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Stability and instability of standing waves for nonlinear Schrödinger equations

by

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A thesis presented

by

Reika Fukuizumi

to

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Sendai, Japan

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CHAPTER 1

Introduction

In this thesis, we study the following nonlinear Schrödinger equations:

$$i\partial_t u = -\Delta u + V(x)u + g(u), \qquad (t,x) \in \mathbb{R}^{1+n}, \tag{1.1}$$

where $V(x)$ is a real valued function and $g(0) = 0$, $g(e^{i\theta}z) = e^{i\theta}g(z)$ for $z \in \mathbb{C}$ and $\theta \in \mathbb{R}$.

When $V(x) \equiv 0$, (1.1) arises in various physical contexts such as nonlinear optics and plasma physics (see, e.g., [**11, 52, 56**]). The nonlinearity $g(u)$ enters due to the effect of changes in the field intensity on the wave propagation characteristics of the medium. When $V(x) \equiv 0$ and $g(u) = -|u|^2u + |u|^4u$ with $n = 3$, (1.1) appears in boson gas interaction, and so on (see, e.g., [**2**] and its references). The potential $V(x)$ can be thought of as modeling inhomogeneities in the medium. In [**48**], Equation (1.1) with a bounded potential $V(x)$ is studied as a model proposed to describe the local dynamics at a nucleation site. Equation (1.1) with a harmonic potential $V(x) = |x|^2$, $g(u) = -|u|^2u$ and $n = 3$ is known as a model for describing the Bose-Einstein condensate with attractive inter-particle interactions under a magnetic trap (see, e.g., [**3, 27, 53**]).

By a standing wave, we mean a solution of (1.1) of the form

$$u_\omega(t,x) = e^{i\omega t}\phi_\omega(x),$$

where $\omega \in \mathbb{R}$, and $\phi_\omega(x)$ is a ground state of the following stationary problem

$$\begin{cases} -\Delta\phi + V(x)\phi + \omega\phi + g(\phi) = 0, & x \in \mathbb{R}^n, \\ \phi \in X, \qquad \phi \not\equiv 0. \end{cases} \tag{1.2}$$

For the terminology "ground state", see Definition I below. Here, X is a real Hilbert space defined by

$$X := \{v \in H^1(\mathbb{R}^n, \mathbb{C})\ ; V(x)|v(x)|^2 \in L^1(\mathbb{R}^n)\}$$

with the inner product

$$(v, w)_X := \mathrm{Re} \int_{\mathbb{R}^n} (v(x)\overline{w(x)} + \nabla v(x) \cdot \overline{\nabla w(x)} + V(x)v(x)\overline{w(x)})dx.$$

The norm of X is denoted by $\|\cdot\|_X$, and from now on, $\|\cdot\|_r$ stands for the norm of $L^r(\mathbb{R}^n)$.

In this thesis, we consider the stability and instability of standing wave solutions of (1.1). We first recall relevant known results. Many authors have been studying the problem of stability and instability of standing waves for nonlinear Schrödinger equations (see, e.g., [**4, 8, 9, 13, 16, 18, 25, 28, 29, 31, 39, 40, 41, 42, 43, 44, 48, 50, 54, 55, 57, 58**]). First, we consider the case $V(x) \equiv 0$ and $g(u) = -|u|^{p-1}u$, namely,

$$i\partial_t u = -\Delta u - |u|^{p-1}u, \qquad (t, x) \in \mathbb{R}^{1+n}, \tag{1.3}$$

where $1 < p < 2^* - 1$. Here, we put $2^* = \infty$ if $n = 1, 2$, and $2^* = 2n/(n-2)$ if $n \geq 3$.

For $\omega > 0$, there exists a unique positive radial solution $\psi_\omega(x)$ of

$$\begin{cases} -\Delta\psi + \omega\psi - |\psi|^{p-1}\psi = 0, & x \in \mathbb{R}^n, \\ \psi \in H^1(\mathbb{R}^n), & \psi \not\equiv 0 \end{cases} \tag{1.4}$$

(see Strauss [**51**] and Berestycki and Lions [**5**] for the existence, and Kwong [**34**] for the uniqueness). It is known that a positive solution of (1.4) is a ground state. In [**9**] Cazenave and Lions proved that if $p < 1 + 4/n$, then the standing wave solution $e^{i\omega t}\psi_\omega(x)$ is stable for any $\omega > 0$. On the other hand, it is shown that if $p \geq 1 + 4/n$, then the standing wave solution $e^{i\omega t}\psi_\omega(x)$ is unstable for any $\omega > 0$ (see Berestycki and Cazenave [**4**] for $p > 1 + 4/n$, and Weinstein [**54**] for $p = 1 + 4/n$).

Regarding nonlinear Schrödinger equations with other nonlinearities, or other nonlinear evolution equations such as nonlinear Klein–Gordon equations, Grillakis, Shatah and Strauss [**28, 29**] gave a virtually necessary and sufficient condition for the stability and instability of stationary states for the Hamiltonian systems under certain assumptions. By virtue of the abstract theory in Grillakis, Shatah and Strauss [**28, 29**], under some assumptions on the spectrum of linearized operators, $e^{i\omega_1 t}\phi_{\omega_1}(x)$ is stable (resp. unstable) if the function $\|\phi_\omega\|_2^2$ is strictly increasing (resp. decreasing) at $\omega = \omega_1$. In the case where $V(x) \equiv 0$ and $g(u) = -|u|^{p-1}u$, by means of the scaling $\psi_\omega(x) = \omega^{1/(p-1)}\psi_1(\sqrt{\omega}x)$, it is easily checked if $\|\psi_\omega\|_2^2$ increases or decreases. However, it seems difficult to check this property of $\|\phi_\omega\|_2^2$ for general $V(x)$ and $g(u)$.

When $-\Delta + V(x)$ has the first eigenvalue λ_1, Rose and Weinstein [**48**] claimed that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1.1) is stable for ω such that $\omega > -\lambda_1$ and sufficiently close to $-\lambda_1$. To verify the stability condition of Grillakis, Shatah and Strauss, they investigated the behavior of the function $\|\phi_\omega\|_2^2$ of ω near $-\lambda_1$ using the standard bifurcation theory. However, it seems likely that there remains a possibility of oscillations of $\|\phi_\omega\|_2^2$ and that the extraction of a sequence $\{\omega_k\}$ is required. For the case of $V(x) = |x|^2$ and $g(u) = -|u|^{p-1}u$, the author [**18**] proved that there exists a sequence $\{\omega_k\}$ approaching $-\lambda_1$, for which $e^{i\omega_k t}\phi_{\omega_k}(x)$ is stable. If we prove the stability for any $\omega > -\lambda_1$ near $-\lambda_1$ without choosing a sequence $\{\omega_k\}$, the standard bifurcation theory would require nonlinearity to be regular enough, for example, $p \geq 3$ (see, Remark 4.4 of [**18**]).

Also, there have been several papers on the stability properties of standing waves for a nonlinear Schrödinger equation with a potential which has a small parameter $h > 0$ (see, Oh [**39**], Example C of Section 6 in Grillakis, Shatah and Strauss [**28**]). However, the argument in [**28**] and [**39**] could not be applied to our present case. Indeed, their perturbation methods require the potential to be bounded at infinity, because they use the convergence of linearized operators in the strong resolvent sense.

When $V(x) \equiv 0$, $g(u) = -|u|^{p-1}u - b|u|^{q-1}u$ with $b \in \mathbb{R}$ and $n = 1$, Ohta [**40**] proved that if $1 < p < 5 < q < \infty$, the standing wave solution of (1.1) is stable for sufficiently small $\omega > 0$. For the case of $n = 1$, Equation (1.2) has explicit solutions and we could analyse ground states easily. For the case where $b > 0$ and $n \geq 2$, under the assumption that $\omega \mapsto \phi_\omega$ is a C^1 mapping, Ohta (Remark 1.7 of [**41**]) showed that if $p < 1 + 4/n$, then there exists a sequence $\{\omega_k\}$ approaching 0, for which $e^{i\omega_k t}\phi_{\omega_k}$ is stable.

Before we state our theorems, we give several precise definitions.

Definition I. We define the action functional on X by

$$S_\omega(v) := \frac{1}{2}\|\nabla v\|_2^2 + \frac{\omega}{2}\|v\|_2^2 + \int_{\mathbb{R}^n} V(x)|v(x)|^2 dx + \int_{\mathbb{R}^n} G(v(x))dx,$$

where $G(z) = \int_0^{|z|} g(s)ds$ for $z \in \mathbb{C}$. We will impose suitable conditions on $g(u)$ in each chapter in order for the last integral of $S_\omega(v)$ to be well-defined.

We denote the set of solutions for (1.2) by

$$\mathcal{X}_\omega = \{v \in H^1(\mathbb{R}^n) \ ; \ S'_\omega(v) = 0, v \neq 0\}$$

and the set of the ground states for (1.2) by

$$\mathcal{G}_\omega = \{\phi \in \mathcal{X}_\omega \ ; \ S_\omega(\phi) \leq S_\omega(v) \text{ for all } v \in \mathcal{X}_\omega\}.$$

The stability and instability of standing wave solutions are formulated as follows.

Definition II. Let T_V be the maximal linear subspace of $\mathbb{R}^n$ contained in $\{y \in \mathbb{R}^n \ ; \ V(x+y) = V(x), \ x \in \mathbb{R}^n\}$, and for $\phi_\omega \in \mathcal{G}_\omega$, we put

$$U^\delta(\phi_\omega) := \left\{v \in X \ ; \ \inf\{\|v - e^{i\theta}\phi_\omega(\cdot + y)\|_X \ ; \ \theta \in \mathbb{R}, \ y \in T_V\} < \delta\right\}.$$

We say that a standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1.1) is stable in X if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for each $u_0 \in U_\delta(\phi_\omega)$, the solution $u(t)$ of (1.1) with $u(0) = u_0$ satisfies $u(t) \in U_\varepsilon(\phi_\omega)$ for all $t \geq 0$. Otherwise, $e^{i\omega t}\phi_\omega(x)$ is said to be unstable in X.

In Chapters 2 and 3, we study (1.1) with $g(u) = -|u|^{p-1}u$. For simplicity, here we state our results only in the case of $V(x) = |x|^2$. In this case, $T_V = \{0\}$ in Definition II. For the case where $V(x)$ is more general, see Theorems 2.1, 3.1 and 3.2 in Chapters 2 and 3. For the case where $V(x) = |x|^2$, the time local well-posedness for the Cauchy problem to (1.1) in X, the conservation of energy and $L^2(\mathbb{R}^n)$-norm, and the virial identity hold (see Oh [**38**] and Section 6.4, Theorem 9.2.5 and Remark 9.2.9 of Cazenave [**7**]). Namely, we have the following proposition.

PROPOSITION I. *For any* $u_0 \in X$, *there exist* $T = T(\|u_0\|_X) > 0$ *and a unique solution* $u(t) \in C([0,T], X)$ *of* (1.1) *with* $u(0) = u_0$ *satisfying*

$$E(u(t)) = E(u_0), \quad Q(u(t)) = Q(u_0), \quad t \in [0,T],$$

where

$$E(v) := \frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{2}\|xv\|_2^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1},$$
$$Q(v) := \frac{1}{2}\|v\|_2^2.$$

In addition, if $u_0 \in X$ *satisfies* $|x|u_0 \in L^2(\mathbb{R}^n)$, *then the virial identity*

$$\frac{d^2}{dt^2}\|xu(t)\|_2^2 = 8P(u(t)) \tag{1.5}$$

holds for $t \in [0,T]$, *where*

$$P(v) := \|\nabla v\|_2^2 - \|xv\|_2^2 - \frac{n(p-1)}{2(p+1)}\|v\|_{p+1}^{p+1}.$$

The existence of ground states is proved by the standard variational argument, since the embedding $X \subset L^2(\mathbb{R}^n)$ is compact (see the author [**17**], Kavian and Weissler [**33**]).

PROPOSITION II. *Let*

$$\lambda_1 := \inf\left\{\|\nabla v\|_2^2 + \|xv\|_2^2 \ ; \ v \in X, \ \|v\|_2 = 1\right\}. \tag{1.6}$$

Then $\mathcal{G}_\omega$ *is not empty for any* $\omega \in (-\lambda_1, \infty)$.

The following three theorems are our main results in Chapters 2 and 3.

THEOREM I. *Assume* $1 < p < 1 + 4/n$. *Let* $\phi_\omega \in \mathcal{G}_\omega$. *Then there exists* $\omega_* = \omega_*(n,p) > 0$ *such that the standing wave solution* $e^{i\omega t}\phi_\omega(x)$ *of* (1.1) *is stable in* X_{rad} *for any* $\omega \in (\omega_*, \infty)$, *where* $X_{\mathrm{rad}} := \{v \in X \ ; \ v(|x|) = v(x), \ x \in \mathbb{R}^n\}$.

THEOREM II. *Assume* $1 < p < 2^* - 1$. *Let* $\phi_\omega \in \mathcal{G}_\omega$. *Then there exists* $\omega^* = \omega^*(n,p) > -\lambda_1$ *such that the standing wave solution* $e^{i\omega t}\phi_\omega(x)$ *of* (1.1) *is stable in* X *for any* $\omega \in (-\lambda_1, \omega^*)$.

Theorem III. *Assume $1+4/n<p<2^*-1$. Let $\phi_\omega\in\mathcal{G}_\omega$. Then there exists $\omega_\star=\omega_\star(n,p)>0$ such that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of* (1.1) *is unstable in X for any $\omega\in(\omega_\star,\infty)$.*

For a bounded potential $V(x)$, Rose and Weinstein [**48**] studied by numerical simulations that if ω is sufficiently large and $p>1+4/n$, then $\|\phi_\omega\|_2^2$ would decrease, while if $p<1+4/n$, then $\|\phi_\omega\|_2^2$ would increase for large ω. From Theorems I and III we can affirm with mathematical precision that this numerical result is true. Furthermore, Theorem II gives an improvement of the results by Rose and Weinstein [**48**] and by the author [**18**], since we do not need to extract a sequence $\{\omega_k\}$. Moreover, for the nonlinear Schrödinger equation (2.24) with a constant magnetic field, Gonçalves Ribeiro [**25**] showed that if $\omega>0$ and $p_0(3):=1+4/3+(4\sqrt{10}-8)/9\le p<5$, then the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (2.24) is unstable in $H^1_{A,0}(\mathbb{R}^3)$ (see Chapter 2, Section 4). In [**18**], the author proved for the case $V(x)=|x|^2$ that if $\omega>0$ and $p\ge p_0(n):=(n^2+4+4\sqrt{n^2+1})/n^2$, then the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1.1) is unstable. Here, we note that $1+4/n<p_0(n)<2^*-1$, so that Theorem III also gives an improvement of the results in [**18**] and [**25**].

In Chapter 4, we consider (1.1) with $V(x)\equiv 0$. For simplicity, here we state the result only in the case of $g(u)=-|u|^{p-1}u-b|u|^{q-1}u$ and $b\in\mathbb{R}$. For the case of more general $g(u)$, see Theorem 4.1 in Chapter 4. In the present case, $T_V=\mathbb{R}^n$. The time local well-posedness for the Cauchy problem to (1.1) in X has been established by Kato [**32**] (see also, [**10, 24**]) and we have the conservation of energy and $L^2(\mathbb{R}^n)$-norm. Namely, the following proposition holds.

Proposition III. *For any $u_0\in X$, there exist $T=T(\|u_0\|_X)>0$ and a unique solution $u(t)\in C([0,T],X)$ of* (1.1) *with $u(0)=u_0$ satisfying*

$$E(u(t))=E(u_0),\quad Q(u(t))=Q(u_0),\quad t\in[0,T],$$

where

$$E(v):=\frac{1}{2}\|\nabla v\|_2^2-\frac{1}{p+1}\|v\|_{p+1}^{p+1}-\frac{b}{q+1}\|v\|_{q+1}^{q+1},$$
$$Q(v):=\frac{1}{2}\|v\|_2^2.$$

The existence of ground states follows from the result by Berestycki and Lions [**5**].

PROPOSITION IV. *Let*

$$\omega_0 := \sup\left\{\omega > 0\ ;\ \frac{\omega}{2}s^2 - \frac{1}{p+1}s^{p+1} - \frac{b}{q+1}s^{q+1} < 0 \quad \textit{for some} \ \ s > 0\right\}.$$

Then $\mathcal{G}_\omega$ is not empty for any $\omega \in (0, \omega_0)$.

Our main result in Chapter 4 is the following

THEOREM IV. *Assume $n \geq 3$ and $1 < p < q < 2^*-1$. Let $\phi_\omega \in \mathcal{G}_\omega$ and $p < 1+4/n$. Then there exists $\omega^\star = \omega^\star(n, p, q, b) \in (0, \omega_0)$ such that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of* (1.1) *is stable in $H^1(\mathbb{R}^n)$ for any $\omega \in (0, \omega^\star)$.*

By Theorem IV, we extend the result of Ohta [**40**] to higher spatial dimensions $n \geq 3$. We can also prove the same result for the generalized Davey-Stewartson system, namely $g(u) = -|u|^{p-1}u - E_1(|u|^2)u$ with $n = 3$, where E_1 is a singular integral operator with symbol $\sigma_1(\xi) = \xi_1^2/|\xi|^2$ for $\xi \in \mathbb{R}^n$, although such nonlinearity is nonlocal. Concerning the Davey-Stewartson system, under the assumption that $\omega \mapsto \phi_\omega$ is a C^1 mapping, Ohta [**41**] showed that if $p < 1 + 4/n$, then there exists a sequence $\{\omega_k\}$ approaching 0, for which $e^{i\omega_k t}\phi_{\omega_k}$ is stable. Our proof gives an improvement of this result in the sense that we have no need to extract a sequence $\{\omega_k\}$ and do not assume the regularity of ϕ_ω with respect to ω. In Section 5 of Chapter 4, we state briefly the result for the generalized Davey-Stewartson system.

For the proof of Theorem III, we use the virial identity (1.5) and the following sufficient condition for instability, which is a modification of Theorem 3 in Ohta [**42**] (see also [**18, 25, 50**]).

PROPOSITION V. *Let $1 < p < 2^* - 1$ and $\phi_\omega \in \mathcal{G}_\omega$. If $\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < 0$, then the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of* (1.1) *is unstable in X. Here, $v^\lambda(x) := \lambda^{n/2}v(\lambda x)$ for $\lambda > 0$.*

Since $\|v^\lambda\|_2^2 = \|v\|_2^2$, the assumption $\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < 0$ means that $\phi_\omega(x)$ is not a local minimizer on $\{v \in X \ ; \ \|v\|_2 = \|\phi_\omega\|_2\}$.

For proving Theorems I, II and IV, we use the following sufficient condition. We emphasize that it suffices to prove the coerciveness of the linearized operators and there is no need to study the spectra of linearized operators, for example, the simpleness of zero and negative eigenvalues.

PROPOSITION VI. *Assume* $1 < p < 2^* - 1$ *and* $\phi_\omega \in \mathcal{G}_\omega$. *If there exists* $\delta > 0$ *such that*

$$\langle S''_\omega(\phi_\omega)v, v\rangle \geq \delta \|v\|_X^2 \tag{1.7}$$

for any $v \in X$ *satisfying* $\mathrm{Re}(\phi_\omega, v)_{L^2} = 0$ *and* $\mathrm{Re}(i\phi_\omega, v)_{L^2} = 0$, (*for Theorem* IV, *in addition,* $\mathrm{Re}(\partial_l\phi_\omega, v)_{L^2} = 0$, $l = 1, \cdots, n$), *then the standing wave solution* $e^{i\omega t}\phi_\omega(x)$ *of* (1.1) *is stable in* X.

In Proposition VI, the condition $\mathrm{Re}(\phi_\omega, v)_{L^2} = 0$ is related to the conservation of the charge Q. Indeed, we have $\langle Q'(\phi_\omega), v\rangle = \mathrm{Re}(\phi_\omega, v)_{L^2}$. Moreover, it follows from $S'_\omega(e^{i\theta}\phi_\omega) = 0$ for $\theta \in \mathbb{R}$ and $S'_\omega(\phi_\omega(\cdot + y)) = 0$ for $y \in \mathbb{R}^n$ that $S''_\omega(\phi_\omega)i\phi_\omega = 0$ and $S''_\omega(\phi_\omega)\partial_l\phi_\omega = 0$ for $l = 1, \cdots, n$. Accordingly, the condition (1.7) does not hold if we do not restrict $v \in X$ to satisfy $\mathrm{Re}(i\phi_\omega, v)_{L^2} = 0$ and $\mathrm{Re}(\partial_l\phi_\omega, v)_{L^2} = 0$ for $l = 1, \cdots, n$.

Proposition VI means that if the action $S_\omega(v)$ is minimized at $v = \phi_\omega$ on the hypersurface $\{v \in X \ ; \|v\|_2 = \|\phi_\omega\|_2\}$, then the standing wave solution $e^{i\omega t}\phi_\omega(x)$ is stable.

By applying this sufficient condition, we may avoid such delicate difficulty that we have to choose a sequence $\{\omega_k\}$. Nevertheless, it is not easy to verify the condition (1.7) directly. Therefore, we first study a limiting problem. We investigate the rescaling limit of $\phi_\omega(x)$ as $\omega \to \infty$ or $\omega \to -\lambda_1$, which is based on the spirit of the analysis in Esteban and Strauss [**16**]. They treated the equation (1.1) with $V(x) \equiv 0$ and $g(u) = -|u|^{p-1}u$ for $|x| > R > 0$ with a Neumann boundary condition on $|x| = R$. Among others, they proved that the instability range of p in the case of the whole space is still valid for the exterior domains when R is sufficiently large or small. We show that as $\omega \to \infty$, the rescaled function $\tilde{\phi}_\omega(x)$ defined by $\phi_\omega(x) = \omega^{1/(p-1)}\tilde{\phi}_\omega(\sqrt{\omega}x)$ tends to a unique positive radial solution $\psi_1(x)$ of (1.4) with $\omega = 1$. While, as $\omega \to -\lambda_1$, $\phi_\omega(x)/\|\phi_\omega\|_2^2$ converges to the first eigenfunction Φ corresponding to the firsteigenvalue λ_1 of (1.6). From known stability properties of $\psi_1(x)$ and Φ, we are able to prove (1.7) as the limit case. Here, we remark that if we let $\omega \to \infty$ for the case $V(x) = |x|^2$ and $g(u) = -|u|^{p-1}u$, we lose control of the nullspace of the linearized operator $S''_\omega(\phi_\omega)$, since (1.1) with $V(x) \equiv 0$ has the invariance of translations, although (1.1) does not have the corresponding invariance in general. This is the reason why we assume the functional space to be radial space X_{rad} in Theorem I. The assumption on the space X_{rad} can be somewhat weakened (see Chapter 3 for details).

The assumptions in Proposition VI are slightly different from those in Lemma 4.5 of [**29**] or Theorem 3.4 of [**28**]. Indeed, applying Lemma 4.5 of [**29**] directly to our case, $\text{Re}(i\phi_\omega, v)_{L^2} = 0$ and $\text{Re}(\partial_l\phi_\omega, v)_{L^2} = 0$ for $l = 1, \cdots, n$ are replaced by $(i\phi_\omega, v)_X = 0$ and $(\partial_l\phi_\omega, v)_X = 0$ for $l = 1, \cdots, n$ in Proposition VI. If we apply Proposition VI with $(i\phi_\omega, v)_X = 0$ and $(\partial_l\phi_\omega, v)_X = 0$ for $l = 1, \cdots, n$, we need more detailed convergence property of $\phi_\omega(x)$ to $\psi_1(x)$ than those of Lemma 3.3 in Chapter 3 and Lemma 4.2 in Chapter 4. That is why such weaker restrictions to $v \in X$ as in Proposition VI is more convenient for us.

Lastly, in Appendix, we consider the case where $V(x) \equiv 0$ and $g(u) = -|u|^{p-1}u + |u|^{q-1}u$. In this case, there exists a ground state for any $\omega \in (0, \omega_0)$ by Proposition IV. We note that $\omega_0 < \infty$. Anderson [**1**] and Shatah [**51**] showed that there are stable standing waves for ω close to ω_0 with $p = 3$, $q = 5$ and $n = 3$. However, it is not apparent from their proof that the standing wave solution is stable for any ω close to ω_0. In that point, Ohta [**40**] proved in one dimensional case that for $1 < p < q < \infty$, the standing wave solution is stable for any ω close to ω_0. However, the case $n \geq 2$ remains open. We consider the limiting problem corresponding to this case and prove that $\phi_\omega(x)$ converges to a constant solution of the limiting problem in a certain sense. We hope that this will be helpful for future study of this problem.

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CHAPTER 2

Instability of standing waves for nonlinear Schrödinger equations with potentials

§ 2.1. Introduction

This chapter is devoted to a joint work with Masahito Ohta. In this chapter, we consider the case $g(u) = -|u|^{p-1}u$ in (1.1), namely,

$$i\partial_t u = -\Delta u + V(x)u - |u|^{p-1}u, \qquad (t,x) \in \mathbb{R}^{1+n}, \tag{2.1}$$

where $1 < p < 2^* - 1$. Here, we put $2^* = \infty$ if $n = 1, 2$, and $2^* = 2n/(n-2)$ if $n \geq 3$. For potential $V(x)$, we assume the following conditions (V0)–(V2).

(V0) There exist real valued functions $V_1(x)$ and $V_2(x)$ such that $V(x) = V_1(x) + V_2(x)$.

(V1.1) $V_1(x) \in C^2(\mathbb{R}^n)$ and there exist positive constants m and C such that $0 \leq V_1(x) \leq C(1 + |x|^m)$ on $\mathbb{R}^n$.

(V1.2) There exists $C_\alpha > 0$ such that $|x^\alpha \partial_x^\alpha V_1(x)| \leq C_\alpha(1 + V_1(x))$ on $\mathbb{R}^n$ for $|\alpha| \leq 2$.

(V2) There exists q such that $q \geq 1$, $q > n/2$ and $x^\alpha \partial_x^\alpha V_2(x) \in L^q(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ for $|\alpha| \leq 2$.

Examples.

(i) (Harmonic potentials) For $c_1, \cdots, c_n \in \mathbb{R}$, $\displaystyle\sum_{j=1}^n c_j^2 x_j^2$ satisfies (V1.1) and (V1.2).

(ii) For $c \in \mathbb{R}$ and $0 < a < \min\{2, n\}$, $c|x|^{-a}$ satisfies (V2).

(iii) (V2) is satisfied if $U(x) \in C^2(\mathbb{R}^n)$ satisfies $|\partial_x^\alpha U(x)| \leq C_\alpha \langle x \rangle^{-|\alpha|}$ for $|\alpha| \leq 2$.

(iv) $1 + \sin x_1$ satisfies (V1.1), but does not satisfy (V1.2) nor (V2).

We define a real Hilbert space X by

$$X := \{v \in H^1(\mathbb{R}^n, \mathbb{C}) \,;\, V_1(x)|v(x)|^2 \in L^1(\mathbb{R}^n)\}$$

with the inner product

$$(v,w)_X := \mathrm{Re}\int_{\mathbb{R}^n}(v(x)\overline{w(x)} + \nabla v(x)\cdot\overline{\nabla w(x)} + V_1(x)v(x)\overline{w(x)})dx.$$

The norm of X is denoted by $\|\cdot\|_X$. Let G be a closed subgroup of $O(n)$ such that $V_1(x)$ and $V_2(x)$ are invariant under G, i.e., $V_j(gx) = V_j(x)$ for $g \in G$, $x \in \mathbb{R}^n$ and $j = 1, 2$. We define a closed subspace X_G of X by

$$X_G := \{v \in X\ ;\ v(gx) = v(x),\ g \in G,\ x \in \mathbb{R}^n\}.$$

We note that $X_G = X$ if $G = \{\mathrm{Id}\ (\text{identity matrix})\}$, and $X_G = X_{\mathrm{rad}}$ if $G = O(n)$, where

$$X_{\mathrm{rad}} = \{v \in X\ ;\ v(x) = v(|x|),\ x \in \mathbb{R}^n\}.$$

Moreover, we define the energy functional E on X_G by

$$E(v) := \frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^n} V(x)|v(x)|^2dx - \frac{1}{p+1}\|v\|_{p+1}^{p+1}.$$

We remark that by the assumptions (V2) and $1 < p < 2^* - 1$, the functional E is well-defined on X_G. We assume that the time local well-posedness for the Cauchy problem to (2.1) in X_G, the conservation of energy and $L^2(\mathbb{R}^n)$-norm, and the virial identity hold.

Assumption (A1). For any $u_0 \in X_G$, there exist $T = T(\|u_0\|_X) > 0$ and a unique solution $u(t) \in C([0,T], X_G)$ of (2.1) with $u(0) = u_0$ satisfying

$$E(u(t)) = E(u_0), \quad \|u(t)\|_2^2 = \|u_0\|_2^2, \quad t \in [0,T].$$

In addition, if $u_0 \in X_G$ satisfies $|x|u_0 \in L^2(\mathbb{R}^n)$, then the virial identity

$$\frac{d^2}{dt^2}\|xu(t)\|_2^2 = 8P(u(t)) \tag{2.2}$$

holds for $t \in [0,T]$, where

$$P(v) := \|\nabla v\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^n} x\cdot\nabla V(x)|v(x)|^2dx - \frac{n(p-1)}{2(p+1)}\|v\|_{p+1}^{p+1}. \tag{2.3}$$

Remark 2.1. The assumption (A1) is verified, if $V(x)$ satisfies the following conditions (A1.1)–(A1.2) with (V0) (see Section 6.4, Theorem 9.2.5 and Remark 9.2.9 of [**7**]).

(A1.1) $V_1(x) \in C^\infty(\mathbb{R}^n)$, $V_1(x) \geq 0$ in $\mathbb{R}^n$, $\partial_x^\alpha V_1(x) \in L^\infty(\mathbb{R}^n)$ for $|\alpha| \geq 2$, and there exists

$C > 0$ such that $|x \cdot \nabla V_1(x)| \leq C(|x|^2 + V_1(x))$ in $\mathbb{R}^n$.

(A1.2) $V_2(x) \in L^{q_0}(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ for some $q_0 \geq 1$, $q_0 > n/2$ and

$x \cdot \nabla V_2(x) \in L^{q_1}(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ for some $q_1 \geq 1$, $q_1 > n/2$.

Next, we consider the stationary problem (1.2) with $g(u) = -|u|^{p-1}u$,

$$\begin{cases} -\Delta\phi + V(x)\phi + \omega\phi - |\phi|^{p-1}\phi = 0, & x \in \mathbb{R}^n, \\ \phi \in X_G, \qquad \phi \not\equiv 0. \end{cases} \tag{2.4}$$

Definition 1. We define two functionals on X_G:

$$S_\omega(v) := E(v) + \frac{\omega}{2}\|v\|_2^2 \qquad \text{(action)},$$
$$I_\omega(v) := \|\nabla v\|_2^2 + \omega\|v\|_2^2 + \int_{\mathbb{R}^n} V(x)|v(x)|^2 dx - \|v\|_{p+1}^{p+1}.$$

Let $\mathcal{M}_\omega^G$ be the set of all minimizers for

$$\inf\{S_\omega(v) \ ; \ v \in X_G \setminus \{0\}, \ I_\omega(v) = 0\}. \tag{2.5}$$

Remark 2.2. (i) We note that

$$P(v) = \partial_\lambda S_\omega(v^\lambda)|_{\lambda=1}, \quad I_\omega(v) = \partial_\lambda S_\omega(\lambda v)|_{\lambda=1},$$

where $v^\lambda(x) := \lambda^{n/2}v(\lambda x)$ for $\lambda > 0$.

(ii) Let $\phi_\omega \in \mathcal{M}_\omega^G$. There exists a Lagrange multiplier $\Lambda \in \mathbb{R}$ such that $S_\omega'(\phi_\omega) = \Lambda I_\omega'(\phi_\omega)$. Taking the pairing of this equation with ϕ_ω, we obtain $\langle S_\omega'(\phi_\omega), \phi_\omega\rangle = \Lambda\langle I_\omega'(\phi_\omega), \phi_\omega\rangle$. Since $\langle S_\omega'(\phi_\omega), \phi_\omega\rangle = I_\omega(\phi_\omega) = 0$ and $\langle I_\omega'(\phi_\omega), \phi_\omega\rangle = -(p-1)\|\phi_\omega\|_{p+1}^{p+1} < 0$, we have $\Lambda = 0$. Namely, ϕ_ω satisfies (2.4). Moreover, for any $v \in X_G \setminus \{0\}$ satisfying $S_\omega'(v) = 0$, we have $I_\omega(v) = 0$. Thus, by the definition of $\mathcal{M}_\omega^G$, we have $S_\omega(\phi_\omega) \leq S_\omega(v)$. That is, $\phi_\omega \in \mathcal{M}_\omega^G$ is a minimal action solution of (2.4) (see more detailed remark in Remark 3.2 (ii)).

We also assume the existence of minimal action solutions of (2.4) for large ω.

Assumption (A2). There exists $\omega_0 \in (0, \infty)$ such that $\mathcal{M}_\omega^G$ is not empty and $\mathcal{M}_\omega^G \subset \{v \in X_G \ ; \ |x|v(x) \in L^2(\mathbb{R}^n)\}$ for any $\omega \in (\omega_0, \infty)$.

Remark 2.3. Some examples of $V(x)$ such that $\mathcal{M}_\omega^G$ is not empty are remarked in Chapter 3, Remark 3.3. The assumption $\mathcal{M}_\omega^G \subset \{v \in X_G \ ; \ |x|v(x) \in L^2(\mathbb{R}^n)\}$ is required to make use of the virial identity (2.2) in the proof of Proposition 1.1 below.

Definition 2. Let T_V be the maximal linear subspace of $\mathbb{R}^n$ contained in $\{y \in \mathbb{R}^n \ ; \ V(x+y) = V(x), \ x \in \mathbb{R}^n\}$, and for $\phi_\omega \in \mathcal{M}_\omega^G$, we put

$$N_\delta(\phi_\omega) := \left\{v \in X_G \ ; \ \inf\{\|v - e^{i\theta}\phi_\omega(\cdot + y)\|_X \ ; \ \theta \in \mathbb{R}, \ y \in T_V\} < \delta\right\}.$$

We say that a standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (2.1) is stable in X_G if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $u_0 \in N_\delta(\phi_\omega)$, the solution $u(t)$ of (2.1) with $u(0) = u_0$ satisfies $u(t) \in N_\varepsilon(\phi_\omega)$ for any $t \geq 0$. Otherwise, $e^{i\omega t}\phi_\omega(x)$ is said to be unstable in X_G.

Remark 2.4. Let $n = 3$, $c > 0$ and $V(x) = c(x_1^2 + x_2^2)$. In this case, we have $T_V = \{(0,0,z) \in \mathbb{R}^3 \ ; \ z \in \mathbb{R}\}$. This example will be used in Section 4.

Our main result in this chapter is the following.

Theorem 2.1. *Assume* $(V0)$–$(V2)$, $(A1)$ *and* $(A2)$. *Let* $1 + 4/n < p < 2^* - 1$ *and* $\phi_\omega(x) \in \mathcal{M}_\omega^G$. *Then there exists* $\omega_* = \omega_*(n,p) \in (\omega_0, \infty)$ *such that the standing wave solution* $e^{i\omega t}\phi_\omega(x)$ *of* (2.1) *is unstable in* X_G *for any* $\omega \in (\omega_*, \infty)$.

For the proof of Theorem 2.1, the following sufficient condition for instability is employed, which is a modification of Theorem 3 in [**42**] (see also [**18, 25, 50**]).

Proposition 2.1. *Assume* $(V0)$–$(V2)$, $(A1)$ *and* $(A2)$. *Let* $1 < p < 2^* - 1$ *and* $\phi_\omega(x) \in \mathcal{M}_\omega^G$. *If* $\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < 0$, *then the standing wave solution* $e^{i\omega t}\phi_\omega(x)$ *of* (2.1) *is unstable in* X_G. *Here,* $v^\lambda(x) := \lambda^{n/2}v(\lambda x)$ *for* $\lambda > 0$.

This chapter is organized as follows. In Section 2.2, we prove Theorem 2.1 using Proposition 2.1. The variational characterization of $\phi_\omega(x) \in \mathcal{M}_\omega^G$ and the rescaled function $\tilde{\phi_\omega}(x)$ defined by $\phi_\omega(x) = \omega^{1/(p-1)}\tilde{\phi_\omega}(\sqrt{\omega}x)$ play an important role in the proof of Theorem 2.1 (see Lemma 2.1). In Section 2.3, we give the proof of Proposition 2.1 following that of Theorem 3 in [**42**]. In Section 2.4, as an application of Theorem 2.1, we study the nonlinear Schrödinger equation (2.24) with a constant magnetic field, and improve the result in Gonçalves Ribeiro [**25**].

§ 2.2. Proof of Theorem 2.1

In this section, we prove Theorem 2.1 using Proposition 2.1, which will be proved in Section 3. By simple computations, we have

$$E(v^\lambda) = \frac{\lambda^2}{2}\|\nabla v\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^n} V\left(\frac{x}{\lambda}\right)|v(x)|^2dx - \frac{\lambda^{n(p-1)/2}}{p+1}\|v\|_{p+1}^{p+1},$$

$$\begin{aligned}\partial_\lambda^2 E(v^\lambda)|_{\lambda=1} &= \|\nabla v\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^n}\left\{2x\cdot\nabla V(x) + \sum_{j,k=1}^n x_jx_k\partial_j\partial_k V(x)\right\}|v(x)|^2dx \\ &\quad -\frac{n(p-1)}{2(p+1)}\left\{\frac{n(p-1)}{2}-1\right\}\|v\|_{p+1}^{p+1}.\end{aligned}$$

Since $P(\phi_\omega) = \partial_\lambda S_\omega(\phi_\omega^\lambda)|_{\lambda=1} = 0$ (see (2.3) and Remark 1.2), if we put

$$V^*(x) = 3x\cdot\nabla V(x) + \sum_{j,k=1}^n x_jx_k\partial_j\partial_k V(x), \tag{2.6}$$

then we have

$$\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} = \frac{1}{2}\int_{\mathbb{R}^n} V^*(x)|\phi_\omega(x)|^2dx - \frac{n(p-1)}{2(p+1)}\left\{\frac{n(p-1)}{2}-2\right\}\|\phi_\omega\|_{p+1}^{p+1}.$$

Thus, we see that the condition $\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < 0$ is equivalent to

$$\frac{\displaystyle\int_{\mathbb{R}^n} V^*(x)|\phi_\omega(x)|^2dx}{\|\phi_\omega\|_{p+1}^{p+1}} < \frac{n(p-1)\{n(p-1)-4\}}{2(p+1)}. \tag{2.7}$$

We remark that the right hand side of (2.7) is a positive constant by the assumption $p > 1+4/n$ in Theorem 2.1. In what follows, we will show that the left hand side of (2.7) converges to 0 as $\omega\to\infty$. To this end, we rescale $\phi_\omega(x)\in\mathcal{M}_\omega^G$ as follows:

$$\phi_\omega(x) = \omega^{1/(p-1)}\tilde{\phi_\omega}(\sqrt{\omega}x), \quad \omega\in(\omega_0,\infty). \tag{2.8}$$

Then, the rescaled function $\tilde{\phi_\omega}(x)$ satisfies

$$-\Delta\phi + \phi + \omega^{-1}V\left(\frac{x}{\sqrt{\omega}}\right)\phi - |\phi|^{p-1}\phi = 0, \quad x\in\mathbb{R}^n. \tag{2.9}$$

Moreover, since we have

$$\frac{\displaystyle\int_{\mathbb{R}^n} V^*(x)|\phi_\omega(x)|^2dx}{\|\phi_\omega\|_{p+1}^{p+1}} = \frac{\displaystyle\omega^{-1}\int_{\mathbb{R}^n} V^*(x/\sqrt{\omega})|\tilde{\phi_\omega}(x)|^2dx}{\|\tilde{\phi_\omega}\|_{p+1}^{p+1}},$$

it suffices to prove

$$\lim_{\omega\to\infty} \frac{\omega^{-1}\displaystyle\int_{\mathbb{R}^n} V^*(x/\sqrt{\omega})|\tilde{\phi_\omega}(x)|^2dx}{\|\tilde{\phi_\omega}\|_{p+1}^{p+1}} = 0. \tag{2.10}$$

When $\omega \to \infty$, the term $\omega^{-1}V\left(x/\sqrt{\omega}\right)\phi$ in (2.9) disappears formally, and we expect that $\tilde{\phi_\omega}(x)$ may converge to the unique positive radial solution $\psi_1(x)$ of (1.4) with $\omega = 1$ in some sense. Since the standing wave solution $e^{it}\psi_1(x)$ of (2.1) with $V(x) \equiv 0$ is unstable in $H^1(\mathbb{R}^n)$ when $p > 1 + 4/n$, we expect that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (2.1) may be also unstable in X_G when $p > 1 + 4/n$ and ω is sufficiently large. This is the reason why we introduce the rescaled function $\tilde{\phi_\omega}(x)$ to prove (2.10). In what follows, we justify this formal argument. First, we put

$$\tilde{I_\omega}(v) := \|\nabla v\|_2^2 + \|v\|_2^2 + \omega^{-1}\int_{\mathbb{R}^n} V\left(\frac{x}{\sqrt{\omega}}\right)|v(x)|^2dx - \|v\|_{p+1}^{p+1},$$
$$I_1^0(v) := \|\nabla v\|_2^2 + \|v\|_2^2 - \|v\|_{p+1}^{p+1}.$$

The following Lemma 2.1 is a key to show (2.10).

LEMMA 2.1. *Let $1 < p < 2^* - 1$ and $\phi_\omega \in \mathcal{M}_\omega^G$ for large ω. Assume $(V0)$, $(V1.1)$ and $V_2(x) \in L^q(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ for some q such that $q > n/2$ and $q \geq 1$. Let $\tilde{\phi_\omega}(x)$ be the rescaled function defined by (2.8), and $\psi_1(x)$ be the unique positive radial solution of (1.4) with $\omega = 1$ in $H^1(\mathbb{R}^n)$. Then, we have*

$$\text{(i)}\ \lim_{\omega\to\infty}\|\tilde{\phi_\omega}\|_{p+1}^{p+1} = \|\psi_1\|_{p+1}^{p+1}, \quad \text{(ii)}\ \lim_{\omega\to\infty} I_1^0(\tilde{\phi_\omega}) = 0,$$
$$\text{(iii)}\ \lim_{\omega\to\infty}\|\tilde{\phi_\omega}\|_{H^1}^2 = \|\psi_1\|_{H^1}^2, \quad \text{(iv)}\ \lim_{\omega\to\infty}\omega^{-1}\int_{\mathbb{R}^n} V\left(\frac{x}{\sqrt{\omega}}\right)|\tilde{\phi_\omega}(x)|^2dx = 0.$$

We prepare one lemma to prove Lemma 2.1.

LEMMA 2.2. *Let $U(x) \in L^q(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ for some q such that $q > n/2$ and $q \geq 1$. Then, there exists a constant $C > 0$ such that*

$$\left|\int_{\mathbb{R}^n} U(x)|v(x)|^2dx\right| \leq C\|U\|_{L^q+L^\infty}\|v\|_{H^1}^2, \quad v \in H^1(\mathbb{R}^n).$$

Lemma 2.2 is easily proved by the Hölder and the Gagliardo-Nirenberg inequalities. So, we omit the proof.

Proof of Lemma 2.1. First of all, we note that $\tilde{\phi_\omega}(x)$ is a minimizer of

$$\inf\left\{\|v\|_{p+1}^{p+1}\ ;\ v\in X_G\setminus\{0\},\ \tilde{I_\omega}(v)\le 0\right\}, \tag{2.11}$$

and $\psi_1(x)$ is a minimizer of

$$\inf\left\{\|v\|_{p+1}^{p+1}\ ;\ v\in H^1(\mathbb{R}^n)\setminus\{0\},\ I_1^0(v)\le 0\right\},$$

(see Lemma 2.3). In order to prove (i), we show that for any $\mu>1$ there exists $\omega(\mu)\in(\omega_0,\infty)$ such that $\tilde{I_\omega}(\mu\psi_1)<0$ and $I_1^0(\mu\tilde{\phi_\omega})<0$ hold for any $\omega\in(\omega(\mu),\infty)$. If this is true, then the above variational characterizations of $\tilde{\phi_\omega}(x)$ and $\psi_1(x)$ yield that

$$\frac{1}{\mu^{p+1}}\|\psi_1\|_{p+1}^{p+1}\le\|\tilde{\phi_\omega}\|_{p+1}^{p+1}\le\mu^{p+1}\|\psi_1\|_{p+1}^{p+1},\quad \omega\in(\omega(\mu),\infty).$$

Since $\mu>1$ is arbitrary, we conclude (i). First, from $I_1^0(\psi_1)=0$, we have

$$\mu^{-2}\tilde{I_\omega}(\mu\psi_1)=-(\mu^{p-1}-1)\|\psi_1\|_{p+1}^{p+1}+\omega^{-1}\int_{\mathbb{R}^n}V\left(\frac{x}{\sqrt{\omega}}\right)|\psi_1(x)|^2dx.$$

Since $\psi_1(x)$ has an exponential decay at infinity (see, e.g., [**5**, Lemma 2]), we have

$$\lim_{\omega\to\infty}\omega^{-1}\int_{\mathbb{R}^n}V\left(\frac{x}{\sqrt{\omega}}\right)|\psi_1(x)|^2dx=0. \tag{2.12}$$

Indeed, from (V1.1) and Lemma 2.2, we have

$$\begin{aligned}&\left|\omega^{-1}\int_{\mathbb{R}^n}V\left(\frac{x}{\sqrt{\omega}}\right)|\psi_1(x)|^2dx\right|\\&\le\omega^{-1}\int_{\mathbb{R}^n}V_1\left(\frac{x}{\sqrt{\omega}}\right)|\psi_1(x)|^2dx+\left|\omega^{-1}\int_{\mathbb{R}^n}V_2\left(\frac{x}{\sqrt{\omega}}\right)|\psi_1(x)|^2dx\right|\\&\le\omega^{-1}C\int_{\mathbb{R}^n}(1+\omega^{-m/2}|x|^m)|\psi_1(x)|^2dx+C(\omega^{-\theta(q)}+\omega^{-1})\|V_2\|_{L^q+L^\infty}\|\psi_1\|_{H^1}^2,\end{aligned}$$

where $\theta(q):=1-n/2q$. Therefore we obtain (2.12) since $|x|^m|\psi_1(x)|^2\in L^1(\mathbb{R}^n)$ and $q>n/2$. Thus, for any $\mu>1$, there exists $\omega_1(\mu)\in(\omega_0,\infty)$ such that $\tilde{I_\omega}(\mu\psi_1)<0$ for any $\omega\in(\omega_1(\mu),\infty)$. Next, from $\tilde{I_\omega}(\tilde{\phi_\omega})=0$, we have

$$\begin{aligned}\mu^{-2}I_1^0(\mu\tilde{\phi_\omega})&=-(\mu^{p-1}-1)\|\tilde{\phi_\omega}\|_{p+1}^{p+1}-\omega^{-1}\int_{\mathbb{R}^n}V\left(\frac{x}{\sqrt{\omega}}\right)|\tilde{\phi_\omega}(x)|^2dx\\&\le-(\mu^{p-1}-1)\|\tilde{\phi_\omega}\|_{p+1}^{p+1}+\omega^{-1}\int_{\mathbb{R}^n}V_-\left(\frac{x}{\sqrt{\omega}}\right)|\tilde{\phi_\omega}(x)|^2dx,\end{aligned}$$

where $V_-(x) = \max\{-V(x), 0\}$. From the assumptions (V0) and (V1.1), we have $V_- \in L^q(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ with $q \geq 1$ and $q > n/2$. Thus, by Lemma 2.2, there exists $C > 0$ such that

$$\omega^{-1} \int_{\mathbb{R}^n} V_- \left(\frac{x}{\sqrt{\omega}}\right) |\tilde{\phi_\omega}(x)|^2 dx \leq C(\omega^{-\theta(q)} + \omega^{-1}) \|V_-\|_{L^q+L^\infty} \|\tilde{\phi_\omega}\|_{H^1}^2.$$

Note that $\theta(q) \in (0, 1]$ since $q > n/2$. Moreover, from $\tilde{I_\omega}(\tilde{\phi_\omega}) = 0$, we have

$$\begin{aligned}
\|\tilde{\phi_\omega}\|_{H^1}^2 &\leq \|\tilde{\phi_\omega}\|_{p+1}^{p+1} + \omega^{-1} \int_{\mathbb{R}^n} V_- \left(\frac{x}{\sqrt{\omega}}\right) |\tilde{\phi_\omega}(x)|^2 dx \\
&\leq \|\tilde{\phi_\omega}\|_{p+1}^{p+1} + C(\omega^{-\theta(q)} + \omega^{-1}) \|V_-\|_{L^q+L^\infty} \|\tilde{\phi_\omega}\|_{H^1}^2,
\end{aligned}$$

which implies

$$\left(1 - C(\omega^{-\theta(q)} + \omega^{-1}) \|V_-\|_{L^q+L^\infty}\right) \|\tilde{\phi_\omega}\|_{H^1}^2 \leq \|\tilde{\phi_\omega}\|_{p+1}^{p+1}.$$

Thus, we have

$$\mu^{-2} I_1^0(\mu\tilde{\phi_\omega}) \leq -\left(\mu^{p-1} - 1 - \frac{C(\omega^{-\theta(q)} + \omega^{-1}) \|V_-\|_{L^q+L^\infty}}{1 - C(\omega^{-\theta(q)} + \omega^{-1}) \|V_-\|_{L^q+L^\infty}}\right) \|\tilde{\phi_\omega}\|_{p+1}^{p+1}. \quad (2.13)$$

Therefore, for any $\mu > 1$, there exists $\omega_2(\mu) \in (\omega_0, \infty)$ such that $I_1^0(\mu\tilde{\phi_\omega}) < 0$ for any $\omega \in (\omega_2(\mu), \infty)$. Hence, we conclude (i).

Secondly, we show (ii). By (4.20) with $\mu = 1$ and (i), we have

$$\limsup_{\omega \to \infty} I_1^0(\tilde{\phi_\omega}) \leq 0.$$

Moreover, for any $\omega \in (\omega_0, \infty)$ there exists $\mu(\omega) > 0$ such that $I_1^0(\mu(\omega)\tilde{\phi_\omega}) = 0$. Thus, we have

$$\|\psi_1\|_{p+1}^{p+1} \leq \|\mu(\omega)\tilde{\phi_\omega}\|_{p+1}^{p+1} = \mu(\omega)^{p+1} \|\tilde{\phi_\omega}\|_{p+1}^{p+1},$$

which together with (i) implies that

$$\liminf_{\omega \to \infty} \mu(\omega) \geq \liminf_{\omega \to \infty} \|\psi_1\|_{p+1} / \|\tilde{\phi_\omega}\|_{p+1} = 1.$$

From $I_1^0(\mu(\omega)\tilde{\phi_\omega}) = 0$ and (i), we have

$$\liminf_{\omega \to \infty} I_1^0(\tilde{\phi_\omega}) = \liminf_{\omega \to \infty} (\mu(\omega)^{p-1} - 1) \|\tilde{\phi_\omega}\|_{p+1}^{p+1} \geq 0.$$

Hence, we conclude (ii).

Next, from (i), (ii) and $I_1^0(\psi_1) = 0$, we have

$$\lim_{\omega \to \infty} \|\tilde{\phi_\omega}\|_{H^1}^2 = \lim_{\omega \to \infty} \|\tilde{\phi_\omega}\|_{p+1}^{p+1} = \|\psi_1\|_{p+1}^{p+1} = \|\psi_1\|_{H^1}^2,$$

which shows (iii).

Finally, from (ii) and $\tilde{I}_\omega(\tilde{\phi}_\omega)=0$, we have

$$\lim_{\omega\to\infty}\omega^{-1}\int_{\mathbb{R}^n} V\left(\frac{x}{\sqrt{\omega}}\right)|\tilde{\phi}_\omega(x)|^2dx=0,$$

which shows (iv). $\square$

We are now in a position to give the proof of Theorem 2.1.

Proof of Theorem 2.1. As stated above, we have only to show (2.10). Recall that $V^*(x)$ is defined by (2.6), and we put

$$V_l^*(x):=3x\cdot\nabla V_l(x)+\sum_{j,k=1}^n x_jx_k\partial_j\partial_k V_l(x),\quad (l=1,2).$$

By the assumption (V2) and Lemma 2.2, we have

$$\omega^{-1}\int_{\mathbb{R}^n}\left|V_2^*\left(\frac{x}{\sqrt{\omega}}\right)\right||\tilde{\phi_\omega}(x)|^2dx\le C(\omega^{-1}+\omega^{-\theta(q)})\|V_2^*\|_{L^q+L^\infty}\|\tilde{\phi_\omega}\|_{H^1}^2,\tag{2.14}$$

$$\omega^{-1}\int_{\mathbb{R}^n}\left|V_2\left(\frac{x}{\sqrt{\omega}}\right)\right||\tilde{\phi_\omega}(x)|^2dx\le C(\omega^{-1}+\omega^{-\theta(q)})\|V_2\|_{L^q+L^\infty}\|\tilde{\phi_\omega}\|_{H^1}^2.\tag{2.15}$$

From Lemma 2.1 (iii), (iv) and (2.15), we have

$$\lim_{\omega\to\infty}\omega^{-1}\int_{\mathbb{R}^n} V_1\left(\frac{x}{\sqrt{\omega}}\right)|\tilde{\phi}_\omega(x)|^2dx=0.\tag{2.16}$$

Now, from the assumption (V1.2), we have

$$\omega^{-1}\int_{\mathbb{R}^n}\left|V_1^*\left(\frac{x}{\sqrt{\omega}}\right)\right||\tilde{\phi_\omega}(x)|^2dx\le C\omega^{-1}\int_{\mathbb{R}^n}\left(1+V_1\left(\frac{x}{\sqrt{\omega}}\right)\right)|\tilde{\phi_\omega}(x)|^2dx.$$

Thus, from (2.16) and Lemma 2.1 (iii), we have

$$\lim_{\omega\to\infty}\omega^{-1}\int_{\mathbb{R}^n}\left|V_1^*\left(\frac{x}{\sqrt{\omega}}\right)\right||\tilde{\phi_\omega}(x)|^2dx=0.\tag{2.17}$$

Since $V^*(x)=V_1^*(x)+V_2^*(x)$, it follows from (2.14) and (2.17) that

$$\lim_{\omega\to\infty}\omega^{-1}\int_{\mathbb{R}^n}\left|V^*\left(\frac{x}{\sqrt{\omega}}\right)\right||\tilde{\phi_\omega}(x)|^2dx=0.$$

Hence, by lemma 2.1 (i), we obtain (2.10). $\square$

Remark 2.5. Let $\phi_\omega(x)\in\mathcal{M}_\omega^G$. Without loss of generality, we may assume that $\phi_\omega(x)$ is positive in $\mathbb{R}^n$. By Lemma 2.1 and the concentration compactness principle, we see that for any sequence $\{\omega_j\}$ with $\omega_j \to \infty$, there exist a subsequence $\{\tilde{\phi_{\omega_{j_k}}}(x)\}$ of $\{\tilde{\phi_{\omega_j}}(x)\}$ and a sequence $\{y_k\} \subset \mathbb{R}^n$ such that

$$\lim_{k\to\infty} \|\tilde{\phi_{\omega_{j_k}}} - \psi_1(\cdot + y_k)\|_{H^1} = 0 \tag{2.18}$$

(see Theorem III.1 in [**37**]). Although (2.18) may give some information on the asymptotic behavior of $\phi_\omega(x) \in \mathcal{M}_\omega^G$ as $\omega \to \infty$, we do not use (2.18) in the proof of Theorem 2.1 directly. We also note that Lemma 2.1 holds for any $p \in (1, 2^* - 1)$. Finally, we remark that, in the case $p = 1 + 4/n$, it follows from (2.18) that $\lim_{\omega\to\infty} \|\phi_\omega\|_2^2 = \|\psi_1\|_2^2$.

§ 2.3. Proof of Proposition 2.1

In this section we give the proof of Proposition 2.1 following that of Theorem 3 in [**42**].

Lemma 2.3. *Let $\phi_\omega \in \mathcal{M}_\omega^G$. Then, we have*

$$\begin{aligned}
\text{(i)} \qquad & \|\phi_\omega\|_{p+1}^{p+1} = \inf\{\|v\|_{p+1}^{p+1}\ ;\ v \in X_G \setminus \{0\},\ I_\omega(v) = 0\} \\
& \qquad\qquad = \inf\{\|v\|_{p+1}^{p+1}\ ;\ v \in X_G \setminus \{0\},\ I_\omega(v) \le 0\}, \\
\text{(ii)} \qquad & S_\omega(\phi_\omega) = \inf\{S_\omega(v)\ ;\ v \in X_G,\ \|v\|_{p+1}^{p+1} = \|\phi_\omega\|_{p+1}^{p+1}\}.
\end{aligned}$$

Proof. (i). Since we have

$$S_\omega(v) = \frac{1}{2} I_\omega(v) + \frac{p-1}{2(p+1)} \|v\|_{p+1}^{p+1}, \quad v \in X_G,$$

we see that

$$\begin{aligned}
d_1(\omega) &:= \inf\{S_\omega(v)\ ;\ v \in X_G \setminus \{0\},\ I_\omega(v) = 0\} \\
&= \inf\left\{\frac{p-1}{2(p+1)} \|v\|_{p+1}^{p+1}\ ;\ v \in X_G \setminus \{0\},\ I_\omega(v) = 0\right\},
\end{aligned}$$

and $d_1(\omega) = S_\omega(\phi_\omega) = [(p-1)/2(p+1)] \|\phi_\omega\|_{p+1}^{p+1}$. We put

$$d_2(\omega) := \inf\left\{\frac{p-1}{2(p+1)} \|v\|_{p+1}^{p+1}\ ;\ v \in X_G \setminus \{0\},\ I_\omega(v) \le 0\right\}.$$

Since it is clear that $d_2(\omega) \le d_1(\omega)$, we show $d_1(\omega) \le d_2(\omega)$. For any $v \in X_G \setminus \{0\}$ satisfying $I_\omega(v) < 0$, there exists $\lambda_0 \in (0, 1)$ such that $I_\omega(\lambda_0 v) = 0$. Thus, we have

$$d_1(\omega) \le \frac{p-1}{2(p+1)} \|\lambda_0 v\|_{p+1}^{p+1} = \frac{(p-1)}{2(p+1)} {\lambda_0}^{p+1} \|v\|_{p+1}^{p+1} < \frac{p-1}{2(p+1)} \|v\|_{p+1}^{p+1}.$$

Hence, we have $d_1(\omega) \leq d_2(\omega)$.

(ii). We put

$$d_3(\omega) := \inf\{S_\omega(v) \ ; \ v \in X_G, \ \|v\|_{p+1}^{p+1} = \|\phi_\omega\|_{p+1}^{p+1}\}.$$

Since $d_3(\omega) \leq S_\omega(\phi_\omega)$, it suffices to prove $S_\omega(\phi_\omega) \leq d_3(\omega)$. By (i), for any $v \in X_G$ satisfying $\|v\|_{p+1}^{p+1} = \|\phi_\omega\|_{p+1}^{p+1}$, we have $I_\omega(v) \geq 0$. Thus, we have

$$S_\omega(v) \geq \frac{p-1}{2(p+1)}\|v\|_{p+1}^{p+1} = \frac{p-1}{2(p+1)}\|\phi_\omega\|_{p+1}^{p+1} = S_\omega(\phi_\omega).$$

Therefore, we obtain $S_\omega(\phi_\omega) \leq d_3(\omega)$. □

Lemma 2.4. *If $\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < 0$, then there exist positive constants ε_1 and δ_1 with the following property: for any $v \in N_{\delta_1}(\phi_\omega)$ satisfying $\|v\|_2^2 = \|\phi_\omega\|_2^2$, there exists $\lambda(v) \in (1-\varepsilon_1, 1+\varepsilon_1)$ such that $E(\phi_\omega) \leq E(v) + (\lambda(v)-1)P(v)$, where $N_{\delta_1}(\phi_\omega)$ is the set defined in Definition* 2.

Proof. From the assumption $\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < 0$ and the continuity of $\partial_\lambda^2 E(v^\lambda)$ in λ and v, there exist $\varepsilon_1, \delta_1 > 0$ such that $\partial_\lambda^2 E(v^\lambda) < 0$ for any $\lambda \in (1-\varepsilon_1, 1+\varepsilon_1)$ and $v \in N_{\delta_1}(\phi_\omega)$. Since $\partial_\lambda E(v^\lambda)|_{\lambda=1} = P(v)$, the Taylor expansion at $\lambda = 1$ gives

$$E(v^\lambda) \leq E(v) + (\lambda - 1)P(v), \quad \lambda \in (1-\varepsilon_1, 1+\varepsilon_1), \quad v \in N_{\delta_1}(\phi_\omega). \tag{2.19}$$

For any $v \in N_{\delta_1}(\phi_\omega)$, we put $\lambda(v) := (\|\phi_\omega\|_{p+1}^{p+1}/\|v\|_{p+1}^{p+1})^{2/n(p-1)}$. Then, we have $\|v^{\lambda(v)}\|_{p+1}^{p+1} = \|\phi_\omega\|_{p+1}^{p+1}$, and we can take δ_1 small enough to have $\lambda(v) \in (1-\varepsilon_1, 1+\varepsilon_1)$. Furthermore, from Lemma 2.3 (ii), if $\|v\|_2^2 = \|\phi_\omega\|_2^2$, we have

$$E(v^{\lambda(v)}) = S_\omega(v^{\lambda(v)}) - \frac{\omega}{2}\|v^{\lambda(v)}\|_2^2 \geq S_\omega(\phi_\omega) - \frac{\omega}{2}\|\phi_\omega\|_2^2 = E(\phi_\omega). \tag{2.20}$$

Therefore, from (2.19) and (2.20), we have $E(\phi_\omega) \leq E(v) + (\lambda(v)-1)P(v)$ for any $v \in N_{\delta_1}(\phi_\omega)$ satisfying $\|v\|_2^2 = \|\phi_\omega\|_2^2$. □

Definition 3. Let δ_1 be the positive constant in Lemma 2.4. We put

$$\mathcal{A} := \{v \in N_{\delta_1}(\phi_\omega) \ ; \ E(v) < E(\phi_\omega), \ \|v\|_2^2 = \|\phi_\omega\|_2^2, \ P(v) < 0\},$$

and for any $u_0 \in N_{\delta_1}(\phi_\omega)$, we define the exit time from $N_{\delta_1}(\phi_\omega)$ by

$$T(u_0) = \sup\{T > 0 \ ; \ u(t) \in N_{\varepsilon_1}(\phi_\omega), \ 0 \leq t \leq T\},$$

where $u(t)$ is a solution of (2.1) with $u(0) = u_0$.

LEMMA 2.5. *If* $\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < 0$, *then for any* $u_0 \in \mathcal{A}$, *there exists* $\varepsilon_0 = \varepsilon_0(u_0) > 0$ *such that* $P(u(t)) \le -\varepsilon_0$ *for* $0 \le t < T(u_0)$.

Proof. Take $u_0 \in \mathcal{A}$ and put $\varepsilon_2 = E(\phi_\omega) - E(u_0) > 0$. From Lemma 2.4 and the conservation laws in the assumption (A1), we have

$$\varepsilon_2 \le (\lambda(u(t)) - 1)P(u(t)), \quad 0 \le t < T(u_0). \tag{2.21}$$

Thus, we see that $P(u(t)) \ne 0$ for $0 \le t < T(u_0)$. Since the function $t \mapsto P(u(t))$ is continuous and $P(u_0) < 0$, we have $P(u(t)) < 0$ for $0 \le t < T(u_0)$. Therefore, from Lemma 2.4 and (2.21), we have

$$-P(u(t)) \ge \frac{\varepsilon_2}{1 - \lambda(u(t))} \ge \frac{\varepsilon_2}{\varepsilon_1}, \quad 0 \le t < T(u_0).$$

Hence, putting $\varepsilon_0 = \varepsilon_2/\varepsilon_1$, we have $P(u(t)) \le -\varepsilon_0$ for $0 \le t < T(u_0)$. □

Proof of Proposition 2.1. Since $\partial_\lambda E({\phi_\omega}^\lambda)|_{\lambda=1} = 0$, $\partial_\lambda^2 E({\phi_\omega}^\lambda)|_{\lambda=1} < 0$ and $P({\phi_\omega}^\lambda) = \lambda \partial_\lambda E({\phi_\omega}^\lambda)$, we have $E({\phi_\omega}^\lambda) < E(\phi_\omega)$ and $P({\phi_\omega}^\lambda) < 0$ for $\lambda > 1$ sufficiently close to 1. Furthermore, since $\|{\phi_\omega}^\lambda\|_2^2 = \|\phi_\omega\|_2^2$ and $\lim_{\lambda\to1} \|{\phi_\omega}^\lambda - \phi_\omega\|_X = 0$, we have ${\phi_\omega}^\lambda \in \mathcal{A}$ for $\lambda > 1$ sufficiently close to 1. Since we assume $|x|{\phi_\omega}^\lambda(x) \in L^2(\mathbb{R}^n)$ in the assumption (A2), it follows from the virial identity (2.2) in the assumption (A1) that

$$\frac{d^2}{dt^2}\|xu_\lambda(t)\|_2^2 = 8P(u_\lambda(t)), \quad 0 \le t < T({\phi_\omega}^\lambda), \tag{2.22}$$

where $u_\lambda(t)$ is the solution of (2.1) with $u_\lambda(0) = {\phi_\omega}^\lambda$. From Lemma 2.5, there exists $\varepsilon_\lambda > 0$ such that

$$P(u_\lambda(t)) \le -\varepsilon_\lambda, \qquad 0 \le t < T(\phi_\omega^\lambda). \tag{2.23}$$

Hence, from (2.22) and (2.23), we can conclude that $T(\phi_\omega^\lambda) < \infty$. Since $\lim_{\lambda\to1}\|{\phi_\omega}^\lambda - \phi_\omega\|_X = 0$, the proof is completed. □

§ 2.4. NLS with a constant magnetic field

In this section, we consider the nonlinear Schrödinger equation with a constant magnetic field $B = (0, 0, b)$:

$$i\partial_t u = -(\nabla + iA(x))^2 u - |u|^{p-1}u, \qquad (t, x) \in \mathbb{R}^{1+3}, \tag{2.24}$$

where $1 < p < 5$ and

$$A(x_1, x_2, x_3) = \frac{b}{2}(-x_2, x_1, 0), \quad b \in \mathbb{R} \setminus \{0\}.$$

Here, we note that $B = \operatorname{rot} A(x) = (0, 0, b)$, $\operatorname{div} A(x) = 0$ and

$$-(\nabla + iA(x))^2 u = -\Delta u - 2iA(x) \cdot \nabla u + |A(x)|^2 u = -\Delta u - bi\frac{\partial u}{\partial \theta} + \frac{b^2}{4}\rho^2 u,$$

where we used the cylindrical coordinates (ρ, θ, z) in $\mathbb{R}^3$:

$$x_1 = \rho \cos\theta, \quad x_2 = \rho \sin\theta, \quad x_3 = z.$$

As in [**25**], we consider (2.24) in the closed subspace

$$H^1_{A,0}(\mathbb{R}^3) = \{v \in H^1(\mathbb{R}^3) \ ; \ \rho v \in L^2(\mathbb{R}^3), \ v = v(\rho, z) \text{ does not depend on } \theta\}$$

of the energy space $H^1_A(\mathbb{R}^3) = \{v \in L^2(\mathbb{R}^3) \ ; \ (\nabla + iA(x))v \in L^2(\mathbb{R}^3)\}$. We note that in $H^1_{A,0}(\mathbb{R}^3)$, equation (2.24) is equivalent to

$$i\partial_t u = -\Delta u + \frac{b^2}{4}\rho^2 u - |u|^{p-1}u, \qquad (t, x) \in \mathbb{R}^{1+3}. \tag{2.25}$$

Let $V_1(x) = (b^2/4)(x_1^2 + x_2^2) = (b^2/4)\rho^2$, $V_2(x) \equiv 0$, and let G be the group of rotations around the x_3-axis in $\mathbb{R}^3$. Then, $V(x) = V_1(x) + V_2(x) = (b^2/4)\rho^2$ satisfies the assumptions (V0)–(V2) in Section 2.1, and we have $X_G = H^1_{A,0}(\mathbb{R}^3)$. For $V(x) = (b^2/4)\rho^2$, the functionals E, S_ω and I_ω on $H^1_{A,0}(\mathbb{R}^3)$ are defined as in Section 2.1. The assumption (A1) is verified by [**8, 26**]. For the assumption (A2), the existence of minimal action solution $\phi_\omega(\rho, z)$ of the stationary problem:

$$-\Delta\phi + \omega\phi + \frac{b^2}{4}\rho^2\phi - |\phi|^{p-1}\phi = 0, \qquad x \in \mathbb{R}^3 \tag{2.26}$$

in $H^1_{A,0}(\mathbb{R}^3)$ was proved by Esteban and Lions [**15**] for $\omega \in (-|b|, \infty)$. More precisely, we have

Lemma 2.6. *Let $1 < p < 5$ and $\omega \in (-|b|, \infty)$. Then, the set $\mathcal{M}^G_\omega$ is not empty, i.e., there exists a minimizer $\phi_\omega(\rho, z)$ of*

$$\inf\{S_\omega(v) \ ; \ v \in H^1_{A,0}(\mathbb{R}^3) \setminus \{0\}, \ I_\omega(v) = 0\}.$$

Proof. Esteban and Lions [**15**] proved that for any $\omega \in (-|b|, \infty)$, there exists a minimizer $\varphi_\omega(x)$ for

$$\alpha_\omega := \inf\{W_\omega(v)\ ;\ v \in H^1_{A,0}(\mathbb{R}^3),\ \|v\|_{p+1}^{p+1} = 1\},$$

where

$$W_\omega(v) = I_\omega(v) + \|v\|_{p+1}^{p+1} = \|\nabla v\|_2^2 + \omega\|v\|_2^2 + \frac{b^2}{4}\int_{\mathbb{R}^3} \rho^2 |v(x)|^2 dx.$$

Here, we put $\phi_\omega(x) = \alpha_\omega^{1/(p-1)}\varphi_\omega(x)$. Then, we have $\phi_\omega \in H^1_{A,0}(\mathbb{R}^3) \setminus \{0\}$ and $I_\omega(\phi_\omega) = 0$. Moreover, for any $v \in H^1_{A,0}(\mathbb{R}^3) \setminus \{0\}$ satisfying $I_\omega(v) = 0$, we have

$$\begin{aligned} S_\omega(\phi_\omega) &= \frac{p-1}{2(p+1)}\alpha_\omega^{(p+1)/(p-1)} \\ &\le \frac{p-1}{2(p+1)} W_\omega\left(\frac{v}{\|v\|_{p+1}}\right)^{(p+1)/(p-1)} = \frac{p-1}{2(p+1)}\|v\|_{p+1}^{p+1} = S_\omega(v). \end{aligned}$$

Hence, we conclude that $\phi_\omega \in \mathcal{M}^G_\omega$. $\square$

The stability of standing wave solutions of (2.24) was studied by Cazenave and Esteban [**8**] for the case $1 < p < 1 + 4/3$. For $\phi_\omega(\rho, z) \in \mathcal{M}^G_\omega$ in Lemma 2.6, Gonçalves Ribeiro [**25**] proved that if $1 + 4/3 + (4\sqrt{10} - 8)/9 \le p < 5$, the standing wave solution $e^{i\omega t}\phi_\omega(\rho, z)$ of (2.24) is unstable in $H^1_{A,0}(\mathbb{R}^3)$ for any $\omega > 0$. Here, we remark that $\phi_\omega(\rho, z)$ exists for $\omega \in (-|b|, \infty)$. Applying Theorem 2.1 to (2.25), we obtain the following theorem, which covers the case $1+4/3 < p < 1+4/3+(4\sqrt{10}-8)/9$ and gives an improvement of the above result by Gonçalves Ribeiro [**25**].

Theorem 2.2. *Let $1 + 4/3 < p < 5$ and $\phi_\omega(\rho, z) \in \mathcal{M}^G_\omega$. Then there exists $\omega_* = \omega_*(p, b) \in (0, \infty)$ such that the standing wave solution $e^{i\omega t}\phi_\omega(\rho, z)$ of (2.24) is unstable in $H^1_{A,0}(\mathbb{R}^3)$ for any $\omega \in (\omega_*, \infty)$.*

Proof. We apply Theorem 2.1 to (2.25). As stated above, $V(x) = V_1(x)+V_2(x) = (b^2/4)\rho^2$ satisfies (V0)–(V2) and (A1). For (A2), by Lemma 2.6, the set $\mathcal{M}^G_\omega$ is not empty for $\omega \in (-|b|, \infty)$. Thus, we have only to show that

$$\mathcal{M}^G_\omega \subset \{v \in H^1_{A,0}(\mathbb{R}^3)\ ;\ |x|v(x) \in L^2(\mathbb{R}^3)\}, \quad \omega \in (0, \infty). \tag{2.27}$$

For any $\omega > 0$, it follows from [**45**, Theorem 2.5] that the operator $-\Delta+(b^2/4)\rho^2+\omega$ is m-accretive in $L^r(\mathbb{R}^n)$ for $1 < r < \infty$. By following the argument of Cazenave [**7**, Theorem 8.1.1], we see that all $v \in \mathcal{M}_\omega^G$ decay exponentially. Therefore, we have (2.27). This completes the proof. $\square$

CHAPTER 3

Stability of standing waves for nonlinear Schrödinger equations with potentials

§ 3.1. Introduction

This chapter is a joint work with Masahito Ohta. We consider the same case as Chapter 2,

$$i\partial_t u = -\Delta u + V(x)u - |u|^{p-1}u, \qquad (t,x) \in \mathbb{R}^{1+n}, \tag{3.1}$$

where $1 < p < 2^* - 1$. Here, we put $2^* = \infty$ if $n = 1, 2$, and $2^* = 2n/(n-2)$ if $n \geq 3$. In this chaper, for potential $V(x)$, we assume the following (I).

(I) There exist real valued functions $V_1(x)$ and $V_2(x)$ such that $V(x) = V_1(x) + V_2(x)$, and the following (V1) and (V2) hold.

(V1) $V_1(x) \in C^\infty(\mathbb{R}^n)$, $V_1(x) \geq 0$ in $\mathbb{R}^n$ and $\partial_x^\alpha V_1(x) \in L^\infty(\mathbb{R}^n)$ for $|\alpha| \geq 2$.

(V2) There exists q such that $q > n/2$, $q \geq 1$ and $V_2(x) \in L^q(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$.

Here, we define a function space X by

$$X := \{v \in H^1(\mathbb{R}^n, \mathbb{C}) \ ; \ V_1(x)|v(x)|^2 \in L^1(\mathbb{R}^n)\}.$$

We regard X as a real Hilbert space with the inner product

$$(v,w)_X := \mathrm{Re} \int_{\mathbb{R}^n} (v(x)\overline{w(x)} + \nabla v(x) \cdot \overline{\nabla w(x)} + V_1(x)v(x)\overline{w(x)})dx.$$

Moreover, we define an operator $H : X \to X^*$ by

$$\langle Hv, w\rangle = \mathrm{Re} \int_{\mathbb{R}^n} (\nabla v(x) \cdot \overline{\nabla w(x)} + V(x)v(x)\overline{w(x)})dx, \quad v, w \in X. \tag{3.2}$$

By (I), H is well-defined. In fact, by (V2) and the Hölder and Sobolev inequalities, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\left| \int_{\mathbb{R}^n} V_2(x)|v(x)|^2 \, dx \right| \leq \|V_2\|_{L^q + L^\infty} \left(C_\varepsilon \|v\|_{L^2}^2 + \varepsilon \|\nabla v\|_{L^2}^2 \right), \quad v \in H^1(\mathbb{R}^n). \tag{3.3}$$

Furthermore, for a subgroup G of $O(n)$ such that $V_1(x)$ and $V_2(x)$ are G-invariant, we define a closed subspace X_G of X by

$$X_G := \{v \in X \ ; \ v(gx) = v(x), \ g \in G, \ x \in \mathbb{R}^n\}.$$

We say that a function $v(x)$ is G-invariant if $v(gx) = v(x)$ holds for any $g \in G$ and $x \in \mathbb{R}^n$. Note that $X_{\{\mathrm{Id}\}} = X$, $X_{O(n)} = X_{\mathrm{rad}} := \{v \in X \ ; \ v(x) = v(|x|), x \in \mathbb{R}^n\}$, and $X_{G_2} \subset X_{G_1}$ if $G_1 \subset G_2$. Similarly, we define a closed subspace $H^1_G(\mathbb{R}^n)$ of $H^1(\mathbb{R}^n)$. In Theorem 3.1, we assume that $V(x)$ is G_R-invariant, where

$$G_R = \{(a_{jk}) \in O(n) \ ; \ a_{11}, \cdots, a_{nn} \in \{1, -1\}, \ a_{jk} = 0 \ \text{if} \ j \neq k\}. \tag{3.4}$$

In Theorem 3.2, we also assume the following (II).

(II) There exists a subgroup G of $O(n)$ such that $V_1(x)$ and $V_2(x)$ are G-invariant, and the following (V3) holds.

(V3) All minimizing sequences for

$$\lambda_1^G = \inf\{\langle Hv, v\rangle \ ; \ v \in X_G, \ \|v\|_{L^2} = 1\} \tag{3.5}$$

are relatively compact in $L^2(\mathbb{R}^n)$.

Remark 3.1. By (I), we see that λ_1^G is finite, and it follows from (II) that there exists a non-negative minimizer Φ of (3.5) (see Proof of Lemma 3.6 below). Moreover, by (I) and (II), we see that Φ is positive in $\mathbb{R}^n$, and we have

$$\lambda_2^G := \inf\{\langle Hv, v\rangle \ ; \ v \in X_G, \ \|v\|_{L^2} = 1, \ \mathrm{Re}(\Phi, v)_{L^2} = 0\} > \lambda_1^G. \tag{3.6}$$

In particular, λ_1^G is simple, and Φ is the unique positive minimizer of (3.5) (see Chapter 8 of [**23**]).

For $\omega > -\lambda_1^G$, we define

$$(v, w)_{X(\omega)} = \langle Hv, w\rangle + \omega\,\mathrm{Re}(v, w)_{L^2}, \quad \|v\|_{X(\omega)} = (v, v)_{X(\omega)}^{1/2}, \quad v, w \in X_G. \tag{3.7}$$

Then, we see that $\|\cdot\|_{X(\omega)}$ is an equivalent norm on X_G to $\|\cdot\|_X$.

Examples. (i) (Harmonic potentials) For $c_1, \cdots, c_n \in \mathbb{R}$, $\sum_{j=1}^n c_j^2 x_j^2$ satisfies (V1) and is G_R-invariant. Moreover, if $c_j > 0$ for $1 \leq j \leq n$, it satisfies (II) with $G = \{\mathrm{Id}\}$.

(ii) For $0 < a < \min\{2, n\}$ and $c \in \mathbb{R}$, $c|x|^{-a}$ satisfies (V2) and is G_R-invariant. Moreover, if $c < 0$, it satisfies (II) with $G = \{\mathrm{Id}\}$.

We define the energy E and the charge Q on X by

$$E(v) := \frac{1}{2}\langle Hv, v\rangle - \frac{1}{p+1}\|v\|_{p+1}^{p+1}, \quad Q(v) := \frac{1}{2}\|v\|_2^2.$$

We remark that by (I) and $1 < p < 2^* - 1$, the functional E is well-defined on X. The following well-posedness of the Cauchy problem for (3.1) in X is already established in Section 6.4 and Theorem 9.2.5 of [**7**].

Proposition 3.1. *Assume* $1 < p < 2^* - 1$ *and (I). For any* $u_0 \in X$, *there exist* $T = T(\|u_0\|_X) > 0$ *and a unique solution* $u(t) \in C([0,T], X)$ *of* (3.1) *with* $u(0) = u_0$ *satisfying*

$$E(u(t)) = E(u_0), \quad Q(u(t)) = Q(u_0), \quad t \in [0, T].$$

Next, we consider the stationary problem (1.2) with $g(u) = -|u|^{p-1}u$.

Definition 4. For $\omega > -\lambda_1^G$, we define two functionals S_ω and I_ω on X_G by

$$\begin{aligned} &S_\omega(v) := E(v) + \omega Q(v) = \frac{1}{2}\|v\|_{X(\omega)}^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1} \qquad \text{(action)},\\ &I_\omega(v) := \|v\|_{X(\omega)}^2 - \|v\|_{p+1}^{p+1}. \end{aligned}$$

Let $\mathcal{M}_\omega^G$ be the set of all non-negative minimizers for

$$\inf\{S_\omega(v)\ ;\ v \in X_G \setminus \{0\},\ I_\omega(v) = 0\}. \tag{3.8}$$

Remark 3.2. (i) Note that $I_\omega(v) = \partial_\lambda S_\omega(\lambda v)|_{\lambda=1} = \langle S_\omega'(v), v\rangle$.
(ii) Let $\phi_\omega \in \mathcal{M}_\omega^G$. Then, there exists a Lagrange multiplier $\Lambda \in \mathbb{R}$ such that $S_\omega'(\phi_\omega) = \Lambda I_\omega'(\phi_\omega)$. Thus, we have $\langle S_\omega'(\phi_\omega), \phi_\omega\rangle = \Lambda\langle I_\omega'(\phi_\omega), \phi_\omega\rangle$. Since $\langle S_\omega'(\phi_\omega), \phi_\omega\rangle$ $= I_\omega(\phi_\omega) = 0$ and $\langle I_\omega'(\phi_\omega), \phi_\omega\rangle = -(p-1)\|\phi_\omega\|_{p+1}^{p+1} < 0$, we have $\Lambda = 0$. Namely, ϕ_ω satisfies (1.2). Moreover, for any $v \in X_G \setminus \{0\}$ satisfying $S_\omega'(v) = 0$, we have $I_\omega(v) = 0$. Thus, by the definition of $\mathcal{M}_\omega^G$, we have $S_\omega(\phi_\omega) \le S_\omega(v)$. Namely, $\phi_\omega \in \mathcal{M}_\omega^G$ is a ground state (minimal action solution) of (1.2) in X_G. It is easy to see that a ground state of (1.2) in X_G is a minimizer of (3.8).

REMARK 3.3. If $V(x) \in C(\mathbb{R}^n)$ satisfies $\lim_{|x|\to\infty} V(x) = +\infty$, it is easy to see that $\mathcal{M}_\omega^G$ is not empty for $\omega > -\lambda_1^G$, since the embedding $X_G \subset L^r(\mathbb{R}^n)$ is compact for $2 \le r < 2^*$. Thus, $\mathcal{M}_\omega^G$ is not empty for $\omega > -\lambda_1^G$ in the case where $V(x) = \sum_{j=1}^n c_j^2 x_j^2$ with $c_1, \cdots, c_n \in \mathbb{R}$. Moreover, noting the proof of Lemma 4.1 of [**20**] (Lemma 2.6 of Chapter 2), it follows from Theorem I.2 of [**37**] that $\mathcal{M}_\omega^G$ is not empty for $\omega > -\lambda_1^G$ in the case where $V(x) = c|x|^{-a}$ with $0 < a < \min\{2, n\}$ and $c < 0$. However, as a general rule, in addition to (I), we may need some more assumptions related to the concentration compactness principle (see, e.g., [**36, 37, 47**]).

DEFINITION 5. For $\phi_\omega \in \mathcal{M}_\omega^G$ and $\delta > 0$, we put

$$U_\delta^G(\phi_\omega) := \left\{ v \in X_G \ ; \ \inf_{\theta\in\mathbb{R}} \|v - e^{i\theta}\phi_\omega\|_X < \delta \right\}.$$

We say that a standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (3.1) is stable in X_G if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $u_0 \in U_\delta^G(\phi_\omega)$, the solution $u(t)$ of (3.1) with $u(0) = u_0$ satisfies $u(t) \in U_\varepsilon^G(\phi_\omega)$ for any $t \ge 0$. Otherwise, $e^{i\omega t}\phi_\omega(x)$ is said to be unstable in X_G.

Our main results in this chapter are the followings.

THEOREM 3.1. *Assume $1 < p < 1 + 4/n$ and (I). Let G be a subgroup of $O(n)$ such that $G_R \subset G$ and that $V_1(x)$ and $V_2(x)$ are G-invariant. Here, G_R is the subgroup of $O(n)$ defined by* (3.4). *Assume that there exists $\omega_0 > 0$ such that $\mathcal{M}_\omega^G$ is not empty for any $\omega \in (\omega_0, \infty)$. Let $\phi_\omega(x) \in \mathcal{M}_\omega^G$. Then, there exists $\omega_* \in (\omega_0, \infty)$ such that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of* (3.1) *is stable in X_G for any $\omega \in (\omega_*, \infty)$.*

THEOREM 3.2. *Assume $1 < p < 2^* - 1$, (I), (II) and that there exists $\omega^0 > -\lambda_1^G$ such that $\mathcal{M}_\omega^G$ is not empty for any $\omega \in (-\lambda_1^G, \omega^0)$. Let $\phi_\omega(x) \in \mathcal{M}_\omega^G$. Then, there exists $\omega^* \in (-\lambda_1^G, \omega^0)$ such that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of* (3.1) *is stable in X_G for any $\omega \in (-\lambda_1^G, \omega^*)$.*

For the proofs of Theorems 3.1 and 3.2, we use the following sufficient condition for stability in X_G.

PROPOSITION 3.2. *Assume* $1 < p < 2^* - 1$ *and* (*I*). *Let* G *be a subgroup of* $O(n)$ *such that* $V_1(x)$ *and* $V_2(x)$ *are* G*-invariant, and* $\phi_\omega \in \mathcal{M}_\omega^G$. *If there exists* $\delta > 0$ *such that*

$$\langle S_\omega''(\phi_\omega)v, v\rangle \geq \delta \|v\|_X^2 \tag{3.9}$$

for any $v \in X_G$ *satisfying* $\mathrm{Re}(\phi_\omega, v)_{L^2} = 0$ *and* $\mathrm{Re}(i\phi_\omega, v)_{L^2} = 0$, *then the standing wave solution* $e^{i\omega t}\phi_\omega(x)$ *of* (3.1) *is stable in* X_G.

By applying this sufficient condition, we may avoid such delicate difficulty that we have to choose a sequence $\{\omega_k\}$. However still, it is not easy to verify condition (3.9) directly. In Section 3.3 and Section 3.4, we first study a limiting problem. We investigate the rescaling limit of $\phi_\omega(x)$ as $\omega \to \infty$ or $\omega \to -\lambda_1^G$, which is based on the spirit of the analysis in Esteban and Strauss [**16**]. We show that as $\omega \to \infty$, the rescaled function $\tilde{\phi}_\omega(x)$ defined by $\phi_\omega(x) = \omega^{1/(p-1)}\tilde{\phi}_\omega(\sqrt{\omega}x)$ tends to the unique positive radial solution $\psi_1(x)$ of (1.4) with $\omega = 1$. While, as $\omega \to -\lambda_1^G$, $\phi_\omega(x)/\|\phi_\omega\|_2^2$ converges to Φ, which is the first eigenfunction of the linear eigenvalue problem (3.5). From known stability properties of $\psi_1(x)$ and Φ, we are able to prove (3.9) in the limit. Here, we remark that if we let $\omega \to \infty$, we lose control of the nullspace of the linearized operator $S_\omega''(\phi_\omega)$, since (3.1) with $V(x) \equiv 0$ has the invariance of translations, although (3.1) may not have the corresponding invariance (see Lemmas 3.2 and 3.4). This is the reason why we need to assume that G contains G_R in Theorem 3.1.

REMARK 3.4. The assumptions in Proposition 3.2 are slightly different from those in Theorems 3.4 and 3.5 of [**28**]. In fact, applying Theorems 3.4 and 3.5 of [**28**] directly to our case, $\mathrm{Re}(i\phi_\omega, v)_{L^2} = 0$ is replaced by $(i\phi_\omega, v)_X = 0$ in Proposition 3.2. For the sake of completeness, we give the proof of Proposition 3.2 in Section 3.2.

In Proposition 3.2, the condition $\mathrm{Re}(\phi_\omega, v)_{L^2} = 0$ is related to the conservation of charge Q. In fact, we have $\langle Q'(\phi_\omega), v\rangle = \mathrm{Re}(\phi_\omega, v)_{L^2}$. Moreover, since it follows from $S_\omega'(e^{i\theta}\phi_\omega) = 0$ for $\theta \in \mathbb{R}$ that $S_\omega''(\phi_\omega)i\phi_\omega = 0$, (3.9) does not hold if we do not restrict $v \in X_G$ to satisfy $\mathrm{Re}(i\phi_\omega, v)_{L^2} = 0$. For $v \in X_G$ with $v_1(x) = \mathrm{Re}\, v(x)$ and

$v_2(x) = \operatorname{Im} v(x)$, we have

$$\langle S''_\omega(\phi_\omega)v, v\rangle = \langle L_{1,\omega}v_1, v_1\rangle + \langle L_{2,\omega}v_2, v_2\rangle, \tag{3.10}$$

$$\langle L_{1,\omega}v_1, v_1\rangle = \|v_1\|^2_{X(\omega)} - p\int_{\mathbb{R}^n} \phi_\omega^{p-1}(x)|v_1(x)|^2\,dx, \tag{3.11}$$

$$\langle L_{2,\omega}v_2, v_2\rangle = \|v_2\|^2_{X(\omega)} - \int_{\mathbb{R}^n} \phi_\omega^{p-1}(x)|v_2(x)|^2\,dx, \tag{3.12}$$

$$\operatorname{Re}(\phi_\omega, v)_{L^2} = (\phi_\omega, v_1)_{L^2}, \quad \operatorname{Re}(i\phi_\omega, v)_{L^2} = (\phi_\omega, v_2)_{L^2}. \tag{3.13}$$

This chapter is organized as follows. In Section 3.2, following Grillakis, Shatah and Strauss [**28**], we give the proof of Proposition 3.2. In Section 3.3, using a convergence lemma (see Lemma 3.3) proved in [**20**] and some properties of the standing wave solution $e^{i\omega t}\psi_\omega(x)$ of (3.1) with $V(x) \equiv 0$ proved in Weinstein [**55**], we prove Theorem 3.1. For a bounded potential $V(x)$, Rose and Weinstein [**48**] studied by numerical simulations that if $p < 1 + 4/n$, $\|\phi_\omega\|_2^2$ would increase for large ω, so that $e^{i\omega t}\phi_\omega(x)$ would be stable. We can affirm that this numerical result is correct by Theorem 3.1. In Section 3.4, we prove Theorem 3.2.

§ 3.2. Proof of Proposition 3.2

In this section, we prove Proposition 3.2. First, following Grillakis, Shatah and Strauss [**28**, Theorem 3.4], we prove the following lemma (see also Iliev and Kirchev [**31**, Proposition 1]).

Lemma 3.1. *Under the assumptions in Proposition 3.2, there exist $C > 0$ and $\varepsilon > 0$ such that*

$$E(u) - E(\phi_\omega) \geq C \inf_{\theta\in\mathbb{R}} \|u - e^{i\theta}\phi_\omega\|_X^2$$

for $u \in U_\varepsilon^G(\phi_\omega)$ with $Q(u) = Q(\phi_\omega)$.

Proof. Let $u \in U_\varepsilon^G(\phi_\omega)$ with $Q(u) = Q(\phi_\omega)$. By the implicit function theorem, if $\varepsilon > 0$ is small, there exists $\theta(u) \in \mathbb{R}$ such that

$$\|u - e^{i\theta(u)}\phi_\omega\|_X^2 = \min_{\theta\in\mathbb{R}} \|u - e^{i\theta}\phi_\omega\|_X^2. \tag{3.14}$$

Let $v := e^{-i\theta(u)}u - \phi_\omega$. Taylor expansion gives

$$S_\omega(u) = S_\omega(e^{-i\theta(u)}u) = S_\omega(\phi_\omega) + \langle S'_\omega(\phi_\omega), v\rangle + \frac{1}{2}\langle S''_\omega(\phi_\omega)v, v\rangle + o(\|v\|_X^2).$$

Since $S'_\omega(\phi_\omega)=0$ and $Q(\phi_\omega)=Q(u)$, we have

$$E(u)-E(\phi_\omega)=\frac{1}{2}\langle S''_\omega(\phi_\omega)v,v\rangle+o(\|v\|_X^2). \tag{3.15}$$

We decompose v as $v=a\phi_\omega+bi\phi_\omega+w$ with $a,b\in\mathbb{R}$, $w\in X_G$ satisfying $\mathrm{Re}(w,\phi_\omega)_{L^2}=0$ and $\mathrm{Re}(w,i\phi_\omega)_{L^2}=0$. Another expansion gives

$$Q(\phi_\omega)=Q(u)=Q(e^{-i\theta(u)}u)=Q(\phi_\omega)+\langle Q'(\phi_\omega),v\rangle+O(\|v\|_X^2),$$
$$\langle Q'(\phi_\omega),v\rangle=\mathrm{Re}(\phi_\omega,v)_{L^2}=\mathrm{Re}(\phi_\omega,a\phi_\omega+bi\phi_\omega+w)_{L^2}=a\|\phi_\omega\|_2^2.$$

Thus, we have $a=O(\|v\|_X^2)$. Moreover, by (3.14) and $(\phi_\omega,i\phi_\omega)_X=0$, we have $0=(v,i\phi_\omega)_X=b\|\phi_\omega\|_X^2+(w,i\phi_\omega)_X$. Thus, we have $|b|\|\phi_\omega\|_X\le\|w\|_X$ and $\|v\|_X\le(|a|+|b|)\|\phi_\omega\|_X+\|w\|_X\le 2\|w\|_X+O(\|v\|_X^2)$. Therefore, we have

$$\|w\|_X^2\ge\frac{1}{4}\|v\|_X^2+O(\|v\|_X^3). \tag{3.16}$$

Furthermore, since $S''_\omega(\phi_\omega)i\phi_\omega=0$, we have

$$\begin{aligned}\langle S''_\omega(\phi_\omega)w,w\rangle &= \langle S''_\omega(\phi_\omega)v,v\rangle-2a\langle S''_\omega(\phi_\omega)\phi_\omega,v\rangle+a^2\langle S''_\omega(\phi_\omega)\phi_\omega,\phi_\omega\rangle\\ &= \langle S''_\omega(\phi_\omega)v,v\rangle+O(\|v\|_X^3).\end{aligned} \tag{3.17}$$

Since $w\in X_G$ satisfies $\mathrm{Re}(w,\phi_\omega)_{L^2}=0$ and $\mathrm{Re}(w,i\phi_\omega)_{L^2}=0$, by (3.9) in Proposition 3.2, there exists $\delta>0$ such that

$$\langle S''_\omega(\phi_\omega)w,w\rangle\ge\delta\|w\|_X^2. \tag{3.18}$$

By (3.15)–(3.18), we have

$$E(u)-E(\phi_\omega)\ge\frac{\delta}{2}\|w\|_X^2+o(\|v\|_X^2)\ge\frac{\delta}{8}\|v\|_X^2+o(\|v\|_X^2).$$

Finally, since $u\in U_\varepsilon^G(\phi_\omega)$ and $\|v\|_X=\|u-e^{i\theta(u)}\phi_\omega\|_X<\varepsilon$, we can take $\varepsilon=\varepsilon(\delta)>0$ such that

$$E(u)-E(\phi_\omega)\ge\frac{\delta}{9}\|u-e^{i\theta(u)}\phi_\omega\|_X^2.$$

This completes the proof. □

Proposition 3.2 follows from Lemma 3.1 and the proof of Theorem 3.5 in [**28**].

§ 3.3. Proof of Theorem 3.1

In this section, following the idea of Esteban and Strauss [**16**], we prove the following Lemma 3.2 to show Theorem 3.1.

LEMMA 3.2. *Assume (I). Let G be as in Theorem 3.1 and* $\phi_\omega \in \mathcal{M}_\omega^G$.
(i) *Let* $1 < p < 1 + 4/n$. *There exists* $\omega_1 > 0$ *with the following property: for any* $\omega \in (\omega_1, \infty)$, *there exists* $\delta_1 > 0$ *such that*

$$\langle L_{1,\omega} v, v \rangle \geq \delta_1 \|v\|_{X(\omega)}^2$$

for any $v \in X_G(\mathbb{R}^n, \mathbb{R})$ *satisfying* $(v, \phi_\omega)_{L^2} = 0$.
(ii) *Let* $1 < p < 2^* - 1$. *There exists* $\omega_2 > 0$ *with the following property: for any* $\omega \in (\omega_2, \infty)$, *there exists* $\delta_2 > 0$ *such that*

$$\langle L_{2,\omega} v, v \rangle \geq \delta_2 \|v\|_{X(\omega)}^2$$

for any $v \in X_G(\mathbb{R}^n, \mathbb{R})$ *satisfying* $(v, \phi_\omega)_{L^2} = 0$.

Proof of Theorem 3.1. Since $\|\cdot\|_{X(\omega)}$ is equivalent to $\|\cdot\|_X$, by (3.10) and Lemma 3.2, there exists $\delta > 0$ such that (3.9) holds for any $v \in X_G$ satisfying $\mathrm{Re}(\phi_\omega, v)_{L^2} = 0$ and $\mathrm{Re}(i\phi_\omega, v)_{L^2} = 0$. Hence, Theorem 3.1 follows from Proposition 3.2. $\square$

To prove Lemma 3.2, we rescale $\phi_\omega(x) \in \mathcal{M}_\omega^G$ as follows:

$$\phi_\omega(x) = \omega^{1/(p-1)} \tilde{\phi}_\omega(\sqrt{\omega} x), \quad \omega \in (\omega_0, \infty). \tag{3.19}$$

The rescaled function $\tilde{\phi}_\omega(x)$ satisfies

$$-\Delta \phi + \omega^{-1} V\left(\frac{x}{\sqrt{\omega}}\right) \phi + \phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^n. \tag{3.20}$$

When $\omega \to \infty$, the term $\omega^{-1} V(x/\sqrt{\omega})\,\phi$ in (3.20) disappears formally, and we expect that $\tilde{\phi}_\omega(x)$ may converge to the unique positive radial solution $\psi_1(x)$ of (1.4) with $\omega = 1$ in some sense. Since the standing wave solution $e^{it}\psi_1(x)$ of (3.1) with $V(x) \equiv 0$ is stable in $H^1(\mathbb{R}^n)$ when $p < 1 + 4/n$, we expect that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (3.1) may be also stable in X_G when $p < 1 + 4/n$ and ω is sufficiently large. This is the reason why we introduce the rescaled function $\tilde{\phi}_\omega(x)$. In fact, we justified this conjecture in [**20**].

LEMMA 3.3. *Assume $1 < p < 2^* - 1$ and (I). Let G be as in Theorem 3.1, $\phi_\omega(x) \in \mathcal{M}_\omega^G$ and ψ_1 be the unique positive radial solution of* (1.4) *with $\omega = 1$. Then, for $\tilde{\phi}_\omega(x)$ defined by* (3.19), *we have*

$$\lim_{\omega\to\infty} \|\tilde{\phi}_\omega - \psi_1\|_{H^1} = 0. \tag{3.21}$$

Proof. By Lemma 2.1 of [**20**] and the concentration compactness principle, for any sequence $\{\omega_j\}$ with $\omega_j \to \infty$, there exist a subsequence of $\{\tilde{\phi}_{\omega_j}(x)\}$ (still denoted by the same letter) and a sequence $\{y_j\} \subset \mathbb{R}^n$ such that

$$\lim_{j\to\infty} \|\tilde{\phi}_{\omega_j}(\cdot + y_j) - \psi_1\|_{H^1} = 0 \tag{3.22}$$

(see Theorem III.1 in [**37**]). Since $\tilde{\phi}_{\omega_j} \in X_G \subset X_{G_R}$ and $\psi_1 \in H^1_{\mathrm{rad}}(\mathbb{R}^n) \subset H^1_{G_R}(\mathbb{R}^n)$, we see that $y_j = 0$ in (3.22), which implies (3.21). $\square$

For $\omega > -\lambda_1^G$, we define the rescaled norm $\|\cdot\|_{\tilde{X}(\omega)}$ by

$$\|v\|_{\tilde{X}(\omega)}^2 = \|v\|_{H^1}^2 + \int_{\mathbb{R}^n} \omega^{-1} V\left(\frac{x}{\sqrt{\omega}}\right) |v(x)|^2\, dx, \quad v \in X_G,$$

and we define the rescaled operators $\tilde{L}_{1,\omega}$ and $\tilde{L}_{2,\omega}$ by

$$\langle \tilde{L}_{1,\omega} v, v\rangle = \|v\|_{\tilde{X}(\omega)}^2 - p \int_{\mathbb{R}^n} \tilde{\phi}_\omega^{p-1}(x)|v(x)|^2\, dx,$$
$$\langle \tilde{L}_{2,\omega} v, v\rangle = \|v\|_{\tilde{X}(\omega)}^2 - \int_{\mathbb{R}^n} \tilde{\phi}_\omega^{p-1}(x)|v(x)|^2\, dx.$$

Then, for $v(x) = \omega^{1/(p-1)}\tilde{v}(\sqrt{\omega}x)$, we have

$$\|v\|_{X(\omega)}^2 = \omega^{1+2/(p-1)-n/2}\|\tilde{v}\|_{\tilde{X}(\omega)}^2, \quad (\phi_\omega, v)_{L^2} = \omega^{2/(p-1)-n/2}(\tilde{\phi}_\omega, \tilde{v})_{L^2},$$
$$\langle L_{j,\omega} v, v\rangle = \omega^{1+2/(p-1)-n/2}\langle \tilde{L}_{j,\omega}\tilde{v}, \tilde{v}\rangle, \quad j = 1, 2$$

(see (3.7), (3.11) and (3.12)). Moreover, we define

$$\langle L_1^0 v, v\rangle = \|v\|_{H^1}^2 - p\int_{\mathbb{R}^n} \psi_1^{p-1}(x)|v(x)|^2\, dx,$$
$$\langle L_2^0 v, v\rangle = \|v\|_{H^1}^2 - \int_{\mathbb{R}^n} \psi_1^{p-1}(x)|v(x)|^2\, dx.$$

Then, we have

Lemma 3.4. (i) *Let G be a subgroup of $O(n)$ such that $G_R \subset G$. If $1 < p < 1+4/n$, then there exists $\delta_1 > 0$ such that $\langle L_1^0 v, v\rangle \ge \delta_1 \|v\|_{L^2}^2$ for any $v \in H_G^1(\mathbb{R}^n, \mathbb{R})$ satisfying $(v, \psi_1)_{L^2} = 0$.*
(ii) *Let $1 < p < 2^* - 1$. There exists $\delta_2 > 0$ such that $\langle L_2^0 v, v\rangle \ge \delta_2 \|v\|_{L^2}^2$ for any $v \in H^1(\mathbb{R}^n, \mathbb{R})$ satisfying $(v, \psi_1)_{L^2} = 0$.*

Proof. We begin with (ii). Since $L_2^0 \psi_1 = 0$ and $\psi_1(x) > 0$ for $x \in \mathbb{R}^n$, ψ_1 is the first eigenfunction of L_2^0 corresponding to the eigenvalue 0. Moreover, by Weyl's theorem, the essential spectrum of L_2^0 are in $[1, \infty)$, since ψ_1 tends to zero at infinity. These conclude (ii). Next, we show (i). By Propositions 3.3 and 4.4 of Weinstein [**55**], if $1 < p < 1 + 4/n$, then there exists $\delta_1 > 0$ such that $\langle L_1^0 v, v\rangle \ge \delta_1 \|v\|_{L^2}^2$ for any $v \in H^1(\mathbb{R}^n, \mathbb{R})$ satisfying $(v, \psi_1)_{L^2} = 0$ and $(v, \psi_1^{p-1} \partial_j \psi_1)_{L^2} = 0$ for $1 \le j \le n$. Note that Proposition 3.2 of [**55**] was completely proved by Kwong [**34**] for any $n \ge 1$. Since $\psi_1 \in H^1_{\mathrm{rad}}(\mathbb{R}^n) \subset H^1_{G_R}(\mathbb{R}^n)$ and $H^1_G(\mathbb{R}^n) \subset H^1_{G_R}(\mathbb{R}^n)$, we see that $(v, \psi_1^{p-1}\partial_j \psi_1)_{L^2} = 0$ for any $v \in H^1_G(\mathbb{R}^n, \mathbb{R})$ and $1 \le j \le n$. This concludes (i). □

Proof of Lemma 3.2. We show (i) by contradiction. Suppose that (i) were false. Then, there would exist $\{\omega_j\}$ and $\{v_j\} \subset X_G(\mathbb{R}^n, \mathbb{R})$ such that $\omega_j \to \infty$,

$$\lim_{j\to\infty} \langle \tilde{L}_{1,\omega_j} v_j, v_j\rangle \le 0, \tag{3.23}$$

$$\|v_j\|_{\tilde{X}(\omega_j)}^2 = 1, \quad (v_j, \tilde{\phi}_{\omega_j})_{L^2} = 0. \tag{3.24}$$

Here, by (3.3), we have

$$\left| \int_{\mathbb{R}^n} {\omega_j}^{-1} V_2\left(\frac{x}{\sqrt{\omega_j}}\right) |v_j(x)|^2 \, dx \right| \le C({\omega_j}^{-\theta(q)} + \omega_j^{-1}) \|V_2\|_{L^q + L^\infty} \|v_j\|_{H^1}^2,$$

where $\theta(q) := 1 - n/2q > 0$. Moreover, since $\|v_j\|_{\tilde{X}(\omega_j)}^2 = 1$ and $V_1(x) \ge 0$, we have

$$1 = \|v_j\|_{\tilde{X}(\omega_j)}^2 \ge \left\{1 - C({\omega_j}^{-\theta(q)} + \omega_j^{-1}) \|V_2\|_{L^q+L^\infty}\right\} \|v_j\|_{H^1}^2.$$

Thus, we see that $\{v_j\}$ is bounded in $H^1(\mathbb{R}^n)$ and

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} {\omega_j}^{-1} V_2\left(\frac{x}{\sqrt{\omega_j}}\right) |v_j(x)|^2 \, dx = 0. \tag{3.25}$$

Since $\{v_j\}$ is bounded in $H^1(\mathbb{R}^n)$, there exists a subsequence of $\{v_j\}$ (still denoted by $\{v_j\}$) and $v_0 \in H_G^1(\mathbb{R}^n, \mathbb{R})$ such that $v_j \to v_0$ weakly in $H_G^1(\mathbb{R}^n, \mathbb{R})$ and $v_j^2 \to v_0^2$

weakly in $L^{(p+1)/2}(\mathbb{R}^n)$. Further, by Lemma 3.3, we see that $\tilde{\phi}_{\omega_j} \to \psi_1$ strongly in $H^1(\mathbb{R}^n)$ and $\tilde{\phi}_{\omega_j}^{p-1} \to \psi_1^{p-1}$ strongly in $L^{(p+1)/(p-1)}(\mathbb{R}^n)$. Thus, we have

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} \tilde{\phi}_{\omega_j}^{p-1}(x)|v_j(x)|^2\,dx = \int_{\mathbb{R}^n} \psi_1^{p-1}(x)|v_0(x)|^2 dx. \tag{3.26}$$

Therefore, by (3.23), (3.24) and (3.26), we have

$$\begin{aligned} 0 &\geq \liminf_{j\to\infty} \langle \tilde{L}_{1,\omega_j} v_j, v_j\rangle \\ &= \liminf_{j\to\infty} \left(\|v_j\|^2_{\tilde{X}(\omega_j)} - p\int_{\mathbb{R}^n} \tilde{\phi}_{\omega_j}^{p-1}(x)|v_j(x)|^2\,dx \right) \\ &= 1 - p\int_{\mathbb{R}^n} \psi_1^{p-1}(x)|v_0(x)|^2 dx. \end{aligned} \tag{3.27}$$

Again, by (3.23), (3.25), (3.26) and $V_1(x) \geq 0$, we have

$$\begin{aligned} 0 &\geq \liminf_{j\to\infty} \langle \tilde{L}_{1,\omega_j} v_j, v_j\rangle \\ &= \liminf_{j\to\infty} \left(\|v_j\|^2_{H^1} + \int_{\mathbb{R}^n} {\omega_j}^{-1} V\left(\frac{x}{\sqrt{\omega_j}}\right)|v_j(x)|^2\,dx - p\int_{\mathbb{R}^n} \tilde{\phi}_{\omega_j}^{p-1}(x)|v_j(x)|^2\,dx \right) \\ &\geq \|v_0\|^2_{H^1} - p\int_{\mathbb{R}^n} \psi_1^{p-1}(x)|v_0(x)|^2 dx = \langle L_1^0 v_0, v_0\rangle. \end{aligned}$$

Moreover, by (3.24), we have $(v_0, \psi_1)_{L^2} = 0$. Therefore, by Lemma 3.4 (i), we have $v_0 = 0$. However, this contradicts (3.27). Hence, we conclude (i). By the analogous argument, we can also prove (ii). □

§ 3.4. Proof of Theorem 3.2

In this section, we prove Theorem 3.2. Throughout this section, for simplicity, we assume $\lambda_1^G = 0$ in (V3). By considering $V(x) - \lambda_1^G$ instead of $V(x)$, without loss of generality, we can assume $\lambda_1^G = 0$. As in the proof of Theorem 3.1, Theorem 3.2 follows from Lemma 3.5.

LEMMA 3.5. *Assume* $1 < p < 2^* - 1$, *(I) and (II), and let* $\phi_\omega \in \mathcal{M}_\omega^G$.

(i) *There exists* $\omega_1 > 0$ *with the following property: for any* $\omega \in (0, \omega_1)$, *there exists* $\delta_1 > 0$ *such that*

$$\langle L_{1,\omega} v, v\rangle \geq \delta_1 \|v\|^2_{X(\omega)}$$

for any $v \in X_G(\mathbb{R}^n, \mathbb{R})$ *satisfying* $(v, \phi_\omega)_{L^2} = 0$.

(ii) *There exists* $\omega_2 > 0$ *with the following property: for any* $\omega \in (0, \omega_2)$*, there exists* $\delta_2 > 0$ *such that*

$$\langle L_{2,\omega} v, v \rangle \geq \delta_2 \|v\|_{X(\omega)}^2$$

for any $v \in X_G(\mathbb{R}^n, \mathbb{R})$ *satisfying* $(v, \phi_\omega)_{L^2} = 0$.

To prove Lemma 3.5, as we saw in Section 3.3, we expect formally the following. When $\omega \to 0$, if the effect of the nonlinear term $|\phi|^{p-1}\phi$ in (1.2) (with $g(\phi) = -|\phi|^{p-1}\phi$) would disappear, we could have the linear equation $-\Delta\phi + V(x)\phi = 0$, which has a solution $\Phi(x)$ with $\|\Phi\|_{L^2} = 1$. So, we expect that $\phi_\omega(x)/\|\phi_\omega\|_{L^2}$ may converge to $\Phi(x)$ as $\omega \to 0$ in some sense. Since the linear mode is stable, we expect that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (3.1) may be also stable in X_G when ω is close to 0 for any $1 < p < 2^* - 1$. In what follows, we justify this intuitive discussion.

Lemma 3.6. *Let* $\phi_\omega \in \mathcal{M}_\omega^G$ *and* $\hat{\phi}_\omega(x) := \phi_\omega(x)/\|\phi_\omega\|_2$*. Then, we have*

$$\lim_{\omega \to 0} \|\hat{\phi}_\omega - \Phi\|_X = 0.$$

Before we show Lemma 3.6, we need some preparations. For simplifying the argument, we use the operator H defined by (3.2) and functionals Q and N on X_G defined by

$$Q(v) = \frac{1}{2}\|v\|_2^2, \quad N(v) = \frac{1}{p+1}\|v\|_{p+1}^{p+1}.$$

Note that $\langle Q'(v), w \rangle = \mathrm{Re}(v, w)_{L^2}$, $\langle (H + Q')(v), w \rangle = (v, w)_{X(1)}$, and

$$\langle N'(v), w \rangle = \mathrm{Re} \int_{\mathbb{R}^n} |v(x)|^{p-1} v(x) \overline{w(x)}\, dx,$$

$$|\langle N'(u) - N'(v), w \rangle| \leq C \left(\|u\|_{p+1}^{p-1} + \|v\|_{p+1}^{p-1} \right) \|u - v\|_{p+1} \|w\|_{p+1}.$$

The proof of the following Lemma 3.7 is based on Section 4 in [**18**].

Lemma 3.7. *Let* $\phi_\omega \in \mathcal{M}_\omega^G$*. Then, we have*

$$\|\phi_\omega\|_{p+1}^{p+1} \leq \omega^{(p+1)/(p-1)} \|\Phi\|_{p+1}^{-2(p+1)/(p-1)}, \quad \omega > 0, \tag{3.28}$$

$$\lim_{\omega \to 0} \omega^{-2/(p-1)} \|\phi_\omega\|_2^2 = \|\Phi\|_{p+1}^{-2(p+1)/(p-1)}, \tag{3.29}$$

$$\lim_{\omega \to 0} \frac{\|\phi_\omega\|_{p+1}^{p+1}}{\|\phi_\omega\|_2^2} = 0. \tag{3.30}$$

Proof. First, we note that ϕ_ω is a minimizer of

$$\inf\{\|v\|_{p+1}^{p+1}\ ;\ v\in X_G\setminus\{0\},\ I_\omega(v)=0\} \tag{3.31}$$

(see Definition 4), and Φ is a minimizer of (3.5) with $\lambda_1^G=0$. For $\mu>0$, we have

$$I_\omega(\mu\Phi)=\mu^2\|\Phi\|_{X(\omega)}^2-\mu^{p+1}\|\Phi\|_{p+1}^{p+1}=\mu^2\omega-\mu^{p+1}\|\Phi\|_{p+1}^{p+1}.$$

Thus, we have $I_\omega(\mu(\omega)\Phi)=0$ for $\mu(\omega)=(\omega/\|\Phi\|_{p+1}^{p+1})^{1/(p-1)}$. Since ϕ_ω is a minimizer of (3.31), we have $\|\phi_\omega\|_{p+1}^{p+1}\le\|\mu(\omega)\Phi\|_{p+1}^{p+1}=\omega^{(p+1)/(p-1)}\|\Phi\|_{p+1}^{-2(p+1)/(p-1)}$, namely we have (3.28). Next, by $\lambda_1^G=0$ and $I_\omega(\phi_\omega)=0$, we have

$$0\le\|\nabla\phi_\omega\|_2^2+\int_{\mathbb{R}^n}V(x)|\phi_\omega(x)|^2dx=-\omega\|\phi_\omega\|_2^2+\|\phi_\omega\|_{p+1}^{p+1}.$$

Combining with (3.28), we obtain

$$\|\phi_\omega\|_2^2\le\omega^{2/(p-1)}\|\Phi\|_{p+1}^{-2(p+1)/(p-1)},\quad \omega>0. \tag{3.32}$$

Furthermore, by $I_\omega(\phi_\omega)=0$, (3.28) and (3.32), we have

$$\begin{aligned}\|\phi_\omega\|_{X(1)}^2 &= (1-\omega)\|\phi_\omega\|_2^2+\|\phi_\omega\|_{p+1}^{p+1}\\ &\le (1-\omega)\omega^{2/(p-1)}\|\Phi\|_{p+1}^{-2(p+1)/(p-1)}+\omega^{(p+1)/(p-1)}\|\Phi\|_{p+1}^{-2(p+1)/(p-1)}\\ &= \omega^{2/(p-1)}\|\Phi\|_{p+1}^{-2(p+1)/(p-1)}.\end{aligned} \tag{3.33}$$

Now we show (3.29). We decompose ϕ_ω as $\phi_\omega=a_\omega\Phi+y_\omega$, where $a_\omega\in\mathbb{R}$ and $y_\omega\in X_G$ with $(y_\omega,\Phi)_{X(1)}=0$. It follows from (3.33) that

$$\max\{|a_\omega|^2\|\Phi\|_{X(1)}^2,\|y_\omega\|_{X(1)}^2\}\le|a_\omega|^2\|\Phi\|_{X(1)}^2+\|y_\omega\|_{X(1)}^2=\|\phi_\omega\|_{X(1)}^2\le C\omega^{2/(p-1)}. \tag{3.34}$$

We investigate the asymptotic behavior of y_ω and a_ω as $\omega\to0$ more precisely. Since ϕ_ω is a solution of (1.2), using H, Q and N, we have

$$(H+\omega Q')(\phi_\omega)=(H+Q')(\phi_\omega)-(1-\omega)Q'(\phi_\omega)=N'(\phi_\omega). \tag{3.35}$$

Note that $H\Phi=0$ and $\mathrm{Re}(\Phi,y_\omega)_{L^2}=\langle Q'(\Phi),y_\omega\rangle=\langle(H+Q')(\Phi),y_\omega\rangle=(\Phi,y_\omega)_{X(1)}$ $=0$. Taking the pairing between (3.35) and y_ω, we have

$$\begin{aligned}&\langle N'(\phi_\omega),y_\omega\rangle=\langle(H+\omega Q')(\phi_\omega),y_\omega\rangle\\ &=(\phi_\omega,y_\omega)_{X(1)}-(1-\omega)\,\mathrm{Re}(\phi_\omega,y_\omega)_{L^2}=\|y_\omega\|_{X(1)}^2-(1-\omega)\|y_\omega\|_2^2.\end{aligned} \tag{3.36}$$

Furthermore, by (3.6), we have $\langle Hy_\omega, y_\omega\rangle \geq \lambda_2^G \|y_\omega\|_2^2$ with $\lambda_2^G > \lambda_1^G = 0$. Namely, we have $\|y_\omega\|_{X(1)}^2 = \langle (H+Q')(y_\omega), y_\omega\rangle \geq (\lambda_2^G + 1)\|y_\omega\|_2^2$. Therefore, for any $0<\omega<1$, we have

$$\|y_\omega\|_{X(1)}^2 - (1-\omega)\|y_\omega\|_2^2 \geq \|y_\omega\|_{X(1)}^2 - \frac{1-\omega}{\lambda_2^G+1}\|y_\omega\|_{X(1)}^2 \geq \frac{\lambda_2^G}{\lambda_2^G+1}\|y_\omega\|_{X(1)}^2. \quad (3.37)$$

While, we have

$$\begin{aligned} |\langle N'(\phi_\omega), y_\omega\rangle| &= |\langle N'(\phi_\omega) - N'(a_\omega\Phi), y_\omega\rangle + \langle N'(a_\omega\Phi), y_\omega\rangle| \\ &\leq C(\|\phi_\omega\|_{p+1}^{p-1} + \|a_\omega\Phi\|_{p+1}^{p-1})\|y_\omega\|_{p+1}^2 + C\|a_\omega\Phi\|_{p+1}^p\|y_\omega\|_{p+1}. \end{aligned} \quad (3.38)$$

We summarize (3.36)–(3.38) to obtain

$$\frac{\lambda_2^G}{\lambda_2^G+1}\|y_\omega\|_{X(1)}^2 \leq C\left(\|\phi_\omega\|_{X(1)}^{p-1} + |a_\omega|^{p-1}\|\Phi\|_{X(1)}^{p-1}\right)\|y_\omega\|_{X(1)}^2 + C|a_\omega|^p\|\Phi\|_{X(1)}^p\|y_\omega\|_{X(1)}.$$

By (3.34), if we take ω so small that $C\left(\|\phi_\omega\|_{X(1)}^{p-1} + |a_\omega|^{p-1}\|\Phi\|_{X(1)}^{p-1}\right) < \lambda_2^G/(\lambda_2^G+1)$, then we have

$$\|y_\omega\|_{X(1)} \leq C|a_\omega|^p\|\Phi\|_{X(1)}^p. \quad (3.39)$$

Here, we note that $a_\omega \neq 0$. Indeed, if $a_\omega = 0$, then it follows from (3.39) that $y_\omega = 0$, so we have $\phi_\omega = 0$, which is a contradiction. Next, taking the pairing between (3.35) and Φ, we have

$$\begin{aligned} &\langle H\phi_\omega, \Phi\rangle + \omega\langle Q'(\phi_\omega), \Phi\rangle = \langle N'(\phi_\omega), \Phi\rangle \\ &= \langle N'(a_\omega\Phi), \Phi\rangle + \langle N'(\phi_\omega) - N'(a_\omega\Phi), \Phi\rangle \\ &= |a_\omega|^{p-1}a_\omega\|\Phi\|_{p+1}^{p+1} + \langle N'(\phi_\omega) - N'(a_\omega\Phi), \Phi\rangle. \end{aligned}$$

Here, by (3.39), we have

$$\begin{aligned} &\langle H\phi_\omega, \Phi\rangle + \omega\langle Q'(\phi_\omega), \Phi\rangle = \langle H\Phi, \phi_\omega\rangle + \omega \operatorname{Re}(a_\omega\Phi + y_\omega, \Phi)_{L^2} = \omega a_\omega\|\Phi\|_2^2 = \omega a_\omega, \\ &|\langle N'(\phi_\omega) - N'(a_\omega\Phi), \Phi\rangle| \leq C\left(\|\phi_\omega\|_{X(1)}^{p-1} + |a_\omega|^{p-1}\|\Phi\|_{X(1)}^{p-1}\right)\|y_\omega\|_{X(1)}\|\Phi\|_{X(1)} \\ &\leq C\left(\|y_\omega\|_{X(1)}^{p-1} + |a_\omega|^{p-1}\right)\|y_\omega\|_{X(1)} \leq C|a_\omega|^{2p-1}. \end{aligned}$$

Since $a_\omega \neq 0$, we have

$$\begin{aligned} \omega &= |a_\omega|^{p-1}\|\Phi\|_{p+1}^{p+1} + \frac{1}{a_\omega}\langle N'(\phi_\omega) - N'(a_\omega\Phi), \Phi\rangle \\ &= |a_\omega|^{p-1}\|\Phi\|_{p+1}^{p+1} + O(|a_\omega|^{2(p-1)}). \end{aligned}$$

Since it follows (3.34) that $|a_\omega| = O(\omega^{1/(p-1)})$, we have

$$|a_\omega|^{p-1} = \omega \|\Phi\|_{p+1}^{-(p+1)} + O(|a_\omega|^{2(p-1)}) = \omega \|\Phi\|_{p+1}^{-(p+1)} + O(\omega^2).$$

Therefore, we obtain

$$|a_\omega| = \omega^{1/(p-1)} \|\Phi\|_{p+1}^{-(p+1)/(p-1)} + O(\omega^{p/(p-1)}). \tag{3.40}$$

From (3.39) and (3.40), we see that

$$\|\phi_\omega\|_2^2 = |a_\omega|^2 + \|y_\omega\|_2^2 = \omega^{2/(p-1)} \|\Phi\|_{p+1}^{-2(p+1)/(p-1)} + o(\omega^{2/(p-1)}),$$

which implies (3.29). Finally, (3.30) follows from (3.28) and (3.29). □

We are now in a position to give a proof of Lemma 3.6.

Proof of Lemma 3.6. First, we note that $\|\hat{\phi}_\omega\|_2 = 1$. Dividing $I_\omega(\phi_\omega) = 0$ by $\|\phi_\omega\|_2^2$ implies

$$\langle H\hat{\phi}_\omega, \hat{\phi}_\omega \rangle + \omega = \|\phi_\omega\|_{p+1}^{p+1} / \|\phi_\omega\|_2^2.$$

By (3.30) in Lemma 3.7, $\{\hat{\phi}_\omega\}$ is a minimizing sequence of (3.5) with $\lambda_1^G = 0$ as $\omega \to 0$. Moreover, by (3.3), we see that $\{\hat{\phi}_\omega\}$ is bounded in X_G. Thus, by (V3), there exist a subsequence $\{\hat{\phi}_{\omega_j}\}$ and $\phi_0 \in X_G$ such that $\omega_j \to 0$ and

$$\hat{\phi}_{\omega_j} \to \phi_0 \text{ weakly in } X_G, \tag{3.41}$$

$$\hat{\phi}_{\omega_j} \to \phi_0 \text{ strongly in } L^2(\mathbb{R}^n). \tag{3.42}$$

By (3.42), we have $1 = \lim_{j\to\infty} \|\phi_{\omega_j}\|_2^2 = \|\phi_0\|_2^2$ and $\phi_0 \neq 0$. Moreover, by the lower semi-continuity of $X(1)$-norm and $\langle H\phi_0, \phi_0\rangle \geq 0$, we have

$$\begin{aligned} 1 &= \liminf_{j\to\infty} \left(\langle H\hat{\phi}_{\omega_j}, \hat{\phi}_{\omega_j} \rangle + \|\hat{\phi}_{\omega_j}\|_2^2 \right) = \liminf_{j\to\infty} \|\hat{\phi}_{\omega_j}\|_{X(1)}^2 \\ &\geq \|\phi_0\|_{X(1)}^2 = \langle H\phi_0, \phi_0 \rangle + \|\phi_0\|_2^2 \geq 1. \end{aligned} \tag{3.43}$$

Thus, we have $\langle H\phi_0, \phi_0\rangle = 0$ and $\|\phi_0\|_2^2 = 1$. Since ϕ_0 is non-negative, by Remark 3.1, we have $\phi_0 = \Phi$. Therefore, by (3.41) and (3.43), we see that $\hat{\phi}_{\omega_j} \to \Phi$ strongly in X_G. This completes the proof. □

Proof of Lemma 3.5. Suppose that (i) were false. Then, there would exist $\{\omega_j\}$ and $\{v_j\} \subset X_G(\mathbb{R}^n, \mathbb{R})$ such that $\omega_j \to 0$ and

$$\lim_{j\to\infty} \langle L_{1,\omega_j} v_j, v_j \rangle \le 0, \tag{3.44}$$

$$\|v_j\|_{X(\omega_j)}^2 = 1, \quad (v_j, \hat{\phi}_{\omega_j})_{L^2} = 0. \tag{3.45}$$

Since $\|v_j\|_{X(\omega_j)}^2 = \langle Hv_j, v_j \rangle + \omega_j \|v_j\|_2^2 = 1$, we have $\langle Hv_j, v_j \rangle \le 1$. We decompose v_j as $v_j = c_j \Phi + w_j$ with $c_j \in \mathbb{R}$ and $w_j \in X_G$ satisfying $\mathrm{Re}(w_j, \Phi)_{L^2} = 0$. Then, by $H\Phi = 0$ and (3.6) with $\lambda_2^G > \lambda_1^G = 0$, we have $1 \ge \langle Hv_j, v_j \rangle = \langle Hw_j, w_j \rangle \ge \lambda_2^G \|w_j\|_{L^2}^2$. Thus, $\{w_j\}$ is bounded in $L^2(\mathbb{R}^n)$, and there exist a subsequence of $\{w_j\}$ (still denoted by $\{w_j\}$) and $w_0 \in L^2(\mathbb{R}^n)$ such that $w_j \to w_0$ weakly in $L^2(\mathbb{R}^n)$. Moreover, by (3.45), we have

$$0 = (v_j, \hat{\phi}_{\omega_j})_{L^2} = c_j \,\mathrm{Re}(\Phi, \hat{\phi}_{\omega_j})_{L^2} + \mathrm{Re}(w_j, \hat{\phi}_{\omega_j})_{L^2}. \tag{3.46}$$

By Lemma 3.6, we have $\mathrm{Re}(\Phi, \hat{\phi}_{\omega_j})_{L^2} \to \|\Phi\|_2^2 = 1$ and $\mathrm{Re}(w_j, \hat{\phi}_{\omega_j})_{L^2} \to \mathrm{Re}(w_0, \Phi)_{L^2}$. Thus, by (3.46), we see that $c_j \to -\mathrm{Re}(w_0, \Phi)_{L^2}$ and $\{c_j\}$ is bounded. Since $\|v_j\|_2 \le |c_j| \|\Phi\|_2 + \|w_j\|_2$, we see that $\{v_j\}$ is bounded in $L^2(\mathbb{R}^n)$. Since $\langle Hv_j, v_j \rangle \le 1$ and $X \hookrightarrow H^1(\mathbb{R}^n)$, $\{v_j\}$ is bounded in $H^1(\mathbb{R}^n)$. Therefore, there exists a subsequence $\{v_j\}$ which converges to some v_0 weakly in $H^1(\mathbb{R}^n)$ and $|v_j|^2 \to |v_0|^2$ weakly in $L^{(p+1)/2}(\mathbb{R}^n)$. Thus, by Lemma 3.6 and (3.29), we have

$$\begin{aligned}
\lim_{j\to\infty} \int_{\mathbb{R}^n} \phi_{\omega_j}^{p-1}(x) |v_j(x)|^2 dx &= \lim_{j\to\infty} \int_{\mathbb{R}^n} \|\phi_{\omega_j}\|_2^{p-1} \hat{\phi}_{\omega_j}^{p-1}(x) |v_j(x)|^2 dx \\
&= \int_{\mathbb{R}^n} \Phi^{p-1}(x) |v_0(x)|^2 dx \lim_{j\to\infty} \|\phi_{\omega_j}\|_2^{p-1} = 0.
\end{aligned} \tag{3.47}$$

From (3.44), (3.45) and (3.47), we obtain

$$0 \ge \lim_{j\to\infty} \langle L_{1,\omega_j} v_j, v_j \rangle = \lim_{j\to\infty} \left(\|v_j\|_{X(\omega_j)}^2 - p \int_{\mathbb{R}^n} \phi_{\omega_j}^{p-1}(x) |v_j(x)|^2 dx \right) = 1,$$

which is a contradiction. Hence, we conclude (i). By the analogous argument, we can also prove (ii). $\square$

CHAPTER 4

Stability of standing waves for nonlinear Schrödinger equations with double power nonlinearity

§ 4.1. Introduction

In this chapter, we consider the case where $g(u) = -|u|^{p-1}u + f(u)$ and $V(x) \equiv 0$ in (1.1), that is,

$$i\partial_t u = -\Delta u - |u|^{p-1}u + f(u), \qquad (t,x) \in \mathbb{R}^{1+n}, \tag{4.1}$$

where $1 < p < 2^* - 1$, $2^* := 2n/(n-2)$ if $n \geq 3$, $2^* := \infty$ if $n = 1, 2$. We always assume that $f(z) = \dfrac{z}{|z|} f_1(|z|)$ for all $z \in \mathbb{C} \setminus \{0\}$, where $f_1 \in C^1(\mathbb{R}, \mathbb{R})$ and $f_1(0) = 0$.

The main purpose of this chapter is to show that under suitable assumptions on $f(u)$, if $p < 1 + 4/n$ and $n \geq 3$, the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (4.1) is stable for sufficiently small $\omega > 0$ (see Theorem 4.1 below).

Remark 4.1. By a series of Ohta's papers [**41, 42, 43, 44**] on the generalized Davey-Stewartson system, we may have instability or strong instability (blow-up) results of standing waves of (4.1) for the case where $f(u) = -|u|^{q-1}u$, if $\omega > 0$ and $1 + 4/n \leq p < q < 2^* - 1$ or sufficiently large $\omega > 0$ and $1 < p < 1 + 4/n < q < 2^* - 1$. We remark that it follows the stability for the case $f(u) = -|u|^{q-1}u$, if $n \geq 3$, $1 < p < q < 1 + 4/n$ and sufficiently large $\omega > 0$ from the analog of Theorem 4.1 below.

In this chapter, we assume the followings for $f(u)$. Let $F(z) = \int_0^{|z|} f(t)dt$ for $z \in \mathbb{C}$. For all $j = 0, 1$,

(f1) $f^{(j)}(z) = o(|z|^{p-j})$, as $|z| \to 0$.
(f2) $f^{(j)}(z) = o(|z|^{(2^*-1)-j})$, as $|z| \to \infty$.
(f3) There exists $s > 0$ such that $\dfrac{\omega}{2}s^2 - \dfrac{1}{p+1}s^{p+1} + F(s) < 0$.

We define the energy E and the charge Q on $H^1(\mathbb{R}^n)$ by

$$E(v) := \frac{1}{2}\|\nabla v\|_2^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1} + \int_{\mathbb{R}^n} F(v)dx, \quad Q(v) := \frac{1}{2}\|v\|_2^2.$$

The following well-posedness of the Cauchy problem for (4.1) in $H^1(\mathbb{R}^n)$ is already established (See, for example, [**7, 10, 24, 32**]).

Proposition 4.1. *For any $u_0 \in H^1(\mathbb{R}^n)$, there exist $T > 0$ and a unique solution $u(t) \in C([0,T), H^1(\mathbb{R}^n))$ of (4.1) with $u(0) = u_0$ such that $T = +\infty$ or else, $T < +\infty$ and $\lim_{t\uparrow T} \|\nabla u(t)\|_2 = +\infty$. Furthermore, $u(t)$ satisfies*

$$E(u(t)) = E(u_0), \quad Q(u(t)) = Q(u_0), \quad t \in [0,T).$$

Next, we consider the stationary problem (1.2) with $V(x) \equiv 0$ and $g(u) = -|u|^{p-1}u + f(u)$.

Definition 6. For $\omega > 0$, we define a functional called action on $H^1(\mathbb{R}^n)$ by

$$S_\omega(v) := E(v) + \omega Q(v) = \frac{1}{2}\|\nabla v\|_2^2 + \frac{\omega}{2}\|v\|_2^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1} + \int_{\mathbb{R}^n} F(v)dx.$$

The existence of ground states for (1.2) in this case was proved in [**5, 37**].

Proposition 4.2. *Let*

$$\omega_0 := \sup\{\omega > 0 \ ; \ \frac{\omega}{2}s^2 - \frac{1}{p+1}s^{p+1} + F(s) < 0 \quad \textit{for some} \ \ s > 0\}.$$

Then, $\mathcal{G}_\omega$ is not empty for any $\omega \in (0, \omega_0)$.

Remark 4.2. Let $\phi_\omega \in \mathcal{G}_\omega$. By the assumptions (f1) and (f2), using the analogous method in Theorem 8.1.1 of [**7**] and Theorem 2.4 of [**12**], we have $\phi_\omega \in W^{3,r}(\mathbb{R}^n)$ for $r \in [2,\infty)$, $\lim_{|x|\to\infty}\{|\phi_\omega(x)| + |\nabla\phi_\omega(x)|\} = 0$ and so on. It also follows from (f1), (f2) and the maximum principle that $\phi_\omega(x) > 0$ in $\mathbb{R}^n$.

Definition 7. For $\phi_\omega \in \mathcal{G}_\omega$ and $\delta > 0$, we put

$$U_\delta(\phi_\omega) := \left\{ v \in H^1(\mathbb{R}^n) \ ; \ \inf_{\theta\in\mathbb{R}, y\in\mathbb{R}^n} \|v - e^{i\theta}\tau_y\phi_\omega\|_{H^1} < \delta \right\},$$

where $\tau_y v(x) = v(x-y)$. We say that a standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (4.1) is stable in $H^1(\mathbb{R}^n)$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $u_0 \in U_\delta(\phi_\omega)$, the solution $u(t)$ of (4.1) with $u(0) = u_0$ satisfies $u(t) \in U_\varepsilon(\phi_\omega)$ for any $t \geq 0$. Otherwise, $e^{i\omega t}\phi_\omega(x)$ is said to be unstable in $H^1(\mathbb{R}^n)$.

Our main result in this chapter is the following.

THEOREM 4.1. *Assume* $n \geq 3$, $1 < p < 1 + 4/n$, $(f1)$, $(f2)$ *and* $(f3)$. *Let* $\phi_\omega(x) \in \mathcal{G}_\omega$. *Then there exists* $\omega^* \in (0, \omega_0)$ *such that the standing wave solution* $e^{i\omega t}\phi_\omega(x)$ *of* (4.1) *is stable in* $H^1(\mathbb{R}^n)$ *for any* $\omega \in (0, \omega^*)$.

For the proof of Theorem 4.1, we use the following sufficient condition for stability in $H^1(\mathbb{R}^n)$.

PROPOSITION 4.3. *Assume* $1 < p < 2^* - 1$, $(f1)$, $(f2)$ *and* $(f3)$. *Let* $\phi_\omega \in \mathcal{G}_\omega$. *If there exists* $\delta > 0$ *such that*

$$\langle S''_\omega(\phi_\omega)v, v\rangle \geq \delta \|v\|^2_{H^1} \tag{4.2}$$

for any $v \in H^1(\mathbb{R}^n)$ *satisfying* $\mathrm{Re}(\phi_\omega, v)_{L^2} = 0$, $\mathrm{Re}(i\phi_\omega, v)_{L^2} = 0$ *and* $\mathrm{Re}(\partial_l \phi_\omega, v)_{L^2} = 0$ *for* $l = 1, \cdots, n$, *then the standing wave solution* $e^{i\omega t}\phi_\omega(x)$ *of* (4.1) *is stable in* $H^1(\mathbb{R}^n)$.

REMARK 4.3. Theorem 4.1 does not seem to follow immediately from the abstract scheme in [**29**] since it would not be easy to compute the spectrum of linearized operators, especially for the case that the nonlinearity is not monotone. This has a close relation to the uniqueness problem for (1.2). Moreover, the assumptions in Proposition 4.3 are slightly different from those in Lemma 4.5 of [**29**] or Theorem 3.4 of [**28**]. In fact, applying Lemma 4.5 of [**29**] directly to our case, $\mathrm{Re}(i\phi_\omega, v)_{L^2} = 0$ and $\mathrm{Re}(\partial_l \phi_\omega, v)_{L^2} = 0$ for $l = 1, \cdots, n$ are replaced by $(i\phi_\omega, v)_{H^1} = 0$ and $(\partial_l \phi_\omega, v)_{H^1} = 0$ for $l = 1, \cdots, n$ in Proposition 4.3. If we apply Proposition 4.3 with $(i\phi_\omega, v)_{H^1} = 0$ and $(\partial_l \phi_\omega, v)_{H^1} = 0$ for $l = 1, \cdots, n$, we need more detailed convergence property of $\phi_\omega(x)$ to $\psi_1(x)$, for instance, strongly in $W^{2,2}(\mathbb{R}^n)$. That is why it is more convenient for us to use such weaker restrictions to $v \in H^1(\mathbb{R}^n)$ as $\mathrm{Re}(i\phi_\omega, v)_{L^2} = 0$ and $\mathrm{Re}(\partial_l \phi_\omega, v)_{L^2} = 0$ for $l = 1, \cdots, n$. For the sake of completeness, we give the proof of Proposition 4.3 in Section 4.2.

It does not seem easy to verify condition (4.2) directly. In Section 4.3, we first study a limiting problem. We show that as $\omega \to 0$, the rescaled function $\tilde{\phi}_\omega(x)$ defined by $\phi_\omega(x) = \omega^{1/(p-1)}\tilde{\phi}_\omega(\sqrt{\omega}x)$ tends to the unique positive radial solution

$\psi_1(x)$ of (1.4) with $\omega = 1$ up to translations. From known stability properties of $\psi_1(x)$, we are able to prove (4.2) in the limit. We remark that the similar idea was also applied to nonlinear Schrödinger equations with potentials in [**21**] (Chapter 3).

For $v \in H^1(\mathbb{R}^n)$ with $v_1(x) = \operatorname{Re} v(x)$ and $v_2(x) = \operatorname{Im} v(x)$, we have

$$\langle S''_\omega(\phi_\omega)v, v\rangle = \langle L_{1,\omega}v_1, v_1\rangle + \langle L_{2,\omega}v_2, v_2\rangle, \tag{4.3}$$

$$\langle L_{1,\omega}v_1, v_1\rangle = \|\nabla v_1\|_2^2 + \omega\|v_1\|_2^2 - p\int_{\mathbb{R}^n} \phi_\omega^{p-1}(x)|v_1(x)|^2 dx \tag{4.4}$$

$$+ \int_{\mathbb{R}^n} f'(\phi_\omega(x))|v_1(x)|^2\, dx, \tag{4.5}$$

$$\langle L_{2,\omega}v_2, v_2\rangle = \|\nabla v_2\|_2^2 + \omega\|v_2\|_2^2 - \int_{\mathbb{R}^n} \phi_\omega^{p-1}(x)|v_2(x)|^2 dx \tag{4.6}$$

$$+ \int_{\mathbb{R}^n} \frac{f(\phi_\omega(x))}{\phi_\omega(x)}|v_2(x)|^2\, dx, \tag{4.7}$$

$$\operatorname{Re}(\phi_\omega, v)_{L^2} = (\phi_\omega, v_1)_{L^2}, \quad \operatorname{Re}(i\phi_\omega, v)_{L^2} = (\phi_\omega, v_2)_{L^2},$$

$$\operatorname{Re}(\partial_l\phi_\omega, v)_{L^2} = (\partial_l\phi_\omega, v_1)_{L^2}, \qquad \text{for } l = 1, \cdots, n.$$

This chapter is organized as follows. In Section 4.2, following Grillakis, Shatah and Strauss [**28**], we give the proof of Proposition 4.3. In Section 4.3, we prove a convergence lemma which plays the most important role in this chapter. Here, we briefly note the difficulty in spatial dimension $n = 2$. We consider a certain minimization problem in Section 4.3. Using the variational characterization of minimizers, we show the convergence lemma. But, a minimizer is not always a solution of (1.2) since the Lagrange multiplier is not always determined (see, proof of Lemma 4.3 (ii)). In Section 4.4, we prove Theorem 4.1, using the convergence lemma of Section 4.3 and some properties of the standing wave solution $e^{i\omega t}\psi_\omega(x)$ of (4.1) with $f(u) \equiv 0$. Lastly, we give a statement for the generalized Davey-Stewartson system in Section 4.5. Throughout this chapter, different positive constants might be denoted by the same letter C.

§ 4.2. Proof of Proposition 4.3

In this section, we prove Proposition 4.3. First, following Grillakis, Shatah and Strauss [**28**, Theorem 3.4] and Iliev and Kirchev [**31**, Proposition 3], we prove the following lemma.

Lemma 4.1. *Under the assumptions in Proposition 4.3, there exist $C > 0$ and $\varepsilon > 0$ such that*

$$E(u) - E(\phi_\omega) \geq C \inf_{\theta \in \mathbb{R}, y \in \mathbb{R}^n} \|u - e^{i\theta}\phi_\omega(\cdot - y)\|_{H^1}^2$$

for $u \in U_\varepsilon(\phi_\omega)$ with $Q(u) = Q(\phi_\omega)$.

Proof. Let $u \in U_\varepsilon(\phi_\omega)$ with $Q(u) = Q(\phi_\omega)$. By the implicit function theorem, if $\varepsilon > 0$ is small, there exists $\theta(u) \in \mathbb{R}$ and $y(u) \in \mathbb{R}^n$ such that

$$\|u - e^{i\theta(u)}\phi_\omega(\cdot - y(u))\|_{H^1}^2 = \min_{\theta \in \mathbb{R}, y \in \mathbb{R}^n} \|u - e^{i\theta}\phi_\omega(\cdot - y)\|_{H^1}^2, \tag{4.8}$$

(see Lemma 3.4 and Lemma 3.5 of [**13**] for details). Let $v := e^{-i\theta(u)}u(\cdot + y(u)) - \phi_\omega$. Taylor expansion gives

$$S_\omega(u) = S_\omega(e^{-i\theta(u)}u(\cdot + y(u))) = S_\omega(\phi_\omega) + \langle S_\omega'(\phi_\omega), v\rangle + \frac{1}{2}\langle S_\omega''(\phi_\omega)v, v\rangle + o(\|v\|_{H^1}^2).$$

Since $S_\omega'(\phi_\omega) = 0$ and $Q(\phi_\omega) = Q(u)$, we have

$$E(u) - E(\phi_\omega) = \frac{1}{2}\langle S_\omega''(\phi_\omega)v, v\rangle + o(\|v\|_{H^1}^2). \tag{4.9}$$

We decompose v as $v = a\phi_\omega + bi\phi_\omega + c_l\partial_l\phi_\omega + w$ with $a, b, c_l \in \mathbb{R}$, $w \in H^1(\mathbb{R}^n, \mathbb{C})$ satisfying $\mathrm{Re}(w, \phi_\omega)_{L^2} = 0$, $\mathrm{Re}(w, i\phi_\omega)_{L^2} = 0$ and $\mathrm{Re}(w, \partial_l\phi_\omega)_{L^2} = 0$ for $l = 1, \cdots, n$. Another expansion gives

$$Q(\phi_\omega) = Q(u) = Q(e^{-i\theta(u)}u(\cdot + y(u))) = Q(\phi_\omega) + \langle Q'(\phi_\omega), v\rangle + O(\|v\|_{H^1}^2),$$
$$\langle Q'(\phi_\omega), v\rangle = \mathrm{Re}(\phi_\omega, v)_{L^2} = \mathrm{Re}(\phi_\omega, a\phi_\omega + bi\phi_\omega + c_l\partial_l\phi_\omega + w)_{L^2} = a\|\phi_\omega\|_2^2.$$

Thus, we have $a = O(\|v\|_{H^1}^2)$. Here, we can have $\mathrm{Re}(\phi_\omega, i\phi_\omega)_{H^1} = 0$, $\mathrm{Re}(\phi_\omega, \partial_l\phi_\omega)_{H^1} = 0$ and $\mathrm{Re}(i\phi_\omega, \partial_l\phi_\omega)_{H^1} = 0$ for $l = 1, \cdots, n$. Moreover, by (4.8), we have $0 = \mathrm{Re}(v, i\phi_\omega)_{H^1} = b\|\phi_\omega\|_{H^1}^2 + \mathrm{Re}(w, i\phi_\omega)_{H^1}$ and $0 = \mathrm{Re}(v, \partial_l\phi_\omega)_{H^1} = c_l\|\partial_l\phi_\omega\|_{H^1}^2 + \mathrm{Re}(w, \partial_l\phi_\omega)_{H^1}$ for $l = 1, \cdots, n$. Thus, we have $|b|\|\phi_\omega\|_{H^1} \leq \|w\|_{H^1}$, $|c_l|\|\partial_l\phi_\omega\|_{H^1} \leq$

$\|w\|_{H^1}$ for $l = 1, \cdots, n$ and $\|v\|_{H^1} \le (|a| + |b|)\|\phi_\omega\|_{H^1} + |c_l|\|\partial_l \phi_\omega\|_{H^1} + \|w\|_{H^1} \le 3\|w\|_{H^1} + O(\|v\|_{H^1}^2)$. Therefore, we have

$$\|w\|_{H^1}^2 \ge \frac{1}{9}\|v\|_{H^1}^2 + O(\|v\|_{H^1}^3). \tag{4.10}$$

Furthermore, since $S''_\omega(\phi_\omega) i\phi_\omega = 0$ and $S''_\omega(\phi_\omega)\partial_l \phi_\omega = 0$ for $l = 1, \cdots, n$, we have

$$\begin{aligned} \langle S''_\omega(\phi_\omega)w, w\rangle &= \langle S''_\omega(\phi_\omega)v, v\rangle - 2a\langle S''_\omega(\phi_\omega)\phi_\omega, v\rangle + a^2\langle S''_\omega(\phi_\omega)\phi_\omega, \phi_\omega\rangle \\ &= \langle S''_\omega(\phi_\omega)v, v\rangle + O(\|v\|_{H^1}^3). \end{aligned} \tag{4.11}$$

Since $w \in H^1(\mathbb{R}^n)$ satisfies $\mathrm{Re}(w, \phi_\omega)_{L^2} = 0$, $\mathrm{Re}(w, i\phi_\omega)_{L^2} = 0$ and $\mathrm{Re}(w, \partial_l \phi_\omega)_{L^2} = 0$ for $l = 1, \cdots, n$, by (4.2) in Proposition 4.3, there exists $\delta > 0$ such that

$$\langle S''_\omega(\phi_\omega)w, w\rangle \ge \delta\|w\|_{H^1}^2. \tag{4.12}$$

By (4.9)–(4.12), we have

$$E(u) - E(\phi_\omega) \ge \frac{\delta}{2}\|w\|_{H^1}^2 + o(\|v\|_{H^1}^2) \ge \frac{\delta}{18}\|v\|_{H^1}^2 + o(\|v\|_{H^1}^2).$$

Finally, since $u \in U_\varepsilon(\phi_\omega)$ and $\|v\|_{H^1} = \|u - e^{i\theta(u)}\phi_\omega(\cdot - y(u))\|_{H^1} < \varepsilon$, we can take $\varepsilon = \varepsilon(\delta) > 0$ such that

$$E(u) - E(\phi_\omega) \ge \frac{\delta}{36}\|u - e^{i\theta(u)}\phi_\omega(\cdot - y(u))\|_{H^1}^2.$$

This completes the proof. $\square$

Proposition 4.3 follows from Lemma 4.1 and the proof of Theorem 3.5 in [**28**] or Theorem 4.1 of [**29**].

§ 4.3. Proof of convergence lemma

To prove Lemma 4.5 in Section 4, we rescale $\phi_\omega \in \mathcal{G}_\omega$ as follows:

$$\phi_\omega(x) = \omega^{1/(p-1)}\tilde{\phi_\omega}(\sqrt{\omega}x), \quad \omega \in (0, \omega_0). \tag{4.13}$$

The rescaled function $\tilde{\phi_\omega}(x)$ satisfies

$$-\Delta\tilde{\phi_\omega} + \tilde{\phi_\omega} - |\tilde{\phi_\omega}|^{p-1}\tilde{\phi_\omega} + \omega^{-p/(p-1)}f(\omega^{1/(p-1)}\tilde{\phi_\omega}) = 0, \quad x \in \mathbb{R}^n. \tag{4.14}$$

From (f1) and (f2) with $j = 0$, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|\omega^{-p/(p-1)}f(\omega^{1/(p-1)}\tilde{\phi_\omega})| \le \varepsilon|\tilde{\phi_\omega}|^p + C_\varepsilon\omega^{\theta(p)}|\tilde{\phi_\omega}|^{2^*-1},$$

where $\theta(p) := \{(2^* - 1) - p\}/(p-1)$. Since $\theta(p) > 0$, when $\omega \to 0$, the term of higher degree than the p th power nonlinearity disappears formally. So, we expect that $\tilde{\phi}_\omega(x)$ may converge to the unique positive radial solution $\psi_1(x)$ of (1.4) with $\omega = 1$ in some sense. Since the standing wave solution $e^{it}\psi_1(x)$ of (4.1) with $f(u) \equiv 0$ is stable in $H^1(\mathbb{R}^n)$ when $p < 1+4/n$, we also expect that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (4.1) may be stable in $H^1(\mathbb{R}^n)$ when $p < 1 + 4/n$ and ω is sufficiently small. This is the reason why we introduce the rescaled function $\tilde{\phi}_\omega(x)$. In fact, we shall justify this observation in this section.

Lemma 4.2. *Assume* $1 < p < 2^*-1$, $(f1)$, $(f2)$ *and* $(f3)$. *Let* $n \geq 3$, $\phi_\omega(x) \in \mathcal{G}_\omega$ *and* $\psi_1(x)$ *be the unique positive radial solution of* (1.4) *with* $\omega = 1$. *Then, for any sequence* $\{\omega_j\}$ *with* $\omega_j \to 0$, *there exist a subsequence of* $\{\tilde{\phi}_{\omega_j}(x)\}$ (*still denoted by the same letter*) *and a sequence* $\{y_j\} \subset \mathbb{R}^n$ *such that*

$$\lim_{j\to\infty} \|\tilde{\phi}_{\omega_j}(\cdot + y_j) - \psi_1\|_{H^1} = 0 \tag{4.15}$$

Before we show Lemma 4.2, we need some preparations. We define the following functionals on $H^1(\mathbb{R}^n)$.

$$\begin{aligned} V_\omega(v) &:= \frac{1}{2}\|\nabla v\|_2^2 - S_\omega(v) \\ &= \frac{1}{p+1}\|v\|_{p+1}^{p+1} - \int_{\mathbb{R}^n} F(v)dx - \frac{\omega}{2}\|v\|_2^2, \\ K_\omega(v) &:= S_\omega(v) - \frac{1}{n}\|\nabla v\|_2^2 \\ &= \left(\frac{1}{2} - \frac{1}{n}\right)\|\nabla v\|_2^2 + \frac{\omega}{2}\|v\|_2^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1} + \int_{\mathbb{R}^n} F(v)dx. \end{aligned}$$

Lemma 4.3. *Assume* $(f1)$, $(f2)$ *and* $(f3)$. *Let* $\phi_\omega \in \mathcal{G}_\omega$, $n \geq 3$ *and* $\mu_\omega = (1/2 - 1/n)\|\nabla \phi_\omega\|_2^2$.

(i) *The variational problem*

$$\inf\left\{\frac{1}{n}\|\nabla v\|_2^2 \ ; \ v \in H^1(\mathbb{R}^n)\setminus\{0\},\ V_\omega(v) \geq \mu_\omega\right\} \tag{4.16}$$

is equivalent to

$$\inf\left\{\frac{1}{n}\|\nabla v\|_2^2 \ ; \ v \in H^1(\mathbb{R}^n)\setminus\{0\},\ K_\omega(v) \leq 0\right\}. \tag{4.17}$$

(ii) *Let $\mathcal{M}_\omega$ be the set of minimizer for (4.17). If $\phi \in \mathcal{M}_\omega$, then $\phi \in \mathcal{X}_\omega$. In particular, we have $\phi \in \mathcal{G}_\omega$ and $\mathcal{G}_\omega \equiv \mathcal{M}_\omega$.*

REMARK 4.4. (i) Note that $K_\omega(v) = \dfrac{1}{n}\partial_\lambda S_\omega(v(\cdot/\lambda))|_{\lambda=1}$ for $v \in H^1(\mathbb{R}^n)$. The functional K_ω is called the Pohozaev functional.
(ii) The existence of minimizers for (4.16) was showed in Theorem II.2 of [**37**].

Proof of Lemma 4.3.

For the proof of (i), see Lemma 2.1 (4) of [**41**]. We show (ii). Let $\phi \in \mathcal{M}_\omega$. Then, there exists a Lagrange multiplier $\Lambda \in \mathbb{R}$ such that $(1/n)(\|\nabla\phi\|_2^2)' = \Lambda K_\omega'(\phi)$. Thus, we have $\langle (1/n)(\|\nabla\phi\|_2^2)',\, x\cdot\phi\rangle = \Lambda\langle K_\omega'(\phi),\, x\cdot\phi\rangle$, which implies

$$\left(\frac{2}{n} - \frac{n-2}{n}\Lambda\right)\left(1-\frac{n}{2}\right)\|\nabla\phi\|_2^2 = n\Lambda\left\{\frac{1}{p+1}\|\phi\|_{p+1}^{p+1} - \int_{\mathbb{R}^n} F(\phi)dx - \frac{\omega}{2}\|\phi\|_2^2\right\}.$$

It follows from $K_\omega(\phi) = 0$ that

$$\left(\Lambda + \frac{2}{n} - \frac{n-2}{n}\Lambda\right)\|\nabla\phi\|_2^2 = 0.$$

Therefore, we have $\Lambda = -1$ and $\phi \in \mathcal{X}_\omega$. Moreover, for any $v \in \mathcal{X}_\omega$, we have $K_\omega(v) = 0$. Thus, by the definition of (4.17), we have $(1/n)\|\nabla\phi\|_2^2 \le (1/n)\|\nabla v\|_2^2$ and then, $S_\omega(\phi) = K_\omega(\phi) + (1/n)\|\nabla\phi\|_2^2 \le K_\omega(v) + (1/n)\|\nabla v\|_2^2 = S_\omega(v)$. Namely, $\phi \in \mathcal{G}_\omega$. It is easy to see that a ground state of (1.2) in $H^1(\mathbb{R}^n)$ is a minimizer of (4.17). $\square$

We consider

$$\begin{aligned}\tilde{K_\omega}(v) &:= \left(\frac{1}{2}-\frac{1}{n}\right)\|\nabla v\|_2^2 + \frac{1}{2}\|v\|_2^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1} \\ &\quad + \omega^{-(p+1)/(p-1)}\int_{\mathbb{R}^n} F(\omega^{1/(p-1)}v)dx, \\ K_1^0(v) &:= \left(\frac{1}{2}-\frac{1}{n}\right)\|\nabla v\|_2^2 + \frac{1}{2}\|v\|_2^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1}.\end{aligned}$$

LEMMA 4.4. *Assume $(f1)$, $(f2)$ and $(f3)$. Let $\phi_\omega \in \mathcal{G}_\omega$ and $n \ge 3$. Then we have,*

$$\text{(i)} \lim_{\omega\to 0}\|\nabla\tilde{\phi}_\omega\|_2^2 = \|\nabla\psi_1\|_2^2, \quad \text{(ii)} \lim_{\omega\to 0} K_1^0(\tilde{\phi}_\omega) = 0.$$

Proof of Lemma 4.4. First of all, we note that $\tilde{\phi_\omega}(x)$ is a minimizer of

$$\inf\left\{\frac{1}{n}\|\nabla v\|_2^2\ ;\ v\in H^1(\mathbb{R}^n)\setminus\{0\},\ \tilde{K_\omega}(v)\le 0\right\}, \tag{4.18}$$

and $\psi_1(x)$ is a minimizer of

$$\inf\left\{\frac{1}{n}\|\nabla v\|_2^2\ ;\ v\in H^1(\mathbb{R}^n)\setminus\{0\},\ K_1^0(v)\le 0\right\}. \tag{4.19}$$

In order to prove (i), we show that for any $\mu>1$ there exists $\omega(\mu)\in(0,\omega_0)$ such that $\tilde{K_\omega}(\mu\psi_1)<0$ and $K_1^0(\mu\tilde{\phi_\omega})<0$ hold for any $\omega\in(0,\omega(\mu))$. If this is true, then the above variational characterizations of $\tilde{\phi_\omega}(x)$ and $\psi_1(x)$ yield that

$$\frac{1}{\mu^2}\|\nabla\psi_1\|_2^2\le\|\nabla\tilde{\phi_\omega}\|_2^2\le\mu^2\|\nabla\psi_1\|_2^2,\quad \omega\in(0,\omega(\mu)).$$

Since $\mu>1$ is arbitrary, we conclude (i). First, from (f1) and (f2) with $j=0$, for any $\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$\int_{\mathbb{R}^n}F(v(x))dx\le\frac{\varepsilon}{p+1}\|v\|_{p+1}^{p+1}+\frac{C_\varepsilon}{2^*}\|v\|_{2^*}^{2^*},$$

for any $v\in H^1(\mathbb{R}^n)$. Using $K_1^0(\psi_1)=0$, we have

$$\begin{aligned}\mu^{-2}\tilde{K_\omega}(\mu\psi_1)&=-(\mu^{p-1}-1)\frac{1}{p+1}\|\psi_1\|_{p+1}^{p+1}+\omega^{-(p+1)/(p-1)}\mu^{-2}\int_{\mathbb{R}^n}F(\omega^{1/(p-1)}\mu\psi_1)dx\\&\le-(\mu^{p-1}-1)\frac{1}{p+1}\|\psi_1\|_{p+1}^{p+1}+\varepsilon\frac{\mu^{p-1}}{p+1}\|\psi_1\|_{p+1}^{p+1}+C_\varepsilon\frac{\mu^{2^*-2}}{2^*}\omega^{\theta(p)}\|\psi_1\|_{2^*}^{2^*},\end{aligned}$$

where $\theta(p):=\{(2^*-1)-p\}/(p-1)$. We take $\varepsilon=(\mu^{p-1}-1)/(2\mu^{p-1})$ to get

$$\begin{aligned}\mu^{-2}\tilde{K_\omega}(\mu\psi_1)&\le-(\mu^{p-1}-1)\frac{1}{2(p+1)}\|\psi_1\|_{p+1}^{p+1}+C_\mu\frac{\mu^{2^*-2}}{2^*}\omega^{\theta(p)}\|\psi_1\|_{2^*}^{2^*}\\&=-\frac{1}{2(p+1)}\|\psi_1\|_{p+1}^{p+1}\left\{(\mu^{p-1}-1)-\omega^{\theta(p)}\frac{2\mu^{2^*-2}C_\mu(p+1)}{2^*}\frac{\|\psi_1\|_{2^*}^{2^*}}{\|\psi_1\|_{p+1}^{p+1}}\right\}.\end{aligned}$$

Thus, for any $\mu>1$, there exists $\omega_1(\mu)\in(0,\omega_0)$ such that $\tilde{K_\omega}(\mu\psi_1)<0$ for any $\omega\in(0,\omega_1(\mu))$. Namely, we have $\|\nabla\tilde{\phi_\omega}\|_2^2\le\mu^2\|\nabla\psi_1\|_2^2$, for any $\omega\in(0,\omega_1(\mu))$. Next, from (f1), (f2) and $\tilde{K_\omega}(\tilde{\phi_\omega}) = 0$, we have

$$\begin{aligned}
\mu^{-2} K_1^0(\mu\tilde{\phi_\omega}) &= -(\mu^{p-1}-1)\frac{1}{p+1}\|\tilde{\phi_\omega}\|_{p+1}^{p+1} - \omega^{-(p+1)/(p-1)}\int_{\mathbb{R}^n} F(\omega^{1/(p-1)}\tilde{\phi_\omega})dx \\
&\le -(\mu^{p-1}-1)\frac{1}{p+1}\|\tilde{\phi_\omega}\|_{p+1}^{p+1} + \frac{\varepsilon}{p+1}\|\tilde{\phi_\omega}\|_{p+1}^{p+1} + \frac{C_\varepsilon}{2^*}\omega^{\theta(p)}\|\tilde{\phi_\omega}\|_{2^*}^{2^*} \\
&= -(\mu^{p-1}-1)\frac{1}{2(p+1)}\|\tilde{\phi_\omega}\|_{p+1}^{p+1} + \frac{C}{2^*}\omega^{\theta(p)}\|\nabla\tilde{\phi_\omega}\|_2^{2^*} \\
&= -(\mu^{p-1}-1)\frac{1}{2(p+1)}\|\tilde{\phi_\omega}\|_{p+1}^{p+1} + \frac{C}{2^*}\omega^{\theta(p)}\|\nabla\tilde{\phi_\omega}\|_2^{2}\|\nabla\tilde{\phi_\omega}\|_2^{2^*-2}.
\end{aligned}$$

We have taken $\varepsilon = (\mu^{p-1}-1)/2$ and used the Sobolev embedding. Here, from $\tilde{K_\omega}(\tilde{\phi_\omega}) = 0$ and by the same argument as above, we have

$$\begin{aligned}
\left(\frac{1}{2}-\frac{1}{n}\right)\|\nabla\tilde{\phi_\omega}\|_2^2 &\le \frac{1}{p+1}\|\tilde{\phi_\omega}\|_{p+1}^{p+1} - \omega^{-(p+1)/(p-1)}\int_{\mathbb{R}^n} F(\omega^{1/(p-1)}\tilde{\phi_\omega})dx \\
&\le \frac{1}{2(p+1)}\|\tilde{\phi_\omega}\|_{p+1}^{p+1} + \frac{C}{2^*}\omega^{\theta(p)}\|\nabla\tilde{\phi_\omega}\|_2^{2}\|\nabla\tilde{\phi_\omega}\|_2^{2^*-2}.
\end{aligned}$$

Since we have already proved that $\|\nabla\tilde{\phi_\omega}\|_2^2 \le 2\|\nabla\psi_1\|_2^2$ for any $\omega \in (0, \omega_1(\sqrt{2}))$, we have

$$\left(\frac{1}{2}-\frac{1}{n}\right)\|\nabla\tilde{\phi_\omega}\|_2^2 \le \frac{1}{2(p+1)}\|\tilde{\phi_\omega}\|_{p+1}^{p+1} + \frac{C}{2^*}\omega^{\theta(p)}\|\nabla\tilde{\phi_\omega}\|_2^{2}\|\nabla\psi_1\|_2^{2^*-2},$$

for any $\omega \in (0, \omega_1(\sqrt{2}))$. Since $n \ge 3$, there exists $\omega_3 > 0$ such that

$$\frac{C}{2^*}\omega^{\theta(p)}\|\nabla\psi_1\|_2^{2^*-2} \le \frac{1}{2}\left(\frac{1}{2}-\frac{1}{n}\right)$$

for any $\omega \in (0, \omega_3)$. So, we have

$$\|\nabla\tilde{\phi_\omega}\|_2^2 \le \frac{2^*}{p+1}\|\tilde{\phi_\omega}\|_{p+1}^{p+1}$$

for any $\omega \in (0, \omega_4)$, where $\omega_4 := \min\{\omega_3, \omega_1(\sqrt{2})\}$. Thus, we have

$$\begin{aligned}
\mu^{-2} K_1^0(\mu\tilde{\phi_\omega}) &\le -(\mu^{p-1}-1)\frac{1}{2(p+1)}\|\tilde{\phi_\omega}\|_{p+1}^{p+1} + \frac{C}{2^*}\omega^{\theta(p)}\|\nabla\tilde{\phi_\omega}\|_2^{2}\|\nabla\psi_1\|_2^{2^*-2} \\
&\le -\frac{1}{2(p+1)}\{(\mu^{p-1}-1) - 2C\omega^{\theta(p)}\|\nabla\psi_1\|_2^{2^*-2}\}\|\tilde{\phi_\omega}\|_{p+1}^{p+1} \qquad (4.20)
\end{aligned}$$

for any $\omega \in (0, \omega_4)$. Therefore, for any $\mu > 1$, there exists $\omega_2(\mu) \in (0, \omega_4)$ such that $K_1^0(\mu\tilde{\phi_\omega}) < 0$ for any $\omega \in (0, \omega_2(\mu))$. Hence we conclude (i).

Secondly, we show (ii). From $\tilde{K}_\omega(\tilde{\phi_\omega})=0$, we have

$$\begin{aligned}\frac{1}{2}\|\tilde{\phi_\omega}\|_2^2 &\le \frac{1}{p+1}\|\tilde{\phi_\omega}\|_{p+1}^{p+1}-\omega^{-(p+1)/(p-1)}\int_{\mathbb{R}^n}F(\omega^{1/(p-1)}\tilde{\phi_\omega})dx\\ &\le \frac{2}{p+1}\|\tilde{\phi_\omega}\|_{p+1}^{p+1}+C\omega^{\theta(p)}\|\tilde{\phi_\omega}\|_{2^*}^{2^*}\\ &\le \frac{2}{p+1}\left\{\frac{p+1}{8}\|\tilde{\phi_\omega}\|_2^2+C\|\tilde{\phi_\omega}\|_{2^*}^{2^*}\right\}+C\omega^{\theta(p)}\|\nabla\tilde{\phi_\omega}\|_2^{2^*}.\end{aligned}$$

Therefore, we have

$$\|\tilde{\phi_\omega}\|_2^2\le C(1+\omega^{\theta(p)})\|\nabla\tilde{\phi_\omega}\|_2^{2^*},$$

$$\|\tilde{\phi_\omega}\|_{p+1}\le C\|\tilde{\phi_\omega}\|_{H^1}\le\{C(1+\omega^{\theta(p)})\|\nabla\tilde{\phi_\omega}\|_2^{2^*}+C\|\nabla\tilde{\phi_\omega}\|_2^2\}^{1/2}. \tag{4.21}$$

By (4.20) with $\mu=1$, (4.21) and (i), we have

$$\limsup_{\omega\to 0}K_1^0(\tilde{\phi_\omega})\le 0.$$

Moreover, for any $\omega\in(0,\omega_0)$ there exists $\mu(\omega)>0$ such that $K_1^0(\mu(\omega)\tilde{\phi_\omega})=0$. Thus, we have

$$\|\nabla\psi_1\|_2^2\le\|\mu(\omega)\nabla\tilde{\phi_\omega}\|_2^2=\mu(\omega)^2\|\nabla\tilde{\phi_\omega}\|_2^2,$$

which together with (i) implies that

$$\liminf_{\omega\to 0}\mu(\omega)\ge\liminf_{\omega\to 0}\|\nabla\psi_1\|_2/\|\nabla\tilde{\phi_\omega}\|_2=1.$$

From $K_1^0(\mu(\omega)\tilde{\phi_\omega})=0$, we have

$$\liminf_{\omega\to 0}K_1^0(\tilde{\phi_\omega})=\liminf_{\omega\to 0}\left\{\frac{1}{p+1}(\mu(\omega)^{p-1}-1)\|\tilde{\phi_\omega}\|_{p+1}^{p+1}\right\}\ge 0.$$

Hence, we conclude (ii). □

Finally, we are in position to prove Lemma 4.2.

Proof of Lemma 4.2. By Lemma 4.4, for any $\{\omega_j\}\to 0$, $\{\tilde{\phi_{\omega_j}}\}$ is a minimizing sequence of (4.19). We know from a similar proof to Lemma 4.3 (i) that (4.19) is equivalent to

$$\inf\left\{\frac{1}{n}\|\nabla v\|_2^2\ ;\ v\in H^1(\mathbb{R}^n)\setminus\{0\},\ V_1^0(v)\ge\mu_1^0\right\}, \tag{4.22}$$

where

$$\begin{aligned}V_1^0(v) &:= \frac{1}{p+1}\|v\|_{p+1}^{p+1} - \frac{1}{2}\|v\|_2^2,\\ \mu_1^0 &= (1/2 - 1/n)\|\nabla\psi_1\|_2^2.\end{aligned}$$

Thus, applying the concentration compactness principle (see Theorem II.2 in [**37**]) to (4.22), we obtain a minimum of (4.22) to which a subsequence of $\{\tilde{\phi_{\omega_j}}\}$ converges. It follows from Lemma 4.3 (ii) that such minimum is a solution of (1.4), namely $\psi_1(x)$. Therefore we have (4.15). □

§ 4.4. Proof of Theorem 4.1

In this section, following the idea of Esteban and Strauss [**16**], we prove the following Lemma 4.5 to show Theorem 4.1.

LEMMA 4.5. *Assume* $(f1)$, $(f2)$ *and* $(f3)$. *Let* $\phi_\omega \in \mathcal{G}_\omega$.
(i) *Let* $1 < p < 1 + 4/n$. *There exists* $\omega_1^* > 0$ *with the following property: for any* $\omega \in (0, \omega_1^*)$, *there exists* $\delta_1 > 0$ *such that*

$$\langle L_{1,\omega} v, v\rangle \geq \delta_1 \|v\|_{H^1}^2$$

for any $v \in H^1(\mathbb{R}^n, \mathbb{R})$ *satisfying* $(v, \phi_\omega)_{L^2} = 0$ *and* $(v, \partial_l \phi_\omega)_{L^2} = 0$ *for* $l = 1, \cdots, n$.
(ii) *There exists* $\omega_2^* > 0$ *with the following property: for any* $\omega \in (0, \omega_2^*)$, *there exists* $\delta_2 > 0$ *such that*

$$\langle L_{2,\omega} v, v\rangle \geq \delta_2 \|v\|_{H^1}^2$$

for any $v \in H^1(\mathbb{R}^n, \mathbb{R})$ *satisfying* $(v, \phi_\omega)_{L^2} = 0$.

Proof of Theorem 4.1. By (4.3) and Lemma 4.5, there exists $\delta > 0$ such that (4.2) holds for any $v \in H^1(\mathbb{R}^n)$ satisfying $\mathrm{Re}(\phi_\omega, v)_{L^2} = 0$, $\mathrm{Re}(\partial_l \phi_\omega, v)_{L^2} = 0$ for $l = 1, \cdots, n$ and $\mathrm{Re}(i\phi_\omega, v)_{L^2} = 0$. Hence, Theorem 4.1 follows from Proposition 4.3. □

For $\omega \in (0, \omega_0)$, we define the rescaled operators $\tilde{L}_{1,\omega}$ and $\tilde{L}_{2,\omega}$ by

$$\begin{aligned}\langle \tilde{L}_{1,\omega} v, v\rangle &= \|v\|_{H^1}^2 - p\int_{\mathbb{R}^n} \tilde{\phi_\omega}^{p-1}(x)|v(x)|^2\,dx \\ &\quad + \int_{\mathbb{R}^n} \omega^{-1} f'(\omega^{1/(p-1)}\tilde{\phi_\omega}(x))|v(x)|^2\,dx, \\ \langle \tilde{L}_{2,\omega} v, v\rangle &= \|v\|_{H^1}^2 - \int_{\mathbb{R}^n} \tilde{\phi_\omega}^{p-1}(x)|v(x)|^2\,dx \\ &\quad + \int_{\mathbb{R}^n} \omega^{-p/(p-1)} \frac{f(\omega^{1/(p-1)}\tilde{\phi_\omega}(x))}{\tilde{\phi_\omega}(x)}|v(x)|^2\,dx.\end{aligned}$$

Then, for $v(x) = \omega^{1/(p-1)}\tilde{v}(\sqrt{\omega}x)$, we have

$$\begin{aligned}&\|v\|_{H^1}^2 = \omega^{1+2/(p-1)-n/2}\|\tilde{v}\|_{H^1}^2, \quad \langle L_{j,\omega} v, v\rangle = \omega^{1+2/(p-1)-n/2}\langle \tilde{L}_{j,\omega}\tilde{v}, \tilde{v}\rangle, \quad j=1,2, \\ &(\phi_\omega, v)_{L^2} = \omega^{2/(p-1)-n/2}(\tilde{\phi_\omega}, \tilde{v})_{L^2}, \\ &(\partial_l\phi_\omega, v)_{L^2} = \omega^{2/(p-1)-n/2+1/2}(\partial_l\tilde{\phi_\omega}, \tilde{v})_{L^2}, \quad l=1,\cdots,n.\end{aligned}$$

(see (4.5) and (4.7)). Moreover, we define

$$\begin{aligned}\langle L_1^0 v, v\rangle &= \|v\|_{H^1}^2 - p\int_{\mathbb{R}^n} \psi_1^{p-1}(x)|v(x)|^2\,dx, \\ \langle L_2^0 v, v\rangle &= \|v\|_{H^1}^2 - \int_{\mathbb{R}^n} \psi_1^{p-1}(x)|v(x)|^2\,dx.\end{aligned}$$

Then, we have

LEMMA 4.6. (i) *If $1 < p < 1+4/n$, then there exists $\delta_1 > 0$ such that $\langle L_1^0 v, v\rangle \geq \delta_1\|v\|_{L^2}^2$ for any $v \in H^1(\mathbb{R}^n, \mathbb{R})$ satisfying $(v, \psi_1)_{L^2} = 0$ and $(v, \partial_l\psi_1)_{L^2} = 0$ for $l = 1, \cdots, n$.*
(ii) *Let $1 < p < 2^* - 1$. There exists $\delta_2 > 0$ such that $\langle L_2^0 v, v\rangle \geq \delta_2\|v\|_{L^2}^2$ for any $v \in H^1(\mathbb{R}^n, \mathbb{R})$ satisfying $(v, \psi_1)_{L^2} = 0$.*

Proof. We begin with (ii). Since $L_2^0\psi_1 = 0$ and $\psi_1(x) > 0$ for $x \in \mathbb{R}^n$, ψ_1 is the first eigenfunction of L_2^0 corresponding to the eigenvalue 0. Moreover, by Weyl's theorem, the essential spectrum of L_2^0 are in $[1, \infty)$, since ψ_1 tends to zero at infinity. These conclude (ii). Next, we show (i). The essential spectrum of L_1^0 are in $[1, \infty)$. Also, Weinstein proved in Appendix A of [**55**] (completed by Kwong [**34**]) that L_1^0 has exactly one simple negative eigenvalue and

$$\ker(L_1^0) := \{v \in H^1(\mathbb{R}^n, \mathbb{R})\ ;\ L_1^0 v = 0\} = \operatorname{span}\{\partial_l\psi_1(x)\ ;\ l = 1, \cdots, n\}.$$

Using these facts and the spectral decomposition, we can have (i) by the argument of Proposition 1 in Iliev and Kirchev [**31**], which is based on the method of Grillakis, Shatah and Strauss [**28**]. $\square$

Proof of Lemma 4.5. We show (i) by contradiction. Suppose that (i) were false. Then, there would exist $\{\omega_j\}$ and $\{v_j\} \subset H^1(\mathbb{R}^n, \mathbb{R})$ such that $\omega_j \to 0$,

$$\lim_{j\to\infty} \langle \tilde{L}_{1,\omega_j} v_j, v_j \rangle \leq 0, \tag{4.23}$$

$$\|v_j\|_{H^1}^2 = 1, \tag{4.24}$$

$$(v_j, \tilde{\phi}_{\omega_j})_{L^2} = 0, \quad (v_j, \partial_l \tilde{\phi}_{\omega_j})_{L^2} = 0, \tag{4.25}$$

for $l = 1, \cdots, n$. By Lemma 4.2, there exist a subsequence of $\{\tilde{\phi}_{\omega_j}\}$ (still denoted by $\{\tilde{\phi}_{\omega_j}\}$) and $\{y_j\} \in \mathbb{R}^n$ such that $\tilde{\phi}_{\omega_j}(\cdot + y_j) \to \psi_1(\cdot)$ strongly in $H^1(\mathbb{R}^n)$ and $\tilde{\phi}_{\omega_j}^{r-1}(\cdot + y_j) \to \psi_1^{r-1}(\cdot)$ strongly in $L^{(r+1)/(r-1)}(\mathbb{R}^n)$, where $1 < r \leq 2^* - 1$. Further, since $\{v_j(\cdot + y_j)\}$ is bounded in $H^1(\mathbb{R}^n)$, there exists a subsequence of $\{v_j(\cdot + y_j)\}$ (still denoted by $\{v_j(\cdot + y_j)\}$) and $v_0(\cdot) \in H^1(\mathbb{R}^n, \mathbb{R})$ such that $v_j(\cdot + y_j) \to v_0(\cdot)$ weakly in $H^1(\mathbb{R}^n)$ and $v_j^2(\cdot + y_j) \to v_0^2(\cdot)$ weakly in $L^{(r+1)/2}(\mathbb{R}^n)$. Thus, we have

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} \tilde{\phi}_{\omega_j}^{r-1}(x + y_j)|v_j(x + y_j)|^2\, dx = \int_{\mathbb{R}^n} \psi_1^{r-1}(x)|v_0(x)|^2 dx. \tag{4.26}$$

Here, by (f1) and (f2) with $j = 1$, we have for any $\varepsilon > 0$,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} {\omega_j}^{-1} f'\left({\omega_j}^{1/(p-1)} \tilde{\phi}_{\omega_j}(x + y_j) \right) |v_j(x + y_j)|^2\, dx \right| \\
&\leq {\omega_j}^{-1} \times \\
&\int_{\mathbb{R}^n} \left(\varepsilon \left| {\omega_j}^{1/(p-1)} \tilde{\phi}_{\omega_j}(x + y_j) \right|^{p-1} + C_\varepsilon \left| {\omega_j}^{1/(p-1)} \tilde{\phi}_{\omega_j}(x + y_j) \right|^{2^*-2} \right) |v_j(x + y_j)|^2\, dx \\
&= \varepsilon \int_{\mathbb{R}^n} \tilde{\phi}_{\omega_j}^{p-1}(x + y_j)|v_j(x + y_j)|^2\, dx \\
&\qquad + C_\varepsilon {\omega_j}^{\theta(p)} \int_{\mathbb{R}^n} \tilde{\phi}_{\omega_j}^{2^*-2}(x + y_j)|v_j(x + y_j)|^2\, dx
\end{aligned}$$

where $\theta(p) := \{(2^* - 1) - p\}/(p - 1) > 0$. Accordingly, using (4.26) with $r = p$ or $2^* - 1$, we have

$$\begin{aligned}
&\limsup_{j\to\infty} \left| \int_{\mathbb{R}^n} {\omega_j}^{-1} f'\left({\omega_j}^{1/(p-1)} \tilde{\phi}_{\omega_j}(x + y_j) \right) |v_j(x + y_j)|^2\, dx \right| \\
&\leq \varepsilon \int_{\mathbb{R}^n} \psi_1^{p-1}(x)|v_0(x)|^2 dx.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} {\omega_j}^{-1} f' \left({\omega_j}^{1/(p-1)} \tilde{\phi}_{\omega_j}(x+y_j) \right) |v_j(x+y_j)|^2 \, dx = 0. \tag{4.27}$$

Therefore, by (4.23), (4.24), (4.26) and (4.27), we have

$$\begin{aligned} 0 &\geq \liminf_{j\to\infty} \langle \tilde{L}_{1,\omega_j} v_j, v_j \rangle \\ &= \liminf_{j\to\infty} \left(\|v_j\|_{H^1}^2 - p \int_{\mathbb{R}^n} \tilde{\phi}_{\omega_j}^{p-1}(x+y_j) |v_j(x+y_j)|^2 \, dx \right) \\ &= 1 - p \int_{\mathbb{R}^n} \psi_1^{p-1}(x) |v_0(x)|^2 dx. \end{aligned} \tag{4.28}$$

Again, by (4.23), (4.26) and (4.27), we have

$$\begin{aligned} 0 &\geq \liminf_{j\to\infty} \langle \tilde{L}_{1,\omega_j} v_j, v_j \rangle \\ &= \liminf_{j\to\infty} \left(\|v_j\|_{H^1}^2 - p \int_{\mathbb{R}^n} \tilde{\phi}_{\omega_j}^{p-1}(x+y_j) |v_j(x+y_j)|^2 \, dx \right) \\ &\geq \|v_0\|_{H^1}^2 - p \int_{\mathbb{R}^n} \psi_1^{p-1}(x) |v_0(x)|^2 dx = \langle L_1^0 v_0, v_0 \rangle. \end{aligned}$$

Moreover, by (4.25), we have $(v_0, \psi_1)_{L^2} = (v_0, \partial_l \psi_1)_{L^2} = 0$, for $l = 1, \cdots, n$. Therefore, by Lemma 4.6 (i), we have $v_0 = 0$. However, this contradicts (4.28). Hence, we conclude (i). By the analogous argument, we can also prove (ii). □

§ 4.5. The generalized Davey-Stewartson System

In this section, we give a remark on the stability of standing waves $e^{i\omega t}\phi_\omega(x)$ for

$$i\partial_t u = -\Delta u - |u|^{p-1}u - E_1(|u|^2)u, \qquad (t,x) \in \mathbb{R}^{1+n}, \tag{4.29}$$

where $1 < p < 2^* - 1$, $n = 2$, or 3 and E_1 is the singular integral operator with symbol $\sigma_1(\xi) = \xi_1^2/|\xi|^2$ for $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$. $\phi_\omega(x)$ is a ground state of

$$-\Delta\phi + \omega\phi - |\phi|^{p-1}\phi - E_1(|\phi|^2)\phi = 0, \quad x \in \mathbb{R}^n. \tag{4.30}$$

For $\omega > 0$, Cipolatti [**12**] showed the existence of ground states for (4.30) by studying the following variational problem:

$$\Sigma_\mu = \inf\{\|\nabla v\|_2^2 \ ; \ v \in H^1(\mathbb{R}^n) \setminus \{0\}, \ V_\omega(v) \geq \mu\}, \quad \mu > 0, \tag{4.31}$$

where $V_\omega(v) := \dfrac{1}{p+1}\|v\|_{p+1}^{p+1} + \dfrac{1}{4}\displaystyle\int_{\mathbb{R}^n} |v|^2 E_1(|v|^2) dx - \dfrac{\omega}{2}\|v\|_2^2$.

Equation (4.29) describes the evolution of weakly nonlinear waves that travel predominantly in one direction (see [**14, 22**]). The unique local existence of H^1 solution to the Cauchy problem of (4.29) has already been established (see [**22**]). There were some papers concerned with the stability and instability of standing waves for (4.29) (see [**13, 41, 42, 43, 44**]).

By applying the argument of Sections 3 and 4 to the above variational characterization (4.31), we have the following improvement of the result in Ohta [**40**].

THEOREM 4.2. *Assume* $n = 3$ *and* $1 < p < 1 + 4/3$. *Let* $\phi_\omega(x)$ *be a ground state for* (4.30). *Then there exists* $\omega_* > 0$ *such that the standing wave solution* $e^{i\omega t}\phi_\omega(x)$ *of* (4.29) *is stable in* $H^1(\mathbb{R}^n)$ *for any* $\omega \in (0, \omega_*)$.

CHAPTER 5

Appendix

In this chapter, we consider (1.1) with $V(x) \equiv 0$ and $g(u) = -|u|^{p-1}u + |u|^{q-1}u$,

$$i\partial_t u = -\Delta u - |u|^{p-1}u + |u|^{q-1}u, \qquad (t,x) \in \mathbb{R}^{1+n}, \tag{5.1}$$

where $1 < p < q < 2^* - 1$. When we study the stability of standing waves $e^{i\omega t}\phi_\omega(x)$ of (5.1), the corresponding stationary problem is the following.

$$\begin{cases} -\Delta\phi + \omega\phi - |\phi|^{p-1}\phi + |\phi|^{q-1}\phi = 0, & x \in \mathbb{R}^n, \\ \phi \in H^1(\mathbb{R}^n), \qquad \phi \not\equiv 0. \end{cases} \tag{5.2}$$

The existence of ground states follows from Proposition IV in Chapter 1 for any $\omega \in (0, \omega_0)$. In this case, we can explicitly have the expression of ω_0, that is,

$$\omega_0 = \frac{2(q-p)}{(p+1)(q-1)} \left\{ \frac{(p-1)(q+1)}{(p+1)(q-1)} \right\}^{(p-1)/(q-p)}.$$

As we mentioned in Chapter 1, this case was numerically studied by Anderson [**1**] and Shatah [**51**] showed that there are stable standing waves for ω close to ω_0 with $p = 3$, $q = 5$ and $n \geq 3$. Namely, there exists a sequence $\{\omega_k\}$ approaching ω_0, for which $e^{i\omega_k t}\phi_{\omega_k}(x)$ is stable. In $n = 1$ case, Ohta [**40**] proved that for $1 < p < q < \infty$, the standing wave solution is stable for *any* ω close to ω_0. For proving the same result in $n \geq 2$, we study the asymptotic behavior of $\phi_\omega(x)$ as $\omega \to \omega_0$, which would be useful to apply this case to Proposition VI.

First, we remark that by the standard elliptic argument, if $\phi_\omega \in \mathcal{G}_\omega$, we have $\phi_\omega(x) > 0$ in $\mathbb{R}^n$, $\phi_\omega \in C^2(\mathbb{R}^n)$, $\lim_{|x|\to\infty} |\phi_\omega(x)| = 0$ and $\phi_\omega(x)$ decays exponentially. Moreover, it is known that $\phi_\omega(x) = \phi_\omega(|x|)$ and $\phi'_\omega(r) < 0$ for all $r > 0$, where $r = |x|$ and a prime denotes the differentiation with respect to r (see, e.g., [**35**]). Thus in order to study the properties of a solution $\phi_\omega(x)$ of (5.2), it is important to investigate the properties of solutions $u \in C^2([0,\infty))$ of the corresponding radial problem

$$\begin{cases} -u'' - \dfrac{n-1}{r}u' = h_\omega(u), \quad r>0, \quad \omega \in (0,\omega_0)\ ; \qquad u>0, \quad r\geq 0, \\ u'(0)=0, \quad \lim\limits_{r\to\infty} u(r)=0, \end{cases} \tag{5.3}$$

where $h_\omega(u) := -\omega u + |u|^{p-1}u - |u|^{q-1}u$. We define for $u \geq 0$ and $\omega \in (0,\omega_0)$,

$$H_\omega(u) = \int_0^u h_\omega(s)ds,$$
$$\beta_\omega = \inf\{u>0\ ;\ H_\omega(u)>0\}.$$

Here, we also define for $u \geq 0$,

$$h_{\omega_0}(u) := -\omega_0 u + |u|^{p-1}u - |u|^{q-1}u,$$
$$H_{\omega_0}(u) = \int_0^u h_{\omega_0}(s)ds.$$

First, we need two basic lemmas in [**46**] concerning solutions of (5.3).

Lemma 5.1. *Let u be a solution of equation* (5.3)*, non-constant on a finite interval $(r_0, r_1) \subset \mathbb{R}^+$. Then, the following inequality holds.*

$$\frac{1}{2}u'(r_1)^2 - \frac{1}{2}u'(r_0)^2 < -\int_{u(r_0)}^{u(r_1)} h_\omega(s)ds. \tag{5.4}$$

Proof. Multiply equation (5.3) by u' and integrate over (r_0, r_1). This yields

$$\frac{1}{2}\left[u'(r)^2\right]_{r_0}^{r_1} + (n-1)\int_{r_0}^{r_1} u'(r)^2\frac{dr}{r} + \int_{u(r_0)}^{u(r_1)} h_\omega(s)ds = 0.$$

Since the second term is positive unless $u(r) \equiv$ constant, the desired inequality follows. $\square$

Lemma 5.2. *Let u be a solution of the problem* (5.3) *satisfying $u'(r) \leq 0$ for all $r>0$. Then, we have*

$$\lim_{r\to\infty} u'(r) = 0 \quad \textit{and} \quad u(0) \geq \beta_\omega.$$

Proof. Since $u'(r) \le 0$ on $(0, \infty)$ and $h_\omega(s) < 0$ for small $s > 0$, for r large enough,

$$u''(r) = -\frac{n-1}{r}u'(r) - h_\omega(u(r)) > 0.$$

Therefore there exists $\lim_{r\to\infty} u'(r)(\le 0)$. Because $u(r) > 0$ for all $r > 0$, it follows that $\lim_{r\to\infty} u'(r) = 0$. Now we apply Lemma 5.1 on $(0, r)$ and let $r \to \infty$. This yields $H_\omega(u(0)) \ge 0$ and hence $u(0) \ge \beta_\omega$. □

REMARK 5.1. We do not actually need to assume that $u'(r) \le 0$ for all $r > 0$. We can prove that a solution u of (5.3) is strictly decreasing (see Lemma 3 of [**46**]).

Our main purpose in this chapter is to show the following proposition. We prove it by using ODE method in [**6**].

PROPOSITION 5.1. *Let $\phi_\omega \in \mathcal{G}_\omega$. Then, we have*

$$\lim_{\omega\to\omega_0} \|\phi_\omega - \phi_0\|_{H^1_{\mathrm{loc}}(\mathbb{R}^n)} = 0,$$

where

$$\phi_0 = \left\{\frac{(p-1)(q+1)}{(p+1)(q-1)}\right\}^{1/(q-p)}.$$

REMARK 5.2. (i) ϕ_0 is a nonzero solution of

$$\begin{aligned} H_{\omega_0}(u) &= -\frac{\omega_0}{2}u^2 + \frac{1}{p+1}u^{p+1} - \frac{1}{q+1}u^{q+1} = 0, \\ h_{\omega_0}(u) &= H'_{\omega_0}(u) = -\omega_0 u + u^p - u^q = 0, \quad \text{for} \quad u \ge 0. \end{aligned}$$

(ii) We note that β_ω converges to ϕ_0 a.e. $x \in \mathbb{R}^n$ as $\omega \to \omega_0$.

Proof of Proposition 5.1. For any $\omega \in (0, \omega_0)$, $-\Delta\phi_\omega(x) \ge 0$ at the point $x = 0$ where $\phi_\omega(x)$ achieves its maximum and hence from (5.2), $-\omega\phi_\omega + \phi_\omega^p - \phi_\omega^q \ge 0$ at $x = 0$. Thus, we have $\phi_\omega(0) \le 1$ since $\omega > 0$, $p < q$ and $\phi_\omega(x) > 0$ in $\mathbb{R}^n$. So, we have $\phi_\omega(x) \le \phi_\omega(0) \le 1$ for any $x \in \mathbb{R}^n$. For any fixed $R > 0$, B_R denote the ball of radius R centered at the origin, and we have $\int_{B_R} |\phi_\omega(x)|^2 dx \le C$, where C is independent of ω. By the elliptic regularity theorem, $\|\phi_\omega\|_{W^{2,2}(B_R)}$ is also bounded. For any $\{\omega_j\}$ with $\omega_j \to \omega_0$, there exist a subsequence $\{\phi_{\omega_j}\}$ (still denoted by the same letter) and $v \in W^{2,2}(B_R)$ such that ϕ_{ω_j} converges strongly to v in $H^1(B_R) \cap L^{r+1}(B_R)$ for $1 < r < 2^* - 1$. Particularly, ϕ_{ω_j} converges to v a.e. $x \in \mathbb{R}^n$ and hence $v(x)$ is positive, radially symmetric, monotone decreasing, $C^2(\mathbb{R}^n)$ function and satisfies the equation

$$-\Delta v + \omega_0 v - |v|^{p-1}v + |v|^{q-1}v = 0.$$

From Lemma 5.2, we have $\phi_\omega(0) \geq \beta_\omega$. Therefore,

$$v(0) = \liminf_{\omega_j \to \omega_0} \phi_{\omega_j}(0) \geq \liminf_{\omega_j \to \omega_0} \beta_{\omega_j} = \phi_0.$$

While, if we suppose that $v(0) > \phi_0$, then we have $h_{\omega_0}(v(0)) < 0$. But, since v achieves its maximum at the origin, it follows that $-\Delta v(0) = h_{\omega_0}(v(0)) \geq 0$, which is a contradiction. Hence, we have $v(0) = \phi_0$. By applying the following Lemma 5.3 to v, we obtain that $v(x) \equiv \phi_0$ for $x \in \mathbb{R}^n$, which concludes the proof. □

LEMMA 5.3. *Let $u \in C^2([0,\infty))$ be a positive solution of the initial value problem*

$$\begin{cases} -u'' - \dfrac{n-1}{r}u' = h_{\omega_0}(u), \quad r > 0\ ; \\ u(0) = \phi_0, \qquad u'(0) = 0. \end{cases} \tag{5.5}$$

Assume that $u'(r) \leq 0$ for all $r > 0$. Then, $u \equiv \phi_0$ (constant).

Proof. For the case $n = 1$, by the uniqueness to the IVP, $u \equiv \phi_0$ since $h_{\omega_0}(\phi_0) = 0$. Therefore, from now on, we discuss the case $n \geq 2$. Since $u(r) > 0$ and $u'(r) \leq 0$ for any $r > 0$, there exists $l \geq 0$ such that $\lim_{r\to\infty} u(r) = l$. If we show that $l = \phi_0$, we have $u \equiv \phi_0$ because $u'(r) \leq 0$ for any $r > 0$. Multiply (5.5) by u' and integrate between 0 and r. This yields

$$\frac{1}{2}\{u'(r)\}^2 + (n-1)\int_0^r \{u'(s)\}^2 \frac{ds}{s} = H_{\omega_0}(\phi_0) - H_{\omega_0}(u(r)). \tag{5.6}$$

We remark that there exists $M > 0$ such that $H_{\omega_0}(u(r)) \geq -M$ and that $H_{\omega_0}(\phi_0) = 0$. Therefore

$$\lim_{r\to\infty} \int_0^r \{u'(s)\}^2 \frac{ds}{s} < \infty$$

and hence $u'(r)$ converges as $r \to \infty$. Since u is bounded, we deduce that

$$\lim_{r\to\infty} u'(r) = 0.$$

Thus, from (5.5), we deduce that $u''(r)$ converges as $r \to \infty$. Again, since u' is bounded,

$$\lim_{r\to\infty} u''(r) = 0.$$

If we now let r tend to infinity in (5.5), we obtain

$$h_{\omega_0}(l) = 0.$$

Let

$$\alpha := \inf\{s > 0 \ ; \ h_{\omega_0}(s) \geq 0\}.$$

The possible value of l are 0, α and ϕ_0. Let us prove that $l \neq \alpha$. Suppose to the contrary that $l = \alpha$. We set

$$w(r) = r^{(n-1)/2}\{u(r) - \alpha\}.$$

Then w satisfies

$$\begin{aligned} -w'' &= \left\{ \frac{h_{\omega_0}(u)}{u-\alpha} - \frac{(n-1)(n-3)}{4r^2} \right\} w, \\ w(r) &> 0 \quad \text{for all} \quad r \geq 0. \end{aligned}$$

Since $u(r) \downarrow \alpha$ as $r \to \infty$ and $h'_{\omega_0}(\alpha) > 0$, we have

$$\lim_{u\to\alpha} \frac{h_{\omega_0}(u)}{u-\alpha} > 0.$$

Therefore, there exist $\delta > 0$ and $R_1 > 0$ such that

$$\frac{h_{\omega_0}(u)}{u-\alpha} - \frac{(n-1)(n-3)}{4r^2} \geq \delta \quad \text{for all} \quad r \geq R_1.$$

Thus $w'' < 0$ for $r \geq R_1$, which implies that for some $L \geq -\infty$, $w'(r) \downarrow L$ as $r \to \infty$. Now, if $L < 0$, then $w(r) \to -\infty$ as $r \to \infty$, which is impossible. On the other hand, if $L \geq 0$, then $w(r) \geq w(R_1) > 0$ for $r \geq R_1$. This implies that $-w'' \geq \delta w(R_1) > 0$ for $r \geq R_1$. Therefore $w'(r) \downarrow -\infty$ as $r \to \infty$ and so $w(r) \downarrow -\infty$ as $r \to \infty$. The contradiction proves that $l \neq \alpha$. Next, we show that $l \neq 0$ by contradiction. Assume that $l = 0$. Let r tend to ∞ in (5.6), then we have

$$(n-1)\int_0^\infty \{u'(s)\}^2 \frac{ds}{s} = 0.$$

Since $n \geq 2$ and u' is continuous, we have $u' \equiv 0$ for any $r \geq 0$. This is impossible because we have assumed that $u(r) \downarrow 0$ and $u(0) = \phi_0 > 0$. Hence, $l \neq 0$ and then we have proved that $l = \phi_0$. $\square$

REMARK 5.3. As we proved the stability of standing waves in Chapters 3 and 4, we hope to show that the standing wave solution $e^{i\omega t}\phi_\omega$ is stable for any ω close to ω_0. However, the difference of topology between the sufficient condition (1.7) in Proposition VI and the convergence of $\phi_\omega(x)$ as $\omega \to \omega_0$ in Proposition 5.1 makes it hard to apply the similar method.

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