\}$ with the standard metric, and X be a tripod with the infinite length. In other words, X is the union of three copies of the half-line $[0, \infty)$ identified at 0, and the distance is defined as the length distance. We denote the half-lines of X by l_1, l_2, and l_3 (see Figure 6).

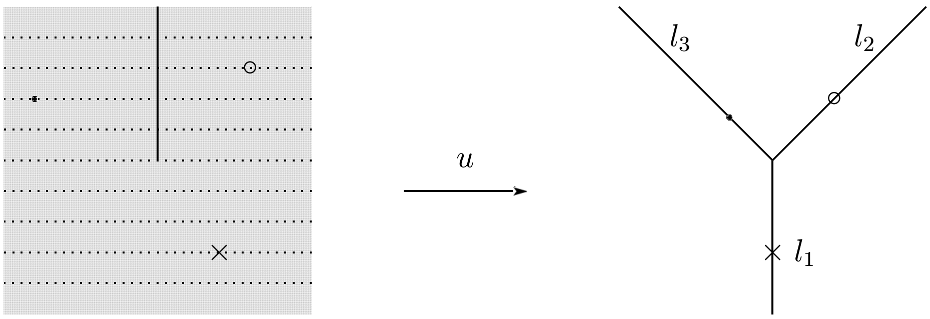

Figure 6

Define the map $u : M \longrightarrow X$ as follows:

- For $(p, 0) \in M$ with $p \neq 0$, $u(p, 0) := 0$.
- For $(p, q) \in M$ with $q < 0$, we define $u(p, q)$ as the point on l_1 whose distance from 0 is $|q|$.
- For $(p, q) \in M$ with $p > 0$ ($p < 0$ respectively) and $q > 0$, we define $u(p, q)$ as the point on l_2 (l_3 respectively) whose distance from 0 is $|q|$.

Clearly u is totally geodesic. If a Finslerian manifold N with the property stated in Theorem 3.3.1 exists, then, since N is of one-dimensional, it is a line segment or a circle. However, there exists no surjective, locally isometric embedding $u_2 : N \longrightarrow X$ for such N.

We assume that (M, g) is geodesically complete and that $r_X(x) > 0$ for any $x \in u(M)$ in this and the next sections. Hence we have $\sigma(p) = r_u(p)$. Then $|du|$ is continuous on TM, so that a_u, b_u, and r_u are continuous on M. Moreover, by Lemma 3.2.4, $\dim V_p$ and $\dim H_p$ are independent of $p \in M$, and a_u and b_u are constant on each connected component of $u^{-1}(x)$ for any $x \in u(M)$. So that we put $m = \dim H_p$.

Definition 3.3.3 (i) Define $N := M/\sim$, where $p \sim q$ holds if

$$\inf\{l(u \circ \gamma) \,|\, \gamma \text{ is a path from } p \text{ to } q\} = 0.$$

(ii) For any $P, Q \in N$, we define

$$d_N(P,Q) := \inf\{l(u \circ \gamma) \mid \gamma \text{ is a path from } p \text{ to } q\},$$

where $p \in P$ and $q \in Q$.

The function d_N is independent of the choice of $p \in P$ and $q \in Q$, and defines a distance function on N. By definition, we have

$$d_N(P,Q) \geq d_X(u(p), u(q)) \quad \text{for any } p \in P \text{ and } q \in Q. \tag{3.2}$$

Define $u_1 : M \longrightarrow N$ as the projection and $u_2 : N \longrightarrow X$ by $u_2(P) := u(p)$ for $p \in P$, which is well-defined by the inequality (3.2). Clearly $u = u_2 \circ u_1$.

From now on, for convenience, we put $P = u_1(p)$, $Q = u_1(q)$, and $R = u_1(r)$ respectively, and $x = u_2(P)$, $y = u_2(Q)$, and $z = u_2(R)$ respectively.

Lemma 3.3.4 *The map u_1 is totally geodesic. Moreover, we have*

$$d_N(P,Q) = |du|\left(\frac{\exp_p^{-1} q}{|\exp_p^{-1} q|}\right) d_M(p,q)$$

for any $q \in B_{r_u(p)}(p) \setminus \{p\}$.

Proof. For any $q \in B_{r_u(p)}(p) \setminus \{p\}$, we have

$$d_X(x,y) \leq d_N(P,Q) \leq l(u \circ \gamma_{pq}) = d_X(x,y).$$

Hence

$$d_N(P,Q) = d_X(x,y) = |du|\left(\frac{\exp_p^{-1} q}{|\exp_p^{-1} q|}\right) d_M(p,q),$$

and $u_1 \circ \gamma_{pq}$ is a minimal geodesic from P to Q. □

The following lemma and proposition are the keys to showing that N is a C^∞-differentiable manifold.

Lemma 3.3.5 *Let $\gamma : [0,1] \longrightarrow M$ be a path contained in $u^{-1}(x)$ for some $x \in u(M)$, and put $r_\gamma := \inf_{t\in[0,1]} r_u(\gamma(t)) > 0$. Then the set $B_{r_\gamma}(\gamma) \cap u^{-1}(y)$ is connected for each $y \in u(B_{r_\gamma}(\gamma))$.*

Proof. By splitting $B_{r_\gamma}(\gamma)$ along γ, there exists a linear isometry $F_t : H_{\gamma(0)} \longrightarrow H_{\gamma(t)}$ such that

$$u \circ \exp_{\gamma(t)} \circ F_t = u \circ \exp_{\gamma(0)} \text{ on } B_{r_\gamma}(0; T_{\gamma(0)}M) \cap H_{\gamma(0)}$$

for any $t \in [0,1]$, and that F_t is continuous in t. Fix a point $y \in u(B_{r_\gamma}(\gamma))$ and put

$$v := \exp_{\gamma(0)}^{-1}[\exp_{\gamma(0)}(B_{r_\gamma}(0) \cap H_{\gamma(0)}) \cap u^{-1}(y)] \in B_{r_\gamma}(0) \cap H_{\gamma(0)}.$$

For any $q_1 \in B_{r_\gamma}(\gamma(s)) \cap u^{-1}(y)$ and $q_2 \in B_{r_\gamma}(\gamma(t)) \cap u^{-1}(y)$ with $s \leq t$, we put

$$q_1' := \exp_{\gamma(s)}(F_s(v)), \quad q_2' := \exp_{\gamma(t)}(F_t(v)).$$

Then q_1' and q_2' are connected by the path $\gamma'(\tau) := \exp_{\gamma(\tau)} \circ F_\tau(v)$, $\tau \in [s,t]$, which is a translation of $\gamma|_{[s,t]}$ by F. Indeed, if $v = 0$, then we find $\gamma' = \gamma$. Therefore q_1 and q_2 are connected by the path which consists of $\gamma_{q_1 q_1'}$, γ', and $\gamma_{q_2' q_2}$. This path is clearly contained in $B_{r_\gamma}(\gamma) \cap u^{-1}(y)$, and hence $B_{r_\gamma}(\gamma) \cap u^{-1}(y)$ is connected. $\square$

We observe that γ' in the proof above can be defined by the assumption that (M,g) is geodesically complete.

Proposition 3.3.6 *For any two points $p, p' \in P \in N$, p and p' can be connected by a path contained in P. In particular, N coincides with the set of all arcwise connected components of $u^{-1}(x)$ for all $x \in u(M)$ as sets.*

Proof. Put $\varepsilon := r_u(p) \inf_{B_{r_u(p)}(p)} b_u > 0$. Since $p, p' \in P$, there exists a unit speed curve $\gamma : [0, l(\gamma)] \longrightarrow M$ satisfying $\gamma(0) = p, \gamma(l(\gamma)) = p'$, and $l(u \circ \gamma) < \varepsilon$. Set $q_t := u^{-1}[u(\gamma(t))] \cap \exp_p \left(B_{r_u(p)}(0; T_pM) \cap H_p\right)$ for $t \in [0, l(\gamma)]$. Note that this set consists of at most a single point. Let $I \subset [0, l(\gamma)]$ be the set of all t's which satisfies the following:

(*) $\gamma(t)$ and q_t can be connected by a path ξ_t which is contained in $u^{-1}[u(\gamma(t))]$ and satisfies $l(\xi_t) \leq t$.

It suffices to show $I = [0, l(\gamma)]$. By the local isometric splitting property for u around p, sufficiently small $t \geq 0$ belongs to I. Applying Lemma 3.3.5 to ξ_t for any $t \in I$, we obtain that $t + \delta \in I$ holds for any $\delta \in (0, \inf_{\xi_t} r_u) \cap (0, l(\gamma) - t]$. In fact, $q_{t+\delta}$ is not empty since, if it is, then we have

$$l(u \circ \gamma) \geq r_u(p) \inf_{B_{r_u(p)}(p)} b_u = \varepsilon,$$

which is a contradiction. We also note that the curve $\xi_{t+\delta}$ which is obtained from ξ_t as in the proof of Lemma 3.3.5 satisfies $l(\xi_{t+\delta}) \leq t + \delta$. Indeed, then we find $q_1 = q_1'$, $l(\gamma') = l(\gamma|_{[s,t]}) = l(\xi_t) \leq t$, and $d_M(q_2, q_2') \leq \delta$. Since the closure of $B_{r_u(p)+l(\gamma)}(p)$ is compact, we have $\inf_{\bigcup_{0<t<l(\gamma)} \xi_t} r_u > 0$. Thus we obtain $I = [0, l(\gamma)]$, which completes the proof. $\square$

By the proposition above, a_u and b_u are constant on each $P \in N$. We denote them by $a_u(P)$ and $b_u(P)$ respectively, and put $r_u(P) := \sup_P r_u \in (0, r_X(x)/a_u(P)]$.

Corollary 3.3.7 (i) *If $Q \in B_{b_u(P) r_u(p)}(P; N)$, then $B_{r_u(p)}(p; M) \cap Q \neq \emptyset$.*

(ii) *The map u_2 is a locally isometric embedding.*

(iii) *The convexity radius $r_N(P)$ is positive at any point $P \in N$. Moreover, we have $r_N(P) \geq b_u(P) r_u(P)$.*

Proof. (i) Take any $p' \in P$. By Proposition 3.3.6 and the local isometric splitting property for u, there exists a linear isometry $F : H_p \longrightarrow H_{p'}$ satisfying that

$$u_1 \circ \exp_{p'} \circ F = u_1 \circ \exp_p \text{ on } B_\varepsilon(0; T_pM) \cap H_p$$

for sufficiently small $\varepsilon > 0$. For any $v \in U_pM \cap H_p$, we put

$$I_v := \{t \in \mathbb{R} \mid u_1(\exp_{p'} \circ F(tv)) = u_1(\exp_p(tv))\}.$$

Clearly the set I_v is closed and not empty. On the other hand, by applying Proposition 3.3.6 to $\exp_{p'}(F(tv))$ and $\exp_p(tv)$ for $t \in I_v$, there exists a path joining them contained in $u^{-1}[u(\exp_p(tv))]$. Splitting the neighborhood of this path like in the proof of Lemma 3.3.5, we obtain that I_v is open. Hence $I_v = \mathbb{R}$, i.e., $u_1 \circ \exp_{p'} \circ F = u_1 \circ \exp_p$ holds on H_p. Therefore $B_{r_u(p)}(p; M) \cap Q = \emptyset$ implies $B_{r_u(p)}(P; M) \cap Q = \emptyset$, and then we find $d_N(P, Q) \geq b_u(P)r_u(p)$.

(ii) By (i), for any $Q, R \in B_{b_u(P)r_u(P)}(P; N)$, there exists a point $p \in P$ such that

$$B_{r_u(p)}(p; M) \cap Q \neq \emptyset \quad \text{and} \quad B_{r_u(p)}(p; M) \cap R \neq \emptyset.$$

Take $q \in B_{r_u(p)}(p; M) \cap Q$ and $r \in B_{r_u(p)}(p; M) \cap R$. Then

$$d_X(y, z) \leq d_N(Q, R) \leq l(u \circ \gamma_{qr}) = d_X(y, z).$$

Hence $u_2|_{B_{b_u(P)r_u(P)}(P;N)}$ preserves the distance.

(iii) Since

$$u_2\big(B_{b_u(P)r_u(P)}(P; N)\big) \subset B_{b_u(P)r_u(P)}(x; X) \subset B_{r_X(x)}(x; X),$$

for any geodesic γ contained in $B_{b_u(P)r_u(P)}(P; N)$, $u_2 \circ \gamma$ is minimal. Combining this with the isometric property of $u_2|_{B_{b_u(P)r_u(P)}(P;N)}$, we obtain that γ is minimal. We next show the strong convexity of the ball $B_\delta(S; N)$ with $d_N(P, S) \leq b_u(P)r_u(P) - \delta$. For any $Q, R \in B_\delta(S; N)$, by virtue of the proof of (ii), a minimal geodesic γ from Q to R exists. Then $l(u_2 \circ \gamma) = l(\gamma) = d_N(Q, R) = d_X(y, z)$, i.e., $u_2 \circ \gamma$ is a minimal geodesic from y to z. On the other hand, since $y, z \in u_2(B_\delta(S; N)) \subset B_\delta(u_2(S); X)$ and

$$d_X(x, u_2(S)) = d_N(P, S) \leq b_u(P)r_u(P) - \delta \leq r_X(x) - \delta,$$

we obtain that $u_2 \circ \gamma$ is a unique minimal geodesic from y to z, and is contained in $B_\delta(u_2(S); X)$. Therefore γ is a unique minimal geodesic from Q to R, and is contained in $B_\delta(S; N)$ since $B_\delta(S; N)$ is a connected component of $u_2^{-1}[B_\delta(u_2(S); X)]$. Consequently, $B_\delta(S; N)$ is strongly convex, and hence $r_N(P) \geq b_u(P)r_u(P)$. □

By Lemma 3.3.4 and Corollary 3.3.7(iii), we can apply all the discussions for u also to u_1. In particular, we have the following:

- u_1 and $|du_1|$ are continuous, $|du_1| = |du|, a_{u_1} = a_u$, and $b_{u_1} = b_u$.
- H_p and V_p with respect to u_1 coincide with H_p and V_p with respect to u respectively.
- $(du_1)_p|_{U_pM \cap H_p} : U_pM \cap H_p \longrightarrow \Sigma'_P$ is injective (see Lemma 3.1.6).
- Any geodesic in N does not branch (see Corollary 3.1.7).

Corollary 3.3.8 (i) *The map* $(du_1)_p|_{U_pM\cap H_p} : U_pM \cap H_p \longrightarrow \Sigma'_P$ *is surjective* (*and hence bijective*)*. Moreover,* u_1 *is an open map.*

(ii) *The map* $u_1 \circ \exp_p : B_{\rho(p)}(0; T_pM) \cap H_p \longrightarrow u_1 \circ \exp_p \big(B_{\rho(p)}(0; T_pM) \cap H_p\big)$ *is homeomorphic, where* $\rho(p) := b_u(p)r_u(p)/a_u(p)$.

Proof. (i) For any $Q \in B_{b_u(P)r_u(P)}(P; N)$, by Corollary 3.3.7(i), there exists a point $p \in P$ satisfying

$$B_{r_u(p)}(p) \cap Q \cap \exp_p \big(B_{r_u(p)}(0) \cap H_p\big) \neq \emptyset.$$

Take a point q in this set. Then $\gamma_{PQ} = u_1 \circ \gamma_{pq} \in (du_1)_p(U_pM \cap H_p)$. Hence $(du_1)_p|_{U_pM\cap H_p}$ is surjective. Moreover, combining this with the continuity of $|du_1|$ and $\inf_{U_pM\cap H_p} |du_1| = b_u(p) > 0$, we obtain that u_1 is open.

(ii) The bijectivity is clear by (i), $\rho(p) \leq r_M(p)$,

$$u_1 \circ \exp_p \big(B_{\rho(p)}(0) \cap H_p\big) \subset B_{b_u(p)r_u(p)}(P; N) \subset B_{r_N(P)}(P; N),$$

and by the fact that any geodesic in N does not branch. The continuity of the inverse map is clear since u_1 is open. $\square$

For each $p \in M$, we set

$$U_p := u_1 \circ \exp_p \big(B_{\rho(p)}(0; T_pM) \cap H_p\big) \subset N,$$
$$\varphi_p := (u_1 \circ \exp_p |_{B_{\rho(p)}(0;T_pM)\cap H_p})^{-1} : U_p \longrightarrow B_{\rho(p)}(0; \mathbb{R}^m).$$

Then $\{(U_p, \varphi_p)\}_{p\in M}$ gives a C^∞-differentiable structure on N. Indeed, if $U_p \cap U_q \neq \emptyset$, then Proposition 3.3.6 shows that, for any $R \in U_p \cap U_q$, the points $r_1 := \exp_p \circ\varphi_p(R)$ and $r_2 := \exp_q \circ\varphi_q(R)$ can be connected by a path γ contained in $(u_1)^{-1}(R)$. By splitting the neighborhood of γ along γ, the map $\varphi_q \circ \varphi_p^{-1}$ is C^∞ in a neighborhood of $\varphi_p(R)$. Therefore $\varphi_q \circ \varphi_p^{-1} : \varphi_p(U_p \cap U_q) \longrightarrow \varphi_q(U_p \cap U_q)$ is C^∞. Clearly u_1 is C^∞ with respect to this structure.

Remark 3.3.9 By Theorem 3.2.1, the distribution V obtained from $\{V_p\}_{p\in M}$ is regular. Hence the set of all leaves of V, say M/V, with the quotient topology is a differentiable manifold ([P, Theorem VIII]). However, it is not a Hausdorff space in general. On the other hand, N is always Hausdorff but does not always coincide with M/V (see Example 3.3.2). Proposition 3.3.6 implies $M/V = N$ as sets, and the continuity and the open property of u_1 together imply that they are homeomorphic.

3.4 Finslerian and Riemannian structures

Next, we define a Finslerian structure on N. For any vector $\xi \in T_PN$ $(P \in N)$, we define

$$|\xi| := |du_1|(v) = |du|(v),$$

where $v \in H_p$ such that $(u_1)_*(v) = \xi$ and $(u_1)_*$ is the differential of u_1 (such v exists uniquely for each $p \in P$). Namely, $|\xi|$ is the speed of the geodesic of direction ξ. This is independent of the choice of $p \in P$ by virtue of the proof of Corollary 3.3.7(i).

Proposition 3.4.1 (i) $|\cdot|$ *satisfies the triangle inequality on each* T_PN.

(ii) *If* (X, d_X) *is an Alexandrov space with local curvature upper bound, then* $|\cdot|$ *satisfies the parallelogram identity on each* T_PN.

Proof. (i) By the continuity of $|du_1|$, we obtain that $\lim_{\varepsilon\to 0}(B_\varepsilon(P;N), d_N/\varepsilon)$ is isometric to $(B_1(0;T_PN), |\cdot|)$. Therefore the triangle inequality for $|\cdot|$ is shown by that for d_N.

(ii) If (X, d_X) is an Alexandrov space with local curvature upper bound, then so is (N, d_N) since u_2 is a locally isometric embedding. Combining this with the isometric property between $(B_1(0;T_PN), |\cdot|)$ and $\lim_{\varepsilon\to 0}(B_\varepsilon(P;N), d_N/\varepsilon)$, we prove that $(T_PN, |\cdot|)$ is a CAT(0)-space. On the other hand, since $(T_PN, |\cdot|)$ is a Banach space, it is a Hilbert space. $\square$

By this proposition, $(N, |\cdot|)$ is a Finslerian manifold. Moreover, if (X, d_X) is an Alexandrov space with local curvature upper bound, then (N, h) is a Riemannian manifold, where we set

$$h(\xi, \eta) := \frac{1}{2}\{|\xi+\eta|^2 - |\xi|^2 - |\eta|^2\} \quad \text{for } \xi, \eta \in T_PN.$$

Clearly these Finslerian and Riemannian structures are C^0 and compatible with d_N. Thus we have Theorem 3.3.1 except the smoothness of the Riemannian structure. There exists an example of u for which N is not a Riemannian but a Finslerian manifold.

Example 3.4.2 Fix a norm $\|\cdot\|$ on $\mathbb{R}^n$ and let $\|\cdot\|_{\mathrm{eucl}}$ be the standard norm on $\mathbb{R}^n$. Since any line segment in $(\mathbb{R}^n, \|\cdot\|)$ is a (not necessarily unique) minimal path, the identity map $u : (\mathbb{R}^n, \|\cdot\|_{\mathrm{eucl}}) \longrightarrow (\mathbb{R}^n, \|\cdot\|)$ is totally geodesic. Then (N, d_N) coincides with $(\mathbb{R}^n, \|\cdot\|)$ and, needless to say, there exist many norms not satisfying the parallelogram identity.

By the example above, since there exist many norms on $\mathbb{R}^n$ which are not C^1, we can also say that C^0 is sharp in the Finslerian case even if u is a diffeomorphism. However, we shall prove that h is C^∞ in the Riemannian case. Before beginning the proof, we define a map which will be proved to coincide with the exponential map. We define $e : TN \longrightarrow N$ as, for $\xi \in T_PN$,

$$e(\xi) := u_1 \circ \exp_p \circ ((u_1)_*|_{H_p})^{-1}(\xi),$$

where $p \in P$. The map e is C^∞ and independent of the choice of $p \in P$ by virtue of the proof of Corollary 3.3.7(i).

Proposition 3.4.3 *Assume that* (X, d_X) *is an Alexandrov space with local curvature upper bound. Then the Riemannian manifold* (N, h) *satisfies the following:*

(i) *Two definitions of a geodesic in* N *(i.e., as a curve in a metric space and as a curve in a Riemannian manifold) coincide, equivalently,* $e = \exp$ *holds.*

(ii) *The map* u_1 *is totally geodesic as a map between Riemannian manifolds.*

(iii) *The metric* h *is* C^∞.

Proof. (i) We first show that h is at least C^1. It is sufficient to show this locally, so that we consider sufficiently small neighborhoods U of $P \in N$ and D of $0 \in T_P N$ respectively instead of N and TN. It follows from $d_N(Q,R) = |(e|_{T_Q N})^{-1}(R)|$ and the smoothness of e that the function $R \longmapsto d_N(Q,R)^2$ is C^∞ on U for an arbitrarily fixed $Q \in U$. This implies that d_N^2 is C^1 on $U \times U$, and hence the function $Q \longmapsto d_N(Q, f(Q))^2$ is C^1 for any C^1-map $f : U \longrightarrow U$. For a C^1-vector field $V \subset D$, by putting $f(Q) := e(V(Q))$, we find that $h(V,V)$ is C^1. Thus h is C^1. In particular, the Christoffel symbols and the exponential map of (N,h) can be defined and C^0.

Let γ be any geodesic in N as a curve in a metric space. Then, since γ is locally the image of a geodesic in M, γ is C^∞ and $\nabla_t \dot\gamma$ is a C^0-vector field along γ. For small $\varepsilon > 0$, let V_ε be a C^∞-vector field along γ with $\sup |V_\varepsilon - \nabla_t \dot\gamma| < \varepsilon$. Define $f(t,s) := e(sV_\varepsilon(t))$. Then f is C^∞ and

$$\frac{\partial f}{\partial s}(t,0) = (u_1)_* \circ ((u_1)_*|_{H_p})^{-1}(V_\varepsilon(t)) = V_\varepsilon(t) \quad \text{for any } t.$$

By the first variation formula for energy and the minimality of γ, we have

$$\begin{aligned}
0 &= \frac{\partial}{\partial s}\Big|_{s=0} \left(\frac{1}{2} \int \left| \frac{\partial f}{\partial t}(t,s) \right|^2 dt \right) \\
&= -\int h(V_\varepsilon(t), \nabla_t \dot\gamma(t))\, dt \\
&= -\int \{ |\nabla_t \dot\gamma(t)|^2 + h(V_\varepsilon(t) - \nabla_t \dot\gamma(t), \nabla_t \dot\gamma(t)) \}\, dt.
\end{aligned}$$

Hence

$$\begin{aligned}
\int |\nabla_t \dot\gamma(t)|^2 dt &= \int h(\nabla_t \dot\gamma(t) - V_\varepsilon(t), \nabla_t \dot\gamma(t))\, dt \\
&\le \int |\nabla_t \dot\gamma(t) - V_\varepsilon(t)|\, |\nabla_t \dot\gamma(t)|\, dt \\
&\to 0 \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

Thus $\nabla_t \dot\gamma \equiv 0$, and hence γ is a geodesic as a curve in a Riemannian manifold. The converse is clear by the definition of a geodesic in a metric space.

(ii) Easily obtained from (i).

(iii) By (ii), ${}'\nabla_{W_1}(u_1)_*(W_2) = (u_1)_*(\nabla_{W_1} W_2)$ for any C^∞-vector fields W_1 and W_2 on N. In particular, the Christoffel symbols of (N,h) are C^∞, and hence h is C^∞. $\square$

Thus we complete the proof of Theorem 3.3.1. By applying this theorem locally, we obtain two corollaries below. We notice that, since the geodesically completeness of (M,g) is necessary only for Lemma 3.3.5 and Proposition 3.3.6, Theorem 3.3.1 is applicable to $u|_{B_{\sigma(p)}(p)}$. The following local version of Theorem 3.3.1 will be used in the final chapter of this thesis as a key tool.

Corollary 3.4.4 *Let (M,g) be a Riemannian manifold, (X,d_X) be an Alexandrov space with local curvature upper bound, and $u : M \longrightarrow X$ be a totally geodesic map. Then,* *for any $p \in M$, the set $u(B_{\rho(p)}(p))$ has a C^∞-Riemannian structure compatible with d_X. Moreover, $u|_{B_{\rho(p)}(p)}$ is C^∞ and totally geodesic with respect to this structure.*

Corollary 3.4.5 *Let (M, g), (X, d_X), and u be as in Corollary* 3.4.4. *If (M, g) is irreducible, then u is a locally homothetic embedding as a map between metric spaces or a constant map. If, in addition, u is injective, then $u(M)$ is a C^∞-Riemannian manifold and u is homothetic as a map between Riemannian manifolds.*

Proof. Fix a parallel vector field ζ along a unit speed curve $c : [0, l] \longrightarrow M$, and put $p = c(0)$. By applying Theorem 3.3.1(ii) to $u|_{B_{\sigma(p)}(p)}$, the map u_1 derived from $u|_{B_{\sigma(p)}(p)}$ is totally geodesic as a map between Riemannian manifolds, so that $((u_1)_*\zeta)|_{[0,\sigma(p)]}$ is a parallel vector field. In particular, $|(u_1)_*(\zeta(t))| = |(u_1)_*(\zeta(0))|$ for any $t \in [0, \sigma(p)]$. Hence $|du|(\zeta(t)) = |du|(\zeta(0))$ for $t \in [0, \sigma(p)]$. Consequently, we have $|du|(\zeta(0)) = |du|(\zeta(l))$. Since (M, g) is irreducible, $|du|$ is a constant function on UM. This is equivalent to the local homothetic property of u. The second part is clear since $N = u(M)$ if u is injective. □

This is a generalization of a well-known fact (Fact 1.0.2) for a totally geodesic map between Riemannian manifolds, which is needed in the proof of Margulis' superrigidity theorem by using a harmonic map. See (III) in Chapter 1.

Remark 3.4.6 Note that the local homothetic property as a map between metric spaces is equivalent to the homothetic property as a map between Riemannian (or Finslerian) manifolds.

Chapter 4

Sobolev spaces

The theory of Sobolev spaces for maps from or into metric spaces is making remarkable progress in these years. There are several definitions of such kind of Sobolev spaces. Among them, we study Cheeger's definition [Ch] most deeply. Although he treats only a Sobolev space for functions, we can naturally generalize his definition to that for maps into an arbitrary metric space. In the first four sections, we treat several topics, including Dirichlet problem, of this type of Sobolev space. In the last section, we recall some other definitions of Sobolev spaces and, by studying the relation between them, will prove some results on what happens to a Sobolev space if we embed its target space into a Banach space isometrically. This chapter is based on [O2].

4.1 Upper gradients

Throughout this chapter, without otherwise indicated, let (Z, d_Z) and (X, d_X) be metric spaces, $U \subset Z$ be an open set, and μ be a Borel regular measure on Z such that any ball with finite positive radius is of finite positive measure. We define $U_\varepsilon := \{z \in U \mid \mathrm{dist}(z, \partial U) \geq \varepsilon\}$ for $\varepsilon > 0$.

Definition 4.1.1 A Borel measurable function $g : U \longrightarrow [0, \infty]$ is called an *upper gradient* for a map $u : U \longrightarrow X$ if, for any unit speed curve $c : [0, l] \longrightarrow U$, we have

$$d_X\big(u(c(0)), u(c(l))\big) \leq \int_0^l g(c(s))\, ds.$$

Remark 4.1.2 (i) The function $g \equiv \infty$ is an upper gradient for any u.

(ii) If (Z, d_Z) and (X, d_X) are Riemannian manifolds and if u is smooth, then the function

$$z \longmapsto \text{ the operator norm of the differential } (u_*)_z : T_z Z \to T_{u(z)} X$$

on U is an upper gradient for u.

(iii) If $\int_0^l g(c(s))\, ds < \infty$, then $u \circ c$ is uniformly continuous on $[0, l]$.

Lemma 4.1.3 (cf. [Ch, Proposition 1.6]) *Let U_1 and U_2 be open sets in Z, $u : U_1 \cup U_2 \longrightarrow X$ be a map, and $g_i : U_i \longrightarrow [0, \infty]$ be an upper gradient for $u|_{U_i}$ for $i = 1, 2$. Then $g_1 \vee g_2$ is an upper gradient for u, where each g_i is extended to $U_1 \cup U_2$ by 0.*

Proof. Fix a unit speed curve $c : [0, l] \longrightarrow U_1 \cup U_2$. Since U_1 and U_2 are open and $[0, l]$ is compact, there exists a sequence $\{l_j\}_{j=0}^N \subset [0, l]$ $(1 \leq N < \infty)$ such that $l_0 = 0$, $l_N = l$ and, for any $0 \leq j < N$, there exists $i_j \in \{1, 2\}$ satisfying $c([l_j, l_{j+1}]) \subset U_{i_j}$. Therefore we obtain

$$\begin{aligned} d_X\big(u(c(0)), u(c(l))\big) &\leq \sum_{j=0}^{N-1} d_X\big(u(c(l_j)), u(c(l_{j+1}))\big) \leq \sum_{j=0}^{N-1} \int_{l_j}^{l_{j+1}} g_{i_j}(c(s))\, ds \\ &\leq \int_0^l (g_1 \vee g_2)(c(s))\, ds. \end{aligned}$$

$\square$

For a continuous map $u : U \longrightarrow X$ and a point $z \in U$, we define

$$\operatorname{Lip} u(z) := \lim_{r \to 0} \sup_{0 < d_Z(z,w) < r} \frac{d_X(u(z), u(w))}{d_Z(z, w)},$$

and we put $\operatorname{Lip} u(z) = 0$ if z is an isolated point. If u is Lipschitz continuous, then this function is not greater than the Lipschitz constant of u.

Lemma 4.1.4 *For any continuous map $u : U \longrightarrow X$, the function $\operatorname{Lip} u$ is Borel measurable.*

Proof. For fixed $r > 0$, the continuity of u implies that the function

$$z \longmapsto \sup_{0 < d_Z(z,w) < r} \frac{d_X(u(z), u(w))}{d_Z(z, w)}$$

is lower semi-continuous. Moreover, this function is monotone non-increasing as r tends to 0 for fixed $z \in U$. Thus the limit function $\operatorname{Lip} u$ is Borel measurable. $\square$

By using Rademacher's theorem, Cheeger proved the following.

Proposition 4.1.5 ([Ch, Proposition 1.11]) *If a function $f : U \longrightarrow \mathbb{R}$ is locally Lipschitz continuous, then $\operatorname{Lip} f$ is an upper gradient for f.*

Proof. Fix a unit speed curve $c : [0, l] \longrightarrow U$. Since $f \circ c$ is Lipschitz continuous, $f \circ c$ is differentiable a.e. on $[0, l]$ by Rademacher's theorem, and it satisfies

$$\big|f(c(0)) - f(c(l))\big| \leq \int_0^l |(f \circ c)'(s)|\, ds.$$

Thus it suffices to show that $|(f \circ c)'(s)| \leq (\operatorname{Lip} f)(c(s))$ holds for any $s \in (0, l)$ for which $(f \circ c)'(s)$ exists. At such s, since c has the unit speed, we have

$$\begin{aligned}|(f \circ c)'(s)| &= \lim_{\delta \to 0} \frac{\big|f(c(s+\delta)) - f(c(s))\big|}{\delta} \leq \liminf_{\delta \to 0} \frac{\big|f(c(s+\delta)) - f(c(s))\big|}{d_Z(c(s+\delta), c(s))} \\ &\leq (\operatorname{Lip} f)(c(s)).\end{aligned}$$

This completes the proof. $\square$

We next define the Cheeger-type Sobolev spaces. Take any point $x_0 \in X$ and fix it as a base point. In the remainder of this section, let $1 \leq p < \infty$. For two measurable maps $u, v : U \longrightarrow X$, we define $d_{L^p}(u, v) := \big(\int_U d_X(u, v)^p \, d\mu\big)^{1/p}$. We also define

$$L^p(U; X) := \{u : U \longrightarrow X \,|\, \text{measurable}, d_{L^p}(u, x_0) < \infty\}/\sim,$$

where x_0 denotes the constant map to x_0, and $u_1 \sim u_2$ holds if $u_1 = u_2$ a.e. on U. The function d_{L^p} defines a distance on $L^p(U; X)$. We remark that, if $\mu(U) < \infty$, then the set $L^p(U; X)$ is independent of the choice of the base point x_0.

Definition 4.1.6 For $u \in L^p(U; X)$, we define the *Cheeger-type p-energy* of u as

$$E_p^C(u) := \inf_{\{(u_i, g_i)\}_{i=1}^\infty} \liminf_{i \to \infty} |g_i|^p_{L^p(U)},$$

where the infimum is taken over all sequences $\{(u_i, g_i)\}_{i=1}^\infty$ such that $u_i \to u$ in $L^p(U; X)$ as $i \to \infty$ and g_i is an upper gradient for u_i for each i. We next define the *Cheeger-type $(1,p)$-Sobolev space* by

$$H^{1,p}(U; X) := \{u \in L^p(U; X) \,|\, E_p^C(u) < \infty\}.$$

Since we consider only the Cheeger-type energy in this and following three sections, we write this energy $E_p(u)$ for simplicity. By definition, if $u = v$ a.e. on U, then we have $E_p(u) = E_p(v)$. Hence $H^{1,p}(U; X)$ is naturally embedded in $L^p(U; X)$. We also note that the p-energy $E_p(u)$ is independent of the choice of the base point x_0.

Remark 4.1.7 If (Z, g) and (X, h) are Riemannian manifolds and $u : U \longrightarrow X$ is smooth, then the energy $E_p(u)$ does not necessarily coincide with the usually defined one (see also Chapter 5 of this thesis). This is caused by the difference between the operator norm of $(u_*)_z$ and the Hilbert-Schmidt norm $|u_*|(z)$, which is defined by

$$|u_*|(z)^2 := \sum_{i,j=1}^n \sum_{\alpha,\beta=1}^m g^{ij}(z) h_{\alpha\beta}(u(z)) \frac{\partial(y^\alpha \circ u)}{\partial x^i}(z) \frac{\partial(y^\beta \circ u)}{\partial x^j}(z),$$

where $n = \dim Z$, $m = \dim X$, $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^m)$ are local coordinate systems on some neighborhoods of z and $u(z)$ respectively,

$$g_{ij} := g\bigg(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\bigg), \quad h_{\alpha\beta} := h\bigg(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}\bigg),$$

and (g^{ij}) denotes the inverse matrix of (g_{ij}).

Theorem 4.1.8 (cf. [Ch, Theorem 2.5]) *If a sequence $\{u_i\}_{i=1}^{\infty}$ converges to u in $L^p(U;X)$, then we have $E_p(u) \le \liminf_{i\to\infty} E_p(u_i)$.*

Proof. Easily derived from the diagonal processes. □

We end this section with two more definitions.

Definition 4.1.9 A function $g \in L^p(U)$ is called a *generalized upper gradient* for $u \in H^{1,p}(U;X)$ if there exists a sequence $\{(u_i, g_i)\}_{i=1}^{\infty}$ such that g_i is an upper gradient for u_i, and $u_i \to u$ in $L^p(U;X)$ and $g_i \to g$ in $L^p(U)$ respectively as $i \to \infty$.

By the definition of the p-energy, it holds that $|g|_{L^p}^p \ge E_p(u)$ for any generalized upper gradient g for u.

Definition 4.1.10 A generalized upper gradient $g \in L^p(U)$ for a map $u \in H^{1,p}(U;X)$ is said to be *minimal* if it satisfies $|g|_{L^p}^p = E_p(u)$.

4.2 Minimal generalized upper gradients

In this section, let (X, d_X) be a geodesic length space such that d_X is *convex*, that is, any two points in X can be connected by a minimal geodesic and, for any two minimal geodesics $\gamma_1, \gamma_2 : [0,1] \longrightarrow X$, the function $t \longmapsto d_X(\gamma_1(t), \gamma_2(t))$ is convex. In particular, every minimal geodesic is unique. Note that all CAT(0)-spaces satisfy this property. On the other hand, there are some metric spaces which is not a CAT(0)-space, but whose distance function is convex, e.g., $L^p(U)$ with $p \in (1,\infty) \setminus \{2\}$. By definition, for any $x, y, x', y' \in X$ and $0 \le \lambda \le 1$, it holds that

$$d_X\big((1-\lambda)x + \lambda y, (1-\lambda)x' + \lambda y'\big) \le (1-\lambda) d_X(x, x') + \lambda d_X(y, y'). \tag{4.1}$$

Lemma 4.2.1 *Let $u_1, u_2 : U \longrightarrow X$ be maps. For any upper gradients g_1 for u_1 and g_2 for u_2, and for $0 \le \lambda \le 1$, the function $g := (1-\lambda)g_1 + \lambda g_2$ is an upper gradient for the map $v := (1-\lambda)u_1 + \lambda u_2$. In particular, for any $u_1, u_2 \in H^{1,p}(U;X)$ with $1 \le p < \infty$ and for any $0 \le \lambda \le 1$, we have*

$$E_p\big((1-\lambda)u_1 + \lambda u_2\big)^{\frac{1}{p}} \le (1-\lambda) E_p(u_1)^{\frac{1}{p}} + \lambda E_p(u_2)^{\frac{1}{p}}.$$

Namely, ${E_p}^{1/p}$ (and hence also E_p) is a convex function on $H^{1,p}(U;X)$.

Proof. Fix a unit speed curve $c : [0,l] \longrightarrow U$. By (4.1), we obtain

$$\begin{aligned} &d_X\big(v(c(0)), v(c(l))\big) \\ &\quad \le (1-\lambda) d_X\big(u_1(c(0)), u_1(c(l))\big) + \lambda d_X\big(u_2(c(0)), u_2(c(l))\big) \\ &\quad \le \int_0^l g(c(s))\, ds. \end{aligned}$$

Hence g is an upper gradient for v. The second part is clear since, by (4.1), if $u_{1,i} \to u_1$ and $u_{2,i} \to u_2$ in $L^p(U;X)$ respectively as $i \to \infty$, then $(1-\lambda)u_{1,i} + \lambda u_{2,i} \to v$ in $L^p(U;X)$ as $i \to \infty$. □

Theorem 4.2.2 (cf. [Ch, Theorem 2.10]) *Let $1 < p < \infty$. Then, for any $u \in H^{1,p}(U;X)$, there exists a unique minimal generalized upper gradient g_u for u.*

Proof. Take a minimizing sequence $\{(u_i, g_i)\}_{i=1}^{\infty}$, i.e., it satisfies that $u_i \to u$ in $L^p(U;X)$ as $i \to \infty$, g_i is an upper gradient for u_i, and that $\lim_{i\to\infty} |g_i|_{L^p}^p = E_p(u)$. We shall show that $\{g_i\}$ is a Cauchy sequence in $L^p(U)$. If $\{g_i\}$ is not a Cauchy sequence, then $E_p(u) > 0$ and there exist $\varepsilon > 0$ and sequences $\{i_n\}_{n=1}^{\infty}$ and $\{j_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} i_n = \lim_{n\to\infty} j_n = \infty$ and $|g_{i_n} - g_{j_n}|_{L^p} > \varepsilon$ for any n. By the uniform convexity of $L^p(U)$, it implies

$$\lim_{n\to\infty} \left| \frac{1}{2}(g_{i_n} + g_{j_n}) \right|_{L^p} < \lim_{n\to\infty} \frac{1}{2}(|g_{i_n}|_{L^p} + |g_{j_n}|_{L^p}) = E_p(u)^{\frac{1}{p}}.$$

On the other hand, since $(1/2)u_{i_n} + (1/2)u_{j_n} \to u$ in $L^p(U;X)$, we have

$$E_p(u) \le \liminf_{n\to\infty} E_p\left(\frac{1}{2}u_{i_n} + \frac{1}{2}u_{j_n}\right) \le \liminf_{n\to\infty} \left|\frac{1}{2}(g_{i_n} + g_{j_n})\right|_{L^p}^p$$

by Theorem 4.1.8 and Lemma 4.2.1. This is a contradiction. Hence $\{g_i\}$ is a Cauchy sequence, so that it converges to some minimal generalized upper gradient $g \in L^p(U)$ for u. The uniqueness also follows from the uniform convexity of $L^p(U)$. $\square$

Lemma 4.2.3 (cf. [Ch, Lemma 1.7]) *Let $u_1, u_2 : U \longrightarrow X$ be maps and $\phi : U \longrightarrow \mathbb{R}$ be a function. For any upper gradients g_1, g_2, and g_3 for u_1, u_2, and ϕ respectively and for any $\varepsilon > 0$, the function*

$$g := g_3 \cdot (d_X(u_1, u_2) + \varepsilon) + (1 - \phi + \varepsilon)g_1 + (\phi + \varepsilon)g_2$$

is an upper gradient for the map $v := (1 - \phi)u_1 + \phi\, u_2$, where $d_X(u_1, u_2)$ denotes the function defined by $d_X(u_1, u_2)(z) := d_X(u_1(z), u_2(z))$.

If, in addition, ϕ is Lipschitz continuous, then, for any generalized upper gradients $g_1, g_2 \in L^p(U)$ for $u_1, u_2 \in H^{1,p}(U;X)$ respectively and $\varepsilon > 0$, the function

$$g' := (\mathrm{Lip}\, \phi) \cdot d_X(u_1, u_2) + (1 - \phi + \varepsilon)g_1 + (\phi + \varepsilon)g_2$$

is a generalized upper gradient for v.

Proof. Fix a unit speed curve $c : [0, l] \longrightarrow U$. If $\int_0^l g_i \circ c\, ds = \infty$ for some i, then $\int_0^l g \circ c\, ds = \infty$. Hence we may assume $\int_0^l g_i \circ c\, ds < \infty$ for all i, and then $u_1 \circ c, u_2 \circ c$, and $\phi \circ c$ are uniformly continuous (see Remark 4.1.2). Take a sufficiently large $n \ge 1$ such that we have $\big|\phi(c(s)) - \phi(c(t))\big| < \varepsilon$ and $d_X\big(u_i(c(s)), u_i(c(t))\big) < \varepsilon/2$ for any $s, t \in [0, l]$ with $|s - t| < l/n$ and for $i = 1, 2$. Put $z_j := c(l_j)$, where we set $l_j := (j/n)l$ for $0 \le j \le n$.

Then, by (4.1), we have

$$
\begin{aligned}
&d_X\bigl(v(c(0)), v(c(l))\bigr) \\
&\le \sum_{j=0}^{n-1} d_X(v(z_j), v(z_{j+1})) \\
&\le \sum_{j=0}^{n-1} \bigl\{d_X\bigl(v(z_j), (1-\phi(z_{j+1}))u_1(z_j) + \phi(z_{j+1})u_2(z_j)\bigr) \\
&\qquad + d_X\bigl((1-\phi(z_{j+1}))u_1(z_j) + \phi(z_{j+1})u_2(z_j), v(z_{j+1})\bigr)\bigr\} \\
&\le \sum_{j=0}^{n-1} \bigl\{|\phi(z_j) - \phi(z_{j+1})|\, d_X(u_1(z_j), u_2(z_j)) \\
&\qquad + (1-\phi(z_{j+1}))d_X(u_1(z_j), u_1(z_{j+1})) + \phi(z_{j+1})d_X(u_2(z_j), u_2(z_{j+1}))\bigr\} \\
&\le \sum_{j=0}^{n-1} \left\{\left(\int_{l_j}^{l_{j+1}} g_3 \circ c\, ds\right) d_X(u_1(z_j), u_2(z_j)) \right. \\
&\qquad \left. + (1-\phi(z_{j+1})) \int_{l_j}^{l_{j+1}} g_1 \circ c\, ds + \phi(z_{j+1}) \int_{l_j}^{l_{j+1}} g_2 \circ c\, ds \right\} \\
&\le \sum_{j=0}^{n-1} \int_{l_j}^{l_{j+1}} \bigl\{g_3 \cdot (d_X(u_1, u_2) + \varepsilon) + (1-\phi+\varepsilon)g_1 + (\phi+\varepsilon)g_2\bigr\} \circ c\, ds \\
&= \int_0^l g \circ c\, ds.
\end{aligned}
$$

Therefore g is an upper gradient for v.

Let ϕ be Lipschitz continuous, and g_1 and g_2 be generalized upper gradients for u_1 and u_2 respectively. Then, by definition, there exist sequences $\{(u_{1,j}, g_{1,j})\}_{j=1}^{\infty}$ and $\{(u_{2,j}, g_{2,j})\}_{j=1}^{\infty}$ such that $u_{i,j} \to u_i$ in $L^p(U;X)$ and $g_{i,j} \to g_i$ in $L^p(U)$ respectively as $j \to \infty$, and that $g_{i,j}$ is an upper gradient for $u_{i,j}$ for any j and $i = 1, 2$. Let L be the Lipschitz constant of ϕ. By (4.1), we obtain $v_j := (1-\phi)\, u_{1,j} + \phi u_{2,j} \ \to v$ in $L^p(U;X)$ as $j \to \infty$. Moreover, by Proposition 4.1.5 and the first part of this lemma,

$$
g'_j := (\operatorname{Lip}\phi) \cdot d_X(u_{1,j}, u_{2,j}) + (1-\phi+\varepsilon)g_{1,j} + (\phi+\varepsilon)g_{2,j}
$$

is an upper gradient for v_j. Note that, since $\int_0^l (\operatorname{Lip}\phi) \circ c\, ds < \infty$, we can let $\varepsilon \to 0$ in the term $(\operatorname{Lip}\phi) \cdot (d_X(u_{1,j}, u_{2.j}) + \varepsilon)$. Then we have

$$
\begin{aligned}
|g'_j - g'|_{L^p} \le L\, &|d_X(u_{1,j}, u_1) + d_X(u_{2,j}, u_2)|_{L^p} \\
&+ (1+\varepsilon)|g_{1,j} - g_1|_{L^p} + (1+\varepsilon)|g_{2,j} - g_2|_{L^p} \\
\to 0 &\quad \text{as } j \to \infty.
\end{aligned}
$$

Hence g' is a generalized upper gradient for v. □

Proposition 4.2.4 (cf. [Ch, Proposition 2.17]) *Let $1 \le p < \infty$, $W \subset U$ be an open set, and $u \in H^{1,p}(U;X)$. If g_U and g_W are generalized upper gradients for $u|_U$ and $u|_W$ respectively, then the function g defined by $g := g_U$ on $U \setminus W$ and $g := g_W$ on W is a generalized upper gradient for u. In particular, if $1 < p < \infty$, then we have $g_u|_W = g_{u|_W}$ a.e. on W.*

Proof. We may assume that the set W is bounded. Take two sequences $\{u_{U,i}, g_{U,i}\}_{i=1}^{\infty}$ and $\{u_{W,i}, g_{W,i}\}_{i=1}^{\infty}$ such that $u_{U,i} \to u|_U$ in $L^p(U;X)$, $u_{W,i} \to u|_W$ in $L^p(W;X)$, $g_{U,i} \to g_U$ in $L^p(U)$, and $g_{W,i} \to g_W$ in $L^p(W)$ respectively as $i \to \infty$, and that $g_{U,i}$ and $g_{W,i}$ are upper gradients for $u_{U,i}$ and $u_{W,i}$ respectively. Take any $\eta > 0$ and let $\phi : U \longrightarrow [0,1]$ be a Lipschitz continuous function such that $\phi \equiv 1$ on W_η, $\phi \equiv 0$ on $U \setminus W$, and its Lipschitz constant is not greater than η^{-1}. Recall that $W_\eta = \{z \in W \mid \mathrm{dist}(z, \partial W) \ge \eta\}$. Put $u_i := (1-\phi)u_{U,i} + \phi\, u_{W,i}$ for each i. Then, by (4.1), we have

$$d_X(u, u_i) \le (1-\phi) d_X(u, u_{U,i}) + \phi\, d_X(u, u_{W,i}) \to 0$$

in $L^p(U)$ as $i \to \infty$. Fix $\varepsilon > 0$ and put

$$G_i := \chi_W \cdot \big\{(\mathrm{Lip}\,\phi) \cdot d_X(u_{U,i}, u_{W,i}) + (1-\phi+\varepsilon) g_{U,i} + (\phi+\varepsilon) g_{W,i}\big\},$$

where χ_W denotes the characteristic function on W. By Lemma 4.2.3, $G_i|_W$ is an upper gradient for $u_i|_W$. On the other hand, $g_{U,i}|_{U\setminus \mathrm{supp}\,\phi}$ is an upper gradient for $u_i|_{U\setminus\mathrm{supp}\,\phi}$. Put $g'_{U,i} := \chi_{U\setminus\mathrm{supp}\,\phi} \cdot g_{U,i}$. By Lemma 4.1.3, $g_i := G_i \vee g'_{U,i}$ is an upper gradient for u_i, and

$$\begin{aligned}
|g'_{U,i} - g_U|_{L^p(U\setminus W)} &= |g_{U,i} - g_U|_{L^p(U\setminus W)} \ \to\ 0 \quad \text{as } i \to \infty,\\
|g'_{U,i} - g_W|_{L^p(W\setminus \mathrm{supp}\,\phi)} &= |g_{U,i} - g_W|_{L^p(W\setminus\mathrm{supp}\,\phi)} \ \to\ 0 \quad \text{as } \eta \to 0,\\
|G_i - g_W|_{L^p(W)} &\le |(\mathrm{Lip}\,\phi)\cdot d_X(u_{U,i}, u_{W,i})|_{L^p(W)} + |(1-\phi+\varepsilon) g_{U,i}|_{L^p(W)}\\
&\quad\ + |(\phi+\varepsilon)(g_{W,i} - g_W)|_{L^p(W)} + |(\phi+\varepsilon-1) g_W|_{L^p(W)}\\
&\le \eta^{-1} |d_X(u_{U,i}, u_{W,i})|_{L^p(W\setminus W_\eta)} + \varepsilon(|g_{U,i}|_{L^p(W_\eta)} + |g_W|_{L^p(W_\eta)})\\
&\quad\ + (1+\varepsilon)(|g_{U,i}|_{L^p(W\setminus W_\eta)} + |g_W|_{L^p(W\setminus W_\eta)} + |g_{W,i} - g_W|_{L^p(W)})\\
&\to 0 \quad \text{as } i \to \infty, \varepsilon \to 0, \eta \to 0 \text{ in this order.}
\end{aligned}$$

Hence $g_i \to g$ in $L^p(U)$ as $i \to \infty, \varepsilon \to 0$ and then $\eta \to 0$. □

Corollary 4.2.5 (cf. [Ch, Theorem 2.18]) *Let $1 < p < \infty$. If $g \in L^p(U)$ is a generalized upper gradient for $u \in H^{1,p}(U;X)$, then $g_u \le g$ holds a.e. on U.*

Proof. Assume that there exists a bounded measurable set $A \subset U$ such that $\mu(A) > 0$ and, for some $\varepsilon > 0$, we have $g(z)^p < g_u(z)^p - \varepsilon$ for any $z \in A$. Take a sequence of open sets $\{W_i\}_{i=1}^{\infty}$ such that $W_i \supset A$ for all i and $\lim_{i\to\infty} \mu(W_i \setminus A) = 0$. By Proposition 4.2.4, for each i, the function g_i defined by $g_i := g$ on W_i and $g_i := g_u$ on $U \setminus W_i$ is a generalized upper gradient for u. Then, by assumption, we obtain

$$\lim_{i\to\infty} |g_i|^p_{L^p(U)} = |g_u|^p_{L^p(U\setminus A)} + |g|^p_{L^p(A)} \le |g_u|^p_{L^p(U)} - \varepsilon\mu(A).$$

Hence there exists i satisfying $|g_i|_{L^p} < |g_u|_{L^p}$, which contradicts the minimality of g_u. Thus we have $g_u \le g$ a.e. on U. □

Corollary 4.2.6 (cf. [Ch, Corollary 2.25]) *Let $1 < p < \infty$. For $u, v \in H^{1,p}(U; X)$, if $u = v$ a.e. on an open set $W \subset U$, then we have $g_u = g_v$ a.e. on W.*

Proof. Similar to the proof of Corollary 4.2.5. We remark that we can let $W_i \subset W$ since W is open. $\square$

Lemma 4.2.7 *Fix three points $x, y_1, y_2 \in X$ and $\varepsilon > 0$, put $d_i := d_X(x, y_i)$, and set*

$$y_i' := \begin{cases} x & \text{if } d_i \leq \varepsilon, \\ (\varepsilon/d_i)x + \big(1 - (\varepsilon/d_i)\big)y_i & \text{if } d_i > \varepsilon \end{cases}$$

for each $i = 1, 2$. Then we have $d_X(y_1', y_2') \leq d_X(y_1, y_2)$.

Proof. (a) If $d_1 \leq \varepsilon$ and $d_2 \leq \varepsilon$, then clearly $d_X(y_1', y_2') = d_X(x, x) = 0$.

(b) If $d_1 \leq \varepsilon$ and $d_2 > \varepsilon$, then we have

$$d_X(y_1', y_2') = d_X(x, y_2') = d_2 - \varepsilon \leq d_2 - d_1 \leq d_X(y_1, y_2).$$

(c) If $d_1 \geq d_2 > \varepsilon$, then we put $x' := (d_2/d_1)x + \big(1 - (d_2/d_1)\big)y_1$. There we have $d_X(x, x') = d_1 - d_2 \leq d_X(y_1, y_2)$. Combining this with (4.1), we obtain

$$d_X(y_1', y_2') \leq d_X(x', x) \vee d_X(y_1, y_2) = d_X(y_1, y_2),$$

for $y_1' = (\varepsilon/d_2)x' + \big(1 - (\varepsilon/d_2)\big)y_1$. $\square$

Proposition 4.2.8 (cf. [Ch, Proposition 2.22]) *Let $1 \leq p < \infty$ and $g \in L^p(U)$ be a generalized upper gradient for $u \in H^{1,p}(U; X)$. Take any $x \in X$ and put $A := u^{-1}(x)$. Then the function g' defined by $g' := g$ on $U \setminus A$ and $g' := 0$ on A is a generalized upper gradient for u. In particular, if $1 < p < \infty$, then $g_u = 0$ holds a.e. on A.*

Proof. We may suppose $\mu(A) > 0$. Furthermore, by Proposition 4.2.4, we can also assume that U is bounded. Take a sequence $\{(u_i, g_i)\}_{i=1}^{\infty}$ such that $u_i \to u$ in $L^p(U; X)$ and $g_i \to g$ in $L^p(U)$ respectively as $i \to \infty$, and that g_i is an upper gradient for u_i. For each $\varepsilon > 0$ and $z \in U$, we set

$$u_{i,\varepsilon}(z) := \begin{cases} x & \text{if } d_X(u_i(z), x) \leq \varepsilon, \\ \big(\varepsilon/d_X(u_i(z), x)\big)x + \Big(1 - \big(\varepsilon/d_X(u_i(z), x)\big)\Big)u_i(z) & \text{if } d_X(u_i(z), x) > \varepsilon. \end{cases}$$

Then, since $d_X(u_{i,\varepsilon}(z), u_i(z)) \leq \varepsilon$ for any $z \in U$, we have

$$\begin{aligned} d_{L^p}(u_{i,\varepsilon}, u) &\leq d_{L^p}(u_{i,\varepsilon}, u_i) + d_{L^p}(u_i, u) \leq \varepsilon\mu(U)^{\frac{1}{p}} + d_{L^p}(u_i, u) \\ &\to 0 \quad \text{as } i \to \infty, \varepsilon \to 0. \end{aligned}$$

Since μ is Borel regular, we can take a sequence of closed sets $\{C_{i,j}^{\varepsilon}\}_{j=1}^{\infty}$ such that $C_{i,j}^{\varepsilon} \subset u_{i,\varepsilon}^{-1}(x)$ and $\lim_{j\to\infty} \mu(C_{i,j}^{\varepsilon}) = \mu(u_{i,\varepsilon}^{-1}(x))$ for each i and ε. Put $g_{i,j}^{\varepsilon} := \chi_{U \setminus C_{i,j}^{\varepsilon}} \cdot g_i$. We shall show that $g_{i,j}^\varepsilon$ is an upper gradient for $u_{i,\varepsilon}$. Fix a unit speed curve $c : [0,l] \longrightarrow U$. If $c^{-1}(C_{i,j}^\varepsilon) = \emptyset$, then we have $g_{i,j}^\varepsilon \circ c = g_i \circ c$. If $c^{-1}(C_{i,j}^\varepsilon) \neq \emptyset$, then we put

$$t_0 := \inf\{t \in [0,l] \mid c(t) \in C_{i,j}^\varepsilon\}, \quad t_1 := \sup\{t \in [0,l] \mid c(t) \in C_{i,j}^\varepsilon\}.$$

Since $C_{i,j}^\varepsilon$ is closed, we have $t_0, t_1 \in c^{-1}(C_{i,j}^\varepsilon)$, so that $u_{i,\varepsilon}(c(t_0)) = u_{i,\varepsilon}(c(t_1)) = x$. Hence we obtain

$$d_X\big(u_{i,\varepsilon}(c(0)), u_{i,\varepsilon}(c(l))\big) \leq d_X\big(u_{i,\varepsilon}(c(0)), u_{i,\varepsilon}(c(t_0))\big) + \big(u_{i,\varepsilon}(c(t_1)), u_{i,\varepsilon}(c(l))\big).$$

Thus it suffices to show that g_i is an upper gradient for $u_{i,\varepsilon}$. For any $z_1, z_2 \in U$, we know $d_X(u_{i,\varepsilon}(z_1), u_{i,\varepsilon}(z_2)) \leq d_X(u_i(z_1), u_i(z_2))$ by Lemma 4.2.7, so that $g_{i,j}^\varepsilon$ is an upper gradient for $u_{i,\varepsilon}$ for any j.

For each $\varepsilon > 0$, it holds that

$$\begin{aligned}
\mu(A \setminus u_{i,\varepsilon}^{-1}(x)) &= \mu(\{z \in U \mid u(z) = x, d_X(u_i(z), x) > \varepsilon\}) \\
&\leq d_{L^p}(u_i, u)^p/\varepsilon^p \\
&\to \ 0 \quad \text{as } i \to \infty.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\limsup_{i\to\infty} \mu(u_{i,\varepsilon}^{-1}(x) \setminus A) &= \limsup_{i\to\infty} \mu(\{z \in U \mid d_X(u_i(z), x) \leq \varepsilon\} \setminus A) \\
&\leq \mu(\{z \in U \mid d_X(u(z), x) \leq 2\varepsilon\} \setminus A) \\
&\to 0 \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
|g_{i,j}^\varepsilon - g|_{L^p(U\setminus A)}^p &\leq |g_i - g|_{L^p(U)}^p + |g|_{L^p(u_{i,\varepsilon}^{-1}(x)\setminus A)}^p \\
&\to 0 \quad \text{as } i \to \infty, \varepsilon \to 0 \text{ in this order,} \\
|g_{i,j}^\varepsilon|_{L^p(A)} &= |g_i|_{L^p(A\setminus C_{i,j}^\varepsilon)} \to 0 \quad \text{as } j \to \infty, i \to \infty \text{ in this order.}
\end{aligned}$$

Hence $g_{i,j}^\varepsilon \to g'$ in $L^p(U)$ as $j \to \infty, i \to \infty$ and then $\varepsilon \to 0$. ☐

We finally observe that all the discussions in this section are applicable to the case where X is a convex subset of a Banach space. For the later convenience, we only state the following:

Theorem 4.2.9 *Let $1 < p < \infty$ and X be a convex subset of some Banach space. Then, for any $u \in H^{1,p}(U; X)$, there exists a unique minimal generalized upper gradient g_u for u.*

We remark that a convex subset of a Banach space is not always a CAT(0)-space, and vice versa. Indeed, minimal geodesics are not unique in some Banach spaces, and even a tripod, which is a typical example of CAT(0)-space, can not be isometrically embedded in any Banach space with a convex image.

4.3 Dirichlet problem

In this section, let (X, d_X) be a complete CAT(0)-space. Then $L^p(U; X)$ is complete for any $1 \le p < \infty$ and, in particular, $L^2(U; X)$ is a complete CAT(0)-space.

For $1 < p < \infty$, we define the distance $d_{H^{1,p}}$ on $H^{1,p}(U; X)$ by

$$d_{H^{1,p}}(u, v) := d_{L^p}(u, v) + |g_u - g_v|_{L^p}$$

for $u, v \in H^{1,p}(U; X)$. There, for $v \in H^{1,p}(U; X)$, we define $H^{1,p}_v(U; X)$ as the $d_{H^{1,p}}$-closure of the set $\{u \in H^{1,p}(U; X) \mid \operatorname{supp} d_X(u, v) \subset U\}$. Note that $H^{1,p}_v(U; X)$ is a convex subset in $H^{1,p}(U; X)$ and that

$$\inf\{E_p(u) \mid u \in H^{1,p}(U; X), \operatorname{supp} d_X(u, v) \subset U\} = \inf_{u \in H^{1,p}_v(U;X)} E_p(u).$$

Definition 4.3.1 A map $v \in H^{1,2}(U; X)$ is said to be E^C-*harmonic* if it satisfies

$$E^C_2(v) = \inf_{u \in H^{1,2}_v(U;X)} E^C_2(u).$$

If $\partial U = \emptyset$, then $H^{1,2}_v(U; X) = H^{1,2}(U; X)$ and hence any constant map is E^C-harmonic. In the remainder of this section, we assume $\partial U \neq \emptyset$ and fix a map $v \in H^{1,2}(U; X)$. For $\lambda > 0$, we put

$$E^\lambda := \inf\{\lambda E_2(u) + d_{L^2}(u, x_0)^2 \mid u \in H^{1,2}_v(U; X)\}.$$

Lemma 4.3.2 *For any $\lambda > 0$, there exists a unique map $u_\lambda \in H^{1,2}_v(U; X)$ which satisfies $E^\lambda = \lambda E_2(u_\lambda) + d_{L^2}(u_\lambda, x_0)^2$.*

Proof. Take a sequence $\{u_i\}_{i=1}^\infty \subset H^{1,2}_v(U; X)$ such that $\lambda E_2(u_i) + d_{L^2}(u_i, x_0)^2 \to E^\lambda$ as $i \to \infty$. By the uniform convexity of $L^2(U)$, as in the proof of Theorem 4.2.2, we obtain that $\{g_{u_i}\}$ is a Cauchy sequence, so that it converges to some $g \in L^2(U)$. On the other hand, it follows from (2.1), Lemma 4.2.1, and the convexity of the set $H^{1,2}_v(U; X)$ that

$$\begin{aligned}
\frac{1}{4} d_{L^2}(u_i, u_j)^2 &\le \frac{1}{2} d_{L^2}(u_i, x_0)^2 + \frac{1}{2} d_{L^2}(u_j, x_0)^2 - d_{L^2}\left(\frac{1}{2}u_i + \frac{1}{2}u_j, x_0\right)^2 \\
&\le \frac{1}{2} d_{L^2}(u_i, x_0)^2 + \frac{1}{2} d_{L^2}(u_j, x_0)^2 - d_{L^2}\left(\frac{1}{2}u_i + \frac{1}{2}u_j, x_0\right)^2 \\
&\qquad + \lambda\left\{\frac{1}{2}E_2(u_i) + \frac{1}{2}E_2(u_j) - E_2\left(\frac{1}{2}u_i + \frac{1}{2}u_j\right)\right\} \\
&\le \frac{1}{2}\{\lambda E_2(u_i) + d_{L^2}(u_i, x_0)^2\} + \frac{1}{2}\{\lambda E_2(u_j) + d_{L^2}(u_j, x_0)^2\} - E^\lambda \\
&\to 0 \quad \text{as } i, j \to \infty.
\end{aligned}$$

Hence $\{u_i\}$ is also a Cauchy sequence in $L^2(U; X)$, so that it converges to some $u \in L^2(U; X)$.

Note that g is a generalized upper gradient for u. Therefore we have $u \in H^{1,2}(U; X)$ and $E_2(u) \le |g|^2_{L^2}$. If $E_2(u) = |g|^2_{L^2}$, then $g = g_u$ a.e. on U by Theorem 4.2.2, and hence $u_i \to u$ as $i \to \infty$ with respect to $d_{H^{1,2}}$, so that $u \in H_v^{1,2}(U;X)$. Assume $E_2(u) < |g|_{L^2}^2$ and take a sequence $\{(u_i', g_i)\}_{i=1}^\infty$ such that $u_i' \to u$ in $L^2(U;X)$ and $g_i \to g_u$ in $L^2(U)$ respectively as $i \to \infty$, and that g_i is an upper gradient for u_i'. We may assume $d_{L^2}(u_i, u) < i^{-2}$ and $d_{L^2}(u_i', u) < i^{-2}$ for all i. Let ϕ_i be a Lipschitz continuous function on U such that $\phi_i \equiv 1$ on $U_{2i^{-1}}$, $\phi_i \equiv 0$ on $U \setminus U_{i^{-1}}$, and its Lipschitz constant is not greater than i. Put $u_i'' := (1-\phi_i)u_i + \phi_i u_i'$. Since $u_i'' = u_i$ on $U \setminus U_{i^{-1}}$, we have $u_i'' \in H_v^{1,2}(U;X)$. Moreover,

$$d_{L^2}(u_i'', u) \le |d_X(u_i, u) \vee d_X(u_i', u)|_{L^2} \to 0 \quad \text{as } i \to \infty.$$

By Lemma 4.2.3, for any $\varepsilon > 0$, we have

$$\begin{aligned} |g_{u_i''}|_{L^2} &\le |(\operatorname{Lip}\phi_i) \cdot d_X(u_i, u_i') + (1-\phi_i+\varepsilon)g_{u_i} + (\phi_i+\varepsilon)g_i|_{L^2} \\ &\le i \cdot d_{L^2}(u_i, u_i') + \varepsilon(|g_{u_i}|_{L^2} + |g_i|_{L^2}) + |g_{u_i}|_{L^2(U \setminus U_{2i^{-1}})} + |g_i|_{L^2(U_{i^{-1}})}. \end{aligned}$$

Letting ε tend to zero, we obtain

$$|g_{u_i''}|_{L^2} \le 2i^{-1} + |g_{u_i}|_{L^2(U \setminus U_{2i^{-1}})} + |g_i|_{L^2(U_{i^{-1}})} \to |g_u|_{L^2(U)} \quad \text{as } i \to \infty.$$

Since $|g_u|_{L^2}^2 = E_2(u) < |g|_{L^2}^2$, for sufficiently large i, it holds that

$$\lambda E_2(u_i'') + d_{L^2}(u_i'', x_0)^2 < \lambda |g|_{L^2}^2 + d_{L^2}(u, x_0)^2 = E^\lambda.$$

This contradicts $u_i'' \in H_v^{1,2}(U;X)$. Therefore $u \in H_v^{1,2}(U;X)$ and it attains E^λ. The uniqueness is easily proved by the strong convexity of $d_{L^2}(\cdot, x_0)^2$ together with the convexity of E_2. ☐

Theorem 4.3.3 *If there exists a sequence $\{\lambda_n\}_{n=1}^\infty$ satisfying that $\lim_{n\to\infty}\lambda_n = \infty$ and that $\{d_{L^2}(u_{\lambda_n}, x_0)\}_{n=1}^\infty$ is bounded, then u_{λ_n} converges to a minimizer of E_2 in $H_v^{1,2}(U;X)$ with respect to $d_{H^{1,2}}$ as $n \to \infty$.*

Proof. By $\lim_{n\to\infty}\lambda_n = \infty$ and the boundedness of $\{d_{L^2}(u_{\lambda_n}, x_0)\}$, the sequence $\{u_{\lambda_n}\}$ minimizes E_2 in $H_v^{1,2}(U;X)$. By the proof of [J3, Theorem 3.1.1], the sequence $\{u_{\lambda_n}\}$ is a Cauchy sequence. Hence it converges to some $u \in L^2(U;X)$ and satisfies

$$E_2(u) \le \liminf_{n\to\infty} E_2(u_{\lambda_n}) = \inf_{H_v^{1,2}(U;X)} E_2.$$

By a discussion similar to the proof of Lemma 4.3.2, we see that $\{g_{u_{\lambda_n}}\}$ is a Cauchy sequence and $g_{u_{\lambda_n}} \to g_u$ in $L^2(U)$, and hence $u_{\lambda_n} \to u$ with respect to $d_{H^{1,2}}$. Consequently, we have $u \in H_v^{1,2}(U;X)$. ☐

We do not know whether the assumption of Theorem 4.3.3 holds or not in general. We see two cases where it holds. Compare these with [Ch, §7] and [KS, §2.2].

Theorem 4.3.4 *If there exists a constant $C > 0$ such that, for any $f \in H_0^{1,2}(U)$, it holds that*

$$\left(\int_U |f|^2 \, d\mu\right)^{\frac{1}{2}} \le C\left(\int_U |g_f|^2 \, d\mu\right)^{\frac{1}{2}},$$

then there exists an E^C-harmonic map in $H_v^{1,2}(U;X)$.

Proof. Fix $\lambda > 0$ and take a sequence of maps $\{u_i\}_{i=1}^{\infty} \subset H^{1,2}(U;X)$ satisfying that $\operatorname{supp} d_X(u_i, v) \subset U$ and $u_i \to u_\lambda$ with respect to $d_{H^{1,2}}$ as $i \to \infty$. It follows from the triangle inequality that the function $g_{u_i} + g_v$ is a generalized upper gradient for $d_X(u_i, v)$, so that $g_{d_X(u_i,v)} \le g_{u_i} + g_v$ holds a.e. on U by Corollary 4.2.5. Since $d_X(u_i, v) \in H_0^{1,2}(U)$, by assumption, we have

$$d_{L^2}(u_i, v) \le C|g_{d_X(u_i,v)}|_{L^2} \le C(|g_{u_i}|_{L^2} + |g_v|_{L^2}).$$

Letting i tend to infinity, we obtain $d_{L^2}(u_\lambda, v) \le C(|g_{u_\lambda}|_{L^2} + |g_v|_{L^2})$. Thus we have

$$\begin{aligned}
&d_{L^2}(u_\lambda, x_0)^2 \\
&\le \{d_{L^2}(u_\lambda, v) + d_{L^2}(v, x_0)\}^2 \\
&\le 2C^2(|g_{u_\lambda}|_{L^2} + |g_v|_{L^2})^2 + 2d_{L^2}(v, x_0)^2 \\
&\le 4C^2(|g_{u_\lambda}|_{L^2}^2 + |g_v|_{L^2}^2) + 2d_{L^2}(v, x_0)^2 \\
&= 4C^2\lambda^{-1}\big[\{\lambda E_2(u_\lambda) + d_{L^2}(u_\lambda, x_0)^2\} - d_{L^2}(u_\lambda, x_0)^2\big] + 4C^2E_2(v) + 2d_{L^2}(v, x_0)^2 \\
&\le 4C^2\lambda^{-1}\big[\{\lambda E_2(v) + d_{L^2}(v, x_0)^2\} - d_{L^2}(u_\lambda, x_0)^2\big] + 4C^2E_2(v) + 2d_{L^2}(v, x_0)^2 \\
&\le 8C^2E_2(v) + (4C^2\lambda^{-1} + 2)\, d_{L^2}(v, x_0)^2.
\end{aligned}$$

Hence the theorem follows from Theorem 4.3.3. $\square$

The inequality assumed in the theorem above is a type of Poincaré inequality which is actually used in the proof of [KS, Theorem 2.2].

Remark 4.3.5 If (Z, d_Z, μ) is complete and satisfies the doubling condition and the weak Poincaré inequality of type $(2, 2)$ (see Definitions 4.4.5 and 4.4.6 below), then, for $R \in (0, (1/3)\operatorname{diam} Z)$, there exists a constant $C = C(\kappa, C_P, R) \ge 1$ such that, for any ball $B = B_r(z) \subset Z$ with $0 < r \le R$ and any function $f \in H_0^{1,2}(B)$, we have

$$\left(\int_B |f|^2\, d\mu\right)^{\frac{1}{2}} \le Cr\left(\int_B |g_f|^2\, d\mu\right)^{\frac{1}{2}}$$

(see [KiSh] and [Bj, Proposition 3.1]). Hence the assumption in Theorem 4.3.4 is satisfied in this case.

Lemma 4.3.6 *Fix three points $x, y_1, y_2 \in X$ and $R > 0$, put $d_i := d_X(x, y_i)$, and set*

$$Ry_i := \begin{cases} y_i & \text{if } d_i \le R, \\ \big(1 - (R/d_i)\big)x + (R/d_i)y_i & \text{if } d_i > R \end{cases}$$

for each $i = 1, 2$. Then we have $d_X(Ry_1, Ry_2) \le d_X(y_1, y_2)$.

Proof. (a) If $d_1 \le R$ and $d_2 \le R$, then $Ry_i = y_i$ for $i = 1, 2$.

(b) If $d_1 \le R$ and $d_2 > R$, then, by $Ry_1 = y_1$ and (2.1), we have

$$\begin{aligned}
d_X(Ry_1, Ry_2)^2 &\le \left(1 - \frac{R}{d_2}\right)d_1^2 + \frac{R}{d_2}d_X(y_1, y_2)^2 - \left(1 - \frac{R}{d_2}\right)\frac{R}{d_2}d_2^2 \\
&= \left(1 - \frac{R}{d_2}\right)(d_1^2 - Rd_2) + \frac{R}{d_2}d_X(y_1, y_2)^2 \\
&\le d_X(y_1, y_2)^2.
\end{aligned}$$

(c) If $d_1 \geq d_2 > R$, then we put $x' := d_2 y_1 = \big(1 - (d_2/d_1)\big)x + (d_2/d_1)y_1$. By a discussion similar to (b), we know $d_X(x', y_2) \leq d_X(y_1, y_2)$. Thus we obtain $d_X(Ry_1, Ry_2) \leq d_X(x', y_2) \leq d_X(y_1, y_2)$ by (4.1), for $Ry_1 = \big(1 - (R/d_2)\big)x + (R/d_2)x'$. □

Lemma 4.3.7 (cf. [Ch, Proposition 2.20]) *Let $1 \leq p < \infty$. For $u \in H^{1,p}(U; X)$ and $R > 0$, if we define $u' := Ru$ as in Lemma 4.3.6 with $x = x_0$, then $E_p(u') \leq E_p(u)$ and, in particular, $u' \in H^{1,p}(U; X)$. Moreover, if $1 < p < \infty$, then we have $g_{u'} \leq g_u$ a.e. on U.*

Proof. Take any sequence $\{(u_i, g_i)\}_{i=1}^{\infty}$ such that $u_i \to u$ in $L^p(U; X)$ as $i \to \infty$ and g_i is an upper gradient for u_i. Set $u_i' := Ru_i$ as in Lemma 4.3.6 with $x = x_0$. Then, for any $z \in U$, we have $d_X(u_i'(z), u'(z)) \leq d_X(u_i(z), u(z))$ by Lemma 4.3.6. Hence we have $u_i' \to u'$ in $L^p(U; X)$. Similarly, it holds that $d_X(u_i'(z_1), u_i'(z_2)) \leq d_X(u_i(z_1), u_i(z_2))$ for any $z_1, z_2 \in U$. Hence g_i is also an upper gradient for u_i', so that we obtain $E_p(u') \leq E_p(u)$. The second part follows from Corollary 4.2.5. □

Theorem 4.3.8 *If $\mu(U) < \infty$ and if there exists a constant $\varepsilon > 0$ such that $d_X(v, x_0)$ is essentially bounded on $U \setminus U_\varepsilon$, then there exists an E^C-harmonic map in $H_v^{1,2}(U; X)$.*

Proof. Put $R := |d_X(v, x_0)|_{L^\infty(U \setminus U_\varepsilon)} < \infty$. Take any $\lambda > 0$ and a sequence $\{u_i\}_{i=1}^{\infty}$ such that $\operatorname{supp} d_X(u_i, v) \subset U$ and $u_i \to u_\lambda$ with respect to $d_{H^{1,2}}$ as $i \to \infty$. Set $u_i' := Ru_i$ as in Lemma 4.3.6 with $x = x_0$ for each i. Then $\operatorname{supp} d_X(u_i', v) \subset U$ and, by Lemmas 4.3.6 and 4.3.7, we obtain that $d_X(u_i'(z), x_0) \leq d_X(u_i(z), x_0)$ for any $z \in U$ and $E(u_i') \leq E(u_i)$. If $|d_X(u_\lambda, x_0)|_{L^\infty} \geq R + \delta$ for some $\delta > 0$, then, since

$$\liminf_{i \to \infty} \mu(\{z \in U \mid d_X(u_i(z), x_0) \geq R + \delta/2\}) \geq \mu(\{z \in U \mid d_X(u_\lambda(z), x_0) \geq R + \delta\}) > 0,$$

we have $\lambda E_2(u_i') + d_{L^2}(u_i', x_0)^2 < \lambda E_2(u_\lambda) + d_{L^2}(u_\lambda, x_0)^2$ for sufficiently large i. This is a contradiction. Thus we obtain $|d_X(u_\lambda, x_0)|_{L^\infty} \leq R$, so that $d_{L^2}(u_\lambda, x_0) \leq R\mu(U)^{1/2}$. Hence the theorem follows from Theorem 4.3.3. □

4.4 Minimality of Lip u

In this section, we show that $\operatorname{Lip} u$ is a minimal generalized upper gradient for any locally Lipschitz continuous map $u \in H^{1,p}(U; X)$ with $1 < p < \infty$. We first prove that $\operatorname{Lip} u$ is at least an upper gradient for u. In the case of $X = \mathbb{R}$, we have already seen this in Proposition 4.1.5. In that proof, we used Rademacher's theorem, so that we can not use the same technique in the general case. However, by the following easily proved lemma (which is a special version of [HKST, Theorem 3.17]), we can directly apply Proposition 4.1.5 to the general case.

Lemma 4.4.1 *A function $g : U \longrightarrow [0, \infty]$ is an upper gradient for a map $u : U \longrightarrow X$ if and only if g is an upper gradient for the function $d_X(u, x)$ for any $x \in X$.*

Proof. The only if part is clear by the triangle inequality. If g is an upper gradient for $d_X(u, x)$ for any $x \in X$, then, for any unit speed curve $c : [0, l] \longrightarrow U$, we have

$$d_X\big(u(c(0)), u(c(l))\big) = \big|d_X\big(u(c(0)), u(c(l))\big) - d_X\big(u(c(l)), u(c(l))\big)\big| \leq \int_0^l g(c(s))\, ds.$$

Hence g is an upper gradient for u. □

Proposition 4.4.2 *If $u : U \longrightarrow X$ is locally Lipschitz continuous, then* $\mathrm{Lip}\, u$ *is an upper gradient for u.*

Proof. For any $x \in X$, since $d_X(u, x)$ is locally Lipschitz continuous, $\mathrm{Lip}\, d_X(u, x)$ is an upper gradient for $d_X(u, x)$ by Proposition 4.1.5. It follows from $\mathrm{Lip}\, u \geq \mathrm{Lip}\, d_X(u, x)$ that $\mathrm{Lip}\, u$ is an upper gradient for $d_X(u, x)$ for any $x \in X$. Hence $\mathrm{Lip}\, u$ is an upper gradient for u by Lemma 4.4.1. □

Let (X, d_X) be a locally compact, locally geodesics extendable Alexandrov space with local curvature upper bound. Recall that the space Σ_x at any $x \in X$ is compact with respect to the angle metric $\angle_x$ in this situation by Proposition 2.2.2.

For a constant map $u : U \longrightarrow X$, it is clear that $\mathrm{Lip}\, u \equiv 0$ and is a minimal generalized upper gradient for u. Take any nonconstant locally Lipschitz continuous map $u : U \longrightarrow X$ and fix it.

Lemma 4.4.3 *For any $z \in U$, there exists a point $x \in X \setminus \{u(z)\}$ in an R_K-domain containing $u(z)$ which satisfies*

$$\mathrm{Lip}\, u(z) = \mathrm{Lip}\, d_X(u, x)(z).$$

Proof. If $\mathrm{Lip}\, u(z) = 0$, then $\mathrm{Lip}\, d_X(u, x)(z) = 0$ for any $x \in X$. So without loss of generality, we can assume $\mathrm{Lip}\, u(z) > 0$, in particular, z is not an isolated point. Take a sequence $\{z_i\}_{i=1}^{\infty} \subset U \setminus \{z\}$ such that $z_i \to z$ as $i \to \infty$ and

$$\lim_{i\to\infty} \frac{d_X(u(z), u(z_i))}{d_Z(z, z_i)} = \mathrm{Lip}\, u(z).$$

Since the metric space $(\Sigma_{u(z)}, \angle_{u(z)})$ is compact and $\mathrm{Lip}\, u(z) > 0$, there exist a nonconstant geodesic $\gamma : [0, \varepsilon] \longrightarrow X$ with $\gamma(0) = u(z)$ and a subsequence $\{z_n\}$ of $\{z_i\}$ satisfying $\lim_{n\to\infty} \angle_{u(z)}(\gamma, \gamma_{u(z)u(z_n)}) = 0$. Since X is locally geodesics extendable, there exists a geodesic $\overline{\gamma} : [-t, \varepsilon] \longrightarrow X$ such that $\overline{\gamma} = \gamma$ on $[0, \varepsilon]$ for sufficiently small $t > 0$. Put $x := \overline{\gamma}(-t)$. Note that

$$\pi \geq \angle x u(z) u(z_n) \geq \angle \overline{\gamma}(-t) u(z) \gamma(\varepsilon) - \angle \gamma(\varepsilon) u(z) u(z_n) = \pi - \angle_{u(z)}(\gamma, \gamma_{u(z)u(z_n)}).$$

Combining this with Theorem 2.2.3 and the local Lipschitz continuity of u, we have

$$\begin{aligned}
&\frac{|d_X(u(z),x) - d_X(u(z_n),x)|}{d_Z(z,z_n)} \\
&= \frac{d_X(u(z),u(z_n))|\cos\angle xu(z)u(z_n)| + o_{x,u(z)}\big(d_X(u(z),u(z_n))\big)}{d_Z(z,z_n)} \\
&\geq \frac{d_X(u(z),u(z_n))}{d_Z(z,z_n)}\cos\angle_{u(z)}(\gamma,\gamma_{u(z)u(z_n)}) + \theta_{x,u(z)}(d_Z(z,z_n)) \\
&\to \operatorname{Lip} u(z) \quad \text{as } n\to\infty.
\end{aligned}$$

This implies $\operatorname{Lip} d_X(u,x)(z) \geq \operatorname{Lip} u(z)$, so that we obtain $\operatorname{Lip} u(z) = \operatorname{Lip} d_X(u,x)(z)$. □

Lemma 4.4.4 *Let z and x be as in Lemma* 4.4.3. *Then, for each $y \in X$ near x, it holds that*

$$\operatorname{Lip} d_X(u,y)(z) = \operatorname{Lip} u(z) + \theta_{x,u(z)}(d_X(x,y)).$$

Proof. By a discussion similar to the proof of Lemma 4.4.3, we find

$$\operatorname{Lip} d_X(u,y)(z) = \operatorname{Lip} u(z) + \theta_{x,u(z)}(\angle xu(z)y).$$

Since $x \neq u(z)$, it follows from $\tilde{\angle} xu(z)y \geq \angle xu(z)y$ that $\angle xu(z)y = \theta_{d_X(x,u(z))}(d_X(x,y))$ holds. □

To recall Cheeger's theorem for locally Lipschitz continuous functions, we need two terminologies.

Definition 4.4.5 A metric measure space (Z, d_Z, μ) is said to satisfy the *doubling condition* if, for any $r > 0$, there exists $\kappa = \kappa(r) \geq 0$ such that

$$\mu(B_{r'}(z)) \leq 2^{\kappa}\mu(B_{r'/2}(z))$$

holds for any $z \in Z$ and $0 < r' \leq r$.

If a metric measure space (Z, d_Z, μ) satisfies the doubling condition, then it satisfies the Vitali and the Besicovitch covering theorems and then, moreover, the set of Lebesgue points of f is dense in Z for any function $f \in L^1(Z)$. See, for example, [F, §2.8], [EG], [Mat, §2], and [He, §1]. In particular, by the Besicovitch covering theorem, any bounded subset of Z is totally bounded. Hence Z is separable, and is complete if and only if it is proper, i.e., any bounded closed subset is compact.

Definition 4.4.6 Let $1 \leq p, q < \infty$. A metric measure space (Z, d_Z, μ) is said to satisfy the *weak Poincaré inequality of type* (q,p) if, for any $r > 0$, there exist constants $C_P = C_P(p,q,r) \geq 1$ and $\Lambda = \Lambda(p,q,r) \geq 1$ such that, for any open ball $B_{r'}(z)$ with $0 < r' \leq r$, any function $f \in L^q(B_{\Lambda r'}(z))$ and upper gradient $g : B_{\Lambda r'}(z) \longrightarrow [0,\infty]$ for f, it holds that

$$\left(\fint_{B_{r'}(z)} |f - f_{B_{r'}(z)}|^q \, d\mu\right)^{\frac{1}{q}} \leq C_P\, r' \left(\fint_{B_{\Lambda r'}(z)} g^p \, d\mu\right)^{\frac{1}{p}}, \tag{4.2}$$

where we set $f_{B_{r'}(z)} := \fint_{B_{r'}(z)} f\, d\mu = \mu(B_{r'}(z))^{-1}\int_{B_{r'}(z)} f\, d\mu$.

There are some works on the relationship between the doubling condition, the (weak) Poincaré inequality, and the other inequalities. See, for instance, [BM1], [BM2], [BM3], and [HaK].

Theorem 4.4.7 ([Ch, Theorem 6.1]) *Let (Z, d_Z, μ) be a complete metric measure space satisfying the doubling condition and the weak Poincaré inequality of type $(1, p)$ for some $1 < p < \infty$. Then, for any locally Lipschitz continuous function $f \in H^{1,p}(U)$, $\mathrm{Lip}\, f$ is the unique minimal generalized upper gradient for f.*

The existence and the uniqueness of the minimal generalized upper gradient for f follows from Theorem 4.2.2 since $\mathbb{R}$ is a CAT(0)-space.

Theorem 4.4.8 *Let (Z, d_Z, μ) and p be as in Theorem 4.4.7 and (X, d_X) be a locally compact, locally geodesics extendable, and separable Alexandrov space with local curvature upper bound. Then, for any locally Lipschitz continuous map $u \in H^{1,p}(U; X)$, $\mathrm{Lip}\, u$ is a minimal generalized upper gradient for u.*

Proof. Take a countable dense set $\{x_i\}_{i=1}^{\infty} \subset X$. By Lemmas 4.4.3, 4.4.4, and Theorem 4.4.7, we know that

$$\mathrm{Lip}\, u = \sup_{x \in X} \mathrm{Lip}\, d_X(u, x) = \sup_i \mathrm{Lip}\, d_X(u, x_i) = \sup_i g_{d_X(u, x_i)}$$

a.e. on U. Thus we obtain $E_p(u) \le |\mathrm{Lip}\, u|_{L^p}^p = |\sup_i g_{d_X(u,x_i)}|_{L^p}^p$.

Fix a sequence $\{(u_k, g_k)\}_{k=1}^{\infty}$ such that $u_k \to u$ in $L^p(U; X)$ as $k \to \infty$ and g_k is an upper gradient for u_k for each k. By Corollary 4.2.5, it holds that $g_k \ge g_{d_X(u_k, x_i)}$ a.e. on U for any i and k, and hence we have

$$\liminf_{k \to \infty} |g_k|_{L^p} \ge \liminf_{k \to \infty} \big| \sup_i g_{d_X(u_k, x_i)} \big|_{L^p}.$$

It suffices to show $|\sup_i g_{d_X(u,x_i)}|_{L^p} \le \liminf_{k \to \infty} |\sup_i g_{d_X(u_k,x_i)}|_{L^p}$.

Fix a point $z_0 \in U$ and a number $n \ge 1$. For each $j \ge 1$, we let $\mathcal{B}_j$ be the family of closed balls $\overline{B}_r(z) \subset U \cap B_n(z_0)$ which satisfies

$$\int_{B_r(z)} g_{d_X(u, x_j)}^p \, d\mu \ge \int_{\overline{B}_r(z)} \sup_i g_{d_X(u, x_i)}^p \, d\mu - \frac{n^{-1} \mu(\overline{B}_r(z))}{\mu(U \cap B_n(z_0))},$$

and $\mathcal{B} := \bigcup_j \mathcal{B}_j$. Then we have $\inf\{r > 0 \,|\, \overline{B}_r(z) \in \mathcal{B}\} = 0$ for a.e. $z \in U \cap B_n(z_0)$. Hence, by the Vitali covering theorem, there exists a subfamily $\{\overline{B}_l\}_{l=1}^{N} \subset \mathcal{B}$ $(1 \le N \le \infty)$ with $\overline{B}_l \in \mathcal{B}_{j_l}$ which consists of mutually disjoint closed balls such that $\mu\big((U \cap B_n(z_0)) \setminus \bigcup_l \overline{B}_l\big) = 0$. For each l, by applying Theorem 4.1.8 to $\{d_X(u_k, x_{j_l})|_{B_l}\}_{k=1}^{\infty}$, we have

$$|g_{d_X(u, x_{j_l})}|_{L^p(B_l)} \le \liminf_{k \to \infty} |g_{d_X(u_k, x_{j_l})}|_{L^p(B_l)} \le \liminf_{k \to \infty} \big| \sup_i g_{d_X(u_k, x_i)} \big|_{L^p(B_l)}.$$

We notice that $g_{d_X(u,x_{j_l})}|_{B_l} = g_{d_X(u,x_{j_l})|_{B_l}}$ holds a.e. on B_l by Proposition 4.2.4. Therefore we obtain

$$\begin{aligned}
\liminf_{k\to\infty} \big|\sup_i g_{d_X(u_k,x_i)}\big|^p_{L^p(U)} &\geq \liminf_{k\to\infty} \big|\sup_i g_{d_X(u_k,x_i)}\big|^p_{L^p(U\cap B_n(z_0))} \\
&\geq \sum_{l=1}^N \liminf_{k\to\infty} \big|\sup_i g_{d_X(u_k,x_i)}\big|^p_{L^p(\overline{B}_l)} \\
&\geq \sum_{l=1}^N \int_{B_l} g^p_{d_X(u,x_{j_l})}\, d\mu \\
&\geq \sum_{l=1}^N \left\{ \int_{\overline{B}_l} \sup_i g^p_{d_X(u,x_i)}\, d\mu - \frac{n^{-1}\mu(\overline{B}_l)}{\mu(U\cap B_n(z_0))} \right\} \\
&= \int_{U\cap B_n(z_0)} \sup_i g^p_{d_X(u,x_i)}\, d\mu - n^{-1}.
\end{aligned}$$

Letting n tend to infinity, we have

$$\liminf_{k\to\infty} \big|\sup_i g_{d_X(u_k,x_i)}\big|_{L^p} \geq \big|\sup_i g_{d_X(u,x_i)}\big|_{L^p}.$$

Consequently, we obtain

$$E_p(u) \leq |\mathrm{Lip}\, u|^p_{L^p} = \big|\sup_i g_{d_X(u,x_i)}\big|^p_{L^p} \leq \liminf_{k\to\infty} \big|\sup_i g_{d_X(u_k,x_i)}\big|^p_{L^p} \leq \liminf_{k\to\infty} |g_k|^p_{L^p}.$$

Taking the infimum over all such sequences $\{(u_k, g_k)\}$, we have $E_p(u) = |\mathrm{Lip}\, u|^p_{L^p}$, so that $\mathrm{Lip}\, u$ is a minimal generalized upper gradient for u. $\square$

Combining this theorem with Theorem 4.2.2, we immediately have the following:

Corollary 4.4.9 *Let (Z, d_Z, μ) and p be as in Theorem 4.4.7 and (X, d_X) be a locally compact, locally geodesics extendable, and separable $\mathrm{CAT}(0)$-space. Then, for any locally Lipschitz continuous map $u \in H^{1,p}(U; X)$, it holds that $g_u = \mathrm{Lip}\, u$ a.e. on U.*

4.5 Comparison of Sobolev spaces and its application

In this final section of the present chapter, we consider the relation between the Cheeger-type Sobolev space for maps into a Banach space V and that for maps into a subset of V. We first recall the several types of Sobolev spaces (cf. [KS] and [Ra]; [Ha] and [K]; [KM]).

Definition 4.5.1 For $1 \leq p < \infty$, $u \in L^p(U; X)$, $\varepsilon > 0$, and $z \in U_\varepsilon$, we define

$$e^u_{p,\varepsilon}(z) := \fint_{B_\varepsilon(z)} \frac{d_X(u(z), u(w))^p}{\varepsilon^p}\, d\mu(w).$$

The *Korevaar-Schoen-type p-energy* of u and the $(1,p)$*-Sobolev space* are defined by

$$E_p^{KS}(u) := \sup_{f \in C_c(U),\ 0 \le f \le 1} \limsup_{\varepsilon \to 0} \int_U f e_{p,\varepsilon}^u \, d\mu,$$
$$KS^{1,p}(U;X) := \{u \in L^p(U;X) \mid E_p^{KS}(u) < \infty\},$$

where $C_c(U)$ denotes the set of all continuous functions on U with compact support. We also define $\mathcal{E}_p^{KS}(u)$ by replacing $C_c(U)$ in $E_p^{KS}(u)$ with the set

$$\{f \in C(U) \mid \operatorname{supp} f \subset U_\eta \text{ for some } \eta > 0\}.$$

Clearly, $E_p^{KS}(u) \le \mathcal{E}_p^{KS}(u)$ holds for any $u \in L^p(U;X)$ and they coincide if Z is proper. As we mentioned in the paragraph after Definition 4.4.5, the proper property of Z is equivalent to the completeness if (Z, d_Z, μ) satisfies the doubling condition.

We should remark the relation between $KS^{1,1}(U;X)$ and $BV(U;X)$ defined in [A] in the case where Z is a Euclidean space. It is easily proved by the definition of $BV(U;X)$ and [KS, Theorem 1.6.2] that $KS^{1,1}(U;X) \subset BV(U;X)$ holds.

Definition 4.5.2 Let $1 \le p < \infty$. For $u \in L^p(U;X)$, we define the *Hajłasz-type p-energy* of u by $\inf_g |g|_{L^p}$, where the infimum is taken over all functions $g \in L^p(U)$ such that there exist an open covering $\{U_i\}_{i=1}^\infty$ of U and a full measure subset $A \subset U$ (i.e., $\mu(U \setminus A) = 0$) satisfying that

$$d_X(u(z), u(w)) \le d_Z(z,w)\big(g(z) + g(w)\big) \tag{4.3}$$

holds for any i and $z, w \in U_i \cap A$. We define the *Hajłasz-type* $(1,p)$*-Sobolev space* $M^{1,p}(U;X)$ as the set of all maps in $L^p(U;X)$ with finite Hajłasz-type p-energy.

We define one more type of Sobolev space for maps into a Banach space.

Definition 4.5.3 Let $1 \le p < \infty$ and V be a Banach space. We define $P^{1,p}(U;V)$ as the set of all maps $u \in L^p(U;V)$ such that there exist an open covering $\{U_i\}_{i=1}^\infty$ of U, a function $g \in L^p(U)$, and constants $C_P \ge 1$ and $\Lambda \ge 1$ satisfying (4.2) with $q = 1$ for any ball $B_r(z)$ with $B_{\Lambda r}(z) \subset U_i$ for some i.

Let (Z, d_Z, μ) be a complete metric measure space satisfying the doubling condition, and V be a Banach space. Then, for the relation between the Sobolev spaces above, it has been already known that

$$M^{1,p}(U;V) \subset P^{1,p}(U;V) \subset KS^{1,p}(U;V) \subset H^{1,p}(U;V)$$

holds for any $1 < p < \infty$. Note that $M^{1,p}(U;V) \subset P^{1,p}(U;V)$ is clear by definition. Indeed, if (u,g) satisfies (4.3), then it satisfies (4.2) with $q = 1$, $\Lambda = 1$, and $C_P = 4$. The second inclusion $P^{1,p}(U;V) \subset KS^{1,p}(U;V)$ follows from [KM, Theorem 4.1]. The third implication $KS^{1,p}(U;V) \subset H^{1,p}(U;V)$ is essentially proved in a discussion in [KM, Theorem 4.5]. We give a precise proof of this implication for completeness. We need to recall a well-known covering lemma (see, for example, [He, Theorem 1.2]). For a ball $B = B_r(z)$ and $a > 0$, we denote $B_{ar}(z)$ by aB.

Lemma 4.5.4 *Every family $\mathcal{B}$ of open balls of uniformly bounded diameter in a separable metric space contains an at most countable, mutually disjoint subfamily $\{B_i\}_{i=1}^N$ $(1 \leq N \leq \infty)$ satisfying*

$$\bigcup_{B \in \mathcal{B}} B \subset \bigcup_{i=1}^{N} 5B_i.$$

From now on, we denote by $C(\alpha, \beta)$ a constant depending only on α and β.

Theorem 4.5.5 *Let $1 < p < \infty$, (Z, d_Z, μ) be a metric measure space satisfying the doubling condition, and X be a convex subset of a Banach space V. Then, for any $u \in KS^{1,p}(U; X)$ with $\mathcal{E}_p^{KS}(u) < \infty$, we have*

$$E_p^C(u) \leq C(\kappa, p)\mathcal{E}_p^{KS}(u).$$

In particular, if Z is complete, then $KS^{1,p}(U; X) \subset H^{1,p}(U; X)$.

Proof. For each $\delta > 0$, by Lemma 4.5.4 together with the separability of Z, there exists a family of open balls $\{B_\delta(z_k)\}_{k=1}^N$ $(1 \leq N \leq \infty)$ such that $z_k \in U$, $U \subset \bigcup_k B_{\delta/2}(z_k)$, and that $\{B_{\delta/10}(z_k)\}_{k=1}^N$ is disjoint. Put $B_k := B_\delta(z_k)$. For any $z \in U$, we have

$$\begin{aligned}
\mu\Big(B_{\frac{11}{10}\delta}(z)\Big) &\geq \sum_{k: z \in B_k} \mu\Big(B_{\frac{1}{10}\delta}(z_k)\Big) \geq c(\kappa) \sum_{k: z \in B_k} \mu\Big(B_{\frac{21}{10}\delta}(z_k)\Big) \\
&\geq c(\kappa)\, \sharp\{k \mid z \in B_k\} \mu\Big(B_{\frac{11}{10}\delta}(z)\Big).
\end{aligned}$$

Thus we obtain $\sharp\{k \mid z \in B_k\} \leq c(\kappa)^{-1}$. Put $M := c(\kappa)^{-1}$. We can take a partition of unity $\{\phi_k\}_{k=1}^N$ of U which is subordinate to $\{B_k\}_{k=1}^N$ such that every ϕ_k is $C_1 M \delta^{-1}$-Lipschitz continuous, where C_1 is a universal constant.

Fix a map $u \in KS^{1,p}(U; X)$ satisfying $\mathcal{E}_p^{KS}(u) < \infty$ and define $u_\delta := \sum_{k=1}^N \phi_k u_{B_k}$. Since $u_{B_k} \in X$ for any k, the image of u_δ is contained in X.

Claim 4.5.6 *$u_\delta \to u$ in $L^p(U_\eta; X)$ as $\delta \to 0$ for any $\eta > 0$. In particular, $u_\delta|_{U_\eta} \in L^p(U_\eta; X)$.*

Proof. Take any $z \in U_\eta$ and $\delta \leq \eta/2$. For any k with $\phi_k(z) > 0$, since $d_Z(z, z_k) < \delta$, we have $B_k \subset B_{2\delta}(z) \subset B_{3\delta}(z_k)$. Hence it follows from the doubling condition that

$$
\begin{aligned}
|u(z) - u_{B_k}| &\leq \fint_{B_k} |u(z) - u(w)| \, d\mu(w) \\
&\leq \frac{1}{\mu(B_k)} \int_{B_{2\delta}(z)} |u(z) - u(w)| \, d\mu(w) \\
&\leq \frac{C(\kappa)}{\mu(B_{3\delta}(z_k))} \int_{B_{2\delta}(z)} |u(z) - u(w)| \, d\mu(w) \\
&\leq C \fint_{B_{2\delta}(z)} |u(z) - u(w)| \, d\mu(w) \\
&\leq 2\delta C \left(\fint_{B_{2\delta}(z)} \frac{|u(z) - u(w)|^p}{(2\delta)^p} \, d\mu(w) \right)^{\frac{1}{p}} \\
&= 2\delta C e^u_{p,2\delta}(z)^{\frac{1}{p}}.
\end{aligned}
$$

Since the cardinality of the set of all such k's is not greater than M, we obtain

$$
\begin{aligned}
|u - u_\delta|^p_{L^p(U_\eta)} &= \Big| \sum_{k=1}^{N} \phi_k \cdot (u - u_{B_k}) \Big|^p_{L^p(U_\eta)} \leq \big| M(2\delta C)(e^u_{p,2\delta})^{\frac{1}{p}} \big|^p_{L^p(U_\eta)} \\
&= (2MC\delta)^p \int_{U_\eta} e^u_{p,2\delta} \, d\mu.
\end{aligned}
$$

Therefore, since $\mathcal{E}_p^{KS}(u) < \infty$, we have

$$
\limsup_{\delta \to 0} |u - u_\delta|^p_{L^p(U_\eta)} \leq (2MC)^p \limsup_{\delta \to 0} \delta^p \int_{U_\eta} e^u_{p,2\delta} \, d\mu = 0.
$$

□

Claim 4.5.7 (cf. [KM, Lemma 4.6]) *For $z \in U_{7\delta}$ and $z' \in B_\delta(z)$, it holds that*

$$
|u_\delta(z) - u_\delta(z')| \leq C_2(\kappa) d_Z(z, z') \left(\fint_{B_{2\delta}(z)} e^u_{p,5\delta} \, d\mu \right)^{\frac{1}{p}}.
$$

In particular, the map u_δ is locally Lipschitz continuous on $U_{7\delta}$ and, for any $z \in U_{7\delta}$, we have

$$
\operatorname{Lip} u_\delta(z) \leq C_2 \left(\fint_{B_{2\delta}(z)} e^u_{p,5\delta} \, d\mu \right)^{\frac{1}{p}}.
$$

Proof. Let $z \in B_{k_0}$. For k with $\phi_k(z) \neq \phi_k(z')$, we have $\{z, z'\} \cap B_k \neq \emptyset$. Then, since $B_{k_0} \subset B_{2\delta}(z) \subset B_{3\delta}(z_{k_0})$ and

$$
B_k \subset B_{2\delta}(z) \cup B_{2\delta}(z') \subset B_{3\delta}(z) \subset B_{5\delta}(w) \subset B_{7\delta}(z) \subset B_{9\delta}(z_k)
$$

for any $w \in B_{2\delta}(z)$, we have

$$\begin{aligned}
|u_{B_{k_0}} - u_{B_k}| &\le \fint_{B_{k_0}} \fint_{B_k} |u(w) - u(w')| \, d\mu(w') \, d\mu(w) \\
&\le C(\kappa) \fint_{B_{2\delta}(z)} \fint_{B_{5\delta}(w)} |u(w) - u(w')| \, d\mu(w') \, d\mu(w) \\
&\le 5C\delta \Big(\fint_{B_{2\delta}(z)} \fint_{B_{5\delta}(w)} \frac{|u(w) - u(w')|^p}{(5\delta)^p} \, d\mu(w') \, d\mu(w) \Big)^{\frac{1}{p}} \\
&= 5C\delta \Big(\fint_{B_{2\delta}(z)} e^u_{p,5\delta} \, d\mu \Big)^{\frac{1}{p}}.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
|u_\delta(z) - u_\delta(z')| &= \Big| \sum_{k=1}^{N} (\phi_k(z) - \phi_k(z'))(u_{B_k} - u_{B_{k_0}}) \Big| \\
&\le \sum_{k=1}^{N} |\phi_k(z) - \phi_k(z')| \, |u_{B_k} - u_{B_{k_0}}| \\
&\le 2M \cdot \big(C_1 M \delta^{-1} d_Z(z, z') \big) \cdot 5C\delta \Big(\fint_{B_{2\delta}(z)} e^u_{p,5\delta} \, d\mu \Big)^{\frac{1}{p}}.
\end{aligned}$$

$\square$

Combining the claims above with Theorems 4.1.8, 4.2.9, and Fubini's theorem, we obtain

$$\begin{aligned}
\int_{U_\eta} g_u^p \, d\mu &\le \liminf_{\delta \to 0} \int_{U_\eta} g_{u_\delta}^p \, d\mu \\
&\le \liminf_{\delta \to 0} \int_{U_\eta} (\operatorname{Lip} u_\delta)^p \, d\mu \\
&\le \liminf_{\delta \to 0} \int_{U_\eta} \Big(C_2^p \fint_{B_{2\delta}(z)} e^u_{p,5\delta} \, d\mu \Big) \, d\mu(z) \\
&\le C_2^p \liminf_{\delta \to 0} \int_{U_\eta} \Big(\frac{C(\kappa)}{\mu(B_{4\delta}(z))} \int_{B_{2\delta}(z)} e^u_{p,5\delta} \, d\mu \Big) \, d\mu(z) \\
&\le C_2^p C \liminf_{\delta \to 0} \int_{U_\eta} \Big(\int_{B_{2\delta}(z)} \frac{e^u_{p,5\delta}(w)}{\mu(B_{2\delta}(w))} \, d\mu(w) \Big) \, d\mu(z) \\
&\le C_2^p C \liminf_{\delta \to 0} \int_{U_{(\eta - 2\delta)}} \Big(\int_{B_{2\delta}(w) \cap U_\eta} \frac{e^u_{p,5\delta}(w)}{\mu(B_{2\delta}(w))} \, d\mu(z) \Big) \, d\mu(w) \\
&\le C_2^p C \liminf_{\delta \to 0} \int_{U_{(\eta - 2\delta)}} e^u_{p,5\delta} \, d\mu \\
&\le C_2^p C \mathcal{E}_p^{KS}(u)
\end{aligned}$$

for any $\eta > 0$. Letting η tend to zero, we have $E_p^C(u) \le C(\kappa, p) \mathcal{E}_p^{KS}(u)$. $\square$

If, in addition, (Z, d_Z, μ) satisfies the weak Poincaré inequality of type $(1, p)$ for some $1 < p < \infty$ (in the sense of Definition 4.4.6), then we have $H^{1,p}(U; V) \subset P^{1,p}(U; V)$ by the existence of a generalized upper gradient for every $u \in H^{1,p}(U; V)$ together with [HKST, Theorem 4.3]. Hence it holds that

$$M^{1,p}(U; V) \subset P^{1,p}(U; V) = KS^{1,p}(U; V) = H^{1,p}(U; V).$$

If, moreover, (Z, d_Z, μ) satisfies the weak Poincaré inequality of type $(1, q)$ for some $1 \leq q < p$, then we have $KS^{1,p}(U; V) \subset M^{1,p}(U; V)$ by [KM, Theorem 4.5]. Thus these four spaces coincide for such (Z, d_Z, μ).

We next consider the case where a metric space (X, d_X) is isometrically embedded in a Banach space V and the base point $x_0 \in X$ corresponds to the origin $0 \in V$, e.g., the Kuratowski embedding:

$$X \ni x \longmapsto \{y \longmapsto d_X(x_0, y) - d_X(x, y)\} \in L^\infty(X)$$

(see [He, p. 99]). Then we have $H^{1,p}(U; X) \subset H^{1,p}(U; V) \cap L^p(U; X)$ while it is clear that $KS^{1,p}(U; X) = KS^{1,p}(U; V) \cap L^p(U; X)$ and $M^{1,p}(U; X) = M^{1,p}(U; V) \cap L^p(U; X)$ hold (and even energies coincide) by definition. Indeed, if we denote by $i : X \longrightarrow V$ an isometric embedding and $u \in L^p(U; X)$, then, since a generalized upper gradient for $i \circ u$ is not always one for u, we know only $E_p^C(i \circ u) \leq E_p^C(u)$ in general.

Example 4.5.8 Let $Z = [-1, 1]$ with the standard metric and measure, $V = \mathbb{R}$ with the standard metric, and $X = \{0, 1\} \subset V$. We define a map $u : Z \longrightarrow X$ by $u(t) := 0$ for $t \in [-1, 0]$ and $u(t) := 1$ for $t \in (0, 1]$. It is easily proved that u is in $H^{1,1}(Z; V)$ (and $KS^{1,1}(Z; V)$), but not in $H^{1,1}(Z; X)$ (nor $M^{1,1}(Z; V)$).

This is, so to speak, the "$p = n$" case, where p is of $H^{1,p}(U; X)$ and n is the dimension of U. We can prove that $H^{1,p}(U; X) = H^{1,p}(U; V) \cap L^p(U; X)$ holds (as sets) in the "$p > n$" case.

Theorem 4.5.9 *Let (X, d_X) be a metric space isometrically embedded in a Banach space V, and the point $x_0 \in X$ corresponding to the origin $0 \in V$ be the base point of X. Assume that there exist constants $C_b \geq 1$, $s \geq 1$, and $R > 0$ such that*

$$\frac{\mu(B_{r'}(z))}{\mu(B_r(z))} \leq C_b \left(\frac{r'}{r}\right)^s \tag{4.4}$$

holds for any $z \in Z$ and $0 < r \leq r' \leq R$. If (Z, d_Z, μ) is complete and satisfies the doubling condition and the weak Poincaré inequality of type $(1, s)$, then $H^{1,p}(U; X) = H^{1,p}(U; V) \cap L^p(U; X)$ holds for any $p \in (s, \infty)$.

Proof. We already know that

$$H^{1,p}(U; X) \subset H^{1,p}(U; V) \cap L^p(U; X) = M^{1,p}(U; V) \cap L^p(U; X) = M^{1,p}(U; X).$$

By [HaK, Theorem 5.1] and [HKST, Theorem 6.2], any $u \in P^{1,p}(U; V)$ has a continuous representative, and hence so does any $u \in M^{1,p}(U; X)$. Then we have $u \in H^{1,p}(U; X)$ as in the proof of [S, Lemma 4.7]. □

We observe that the inequality (4.4) holds with $s = \kappa(R)$ and $C_b = 2^s$ if (Z, d_Z, μ) satisfies the doubling condition. We can apply the strategy of proving $M^{1,p}(U;X) \subset H^{1,p}(U;X)$ to the case of $p = s = 1$, although $H^{1,1}(U;V) \subset M^{1,1}(U;V)$ does not hold in general as we saw in Example 4.5.8.

Theorem 4.5.10 *Let (Z, d_Z, μ) be a geodesic length space satisfying (4.4) with $s = 1$, and $(V, |\cdot|)$ be a Banach space. Then, for any $(u, g) \in P^{1,1}(U;V) \times L^1(U)$ satisfying (4.2) with $p = q = 1$ and any ball $B_{r_0}(z_0)$ with $\Lambda r_0 \le R$ and with $B_{\Lambda r_0}(z_0) \subset U_i$ for some i, we have*

$$|u(z) - u_{B_r(z_0)}| \le C(C_b, C_P, \Lambda)\, r_0 \fint_{B_{\Lambda r_0}(z_0)} g\, d\mu$$

for any Lebesgue point $z \in B_{r_0}(z_0)$ of u.

Proof. We first assume that $\Lambda = 1$. Put $r := d_Z(z, z_0)$ and let $\gamma : [0,1] \longrightarrow U$ be a minimal geodesic from z to z_0. We set $B_0 := B_{r/4}(\gamma(1))$ $(= B_{r/4}(z_0))$ and $B_{4k+l} := B_{2^{-(k+3)}r}\big(\gamma\big((8-l)2^{-(k+3)}\big)\big)$ for each $k \ge 0$ and $1 \le l \le 4$. Since z is a Lebesgue point of u, it follows from the doubling condition and $B_{4k+l} \subset B_{2^{-k}r}(z) \subset 16B_{4k+l}$ that $\lim_{n\to\infty} u_{B_n} = u(z)$. Hence we have

$$\begin{aligned}
|u(z) - u_{B_r(z_0)}| &= \lim_{n\to\infty} |u_{B_n} - u_{B_r(z_0)}| \\
&\le \sum_{k=0}^{\infty} \left\{ \sum_{l=1}^{3} |u_{B_{4k+l}} - u_{B_{4k+l+1}}| + |u_{B_{4k}} - u_{B_{4k+1}}| \right\} + |u_{B_0} - u_{B_r(z_0)}|.
\end{aligned}$$

Fix any $k \ge 0$ and $1 \le l \le 3$, and put

$$B := B_{2^{-(k+4)}r}\big(\gamma\big(\{(15/2) - l\}2^{-(k+3)}\big)\big).$$

Then we have

$$\begin{aligned}
|u_{B_{4k+l}} - u_{B_{4k+l+1}}| &\le \fint_{B_{4k+l}} |u - u_{3B}|\, d\mu + \fint_{B_{4k+l+1}} |u - u_{3B}|\, d\mu \\
&\le \frac{2}{\mu(B)} \int_{3B} |u - u_{3B}|\, d\mu \\
&\le 6C_b \fint_{3B} |u - u_{3B}|\, d\mu \\
&\le 6C_bC_P \cdot (3 \cdot 2^{-(k+4)}r) \fint_{3B} g\, d\mu \\
&\le 9 \cdot 2^{-(k+3)} C_b^2 C_P r \,\frac{2^{k+5}/3}{\mu(2^{k+5}B)} \int_{2B_{4k+l}} g\, d\mu \\
&\le \frac{12C_b^2C_P r}{\mu(B_r(z_0))} \int_{2B_{4k+l}} g\, d\mu.
\end{aligned}$$

Similarly, we obtain that

$$\begin{aligned}|u_{B_{4k}} - u_{B_{4k+1}}| &\le \fint_{B_{4k+1}} |u - u_{B_{4k}}|\, d\mu \\ &\le 3C_b \fint_{B_{4k}} |u - u_{B_{4k}}|\, d\mu \\ &\le 3C_bC_P \cdot (2^{-(k+2)}r) \fint_{B_{4k}} g\, d\mu \\ &\le 3 \cdot 2^{-(k+2)} C_b^2 C_P r \, \frac{2^{k+3}}{\mu(2^{k+3}B_{4k})} \int_{B_{4k}} g\, d\mu \\ &\le \frac{6C_b^2C_P r}{\mu(B_r(z_0))} \int_{B_{4k}} g\, d\mu\end{aligned}$$

for any $k \ge 0$, and that

$$\begin{aligned}|u_{B_0} - u_{B_r(z_0)}| &\le \fint_{B_0} |u - u_{B_r(z_0)}|\, d\mu \le 4C_b \fint_{B_r(z_0)} |u - u_{B_r(z_0)}|\, d\mu \\ &\le 4C_bC_P r \fint_{B_r(z_0)} g\, d\mu.\end{aligned}$$

Therefore we have

$$\begin{aligned}&|u(z) - u_{B_r(z_0)}| \\ &\quad \le \frac{C(C_b, C_P)r}{\mu(B_r(z_0))} \left[\sum_{k=0}^{\infty} \left\{ \sum_{l=1}^{3} \int_{2B_{4k+l}} g\, d\mu + \int_{B_{4k}} g\, d\mu \right\} + \int_{B_r(z_0)} g\, d\mu \right].\end{aligned}$$

Since the multiplicity of the family $\{2B_{4k+l}\}_{k\ge 0, 1\le l\le 3} \cup \{B_{4k}\}_{k\ge 0}$ is at most nine, we consequently obtain

$$|u(z) - u_{B_r(z_0)}| \le 10Cr \fint_{B_r(z_0)} g\, d\mu \le 10CC_b r_0 \fint_{B_{r_0}(z_0)} g\, d\mu,$$

where $C = C(C_b, C_P)$.

A proof in the case of $\Lambda > 1$ is done by a discussion similar to that above by setting

$$B_{4km+l} := B_{2^{-(k+3)}m^{-1}r}\big(\gamma\big((8 - l/m)2^{-(k+3)}\big)\big)$$

for $k \ge 0$ and $1 \le l \le 4m$, where m denotes the smallest integer not smaller than Λ. $\square$

Corollary 4.5.11 *Let (Z, d_Z, μ) be as in Theorem 4.5.10. If (X, d_X) is complete and can be isometrically embedded in some Banach space V such that $(U; V)$ has the Lipschitz extension property in the sense of* [HKST]*, then any $u \in M^{1,1}(U; X)$ has a continuous representative. In particular, we have $M^{1,1}(U; X) \subset H^{1,1}(U; X)$.*

Proof. Take any $u \in M^{1,1}(U;X)$. Since $(U;V)$ has the Lipschitz extension property, the set of continuous maps is dense in $M^{1,1}(U;V)$ (see [Ha, Theorem 5]). Let $\{u_i\}_{i=1}^{\infty} \subset M^{1,1}(U;V)$ be a sequence of continuous maps which converges to u in $M^{1,1}(U;V)$. By Theorem 4.5.10, we obtain that u_i converges to u uniformly on any compact set, so that its limit is a continuous representative of u. □

Remark 4.5.12 The *Newtonian space* $N^{1,p}(U;V)$ defined in [S] coincides with $H^{1,p}(U;V)$ for any $1 < p < \infty$ and any Banach space V ([S, Theorem 4.10]). The Newtonian space for maps into an arbitrary metric space X is also defined in [HKST] through the Kuratowski embedding of X, but we do not know whether $H^{1,p}(U;X) = N^{1,p}(U;X)$ or not any more.

Chapter 5

Harmonicity of totally geodesic maps

In this last chapter, by using the results proved in the preceding chapters, we shall obtain results on the relationship between the harmonicity and the totally geodesic property of maps, that is, the harmonicity of totally geodesic maps. This chapter is based on [O3].

5.1 Harmonicity

Let (M, g) be a Riemannian manifold such that $\overline{M}$ is compact, μ_g be its volume element, and (X, d_X) be a connected, complete Alexandrov space of curvature ≤ 0. We write $E^{KS} = E_2^{KS}$ and $E^C = E_2^C$ in this chapter for short.

Definition 5.1.1 A continuous map $u : M \longrightarrow X$ is said to be E^{KS}*-continuously harmonic* if $E^{KS}(u) \leq E^{KS}(\alpha_1)$ holds for any continuous variation $\alpha : M \times [0, 1] \longrightarrow X$ which satisfies the following boundary condition:

(i) $\alpha_0 = u$,

(ii) there exists $\varepsilon > 0$ such that $\alpha_t|_{M \setminus M_\varepsilon} = u|_{M \setminus M_\varepsilon}$ for any $t \in [0, 1]$,

where we denote $\alpha(\cdot, t)$ by α_t. We define the E^C*-continuous harmonicity* in the same manner.

If $\partial M\ (= \overline{M} \setminus M) = \emptyset$, i.e., if M itself is compact, then (ii) above places no restriction on α.

Remark 5.1.2 It is not so unnatural that we assume the continuity of u in Definition 5.1.1. In fact, Korevaar and Schoen prove (the existence and) the local Lipschitz continuity of an energy minimizing map in the class

$$\{u \in KS^{1,2}(M; X) \,|\, d_X(u, u_0) \in W_0^{1,2}(M)\}$$

for an arbitrarily fixed $u_0 \in KS^{1,2}(M; X)$ in the case where (M, g) is a Lipschitz Riemannian domain and (X, d_X) is a complete CAT(0)-space ([KS, Theorem 2.4.6]). Here, as usual, $W_0^{1,2}(M)$ denotes the closure of $C_0^\infty(M)$ with respect to the Sobolev norm. There are also some works on the Hölder continuity of harmonic maps (e.g., [BM2], [J2], and [KiSh]).

For $x \in X$, we define

$$r_A(x) := \sup\{r > 0 \,|\, (B_r(x), d_X) \text{ is an } R_K\text{-domain}\}.$$

By definition, it clearly holds that $r_A(x) \leq r_X(x)$. Recall that $r_X(x)$ is the convex radius at x. We also note that both functions are 1-Lipschitz continuous.

We first consider the harmonicity for the Korevaar-Schoen-type energy. Note that, if (X, d_X) is a Riemannian manifold and u is smooth, then

$$E^{KS}(u) = (\dim M + 2)^{-1} \int_M |u_*|^2 \, d\mu_g.$$

The following lemma allows us to consider only the variations which are close to u.

Lemma 5.1.3 *Fix a continuous variation $\alpha : M \times [0,1] \longrightarrow X$ which satisfies $\alpha_t|_{M \setminus M_\varepsilon} = \alpha_0|_{M \setminus M_\varepsilon}$ for any $t \in [0,1]$. Then there exists a continuous variation $\beta : M \times [0,1] \longrightarrow X$ such that $\beta_0 = \alpha_0$, $\beta_1 = \alpha_1$, $\beta_t|_{M \setminus M_\varepsilon} = \alpha_0|_{M \setminus M_\varepsilon}$ holds for any $t \in [0,1]$, and that the function $[0,1] \ni t \longmapsto E^{KS}(\beta_t)$ is convex.*

Proof. This lemma is easily deduced from the fact that a universal covering space $(\widetilde{X}, d_{\widetilde{X}})$ of (X, d_X) with the induced length metric is a complete CAT(0)-space (cf. [BH, Chapter II.4]). We give a proof for completeness.

Let $(\widetilde{M}, \tilde{g})$ be a Riemannian universal covering space of (M, g) and $\tilde{\alpha} : \widetilde{M} \times [0,1] \longrightarrow \widetilde{X}$ be a ξ-equivariant lift of α. Here $\xi : \pi_1(M) \longrightarrow \pi_1(X)$ denotes the homomorphism induced from α_0. Set

$$\tilde{\beta}_t := (1-t)\tilde{\alpha}_0 + t\tilde{\alpha}_1$$

for $t \in [0,1]$ and let $\beta_t : M \longrightarrow X$ be its projection. It is clear that β satisfies $\beta_0 = \alpha_0$, $\beta_1 = \alpha_1$, and $\beta_t|_{M \setminus M_\varepsilon} = \alpha_0|_{M \setminus M_\varepsilon}$ for any $t \in [0,1]$. Take any $t_0, t_1, \lambda \in [0,1]$ with $t_0 \leq t_1$, and put $t_\lambda := (1-\lambda)t_0 + \lambda t_1$. Since $M \times [0,1]$ is compact and β and r_A are uniformly continuous, for sufficiently small $\delta > 0$, we have

$$\begin{aligned} d_X(\beta_{t_\lambda}(p), \beta_{t_\lambda}(q)) &= d_{\widetilde{X}}(\tilde{\beta}_{t_\lambda}(p), \tilde{\beta}_{t_\lambda}(q)) \\ &\leq (1-\lambda) d_{\widetilde{X}}(\tilde{\beta}_{t_0}(p), \tilde{\beta}_{t_0}(q)) + \lambda d_{\widetilde{X}}(\tilde{\beta}_{t_1}(p), \tilde{\beta}_{t_1}(q)) \\ &= (1-\lambda) d_X(\beta_{t_0}(p), \beta_{t_0}(q)) + \lambda d_X(\beta_{t_1}(p), \beta_{t_1}(q)) \end{aligned}$$

for every $p, q \in M$ with $d_M(p,q) < \delta$. Therefore we obtain $E^{KS}(\beta_{t_\lambda}) \leq (1-\lambda)E^{KS}(\beta_{t_0}) + \lambda E^{KS}(\beta_{t_1})$. □

Before giving the proof of the harmonicity of a totally geodesic map, we recall some facts for totally geodesic maps (see Chapter 3 of this thesis). For a totally geodesic map $u : M \longrightarrow X$, we put

$$a := \sup_{U_p M} |du|, \qquad b := \inf_{U_p M \cap H_p} |du|.$$

By Corollary 3.4.4, they are independent of the choice of $p \in M$. In particular, u is a-Lipschitz continuous. If $a = 0$, then u is a constant map. So that, without loss of generality, we assume $a > 0$ from now on, and then $b > 0$.

Theorem 5.1.4 *Any totally geodesic map $u : M \longrightarrow X$ is E^{KS}-continuously harmonic.*

Proof. Let $\alpha : M \times [0,1] \longrightarrow X$ be a continuous variation as in Definition 5.1.1. If $E^{KS}(\alpha_1) < E^{KS}(u)$, and if we denote by β the continuous variation constructed from α by Lemma 5.1.3, then we have $E^{KS}(\beta_t) < E^{KS}(u)$ for all $t \in (0,1]$. Thus we need only to show that, for every α as in Definition 5.1.1, there exists some $t > 0$ satisfying $E^{KS}(u) \le E^{KS}(\alpha_t)$.

Since u is a-Lipschitz continuous and M_ε is compact, α is uniformly continuous on $M \times [0,1]$. Hence we can take $\delta > 0$ such that $d_X(\alpha_t(p), u(p)) < r_A(u(p))$ holds for any $p \in M_\varepsilon$ and $t \in [0,\delta]$. We remark that, by assumption, we know $\alpha_t(p) = u(p)$ for all $p \in M \setminus M_\varepsilon$ and $t \in [0,1]$. Put

$$\rho'(p) := \frac{b}{a}\left\{ r_M(p) \wedge \frac{r_A(u(p))}{a} \right\}, \quad r(p) := \frac{1}{3}\{\rho'(p) \wedge \mathrm{dist}(p, \partial M)\}$$

for $p \in M$. Note that r is (1/3)-Lipschitz continuous. For each $t \in [0,\delta]$ and $p \in M$, we define $\beta_t(p) := F[u(\overline{B}_{r(p)}(p))](\alpha_t(p))$, where the foot-point is taken as an element in $B_{r_A(u(p))}(u(p))$. This makes sense. Indeed, we have $\alpha_t(p) \in B_{r_A(u(p))}(u(p))$,

$$u(\overline{B}_{r(p)}(p)) \subset u(B_{\rho'(p)}(p)) \subset B_{a\rho'(p)}(u(p)) \subset B_{r_A(u(p))}(u(p)),$$

and $u(\overline{B}_{r(p)}(p))$ is compact and geodesically convex since u is totally geodesic.

For each $\eta > 0$, we can take $\delta_\eta \in (0, \delta \wedge \inf_{M_\eta} r]$ such that $d_X(\alpha_t(p), u(p)) < br(p)$ holds for any $p \in M_\eta$ and $t \in [0, \delta_\eta]$.

Claim 5.1.5 *For every $t \in [0,\delta_\eta]$ and $p, q \in M_\eta$ with $d_M(p,q) < \delta_\eta/2$, we have*

$$d_X(\beta_t(p), \beta_t(q)) \le d_X(\alpha_t(p), \alpha_t(q)).$$

Proof. By assumption, we know

$$u(\overline{B}_{r(p)}(p)) \subset u(B_{r(p)+\delta_\eta/2}(q)) \subset u(B_{2r(q)}(q)) \subset B_{r_A(u(q))}(u(q)).$$

Here the second inclusion is derived from

$$r(p) + \delta_\eta/2 < \{r(q) + \delta_\eta/6\} + \delta_\eta/2 < 2r(q).$$

Moreover, we have

$$\alpha_t(p) \in B_{br(p)}(u(p)) \subset B_{br(p)+a\delta_\eta/2}(u(q)) \subset B_{r_A(u(q))}(u(q))$$

since

$$br(p) + a\delta_\eta/2 < b\{r(q) + \delta_\eta/6\} + a\delta_\eta/2 < 2ar(q) < r_A(u(q)).$$

By Lemma 3.2.2, for any $v \in B_{3r(p)}(0) \subset T_pM$ and its projection to H_p, say $v' \in H_p$, we find $u(\exp_p v) = u(\exp_p v')$. It follows from $\overline{B}_{2r(q)}(q) \subset B_{3r(p)}(p)$ that

$$\begin{aligned}
&\mathrm{dist}\big(u(\overline{B}_{2r(q)}(q)) \setminus u(\overline{B}_{r(p)}(p)), u(p)\big) \\
&\ge \mathrm{dist}\big(u(B_{3r(p)}(p)) \setminus u(\overline{B}_{r(p)}(p)), u(p)\big) \\
&= \mathrm{dist}\big(u \circ \exp_p(\{B_{3r(p)}(0;T_pM) \setminus \overline{B}_{r(p)}(0;T_pM)\} \cap H_p), u(p)\big) \\
&\ge br(p) > d_X(\alpha_t(p), u(p)) \\
&\ge d_X\big(F[u(\overline{B}_{2r(q)}(q))](\alpha_t(p)), u(p)\big),
\end{aligned}$$

where the foot-point is taken as an element in $B_{r_A(u(q))}(u(q))$. This implies that

$$F[u(\overline{B}_{2r(q)}(q))](\alpha_t(p)) \in u(\overline{B}_{r(p)}(p)),$$

so that it coincides with $\beta_t(p)$. Putting $p = q$, we also have $F[u(\overline{B}_{2r(q)}(q))](\alpha_t(q)) = \beta_t(q)$. Consequently, we obtain

$$\begin{aligned} d_X(\beta_t(p), \beta_t(q)) &= d_X\big(F[u(\overline{B}_{2r(q)}(q))](\alpha_t(p)), F[u(\overline{B}_{2r(q)}(q))](\alpha_t(q))\big) \\ &\leq d_X(\alpha_t(p), \alpha_t(q)) \end{aligned}$$

by Proposition 2.3.2 with $A = u(\overline{B}_{2r(q)}(q))$. □

Combining this claim with (ii) in Definition 5.1.1, we have $E^{KS}(\beta_t) \leq E^{KS}(\alpha_t)$ for any $t \in [0, \delta_{\varepsilon'}]$ with $\varepsilon' \in (0, \varepsilon)$. Fix $\varepsilon' \in (0, \varepsilon)$ and put $\delta' := \delta_{\varepsilon'}$. Note that

$$\beta_{\delta'}(q) \in u(\overline{B}_{r(q)}(q)) \subset u(B_{2r(p)}(p)) \subset u(B_{\rho(p)}(p))$$

holds for every $p \in M_{\varepsilon'}$ and $q \in B_{\delta'/2}(p)$. Hence we may assume that

$$\beta_{\delta'}|_{B_{\delta'/2}(p)} : B_{\delta'/2}(p) \longrightarrow u(B_{\rho(p)}(p))$$

is smooth for all $p \in M$ with respect to the Riemannian structure derived from Corollary 3.4.4. We know that u is smooth with respect to that structure, and hence so is $\beta_{\delta'}|_{M \setminus M_\varepsilon}$. Put

$$\beta'_t := \left(1 - \frac{t}{\delta'}\right) u + \frac{t}{\delta'} \beta_{\delta'}$$

for $t \in [0, \delta']$. Then β'_t is smooth and the function $[0, \delta'] \ni t \longmapsto E^{KS}(\beta'_t)$ is convex. Applying the first variation formula for energy (cf., for example, [N, §3.3]), by the totally geodesic property of u and the boundary condition, we immediately obtain

$$\frac{d}{dt}\bigg|_{t=0+} E^{KS}(\beta'_t) = 0.$$

Here we denote by $(d/dt)|_{t=0+} f$ the right derivative of a function f at $t = 0$. Therefore we have $E^{KS}(u) \leq E^{KS}(\beta'_{\delta'}) = E^{KS}(\beta_{\delta'}) \leq E^{KS}(\alpha_{\delta'})$. This completes the proof. □

Almost all discussions above for the Korevaar-Schoen-type energy are also valid if we consider the Cheeger-type energy. The only one obstruction is that we do not have the first variation formula for this energy. We will overcome this difficulty by considering a representation of $E^C(u)$ for a map u between Riemannian manifolds.

Let (N, h) be a Riemannian manifold and $u : M \longrightarrow N$ be a smooth map. For $p \in M$, we denote the operator norm of the differential operator at p, $(u_*)_p : T_pM \longrightarrow T_{u(p)}N$, by $|u_*|_{\mathrm{op}}(p)$. Clearly $\operatorname{Lip} u = |u_*|_{\mathrm{op}}$ holds. By Theorem 4.4.8, we find

$$\int_U |u_*|^2_{\mathrm{op}}\, d\mu_g = E^C(u|_U) \leq E^C(u) \leq \int_M |u_*|^2_{\mathrm{op}}\, d\mu_g$$

for any relatively compact domain $U \subset M$ with smooth boundary. Letting $U \to M$, we obtain $E^C(u) = \int_M |u_*|^2_{\mathrm{op}}\, d\mu_g$.

Lemma 5.1.6 *For a smooth map $u : M \longrightarrow N$, we have*

$$E^C(u) = \sup_W \int_M |u_*(W)|^2 \, d\mu_g,$$

where the supremum is taken over all smooth vector fields W on M satisfying $|W(p)| \le 1$ for every $p \in M$.

Proof. Note that

$$E^C(u) = \int_M |u_*|_{\mathrm{op}}^2 \, d\mu_g \ge \sup_W \int_M |u_*(W)|^2 \, d\mu_g.$$

We prove the reverse inequality.

For each $p \in M$, we can take a smooth vector field W_p on M such that $|W_p(q)| \le 1$ holds for all $q \in M$ and that $|u_*|_{\mathrm{op}}(p) = |u_*(W_p(p))|$. Take any $\varepsilon > 0$. Since

$$\lim_{r \to 0} \fint_{B_r(p)} |u_*(W_p)|^2 \, d\mu_g = |u_*|_{\mathrm{op}}(p)^2 = \lim_{r \to 0} \fint_{B_r(p)} |u_*|_{\mathrm{op}}^2 \, d\mu_g,$$

the Vitali covering theorem yields a family of at most countably many mutually disjoint balls $\{B_{r_i}(p_i)\}_i \subset M$ which satisfies $\mu_g(M \setminus \bigcup_i B_{r_i}(p_i)) = 0$ and

$$\fint_{B_{r_i}(p_i)} |u_*(W_{p_i})|^2 \, d\mu_g \ge \fint_{B_{r_i}(p_i)} |u_*|_{\mathrm{op}}^2 \, d\mu_g - \varepsilon.$$

For each i, we take $r_i' \in (0, r_i)$ such that

$$\frac{1}{\mu_g(B_{r_i}(p_i))} \int_{B_{r_i'}(p_i)} |u_*(W_{p_i})|^2 \, d\mu_g \ge \fint_{B_{r_i}(p_i)} |u_*|_{\mathrm{op}}^2 \, d\mu_g - 2\varepsilon.$$

Let W_ε be a smooth vector field satisfying $|W_\varepsilon(p)| \le 1$ for all $p \in M$ and $W_\varepsilon = W_{p_i}$ on $B_{r_i'}(p_i)$ for all i. Then we have

$$\begin{aligned}
\int_M |u_*(W_\varepsilon)|^2 \, d\mu_g &\ge \sum_i \int_{B_{r_i'}(p_i)} |u_*(W_{p_i})|^2 \, d\mu_g \\
&\ge \sum_i \left\{ \int_{B_{r_i}(p_i)} |u_*|_{\mathrm{op}}^2 \, d\mu_g - 2\varepsilon \mu_g(B_{r_i}(p_i)) \right\} \\
&= \int_M |u_*|_{\mathrm{op}}^2 \, d\mu_g - 2\varepsilon \mu_g(M) \\
&= E^C(u) - 2\varepsilon \mu_g(M).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this completes the proof. $\square$

Proposition 5.1.7 *If $\overline{N}$ is an Alexandrov space of curvature ≤ 0, then any harmonic map $u : M \longrightarrow N$ (in the classical sense) with finite energy is E^C-continuously harmonic.*

Proof. Fix a continuous variation α satisfying the condition in Definition 5.1.1 and take β as in the proof of Lemma 5.1.3. We may assume that $\beta : M \times [0,1] \longrightarrow N$ is smooth. It is easily proved that the function $[0,1] \ni t \longmapsto \int_M |(\beta_t)_*(W)|^2 \, d\mu_g$ is convex for any smooth vector field W on M. Moreover, by the same calculation as that deriving the usual first variation formula, we obtain

$$\frac{d}{dt}\Big|_{t=0+} \int_M |(\beta_t)_*(W)|^2 \, d\mu_g = 0.$$

Hence we have $\int_M |u_*(W)|^2 \, d\mu_g \leq \int_M |(\beta_1)_*(W)|^2 \, d\mu_g$. By Lemma 5.1.6, this implies $E^C(u) \leq E^C(\beta_1) = E^C(\alpha_1)$. Therefore u is E^C-continuously harmonic. □

We remark that the nonpositivity of the sectional curvature of N does not always imply that $\overline{N}$ is an Alexandrov space of curvature ≤ 0. For instance, the completion of $\mathbb{R}^2 \setminus \overline{B}_1(0)$ is not an Alexandrov space of curvature ≤ 0. Now we show the latter half of our main theorem in this chapter. Recall that we consider a Riemannian manifold (M, g) such that $\overline{M}$ is compact, and a connected, complete Alexandrov space (X, d_X) of curvature ≤ 0.

Theorem 5.1.8 *Any totally geodesic map $u : M \longrightarrow X$ is E^C-continuously harmonic.*

Proof. All discussions in the proof of Theorem 5.1.4 before using the first variation formula are also applicable to this case. Using Proposition 5.1.7 in place of that formula, we have $E^C(u) \leq E^C(\beta'_{\delta'})$, which completes the proof. □

5.2 Examples

We can construct many examples of harmonic maps whose images are not manifolds by applying Theorems 5.1.4 and 5.1.8.

Example 5.2.1 Let (M, g) be a Riemannian manifold with nonpositive sectional curvature such that $\overline{M}$ is an Alexandrov space of curvature ≤ 0. Take two totally geodesic submanifolds S_1 and S_2 of M with $\mathrm{dist}(S_1, S_2) > 0$ such that there exists an isometry $i : S_1 \longrightarrow S_2$ between them. Let $\bar{i} : \overline{S_1} \longrightarrow \overline{S_2}$ be the canonical extension and set $X := \overline{M}/\bar{i}$. Then X is a complete Alexandrov space of curvature ≤ 0 by Reshetnyak's gluing theorem ([Re], see also [BBI, Theorem 9.1.21]), and the projection $P : M \longrightarrow X$ is totally geodesic. Thus P is E^{KS}- and E^C-continuously harmonic.

We finally give two examples of totally geodesic maps which are not harmonic.

Example 5.2.2 Let $(X, d_X) = (\mathbb{R}^2, \|\cdot\|)$, where we set, for $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$,

$$\|(x_1, y_1) - (x_2, y_2)\| := |x_1 - x_2| + |y_1 - y_2|.$$

This is not a CAT(0)-space, but the conclusion of Lemma 5.1.3 remains true by setting $\beta_t = (1-t)\alpha_0 + t\alpha_1$. However, there exist nonconstant closed geodesics in X (e.g., squares) and they are not minimal.

Example 5.2.3 Let $B_r(p_0;\mathbb{R}^2)$ and $B_r(p_1;\mathbb{R}^2)$ be open disks in $\mathbb{R}^2$ and set

$$X := B_r(p_0;\mathbb{R}^2) \cup [0,1] \cup B_r(p_1;\mathbb{R}^2)/\sim,$$

where $0 \sim p_0$ and $1 \sim p_1$. Intuitively, X is in the shape of a barbell. Let d_X be the induced length metric on X, then (X, d_X) is a CAT(0)-space. Define a measure μ on X by $\mu = \mathcal{L}^2$ on $B_r(p_0;\mathbb{R}^2) \cup B_r(p_1;\mathbb{R}^2)$ and $\mu = \mathcal{L}^1$ on $[0,1]$, where $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure. We shall prove that the identity map $\mathrm{id}_X : (X, d_X, \mu) \longrightarrow (X, d_X)$ is not E^{KS}- nor E^C-continuously harmonic although id_X is clearly totally geodesic. Here we consider $\partial X = \partial B_r(p_0;\mathbb{R}^2) \cup \partial B_r(p_1;\mathbb{R}^2)$.

Claim 5.2.4 *The map* id_X *is not* E^{KS}*- nor* E^C*-continuously harmonic. Furthermore, there exists no* E^{KS}*- nor* E^C*-continuous harmonic map in the homotopy class fixing boundary values to which* id_X *belongs.*

Proof. Fix a small $\varepsilon \in (0, r)$ and set

$$X' := B_\varepsilon(p_0;\mathbb{R}^2) \cup [0,1] \cup B_\varepsilon(p_1;\mathbb{R}^2)/\sim \subset X.$$

For $(a,b) \in [0,1) \times (0,1) \cup \{1\} \times [0,1)$, we define a map $u_{a,b} : X \longrightarrow X$ by (see Figure 7)

(i) $u_{a,b}(t) := at + \dfrac{1-a}{2} \in [0,1]$ for $t \in [0,1]$;

(ii) $u_{a,b}(p_0 + v) := \dfrac{1-a}{2} - \dfrac{1-a}{2b\varepsilon}|v| \in [0,1]$ for $v \in B_{b\varepsilon}(0;\mathbb{R}^2)$;
$u_{a,b}(p_1 + v) := \dfrac{1+a}{2} + \dfrac{1-a}{2b\varepsilon}|v| \in [0,1]$ for $v \in B_{b\varepsilon}(0;\mathbb{R}^2)$;

(iii-1) $u_{a,b}(p_j + v) := p_j + \left\{\varepsilon - \dfrac{\varepsilon - |v|}{1-b}\right\}\dfrac{v}{|v|} \in B_\varepsilon(p_j;\mathbb{R}^2)$ for $v \in B_\varepsilon(0;\mathbb{R}^2) \setminus B_{b\varepsilon}(0;\mathbb{R}^2)$, $j = 0, 1$;

(iii-2) $u_{a,b} := \mathrm{id}_X$ on $X \setminus X'$.

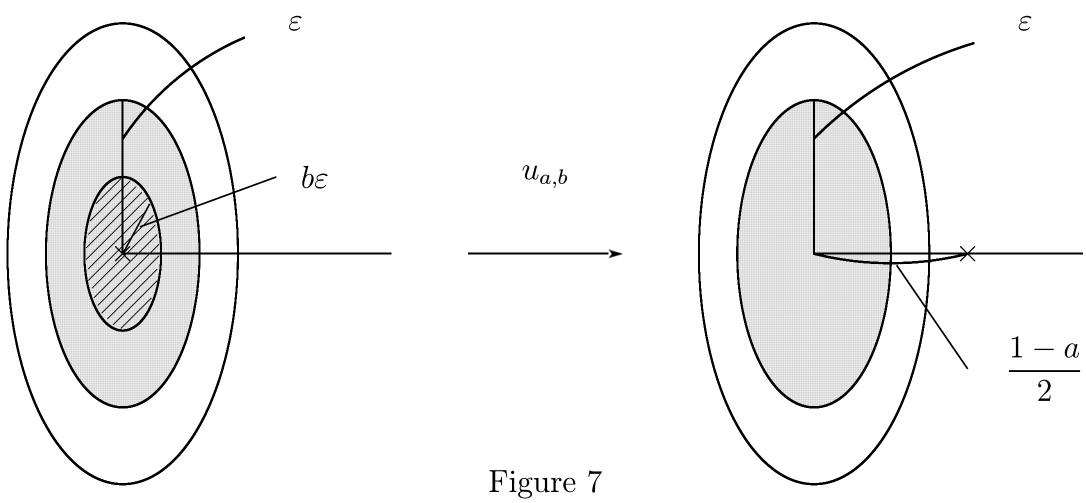

Figure 7

Note that $u_{1,0} = \mathrm{id}_X$ and the map $[0,1] \ni s \longmapsto u_{1-(1-a)s,bs}$ gives a homotopy between id_X and $u_{a,b}$ fixing boundary values. It holds that

$$E^{KS}(\mathrm{id}_X|_{X'}) = \frac{1}{3} + \frac{1}{4} \cdot 2 \cdot 2\pi\varepsilon^2 = \frac{1}{3} + \pi\varepsilon^2$$

and $E^C(\mathrm{id}_X|_{X'}) = 1 + 2\pi\varepsilon^2$. On the other hand,

$$\begin{aligned} E^{KS}(u_{a,b}|_{X'}) &\le \frac{1}{3}a^2 + \frac{1}{4}\left(\frac{1-a}{2b\varepsilon}\right)^2 \cdot 2\pi(b\varepsilon)^2 + \frac{1}{4}\left\{\left(\frac{1}{1-b}\right)^2 + 1\right\} \cdot 2\pi\varepsilon^2(1-b^2) \\ &= \frac{1}{3}a^2 + \frac{\pi}{8}(1-a)^2 + \frac{\pi\varepsilon^2}{2}\left\{\frac{1+b}{1-b} + (1-b^2)\right\}, \\ E^C(u_{a,b}|_{X'}) &= a^2 + \left(\frac{1-a}{2b\varepsilon}\right)^2 \cdot 2\pi(b\varepsilon)^2 + \left(\frac{1}{1-b}\right)^2 \cdot 2\pi\varepsilon^2(1-b^2) \\ &= a^2 + \frac{\pi}{2}(1-a)^2 + 2\pi\varepsilon^2\frac{1+b}{1-b}. \end{aligned}$$

Putting $u = u_{1/2,1/2}$, we have, for sufficiently small $\varepsilon > 0$,

$$E^{KS}(u|_{X'}) \le \frac{1}{12} + \frac{\pi}{32} + \frac{15}{8}\pi\varepsilon^2 < \frac{1}{3}, \quad E^C(u|_{X'}) = \frac{1}{4} + \frac{\pi}{8} + 6\pi\varepsilon^2 < 1.$$

Thus id_X is not E^{KS}- nor E^C-continuously harmonic.

We suppose that there exists a continuous harmonic map $u : X \longrightarrow X$ which is homotopic to id_X fixing boundary values, and will derive a contradiction. The following discussion is common to both E^{KS} and E^C, so that we write E simply. By the harmonicity of $\mathrm{id}_{B_r(p_j;\mathbb{R}^2)}$ (and Proposition 5.1.7), $E(u|_{B_r(p_j;\mathbb{R}^2)})$ takes its minimum if $u|_{B_r(p_j;\mathbb{R}^2)} = \mathrm{id}_{B_r(p_j;\mathbb{R}^2)}$. This implies that $u([0,1]) \subset [0,1]$. Let $A_j := u^{-1}([0,1]) \cap B_r(p_j;\mathbb{R}^2)$ for $j = 0,1$ and divide u into five parts:

$$u_1 : (0,1) \longrightarrow (0,1), \quad u_{2,j} : A_j \longrightarrow [0,1], \quad u_{3,j} : B_r(p_j;\mathbb{R}^2) \setminus A_j \longrightarrow B_r(p_j;\mathbb{R}^2).$$

Note that $u_1, u_{2,j}$, and $u_{3,j}$ correspond to (i), (ii), and (iii) respectively in the first part of this proof. For $b \in (0,1)$, we set $bA_j := \{p_j + bv \mid p_j + v \in A_j\}$, and $u_{2,j}^b(p_j + v) := u_{2,j}(p_j + v/b)$ for $p_j + v \in bA_j$. Then we have $E(u_{2,j}|_{A_j}) = E(u_{2,j}^b|_{bA_j})$. (Recall that $E(u_{a,b}|_{B_{b\varepsilon}(p_j;\mathbb{R}^2)})$ is independent of b.) On the other hand, if we put

$$u'_{3,j} := \begin{cases} u_{3,j} & \text{on } B_r(p_j;\mathbb{R}^2) \setminus A_j, \\ p_j & \text{on } A_j, \end{cases}$$

then we find $E(u'_{3,j}) = E(u_{3,j})$. Hence $E(u_{3,j})$ is minimal if $A_j = \{p_j\}$ and $u_{3,j} = \mathrm{id}_{B_r(p_j;\mathbb{R}^2)\setminus\{p_j\}}$. (Recall that $E(u_{a,b}|_{B_\varepsilon(p_j;\mathbb{R}^2)\setminus B_{b\varepsilon}(p_j;\mathbb{R}^2)})$ is minimal at $b = 0$.) Combining this with $E(u_{2,j}|_{A_j}) = E(u_{2,j}^b|_{bA_j})$, we conclude that $A_j = \{p_j\}$ and $u_{3,j} = \mathrm{id}_{B_r(p_j;\mathbb{R}^2)\setminus\{p_j\}}$. However, then u must be id_X. This is a contradiction. $\square$

The second part of the claim above asserts that a harmonic map in a class containing id_X does not have a continuous representation. This should be compared to the works on the regularity of harmonic maps (see Remark 5.1.2). If we consider $B_r(p_j;\mathbb{R}^2)$ as an open ball in a flat torus, then (X, d_X) is a compact, geodesically complete Alexandrov space of curvature ≤ 0.

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