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Self-dual Kähler metrics of neutral signature on complex surfaces

by

Hiroyuki KAMADA

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A thesis presented

by

Hiroyuki Kamada

to

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Contents

1 Introduction

Geometric notions for Riemannian metrics are often generalized to their counterparts for pseudo-Riemannian metrics of certain signature. For instance, the notion of Einstein metrics and that of Kähler metrics are straightforwardly extended to indefinite ones. Though Riemannian and Lorentzian metrics are principal objects in pseudo-Riemannian geometry, indefinite metrics of other signatures have also their own significance. Indeed, indefinite Kähler metrics possess non-Lorentzian signature, and in the lowest dimensional case, these metrics are of neutral signature (i.e., of type (2,2)).

Throughout this thesis, we call an indefinite metric of neutral signature simply a *neutral metric*. After Ooguri-Vafa [79], Ricci-flat neutral Kähler metrics on complex surfaces have drawn considerable attention in mathematical physics. Motivated by their result, recently Petean [82] studied neutral Kähler Einstein metrics on compact complex surfaces and obtained a remarkable classification of compact Ricci-flat neutral Kähler surfaces.

Next to the Ricci-flat case, then interesting are scalar-flat neutral Kähler metrics on compact complex surfaces. Note that, as in the Riemannian case, the notion of self-duality is similarly defined for neutral metrics on oriented four-manifolds, and the scalar-flatness of a neutral Kähler metric on a complex surface is equivalent to its self-duality. Hence the existence of scalar-flat neutral Kähler metrics is closely related to that of self-dual neutral metrics on oriented four-manifolds. The objective of this thesis is to study the existence problem of self-dual neutral Kähler metrics on complex surfaces. In particular, we are most concerned with the explicit construction of these metrics.

Since the work of Atiyah-Hitchin-Singer [4], self-dual Riemannian metrics, together with the twistor theory, have been extensively studied in various context. It is known that many four-manifolds admit self-dual Riemannian metrics (cf. Taubes [87]). In particular, LeBrun [60] invented a remarkable method, called the hyperbolic ansatz, of constructing self-dual Riemannian metrics, and then provided us with explicit self-dual Riemannian metrics on a variety of compact oriented four-manifolds, in particular, on the connected sum $k\mathbb{CP}^2 = \mathbb{CP}^2\# \cdots \#\mathbb{CP}^2$ of arbitrary k-copies of the complex projective plane $\mathbb{CP}^2$ (see [60], LeBrun [61], [62], and Kim [49]). It should be remarked, however, that not all four-manifolds admit self-dual Riemannian metrics; for example, the product $S^2 \times S^2$ of two-spheres admits no self-dual Riemannian metric.

In recent years, self-dual neutral metrics on four-manifolds have also been studied from several different points of view. For instance, twistor theories relevant to neutral metrics have been studied by Blair [8], Jensen-Rigoli [40], Machida-Sato [68], Mason-Woodhouse [69] and Vaisman [89]. Also, it is known that a neutral metric on a four-manifold whose Jacobi operator satisfies a certain condition, called the (pointwise) Osserman condition, should be self-dual (and Einstein). Works on this result and related topics can be seen in Alekseevsky et al. [2], Blažić et al. [10], Bonome et al. [13], García-Río et al. [27], and Garía-Río et al. [29]. Moreover, self-dual neutral metrics are regarded as appropriate generalizations of indefinite analogues of such metrics as conformally-flat metrics, hyperhermitian metrics, scalar-flat Kähler metrics, and hyperkähler metrics (see Sections 2.1, 2.2, 3.1 and 3.2). However, compared with the Riemannian case, the existence problem of self-dual neutral metrics has not been well explored.

Regarding the existence of such metrics, there have been known many examples of non-compact four-manifolds with self-dual metric. For instance, the complete lift of any Kähler metric on a Riemann surface is a self-dual neutral metric (more precisely, a scalar-flat neutral Kähler metric) on its tangent bundle (cf. Cendan-Verdes et al. [17], Yano-Kobayashi [93], Yano-Ishihara [92]). For other non-compact examples, see Section 2.1 and Chapter 4.

As for compact examples, a flat torus and the product $S^2 \times S^2$ of unit round two-spheres with the product neutral metric have been known. Since these examples are not only self-dual but also conformally-flat, it is natural to ask the following question which is underlying in our study.

Question 1.1 *Does there exist a (simply-connected) compact four-manifold admitting a non-conformally-flat, self-dual neutral metric?*

It should be remarked that, in Petean [82], several examples of non-flat, Ricci-flat neutral Kähler metrics (thus non-conformally-flat, self-dual neutral metrics) were constructed on certain compact complex surfaces, but they are not simply-connected.

Note that, on a compact four-manifold, the conformal-flatness of self-dual Riemannian metric is completely controlled by Hirzebruch signature formula (cf. Besse [7]). In particular, one can see that any self-dual Riemannian metric on the four-sphere S^4 is always conformally-flat. By virtue of Kuiper's theorem ([55]), a compact simply-connected four-manifold with a conformally-flat Riemannian metric is conformally equivalent to the unit round sphere S^4, and hence a (positive-definite) self-dual conformal structure is unique on S^4.

Since the signature formula is also established for compact four-manifolds with neutral metric (see Matsushita-Law [73], Appendix 5.2), it is then natural to ask whether a similar result holds in the neutral case. With the help of Liouville's theorem (see Appendix 5.3), Kuiper [55] also proved that a compact simply-connected four-manifold with a conformally-flat indefinite metric is conformally equivalent to the product $S^2 \times S^2$ with the product neutral metric of unit round metrics. Hence the conformal structure of a conformally-flat neutral metric on $S^2 \times S^2$ is also unique. Then, as a special case of Question 1.1, we are interested in the following

Question 1.2 *Is the conformal class of a self-dual neutral metric on $S^2 \times S^2$ unique? In other words, does there exist a non-conformally-flat, self-dual neutral metric on $S^2 \times S^2$?*

Regarding Question 1.2, we will give the following answer, which is one of our main results in this thesis:

Theorem 1.3 ([45], [46]) *There exists a family of explicit self-dual neutral metrics on $S^2 \times S^2$, which contains non-conformally-flat ones. Furthermore, for each self-dual metric $\bar{g}$ in this family, there exists an almost complex structure I on $S^2 \times S^2$ such that $(\bar{g}, I)$ is a neutral Kähler structure on the product $\mathbb{CP}^1 \times \mathbb{CP}^1$ of complex projective lines.*

We will construct these self-dual (Kähler) metrics by employing an indefinite analogue of LeBrun's hyperbolic ansatz introduced in [47], which is referred to as the *de Sitter ansatz* for brevity. We will also characterize these metrics as self-dual neutral Kähler metrics with an S^1-symmetry of certain type (see Theorem 2.30). It should be remarked that neutral Kähler surfaces $\mathbb{CP}^1 \times \mathbb{CP}^1$ endowed with these self-dual neutral metrics are isomorphic to each other as symplectic manifolds, but not isometric as pseudo-Riemannian manifolds in general. Indeed, a conformally-flat metric and a non-conformally-flat one are never isometric, and hence at least two isometry classes of self-dual neutral Kähler metrics are defined on $\mathbb{CP}^1 \times \mathbb{CP}^1$. Furthermore, we will prove that self-dual metrics in Theorem 1.3 yield infinitely many different isometry classes on $S^2 \times S^2$.

As a special class of self-dual neutral metrics, we are also interested in *neutral hyperkähler structures*, which are indefinite counterparts of hyperkähler structures (see Section 3). Note that any neutral hyperkähler metric is Ricci-flat and neutral Kähler (and hence self-dual). Then, similar to Question 1.1, we can ask the following

Question 1.4 *What kind of compact complex surface does admit a non-flat neutral hyperkähler metric?*

In the positive-definite case, it is known that a compact complex surface admitting a hyperkähler metric is biholomorphic to either a complex torus or a $K3$ surface. Since any hyperkähler metric is Ricci-flat and Kähler (thus anti-self-dual), it follows from Gauss-Bonnet formula that the flatness of a compact hyperkähler surface is determined by its topology. For instance, any hyperkähler metric is flat on a complex torus, but not on a $K3$ surface (see Hitchin [35], cf. [7]).

In the indefinite case, it is also known that a compact complex surface with neutral hyperkähler metric is biholomorphic to either a complex torus or a primary Kodaira surface (see [43], cf. Section 3.2). Then, Question 1.4 reduces to the following

Question 1.5 *Does there exist a non-flat neutral hyperkähler metric on a primary Kodaira surface or a complex torus?*

Regarding Question 1.5, one might expect that any neutral hyperkähler metric on a complex torus or a primary Kodaira surface is flat, since their Euler characteristics are zero and Gauss-Bonnet formula also holds for compact four-manifolds with neutral metric (see [73], Appendix 5.2). In fact, for an arbitrary neutral hyperkähler metric, we obtain a canonical expression of its fundamental form, and a characterization of its flatness. For example, on a primary Kodaira surface, any neutral hyperkähler structure is obtained in the following fashion (see [43]):

Theorem 1.6 ([43]) *Let $X = \mathbb{C}^2/G$ be a primary Kodaira surface. Then, with respect to suitable complex coordinates (w_1, w_2) of $\mathbb{C}^2$, the fundamental form $(\Omega_I, \Omega_{\mathcal{J}}, \Omega_{'K})$ of any neutral hyperkähler structure $(g, I, \mathcal{J}, 'K)$ on X is expressed as*

$$\Omega_I = \mathrm{Im}(dw_1 \wedge d\overline{w_2}) + \sqrt{-1}\mathrm{Re}(w_1)dw_1 \wedge d\overline{w_1} + (\sqrt{-1}/2)\ \partial\overline{\partial}\varphi,$$
$$\Omega_{\mathcal{J}} = \mathrm{Re}(e^{\sqrt{-1}\theta}dw_1 \wedge dw_2), \quad \Omega_{'K} = \mathrm{Im}(e^{\sqrt{-1}\theta}dw_1 \wedge dw_2),$$

where θ is a real constant and φ is a solution of the equation

$$4\sqrt{-1}(\mathrm{Im}(dw_1 \wedge d\overline{w_2}) + \sqrt{-1}\mathrm{Re}(w_1)dw_1 \wedge d\overline{w_1}) \wedge \partial\overline{\partial}\varphi = \partial\overline{\partial}\varphi \wedge \partial\overline{\partial}\varphi.$$

Furthermore, g is flat if and only if φ is constant.

Since one can easily find a nonconstant solution of this equation, we have the following answer to Question 1.5:

Corollary 1.7 ([43]) *There exists a non-flat neutral hyperkähler structure on any primary Kodaira surface.*

We also see that there exist non-flat neutral hyperkähler structures on a complex torus, since the fundamental form of a neutral hyperkähler structure is expressed similarly (see Section 3.4). In consequence, the flatness of compact neutral hyperkähler surfaces cannot be determined by the topology of underlying complex surfaces.

Since a neutral hyperkähler metric is Ricci-flat and self-dual, non-flat neutral hyperkähler metrics provide us with examples of non-conformally-flat, self-dual neutral metrics. From those non-conformally-flat metrics in Theorem 1.3 and Corollary 1.7, we observe that, though self-dual neutral metrics are defined similarly, their global properties are rather different from Riemannian ones. Note that Petean [82] also obtained non-flat, Ricci-flat neutral Kähler metrics on primary Kodaira surfaces and complex tori, each of which is indeed a neutral hyperkähler one (see [43]).

The present thesis is organized as follows: In Chapter 2, we first review certain conditions which assure the existence of neutral metrics on four-manifolds, and illustrate several examples of compact four-manifolds that admit these metrics. We next recall briefly relevant basic definitions and properties of self-dual neutral metrics and neutral Kähler metrics, and then examine the existence of self-dual neutral Kähler structures on compact complex surfaces. Namely, we distinguish compact complex surfaces that can admit self-dual neutral Kähler structures (Proposition 2.7).

In Sections 2.3, 2.4 and 2.5, we study self-dual neutral Kähler surfaces with a certain isometric S^1-action, based on the results in [46] (cf. [45]). An indefinite analogue of the generalized Gibbons-Hawking ansatz, which provides a method of constructing self-dual neutral Kähler metrics, and its generalization are introduced. In particular, we reexamine self-dual neutral metrics constructed by the de Sitter ansatz, the indefinite analogue of LeBrun's hyperbolic ansatz, and also obtain a characterization of this ansatz. Indeed, we prove that a compact self-dual neutral Kähler surface with a certain S^1-symmetry is biholomorphically isometric to the product $\mathbb{CP}^1 \times \mathbb{CP}^1$ of complex projective lines with a metric constructed by the de Sitter ansatz (see Theorem 2.30). In particular, by employing the de Sitter ansatz, we construct a wealth of explicit self-dual neutral Kähler metrics on $\mathbb{CP}^1 \times \mathbb{CP}^1$. Section 2.5 is devoted to the study of the isometry classes of self-dual neutral metrics on $S^2 \times S^2$ constructed by the de Sitter ansatz. We prove a necessary and sufficient condition for these metrics to be isometric to each other. As a consequence, we see that there exist infinitely many different isometry classes of self-dual neutral metrics on $S^2 \times S^2$ (Theorem 2.31).

Chapter 3 is devoted to the study of neutral hyperkähler structures on four-manifolds (cf. [42] and [43]). We begin by recalling several basic properties of split-quaternion structures and neutral hyperkähler ones on four-manifolds. For example, we prove that the integrability of a split-quaternion structure implies the self-duality of a compatible neutral metric (Proposition 3.10). Furthermore, we obtain a characterization of a neutral hyperkähler structure on a four-manifold in terms of the triplet of symplectic forms satisfying certain algebraic identities (Proposition 3.6). In particular, from this characterization, we see that a nonvanishing holomorphic two-form exists on any neutral hyperkähler surface. Then we show that a compact neutral hyperkähler surface should be biholomorphic to either a complex torus or a primary Kodaira surface. In Section 3.3, we study neutral hyperkähler structures on primary Kodaira surfaces, and obtain a canonical expression of the fundamental form of any neutral hyperkähler structure with respect to suitable complex coordinates. By making use of this canonical expression, we prove that every primary Kodaira surface admits non-flat neutral hyperkähler structures (Theorems 3.16 and 3.17). We also prove the corresponding results for complex tori in Section 3.4 (Theorem 3.19).

In Chapter 4, as examples of self-dual neutral Kähler metrics on noncompact complex surfaces studied in [47], we reexamine neutral metrics of Fubini-Study type on the indefinite complex projective space $\mathbb{CP}^2_1$ and of LeBrun type on complex line bundles (e.g., the cotangent bundle T^*H^2) over the real hyperbolic plane H^2. In particular, the Fubini-Study type neutral metric on $\mathbb{CP}^2_1$ is investigated from a point of view of the de Sitter ansatz.

In Appendices, we first prove Proposition 2.14, the Jones-Tod correspondence. In the course of its proof, we find a certain explicit expression of the self-dual part of the Weyl conformal curvature tensor of a metric constructed by the de Sitter ansatz, and prove Proposition 2.25. The second appendix is devoted to Hirzebruch signature formula and Gauss-Bonnet formula for a compact four-manifold with neutral metric in terms of its curvature tensor (see Matsushita [70], Avez [5] and Chern [18], Matsushita-Law [73]). Then we apply these formulas to compact neutral Kähler surfaces (cf. [47]). In the last appendix, Liouville's theorem for indefinite metrics is proved, which plays an essential role in the proof of Kuiper's theorem ([55]).

2 Neutral Kähler surfaces

2.1 Geometry of four-manifolds with neutral metric

We begin by recalling several conditions for the existence of neutral metrics on four-manifolds (see Matsushita [70] and [71]). Let M be a smooth connected four-manifold. Then M admits a neutral metric (i.e., a pseudo-Riemannian metric of type (2,2)) if and only if there exists a two-dimensional tangential distribution on M. If M is simply-connected, then this condition is equivalent to the existence of a pair $(I, \bar{I})$ of almost complex structures on M, where I is compatible with an orientation and $\bar{I}$ with its opposite.

Concerning the existence of almost complex structures, it is known by Wu ([91]) that a compact oriented four-manifold M admits an almost complex structure if and only if in the second cohomology group $H^2(M;\mathbb{Z})$ of M there exists an element c satisfying

$$c \equiv w_2(M) \bmod 2, \tag{2.1}$$

$$c^2 = 2\chi(M) + 3\tau(M), \tag{2.2}$$

where $w_2(M)$, $\chi(M)$ and $\tau(M)$ denote the second Stiefel-Whitney class, the Euler characteristic and the Hirzebruch signature of M, respectively (see also Barth et al. [6]). The first relation means that c is an integral lift of the second Stiefel-Whitney class $w_2(M)$, and hence implies that c is a characteristic element, that is, c satisfies

$$\alpha^2 = \alpha{\cdot}\alpha \equiv c{\cdot}\alpha \mod 2$$

for any $\alpha \in H^2(M;\mathbb{Z})$, where $\cdot$ stands for the cup product. Then, by an algebraic argument, we obtain $c^2 \equiv \tau(M) \bmod 8$, which is equivalent to

$$\chi(M) + \tau(M) \equiv 0 \bmod 4 \tag{2.3}$$

under the condition (2.2) (see Serre [86], Gompf-Stipsicz [32]). Note that if M admits an integrable complex structure I, then the first Chern class $c_1(M, I)$ of (M, I) gives a characteristic element c, and that the relation (2.3) follows from Noether's formula:

$$c_1^2(M,I) + c_2(M,I) = 12(1 - q(M,I) + p_{\mathrm{g}}(M,I)) \equiv 0 \mod 12, \tag{2.4}$$

where $c_2(M, I)$ is the second Chern class, and $q(M, I)$ and $p_{\mathrm{g}}(M, I)$ denote the irregularity and the geometric genus of (M, I), respectively.

The existence of almost complex structures I and $\bar{I}$ mentioned above implies the following necessary conditions:

$$\chi(M) + \tau(M) \equiv 0, \quad \chi(M) - \tau(M) \equiv 0 \bmod 4. \tag{2.5}$$

On the other hand, Hirzebruch-Hopf [34] obtained a necessary and sufficient condition for the existence of an oriented two-dimensional tangential distribution on a compact four-manifold. Taking account of their result, Matsushita [70] showed that the condition (2.5) is also sufficient when M is simply-connected (For general case, see also Matsushita [71]). Therefore we see that the product $S^2 \times S^2$, the connected sum $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ and a $K3$ surface admit neutral metrics, but no connected sum $k\mathbb{CP}^2 = \mathbb{CP}^2 \# \cdots \# \mathbb{CP}^2$ admits a neutral metric, where $\overline{\mathbb{CP}^2}$ denotes the complex projective plane $\mathbb{CP}^2$ with the orientation opposite to that defined by its complex structure (and $0\mathbb{CP}^2$ stands for S^4).

Next, we recall the fundamental properties of the curvature tensor of the Levi-Civita connection of a pseudo-Riemannian four-manifold with neutral metric (see [42], Akivis-Goldberg [1], Besse [7], Mason-Woodhouse [69]). Let (M, g) be a smooth four-manifold with neutral metric g and ∇ the Levi-Civita connection of $M = (M, g)$. Let X, Y, Z, Z' denote arbitrary vector fields on M. Then the curvature tensor R, the Ricci tensor Ric, the scalar curvature s, the traceless Ricci tensor Ric_0 and the Weyl conformal curvature tensor W are defined respectively as follows:

$$
\begin{aligned}
R(X,Y)Z &:= \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z, \\
\mathrm{Ric}(Y,Z) &:= \mathrm{tr}(X \mapsto R(X,Z)Y), \\
s &:= \mathrm{tr}_g \mathrm{Ric}, \\
\mathrm{Ric}_0(Y,Z) &:= \mathrm{Ric}(Y,Z) - \frac{s}{4} g(Y,Z), \\
g(W(X,Y)Z,Z') &:= g(R(X,Y)Z,Z') \\
&\quad -\frac{1}{2}\Big(\mathrm{Ric}_0(Y,Z)g(X,Z') - \mathrm{Ric}_0(X,Z)g(Y,Z') \\
&\quad +\mathrm{Ric}_0(X,Z')g(Y,Z) - \mathrm{Ric}_0(Y,Z')g(X,Z)\Big) \\
&\quad -\frac{s}{12}\Big(g(X,Z')g(Y,Z) - g(Y,Z')g(X,Z)\Big),
\end{aligned}
$$

where tr (resp. tr_g) means the trace (resp. the g-trace) of a (1,1)-tensor (resp. a (2,0)-tensor). Let T be a curvature-like (4,0)-tensor field on M, that is, a section of $\otimes^4 T^*M$ satisfying

$$T(X,Y,Z',Z) = -T(X,Y,Z,Z') = T(Y,X,Z,Z'),$$
$$T(X,Y,Z,Z') + T(X,Z,Z',Y) + T(X,Z',Y,Z) = 0.$$

From T we define an endomorphism, called also T, on the space of two-forms $\Lambda^2 = \Lambda^2 T^*M$ by

$$g(T(X^\flat \wedge Y^\flat), Z^\flat \wedge Z'^\flat) = T(X,Y,Z,Z'), \tag{2.6}$$

where $X^\flat, \ldots, Z'^\flat$ denote the metric-duals of $X, \ldots, Z'$ (e.g., $X^\flat := g(X,\cdot)$), respectively. Define (4,0)-tensor fields corresponding to R, Ric_0 and W by

$$\begin{aligned} R(X,Y,Z,Z') &:= g(R(Z,Z')Y,X) \\ \mathrm{Ric}_0(X,Y,Z,Z') &:= \mathrm{Ric}_0(Z',Y)g(Z,X) - \mathrm{Ric}_0(Z,Y)g(Z',X) \\ &\quad + \mathrm{Ric}_0(Z,X)g(Z',Y) - \mathrm{Ric}_0(Z',X)g(Z,Y), \\ W(X,Y,Z,Z') &:= g(W(Z,Z')Y,X). \end{aligned}$$

Then, from the first Bianchi identity, these are curvature-like tensor fields, since R satisfies $g(R(X,Y)Z,Z') = -g(Z,R(X,Y)Z')$. Furthermore, R, Ric_0 and W also give rise to the corresponding endomorphisms on Λ^2 defined respectively by setting $T = R, \mathrm{Ric}_0$ and W in (2.6). Then we have

$$R = W \oplus \frac{1}{2}\mathrm{Ric}_0 \oplus \frac{s}{12}\mathrm{Id} : \Lambda^2 \to \Lambda^2. \tag{2.7}$$

In what follows, we assume that M is an oriented four-manifold. Then the Hodge star operator $*$ on Λ^2 is defined by

$$\alpha \wedge * \beta = g(\alpha, \beta) * 1$$

for arbitrary two-forms α and β, where $*1$ is the volume form of $M = (M,g)$. Hence $*$ satisfies

$$*^2 = \mathrm{Id}.$$

Then the space Λ^2 splits as

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-, \tag{2.8}$$

where $\Lambda^2_\pm$ denote the ± 1-eigenspaces of the Hodge star operator $*$, that is, $\Lambda^2_\pm := \{\alpha \in \Lambda^2 | \ * \alpha = \pm\alpha\}$.

Since $*W = W*$ and $*\mathrm{Ric}_0 = -\mathrm{Ric}_0*$, the curvature operator R splits into

$$R = W_+ \oplus W_- \oplus \frac{1}{2}\mathrm{Ric}_0 \oplus \frac{s}{12}\mathrm{Id}, \tag{2.9}$$

where $W_\pm := (W \pm *W)/2$. This splitting (2.9) then leads us to the following

Definition 2.1 An oriented pseudo-Riemannian four-manifold M with neutral metric g is said to be *self-dual* (resp. *anti-self-dual*) if $*W = W$, that is, $W_- \equiv 0$ (resp. $*W = -W$, that is, $W_+ \equiv 0$). In particular, if $W \equiv 0$ on M, M is said to be *conformally-flat*.

Most typical examples of four-manifolds with self-dual neutral metric are the spaces of constant curvature $\mathbb{R}^4_2$, S^4_2 and H^4_2. These are not only self-dual but also conformally-flat (and Einstein). Note that $\mathbb{R}^n_\nu$, S^n_ν and H^n_ν are defined by

$$\begin{aligned}
\mathbb{R}^n_\nu &= (\mathbb{R}^n, g_{\mathbb{R}^n_\nu}), \quad g_{\mathbb{R}^n_\nu} := -\sum_{k=0}^{\nu-1} dx_k^2 + \sum_{l=\nu}^{n-1} dx_l^2, \\
S^n_\nu &= \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}_\nu \;\middle|\; -\sum_{k=0}^{\nu-1} x_k^2 + \sum_{l=\nu}^{n} x_l^2 = +1 \right\}, \\
&\quad g_{S^n_\nu} := (g_{\mathbb{R}^{n+1}_\nu})|_{S^n_\nu}, \\
H^n_\nu &= \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}_{\nu+1} \;\middle|\; -\sum_{k=0}^{\nu} x_k^2 + \sum_{l=\nu+1}^{n} x_l^2 = -1 \right\}, \\
&\quad g_{H^n_\nu} := (g_{\mathbb{R}^{n+1}_{\nu+1}})|_{H^n_\nu},
\end{aligned}$$

and have constant curvature 0, $+1$ and -1, respectively.

The indefinite complex projective space $\mathbb{CP}^2_1$ and the indefinite complex hyperbolic space $\mathbb{CH}^2_1$ are also examples of four-manifolds with (anti-)self-dual (non-conformally-flat) metric (see Chapter 4). Concerning their compact quotients, $\mathbb{R}^4_2$ admits a compact quotient $\mathbb{R}^4_2/\Gamma$ by any lattice Γ in $\mathbb{R}^4_2$. Namely, $\mathbb{R}^4_2/\Gamma$ is a flat torus. However, it is known that none of S^4_2, H^4_2, $\mathbb{CP}^2_1$, $\mathbb{CH}^2_1$ admit compact quotients by discrete subgroups of their isometry groups (see O'Neill [78], Wolf [90]).

As a compact example, we obtain the product $S^2 \times S^2$ of unit round two-spheres with the product neutral metric $g_{S^2 \times S^2} = -h_{S^2} \oplus h_{S^2}$, which is conformally-flat (thus self-dual). The product $\Sigma_- \times \Sigma_+$ of hyperbolic spaces with metric $g_{\Sigma_- \times \Sigma_+} = -h_- \oplus h_+$ is also conformally-flat, where each $h_\pm$

denotes a Riemannian metric on $\Sigma_\pm$ of constant curvature -1, respectively. But these examples are not Einstein.

It is also known that the conformal-flatness ($W \equiv 0$) is equivalent to the condition that, around each point, there exist a neighborhood U and a function f on U such that $(U, e^{2f}\cdot g)$ is isometrically embedded into $\mathbb{R}^4_2$ (cf. Besse [7], Lafontaine [56]). By virtue of a result of Kuiper [55], any compact, simply-connected pseudo-Riemannian four-manifold with conformally-flat metric is conformally equivalent to either the unit round sphere S^4 or the product $S^2 \times S^2$ of two spheres with the product neutral metric of unit round metrics. We will give other examples later.

We next recall the significance of the self-duality of neutral metrics in relation to totally null planes, and give, in terms of totally null plane fields, a sufficient condition for neutral metrics on four-manifold to be self-dual. Let (M, g) be an oriented pseudo-Riemannian four-manifold with neutral metric g. A tangential distribution $\mathcal{F}$ on (M, g) is called a *totally null plane field* (or a *maximal isotropic subbundle*) if, at each point of M, $\mathcal{F}$ is a two-dimensional distribution consisting of null vectors with respect to g (i.e., the orthogonal complement $\mathcal{F}^\perp$ of $\mathcal{F}$ with respect to g coincides with $\mathcal{F}$ itself). Note that the condition for $\mathcal{F}$ to be totally null is clearly conformally invariant. Let $\mathrm{Gr}_2(TM)$ be the Grassmannian bundle over M whose fiber at $x \in M$ consists of two-dimensional subspaces in T_xM, and $\mathbb{P}(\Lambda^2)$ denote the bundle over M with the real projective spaces of one-dimensional subspaces in $\Lambda^2 T^*_x M$ as fibers. We now define a map $\Phi : \mathrm{Gr}_2(TM) \longrightarrow \mathbb{P}(\Lambda^2)$ by $\Phi : \mathrm{span}\{u, v\} \mapsto [u^\flat \wedge v^\flat]$, where $[\cdot]$ denotes the equivalence class in $\mathbb{P}(\Lambda^2 T^*_x M)$. Then the following fact is known.

Proposition 2.2 *$\Phi(\mathcal{F})$ belongs to $\mathbb{P}(\Lambda^2_\pm)$ for any totally null plane field $\mathcal{F}$.*

Hence we have the following

Definition 2.3 A totally null plane field $\mathcal{F}$ is said to be *self-dual* (resp. *anti-self-dual*) if $\Phi(\sigma)$ belongs to $\mathbb{P}(\Lambda^2_+)$ (resp. $\mathbb{P}(\Lambda^2_-)$).

By $\kappa(\sigma)$ we denote the sign of $g(R(u, v)v, u)$, which is well-defined for any plane $\sigma = \mathrm{span}\{u, v\}$. It is proved by Dajczer-Nomizu [19] that if $\kappa(\sigma) = 0$ for any degenerate plane σ on (M, g), then all nondegenerate planes have the same sectional curvatures, that is, (M, g) has constant curvature. Noting that $g(R(\xi, \eta)\eta, \xi) = g(W(\xi, \eta)\eta, \xi)$ for any totally null plane $\sigma = \mathrm{span}\{\xi, \eta\}$, we can obtain a conformal analogue of the result in [19]. More precisely, in the four-dimensional case, we can prove the following

Proposition 2.4 *A neutral metric g on an oriented four-manifold M is self-dual (resp. anti-self-dual) if and only if $\kappa(\sigma) = 0$ for any anti-self-dual (resp. self-dual) totally null plane σ at each point of M. In particular, g is conformally-flat if and only if $\kappa(\sigma) = 0$ for any totally null plane σ.*

Concerning the signature of a totally null plane field, we recall the following lemma for later use (see [42]).

Lemma 2.5 *Let (M, g) be an oriented pseudo-Riemannian four-manifold with neutral metric g and $\mathcal{F}$ a totally null plane field on (M, g). If $\mathcal{F}$ is integrable, then $\kappa(\mathcal{F}) = 0$.*

2.2 Hermitian geometry of neutral metrics

Let (M, g) be a real four-manifold with a neutral metric g and I an almost complex structure on M (i.e., $I^2 = -\mathrm{Id}$ on TM). A triplet (M, g, I) is called a neutral Hermitian surface if I is integrable and g is invariant by I, that is, (g, I) satisfies

$$N_I(X,Y) := [IX, IY] - [X, Y] - I[IX, Y] - I[X, IY] \equiv 0, \tag{2.10}$$

and

$$g(IX, IY) = g(X, Y) \tag{2.11}$$

for arbitrary vector fields X and Y on M. Namely, a neutral Hermitian surface (M, g, I) is a complex surface (M, I) with an I-invariant neutral metric g. For a neutral Hermitian surface (M, g, I), the fundamental form Ω_I of (M, g, I) is defined to be a differential two-form given by

$$\Omega_I(X, Y) := g(IX, Y). \tag{2.12}$$

A neutral Hermitian surface (M, g, I) is said to be *neutral Kähler* or a *neutral Kähler surface* if Ω_I is closed.

Via the usual identification $\mathbb{R}^4 = \mathbb{C}^2$, we can regard $\mathbb{R}^4_2$ as a neutral (hyper)Kähler surface, which is denoted by $\mathbb{C}^2_1$. The indefinite complex projective plane $\mathbb{CP}^2_1$ and the indefinite complex hyperbolic plane $\mathbb{CH}^2_1$ are also neutral Kähler surfaces. A complex torus $\mathbb{C}^2_1/\Gamma$ with a flat neutral metric and $(\mathbb{CP}^1 \times \mathbb{CP}^1, g_{S^2 \times S^2})$ and $(\Sigma_- \times \Sigma_+, g_{\Sigma_- \times \Sigma_+})$ with the product complex structures are examples of compact neutral Kähler surfaces.

It should be remarked that the fundamental form Ω_I is a symplectic form on M compatible with the orientation opposite to that determined by I. To see this, let $\{u_1, u_2\}$ be a (local) unitary frame field, that is, $\{u_1, u_2\}$ satisfies

$$Iu_1 = \sqrt{-1}u_1, \quad Iu_2 = \sqrt{-1}u_2,$$

and

$$\begin{gathered} g(u_1, u_1) = g(u_1, u_2) = g(u_2, u_2) = 0, \\ g(u_1, \overline{u_1}) = -1, \quad g(u_2, \overline{u_2}) = +1, \quad g(u_1, \overline{u_2}) = 0. \end{gathered}$$

Then the metric g and the fundamental form Ω_I are given respectively as

$$g = 2(-u^1\overline{u^1} + u^2\overline{u^2}), \quad \Omega_I = \sqrt{-1}(-u^1 \wedge \overline{u^1} + u^2 \wedge \overline{u^2}),$$

where $\{u^1, u^2\}$ denotes the dual unitary coframe field of $\{u_1, u_2\}$. Hence Ω_I satisfies

$$\Omega_I^2 = \Omega_I \wedge \Omega_I = -2(\sqrt{-1}u^1 \wedge \overline{u^1}) \wedge (\sqrt{-1}u^2 \wedge \overline{u^2}).$$

Therefore Ω_I^2 is compatible with the orientation opposite to that defined by I.

Throughout this thesis, we always regard a neutral Kähler surface as being oriented by its complex structure. For later convenience, we denote by $\bar{M}$ the manifold M with the opposite orientation, and simply call a symplectic form compatible with the orientation of $\bar{M}$ an opposite symplectic form on M.

The space Λ^2 of two-forms on a neutral Hermitian surface $M = (M, g, I)$ decomposes into the I-invariant and the I-anti-invariant subspaces as

$$\Lambda^2 = \Lambda^{\mathrm{inv}} \oplus \Lambda^{\mathrm{anti}},$$

where Λ^{inv} and Λ^{anti} are given respectively by

$$\begin{aligned} \Lambda^{\mathrm{inv}} &= \{\alpha \in \Lambda^2 \mid \alpha(IX, IY) = \alpha(X, Y)\}, \\ \Lambda^{\mathrm{anti}} &= \{\alpha \in \Lambda^2 \mid \alpha(IX, IY) = -\alpha(X, Y)\}. \end{aligned}$$

According to the splitting (2.8), the self-dual and the anti-self-dual subspaces $\Lambda^2_\pm$ are also decomposed in the following fashion:

$$\Lambda^2_+ = \Lambda^{\mathrm{inv}}_0, \quad \Lambda^2_- = \mathbb{R}\Omega_I \oplus \Lambda^{\mathrm{anti}}, \tag{2.13}$$

where Λ_0^{inv} consists of I-invariant two-forms on M orthogonal to Ω_I.

As in the Riemannian case, the Ricci tensor Ric of a neutral Kähler surface (M, g, I) is known to be an I-invariant symmetric bilinear form on M. We then define the Ricci form γ by

$$\gamma(X, Y) := \mathrm{Ric}(IX, Y), \tag{2.14}$$

which is a closed real (1,1)-form, and hence determines the first Chern class $c_1(M, I)$ by

$$c_1(M, I) = \frac{1}{2\pi}[\gamma] \in H^{1,1}(M; \mathbb{R}),$$

since $\nabla I \equiv 0$. The scalar curvature s is also determined by the Kähler form Ω_I and the Ricci form γ. In particular, on a neutral Kähler surface (M, g, I), its scalar curvature s vanishes everywhere on M if and only if $\gamma \wedge \Omega_I \equiv 0$. In regard to the self-duality of a neutral Kähler metric g, the following expression holds for the anti-self-dual part W_- of the Weyl conformal tensor W with respect to a suitable real unitary frame field:

$$W_- = \frac{s}{12}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.15}$$

Thus we have the following well-known result (cf. [47]. See Derdziński [21], Itoh [39], LeBrun [58] for the Riemannian analogue).

Proposition 2.6 *Let (M, g, I) be a neutral Kähler surface. Then g is self-dual if and only if g is scalar-flat, that is, $\Omega_I \wedge \gamma \equiv 0$.*

We next write the curvature form of a neutral Kähler surface (M, g, I) in terms of local holomorphic coordinates. Let (w_1, w_2) be local holomorphic coordinates of a complex surface (M, I) with neutral Kähler metric g. For simplicity, we set

$$\partial_\alpha := \partial/\partial w_\alpha, \quad \partial_{\bar\alpha} := \partial/\partial\overline{w_\alpha} \quad \text{and} \quad g_{\alpha\bar\beta} := 2g(\partial_\alpha, \partial_{\bar\beta})$$

$(\alpha, \beta = 1, 2)$. Let ∇ be the Levi-Civita connection of (M, g) and $\{\omega^A_B\}$ the connection form of ∇ with respect to $\{\partial_A\}$ $(A, B = 1, 2, \bar1, \bar2)$. Then $\omega^{\bar\alpha}_\beta = \omega^\alpha_{\bar\beta} \equiv 0$, since $\nabla I \equiv 0$. Moreover ω^α_β (resp. $\omega^{\bar\alpha}_{\bar\beta}$) is a local $(1,0)$-(resp. $(0,1)$-) form, since ∇ is torsion-free. Hence, except for $\{\omega^\alpha_\beta(\partial_\gamma)\}$ and $\{\omega^{\bar\alpha}_{\bar\beta}(\partial_{\bar\gamma})\}$, the components of $\{\omega^A_B\}$ must vanish. Since the Levi-Civita connection ∇ preserves the metric g, we have

$$\omega^\alpha_\beta = \sum_\epsilon g^{\bar\epsilon\alpha}\partial g_{\beta\bar\epsilon}, \qquad \omega^{\bar\alpha}_{\bar\beta} = \sum_\epsilon g^{\bar\alpha\epsilon}\overline{\partial} g_{\epsilon\bar\beta}, \tag{2.16}$$

where $g^{\bar{\alpha}\beta}$ is defined by $\sum_\epsilon g_{\alpha\bar{\epsilon}} g^{\bar{\epsilon}\beta} = \sum_\epsilon g^{\bar{\beta}\epsilon} g_{\epsilon\bar{\alpha}} = \delta_\alpha^\beta$. The curvature form $\{R_B^A\}$ of ∇ is given by

$$R_\beta^\alpha = \overline{\partial}\omega_\beta^\alpha, \qquad R_{\bar{\beta}}^{\bar{\alpha}} = \partial\omega_{\bar{\beta}}^{\bar{\alpha}}. \tag{2.17}$$

In particular, we see that g is flat (i.e., $R \equiv 0$) if and only if every ω_β^α is a local holomorphic one-form on (M, I).

The Ricci form γ is expressed as

$$\gamma = -\sqrt{-1}\partial\overline{\partial}\log|\det(g_{\alpha\bar{\beta}})|, \tag{2.18}$$

which is verified in the same way as in the Riemannian case. In particular, if there exist local holomorphic coordinates (w_1, w_2) at each point of a neutral Kähler surface (M, g, I) such that $|\det(g_{\alpha\bar{\beta}})| \equiv 1$, then (M, g, I) is Ricci-flat.

We now focus our attention on the compact case. Concerning compact self-dual neutral Kähler surfaces, we first show the following

Proposition 2.7 *Let (M, g, I) be a compact self-dual neutral Kähler surface, and $\kappa(M, I)$ denote the Kodaira dimension of (M, I). Then (M, I) is biholomorphic to one of the following surfaces*:

(1) *If $c_1^2(M, I) < 0$, then (M, I) is a surface of class* VII_0 *with no global spherical shell and with positive even second Betti number, and hence $\kappa(M, I) = -\infty$.*

(2) *If $c_1^2(M, I) > 0$ and $\tau(M) > 0$, then (M, I) is a minimal surface of general type with positive even signature, and hence $\kappa(M, I) = 2$.*

(3) *If $c_1^2(M, I) > 0$ and $\tau(M) = 0$, then (M, I) is either a Hirzebruch surface when $\kappa(M, I) = -\infty$, or a minimal surface of general type uniformized by the polydisc when $\kappa(M, I) = 2$.*

(4) *If $c_1^2(M, I) = 0$, then (M, I) is either a hyperelliptic surface, a primary Kodaira surface or a complex torus when $\kappa(M, I) = 0$.*

(5) *If $c_1^2(M, I) = 0$, then (M, I) is a minimal properly elliptic surface with zero signature when $\kappa(M, I) = 1$.*

No surface of type (1) has been known. Several examples of surfaces in the case (2) are known (see Atiyah [3], Kodaira [52]), and the products $\Sigma \times T^2$ of compact Riemann surfaces Σ of genera ≥ 2 and elliptic curves

T^2 are examples in the case (5). However, the author knows no self-dual neutral Kähler metrics on surfaces in the cases (2) and (5). In the case (4), all surfaces admit Ricci-flat neutral Kähler (thus self-dual) metrics (see Petean [82]). In the case (3), the products $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\Sigma_- \times \Sigma_+$ with the standard (indefinite) product metrics are conformally-flat neutral Kähler surfaces, where $\Sigma_\pm$ are compact Riemann surfaces of genera ≥ 2 endowed with Riemannian metrics of curvature -1. We will discuss self-dual neutral Kähler metrics on $\mathbb{CP}^1 \times \mathbb{CP}^1$ later (see Sections 2.2–2.4).

Proof. We first recall the following result due to Petean [82].

Theorem 2.8 (Petean [82]) *Let (M, g, I) be a compact neutral Kähler surface.*

(i) *If $\kappa(M, I) = -\infty$, then (M, I) is either a minimal ruled surface, the blow-up of $\mathbb{CP}^2$ at a point, or a surface of class* VII_0 *with no global spherical shell and with positive even second Betti number.*

(ii) *If $\kappa(M, I) = 0$, then (M, I) is either a hyperelliptic surface, a primary Kodaira surface or a complex torus.*

(iii) *If $\kappa(M, I) = 1$, then (M, I) is a minimal properly elliptic surface with zero signature.*

(iv) *If $\kappa(M, I) = 2$, then (M, I) is a minimal surface of general type with nonnegative even signature.*

It should be noted that the following result of Taubes ([88]) regarding the existence of an opposite symplectic form plays an essential role in obtaining the list above.

Theorem 2.9 (Taubes [88]) *Let $\bar{M}$ be a compact symplectic four-manifold with $b_2^+(\bar{M}) > 1$, oriented by its symplectic structure Ω_I, and let $\bar{I}$ be an Ω_I-compatible almost complex structure on $\bar{M}$, that is, $\Omega_I(\cdot, \bar{I}\cdot)$ is a positive-definite almost Kähler metric on $(\bar{M}, \bar{I})$. Then the first Chern class $c_1(\bar{M}, \bar{I})$ has nonzero Seiberg-Witten invariant.*

Let (M, g, I) be a compact self-dual neutral Kähler surface. Then $\Omega_I \wedge \gamma \equiv 0$. In particular, $[\Omega_I]$ and $c_1(M, I) = [\gamma]/2\pi$ are orthogonal to each other in the cohomology group $H^{1,1}(M; \mathbb{R})$ with respect to the cup product.

In the case where $c_1^2(M, I) < 0$, it follows from Petean's list above (Theorem 2.8) that (M, I) is biholomorphic to either a minimal ruled surface with genus $g \geq 2$ or a surface of class VII_0 with certain properties. Since $c_1(M, I)$ and $[\Omega_I]$ are orthogonal in $H^{1,1}(M; \mathbb{R})$, the assumption $c_1^2(M, I) < 0$ (and $\Omega_I^2 < 0$) also implies that $b_2^-(M) \geq 2$. Hence M should be biholomorphic to the latter, that is, to a certain surface of class VII_0. Indeed, suppose that (M, I) would be biholomorphic to a minimal ruled surface with genus $g \geq 2$. Then, by Theorem 2.9, the Seiberg-Witten invariant for an almost complex structure associated with $(\bar{M}, \Omega_I)$ would not vanish, since $b_2^+(\bar{M}) = b_2^-(M) \geq 2$. On the other hand, since $\bar{M}$ admits a Riemannian metric with positive scalar curvature (see, e.g., LeBrun [64]), all Seiberg-Witten invariants on $\bar{M}$ would vanish, which is a contradiction. Therefore (M, I) is biholomorphic to a surface of class VII_0 in Petean's list, if $c_1^2(M, I) < 0$. (Unfortunately, such a surface is not known at present.)

In the case where $c_1^2(M, I) > 0$, the underlying complex surface (M, I) is biholomorphic to either a rational ruled surface or a minimal surface of general type with nonnegative even signature. Note that there exists a positive-definite Kähler metric h on (M, I) in both cases. Suppose that $\tau(M) > 0$. Then $b_+^2(M) > b_-^2(M) \geq 1$ and hence M cannot admit any Riemannian metric with positive scalar curvature. Therefore M should be biholomorphic to the second candidate, that is, a minimal surface of general type with positive even signature. Suppose that $\tau(M) = 0$. This implies that $c_1^2(M, I) = 2c_2(M, I)(> 0)$. If $b_2^+(M) = b_2^-(M) \geq 2$, then the Seiberg-Witten invariant for $(\bar{M}, \Omega_I)$ does not vanish by Theorem 2.9. Hence $\bar{M}$ cannot admit any positive scalar curvature metric. Therefore it follows from LeBrun's result [64] that the Kodaira dimension of (M, I) should be nonnegative, since M admits a positive-definite Kähler metric. Hence (M, I) is biholomorphic to a minimal surface of general type with $\tau(M) = 0$. Furthermore, M is uniformized by the product of unit discs $D^2 \times D^2$ (see Leung [66], Kotschick [54]). If $b_2^+(M) = b_2^-(M) = 1$, then it follows from Petean's list (Theorem 2.8) again that M is biholomorphic to either a minimal rational ruled surface, the one-point blowing-up of the complex projective plane, or a minimal surface of general type with $b_1(M) = 0$ and $b_2^+(M) = b_2^-(M) = 1$. In the first and second cases, (M, I) is biholomorphic to one of the Hirzebruch surfaces by a result due to Qin [84]. In the third case, (M, I) is also uniformized by the polydisc $D^2 \times D^2$ (see [54]).

The cases (4) and (5) follow from Petean's list (Theorem 2.8). $\square$

2.3 Kähler surfaces with time-like S^1-symmetry

Let (M, g, I) be a neutral Kähler surface. Suppose that (M, g) admits a nontrivial isometric S^1-action generated by a Killing vector field ξ. Then the fixed point set F of this action coincides with $\mathrm{Zero}(\xi)$, the set of zeros of ξ, and becomes a compact complex submanifold of (M, I). We first recall the following

Proposition 2.10 *Let ξ be a Killing vector field on a neutral Kähler surface (M, g, I) (i.e., $\mathcal{L}_\xi g \equiv 0$). Then ξ is a real holomorphic vector field on (M, I) (i.e., $\mathcal{L}_\xi I = 0$).*

Proof. Let ∇ denote the Levi-Civita connection of (M, g). Then, since ξ is a Killing vector field, ξ yields an infinitesimal affine transformation on (M, ∇). Define an endomorphism $A_\xi : T_xM \to T_xM$ by $A_\xi v := -\nabla_v \xi$ for $v \in T_xM$ $(x \in M)$. Since (M, g, I) is a neutral Kähler surface, the holonomy algebra $\mathfrak{h}$ is isomorphic to $\mathfrak{u}(1,1)$. Furthermore, its normalizer in $\mathfrak{so}(2,2)$ is isomorphic to $\mathfrak{u}(1,1)$ itself. It then follows from Corollary 4.3 in Kobayashi-Nomizu [50] that the endomorphism A_ξ belongs to the normalizer of the holonomy algebra $\mathfrak{u}(1,1)$ and hence commutes with I, which means that ξ is a holomorphic vector field on (M, I). $\square$

An isometric action is said to be *time-like* (resp. *space-like*) if ξ satisfies $g(\xi, \xi) < 0$ (resp. $g(\xi, \xi) > 0$) on M, outside F. We will see in Section 2.5 that, for any neutral Kähler metric g on $\mathbb{CP}^1 \times \mathbb{CP}^1$, the fixed point set of a time-like isometric S^1-action on $(\mathbb{CP}^1 \times \mathbb{CP}^1, g)$ has no isolated point, and hence contains two two-dimensional connected components (see Proposition 2.32). We prove here a converse result.

Theorem 2.11 *Let (M, g, I) be a compact neutral Kähler surface. Suppose that (M, g) admits a semi-free isometric S^1-action whose fixed point set F has at least two two-dimensional connected components. If (M, g) is also self-dual, then (M, I) should be biholomorphic to one of the Hirzebruch surfaces. Moreover, F is a disjoint union of two holomorphic spheres. In particular, F has no isolated fixed points.*

Proof. We first recall that $b_2^-(M) = b_2^+(\bar{M}) \geq 1$, since the Kähler form Ω_I of (M, g, I) gives rise to an opposite symplectic structure on M. Let $\bar{I}$ be an Ω_I-compatible almost complex structure on $\bar{M}$. By Theorem 2.9, the first Chern class $c_1(\bar{M}, \bar{I})$ has nonzero Seiberg-Witten invariant, if $b_2^+(\bar{M}) \geq 2$.

Let $\Sigma_1, \ldots, \Sigma_k$ and $\{q_1\}, \ldots, \{q_m\}$ be two-dimensional connected components and isolated fixed points in F, respectively:

$$F = \Sigma_1 \coprod \Sigma_2 \coprod \cdots \coprod \Sigma_k \coprod \{q_1\} \coprod \cdots \coprod \{q_m\} \quad (k \geq 2). \tag{2.19}$$

Since the S^1-action is semi-free, the orbit space $Y := M/S^1$ is a compact, connected three-manifold with boundary $\partial Y \cong \Sigma_1 \coprod \Sigma_2 \coprod \cdots \coprod \Sigma_k$. Let $\pi : M \to Y$ denote the natural projection. For $p_j \in \Sigma_j$ ($j = 1, 2$), there is a smooth path c in Y connecting $\pi(p_1)$ and $\pi(p_2)$ with $c \bigcap \pi(F) = \{\pi(p_1), \pi(p_2)\}$ such that the inverse image $S := \pi^{-1}(c)$ is an embedded two-sphere. Then the self-intersection number of S is zero in both M and $\bar{M}$, and its homology class is non-trivial. Regarding the existence of such an embedded two-sphere S, we recall the following result.

Theorem 2.12 (Kotschick [54]) *Let $\bar{M}$ be a compact oriented four-manifold with $b_2^+(\bar{M}) > 1$ which contains a smoothly embedded two-sphere S of non-negative self-intersection with $S(\neq 0) \in H_2(\bar{M}; \mathbb{Q})$. Then all Seiberg-Witten invariants of $\bar{M}$ vanish.*

Combining Kotschick's result (Theorem 2.12) with a result due to Taubes (Theorem 2.9), we obtain $b_2^-(M) = b_2^+(\bar{M}) \leq 1$, and hence $b_2^-(M) = b_2^+(\bar{M}) = 1$, since $b_2^-(M) \geq 1$ by assumption. For a compact symplectic four-manifold $\bar{M}$ with symplectic orientation, there exists an almost complex structure $\bar{I}$ compatible with the orientation of $\bar{M}$. Then it follows from (2.3) that $\chi(\bar{M}) + \tau(\bar{M}) \equiv 0 \bmod 4$, which is equivalent to that $1 - b_1(\bar{M}) + b_2^+(\bar{M})$ is even. By $b_2^+(\bar{M}) = b_2^-(M) = 1$, the first Betti number $b_1(M) = b_1(\bar{M})$ is also even. It is well-known that a compact complex surface admits a positive-definite Kähler metric if and only if the first Betti number is even (cf. Barth et al. [6]). Therefore M admits a positive-definite Kähler metric, and hence $b_2^+(M) \geq 1$. By a similar argument, we also obtain $b_2^+(M) = 1$. (For compact symplectic four-manifolds with $b_2^+ = 1$, see Ohta-Ono [77] and also McDuff-Salamon [74].)

Let ξ be the Killing vector field generating the S^1-action. Then, from Proposition 2.10, $\xi^{\mathbb{C}} := \xi - \sqrt{-1} I\xi$ is a non-trivial holomorphic vector field on (M, I) with zeros, under the same assumptions as in Theorem 2.11. Owing to Carrell-Howard-Kosniowski [16], any compact complex surface admitting such a vector field has negative Kodaira dimension. Then, from Petean's list (Theorem 2.8), we see that (M, I) should be biholomorphic to a ruled surface, since $b_2^+(M) = 1$.

We next assume that (M, g, I) is also a self-dual neutral Kähler surface. Then, by Proposition 2.7, (M, I) should be biholomorphic to one of the Hirzebruch surfaces.

Finally, we show that the fixed point set F is obtained as $F = \Sigma_1 \coprod \Sigma_2$ for holomorphic spheres Σ_1 and Σ_2. Note first that, by Proposition 2.10, the Killing vector field ξ is also a symplectic vector field with respect to Ω_I (i.e., $\mathcal{L}_\xi \Omega_I \equiv 0$). Since a rational ruled surface M is simply-connected, there exists a moment map $z : M \to \mathbb{R}$ for the given S^1-action, that is, a function z on M satisfying $\iota_\xi \Omega_I = dz$. It is well-known that a moment map $z : M \to \mathbb{R}$ is a perfect Bott-Morse function. Hence we obtain $b_1(F) = b_1(M) = 0$ (cf. Frankel [26]). Therefore each component Σ_j in (2.19) is of genus zero, that is, a holomorphic sphere. On the other hand, the Euler characteristics $\chi(M)$ and $\chi(F)$ satisfy $\chi(F) = \chi(M)$ in general. Then we obtain $2k + m = \chi(F) = \chi(M) = 4$. From the assumption $k \geq 2$, it follows that $k = 2$ and $m = 0$. The proof is now complete. □

We next study a self-dual neutral Kähler surface with a time-like isometric S^1-action in a general setting. (For a Riemannian analogue of the arguments below, see [60] and [63].)

Let (M, g, I) be a neutral Kähler surface and Ω_I denote its Kähler form. Assume that (M, g) admits a time-like isometric S^1-action, and denote by F its fixed point set and by ξ the Killing vector field generating the S^1-action. Assume also that there exists a moment map $z : M \to \mathbb{R}$ for the action. Note that z is unique up to an additive constant and its critical submanifold coincides with F. By virtue of Proposition 2.10, we see that ξ and $I\xi$ satisfy $[\xi, I\xi] \equiv 0$, and hence define a holomorphic foliation $\mathcal{F}$ on $M \setminus F$. Since the holomorphic structure of this foliation $\mathcal{F}$ is compatible with that of (M, I), we can introduce, at least locally, a holomorphic structure on the leaf space $(M \setminus F)/\mathcal{F}$. Let $x + \sqrt{-1}y$ be a local holomorphic coordinate of the leaf space. We now define data w and θ by

$$w := -g(\xi, \xi)^{-1}, \quad \theta := -wg(\xi, \cdot) = g(\xi, \xi)^{-1}\xi^\flat. \tag{2.20}$$

It then follows from the definition of z (i.e., $\iota_\xi \Omega_I = dz$) that

$$Idz = -w^{-1}\theta \quad \text{and} \quad Idx = -dy. \tag{2.21}$$

Since ξ is a Killing vector field on (M, g) and $\theta(\xi) \equiv 1$, the one-form θ is regarded as a connection form of an S^1-bundle $M \setminus F \to (M \setminus F)/S^1$. In particular, $d\theta$ is a basic two-form. Taking account of (2.20) and (2.21), we can express g and Ω_I respectively as

$$\begin{gathered} g = -(wdz^2 + w^{-1}\theta^2) + we^u(dx^2 + dy^2), \\ \Omega_I = -dz \wedge \theta + we^u dx \wedge dy, \end{gathered} \tag{2.22}$$

which are understood to define u. Let t be a fiber-coordinate satisfying $\xi = \partial/\partial t$. Then x, y, z, t are regarded as local coordinates of $M \setminus F$. Note that u and w are independent of t.

The integrability condition of I is equivalent to that

$$\begin{aligned} 0 &\equiv (dx + \sqrt{-1}dy) \wedge (wdz + \sqrt{-1}\theta) \wedge (dw \wedge dz + \sqrt{-1}d\theta) \\ &= ([w_x - d\theta(\partial_y, \partial_z)] + \sqrt{-1}[w_y - d\theta(\partial_z, \partial_x)])dx \wedge dy \wedge dz \wedge dt, \end{aligned} \tag{2.23}$$

where $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$ and $\partial_z = \partial/\partial z$. On the other hand, the closedness of Ω_I is equivalent to

$$
\begin{aligned}
(2.24) \qquad 0 &\equiv d\Omega_I = dz \wedge d\theta + (we^u)_z dz \wedge dx \wedge dy \\
&= [d\theta(\partial_x, \partial_y) + (we^u)_z] dz \wedge dx \wedge dy.
\end{aligned}
$$

From (2.23) and (2.24), we obtain

$$
(2.25) \qquad d\theta = w_x dy \wedge dz + w_y dz \wedge dx - (we^u)_z dx \wedge dy.
$$

Note that $[d\theta]/2\pi$ determines an integral cohomology class on the base space, since θ is a connection form. By $d(d\theta) = 0$, we have the equation:

$$
(2.26) \qquad w_{xx} + w_{yy} - (we^u)_{zz} \equiv 0.
$$

From an argument analogous to that in [60], it follows that the Ricci form γ of (M, g, I) is given by

$$
\gamma = \frac{1}{2} d(Idu),
$$

which is rewritten as

$$
(2.27) \qquad \gamma = -\frac{1}{2}\Big(u_{xx} + u_{yy} - (e^u)_{zz}\Big) dx \wedge dy - (d(w^{-1}u_z) \wedge \theta)_+,
$$

where $(d(w^{-1}u_z) \wedge \theta)_+$ denotes the self-dual part of $d(w^{-1}u_z) \wedge \theta$. Then $(d(w^{-1}u_z) \wedge \theta)_+ \wedge \Omega_I$ vanishes, since $\Omega_I = -dz \wedge \theta + we^u dx \wedge dy$ is an anti-self-dual two-form on (M, g). Recalling that the self-duality of g is equivalent to its scalar-flatness (i.e., $\gamma \wedge \Omega_I \equiv 0$), we obtain the equation:

$$
(2.28) \qquad u_{xx} + u_{yy} - (e^u)_{zz} \equiv 0.
$$

Conversely, given such data $w > 0$ and u on a three-manifold $\mathcal{U}$, we can reconstruct a self-dual neutral Kähler metric on an S^1-bundle over $\mathcal{U}$, by an argument similar to that in [60]. Summarizing these, we obtain an indefinite analogue of the generalized Gibbons-Hawking ansatz (cf. [60]):

Proposition 2.13 *Let $w > 0$ and u be smooth functions on an open set $\mathcal{U}$ in $\mathbb{R}^3$ satisfying* (2.26) *and* (2.28). *Suppose that*

$$
(2.29) \qquad \frac{1}{2\pi}\alpha := \frac{1}{2\pi}\Big(w_x dy \wedge dz + w_y dz \wedge dx - (we^u)_z dx \wedge dy\Big)
$$

determines an integral cohomology class in $H^2(\mathcal{U};\mathbb{R})$. *Let* $\pi : \mathcal{M} \to \mathcal{U}$ *be an* S^1*-bundle over* $\mathcal{U}$ *with connection form* θ *whose curvature is given by* $d\theta = \alpha$. *Define a metric* g *and an almost complex structure* I *by*

$$\begin{gathered} g := -(wdz^2 + w^{-1}\theta^2) + we^u(dx^2+dy^2), \\ Idz := -w^{-1}\theta, \quad Idx := -dy. \end{gathered} \tag{2.30}$$

Then (g, I) *is a self-dual neutral Kähler structure on* $\mathcal{M}$.

Moreover, every self-dual neutral Kähler surface with a time-like isometric S^1*-action is obtained, at least locally, by this construction.*

We now introduce a generalization of this ansatz (Proposition 2.13), based on the Jones-Tod correspondence (cf. Jones-Tod [41], Appendix 5.1).

Let (M, g) be an oriented pseudo-Riemannian four-manifold with neutral metric g admitting a time-like isometric S^1-action. If the S^1-action under consideration is fixed-point free, then the orbit space $Y := M/S^1$ is a smooth three-manifold. Let $\check{g}$ be a Lorentzian metric on Y defined by

$$\pi^*\check{g} = g - \frac{\xi^\flat \otimes \xi^\flat}{g(\xi,\xi)}, \tag{2.31}$$

where $\pi : M \to Y$ is the natural projection and $\xi^\flat := g(\xi, \cdot)$ denotes the metric-dual of ξ. Then $\pi : (M, g) \to (Y, \check{g})$ is a pseudo-Riemannian submersion. Let $\check{\beta}$ be a one-form on Y satisfying

$$\pi^*\check{\beta} = \frac{-dg(\xi,\xi) - 2 *_g (\xi^\flat \wedge d\xi^\flat)}{2g(\xi,\xi)}. \tag{2.32}$$

Then $(\check{g}, -2\check{\beta})$ determines a unique torsion-free affine connection D on Y such that

$$D\check{g} = -2\check{\beta} \otimes \check{g}, \tag{2.33}$$

and gives a (Lorentzian) Weyl structure $([\check{g}], D)$ on Y, where $[\check{g}]$ denotes the conformal class of $\check{g}$ (see Appendix 5.1). Let g' be another metric on M defined by $g' := e^{2\check{f}\circ\pi}g$ for some smooth function $\check{f}$ on Y, and $(\check{g}', -2\check{\beta}')$ the corresponding pair obtained by substituting g' for g in (2.31) and (2.32). Then $(\check{g}, -2\check{\beta})$ and $(\check{g}', -2\check{\beta}')$ determine the same Weyl structure $([\check{g}], D)$. A Weyl structure $([\check{g}], D)$ is said to be *Einstein-Weyl* if the symmetrized Ricci tensor of D is proportional to $\check{g}$. For later convenience, we recall the following (see Appendix 5.1, cf. Hitchin [36], Jones-Tod [41]):

Proposition 2.14 *(M, g) is self-dual if and only if $(Y, [\check{g}], D)$ is Einstein-Weyl.*

Concerning the converse construction of self-dual neutral metrics, we can show the following (cf. [65]):

Proposition 2.15 *Let $(Y, [\check{g}], D)$ be an oriented Lorentzian Einstein-Weyl three-manifold, $\check{\beta}$ a one-form defined by $D\check{g} = -2\check{\beta} \otimes \check{g}$ for a Lorentzian metric $\check{g}$ in the conformal class $[\check{g}]$, and V a smooth positive function on Y satisfying $d\check{*}(d - \check{\beta})V \equiv 0$, where $\check{*}$ denotes the Hodge star operator of $(Y, \check{g})$. Suppose that the cohomology class $[\check{*}(d - \check{\beta})V]/2\pi$ is integral, that is, contained in the image of $H^2(Y; \mathbb{Z}) \to H^2(Y; \mathbb{R})$. Let $\pi : M \to Y$ denote an S^1-bundle over Y with connection form θ whose curvature is given by*

$$d\theta = \check{*}(d - \check{\beta})V. \tag{2.34}$$

Then, for any non-vanishing function f on M, a neutral metric g defined by

$$g = f(-V^{-1}\theta \otimes \theta + V\pi^*\check{g}) \tag{2.35}$$

is self-dual with respect to a suitable orientation.

Proof. We first note that $\pi : (M, (fV)^{-1}g) \to (Y, \check{g})$ is a pseudo-Riemannian submersion, and the standard S^1-action along the fiber yields an isometry of $(M, (fV)^{-1}g)$. The Killing vector field ξ generating the S^1-action on $(M, (fV)^{-1}g)$ satisfies

$$(fV)^{-1}g(\xi, \xi) = -V^{-2}, \quad \xi^\flat := (fV)^{-1}g(\xi, \cdot) = -V^{-2}\theta.$$

Define a one-form $\beta_{(fV)^{-1}g}$ by

$$\beta_{(fV)^{-1}g} = \frac{-d((fV)^{-1}g(\xi, \xi)) - 2 *_{(fV)^{-1}g} (\xi^\flat \wedge d\xi^\flat)}{2(fV)^{-1}g(\xi, \xi)}.$$

Then we see that $\beta_{(fV)^{-1}g}$ coincides with $\check{\beta}$. Indeed, noting that

$$*_{(fV)^{-1}g}(\theta \wedge \pi^*\alpha) = V\pi^*(\check{*}\alpha)$$

for any two-form α on Y, we have

$$\begin{aligned}
\beta_{(fV)^{-1}g} &= \frac{-d((fV)^{-1}g(\xi,\xi)) - 2*_{(fV)^{-1}g}(\xi^\flat \wedge d\xi^\flat)}{2(fV)^{-1}g(\xi,\xi)} \\
&= \frac{dV^{-2} - 2*_{(fV)^{-1}g}(V^{-2}\theta \wedge V^{-2}d\theta)}{-2V^{-2}} \\
&= \frac{-2V^{-3}dV - 2V^{-3}\check{*}(d\theta)}{-2V^{-2}} \\
&= \frac{-2V^{-3}dV + 2V^{-3}(d-\check{\beta})V)}{-2V^{-2}} \\
&= \check{\beta}.
\end{aligned}$$

Thus it is verified that $(\check{g}, -2\beta_{(fV)^{-1}g})$ and $(\check{g}, -2\check{\beta})$ define the same Weyl structure. By Proposition 2.14, $(fV)^{-1}g$ is a self-dual metric, and hence so is g. $\square$

***Remark* 2.16** If we take another connection form θ' satisfying $d\theta' = \check{*}(dV - \check{\beta}V) = d\theta$, then

$$g := -V^{-1}\theta \otimes \theta + V\pi^*\check{g} \quad \text{and} \quad g' := -V^{-1}\theta' \otimes \theta' + V\pi^*\check{g}$$

are isometric to each other. Indeed, there exists a map $e^{\sqrt{-1}\phi} : Y \to S^1$ satisfying $\theta' - \theta = d\phi$. Then the map $e^{\sqrt{-1}\phi}$ induces a bundle isomorphism (gauge transformation) $\Phi : M \to M$ such that $g' = \Phi^* g$. Namely, (M, g) and (M, g') are isometric via the map Φ.

If we take the de Sitter three-space S^3_1 as an Einstein(-Weyl) manifold Y in Proposition 2.15, then we obtain an analogue of LeBrun's hyperbolic ansatz. Recall that the de Sitter space S^3_1 can be realized as a hyperquadric in the Minkowski space-time $\mathbb{R}^4_1$ as follows:

$$\begin{aligned} S^3_1 &:= \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4_1 \mid -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}, \\ g_{S^3_1} &:= (-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2)|_{S^3_1}. \end{aligned} \tag{2.36}$$

It is well-known that the de Sitter space $S^3_1 = (S^3_1, g_{S^3_1})$ is a Lorentzian space-form of constant curvature $+1$ (cf. Wolf [90]). Then an ansatz given in Proposition 2.15 is rewritten in the following fashion (see [47], cf. [60]):

Proposition 2.17 *Let V be a smooth positive function on the de Sitter three-space S^3_1 such that $\check{*}dV/2\pi$ is a closed two-form on S^3_1 that determines an* *integral class in $H^2(S^3_1;\mathbb{R})$. Let $\mathcal{M} \to S^3_1$ denote an S^1-bundle over S^3_1 with connection form θ whose curvature is given by*

$$d\theta = \check{*}dV, \tag{2.37}$$

where $\check{*}$ denotes the Hodge star operator of S^3_1. Define a metric $g_{V,\theta}$ on $\mathcal{M}$ by

$$g_{V,\theta} := -V^{-1}\theta\otimes\theta + V\, g_{S^3_1}. \tag{2.38}$$

Then $g_{V,\theta}$ is a self-dual neutral metric on $\mathcal{M}$, and, at least locally, is conformal to a scalar-flat neutral Kähler metric with respect to a suitable complex structure on $\mathcal{M}$.

***Remark* 2.18** Although $g_{V,\theta}$ depends on V and θ, the isometry class of $g_{V,\theta}$ is independent of the choice of θ (see Remark 2.16). Unless otherwise stated, we shall write g_V in place of $g_{V,\theta}$ for brevity.

2.4 Construction of self-dual Kähler metrics

Let (V,θ) be a solution of (2.37) satisfying that $V > 0$ and $[\check{*}dV]/2\pi = 0$ in the image $\mathrm{Im}(H^2(S^3_1;\mathbb{Z}) \to H^2(S^3_1;\mathbb{R}))$. Then we obtain a self-dual neutral metric g_V on $\mathcal{M}(\cong S^1 \times S^3_1)$, the total space of a trivial S^1-bundle over S^3_1. In this section, we first study several conditions for the existence of a neutral metric $\bar{g}_V$ conformal to g_V such that $\bar{g}_V$ can extend smoothly to $\overline{\mathcal{M}} := S^2\times S^2$.

We first identify S^3_1 with $\mathbb{R} \times S^2$ via the map

$$S^3_1 \ni (x_0, x_1, x_2, x_3) = (\sinh\rho, (\cosh\rho)v) \mapsto (\rho, v) \in \mathbb{R} \times S^2,$$

where $v \in S^2 \subset \mathbb{R}^3$. Let h_{S^2} denote the standard unit round metric on $S^2 = \{\rho = 0\}$ and ω_{S^2} its volume form. Then $g_{S^3_1}$ is expressed as

$$g_{S^3_1} = -d\rho^2 + \cosh^2\!\rho\; h_{S^2} \quad (-\infty < \rho < +\infty),$$

and the Hodge star operator $\check{*}$ of S^3_1 is given by

$$\check{*}d\rho = -\cosh^2\!\rho\,\omega_{S^2}, \quad \check{*}d\zeta = \sqrt{-1}d\rho\wedge d\zeta,$$

where ζ is a complex coordinate of S^2.

Concerning smooth extensions of neutral metrics, we show the following

Proposition 2.19 *Let V be a smooth positive function on $S^3_1 = \mathbb{R}\times S^2$ such that $\ast dV$ is an exact two-form on S^3_1 and (V,θ) is a smooth solution of* (2.37)*. Assume that θ has no $d\rho$-component. Define a metric $\bar{g}_V$ on $\mathcal{M} = S^1\times\mathbb{R}\times S^2$ by*

$$\bar{g}_V := -\frac{V d\rho^2 + V^{-1}\theta^2}{\cosh^2\rho} + V\, h_{S^2}. \tag{2.39}$$

Then $\bar{g}_V$ extends smoothly to $\overline{\mathcal{M}} \cong S^2\times S^2$, via polar coordinates at $r := e^{\rho} = 0$ and at $q := e^{-\rho} = 0$, if and only if V satisfies the following conditions:

$$\begin{aligned} V &= 1 + r^2 F_-(r^2,\zeta) \text{ as } r \to +0,\\ V &= 1 + q^2 F_+(q^2,\zeta) \text{ as } q \to +0 \end{aligned} \tag{2.40}$$

for smooth functions $F_\pm$ on $\mathbb{R}\times S^2$ in variables r^2, q^2 and ζ.

Proof. Let t be a fiber-coordinate of the trivial S^1-bundle $\mathcal{M}$. Since the situations around $r=0$ and $q=0$ are similar, we discuss only the case near $r=0$. Set $\hat{x}+\sqrt{-1}\hat{y} := re^{\sqrt{-1}t}$. Then the following relations hold:

$$\begin{aligned} r^2 = \hat{x}^2+\hat{y}^2, \quad rdr = \hat{x}d\hat{x}+\hat{y}d\hat{y}, \quad r^2dt = -\hat{y}d\hat{x}+\hat{x}d\hat{y},\\ dr^2 + r^2dt^2 = d\hat{x}^2+d\hat{y}^2, \quad rdr\wedge dt = d\hat{x}\wedge d\hat{y}. \end{aligned} \tag{2.41}$$

We first verify that the condition (2.40) is necessary. Suppose that $\bar{g}_V$ extends smoothly to $\overline{\mathcal{M}}$ via polar coordinate $\hat{x}+\sqrt{-1}\hat{y} = re^{\sqrt{-1}t}$. The restriction of $\bar{g}_V$ to $S^2\times\{\zeta\}$ ($\zeta\in S^2$) is also smooth in $(\hat{x},\hat{y})$. In general, a metric $a(r)dr^2 + 2b(r)rdrdt + c(r)r^2dt^2$ on $\mathbb{R}_+\times S^1 = \{(r, e^{\sqrt{-1}t})\}$ extends smoothly to $\mathbb{R}^2$ via $\hat{x}+\sqrt{-1}\hat{y} = re^{\sqrt{-1}t}$ if and only if $a(r), b(r)$ and $c(r)$ are smooth even functions in r satisfying $a(0) = c(0) (\neq 0)$ and $b(0) = 0$ (cf. Kazdan-Warner [48], Besse [7]). In our case, $\bar{g}_V|_{S^2\times\{\zeta\}}$ is given as

$$\bar{g}_V|_{S^2\times\{\zeta\}} = -\left(\frac{4V}{(1+r^2)^2}dr^2 + \frac{4V^{-1}}{(1+r^2)^2}r^2dt^2\right).$$

Therefore V should be a smooth even function in r satisfying $V(0,\zeta) = V(0,\zeta)^{-1}$, that is, $V(0,\zeta)=1$. Then V should satisfy the condition (2.40).

For sufficiency, we recall that dV is given as

$$\begin{aligned} dV &= 2(F(r^2,\zeta) + r^2\partial_{r^2}F(r^2,\zeta))rdr\\ &\quad + r^2(\partial_\zeta F(r^2,\zeta)d\zeta + \partial_{\bar{\zeta}}F(r^2,\zeta)d\bar{\zeta}), \end{aligned}$$

where $F := F_-$, $\partial_{r^2} F := \partial F/\partial r^2$, $\partial_\zeta F := \partial F/\partial\zeta$ and $\partial_{\bar\zeta} F := \partial F/\partial\overline{\zeta}$. By (2.41), dV is a smooth one-form near $r^2 = 0$ in variables $\hat{x}, \hat{y}$ and ζ. In terms of (r, ζ), the Hodge star operator $\check{*}$ on S^3_1 satisfies

$$\check{*} r dr = -\frac{\sqrt{-1}}{2} \frac{(1 + r^2)^2}{(1 + |\zeta|^2)^2} d\zeta \wedge d\overline{\zeta}, \quad \check{*} d\zeta = \frac{\sqrt{-1}}{r} dr \wedge d\zeta. \tag{2.42}$$

Thus we have

$$\begin{aligned} \check{*} dV &= -\sqrt{-1}(F(r^2, \zeta) + r^2 \partial_{r^2} F(r^2, \zeta)) \frac{(1 + r^2)^2}{(1 + |\zeta|^2)^2} d\zeta \wedge d\overline{\zeta} \\ &\quad + \sqrt{-1} r dr \wedge (\partial_\zeta F(r^2, \zeta) d\zeta - \partial_{\bar\zeta} F(r^2, \zeta) d\overline{\zeta}). \end{aligned}$$

Then the pull-back of $\check{*} dV$ onto $\mathcal{M}$ is regarded as a smooth two-form near $r^2 = 0$. By assumption, there exists a connection form $\theta = dt + A$ such that $\check{*} dV = d\theta = dA$ for some (real) one-form A on S^3_1. Now, comparing both sides of $\check{*} dV = dA$, we see that A is also smooth near $r^2 = 0$. Then $\bar{g}_V$ near $r^2 = 0$ is expressed as

$$\begin{aligned} \bar{g}_V &= -\frac{4\{(1 + r^2 \widetilde{F}(r^2, \zeta)) r^2 (dt + A)^2 + (1 + r^2 F(r^2, \zeta)) dr^2\}}{(1 + r^2)^2} \\ &\quad + (1 + r^2 F(r^2, \zeta)) h_{S^2}, \\ &= -\frac{4(dr^2 + r^2 dt^2)}{(1 + r^2)^2} - \frac{4\{\widetilde{F}(r^2, \zeta)(r^2 dt)^2 + F(r^2, \zeta)(r dr)^2\}}{(1 + r^2)^2} \\ &\quad -\frac{4\{1 + r^2 \widetilde{F}(r^2, \zeta)\}(2(r^2 dt)A + r^2 A^2)}{(1 + r^2)^2} + (1 + r^2 F(r^2, \zeta)) h_{S^2}, \end{aligned}$$

where $1 + r^2 \widetilde{F}(r^2, \zeta) := (1 + r^2 F(r^2, \zeta))^{-1}$ near $r^2 = 0$. Recall that $r dr$, $r^2 dt$, $dr^2 + r^2 dt^2$ on $\mathbb{R}^2 \backslash \{(0, 0)\}$ extends smoothly to $\mathbb{R}^2$ via the coordinates $(\hat{x}, \hat{y}) = r(\cos t, \sin t)$. We can therefore regard $\bar{g}_V$ as a smooth neutral metric on $\mathbb{R}^2 \times S^2$.

Similarly, we also see that $\bar{g}_V$ extends smoothly to a neighborhood of $q^2 = 0$. Thus $\bar{g}_V$ is regarded as a smooth metric on $\overline{\mathcal{M}}$. □

***Remark* 2.20** After a gauge transformation, we may assume that θ has no $d\rho$-component (cf. Remark 2.16).

We next prove that there exists an almost complex structure I_V on $\overline{\mathcal{M}} = S^2 \times S^2$ such that $(\bar{g}_V, I_V)$ is a neutral Kähler structure, if V satisfies the same assumptions as those in Proposition 2.19.

Proposition 2.21 *Let $\bar{g}_V$ be a self-dual neutral metric on $\overline{\mathcal{M}} = S^2 \times S^2$ as in Proposition 2.19. Define an almost complex structure I_V and a two-form $\bar{\Omega}_V$ on $\mathcal{M} = S^1 \times S^3_1 = S^1 \times \mathbb{R} \times S^2$ respectively by*

$$I_V d\rho := -V^{-1}\theta, \quad I_V d\zeta := \sqrt{-1}d\zeta, \tag{2.43}$$

$$\bar{\Omega}_V := \bar{g}_V(I_V\cdot, \cdot). \tag{2.44}$$

Then I_V is integrable on $\mathcal{M}$ and $\bar{\Omega}_V$ extends smoothly to a symplectic form on $\overline{\mathcal{M}}$. Thus $(\bar{g}_V, I_V)$ is regarded as a self-dual neutral Kähler structure on $\overline{\mathcal{M}}$.

Proof. Let $\Lambda^{p,q}$ denote the space of (p,q)-forms on $\mathcal{M}$ with respect to I_V. Then I_V is integrable if and only if $d\Lambda^{1,0}\subset\Lambda^{2,0}\oplus\Lambda^{1,1}$, or equivalently,

$$(d\rho + \sqrt{-1}V^{-1}\theta)\wedge d\zeta\wedge d(d\rho + \sqrt{-1}V^{-1}\theta)\equiv 0, \tag{2.45}$$

since $\Lambda^{1,0}$ is generated by $d\rho + \sqrt{-1}V^{-1}\theta$ and $d\zeta$. By using (2.37), the integrability condition (2.45) is verified as follows:

$$\begin{aligned}
&(d\rho + \sqrt{-1}V^{-1}\theta)\wedge d\zeta\wedge d(d\rho + \sqrt{-1}V^{-1}\theta)\\
&= \sqrt{-1}(d\rho + \sqrt{-1}V^{-1}\theta)\wedge d\zeta\wedge(-V^{-2}dV\wedge\theta + V^{-1}d\theta)\\
&= \sqrt{-1}(d\rho + \sqrt{-1}V^{-1}\theta)\wedge d\zeta\wedge(-V^{-2}dV\wedge\theta + V^{-1}\check{*}dV)\\
&= \sqrt{-1}(-V^{-2}d\rho\wedge d\zeta\wedge dV\wedge\theta + \sqrt{-1}V^{-2}\theta\wedge d\zeta\wedge\check{*}dV)\\
&= -\sqrt{-1}V^{-2}(\partial_{\bar{\zeta}}V\,d\rho\wedge d\zeta\wedge d\bar{\zeta}\wedge\theta - \sqrt{-1}\partial_{\bar{\zeta}}V\,\theta\wedge d\zeta\wedge(-\sqrt{-1}d\rho\wedge d\bar{\zeta}))\\
&\equiv 0.
\end{aligned}$$

We next examine the fundamental form $\bar{\Omega}_V$ of $(\bar{g}_V, I_V)$. By definition, $\bar{\Omega}_V$ is expressed as

$$\bar{\Omega}_V = \bar{g}_V(I_V\cdot, \cdot) = -d\tanh\rho\wedge\theta + V\omega_{S^2}. \tag{2.46}$$

By the coordinate change $r = e^\rho$, we have

$$\bar{\Omega}_V = -\frac{4r dr\wedge\theta}{(1+r^2)^2} + V\omega_{S^2}.$$

From (2.41), we see that $\bar{\Omega}_V$ is smooth and nondegenerate near $r^2 = 0$, and near $q^2 = 0$ as well. Then we can regard $\bar{\Omega}_V$ as a nondegenerate two-form on the whole $\overline{\mathcal{M}}$. By definition, we can also regard I_V as a complex structure on the whole $\overline{\mathcal{M}}$.

The exterior derivative of $\bar{\Omega}_V$ is computed as follows:

$$\begin{aligned} d\bar{\Omega}_V &= -d(d\tanh\rho\wedge\theta) + d(V\omega_{S^2}) \\ &= d\tanh\rho\wedge d\theta + dV\wedge\omega_{S^2} \\ &= \operatorname{sech}^2\rho\, d\rho\wedge\check{*}dV + dV\wedge\omega_{S^2} \\ &= -\operatorname{sech}^2\rho\, d\rho\wedge\partial_\rho V\cosh^2\rho\,\omega_{S^2} + \partial_\rho V d\rho\wedge\omega_{S^2} \\ &\equiv 0. \end{aligned}$$

Thus $(\bar{g}_V, I_V)$ is a neutral Kähler structure on $\overline{\mathcal{M}}$. $\square$

***Remark* 2.22** Note that the Ricci form γ_V is given by

$$\gamma_V = d(V^{-1}\tanh\rho\ \theta) + \omega_{S^2}. \tag{2.47}$$

It is easy to verify the scalar-flatness of $\bar{g}_V$ by checking $\gamma_V\wedge\bar{\Omega}_V \equiv 0$.

***Remark* 2.23** Given two solutions (V,θ) and (V',θ') of (2.37) satisfying the conditions in Proposition 2.19, define a one-parameter family $\{(V_\lambda,\theta_\lambda)\}$ by

$$V_\lambda := \lambda V + (1-\lambda)V', \qquad \theta_\lambda := \lambda\theta + (1-\lambda)\theta'.$$

Then $(V_\lambda,\theta_\lambda)$ is also a solution of (2.37) for each λ, and hence $(\bar{g}_{V_\lambda}, I_{V_\lambda})$ determines a self-dual neutral Kähler structure on $\overline{\mathcal{M}} = S^2\times S^2$. Taking $V'\equiv 1$, we see that I_V is obtained as a smooth deformation of the standard product complex structure $I_0 = I_{S^2}\oplus I_{S^2}$ on $S^2\times S^2 = \mathbb{CP}^1\times\mathbb{CP}^1$, the product of two complex projective lines. Therefore it follows from Kodaira-Spencer theory [53] that $(\overline{\mathcal{M}}, I_{V_\lambda})$ is biholomorphic to $\mathbb{CP}^1\times\mathbb{CP}^1$ for sufficiently small λ. Furthermore, by using results in this section, one can prove that $(\overline{\mathcal{M}}, I_V)$ is biholomorphic to $\mathbb{CP}^1\times\mathbb{CP}^1$.

***Remark* 2.24** The Kähler form Ω_λ of $\bar{g}_{V_\lambda}$ corresponding to $V_\lambda = \lambda V + (1-\lambda)1$ is given by $\Omega_\lambda = \lambda\bar{\Omega}_V + (1-\lambda)\Omega_0$, where Ω_0 is the standard symplectic structure: $\Omega_0 = -\omega_{S^2}(z)\oplus\omega_{S^2}(\zeta)$. By the self-duality of $\bar{g}_V$, it is verified that $[\Omega_\lambda]\cdot c_1(\mathbb{CP}^1\times\mathbb{CP}^1) = 0$, that is, the cohomology class $[\Omega_\lambda]$ is orthogonal to the first Chern class $c_1(\mathbb{CP}^1\times\mathbb{CP}^1)$ with respect to the cup product in $H^2(\mathbb{CP}^1\times\mathbb{CP}^1;\mathbb{R})$. By (2.46), it is also verified that $[\Omega_\lambda]\cdot[\omega_{S^2}(\zeta)] = [\Omega_0]\cdot[\omega_{S^2}(\zeta)]$. Then we see that $[\Omega_\lambda]$ is independent of λ, that is, $[\Omega_\lambda] = [\Omega_0]$ for any λ $(0\le\lambda\le 1)$. It follows from Moser's theorem [76] that $(\overline{\mathcal{M}}, \bar{\Omega}_V)$ is symplectomorphic to $(S^2\times S^2, \Omega_0)$.

We next examine the Weyl conformal tensor W of $\bar{g}_V$. For convenience, we first recall the following proposition, which is verified by a direct computation (see Appendix 5.1).

Proposition 2.25 *Let $\bar{g}_V$ be a self-dual neutral metric on $\overline{\mathcal{M}} = S^2 \times S^2$ defined by* (2.39). *Then the Weyl conformal tensor W of $\bar{g}_V$ is completely determined by the following quadratic form Q_V:*

$$Q_V := VDdV - 3dV \otimes dV + \|dV\|^2 g_{S^3_1},$$

where D is the Levi-Civita connection of $g_{S^3_1}$ and $\|\cdot\|^2$ denotes the indefinite squared norm with respect to $g_{S^3_1}$. In particular, $\bar{g}_V$ is conformally-flat if and only if Q_V vanishes identically.

We shall next examine the conformal-flatness of $\bar{g}_V$. Let g'_V be a metric on $\mathcal{M} = S^1 \times S^3_1$ defined by $g'_V := -\theta^2 + V^2 g_{S^3_1}$ and D' the Levi-Civita connection of $g'_{S^3_1} := V^2 g_{S^3_1}$. Then D' and D satisfy the following relation:

$$D'_X Y = D_X Y + d\log V(X)Y + d\log V(Y)X - g_{S^3_1}(X,Y) D\log V,$$

where $D\log V$ denotes the gradient vector field of $\log V$ with respect to $g_{S^3_1}$. From this relation, we can verify that

$$D' d\mathrm{log}V = V^{-2}(VDdV - 3dV \otimes dV + \|dV\|^2 g_{S^3_1}) = V^{-2} Q_V.$$

Thus, $\bar{g}_V$ is conformally-flat if and only if $D' d\log V \equiv 0$. Since V satisfies $V > 0$ and $V \to 1$ as $\rho \to \pm\infty$, the condition $D' d\log V \equiv 0$ implies that $\log V$ is constant, thus $V \equiv 1$. Summarizing these, we obtain the following

Theorem 2.26 *Let $\bar{g}_V$ be a self-dual neutral Kähler metric on $\overline{\mathcal{M}} = S^2 \times S^2$ defined by* (2.39). *Then $\bar{g}_V$ is conformally-flat if and only if $V \equiv 1$.*

In the case where $V \equiv 1$, $\bar{g}_V$ is not only conformally-flat but also coincides with the standard product metric g_0 on $S^2 \times S^2$. Indeed, take a connection form $\theta = dt$ and set $r = e^\rho$. Then $\bar{g}_V$ is given as

$$\bar{g}_V = -\frac{d\rho^2 + dt^2}{\cosh^2 \rho} + h_{S^2} = -\frac{4(dr^2 + r^2 dt^2)}{(1 + r^2)^2} + h_{S^2},$$

which is just the product metric $g_0 = -h_{S^2} \oplus h_{S^2}$ restricted to $\mathcal{M} = S^1 \times S^3_1 = S^1 \times \mathbb{R} \times S^2$.

For a nonconstant solution V of (2.37) satisfying the conditions in Proposition 2.19, we obtain a non-conformally-flat, self-dual neutral Kähler metric on $S^2 \times S^2$. Next, we construct a family of self-dual neutral Kähler metrics on $S^2 \times S^2$ from some explicit solutions (V, θ) of (2.37).

Let G_0 be a smooth function on S^3_1 defined by

$$G_0 := \frac{1 - \tanh \rho}{2}.$$

Then G_0 satisfies

$$\check{*} dG_0 = \frac{1}{2} \omega_{S^2},$$

and hence

$$\frac{1}{2\pi}[\check{*} dG_0] = 1 \in \mathrm{Im}(H^2(S^3_1; \mathbb{Z}) \to H^2(S^3_1; \mathbb{R})) = \mathbb{Z}.$$

From Proposition 2.17, we thus obtain a self-dual neutral metric g_{G_0} on $S^3 \times \mathbb{R}$, the total space of the Hopf bundle $S^3 \times \mathbb{R} \to S^3_1 = S^2 \times \mathbb{R}$. It should be remarked that g_{G_0} is conformal to a restriction of the Fubini-Study type metric on the indefinite complex projective space $\mathbb{CP}^2_1$ (see Chapter 4).

Let $\{\sigma_j\}_{j=1}^N$ (resp. $\{\tau_j\}_{j=1}^N$) be a family of orientation-preserving isometries on S^3_1 such that each σ_j (resp. τ_j) preserves (resp. reverses) the time-orientation. If we set

$$V := \frac{1}{N} \sum_{j=1}^{N} (G_0 \circ \sigma_j + G_0 \circ \tau_j), \tag{2.48}$$

then V satisfies $[\check{*} dV]/2\pi = 0$ in $\mathrm{Im}(H^2(S^3_1; \mathbb{Z}) \to H^2(S^3_1; \mathbb{R}))$. Thus we obtain a self-dual neutral metric g_V on the total space $\mathcal{M}$ of a trivial S^1-bundle over S^3_1. We can verify that V satisfies the conditions in Proposition 2.19 as follows: Recall that $\mathrm{Isom}^+(S^3_1)$, the group of orientation-preserving isometries of S^3_1, is isomorphic to $SO(1,3)$. Let φ be an orientation-preserving isometry of S^3_1 and φ^{-1} denote its inverse. Then φ and φ^{-1} are expressed as

$$\varphi = \begin{pmatrix} a & b^* \\ c & D \end{pmatrix}, \qquad \varphi^{-1} = \begin{pmatrix} a & -c^* \\ -b & D^* \end{pmatrix},$$

where $a \in \mathbb{R}$, $b, c \in \mathbb{R}^3$ and D is a real 3×3-matrix such that

$$\begin{aligned} a^2 - |c|^2 = 1, \quad -ba + D^* c = 0, \quad -bb^* + D^* D = \mathbb{E}, \\ a^2 - |b|^2 = 1, \quad ca - Db = 0, \quad -cc^* + DD^* = \mathbb{E}. \end{aligned}$$

Here $\mathbb{E}$ is the identity matrix, and * stands for the transpose. Then $\rho\circ\varphi$ satisfies

$$\begin{aligned}
\sinh(\rho\circ\varphi) &= a\sinh\rho + \cosh\rho\, b^*v, \\
\cosh(\rho\circ\varphi) &= \sqrt{1+(a\sinh\rho+\cosh\rho\, b^*v)^2}, \\
\tanh(\rho\circ\varphi) &= \frac{a\sinh\rho+\cosh\rho\, b^*v}{\sqrt{1+(a\sinh\rho+\cosh\rho\, b^*v)^2}}, \\
v\circ\varphi &= \frac{c\sinh\rho+\cosh\rho\, D^*v}{\sqrt{1+(a\sinh\rho+\cosh\rho\, b^*v)^2}}
\end{aligned}$$

for $v \in S^2 \subset \mathbb{R}^3$. Then $G_0\circ\varphi$ is expressed as

$$G_0\circ\varphi = \frac{1-\tanh(\rho\circ\varphi)}{2} = \frac{1}{2}\left(1 - \frac{a\sinh\rho+\cosh\rho\, b^*v}{\sqrt{1+(a\sinh\rho+\cosh\rho\, b^*v)^2}}\right).$$

By using the coordinates $r = e^{\rho}$ and $q = e^{-\rho}$, we have

$$\begin{aligned}
G_0\circ\varphi &= \frac{1}{2}\left(1 - \frac{a(r^2-1)+b^*v(r^2+1)}{\sqrt{4r^2+(a(r^2-1)+b^*v(r^2+1))^2}}\right) \\
&= \frac{1}{2}\left(1 - \frac{a(1-q^2)+b^*v(1+q^2)}{\sqrt{4q^2+(a(1-q^2)+b^*v(1+q^2))^2}}\right).
\end{aligned}$$

Hence $G_0\circ\varphi$ is smooth near both $r^2 = 0$ and $q^2 = 0$. If φ preserves (resp. reverses) the time-orientation of S^3_1, then we have

$$\begin{aligned}
&G_0\circ\varphi \to 1 \quad (\text{resp.}\, G_0\circ\varphi \to 0) \quad \text{as } r^2 \to 0, \\
&G_0\circ\varphi \to 0 \quad (\text{resp.}\, G_0\circ\varphi \to 1) \quad \text{as } q^2 \to 0,
\end{aligned}$$

so that $V = (1/N)\sum_{j=1}^{N}(G_0\circ\sigma_j + G_0\circ\tau_j) > 0$ satisfies the conditions in Proposition 2.19. Therefore $(\bar{g}_V, I_V)$ is a self-dual neutral Kähler structure on $\overline{\mathcal{M}} = S^2\times S^2$. It follows from Remark 2.23 that $(\overline{\mathcal{M}}, I_V)$ is biholomorphic to $\mathbb{CP}^1\times\mathbb{CP}^1$. Noting Theorem 2.26, we obtain the following result, which was referred as Theorem 1.3.

Corollary 2.27 *There exists a family of self-dual neutral Kähler metrics, which includes non-conformally-flat metrics, on* $\mathbb{CP}^1\times\mathbb{CP}^1$.

***Remark* 2.28** In the argument above, ρ is regarded as the signed distance function from the totally geodesic sphere

$$\Sigma := \{\rho = 0\} = \{(0, x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\} \subset S_1^3.$$

In general, an oriented totally geodesic sphere in S_1^3 is determined by a point in $H_+^3 \coprod H_-^3$. The sphere $\Sigma = \{\rho = 0\}$ is indeed corresponding to a point $(1, 0, 0, 0) \in H_+^3$. Take another totally geodesic sphere Σ' corresponding to $p \in H_+^3$ and denote by ρ' the signed distance function from Σ'. Then $g_{S_1^3}$ is also expressed as

$$g_{S_1^3} = -d\rho'^2 + \cosh^2\rho' h_{\Sigma'},$$

where $h_{\Sigma'}$ denotes the unit round metric on Σ'. Let σ be an element in $\mathrm{Isom}^+(H^3) = SO_+(1, 3)$ with $\sigma(1, 0, 0, 0) = p$. Then $\Sigma' = \sigma(\Sigma)$ and $\rho' \circ \sigma = \rho$. For a solution (V, θ) of (2.37), the metrics

$$\bar{g}_V := -\frac{V d\rho^2 + V^{-1}\theta^2}{\cosh^2\rho} + V\, h_\Sigma, \quad \bar{g}'_V := -\frac{V d\rho'^2 + V^{-1}\theta^2}{\cosh^2\rho'} + V\, h_{\Sigma'}$$

on $\overline{\mathcal{M}}$ are both self-dual. Furthermore, $\bar{g}'_V$ is conformal to $\bar{g}_V$. Indeed, $\bar{g}'_V$ is rewritten as

$$\bar{g}'_V = \mathrm{sech}^2\rho'(-V^{-1}\theta^2 + V\, g_{S_1^3}) = \frac{\cosh^2\rho}{\cosh^2\rho'}\bar{g}_V.$$

Thus the isometry class of $\bar{g}_V$ depends on V and the identification $S_1^3 = \mathbb{R} \times S^2$. However, its conformal class is independent of the identification $S_1^3 = \mathbb{R} \times S^2$, and depends only on V. For a metric $\bar{g}_V = (\mathrm{sech}^2\rho)g_V$, we shall call the totally geodesic sphere $\Sigma = \{\rho = 0\}$ in S_1^3 the *neck sphere* (or the *equatorial sphere*).

***Remark* 2.29** For a function V defined by (2.48), let $\{p_j\}_{j=1}^N$ and $\{q_j\}_{j=1}^N$ be the points in H_+^3 and H_-^3 corresponding to the totally geodesic spheres $\{\sigma_j^{-1}(S^2)\}_{j=1}^N$ and $\{\tau_j^{-1}(S^2)\}_{j=1}^N$, respectively. Here S^2 denotes the fixed neck sphere. Then $\bar{g}_V$ depends on the configuration of $\{p_j; q_j\}_{j=1}^N$, rather than on $\{\sigma_j; \tau_j\}_{j=1}^N$.

Each metric $\bar{g}_V$ has an obvious S^1-symmetry coming from the S^1-bundle structure. According to the configuration of $\{p_j; q_j\}_{j=1}^N$, the corresponding metric $\bar{g}_V$ may have other extra symmetries. For example, if $\{q_j\}_{j=1}^N$ consist of the antipodal points of $\{p_j\}_{j=1}^N$, that is, $q_j = -p_j$ $(j = 1, \dots, N)$, then $V \equiv 1$. Hence $\bar{g}_V$ is the standard metric g_0, which has a natural $S(O(3)\times O(3))$-symmetry. If $\{p_j; q_j\}_{j=1}^N$ are simultaneously collinear, that is, if they lie on a common two-dimensional subspace Π in $\mathbb{R}^4_1$, then $\bar{g}_V$ has a $T^2(= S^1\times S^1)$-symmetry. Indeed, the extra S^1-symmetry is given by the rotation around the intersection of the subspace Π and the neck sphere $S^2 = \{\rho = 0\}$. In particular, if $N = 1$, then $\bar{g}_V$ always has a T^2-symmetry (cf. Poon [83]).

Let $G(x, y)$ be a smooth function on $S^3_1 \times (H^3_+ \coprod H^3_-)$ defined by

$$G(x, y) := G_0 \circ \varphi_y(x)$$

for an isometry φ_y on S^3_1 satisfying $\varphi_y(y) = \mathbf{e}_0 = (1, 0, 0, 0)$ $(y \in H^3_+ \coprod H^3_-)$. Then $G(x, y)$ is rewritten as

$$G(x, y) = \frac{1}{2}\left(1 + \frac{\langle \varphi_y(x), \mathbf{e}_0\rangle}{\sqrt{1 + \langle \varphi_y(x), \mathbf{e}_0\rangle^2}}\right) = \frac{1}{2}\left(1 + \frac{\langle x, y\rangle}{\sqrt{1 + \langle x, y\rangle^2}}\right).$$

Setting $\varphi_{p_i} = \sigma_i$ and $\varphi_{q_i} = \tau_i$ $(1 \leq i \leq N)$, we can also express the data V given by (2.48) as

$$V(x) = \frac{1}{N}\sum_{i=1}^{N}[G(x, p_i) + G(x, q_i)].$$

Motivated by this expression, we obtain the following generalization: Let μ_+ and μ_- be probability measures on H^3_+ and H^3_- with compact support, respectively. Define a smooth function V on S^3_1 by

$$V(x) = \int_{H^3_+} G(x, y) d\mu_+(y) + \int_{H^3_-} G(x, y) d\mu_-(y),$$

which satisfies the conditions in Proposition 2.19. Then the corresponding metric $\bar{g}_V$ is self-dual neutral metric on $\overline{\mathcal{M}}$.

For a solution (V, θ) of (2.37), each metric $\bar{g}_V$ defined by (2.39) is neutral Kähler. Therefore we can express $\bar{g}_V$ as in (2.30):

$$\bar{g}_V = -(wdz^2 + w^{-1}\theta^2) + we^u(dx^2 + dy^2),$$

by setting

$$dz = d\tanh\rho, \quad w = V\cosh^2\rho, \quad e^u = \frac{4}{\cosh^2\rho(1 + x^2 + y^2)^2}.$$

Since we may assume that $z = \tanh \rho$, the data u and w are rewritten as

$$w = \frac{V}{1 - z^2}, \quad e^u = \frac{4(1 - z^2)}{(1 + x^2 + y^2)^2}, \tag{2.49}$$

and hence satisfy (2.26) and (2.28). The corresponding Einstein-Weyl structure on the quotient space is indeed induced from that of the de Sitter space S^3_1, and then (2.25) is equivalent to (2.37) under the substitution (2.49). (In the Riemannian case, such a solution as (2.49) appears in, e.g., Calderbank-Pedersen [15].)

For a self-dual neutral Kähler structure (g, I) given by (2.13), we can find, by virtue of Proposition 2.14, an Einstein-Weyl structure on $\mathcal{U}$. Indeed, the structure is determined by a pair $(\check{g}, -2\check{\beta})$ defined to be

$$\check{g} := -dz^2 + e^u(dx^2 + dy^2) \quad \text{and} \quad \check{\beta} := -u_z dz. \tag{2.50}$$

We now prove the following result, which characterizes self-dual neutral Kähler metrics constructed on $\mathbb{CP}^1 \times \mathbb{CP}^1$ by the de Sitter ansatz.

Theorem 2.30 *Let (M, g, I) be a compact self-dual neutral Kähler surface with a time-like S^1-action satisfying the same condition assumed in Theorem* 2.11*, and F denote its fixed point set. Suppose that the Einstein-Weyl structure determined by $(\check{g}, -2\check{\beta})$ in* (2.50) *is closed on the quotient space $(M \setminus F)/S^1$ (i.e., $d\check{\beta} \equiv 0$). Then (M, I) is biholomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ and (M, g, I) is isomorphic to $(\mathbb{CP}^1 \times \mathbb{CP}^1, \bar{g}_V)$ given in Proposition* 2.21*.*

Proof. Recall that the orbit space $Y := M/S^1$ is a compact three-manifold with boundary $\partial Y \cong \Sigma_1 \coprod \Sigma_2$, where Σ_1 and Σ_2 denote the connected components of F. If necessary, by rescaling and adding a constant, we may assume that a moment map $z : M \to \mathbb{R}$ satisfies $z(M) = [-1, 1]$ with $z^{-1}(-1) = \Sigma_1$ and $z^{-1}(+1) = \Sigma_2$. Then z induces a smooth function $\check{z} : Y \to [-1, 1]$, since z is constant along each orbit of the action. By the assumption that ξ is time-like, $I\xi$ is a gradient-like vector field of $\check{z} : Y \to [-1, 1]$, that is,

$$d\check{z}(\pi_* I\xi) = dz(I\xi) = \Omega_I(\xi, I\xi) = g(\xi, \xi) < 0 \quad \text{on} \quad Y \setminus \partial Y.$$

Thus $\check{z}$ has no critical points in the interior of Y, and hence Y is identified with $[-1, 1] \times S^2 = \{(z, x, y) \mid -1 \leq z \leq 1, (x, y) \in S^2\}$, where $S^2 \cong \Sigma_j$ $(j = 1, 2)$ is a two-sphere endowed with a holomorphic structure. In this description, $x + \sqrt{-1}y$ is a holomorphic coordinate of S^2.

It follows from (2.22) and the smoothness of g on M that u and w on $(-1,1)\times S^2$ satisfy

$$(1-z^2)w \to 1,\ we^u \to \text{finite}(>0) \quad \text{as } z\to \pm 1 \tag{2.51}$$

by an argument similar to that in Proposition 2.19. Hence it is verified that $e^u \to +0$ as $z\to\pm 1$. By the integrality of $[d\theta]/2\pi$, we obtain

$$-\frac{1}{2\pi}\frac{d}{dz}\int_{\{z\}\times S^2} we^u dx{\wedge}dy = \frac{1}{2\pi}\int_{\{z\}\times S^2}(-we^u)_z dx{\wedge}dy =: n\ \in \mathbb{Z}.$$

Thus there exists a real constant c such that

$$\int_{\{z\}\times S^2} we^u dx{\wedge}dy = -2\pi n z + c. \tag{2.52}$$

On the other hand, by (2.28), we have

$$\begin{aligned}\frac{d^2}{dz^2}\int_{\{z\}\times S^2} e^u dx{\wedge}dy &= \int_{\{z\}\times S^2}(e^u)_{zz}dx{\wedge}dy \\ &= \int_{\{z\}\times S^2}(u_{xx}+u_{yy})dx{\wedge}dy = -8\pi\end{aligned}$$

for any fixed $z\in(-1,1)$. The last equality follows from the Gauss-Bonnet theorem for S^2 with a z-depending metric $e^u(dx^2+dy^2)$, since its Ricci form is given by $-(1/2)(u_{xx}+u_{yy})dx{\wedge}dy$. From the asymptotic behavior of e^u, we also obtain

$$\frac{1}{1-z^2}\int_{\{z\}\times S^2} e^u dx{\wedge}dy = 4\pi. \tag{2.53}$$

In what follows, we suppose that the Einstein-Weyl structure on $Y\backslash\partial Y = (-1,1)\times S^2$ determined by $(\check{g},-2\check{\beta})$ in (2.50) is closed, that is, $d\check{\beta}\equiv 0$. By arguments similar to those in [63] and [15], we can express u as $u = a(z)+b(x,y)$. Here $a(z)\in C^\infty([-1,1])$ and $b(x,y)\in C^\infty(S^2)$ satisfy

$$b_{xx}+b_{yy} = ke^b, \quad (e^a)_{zz} = k \tag{2.54}$$

for some negative constant k. Without loss of generality, we may assume that $k=-2$. Taking account of (2.53) and (2.54), we obtain $e^u = (1-z^2)e^b$, since $b_{xx}+b_{yy} = -2e^b$ is equivalent to that a Riemannian metric $e^b(dx^2+dy^2)(=: h_{S^2})$ on S^2 is of constant curvature $+1$. Define a function V on $[-1,1]\times S^2$ by $V := we^u e^{-b} = w(1-z^2)$. It follows from (2.51) that

$$\begin{aligned}&\lim_{z\to\pm1}\int_{\{z\}\times S^2} we^u dx\wedge dy \\ &\quad = \lim_{z\to\pm1}\int_{\{z\}\times S^2} w(1-z^2)e^b dx\wedge dy = 4\pi.\end{aligned} \tag{2.55}$$

From (2.52), it also follows that

$$\text{(2.56)} \qquad \lim_{z\to\pm 1}\int_{\{z\}\times S^2} we^u dx\wedge dy = \mp 2\pi n + c.$$

Comparing (2.55) with (2.56), we obtain

$$4\pi = -2\pi n + c, \quad 4\pi = +2\pi n + c,$$

which imply that $n = 0$ and $c = 4\pi$. Therefore, $d\theta$ is an exact two-form on $(-1,1)\times S^2$, so that the corresponding S^1-bundle is trivial. Hence M is biholomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Recalling $w = V(1-z^2)^{-1}$ and $we^u = Ve^b$, and setting $z = \tanh\rho$, we can rewrite g as

$$\begin{aligned} g &= -\frac{Vdz^2 + V^{-1}(1-z^2)^2\theta^2}{1-z^2} + Ve^b(dx^2+dy^2) \\ &= -\frac{Vd\rho^2 + V^{-1}\theta^2}{\cosh^2\rho} + V\,h_{S^2}. \end{aligned}$$

Note that $g_{S^3_1}$ is expressed, via the identification $S^3_1 = (-1,1)\times S^2$, as

$$g_{S^3_1} = -\frac{dz^2}{(1-z^2)^2} + \frac{h_{S^2}}{1-z^2}.$$

It is then verified that (V,θ) satisfies (2.37). Indeed,

$$d\theta = \frac{V_x dy\wedge dz}{1-z^2} + \frac{V_y dz\wedge dx}{1-z^2} - V_z e^b dx\wedge dy = \check{*}dV.$$

Thus we have reexamined an analogue of LeBrun's hyperbolic ansatz. $\square$

2.5 Isometry classes

In this section, we investigate the isometry classes of our self-dual neutral Kähler metrics on $\mathbb{CP}^1\times\mathbb{CP}^1$. Temporarily, we denote a self-dual neutral metric $\bar{g}_V$ defined by (2.39) as $\bar{g}_{V,\theta}$:

$$\bar{g}_{V,\theta} := \operatorname{sech}^2\rho(-V^{-1}\theta\otimes\theta + V\,g_{S^3_1}).$$

Let ϕ be a smooth function on S^3_1. If (V,θ) is a solution of (2.37), then so is $(V,\theta+d\phi)$. Recall that such a modification stems from a gauge transformation $\Phi = e^{\sqrt{-1}\phi} : \mathcal{M}\to\mathcal{M}$ with $\Phi^*\bar{g}_{V,\theta} = \bar{g}_{V,\theta+d\phi}$. Thus $\bar{g}_{V,\theta}$ and $\bar{g}_{V,\theta+d\phi}$

are isometric via the map Φ. Let φ be an orientation-preserving isometry of S^3_1. Given a solution (V, θ) of (2.37), it is easy to see that $(V\circ\varphi, \varphi^*\theta)$ is also a solution, and $\bar{g}_{V\circ\varphi,\varphi^*\theta}$ and $\bar{g}_{V,\theta}$ are related by

$$\bar{g}_{V\circ\varphi,\varphi^*\theta} = \frac{\cosh^2(\rho\circ\varphi)}{\cosh^2\rho}\varphi^*\bar{g}_{V,\theta}.$$

Note that $\bar{g}_{V,\theta}$ depends also on the choice of a totally geodesic neck sphere (see Remark 2.28). If φ preserves the neck sphere $S^2 = \{\rho = 0\}$, then $\varphi^*\bar{g}_{V,\theta} = \bar{g}_{V\circ\varphi,\varphi^*\theta}$, that is, $\bar{g}_{V,\theta}$ and $\bar{g}_{V\circ\varphi,\varphi^*\theta}$ are isometric.

It is natural to ask, for solutions (V, θ) and (V', θ') of (2.37), when the corresponding metrics $\bar{g}_{V,\theta}$ and $\bar{g}_{V',\theta'}$ are isometric. The main goal in this section is to prove the following

Theorem 2.31 *Let $\bar{g}_{V,\theta}$ and $\bar{g}_{V',\theta'}$ be non-conformally-flat, self-dual neutral Kähler metrics on $(\overline{\mathcal{M}}, I) = \mathbb{CP}^1 \times \mathbb{CP}^1$ corresponding respectively to (V, θ) and (V', θ'), which are solutions of* (2.37). *Let φ be an orientation-preserving diffeomorphism on $\overline{\mathcal{M}}$ and suppose $\varphi^*\bar{g}_{V',\theta'} = \bar{g}_{V,\theta}$. Then φ should be induced from an isometry of S^3_1 preserving the neck sphere S^2. In particular, $V'\circ\varphi = V$ holds.*

From Theorem 2.31 above, we see that self-dual metrics obtained in Section 2.4 give rise to infinitely many different isometry classes on $S^2 \times S^2$. For example, let q be a point in H^3_-. Then the isometry class of the metric $\bar{g}_V$ corresponding to $\{\mathbf{e}_0 = (1, 0, 0, 0); q\}$ is parameterized by the hyperbolic distance between $\mathbf{e}_0$ and $-q$ in H^3_+.

Before proving Theorem 2.31, we first recall basic properties of holomorphic vector fields on $\mathbb{CP}^1\times\mathbb{CP}^1$. Let (U_0, z) and (U_∞, z') be local holomorphic coordinate charts of the first $\mathbb{CP}^1$ satisfying $z' = 1/z$ on $U_0 \bigcap U_\infty$, and (V_0, ζ) and (V_∞, ζ') be local holomorphic coordinate charts of the second $\mathbb{CP}^1$ satisfying $\zeta' = 1/\zeta$ on $V_0 \bigcap V_\infty$. It is well-known that any holomorphic vector field on $\mathbb{CP}^1 \times \mathbb{CP}^1$ can be expressed in terms of (z, ζ) as $\alpha(z)\partial_z + \beta(\zeta)\partial_\zeta$, where $\partial_z := \partial/\partial z$ and $\partial_\zeta := \partial/\partial\zeta$. Here $\alpha(z)$ and $\beta(\zeta)$ are polynomials in z and ζ of degree at most two, respectively.

In regard to time-like Killing vector fields, we first prove the following

Proposition 2.32 *Let g be a neutral Kähler metric on $\mathbb{CP}^1\times\mathbb{CP}^1$ and $\xi \not\equiv 0$ a time-like Killing vector field on $(\mathbb{CP}^1\times\mathbb{CP}^1, g)$. Then ξ has no isolated zero. Moreover, taking suitable holomorphic coordinates, we may regard ξ as the real part of $\xi - \sqrt{-1}I\xi = \sqrt{-1}az\partial_z$ for some $a \in \mathbb{R}$. In particular, we can identify* $\mathrm{Zero}(\xi)$ *with* $\{0, \infty\} \times \mathbb{CP}^1$.

Proof. Let $\xi^{\mathbb{C}}$ denote the holomorphic vector field on $\mathbb{CP}^1 \times \mathbb{CP}^1$ associated with ξ, that is, $\xi^{\mathbb{C}} := \xi - \sqrt{-1}I\xi$. At a point $p \in \mathrm{Zero}(\xi)$, taking suitable holomorphic coordinates (z, ζ) with $(z(p), \zeta(p)) = (0, 0)$, we may assume that $\xi^{\mathbb{C}} = \alpha(z)\partial_z + \beta(\zeta)\partial_\zeta$ for $\alpha(z) = z(a_1 + a_2 z)$ and $\beta(\zeta) = \zeta(b_1 + b_2\zeta)$, where $a_1, a_2, b_1, b_2 \in \mathbb{C}$. Since ξ is a time-like vector field, we have

$$(2.57) \qquad \begin{aligned} 2g(\xi, \xi) &= |\alpha(z)|^2 g_{1\bar{1}}(z, \zeta) + \alpha(z)\overline{\beta(\zeta)} g_{1\bar{2}}(z, \zeta) \\ &\quad + \beta(\zeta)\overline{\alpha(z)} g_{2\bar{1}}(z, \zeta) + |\beta(\zeta)|^2 g_{2\bar{2}}(z, \zeta) < 0 \end{aligned}$$

for $(z, \zeta) \notin \mathrm{Zero}(\xi)$. Here $g_{j\bar{l}} = g(\partial_{z^j}, \overline{\partial_{z^l}})$ $(j, l = 1, 2)$ denote the components of g with respect to $\{\partial_{z^1} := \partial_z, \partial_{z^2} := \partial_\zeta\}$. Then, since g is an I-invariant neutral metric, the determinant of the matrix $(g_{j\bar{l}})$ is negative, that is,

$$(2.58) \qquad g_{1\bar{1}}(z, \zeta) g_{2\bar{2}}(z, \zeta) - |g_{1\bar{2}}(z, \zeta)|^2 < 0.$$

The assumption that ξ is Killing (i.e., $\mathcal{L}_\xi g \equiv 0$) is equivalent to

$$(2.59) \qquad \begin{gathered} 2\xi g_{1\bar{1}} + (\alpha'(z) + \overline{\alpha'(z)}) g_{1\bar{1}} = 0, \; 2\xi g_{1\bar{2}} + (\alpha'(z) + \overline{\beta'(\zeta)}) g_{1\bar{2}} = 0, \\ 2\xi g_{2\bar{2}} + (\beta'(\zeta) + \overline{\beta'(\zeta)}) g_{2\bar{2}} = 0. \end{gathered}$$

In particular, we obtain

$$(2.60) \qquad (a_1 + \overline{a_1}) g_{1\bar{1}} = (a_1 + \overline{b_1}) g_{1\bar{2}} = (b_1 + \overline{b_1}) g_{2\bar{2}} = 0$$

at $(z, \zeta) = (0, 0)$.

First, we show that $|a_1|^2 + |b_1|^2 > 0$. Indeed, if we suppose $a_1 = b_1 = 0$, then (2.59) is rewritten as

$$\begin{aligned} \mathrm{Re}(a_2 z^2 \partial_z + b_2 \zeta^2 \partial_\zeta) g_{1\bar{1}} &= -(a_2 z + \overline{a_2 z}) g_{1\bar{1}}, \\ \mathrm{Re}(a_2 z^2 \partial_z + b_2 \zeta^2 \partial_\zeta) g_{1\bar{2}} &= -(a_2 z + \overline{b_2 \zeta}) g_{1\bar{2}}, \\ \mathrm{Re}(a_2 z^2 \partial_z + b_2 \zeta^2 \partial_\zeta) g_{2\bar{2}} &= -(b_2 \zeta + \overline{b_2 \zeta}) g_{2\bar{2}}. \end{aligned}$$

If $(z, \zeta) \to (0, 0)$ along $z = \zeta \in \mathbb{R}$ and along $z = \zeta \in \sqrt{-1}\mathbb{R}$, then we have

$$\begin{aligned} (a_2 + \overline{a_2}) g_{1\bar{1}} = (a_2 + \overline{b_2}) g_{1\bar{2}} = (b_2 + \overline{b_2}) g_{2\bar{2}} = 0, \\ (a_2 - \overline{a_2}) g_{1\bar{1}} = (a_2 - \overline{b_2}) g_{1\bar{2}} = (b_2 - \overline{b_2}) g_{2\bar{2}} = 0 \end{aligned}$$

at $(z, \zeta) = (0, 0)$. From (2.58), it follows that $a_2 = b_2 = 0$, which contradicts $\xi \not\equiv 0$. Thus we obtain that $|a_1|^2 + |b_1|^2 > 0$.

Setting $\zeta = \lambda z$ in (2.57) for an arbitrary $\lambda \in \mathbb{C}$ and $z \to 0$, we have

$$(2.61) \qquad |a_1|^2 g_{1\bar{1}} + \overline{\lambda} a_1 \overline{b_1} g_{1\bar{2}} + \lambda \overline{a_1} b_1 g_{2\bar{1}} + |\lambda|^2 |b_1|^2 g_{2\bar{2}} \leq 0$$

at $(z,\zeta)=(0,0)$. In particular, we obtain

$$|a_1|^2 g_{1\bar{1}}(0,0)\leq 0 \quad \text{and} \quad |b_1|^2 g_{2\bar{2}}(0,0)\leq 0. \tag{2.62}$$

Furthermore, (2.61) also implies that

$$|a_1\overline{b_1}|^4|g_{1\bar{2}}(0,0)|^2(|g_{1\bar{2}}(0,0)|^2-g_{1\bar{1}}(0,0)g_{2\bar{2}}(0,0))\leq 0.$$

Then it follows from (2.58) that $|a_1\overline{b_1}|^4|g_{1\bar{2}}(0,0)|^2=0$. By (2.60), we obtain

$$g_{1\bar{2}}(0,0)=0, \quad \text{and hence} \quad g_{1\bar{1}}(0,0)g_{2\bar{2}}(0,0)<0, \tag{2.63}$$

since $|a_1|^2+|b_1|^2>0$. Then $|a_1|^2|b_1|^2=0$ follows from (2.62). Furthermore, (2.63) implies that ξ has no isolated zero. To show this, we would suppose the contrary, that is, $(|a_1|^2+|a_2|^2)(|b_1|^2+|b_2|^2)>0$. Setting either $z=0$ or $\zeta=0$ in (2.57), we would have $g_{1\bar{1}}(z,0)\leq 0$ and $g_{2\bar{2}}(0,\zeta)\leq 0$, which contradicts (2.63). Therefore $(z,\zeta)=(0,0)$ is not an isolated zero of ξ.

Thus we may assume that $\xi^{\mathbb{C}}=z(a_1+a_2z)\partial_z$ for $a_1\neq 0$. Setting $\tilde{z}:=z/(a_1+a_2z)$, we have $\xi^{\mathbb{C}}=a_1\tilde{z}\partial_{\tilde{z}}$. Here $a_1+\overline{a_1}=0$ follows from (2.60). □

Next, we study the case where two linearly independent time-like Killing vector fields exist.

Proposition 2.33 *Let ξ_1 and ξ_2 be time-like Killing vector fields on a neutral Kähler surface $(\mathbb{CP}^1\times\mathbb{CP}^1,g)$. Suppose that ξ_1 and ξ_2 are linearly independent over $\mathbb{R}$. Then $[\xi_1,\xi_2]\neq 0$, and $\xi_1,\xi_2,[\xi_1,\xi_2]$ are linearly independent over $\mathbb{R}$.*

Proof. We may assume that $\xi_1^{\mathbb{C}}=\sqrt{-1}az\partial_z$ for some real number $a\neq 0$. Since ξ_2 is also a Killing vector field on $(\mathbb{CP}^1\times\mathbb{CP}^1,g)$, the holomorphic vector field $\xi_2^{\mathbb{C}}$ is expressed as either

$$\xi_2^{\mathbb{C}}=(a_0+a_1z+a_2z^2)\partial_z \quad \text{or} \quad \xi_2^{\mathbb{C}}=(b_0+b_1\zeta+b_2\zeta^2)\partial_\zeta.$$

In the second case, ξ_1 and ξ_2 clearly commute, and we may also assume that $\xi_2^{\mathbb{C}}=\sqrt{-1}b\zeta\partial_\zeta$ for some real constant $b\neq 0$. It follows from (2.60) that $g_{1\bar{2}}(0,0)=0$, and hence from (2.57) that $g_{1\bar{1}}(0,0)<0$ and $g_{2\bar{2}}(0,0)<0$. Then we have

$$g_{1\bar{1}}(0,0)g_{2\bar{2}}(0,0)-|g_{1\bar{2}}(0,0)|^2=g_{1\bar{1}}(0,0)g_{2\bar{2}}(0,0)>0.$$

However, this contradicts (2.58).

Now, consider the case where $\xi_2^{\mathbb{C}} = (a_0 + a_1 z + a_2 z^2)\partial_z$. We first notice that $[\xi_1, \xi_2] \equiv 0$ if and only if $[\xi_1^{\mathbb{C}}, \xi_2^{\mathbb{C}}] \equiv 0$. Here $[\xi_1^{\mathbb{C}}, \xi_2^{\mathbb{C}}]$ is computed as

$$[\xi_1^{\mathbb{C}}, \xi_2^{\mathbb{C}}] = \sqrt{-1}a(a_2 z^2 - a_0)\partial_z.$$

Then we have $\xi_2^{\mathbb{C}} = a_1 z \partial_z$ if $[\xi_1, \xi_2] \equiv 0$ everywhere on $\mathbb{CP}^1 \times \mathbb{CP}^1$. By (2.60), we also see that a_1 should be a nonzero pure imaginary. This means that ξ_1 and ξ_2 are linearly dependent over $\mathbb{R}$. If $[\xi_1, \xi_2] \not\equiv 0$, then it can be verified that $\xi_1, \xi_2, [\xi_1, \xi_2]$ are linearly independent over $\mathbb{R}$. $\square$

Proof of Theorem 2.31: Let $\bar{g}_V := \bar{g}_{V,\theta}$ and $\bar{g}_{V'} := \bar{g}_{V',\theta'}$ be non-conformally-flat, self-dual neutral Kähler metrics on $\overline{\mathcal{M}} = \mathbb{CP}^1 \times \mathbb{CP}^1$ given by

$$\bar{g}_V = \mathrm{sech}^2\rho(-V^{-1}\theta^2 + V\, g_{S_1^3}), \quad \bar{g}_{V'} = \mathrm{sech}^2\rho(-V'^{-1}\theta'^2 + V'\, g_{S_1^3}),$$

respectively. Then the pull-back metric $\varphi^*\bar{g}_{V'}$ is written as

$$\varphi^*\bar{g}_{V'} = \mathrm{sech}^2(\rho\circ\varphi)\Big(-(V'\circ\varphi)^{-1}(\varphi^*\theta')^2 + (V'\circ\varphi)\,\varphi^* g_{S_1^3}\Big). \tag{2.64}$$

We now suppose that there exists an orientation-preserving isometry $\varphi : (\overline{\mathcal{M}}, \bar{g}_V) \to (\overline{\mathcal{M}}, \bar{g}_{V'})$. Let ξ and ξ' be the Killing vector fields tangent to the fibers on $(\overline{\mathcal{M}}, \bar{g}_V)$ and $(\overline{\mathcal{M}}, \bar{g}_{V'})$ satisfying $\theta(\xi) = \theta'(\xi') = 1$, respectively. Since $\varphi^*\bar{g}_{V'} = \bar{g}_V$, the pull-back vector field $\varphi^*\xi'$ of ξ' is also a time-like Killing vector field on $(\overline{\mathcal{M}}, \bar{g}_V)$. We have to consider the following two cases:

$$(1)\ \ [\xi, \varphi^*\xi'] \equiv 0, \quad (2)\ \ [\xi, \varphi^*\xi'] \not\equiv 0.$$

In the case (1), we see from Proposition 2.33 that ξ and $\varphi^*\xi'$ are linearly dependent, that is, $\xi' = k\varphi_*\xi$ for some real constant $k\neq 0$. It is clear that

$$\varphi^*\theta'(\xi) = k^{-1}\theta(\xi), \quad \varphi^*\bar{g}_{V'}(\varphi^*\xi', \cdot) = k\bar{g}_V(\xi, \cdot). \tag{2.65}$$

Comparing the same quantity

$$\varphi^*\bar{g}_{V'}(\varphi^*\xi', \varphi^*\xi') = -\frac{(V'\circ\varphi)^{-1}}{\cosh^2(\rho\circ\varphi)} \quad \text{and} \quad k^2\bar{g}_V(\xi, \xi) = -\frac{k^2 V^{-1}}{\cosh^2\rho},$$

we then have

$$k^2(V'\circ\varphi)\cosh^2(\rho\circ\varphi) = V\cosh^2\rho. \tag{2.66}$$

Thus $\varphi^*\bar{g}_{V'}$ is rewritten as

$$\varphi^*\bar{g}_{V'} = -\frac{V^{-1}}{\cosh^2\rho}(k\varphi^*\theta')^2 + \frac{V'\circ\varphi}{\cosh^2(\rho\circ\varphi)}\,\varphi^* g_{S_1^3}. \tag{2.67}$$

It follows from (2.65), (2.66) and (2.67) that

$$\varphi^*\theta' = k^{-1}\theta, \quad \varphi^* g_{S^3_1} = \frac{k^2\cosh^4(\rho\circ\varphi)}{\cosh^4\rho} g_{S^3_1}. \tag{2.68}$$

In particular, φ determines a conformal transformation of the de Sitter space S^3_1.

It is well-known that $\mathrm{Conf}(S^3_1)$, the group of (orientation-preserving) conformal transformations of S^3_1, is isomorphic to $SO(2,3)$. Indeed, if we realize S^3_1 as a hypersurface of $\mathbb{RP}^4$:

$$S^3_1 = \left\{ (x_0 : x_1 : x_2 : x_3 : x_4) \;\middle|\; \begin{array}{c} -x_0^2 + x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0, \\ x_4 = 1 \end{array} \right\},$$

then the action of $\mathrm{Conf}(S^3_1)$ on S^3_1 is induced from a linear transformation on $\mathbb{R}^5$ preserving the quadratic form $-x_0^2 + x_1^2 + x_2^2 + x_3^2 - x_4^2$. This action is also given by a linear fractional transformation as follows:

$$\varphi(x) = \frac{Px + q}{r^*x + s} \quad (x \in S^3_1 \subset \mathbb{R}^4_1), \tag{2.69}$$

where P is a 4×4-matrix, q, r are column vectors of $\mathbb{R}^4$ and $s \in \mathbb{R}$ such that

$$\begin{array}{lll} P^*P - rr^* = \mathbb{E}, & P^*q - rs = 0, & -q^*q + s^2 = 1, \\ PP^* - qq^* = \mathbb{E}, & -Pr + qs = 0, & -r^*r + s^2 = 1. \end{array} \tag{2.70}$$

Here * means the metric dual in $\mathbb{R}^4_1$. Then we have

$$\varphi^* g_{S^3_1} = (r^*x + s)^{-2} g_{S^3_1}. \tag{2.71}$$

In (2.69), we may express P, q, r, x respectively as

$$P = \begin{pmatrix} a & b^* \\ c & D \end{pmatrix}, \; q = \begin{pmatrix} q_0 \\ \underline{q} \end{pmatrix}, \; r = \begin{pmatrix} -r_0 \\ \underline{r} \end{pmatrix}, \; x = \begin{pmatrix} \sinh\rho \\ (\cosh\rho)v \end{pmatrix},$$

where $a, q_0, r_0 \in \mathbb{R}$, $b, c, \underline{q}, \underline{r} \in \mathbb{R}^3$ and D is a 3×3-matrix and $x \in S^3_1$. Then, from (2.69), we obtain

$$\cosh^2(\rho\circ\varphi) = 1 + \left(\frac{a\sinh\rho + \cosh\rho\, b^*v + q_0}{r_0\sinh\rho + \cosh\rho\, \underline{r}^*v + s} \right)^2. \tag{2.72}$$

On the other hand, comparing (2.68) and (2.71), we also obtain

$$\cosh^2(\rho\circ\varphi) = \frac{\cosh^2\rho}{|k(r_0\sinh\rho + \cosh\rho\, \underline{r}^*v + s)|}. \tag{2.73}$$

Therefore we have

$$\begin{aligned} & 1+\left(\frac{a\sinh\rho+\cosh\rho\, b^* v+q_0}{r_0\sinh\rho+\cosh\rho\,\underline{r}^* v+s}\right)^2 \\ (2.74) \qquad & = \frac{\cosh^2\!\rho}{|k(r_0\sinh\rho+\cosh\rho\,\underline{r}^* v+s)|}. \end{aligned}$$

As $\rho\to\pm\infty$, we can see that $r_0=0$ and $\underline{r}=0$, that is, $r=0$. Indeed, unless $r=0$, the limit of the left hand side is finite for some $v\in S^2$, but that of the right hand side is always infinite. This is a contradiction. Since $r=0$, it follows from (2.70) that $q=0$, $s^2=1$ and $P\in O(1,3)$. Then (2.74) is equivalent to

$$k^2(1+(a\sinh\rho+\cosh\rho\, b^* v)^2)^2=\cosh^4\!\rho. \tag{2.75}$$

Dividing both sides of (2.75) by $\cosh^4\!\rho$ and taking $\rho\to\pm\infty$, and setting $\rho=0$, we have

$$k^2(\pm a+b^* v)^4=\lim_{\rho\to\pm\infty}k^2(\mathrm{sech}^2\rho+(a\tanh\rho+b^* v)^2)^2=1, \tag{2.76}$$

$$k^2(1+(b^* v)^2)^2=1 \tag{2.77}$$

for any $v\in S^2$. Then (2.76) and (2.77) imply that $k^2=1$, $b=0$ and $a^2=1$, and hence $c=0$ and $D\in O(3)$. It turns out that φ is induced from an element of $S(O(1)\times O(3)\times O(1))$. Namely,

$$\left(\begin{array}{c|c} P & q \\ \hline r^* & s \end{array}\right)=\left(\begin{array}{cc|c} \pm1 & 0 & 0 \\ 0 & D & 0 \\ \hline 0 & 0 & \pm1 \end{array}\right), \text{ or equivalently, } \varphi(x)=\frac{1}{\pm1}\begin{pmatrix} \pm1 & 0 \\ 0 & D \end{pmatrix}x$$

for $x\in S^3_1$. Therefore $\bar g_V$ and $\bar g_{V'}$ are isometric if and only if φ belongs to $S(O(1)\times O(3))$, since φ is orientation-preserving. Thus, we have verified Theorem 2.31 in the case (1).

We next consider the case (2). In this case, since $\xi_1:=\xi$ and $\xi_2:=\varphi^*\xi'$ do not commute, ξ_1,ξ_2 and $\xi_3:=[\xi_1,\xi_2](\not\equiv 0)$ are linearly independent, and the corresponding holomorphic vector fields may be given by

$$\xi_1^{\mathbb{C}}=\sqrt{-1}az\partial_z,\quad \xi_2^{\mathbb{C}}=(a_0+a_1z+a_2z^2)\partial_z,\quad \xi_3^{\mathbb{C}}=\sqrt{-1}a(a_2z^2-a_0)\partial_z.$$

The restrictions of ξ_1,ξ_2,ξ_3 to the first factor $S^2\times\{\zeta\}$, the z-sphere, are linearly independent Killing vector fields on $S^2\times\{\zeta\}$ with a negative definite metric $\bar g_V|_{S^2\times\{\zeta\}}$. It is well-known that if a two-dimensional Riemannian manifold admits three linearly independent Killing vector fields, then it should be of constant curvature. Thus $(S^2\times\{\zeta\},\bar g_V|_{S^2\times\{\zeta\}})$ is of constant curvature.

Let $\bar{\Omega}_V$ denote the Kähler form of $(\bar{g}_V, I_V)$, and $\omega_{S^2}(z)$ and $\omega_{S^2}(\zeta)$ the volume forms of the unit round spheres $S^2 \times \{\zeta\}$ and $\{z\} \times S^2$, respectively. Then the Lie derivatives $\mathcal{L}_{\xi_a}\omega_{S^2}(\zeta)$, $\mathcal{L}_{\xi_a}\omega_{S^2}(z)$ and $\mathcal{L}_{\xi_a}\bar{\Omega}_V$ vanish identically $(a = 1, 2, 3)$. In particular, we see that $\mathcal{L}_{\xi_a}(\omega_{S^2}(z) \wedge \bar{\Omega}_V) = (\xi_a V)\omega_{S^2}(z) \wedge \omega_{S^2}(\zeta) \equiv 0$ $(a = 1, 2, 3)$. Thus V is independent of z, so that the equation $d\check{*}dV \equiv 0$ is reduced to the Laplace equation on $\{z\} \times S^2$. It then follows that V is a constant function, namely, $V \equiv 1$, and hence that $\bar{g}_V$ is the standard product metric on $S^2 \times S^2$. However, this contradicts the assumption that $\bar{g}_V$ is non-conformally-flat. $\square$

3 Neutral hyperkähler surfaces

3.1 Split-quaternions

We will introduce the notion of a *split-quaternion* structure, which is also called under several different names such as an *almost quaternionic structure of the second kind*, a *paraquaternionic* structure or a *biparacomplex structure*, on a smooth manifold (see [42], cf. Libermann [67], Ianus [38]; Blažić [9], García-Río et al. [28]; Etayo-Santamaria [24]). We begin by recalling the definition of the split-quaternion algebra ${}'\mathbb{H}$.

The split-quaternion algebra ${}'\mathbb{H}$ is an $\mathbb{R}$-algebra with the unit 1, generated by $1, \boldsymbol{i}, {}'\boldsymbol{j}, {}'\boldsymbol{k}$ as a vector space over $\mathbb{R}$:

$$
{}'\mathbb{H} = \{p + q\boldsymbol{i} + r{}'\boldsymbol{j} + s{}'\boldsymbol{k} \mid p, q, r, s \in \mathbb{R}\}.
$$

Here $\boldsymbol{i}, {}'\boldsymbol{j}, {}'\boldsymbol{k}$ are assumed to satisfy the following relations:

$$
\begin{gathered}
\boldsymbol{i}^2 = -1, \quad {}'\boldsymbol{j}^2 = {}'\boldsymbol{k}^2 = 1, \\
\boldsymbol{i}{}'\boldsymbol{j} = -{}'\boldsymbol{j}\boldsymbol{i} = {}'\boldsymbol{k},\ {}'\boldsymbol{j}{}'\boldsymbol{k} = -{}'\boldsymbol{k}{}'\boldsymbol{j} = -\boldsymbol{i},\ {}'\boldsymbol{k}\boldsymbol{i} = -\boldsymbol{i}{}'\boldsymbol{k} = {}'\boldsymbol{j}.
\end{gathered}
$$

It is known that ${}'\mathbb{H}$ can be realized as the Clifford algebra $\mathrm{Cliff}(\mathbb{R}^2)$ associated with the usual Euclidean space $\mathbb{R}^2$. To be precise, let e_1 and e_2 be an orthonormal basis for $\mathbb{R}^2$. Then $(e_1)^2 = (e_2)^2 = 1$ and $(e_1{\cdot}e_2)^2 = -1$ in $\mathrm{Cliff}(\mathbb{R}^2)$, where $\cdot$ denotes the Clifford multiplication. Therefore the map

$$
p + q\boldsymbol{i} + r{}'\boldsymbol{j} + s{}'\boldsymbol{k} \mapsto p + q(-e_1{\cdot}e_2) + re_1 + se_2
$$

gives an isomorphism from ${}'\mathbb{H}$ to $\mathrm{Cliff}(\mathbb{R}^2)$. Note that ${}'\mathbb{H}$ is also identified with the Clifford algebra $\mathrm{Cliff}(\mathbb{R}^2_1)$ associated with the pseudo-Euclidean space $\mathbb{R}^2_1$ of type (1,1), via the map

$$
p + q\boldsymbol{i} + r{}'\boldsymbol{j} + s{}'\boldsymbol{k} \mapsto p + q\varepsilon_1 + r\varepsilon_2 + s\varepsilon_1{\cdot}\varepsilon_2,
$$

where $\{\varepsilon_1, \varepsilon_2\}$ is an orthonormal basis for $\mathbb{R}^2_1$ which satisfies $\langle \varepsilon_1, \varepsilon_1 \rangle = -1$, $\langle \varepsilon_2, \varepsilon_2 \rangle = +1$ and $\langle \varepsilon_1, \varepsilon_2 \rangle = 0$. Then ${}'\mathbb{H}$ is also isomorphic to the algebra $\mathrm{M}_2(\mathbb{R})$ of real $2{\times}2$-matrices. For example, an isomorphism ${}'\mathbb{H}{\cong}\mathrm{M}_2(\mathbb{R})$ is given by

$$
\iota : p + q\boldsymbol{i} + r{}'\boldsymbol{j} + s{}'\boldsymbol{k} \ \mapsto \begin{pmatrix} p+r & -q+s \\ q+s & p-r \end{pmatrix}.
$$

A geometric structure corresponding to ${}'\mathbb{H}$ is defined in the following way.

Definition 3.1 Let M be an n-dimensional manifold. A triplet $(I, 'J, 'K)$ of endomorphisms on the tangent bundle TM of M is called a *split-quaternion structure* if $I, 'J, 'K$ satisfy the following relation:

$$I^2 = -\mathrm{Id},\ 'J^2 = 'K^2 = \mathrm{Id} \quad \text{and} \quad I'J = -'JI = 'K.$$

A metric g on M with a split-quaternion structure $(I, 'J, 'K)$ is said to be *compatible* if the quadruplet $(g, I, 'J, 'K)$ satisfies the following I-invariance and $'J, 'K$-skew-invariance:

$$g(X, Y) = g(IX, IY) = -g('JX, 'JY) = -g('KX, 'KY) \tag{3.1}$$

for arbitrary vector fields X and Y on M.

Let (M, g) be a pseudo-Riemannian manifold compatible with a split-quaternion structure $(I, 'J, 'K)$. Then, such a quadruplet $(g, I, 'J, 'K)$ is called a *neutral almost hyperhermitian* structure (or *an almost paraquaternionic Hermitian* structure) on M. By (3.1), we can define three kinds of two-forms Ω_I, $\Omega_{'J}$, $\Omega_{'K}$ as follows:

$$\Omega_I := g(I\cdot, \cdot),\ \Omega_{'J} := g('J\cdot, \cdot),\ \Omega_{'K} := g('K\cdot, \cdot). \tag{3.2}$$

The triplet $(\Omega_I, \Omega_{'J}, \Omega_{'K})$ is called the fundamental form of $(g, I, 'J, 'K)$.

We now examine the existence of a neutral almost hyperhermitian structure $(g, I, 'J, 'K)$ on a smooth manifold M. We first show that there exists a suitable orthonormal frame field associated with the given neutral almost hyperhermitian structure. In particular, we see that the dimension of M is divisible by 4.

Proposition 3.2 *Let $(M, g, I, 'J, 'K)$ be an n-dimensional neutral almost hyperhermitian manifold ($n \geq 1$). Then $n = 4k$ for some positive integer k, and there exists a local oriented frame field $\{e_1^{\pm}, \ldots, e_{2k}^{\pm}\}$ on M such that*

$$Ie_A^+ = e_A^- \quad (1 \leq A \leq 2k), \tag{3.3}$$

$$g(e_A^+, e_B^+) = g(e_A^-, e_B^-) = 0 \quad (1 \leq A, B \leq 2k), \tag{3.4}$$

$$g(e_{2i-1}^-, e_{2j}^+) = -g(e_{2i-1}^+, e_{2j}^-) = \delta_{ij} \quad (1 \leq i, j \leq k). \tag{3.5}$$

With respect to this local frame field $\{e_1^+, \ldots, e_{2k}^+; e_1^-, \ldots, e_{2k}^-\}$, we can express $I, 'J, 'K$ and g respectively as

$$I = \begin{pmatrix} O_{2k} & -E_{2k} \\ E_{2k} & O_{2k} \end{pmatrix}, \; 'J = \begin{pmatrix} E_{2k} & O_{2k} \\ O_{2k} & -E_{2k} \end{pmatrix}, \; 'K = \begin{pmatrix} O_{2k} & E_{2k} \\ E_{2k} & O_{2k} \end{pmatrix},$$
$$g = \begin{pmatrix} O_{2k} & -{}^tJ_{2k} \\ -J_{2k} & O_{2k} \end{pmatrix} = \begin{pmatrix} O_{2k} & J_{2k} \\ -J_{2k} & O_{2k} \end{pmatrix},$$

where E_{2k} and J_{2k} denote respectively the following $2k\times 2k$-matrices:

$$E_{2k} := \begin{pmatrix} 1 & 0 & & & \\ 0 & 1 & & & \\ & & \ddots & & \\ & & & 1 & 0 \\ & & & 0 & 1 \end{pmatrix}, \quad J_{2k} := \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}.$$

Proof. Let $\mathcal{F}_{'J}^{\pm}$ (resp. $\mathcal{F}_{'K}^{\pm}$) be a tangential distribution on M defined by

$$\mathcal{F}_{'J}^{\pm} := \{v \in TM \mid 'Jv = \pm v\} \text{ (resp. } \mathcal{F}_{'K}^{\pm} := \{v \in TM \mid 'Kv = \pm v\}).$$

Then $\mathcal{F}_{'J}^+$ and $\mathcal{F}_{'J}^-$ are isotropic with respect to g, that is, $g = 0$ on $\mathcal{F}_{'J}^+ \times \mathcal{F}_{'J}^+$ and on $\mathcal{F}_{'J}^- \times \mathcal{F}_{'J}^-$. Note that $\mathcal{F}_{'J}^+$ and $\mathcal{F}_{'J}^-$ are isomorphic to each other, as real vector bundles, since the almost complex structure I gives an isomorphism $I : \mathcal{F}_{'J}^+ \to \mathcal{F}_{'J}^-$. If we identify $\mathcal{F}_{'J}^-$ with $\mathcal{F}_{'J}^+$ by I and if we set $\mathcal{F}_{'J}^+ (\cong \mathcal{F}_{'J}^-) =: E$, then the tangent bundle TM is isomorphic to $E \oplus E$.

We can construct an orthonormal frame field required above by using a modified Gram-Schmidt process: Take a nonzero vector $v_1^+ \in \mathcal{F}_{'J}^+$. Set $e_1^+ := v_1^+$ and $e_1^- := Ie_1^+$. Then $g(e_1^+, e_1^+) = g(e_1^-, e_1^-) = 0$ and $g(e_1^+, e_1^-) = 0$. Hence there exists a vector $v_2^+ \in \mathcal{F}_{'J}^+$ such that $g(e_1^-, v_2^+) \neq 0$. Indeed, if we would suppose that $g(e_1^-, v^+) = 0$ for any $v^+ \in \mathcal{F}_{'J}^+$, then $g(e_1^-, v) = 0$ for any $v \in TM$, since $\mathcal{F}_{'J}^-$ is an isotropic subspace. By the nondegeneracy of g, we would have $e_1^- = 0$, which contradicts $e_1^- \neq 0$. Set $e_2^+ := (1/g(e_1^-, v_2^+))v_2^+$ and $e_2^- := Ie_2^+$. Then $g(e_1^-, e_2^+) = 1$ and $g(e_1^+, e_2^-) = -1$. If the dimension is greater than four, we can take a nonzero vector $v_3^+ \in \mathcal{F}_{'J}^+$ such that $\{e_1^+, e_2^+, v_3^+\}$ is linearly independent. Set $e_3^+ := v_3^+ + g(v_3^+, e_2^-)e_1^+ - g(v_3^+, e_1^-)e_2^+$ and $e_3^- := Ie_3^+$. Then $g(e_3^+, e_1^-) = g(e_3^+, e_2^-) = g(e_3^-, e_1^+) = g(e_3^-, e_2^+) = 0$. By a similar argument, we can show that there exists a vector $v_4^+ \in \mathcal{F}_{'J}^+$ with $g(e_3^-, v_4^+) = 1$. Set $e_4^+ := v_4^+ + g(v_4^+, e_2^-)e_1^+ - g(v_4^+, e_1^-)e_2^+$. Then $g(e_4^+, e_1^-) = g(e_4^+, e_2^-) = 0$ and $g(e_4^+, e_3^-) = 1$. Repeating this process, we see that the dimension is divisible by four, so we set $\dim_{\mathbb{R}} M = 4k$. Then we obtain a basis $\{e_1^{\pm}, \dots, e_{2k}^{\pm}\}$ satisfying the required properties. $\square$

As a corollary of Proposition 3.2, we have the following

Corollary 3.3 *There exists a local orthonormal frame field* $\{e_1, \dots, e_{4k}\}$ *on a* $4k$*-dimensional neutral almost hyperhermitian manifold* $(M, g, I, \prime J, \prime K)$ *such that*

$$e_1,\ \ e_2 := Ie_1,\ \ e_3 := \prime Je_1,\ \ e_4 := \prime Ke_1,\ \dots$$
$$\dots,\ e_{4k-3},\ e_{4k-2} := Ie_{4k-3},\ e_{4k-1} := \prime Je_{4k-3},\ e_{4k} := \prime Ke_{4k-3}$$

satisfy

$$g(e_1, e_1) = g(e_2, e_2) = \cdots = g(e_{4k-3}, e_{4k-3}) = g(e_{4k-2}, e_{4k-2}) = -1,$$
$$g(e_3, e_3) = g(e_4, e_4) = \cdots = g(e_{4k-1}, e_{4k-1}) = g(e_{4k}, e_{4k}) = +1.$$

Proof. Let $\{e_1^{\pm}, \dots, e_{2k}^{\pm}\}$ be as in Proposition 3.2. Define a local orthonormal frame field on (M, g) by

$$\begin{array}{llll}
e_1 := \dfrac{e_2^+ - e_1^-}{\sqrt{2}}, & & e_2 := \dfrac{e_2^- + e_1^+}{\sqrt{2}} = Ie_1, & \\
e_3 := \dfrac{e_1^- + e_2^+}{\sqrt{2}} = \prime Je_1, & & e_4 := \dfrac{e_2^- - e_1^+}{\sqrt{2}} = Ie_3 = \prime Ke_1, & \\
\dots & & \dots & \\
e_{4k-3} := \dfrac{e_{2k}^+ - e_{2k-1}^-}{\sqrt{2}}, & & e_{4k-2} := \dfrac{e_{2k}^- + e_{2k-1}^+}{\sqrt{2}} = Ie_{4k-3} & \\
e_{4k-1} := \dfrac{e_{2k-1}^- + e_{2k}^+}{\sqrt{2}} = \prime Je_{4k-3}, & & e_{4k} := \dfrac{e_{2k}^- - e_{2k-1}^+}{\sqrt{2}} = Ie_{4k-1}, & \\
& & \qquad = \prime Ke_{4k-3}. &
\end{array}$$

Then $\{e_1, \dots, e_{4k}\}$ satisfies the required conditions. $\square$

Let E be a subbundle of the tangent bundle TM consisting of $(+1)$-eigenvectors of $\prime J$ and ω the fundamental form Ω_I restricted to E. Then ω is a nondegenerate smooth section of $\Lambda^2 E^*$, that is, $\mathcal{E} := (E, \omega)$ is a symplectic vector bundle over M. Indeed, the nondegeneracy of ω is verified in the following way: Suppose that $\omega(X, Y) = 0$ for arbitrary vector field Y tangent to E (X is also a vector field tangent to E). Then $g(IX, Y) = 0$ for arbitrary $Y \in E$. On the other hand, if Y is tangent to $I(E)$, then $g(IX, Y) = -g(X, IY) = 0$, since X and IY are tangent to a totally null distribution E. Thus we have $g(IX, Y) = 0$ for any vector field Y on M. Therefore $X \equiv 0$ by the nondegeneracy of g. Summarizing the above, we have the following

Proposition 3.4 *Let $(g, I, 'J, 'K)$ be a neutral almost hyperhermitian structure on a real $n = 4k$-dimensional manifold M. Then there exists a $2k$-dimensional symplectic vector subbundle $\mathcal{E} = (E, \omega)$ of TM such that*

$$(TM, \Omega_I) \cong (E \oplus E, \omega \oplus \omega). \tag{3.6}$$

Furthermore, via the identification (3.6)*, we can express $I, 'J, 'K, g, \Omega_I, \Omega_{'J}, \Omega_{'K}$ respectively as*

$$I = \begin{pmatrix} O & -\mathrm{Id} \\ \mathrm{Id} & O \end{pmatrix}, \ 'J = \begin{pmatrix} \mathrm{Id} & O \\ O & -\mathrm{Id} \end{pmatrix}, \ 'K = \begin{pmatrix} O & \mathrm{Id} \\ \mathrm{Id} & O \end{pmatrix}, \tag{3.7}$$

$$g = \begin{pmatrix} O & \omega \\ -\omega & O \end{pmatrix}, \tag{3.8}$$

$$\Omega_I = \begin{pmatrix} \omega & O \\ O & \omega \end{pmatrix}, \ \Omega_{'J} = \begin{pmatrix} O & -\omega \\ -\omega & O \end{pmatrix}, \ \Omega_{'K} = \begin{pmatrix} \omega & O \\ O & -\omega \end{pmatrix}. \tag{3.9}$$

Conversely, if there exists a subbundle E with a symplectic structure ω of the tangent bundle TM such that $TM \cong E \oplus E$, then M admits a neutral almost hyperhermitian structure $(g, I, 'J, 'K)$ defined by (3.7) *and* (3.8)*.*

Proposition 3.2 allows us to give a description of Proposition 3.4, the existence of neutral almost hyperhermitian structures, from the viewpoint of G-structures as follows: Let $\{e_1^\pm, \ldots, e_{2k}^\pm\}$ and $\{\tilde{e}_1^\pm, \ldots, \tilde{e}_{2k}^\pm\}$ be local oriented frame fields satisfying the conditions (3.3), (3.4) and (3.5). Then there exists a local matrix-valued function T such that

$$(\tilde{e}_1^+, \ldots, \tilde{e}_{2k}^+, \tilde{e}_1^-, \ldots, \tilde{e}_{2k}^-) = (e_1^+, \ldots, e_{2k}^+, e_1^-, \ldots, e_{2k}^-) \begin{pmatrix} T & O_{2k} \\ O_{2k} & T \end{pmatrix},$$

where T takes values in $2k \times 2k$-matrices and satisfies ${}^tT J_{2k} T = J_{2k}$. Therefore the existence of neutral almost hyperhermitian structure is equivalent to that of $\Delta(\mathrm{Sp}(k, \mathbb{R}))$-structure, where $\Delta(\mathrm{Sp}(k, \mathbb{R}))$ denotes the image of the diagonal embedding of the real symplectic group $\mathrm{Sp}(k, \mathbb{R})$ into $\mathrm{Sp}(k, \mathbb{R}) \times \mathrm{Sp}(k, \mathbb{R})$. In other words, a $4k$-dimensional manifold M admits a neutral almost hyperhermitian structure $(g, I, 'J, 'K)$ if and only if there exists a symplectic vector bundle $\mathcal{E} = (E, \omega)$ over M of rank $2k$ such that $TM \cong E \oplus E$ with three almost symplectic structures:

$$\begin{array}{rcrcl} \Omega_1(X, Y) & := & \omega(X^+, Y^+) & + & \omega(X^-, Y^-), \\ \Omega_2(X, Y) & := & -\omega(X^+, Y^-) & + & \omega(Y^+, X^-), \\ \Omega_3(X, Y) & := & \omega(X^+, Y^+) & - & \omega(X^-, Y^-) \end{array}$$

for arbitrary vector fields $X = (X^+, X^-), Y = (Y^+, Y^-) \in E \oplus E$. In Hitchin [37], such a structure $(\Omega_1, \Omega_2, \Omega_3)$ is called a *hypersymplectic structure*, if Ω_1, Ω_2 and Ω_3 are closed forms.

***Remark* 3.5** The dimension of a manifold M admitting a split-quaternion structure $(I, 'J, 'K)$ should be even, since I is an almost complex structure on M. However, the dimension is not necessarily divisible by 4. In fact, the Euclidean space $\mathbb{R}^2$ admits a split-quaternion structure:

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ 'J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ 'K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In this case, there exists no metric on $\mathbb{R}^2$ compatible with the split-quaternion structure $(I, 'J, 'K)$ above.

We can generalize this example to arbitrary even-dimensional Euclidean space $\mathbb{R}^{2m}$, by replacing 1 with the identity matrix E_m. In general, a $2m$-dimensional manifold M admits a split-quaternion structure $(I, 'J, 'K)$ if and only if there exists a subbundle E such that $TM \cong E \oplus E (\cong E \otimes_{\mathbb{R}} \mathbb{C})$. Hence we see that all odd Chern classes $c_{2j+1}(TM, I)$ of a split-quaternion manifold $(M, I, 'J, 'K)$ are two-torsion elements in the cohomology groups $H^{4j+2}(M; \mathbb{Z})$, that is, $2c_{2j+1}(TM, I) = 0$ (see Milnor-Stasheff [75]). In particular, $c_{2j+1}(TM, I) = 0$ in the de Rham cohomology group $H^{4j+2}(M; \mathbb{R})$ $(2 \leq 4j + 2 \leq 2m)$. Furthermore, it follows that any split-quaternion manifold $(M, I, 'J, 'K)$ admits a Norden metric g compatible with I (see Bonome et al. [12]).

Suppose that E has an almost complex structure J_E (e.g., E is a symplectic vector bundle). Then the given almost complex structure I is homotopic to $J_E \oplus (-J_E)$, via the identification $TM = E \oplus E$. Indeed,

$$I(t) := \cos t \begin{pmatrix} O & -\mathrm{Id} \\ \mathrm{Id} & O \end{pmatrix} + \sin t \begin{pmatrix} J_E & O \\ O & -J_E \end{pmatrix}$$

gives a smooth family of almost complex structures on M with $I(0) = I$ and $I(\pi/2) = J_E \oplus (-J_E)$. If $m = 2k$ and k is odd, then $c_{2j+1}(E, -J_E) = -c_{2j+1}(E, J_E)$. Therefore $c_{2j+1}(TM, I) = c_{2j+1}(E \oplus E, J_E \oplus (-J_E)) = 0$ in $H^{4j+2}(M; \mathbb{Z})$ $(1 \leq 2j+1 \leq m)$. In particular, the first Chern class $c_1(TM, I)$ of a neutral almost hyperhermitian four-manifold $(M, g, I, 'J, 'K)$ is zero in the cohomology group $H^2(M; \mathbb{Z})$.

We now focus our attention on the four-dimensional case. A four-manifold M with a split-quaternion structure $(I, 'J, 'K)$ admits a compatible metric g if and only if the structure group of TM can reduce to $\mathrm{GL}_1^+('\mathbb{H}) := \{p + q\boldsymbol{i} + r\boldsymbol{'j} + s\boldsymbol{'k} \mid p^2 + q^2 > r^2 + s^2,\ p, q, r, s \in \mathbb{R}\}$. Note that $\mathrm{GL}_1^+('\mathbb{H})$ is isomorphic to the general linear group $\mathrm{GL}_2^+(\mathbb{R})$ with positive determinants. When we regard $\mathrm{GL}_1^+('\mathbb{H})$ as a subgroup of $\mathrm{GL}_2(\mathbb{C})(\subset \mathrm{GL}_4(\mathbb{R}))$ by a homomorphism

$$\begin{aligned}&\iota : p + q\boldsymbol{i} + r\boldsymbol{'j} + s\boldsymbol{'k} \mapsto \\ &\sqrt{p^2+q^2-r^2-s^2} \cdot \frac{1}{\sqrt{p^2+q^2-r^2-s^2}} \begin{pmatrix} p+\sqrt{-1}q & r+\sqrt{-1}s \\ r-\sqrt{-1}s & p-\sqrt{-1}q \end{pmatrix},\end{aligned}$$

the image of ι is isomorphic to $\mathbb{R}_+ \times \mathrm{SU}(1,1)$, which is contained in the conformal group $\mathrm{CO}(2,2)$. Therefore we see that the conformal class of a neutral metric g on a four-manifold compatible with $(I, 'J, 'K)$ is uniquely determined by the triplet $(I, 'J, 'K)$.

Now, we give a characterization of the fundamental form $(\Omega_I, \Omega_{'J}, \Omega_{'K})$ of a neutral almost hyperhermitian structure $(M, g, I, 'J, 'K)$ (cf. Geiges [30], Geiges-Gonzalo [31]):

Proposition 3.6 *Let $(M, g, I, 'J, 'K)$ be a neutral almost hyperhermitian four-manifold. Then the fundamental form $(\Omega_1, \Omega_2, \Omega_3) := (\Omega_I, \Omega_{'J}, \Omega_{'K})$ satisfies the following relation*:

$$-\Omega_1^2 = \Omega_2^2 = \Omega_3^2, \quad \Omega_l \wedge \Omega_m \equiv 0 \ (l \neq m; l, m = 1, 2, 3). \tag{3.10}$$

Conversely, for any triplet $(\Omega_1, \Omega_2, \Omega_3)$ satisfying (3.10)*, there exists a unique neutral almost hermitian structure $(g, I, 'J, 'K)$ such that $\Omega_I = \Omega_1, \Omega_{'J} = \Omega_2$ and $\Omega_{'K} = \Omega_3$.*

Proof. Let $(M, g, I, 'J, 'K)$ be a neutral almost hyperhermitian four-manifold. Proposition 3.2 implies the existence of a local orthonormal coframe field $\{e^1, e^2, e^3, e^4\}$ on (M, g) satisfying

$$\begin{gathered} e^2 = -Ie^1,\ e^3 = 'Je^1,\ e^4 = 'Ke^1, \\ g = -(e^1)^2 - (e^2)^2 + (e^3)^3 + (e^4)^2. \end{gathered}$$

Then $\Omega_I, \Omega_{'J}, \Omega_{'K}$ are expressed respectively as

$$\Omega_I := -e^1 \wedge e^2 + e^3 \wedge e^4,\ \Omega_{'J} := e^1 \wedge e^3 - e^2 \wedge e^4,\ \Omega_{'K} := e^1 \wedge e^4 - e^3 \wedge e^2,$$

which satisfy the required condition (3.10).

Conversely, we suppose that $(\Omega_1, \Omega_2, \Omega_3)$ satisfies (3.10). Then we define $(I, 'J, 'K)$ by

$$\Omega_3(I\cdot, \cdot) = \Omega_2, \ \Omega_1('J\cdot, \cdot) = \Omega_3, \ \Omega_2('K\cdot, \cdot) = -\Omega_1. \tag{3.11}$$

It follows from $\Omega_2^2 = \Omega_3^2$ and $\Omega_2 \wedge \Omega_3 \equiv 0$ that $I^2 = -\mathrm{Id}$. Similarly, it also follows from $-\Omega_1^2 = \Omega_3^2$ and $\Omega_1 \wedge \Omega_3 \equiv 0$ that $'J^2 = \mathrm{Id}$. By definition, we see that $(I, 'J, 'K)$ is a split-quaternion structure. It can be also verified that Ω_1 is invariant by I, $\Omega_{'J}$ and $\Omega_{'K}$ are skew-invariant by $'J$ and $'K$, respectively. Indeed, by (3.11), we have

$$\begin{aligned}
\Omega_1(IX, IY) &= -\Omega_3(IX, 'KY) = -\Omega_2(X, 'KY) = \Omega_1(X, Y), \\
\Omega_2('JX, 'JY) &= -\Omega_1(IX, 'JY) = -\Omega_3(IX, Y) = -\Omega_2(X, Y), \\
\Omega_3('KX, 'KY) &= \Omega_2('JX, 'KY)) = -\Omega_1('JX, Y) = -\Omega_3(X, Y).
\end{aligned}$$

By a similar computation, we also obtain

$$\Omega_1(\cdot, I\cdot) = -\Omega_2(\cdot, 'J\cdot) = -\Omega_3(\cdot, 'K\cdot) =: g.$$

Therefore g is compatible with $(I, 'J, 'K)$ and hence $(g, I, 'J, 'K)$ is a neutral almost hyperhermitian structure with the desired properties. $\square$

The fundamental form $(\Omega_1, \Omega_2, \Omega_3) := (\Omega_I, \Omega_{'J}, \Omega_{'K})$ of a neutral almost hyperhermitian four-manifold $(M, g, I, 'J, 'K)$ gives rise to three isomorphisms

$$\wedge\Omega_l : \Lambda^1 \longrightarrow \Lambda^3 \quad (l = 1, 2, 3).$$

Then we define one-forms $\beta_1, \beta_2, \beta_3$ by

$$d\Omega_l = \beta_l \wedge \Omega_l \quad (l = 1, 2, 3).$$

These one-forms $\beta_1, \beta_2, \beta_3$, called the *Lee forms*, are related to the integrability of $I, 'J, 'K$. An almost product structure (or an involution) S on TM (i.e., $S^2 = \mathrm{Id}, S \neq \mathrm{Id}$) is said to be *integrable* if the bundles $\mathcal{F}_S^{\pm} := \{v \in TM |\ Sv = \pm v\}$ are integrable. The integrability of S is equivalent to $N_S \equiv 0$, where N_S is the *Nijenhuis tensor* of S defined by

$$N_S(X, Y) := [SX, SY] + S^2[X, Y] - S[SX, Y] - S[X, SY],$$

for arbitrary vector fields X, Y on M. This is also true for the integrability of an almost complex structure I. Namely, I is integrable if and only if the Nijenhuis tensor N_I of I, which is defined by replacing S with I in the definition above, satisfies $N_I \equiv 0$. A neutral almost hyperhermitian four-manifold $(M, g, I, 'J, 'K)$ is called a *neutral hyperhermitian* surface if $I, 'J$ and $'K$ are integrable. We show the following proposition for later use (cf. Boyer [14]).

Proposition 3.7 *Let $(g, I, {}'\!J, {}'\!K)$ be a neutral almost hyperhermitian structure on a four-manifold M. Then I, ${}'\!J$ and ${}'\!K$ are integrable if and only if the Lee forms satisfy $\beta_1 = \beta_2 = \beta_3$.*

***Remark* 3.8** When $I, {}'\!J, {}'\!K$ are integrable, we set $\beta := \beta_1 = \beta_2 = \beta_3$, and call β the *Lee form*.

To show Proposition 3.7, we need the following

Lemma 3.9 *Let $(\Omega_1, \Omega_2, \Omega_3) := (\Omega_I, \Omega_{'J}, \Omega_{'K})$ be the fundamental form of a neutral almost hyperhermitian structure $(g, I, {}'\!J, {}'\!K)$ on M and $(N_1, N_2, N_3) := (N_I, N_{'J}, N_{'K})$ a triplet of the three kinds of the Nijenhuis tensors. Then they satisfy* (3.11) *and the following equations*:

$$\begin{array}{rcrcrcr}
\Omega_1(X,Y) & = & \Omega_1(IX,IY) & = & \Omega_1({}'\!JX,{}'\!JY) & = & \Omega_1({}'\!KX,{}'\!KY), \\
\Omega_2(X,Y) & = & -\Omega_2(IX,IY) & = & -\Omega_2({}'\!JX,{}'\!JY) & = & \Omega_2({}'\!KX,{}'\!KY), \\
\Omega_3(X,Y) & = & -\Omega_3(IX,IY) & = & \Omega_3({}'\!JX,{}'\!JY) & = & -\Omega_3({}'\!KX,{}'\!KY),
\end{array}$$

$$\begin{aligned}
d\Omega_2(X,Y,Z) \; + & \;\; d\Omega_2(IX,IY,Z) \\
= & \;\; d\Omega_3(X,Y,IZ) + d\Omega_3(IX,IY,IZ) \\
& \;\; +\Omega_3(N_1(Y,Z),IX) + \Omega_3(N_1(Z,X),IY), \\
d\Omega_3(X,Y,Z) \; - & \;\; d\Omega_3({}'\!JX,{}'\!JY,Z) \\
= & \;\; d\Omega_1(X,Y,{}'\!JZ) - d\Omega_1({}'\!JX,{}'\!JY,{}'\!JZ) \\
& \;\; -\Omega_1(N_2(Y,Z),{}'\!JX) - \Omega_1(N_2(Z,X),{}'\!JY), \\
-d\Omega_1(X,Y,Z) \; + & \;\; d\Omega_1({}'\!KX,{}'\!KY,Z) \\
= & \;\; d\Omega_2(X,Y,{}'\!KZ) - d\Omega_2({}'\!KX,{}'\!KY,{}'\!KZ) \\
& \;\; -\Omega_2(N_3(Y,Z),{}'\!KX) - \Omega_2(N_3(Z,X),{}'\!KY),
\end{aligned}$$

where X, Y, Z are arbitrary vector fields on M.

Lemma 3.9 can be verified by a direct computation.

Proof of Proposition 3.7. We only show that I is integrable if and only if $\beta_2 = \beta_3$. Set $\beta_{23} := \beta_2 - \beta_3$ and define $B_{23}(X,Y) := \beta_{23}(X)Y - \beta_{23}(Y)X$. From Lemma 3.9, we see that

$$\Omega_2(B_{23}(X,Y) + B_{23}(IX,IY), Z) = \Omega_2(N_1(Y,Z),X) + \Omega_2(N_1(Z,X),Y),$$

where X, Y, Z are arbitrary vector fields on M.

If $\beta_2 = \beta_3$, then we have $\Omega_2(N_1(Y,Z),X) + \Omega_2(N_1(Z,X),Y) \equiv 0$. By changing X, Y, Z cyclically, we also have

$$\begin{aligned} \Omega_2(N_1(Z,X),Y) + \Omega_2(N_1(X,Y),Z) &\equiv 0, \\ \Omega_2(N_1(X,Y),Z) + \Omega_2(N_1(Y,Z),X) &\equiv 0. \end{aligned}$$

So we see that $\Omega_2(N_1(X,Y),Z) \equiv 0$ for any vector fields X, Y, Z on M. Therefore $N_1 \equiv 0$, that is, I is integrable.

Conversely, if I is integrable, then we have $B_{23}(X,Y) + B_{23}(IX,IY) \equiv 0$. If we set $Y = IX$, then $B_{23}(X,IX) = \beta_{23}(X)IX - \beta_{23}(IX)X \equiv 0$. Note that if X is nonzero at a point $x \in M$, then $\{X, IX\}$ is linearly independent in T_xM and hence $\beta_{23}(X) = \beta_{23}(IX) = 0$ at x. Since X is arbitrary, we see that $\beta_{23} \equiv 0$, namely, $\beta_2 = \beta_3$. This completes the proof. $\square$

In regard to the self-duality of neutral hyperhermitian metrics, we prove the following (see [42]. For the Riemannian analogue, see, e.g., Pedersen-Swann [80]):

Proposition 3.10 *Let $(g, I, {}'J, {}'K)$ be a neutral almost hyperhermitian structure on a four-manifold M. If $I, {}'J$ and ${}'K$ are integrable, then g is self-dual.*

Proof. For any constant θ, set ${}'J_\theta := \cos\theta \cdot {}'J + \sin\theta \cdot {}'K$ and ${}'K_\theta := -\sin\theta \cdot {}'J + \cos\theta \cdot {}'K$. Then $(g, I, {}'J_\theta, {}'K_\theta)$ is also a neutral almost hyperhermitian structure on M. Furthermore, its fundamental form $(\Omega_I, \Omega_{'J_\theta}, \Omega_{'K_\theta})$ satisfies

$$d\Omega_I = \beta \wedge \Omega_I, \quad d\Omega_{'J_\theta} = \beta \wedge \Omega_{'J_\theta}, \quad d\Omega_{'K_\theta} = \beta \wedge \Omega_{'K_\theta},$$

since $I, {}'J, {}'K$ are integrable. It then follows from Proposition 4.3 that these $I, {}'J_\theta, {}'K_\theta$ are integrable.

Since ${}'K_\theta = {}'J_{(\theta+\pi/2)}$ for any θ, we may consider only ${}'J_\theta$. Setting $\mathcal{F}^\pm_\theta := \mathcal{F}^\pm_{'J_\theta}$ for simplicity, we see that each $\mathcal{F}^\pm_\theta$ is an integrable totally null plane field (a completely integrable distribution consisting of maximal isotropic planes) on (M, g). From Lemma 2.5, the signature $\kappa(\mathcal{F}^\pm_\theta)$ vanishes for each θ. Moreover, it is verified that each $\mathcal{F}^\pm_\theta$ is an anti-self-dual totally null plane field, since $\Phi(\mathcal{F}^\pm_{'J}) = [-\Omega_I \pm \Omega_{'K}] \in \mathbb{P}(\Lambda^2_-)$ and $\Phi(\mathcal{F}^\pm_\theta)$ depends continuously on θ. Note that for any anti-self-dual totally null plane σ_x ($x \in M$), there exists a constant θ such that $(\mathcal{F}^\pm_\theta)_x = \sigma_x$. Thus we see that $\kappa(\sigma) = 0$ for any anti-self-dual totally null plane σ, and from Proposition 2.4, that g is self-dual. $\square$

We remark that if two of $I, {}'J, {}'K$ (e.g., I and ${}'J$) are integrable, then so is the other one (e.g., ${}'K$). This can be verified from the proof of Proposition 3.7. Here, we give another proof, which is valid for higher dimensions. Let A, B, C

be endomorphisms on the tangent bundle TM satisfying $AB = -BA = C$, and N_A, N_B, N_C denote tensor fields defined by replacing S in N_S with A, B, C, respectively. Then we have the following identity:

$$\begin{aligned} (3.12)\quad & 2N_C(X,Y) = \\ & N_A(BX,BY) - B^2N_A(X,Y) - BN_A(BX,Y) - BN_A(X,BY) \\ & +N_B(AX,AY) - A^2N_B(X,Y) - AN_B(AX,Y) - AN_B(X,AY) \end{aligned}$$

for arbitrary vector fields X, Y on M. Hence if $N_A = N_B \equiv 0$, then $N_C \equiv 0$. By setting $(A,B,C) = (I, 'J, 'K), ('J, 'K, I), ('K, I, 'J)$, we obtain the desired result.

3.2 Neutral hyperkähler structures

The notion of neutral hyperkähler structures is defined as follows:

Definition 3.11 A neutral almost hyperhermitian manifold $(M, g, I, 'J, 'K)$ is called a *neutral hyperkähler* surface if $I, 'J$ and $'K$ are parallel with respect to the Levi-Civita connection ∇ of (M, g).

It is easy to see that this definition is equivalent to the following

Proposition 3.12 *A neutral almost hyperhermitian structure* $(g, I, 'J, 'K)$ *on a manifold* M *is neutral hyperkähler* (*i.e.*, $\nabla I = \nabla 'J = \nabla 'K = 0$) *if and only if* $I, 'J, 'K$ *are integrable and* $\Omega_I, \Omega_{'J}, \Omega_{'K}$ *are closed.*

Proof. We only show that $\nabla 'J \equiv 0$ if and only if $\Omega_{'J} = N_{'J} \equiv 0$. First, note that for arbitrary vector fields X, Y, Z on M, the following identities hold:

$$\begin{aligned} (3.13)\qquad \nabla\Omega_{'J}(X,Y) &= g((\nabla 'J)X, Y), \\ (3.14)\qquad 2g((\nabla_X 'J)Y, Z) &= d\Omega_{'J}(X,Y,Z) + d\Omega_{'J}(X, 'JY, 'JZ) \\ &\quad -g('JX, N_{'J}(Y,Z)). \end{aligned}$$

If $\nabla 'J \equiv 0$, then we see from (3.13) that $\Omega_{'J}$ is closed, since ∇ is torsion-free. From (3.14), we also see that $N_{'J} \equiv 0$. Conversely, if $\Omega_{'J} = N_{'J} \equiv 0$, then it follows from (3.14) that $\nabla 'J \equiv 0$. $\square$

Remark **3.13** A neutral hyperhermitian surface is hyperkähler if and only if its Lee form vanishes identically. Furthermore, if the Lee form is closed (resp. exact), then a neutral hyperhermitian surface is locally (resp. globally) conformal to a neutral hyperkähler surface.

Noting the remark above and Proposition 3.6, we obtain the following

Proposition 3.14 *Let* $(\Omega_1, \Omega_2, \Omega_3)$ *be a triplet of symplectic structures on a four-manifold* M *satisfying the relation* (3.10). *Then there exists a unique neutral hyperkähler structure* $(g, I, 'J, 'K)$ *on* M *such that*

$$\Omega_I = \Omega_1, \quad \Omega_{'J} = \Omega_2, \quad \Omega_{'K} = \Omega_3.$$

Proof. We can define the neutral hyperhermitian structure $(g, I, 'J, 'K)$ by (3.11). It follows from the closedness of Ω_l that $\beta_l \equiv 0$ $(l = 1, 2, 3)$. By Proposition 3.7, we see that $I, 'J, 'K$ are integrable, and hence that $(g, I, 'J, 'K)$ is a neutral hyperkähler structure on M satisfying the required conditions. $\square$

Let $(M, g, I, 'J, 'K)$ be a neutral hyperkähler surface. Then $(\Omega_{'J}, \Omega_{'K})$ is a conformal symplectic couple on M in the sense of Geiges [30]. Therefore the complex-valued two-form $\Omega_{'J} + \sqrt{-1}\Omega_{'K}$ is a nonvanishing closed $(2,0)$-form (i.e., holomorphic two-form) on (M, I), which gives a trivialization of the canonical bundle $K_{(M,I)}$, as a holomorphic vector bundle. In particular, we have $c_1(M, I) = 0$.

We can also prove that any neutral hyperkähler metric g is Ricci-flat and self-dual. Indeed, the self-duality of g follows from Proposition 3.10 or Proposition 2.6. To show the Ricci-flatness, set $(\lambda_-^1, \lambda_-^2, \lambda_-^3) := (-\Omega_I, \Omega_{'J}, \Omega_{'K})$. Then $\lambda_-^1, \lambda_-^2, \lambda_-^3$ form a basis of Λ_-^2. From Proposition 3.12, it follows that $\lambda_-^1, \lambda_-^2, \lambda_-^3$ are parallel with respect to ∇. On the other hand, for a neutral Kähler surface (M, I), the Ricci form of (M, g, I) is determined by the curvature form R_∇ of the connection on Λ_-^2 induced by the Levi-Civita connection ∇ of (M, g). Thus we see that the Ricci curvature of (M, g) vanishes. Summarizing these, we obtain the following

Proposition 3.15 *Any neutral hyperkähler surface* $(M, g, I, 'J, 'K)$ *is Ricci-flat and self-dual, and possesses a nonvanishing holomorphic two-form* $\Omega_{'J} + \sqrt{-1}\Omega_{'K}$ *with respect to* I. *In particular, the canonical bundle* $K_{(M,I)}$ *is trivial as a holomorphic vector bundle.*

To close this section, we remark the following matters on a compact neutral hyperkähler surface $(M, g, I, 'J, 'K)$. It follows from Proposition 3.15 that there exists a nonvanishing holomorphic two-form $\Omega_{'J} + \sqrt{-1}\Omega_{'K}$ on (M, I). In consequence, owing to the Enriques-Kodaira classification (cf. Barth et al. [6]), a compact complex surface (M, I) admitting a neutral hyperkähler structure $(g, I, 'J, 'K)$ is biholomorphic to one of the following possibilities:

(a) a complex torus, (b) a $K3$ surface, (c) a primary Kodaira surface.

It is clear that any complex torus has the standard flat neutral hyperkähler structure induced from that of the complex plane $\mathbb{C}^2$. As mentioned in Section 2.1, Matsushita's result ([70]) implies that $K3$ surfaces admit many neutral metrics (see also Bonome et al. [11]). However, it is known that no $K3$ surface admits a neutral Kähler structure, so the case (b) does not occur (see Draghici [22], Kotschick [54] and Petean [82]). For the case (c), Fernández et al. [25] and de Andrés et al. [20] constructed examples of flat neutral Kähler structures on primary Kodaira surfaces of particular type. We will see later that any primary Kodaira surface admits a neutral hyperkähler structure. Furthermore, we will also discuss the existence of non-flat neutral hyperkähler structures on a primary Kodaira surface and a complex torus.

3.3 Primary Kodaira surfaces

A primary Kodaira surface $X = (M, I)$ is a compact complex surface with $\kappa(X) = 0$, $b_1(X) = 3$, $c_1(X) = 0$, $c_2(X) = 0$. Moreover the other numerical characters of X are given as follows:

$$h^{1,0}(X) = 1,\ q(X) = 2,\ p_{\mathrm{g}}(X) = 1,\ b_2^+(X) = b_2^-(X) = 2,$$

where $h^{1,0}(X), q(X)$ and $p_{\mathrm{g}}(X)$ denote the complex dimension of the space of holomorphic one-forms, the irregularity and the geometric genus of X, respectively (see Barth et al. [6]). Any primary Kodaira surface admits no positive-definite Kähler metric, since its first Betti number $b_1(X)$ is three. It is well-known that every primary Kodaira surface X is covered by the complex plane $\mathbb{C}^2$ and its fundamental group $\pi_1(X)$ is represented injectively into the complex affine transformation group $\mathrm{Affine}(\mathbb{C}^2)$ on $\mathbb{C}^2$:

$$\begin{aligned} \rho : \pi_1(X) &\longrightarrow \mathrm{Affine}(\mathbb{C}^2),\ \gamma \mapsto \rho_\gamma, \\ \rho_\gamma(z_1, z_2) &= (z_1 + \alpha_\gamma, z_2 + \overline{\alpha_\gamma} z_1 + \beta_\gamma), \end{aligned}$$

where (z_1, z_2) are the standard complex coordinates of $\mathbb{C}^2$ and $\alpha_\gamma, \beta_\gamma$ are constants in $\mathbb{C}$ depending only on γ. Setting $G := \rho(\pi_1(X))$, we can then identify X with $\mathbb{C}^2/G$, as a complex surface (see Kodaira [51]).

We are now in a position to state one of our main results in this chapter.

Theorem 3.16 *Let $X = \mathbb{C}^2/G$ be a primary Kodaira surface. Then the following two-forms $\Omega_1, \Omega_2, \Omega_3$ define a neutral hyperkähler structure on X:*

$$\begin{aligned} \Omega_1 &= \mathrm{Im}(dw_1 \wedge d\overline{w_2}) + \sqrt{-1}\mathrm{Re}(w_1) dw_1 \wedge d\overline{w_1} + (\sqrt{-1}/2)\partial\overline{\partial}\varphi, \\ \Omega_2 &= \mathrm{Re}(e^{\sqrt{-1}\theta} dw_1 \wedge dw_2), \quad \Omega_3 = \mathrm{Im}(e^{\sqrt{-1}\theta} dw_1 \wedge dw_2), \end{aligned} \tag{3.15}$$

where (w_1, w_2) *is the standard complex coordinate system of* $\mathbb{C}^2$, θ *is a real constant and* φ *is a solution of the equation*

$$(3.16)\ 4\sqrt{-1}(\mathrm{Im}(dw_1 \wedge d\overline{w_2}) + \sqrt{-1}\mathrm{Re}(w_1)dw_1 \wedge d\overline{w_1}) \wedge \partial\overline{\partial}\varphi = \partial\overline{\partial}\varphi \wedge \partial\overline{\partial}\varphi.$$

In particular, any primary Kodaira surface admits a neutral hyperkähler structure. Conversely, under suitable complex coordinates (w_1, w_2) *of* $\mathbb{C}^2$, *the fundamental form of any neutral hyperkähler structure on* X *is expressed as* (3.15).

Proof. Let $\Psi : X \to \Delta$ be an elliptic fiber bundle structure over the base elliptic curve Δ. Then we have the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{C}^2 & \xrightarrow{\widetilde{\varpi}} & X \\
{\scriptstyle \widetilde{\Psi}}\big\downarrow & & \big\downarrow{\scriptstyle \Psi} \\
\mathbb{C} & \xrightarrow[\varpi]{} & \Delta
\end{array}$$

where $\widetilde{\Psi}$ is the projection from $\mathbb{C}^2$ to the first factor $\mathbb{C}$, and $\widetilde{\varpi}, \varpi$ are the covering maps. Let (z_1, z_2) denote the standard complex coordinate system of $\mathbb{C}^2$ above. Then $\phi := dz_1$ gives rise to a nonvanishing holomorphic one-form on X, and generates the cohomology group $H^0(X; \Omega^1_X) \cong H^{1,0}_{\overline{\partial}}(X)$, where Ω^1_X denotes the sheaf of germs of holomorphic one-forms on X. Furthermore, $\sigma^{0,1} := d\overline{z_2} - z_1 d\overline{z_1}$ is a $\overline{\partial}$-closed $(0,1)$-form on X, and the $\overline{\partial}$-cohomology classes of $\bar{\phi}$ and $\sigma^{0,1}$ generate the Dolbeault cohomology group $H^1(X; \mathcal{O}_X) \cong H^{0,1}_{\overline{\partial}}(X)$, where $\mathcal{O}_X$ denotes the structure sheaf of X. Since $d\sigma^{0,1} = -dz_1 \wedge d\overline{z_1}$, a real one-form $\sigma := \overline{\sigma^{0,1}} + \sigma^{0,1}$ on X is d-closed. Furthermore, we see that the cohomology classes of $\phi, \bar{\phi}$ and σ generate $H^1(X; \mathbb{C})$. Note that $dz_1 \wedge (dz_2 - \overline{z_1}dz_1)$ yields a nonvanishing holomorphic two-form on X.

Let $(\Omega_1, \Omega_2, \Omega_3) := (\Omega_I, \Omega_{\mathcal{J}}, \Omega_{\mathcal{K}})$ be the fundamental form of a neutral hyperkähler structure $(g, I, \mathcal{J}, \mathcal{K})$ on $X = \mathbb{C}^2/G$. As mentioned in Proposition 3.15, $\Omega_2 + \sqrt{-1}\Omega_3$ is a nonvanishing holomorphic two-form on X, and hence defines a global section of the canonical bundle K_X. Therefore there exists a nonzero constant $c_0 = |c_0|e^{\sqrt{-1}\psi} \in \mathbb{C}$ $(\psi \in \mathbb{R})$ such that

$$\Omega_2 + \sqrt{-1}\Omega_3 = c_0 dz_1 \wedge (dz_2 - \overline{z_1}dz_1),$$

since X is compact.

Now, define real d-closed two-forms Ω_2^- and Ω_3^- respectively by

$$\Omega_2^- + \sqrt{-1}\Omega_3^- := \sqrt{-1}d\overline{z_1} \wedge (dz_2 - \overline{z_1}dz_1).$$

It follows from Proposition 3.6 and the definitions of Ω_2^- and Ω_3^- that

$$
\begin{aligned}
-\Omega_1^2 = \Omega_2^2 = \Omega_3^2 = -|c_0|^2(\Omega_2^-)^2 = -|c_0|^2(\Omega_3^-)^2, \\
\Omega_2^- \wedge \Omega_3^- \equiv 0, \quad \Omega_a \wedge \Omega_b^- \equiv 0 \ (2 \le a, b \le 3).
\end{aligned}
\tag{3.17}
$$

We then verify that $(|c_0|\Omega_2^-, \Omega_2, \Omega_3)$ and $(|c_0|\Omega_3^-, \Omega_2, \Omega_3)$ define neutral hyperkähler structures on X, respectively. Note that the cohomology classes of $\Omega_2, \Omega_3, \Omega_2^-, \Omega_3^-$ generate the cohomology group $H^2(X;\mathbb{R})$ and satisfy relations similar to those in (3.17). Recall that the Kähler form Ω_1 is a closed real $(1,1)$-form on X and its cohomology class $[\Omega_1]$ in $H^2(X;\mathbb{R})$ is orthogonal to $[\Omega_2]$ and $[\Omega_3]$ with respect to the cup product. Hence there exist a real one-form η and real constants a, b such that

$$\Omega_1 = |c_0|(a\Omega_2^- + b\Omega_3^-) + d\eta.$$

It then follows from (3.17) that

$$(1 - a^2 - b^2){\Omega_1}^2 = d(\eta \wedge [2|c_0|(a\Omega_2^- + b\Omega_3^-) + d\eta]).$$

By integrating the equation above, we obtain $a^2 + b^2 = 1$, so we may set $a = \cos\epsilon$ and $b = \sin\epsilon$ for some real constant ϵ.

Recalling the decomposition $\eta = \eta^{1,0} + \eta^{0,1}$ ($\overline{\eta^{0,1}} = \eta^{1,0}$), we see that $\eta^{0,1}$ is $\overline{\partial}$-closed, since $\Omega_1, \Omega_2^-, \Omega_3^-$ are real $(1,1)$-forms, and hence that

$$\eta^{0,1} = k\overline{\phi} + l\sigma^{0,1} + \overline{\partial}\mu, \quad d\eta = (\bar{l} - l)dz_1 \wedge d\overline{z_1} + \partial\overline{\partial}(\mu - \bar{\mu}),$$

where k and l are constants, and μ is a complex-valued function on X. Setting $\sqrt{-1}c|c_0|^{2/3} := \bar{l} - l$ ($c \in \mathbb{R}$) and $\sqrt{-1}\varphi := 2(\mu - \bar{\mu})$, we then see that

$$\Omega_1 = |c_0|(\cos\epsilon\ \Omega_2^- + \sin\epsilon\ \Omega_3^-) + \sqrt{-1}c|c_0|^{2/3}\ dz_1 \wedge d\overline{z_1} + (\sqrt{-1}/2)\partial\overline{\partial}\varphi.$$

By making use of the coordinates

$$(w_1, w_2) := (|c_0|^{1/3}e^{\sqrt{-1}\epsilon}z_1 + c, |c_0|^{2/3}z_2),$$

we can express $\Omega_1, \Omega_2, \Omega_3$ as

$$
\begin{gathered}
\Omega_1 = \Omega_0 + (\sqrt{-1}/2)\partial\overline{\partial}\varphi, \\
\Omega_2 + \sqrt{-1}\Omega_3 = e^{\sqrt{-1}(\psi-\epsilon)}dw_1 \wedge dw_2 =: e^{\sqrt{-1}\theta}dw_1 \wedge dw_2,
\end{gathered}
$$

where Ω_0 is given by

$$\Omega_0 := (\sqrt{-1}/2)(d\overline{w_1} \wedge dw_2 - dw_1 \wedge d\overline{w_2} + (w_1 + \overline{w_1})dw_1 \wedge d\overline{w_1}).$$

Therefore we see that $(\Omega_1, \Omega_2, \Omega_3)$ defines a neutral hyperkähler structure on X if and only if φ satisfies the following equation:

$$4\sqrt{-1}\Omega_0 \wedge \partial\overline{\partial}\varphi = \partial\overline{\partial}\varphi \wedge \partial\overline{\partial}\varphi.$$

This completes the proof. $\square$

We note that the corresponding metric $g = g_\varphi$ is explicitly given by

$$g_\varphi = (w_1 + \overline{w_1})|dw_1|^2 - (dw_1 d\overline{w_2} + d\overline{w_1} dw_2) + D^2\varphi, \tag{3.18}$$

where $D^2\varphi$ denotes the complex Hessian of φ. Clearly, the pull-back of any function on the base torus Δ is a solution of (3.16).

By using the expression (3.15), we may give a characterization of flat neutral hyperkähler structures on a primary Kodaira surface in terms of the potential function φ, which shows that each nonconstant function φ on the base torus of any primary Kodaira surface defines a non-flat neutral hyperkähler metric g_φ (cf. Petean [82]).

Theorem 3.17 *Let g_φ be the neutral hyperkähler metric on a primary Kodaira surface X defined by* (3.18), *where φ is a solution of* (3.16). *Then g_φ is flat if and only if φ is constant.*

Proof. Let $X = \mathbb{C}^2/G$ be a primary Kodaira surface, g a neutral hyperkähler metric on X, and $(\Omega_1, \Omega_2, \Omega_3)$ the fundamental form. In terms of complex coordinates (w_1, w_2) satisfying $\Omega_2 + \sqrt{-1}\Omega_3 = e^{\sqrt{-1}\theta} dw_1 \wedge dw_2$ (θ is a real constant), the condition $-\Omega_1^2 = \Omega_2^2 = \Omega_3^2$ is written as

$$g_{1\bar{1}} g_{2\bar{2}} - g_{1\bar{2}} g_{2\bar{1}} \equiv -1. \tag{3.19}$$

Thus the components $g^{\bar{\alpha}\beta}$ satisfy

$$g^{\bar{1}1} = -g_{2\bar{2}}, \quad g^{\bar{1}2} = g_{1\bar{2}}, \quad g^{\bar{2}1} = g_{2\bar{1}}, \quad g^{\bar{2}2} = -g_{1\bar{1}}.$$

The connection form $\{\omega_\beta^\alpha\}$ is given by

$$\begin{aligned} \omega_1^1 &= -g_{2\bar{2}}\partial g_{1\bar{1}} + g_{2\bar{1}}\partial g_{1\bar{2}}, & \omega_2^1 &= -g_{2\bar{2}}\partial g_{2\bar{1}} + g_{2\bar{1}}\partial g_{2\bar{2}}, \\ \omega_1^2 &= g_{1\bar{2}}\partial g_{1\bar{1}} - g_{1\bar{1}}\partial g_{1\bar{2}}, & \omega_2^2 &= g_{1\bar{2}}\partial g_{2\bar{1}} - g_{1\bar{1}}\partial g_{2\bar{2}}. \end{aligned} \tag{3.20}$$

In particular, it follows from (3.19) that

$$\omega_1^1 + \omega_2^2 \equiv 0. \tag{3.21}$$

Recall that the fundamental form Ω_1 may be written as

$$\Omega_1 = (\sqrt{-1}/2)(-dw_1 \wedge d\overline{w_2} - dw_2 \wedge d\overline{w_1} + (w_1 + \overline{w_1})dw_1 \wedge d\overline{w_1} + \partial\overline{\partial}\varphi),$$

where φ is a certain smooth function on X. The components $g_{\alpha\bar{\beta}}$ are given explicitly by

$$g_{1\bar{1}} = w_1 + \overline{w_1} + \frac{\partial^2\varphi}{\partial w_1 \partial\overline{w_1}}, \quad g_{1\bar{2}} = -1 + \frac{\partial^2\varphi}{\partial w_1 \partial\overline{w_2}} (= \overline{g_{2\bar{1}}}), \quad g_{2\bar{2}} = \frac{\partial^2\varphi}{\partial w_2 \partial\overline{w_2}}.$$

From (3.20) and (2.17), we see that g is flat if φ is constant.

For any $\gamma \in G$, we define $\rho_\gamma : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ by

$$\rho_\gamma(w_1, w_2) = (w_1 + \alpha_\gamma, w_2 + \overline{\alpha_\gamma} w_1 + \beta_\gamma).$$

It then follows that

$$\rho_\gamma^*(dw_1) = dw_1, \quad \rho_\gamma^*(dw_2) = dw_2 + \overline{\alpha_\gamma} dw_1, \tag{3.22}$$

$$\rho_{\gamma*}(\partial_1) = \partial_1 + \overline{\alpha_\gamma}\partial_2, \quad \rho_{\gamma*}(\partial_2) = \partial_2. \tag{3.23}$$

Then we can verify the following relations:

$$\begin{aligned} g_{1\bar{1}} \circ \rho_\gamma = g_{1\bar{1}} - \alpha_\gamma g_{1\bar{2}} - \overline{\alpha_\gamma} g_{2\bar{1}} + |\alpha_\gamma|^2 g_{2\bar{2}}, \quad g_{2\bar{2}} \circ \rho_\gamma = g_{2\bar{2}}, \\ g_{1\bar{2}} \circ \rho_\gamma = g_{1\bar{2}} - \overline{\alpha_\gamma} g_{2\bar{2}}, \quad g_{2\bar{1}} \circ \rho_\gamma = g_{2\bar{1}} - \alpha_\gamma g_{2\bar{2}}. \end{aligned} \tag{3.24}$$

By making use of these relations, we also have

$$\rho_\gamma^*\omega_1^1 = \omega_1^1 - \overline{\alpha_\gamma}\omega_2^1, \ \rho_\gamma^*\omega_2^1 = \omega_2^1, \ \rho_\gamma^*\omega_1^2 = \omega_1^2 + 2\overline{\alpha_\gamma}\omega_1^1 - \overline{\alpha_\gamma}^2\omega_2^1. \tag{3.25}$$

If we set

$$\eta_1 := \omega_1^1 + \overline{w_1}\omega_2^1, \quad \eta_2 := \omega_2^1, \quad \eta_3 := \omega_1^2 - 2\overline{w_1}\omega_1^1 - \overline{w_1}^2\omega_2^1,$$

then η_1, η_2, η_3 may be regarded as one-forms on $X = \mathbb{C}^2/G$.

In what follows, we suppose that g is flat. Then η_2 is a holomorphic one-form on X. Since $h^{1,0}(X) = 1$, we can write η_2 as

$$\eta_2 = A dw_1,$$

where A is a constant. In particular,

$$d\eta_2 = \partial\eta_2 = \overline{\partial}\eta_2 \equiv 0.$$

Lemma 3.18 $\eta_2 \equiv 0$.

Proof. From the flatness of g and (3.21), we have

$$0 \equiv d\eta_2 = d\omega_2^1 = -(\omega_1^1 \wedge \omega_2^1 + \omega_2^1 \wedge \omega_2^2) = -2\omega_1^1 \wedge \omega_2^1.$$

Thus we also have

$$\eta_1 \wedge \eta_2 = (\omega_1^1 + \overline{w_1}\omega_2^1) \wedge \omega_2^1 \equiv 0.$$

If $A \neq 0$, then $\eta_1 \wedge dw_1 \equiv 0$. Since η_1 is a $(1,0)$-form on X, we have a function F on X such that

$$\eta_1 = F dw_1, \quad \text{i.e.,} \quad \omega_1^1 = (F - A\overline{w_1}) dw_1.$$

By the flatness of g again, we then obtain

$$0 \equiv \overline{\partial}\omega_1^1 = (\overline{\partial}F - A d\overline{w_1}) \wedge dw_1.$$

Namely, we see that $\overline{\partial}F = A d\overline{w_1}$ and hence $\partial\overline{\partial}F \equiv 0$. From the mean value property for the operator $\partial\overline{\partial}$, we then conclude that F must be constant. Thus $A d\overline{w_1} = \overline{\partial}F \equiv 0$, that is, $A = 0$. This contradicts the assumption $A \neq 0$. $\square$

It follows from Lemma 3.18 and (3.25) that there exists a constant B such that

$$\eta_1 = B dw_1.$$

Then it is easy to see that

$$\overline{\partial}\eta_3 = 2B dw_1 \wedge d\overline{w_1}, \quad \partial\eta_3 = 2B dw_1 \wedge \eta_3. \tag{3.26}$$

We may assume that η_3 is expressed as

$$\eta_3 = f_1 dw_1 + f_2(dw_2 - \overline{w_1} dw_1)$$

for smooth functions f_1, f_2 on X. It then follows from (3.26) that

$$\overline{\partial}(f_1 - \overline{w_1} f_2) + 2B d\overline{w_1} \equiv 0, \quad \overline{\partial} f_2 \equiv 0. \tag{3.27}$$

In particular, f_2 is a holomorphic function on X, and must be a constant, say C. It follows from (3.27) that

$$\partial\overline{\partial} f_1 = \partial((-2B + C) dw_1) \equiv 0.$$

From the mean value property for $\partial\overline{\partial}$ again, we see that f_1 is also constant, say K. It is then easy to see from (3.26) that

$$2B dw_1 \wedge d\overline{w_1} = \overline{\partial}\eta_3 = \overline{\partial}(K dw_1 + C(dw_2 - \overline{w_1} dw_1)) = C dw_1 \wedge d\overline{w_1},$$
$$2BC dw_1 \wedge dw_2 = \partial\eta_3 = \partial(K dw_1 + C(dw_2 - \overline{w_1} dw_1)) \equiv 0,$$

and hence $B = C = 0$. Thus we obtain

$$\eta_1 = \eta_2 \equiv 0, \quad \eta_3 = K dw_1.$$

Using (3.19) and (3.20), we also have

$$\partial g_{2\bar{2}} = -\partial g_{2\bar{1}} \equiv 0, \quad \partial g_{1\bar{2}} = -K g_{2\bar{2}} dw_1, \quad \partial g_{1\bar{1}} = -K g_{2\bar{1}} dw_1.$$

In particular, $g_{2\bar{2}}$ is a constant, since $\partial g_{2\bar{2}} = \overline{\partial} g_{2\bar{2}} \equiv 0$.

By integrating $g_{2\bar{2}} = \partial^2\varphi/\partial w_2 \partial\overline{w_2}$ on each fiber T of $\Psi : X \longrightarrow \Delta$, we obtain

$$g_{2\bar{2}} \int_T dw_2 \wedge d\overline{w_2} = \int_T \frac{\partial^2\varphi}{\partial w_2 \partial\overline{w_2}} dw_2 \wedge d\overline{w_2} = 0,$$

and hence $g_{2\bar{2}} \equiv 0$. Thus φ depends only on the variable w_1, so that φ may be regarded as a function on Δ. In particular, $g_{1\bar{2}} = g_{2\bar{1}} \equiv -1$. On the other hand, we can regard $g_{1\bar{1}} - (w_1 + \overline{w_1})$ as a function on X, satisfying

$$\partial\overline{\partial}(g_{1\bar{1}} - (w_1 + \overline{w_1})) = -\overline{\partial}(\partial g_{1\bar{1}} - dw_1) = -\overline{\partial}(K-1) dw_1 \equiv 0.$$

Hence $g_{1\bar{1}} - (w_1 + \overline{w_1})$ must be constant, say L. Integrating $L = \partial^2\varphi/\partial w_1 \partial\overline{w_1}$ on Δ, we also have $L = 0$. Therefore φ is constant. Namely, g must coincide with g_0. □

3.4 Complex tori

In this section, we give a description of a neutral hyperkähler structure on a complex torus, by arguments similar to those in Section 3.3. We also obtain an analogous characterization of flat neutral hyperkähler structures on complex tori. In particular, we will see that complex tori of particular type (e.g., the product of elliptic curves) admit non-flat neutral hyperkähler structures (see Petean [82]).

Let $X = \mathbb{C}^2/\Gamma$ be a complex torus, where Γ is a lattice in $\mathbb{C}^2$. Then it is well-known that

$$b_1(X) = 4, \ b_2^+(X) = b_2^-(X) = 3, \ c_1(X) = c_2(X) = 0,$$
$$h^{1,0}(X) = q(X) = 2, \ p_{\mathrm{g}}(X) = 1.$$

By a similar but slightly different argument from that in the previous section, we can show the following result.

Theorem 3.19 *Let $X = \mathbb{C}^2/\Gamma$ be a complex torus and (w_1, w_2) the standard complex coordinate system of $\mathbb{C}^2$. Define a triplet $(\Omega_1, \Omega_2, \Omega_3)$ of three symplectic forms on X by*

$$\begin{aligned} \Omega_1 &= \mathrm{Im}(dw_1 \wedge d\overline{w_2}) + (\sqrt{-1}/2)\partial\overline{\partial}\varphi, \\ \Omega_2 &= \mathrm{Re}(dw_1 \wedge dw_2), \quad \Omega_3 = \mathrm{Im}(dw_1 \wedge dw_2). \end{aligned} \tag{3.28}$$

If φ is a solution of the equation

$$4\sqrt{-1}\mathrm{Im}(dw_1 \wedge d\overline{w_2}) \wedge \partial\overline{\partial}\varphi = \partial\overline{\partial}\varphi \wedge \partial\overline{\partial}\varphi, \tag{3.29}$$

then Ω_1, Ω_2 and Ω_3 give rise to a neutral hyperkähler structure on X. Conversely, under suitable complex coordinates (w_1, w_2) of $\mathbb{C}^2$, the fundamental form of any neutral hyperkähler structure on X is expressed as (3.28)*.*

Furthermore, a neutral hyperkähler metric g determined by the triplet $(\Omega_1, \Omega_2, \Omega_3)$ in (3.28) *is flat if and only if φ is constant.*

Proof. Let (z_1, z_2) denote the standard holomorphic coordinates of $\mathbb{C}^2$. Then dz_1 and dz_2 generate the cohomology group $H^0(X; \Omega_X^1) \cong H_{\overline{\partial}}^{1,0}(X)$, and $dz_1, dz_2, d\overline{z_1}, d\overline{z_2}$ generate $H^1(X; \mathbb{C})$. Note that $dz_1 \wedge dz_2$ is a nonvanishing holomorphic two-form on X. Define a triplet $(\Omega_1^-, \Omega_2^-, \Omega_3^-)$ of opposite symplectic forms on X by

$$\Omega_1^- := (\sqrt{-1}/2)(-dz_1 \wedge d\overline{z_1} + dz_2 \wedge d\overline{z_2}), \quad \Omega_2^- + \sqrt{-1}\Omega_3^- := \sqrt{-1}d\overline{z_2} \wedge dz_1,$$

and a positive-definite Kähler form Ω_1^+ by

$$\Omega_1^+ := (\sqrt{-1}/2)(dz_1 \wedge d\overline{z_1} + dz_2 \wedge d\overline{z_2}).$$

Let $(\Omega_1, \Omega_2, \Omega_3) = (\Omega_I, \Omega_{\prime J}, \Omega_{\prime K})$ be the fundamental form of an arbitrary neutral hyperkähler structure on X. By Proposition 3.15, a nonvanishing holomorphic two-form $\Omega_2 + \sqrt{-1}\Omega_3$ on X is given by

$$\Omega_2 + \sqrt{-1}\Omega_3 = c_0 dz_1 \wedge dz_2 = |c_0| e^{\sqrt{-1}\theta} dz_1 \wedge dz_2$$

for some nonzero constant $c_0 \in \mathbb{C}$. Taking account of the cohomology class of Ω_1, we may express Ω_1 as

$$\Omega_1 = |c_0|(a_0\Omega_1^+ + a_1\Omega_1^- + a_2\Omega_2^- + a_3\Omega_3^-) + (\sqrt{-1}/2)\partial\overline{\partial}\varphi$$

for some real constants a_0, a_1, a_2, a_3. Since $(|c_0|\Omega_1^-, \Omega_2, \Omega_3)$, $(|c_0|\Omega_2^-, \Omega_2, \Omega_3)$ and $(|c_0|\Omega_3^-, \Omega_2, \Omega_3)$ are neutral hyperkähler structures on X, we have

$$-a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1,$$

that is, (a_0, a_1, a_2, a_3) is a point on S^3_1. Identifying S^3_1 with $\mathfrak{u}(2)\bigcap \mathrm{SL}_2(\mathbb{C})$ equipped with the metric induced from the determinant det, we obtain $\mathrm{Isom}(S^3_1) \cong \mathrm{PSL}_2(\mathbb{C})$, and hence $S^3_1 = \mathrm{SL}_2(\mathbb{C})/\mathrm{SU}(1,1)$ as a homogeneous space. Indeed, this identification is given by

$$S^3_1 \ni (a_0, a_1, a_2, a_3) \mapsto \sqrt{-1}\begin{pmatrix} a_0 + a_1 & a_2 - \sqrt{-1}a_3 \\ a_2 + \sqrt{-1}a_3 & a_0 - a_1 \end{pmatrix} \in \mathfrak{u}(2)\bigcap \mathrm{SL}_2(\mathbb{C}),$$

and the action of $\mathrm{SL}_2(\mathbb{C})$ on S^3_1 is given by

$$A{\cdot}T := \begin{pmatrix} \bar{s} & -\bar{q} \\ -\bar{r} & \bar{p} \end{pmatrix} \sqrt{-1}\begin{pmatrix} a_0 + a_1 & a_2 - \sqrt{-1}a_3 \\ a_2 + \sqrt{-1}a_3 & a_0 - a_1 \end{pmatrix} \begin{pmatrix} s & -r \\ -q & p \end{pmatrix}$$

for

$$T = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad A = \sqrt{-1}\begin{pmatrix} a_0 + a_1 & a_2 - \sqrt{-1}a_3 \\ a_2 + \sqrt{-1}a_3 & a_0 - a_1 \end{pmatrix}.$$

This action $A{\cdot}T$ induces a natural linear action, say $\rho(T)$, on $\mathbb{R}^4$. Let T be an element in $\mathrm{SL}_2(\mathbb{C})$ such that

$$\rho(T)\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Then, introducing the new coordinates (w_1, w_2) by

$$|c_0|^{1/2} e^{\sqrt{-1}\theta/2} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = T \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

we obtain the following expression of Ω_1:

$$\Omega_1 = (-\sqrt{-1}/2)(dw_1 \wedge d\overline{w_2} + dw_2 \wedge d\overline{w_1}) + (\sqrt{-1}/2)\partial\overline{\partial}\varphi,$$

and also

$$\Omega_2 + \sqrt{-1}\Omega_3 = dw_1 \wedge dw_2,$$

since $T \in \mathrm{SL}_2(\mathbb{C})$ preserves the complex volume on $\mathbb{C}^2$. The equation (3.29) follows from the characterization result (Proposition 3.14) of a neutral hyperkähler structure.

We now examine the flatness of a neutral hyperkähler metric g on a complex torus X. Assume that $(\Omega_1, \Omega_2, \Omega_3)$ is expressed as (3.28) in terms of holomorphic coordinates (w_1, w_2) of $\mathbb{C}$:

$$\Omega_1 = (-\sqrt{-1}/2)(dw_1 \wedge d\overline{w_2} + dw_2 \wedge d\overline{w_1}) + (\sqrt{-1}/2)\partial\overline{\partial}\varphi,$$

$$\Omega_2 + \sqrt{-1}\Omega_3 = dw_1 \wedge dw_2$$

for a smooth function φ satisfying (3.29). Let $g_{\alpha\bar{\beta}} = 2g(\partial_\alpha, \overline{\partial_\beta})$ be the components of g with respect to (w_1, w_2) $(\alpha, \beta = 1, 2)$, and $\{\omega^A_B\}$ the connection form of the Levi-Civita connection ∇ with respect to $\{\partial_A\}$ $(A, B = 1, 2, \bar{1}, \bar{2})$. Recalling (2.17):

$$R^\alpha_\beta = \overline{\partial}\omega^\alpha_\beta, \ \ R^{\bar{\alpha}}_{\bar{\beta}} = \partial\omega^{\bar{\alpha}}_{\bar{\beta}},$$

we see that g is flat if and only if every ω^α_β is a global holomorphic one-form on X.

Now, suppose that g is flat. Then $\omega^1_1, \omega^1_2, \omega^2_1$ are holomorphic, and hence d-closed, one-forms on X. It follows from the flatness of g that

$$\omega^1_1 \wedge \omega^1_2 = \omega^1_2 \wedge \omega^2_1 = \omega^2_1 \wedge \omega^1_1 \equiv 0.$$

Then there exists a nonzero holomorphic one-form ϕ such that

$$\omega^\alpha_\beta = A^\alpha_\beta \phi, \quad A^1_1 + A^2_2 = 0$$

for suitable constants A^α_β $(\alpha, \beta = 1, 2)$. By (2.16), we then obtain

$$\partial g_{\alpha\bar{\beta}} = A_{\alpha\bar{\beta}}\,\phi, \ \text{ or equivalently, } \overline{\partial} g_{\alpha\bar{\beta}} = \overline{A_{\beta\bar{\alpha}}}\,\overline{\phi}.$$

where $A_{\alpha\bar{\beta}} := \sum_{\gamma=1}^2 A^\gamma_\alpha g_{\gamma\bar{\beta}}$. In the Dolbeault cohomology group $H^{0,1}_{\overline{\partial}}(X)$, we see that $0 = [\overline{\partial} g_{\alpha\bar{\beta}}] = \overline{A_{\beta\bar{\alpha}}}\,[\overline{\phi}]$ $(\alpha, \beta = 1, 2)$, which imply that all the coefficients $A_{\alpha\bar{\beta}}$, and hence A^α_β, vanish. Thus all components $g_{\alpha\bar{\beta}}$ are constants. Then we can write $\partial\overline{\partial}\varphi$ as

$$\partial\overline{\partial}\varphi = \sum_{\alpha,\beta} C_{\alpha\bar{\beta}} dw_\alpha \wedge d\overline{w_\beta}$$

for constants $C_{\alpha\bar{\beta}}$. In the second cohomology group $H^2(X; \mathbb{C})$, the left hand side is clearly zero, so that all $C_{\alpha\bar{\beta}}$ vanish. Thus φ should also be constant. With respect to (w_1, w_2), we have the required expression of the flat metric g. $\square$

From Theorem 3.19, we see that there exist non-flat neutral hyperkähler structures on $X = E_1 \times E_2$, the product of elliptic curves E_1 and E_2. Indeed, let w_1 and w_2 be holomorphic coordinates of E_1 and E_2, respectively, and let φ be the pull-back of any nonconstant smooth function on each factor of $X = E_1 \times E_2$, that is, $\varphi = \varphi(w_1)$ or $\varphi = \varphi(w_2)$. Then, since φ is a nonconstant solution of (3.29), the triplet $(\Omega_1, \Omega_2, \Omega_3)$ defined by (3.28) yields a non-flat neutral hyperkähler structure on $X = E_1 \times E_2$ (cf. Petean [82]).

4 Examples

We give two different types of examples of self-dual neutral Hermitian surfaces, that is, the indefinite complex projective space $\mathbb{CP}^2_1$ with the Fubini-Study type metric and complex line bundles over the real hyperbolic plane with LeBrun type neutral metrics.

The Fubini-Study type metric on $\mathbb{CP}^2_1$ is known as a homogeneous pseudo-Riemannian metric with constant holomorphic sectional curvature $+1$. Thus its curvature operator restricted to $\Lambda^2_+ \cong \Lambda^{\mathrm{inv}}_0$ is a constant multiple of the identity map. Therefore the self-dual part W_+ of the Weyl conformal curvature tensor vanishes everywhere, that is, the metric is anti-self-dual. This metric has also been studied in [47] as a metric of Bianchi type VIII. However, we examine here another description of the Fubini-Study type metric from a point of view of the de Sitter ansatz. In [47], LeBrun type neutral metrics were already treated as those of Bianchi type VIII. We hope that these examples will be helpful for finding other construction of self-dual neutral metrics. For further examples, see also [13], [17], [42] and references therein.

Fubini-Study type metric The indefinite complex projective space $\mathbb{CP}^2_1$ is defined as a homogeneous space $\mathrm{U}(1,2)/\mathrm{U}(1,1)\times\mathrm{U}(1)$, where $\mathrm{U}(p,q)$ denotes the indefinite unitary group. We can also describe $\mathbb{CP}^2_1$ as

$$\mathbb{CP}^2_1 = \{(z_0 : z_1 : z_2) \in \mathbb{CP}^2 \mid -|z_0|^2 + |z_1|^2 + |z_2|^2 = +1\},$$

since $\mathrm{U}(1,2)$ acts transitively on $\mathbb{CP}^2_1$ in a natural way and $\mathrm{U}(1,1)\times\mathrm{U}(1)$ is the isotropy subgroup of this action at $(0 : 0 : 1)$. Therefore $\mathbb{CP}^2_1$ is diffeomorphic to $\mathbb{CP}^2\backslash\{|z_1/z_0|^2 + |z_2/z_0|^2 < 1\}$. Let $\varpi : S^5_2 \to \mathbb{CP}^2_1$ be the natural projection $(z_0, z_1, z_2) \mapsto (z_0 : z_1 : z_2)$, which is an indefinite analogue of the Hopf fibration. We can define a pseudo-Riemannian metric g on $\mathbb{CP}^2_1$ such that $\varpi : (S^5_2, g_{S^5_2}) \to (\mathbb{CP}^2_1, g)$ is a pseudo-Riemannian submersion. Then a metric $g_{\mathrm{FS}} := g$ on $\mathbb{CP}^2_1$ is called the Fubini-Study type neutral metric. It is known that g_{FS} is an anti-self-dual, Einstein neutral Kähler metric with respect to the natural complex orientation (cf. [47]).

In terms of the homogeneous coordinates $(z_0 : z_1 : z_2)$, we can express g_{FS} as

$$\varpi^* g_{\mathrm{FS}} = -|dz_0|^2 + |dz_1|^2 + |dz_2|^2 - |-z_0 d\overline{z_0} + z_1 d\overline{z_1} + z_2 d\overline{z_2}|^2,$$

where $-|z_0|^2 + |z_1|^2 + |z_2|^2 = 1$. Setting

$$\zeta_0 := z_0, \quad (\zeta_1, \zeta_2) := (1 + |z_0|^2)^{-1/2}(z_1, z_2)$$

for $(z_0, z_1, z_2) \in S^5_3$ and noting the following diagram:

$$\begin{array}{ccccccc}
S^5_3 & \xrightarrow{\cong} & \mathbb{C}\times S^3 & (z_0,z_1,z_2) & \mapsto & (\zeta_0,(\zeta_1,\zeta_2)) \\
{\scriptstyle e^{\sqrt{-1}t}}\big\downarrow & & \big\downarrow{\scriptstyle e^{\sqrt{-1}t}} & {\scriptstyle e^{\sqrt{-1}t}}\big\downarrow & & \big\downarrow{\scriptstyle e^{\sqrt{-1}t}} \\
S^5_3 & \xrightarrow[\cong]{} & \mathbb{C}\times S^3 & e^{\sqrt{-1}t}(z_0,z_1,z_2) & \mapsto & (e^{\sqrt{-1}t}\zeta_0, e^{\sqrt{-1}t}(\zeta_1,\zeta_2)),
\end{array}$$

we can identify $\mathbb{CP}^2_1$ with the total space of the tautological line bundle $L \to \mathbb{CP}^1$, as smooth manifolds. Let $\sigma_1, \sigma_2, \sigma_3$ be the left-invariant one-forms on $\mathrm{SU}(2) = S^3$ satisfying

$$d\sigma_1 = 2\sigma_2 \wedge \sigma_3,\ d\sigma_2 = 2\sigma_3 \wedge \sigma_1,\ d\sigma_3 = 2\sigma_1 \wedge \sigma_2. \tag{4.1}$$

Set $z_0 := re^{\sqrt{-1}\sigma}$ and $\tilde\sigma_1 := d\sigma + \sigma_1, \tilde\sigma_2 := \sigma_2, \tilde\sigma_3 := \sigma_3$. Then $\tilde\sigma_1, \tilde\sigma_2, \tilde\sigma_3$ satisfy the same condition as (4.1) and g_{FS} is expressed in terms of $r, \tilde\sigma_1, \tilde\sigma_2, \tilde\sigma_3$ as

$$g_{\mathrm{FS}} = -\frac{dr^2}{1+r^2} - r^2(1+r^2)\tilde\sigma_1^2 + (1+r^2)(\tilde\sigma_2^2 + \tilde\sigma_3^2).$$

Let $\bar I$ be an almost complex structure defined by $\bar I dr = r(1+r^2)\tilde\sigma_1, \bar I\tilde\sigma_2 = -\tilde\sigma_3$. Then $\bar I$ is integrable, and moreover $(g_{\mathrm{FS}}, \bar I)$ defines a neutral Kähler structure on $\mathbb{CP}^2_1$.

Setting $r = e^\rho$ and noting $\sigma_2^2 + \sigma_3^2 = h_{S^2}/4$, we obtain the following expression of g_{FS}:

$$\begin{aligned} g_{\mathrm{FS}} &= e^{2\rho}\Big(-(V d\rho^2 + V^{-1}\theta^2) + V\cosh^2\rho\, h_{S^2}\Big) \\ &= e^{2\rho}(-V^{-1}\theta^2 + V\, g_{S^3_1}), \end{aligned} \tag{4.2}$$

where h_{S^2} denotes the unit round metric on S^2, $V := (1+e^{2\rho})^{-1}$, and $\theta := \tilde\sigma_1$ being the connection form of the Hopf fibration $S^3 \to S^2$. It should be remarked that (V, θ) satisfies (2.37): $\ast dV = d\theta$. Let I be an almost complex structure defined by $Id\rho = -V^{-1}\theta, Id\zeta = \sqrt{-1}d\zeta$, where ζ denotes a holomorphic coordinate of $S^2 = \mathbb{CP}^1$. From Proposition 2.17, it follows that g_{FS} is a self-dual neutral metric with respect to the orientation determined by I. (Note that g_{FS} is an anti-self-dual neutral metric with respect to the orientation defined by $\bar I$.) In Section 2.4, this function V was denoted by G_0, and used for constructing self-dual neutral metrics on $S^2 \times S^2$. By a similar argument in Section 2.4, we see that I is integrable and (g_{FS}, I) is locally conformal neutral Kähler. However, g_{FS} itself is not neutral Kähler with respect to I.

Note that the indefinite complex hyperbolic space $\mathbb{CH}^2_1$ is identified with $\mathbb{CH}^2_1 = (\mathbb{CP}^2_1, -g_{\mathrm{FS}})$. At least locally, we can also express g_{FS} as a neutral metric of Bianchi type VIII (see [47]).

LeBrun type neutral metrics LeBrun type neutral metrics, which we introduce here, are indefinite counterparts of positive-definite anti-self-dual Kähler metrics on the total spaces L of complex vector bundles $L \to \mathbb{CP}^1$ constructed in LeBrun [59]. For details, see [47].

Let τ_1, τ_2, τ_3 be left-invariant one-forms on the special linear group $\mathrm{SL}_2(\mathbb{R})$ such that

$$d\tau_1 = -2\tau_2 \wedge \tau_3, \quad d\tau_2 = 2\tau_3 \wedge \tau_1, \quad d\tau_3 = 2\tau_1 \wedge \tau_2. \tag{4.3}$$

LeBrun type neutral metrics are defined to be

$$\begin{aligned} g_{\mathrm{LB}} \;=\; & -\frac{dr^2}{(1-(a/r)^2)(1+k(a/r)^2)} \\ & -r^2(1-(a/r)^2)(1+k(a/r)^2)\tau_1^2 + r^2(\tau_2^2+\tau_3^2) \end{aligned} \tag{4.4}$$

for $r \in (a, +\infty)$, $a > 0$ and $k \in \mathbb{Z}_{\geq 0}$. Then each g_{LB} is a self-dual neutral Kähler metric on $(a,\infty)\times \mathrm{SL}_2(\mathbb{R})$. Taking its quotient by $\mathbb{Z}_{k+1}$, we can regard g_{LB} as a neutral metric on the total space of the complex line bundle $L_{H^2}^{\otimes(k+1)} \to H^2$, where $L_{H^2} \to H^2$ denotes a complex line bundle induced from the indefinite Hopf bundle $H_1^3 = \mathrm{SL}_2(\mathbb{R}) \to H^2$. Since g_{LB} has an $\mathrm{SL}_2(\mathbb{R})$-symmetry, it is also regarded as a metric on the quotient of $L_{H^2}^{\otimes(k+1)} \to H^2$ by a Fuchsian group Γ. When $k = 1$, we can show that g_{LB} is a Ricci-flat neutral Kähler (thus self-dual) metric on T^*H^2 or $T^*(H^2/\Gamma)$, which is an indefinite analogue of the Eguchi-Hanson metric on the cotangent bundle $T^*\mathbb{CP}^1$ (cf. Eguchi-Hanson [23]). In [47], this metric is called the Eguchi-Hanson type neutral metric and is also denoted by g_{EH}. For this metric g_{EH}, see also Ooguri-Vafa [79]. When $k = 0$, we can show that g_{LB} is conformal to the Fubini-Study type neutral metric $-g_{\mathrm{FS}}$ on $\mathbb{CH}_1^2$ (see [47]).

Note that Riemannian analogues of these self-dual neutral metrics of Bianchi type VIII are obtained as Riemannian metrics of Bianchi type IX. Between neutral metrics of Bianchi type VIII and Riemannian metrics of Bianchi type IX, we obtain the following correspondence in general (see [47]):

Theorem 4.1 *Let g (resp. h) be a neutral (resp. Riemannian) metric on $\mathbb{R}_+ \times \mathrm{SL}_2(\mathbb{R})$ (resp. $\mathbb{R}_+ \times \mathrm{SU}(2)$) defined by*

$$g = -f(r)^2 dr^2 - a(r)^2\tau_1^2 + b(r)^2\tau_2^2 + c(r)^2\tau_3^2$$
$$(\textit{resp. } h := f(r)dr^2 + a(r)^2\sigma_1^2 + b(r)^2\sigma_2^2 + c(r)^2\sigma_3^2)$$

for the same data $f(r), a(r), b(r), c(r)$. Define an almost complex structure I

(resp. J) by

$$If(r)dr = -a(r)\tau_1, \quad Ib(r)\tau_2 = -c(r)\tau_3$$
$$(resp.\ Jf(r)dr = -a(r)\sigma_1, \quad Jb(r)\sigma_2 = -c(r)\sigma_3).$$

Then the following correspondences hold:

(1) *g is self-dual if and only if h is anti-self-dual.*

(2) *g is Einstein if and only if so is h.*

(3) *(g, I) is neutral Kähler if and only if (h, J) is Kähler.*

5 Appendices

5.1 The Jones-Tod correspondence

We here give a proof of Proposition 2.14, by using O'Neill's formula for pseudo-Riemannian submersions induced by time-like S^1-symmetries, and also prove Proposition 2.25.

We first recall the assumption of Proposition 2.14: Let (M, g) be an oriented pseudo-Riemannian manifold with neutral metric g admitting a time-like isometric S^1-action. Suppose that the S^1-action is fixed-point free. Then the orbit space $N := M/S^1$ is a smooth pseudo-Riemannian manifold with metric $\check{g}$ defined by

$$\pi^*\check{g} = g - \frac{\xi^\flat \otimes \xi^\flat}{g(\xi, \xi)}. \tag{5.1}$$

where $\pi : M \to N$ is the natural projection, ξ is the Killing vector field on (M, g) generating the S^1-action, and $\xi^\flat := g(\xi, \cdot)$ denotes the metric-dual of ξ. Hence $\pi : (M, g) \to (N, \check{g})$ is a pseudo-Riemannian submersion. (Note that the orbit space M/S^1 was denoted by Y in Chapter 2. However, to avoid confusion, it is denoted by N in this appendix.)

Let g' be another neutral metric on M defined by $g' := |g(\xi, \xi)|^{-1} g$, and let $\check{g}'$ denote the corresponding Lorentzian metric on N defined as (5.1) by replacing g with g'. Then $\pi : (M, g') \to (N, \check{g}')$ is a pseudo-Riemannian submersion with $g'(\xi, \xi) \equiv -1$, and ξ becomes a Killing vector field with respect to g'. Furthermore, it is easy to see that all the fibers of $\pi : M \to N$ are totally geodesic with respect to g'. Let θ be a one-form on M defined by

$$\theta := -g'(\xi, \cdot).$$

Note that θ satisfies the conditions:

$$\iota_\xi \theta \equiv 1, \quad \mathcal{L}_\xi \theta \equiv 0,$$

where ι_ξ and $\mathcal{L}_\xi$ denote the inner derivation and the Lie derivative with respect to ξ, respectively. These conditions imply that $d\theta$ is a basic two-form on $\pi : M \to N$, that is, $d\theta$ satisfies

$$\iota_\xi d\theta \equiv 0, \quad \mathcal{L}_\xi d\theta \equiv 0.$$

Hence there exists a closed two-form Ω on N such that $d\theta = \pi^*\Omega$. Recall that the O'Neill tensor field A is defined by

$$A_E F := (\nabla'_{E^h} F^v)^h + (\nabla'_{E^h} F^h)^v,$$

where ∇' is the Levi-Civita connection of g', E and F are vector fields on M, and E^v (resp. E^h) denotes the vertical (resp. horizontal) component of E. Note that in our case the O'Neill tensor field A satisfies

$$g'(A_X Y, \xi) = \frac{1}{2} d\theta(X, Y), \text{ or equivalently, } A_X Y = -\frac{1}{2} d\theta(X, Y)\xi$$

and

$$g'((\nabla'_\xi A)_X Y, \xi) \equiv 0$$

for horizontal vector fields X, Y. Regarding the curvature tensors of (M, g') and $(N, \check{g}')$, we have the following O'Neill's formula (cf. Besse [7]).

Proposition 5.1 *Let $\pi : (M, g') \to (N, \check{g}')$ be a pseudo-Riemannian submersion with totally geodesic fibers, where M is an $(n+1)$-dimensional manifold and N is an n-dimensional manifold, and ξ a Killing vector field on (M, g') tangent to the fibers with $g'(\xi, \xi) \equiv -1$. Let R' and $\check{R}'$ denote the curvature tensors of g' and $\check{g}'$, respectively. Then, for arbitrary vector fields X, Y, Z, Z' on M orthogonal to the fibers, the following hold*:

$$g'(R'(\xi, X)Y, \xi) = \frac{1}{4} \mathrm{tr}_{g'}(d\theta \otimes d\theta)(X, Y), \tag{5.2}$$

$$g'(R'(X, Y)Z, \xi) = -\frac{1}{2} (\nabla'_Z d\theta)(X, Y), \tag{5.3}$$

$$\begin{aligned} g'(R(X, Y)Z, Z') &= g'(\check{R}'(X, Y)Z, Z') - \frac{1}{2} d\theta(X, Y) d\theta(Z, Z') \\ &\quad -\frac{1}{4} d\theta(X, Z) d\theta(Y, Z') + \frac{1}{4} d\theta(Y, Z) d\theta(X, Z'), \end{aligned} \tag{5.4}$$

where $\mathrm{tr}_{g'}(d\theta \otimes d\theta)(X, Y) = g'(\iota_X d\theta, \iota_Y d\theta)$, and $\check{R}'(X, Y)Z$ denotes the horizontal lift of $\check{R}'(\pi_* X, \pi_* Y)\pi_* Z$.

It should be remarked here that our definition of the curvature tensor R is different in sign from that in Besse [7].

By taking contraction, we obtain the following formula for the Ricci curvatures.

Proposition 5.2 *Let $\pi : (M, g') \to (N, \check{g}')$ and ξ be as in Proposition 5.1, and $r' := \mathrm{Ric}'$ and $\check{r}' := \check{\mathrm{Ric}}'$ be the Ricci curvature tensors of g' and $\check{g}'$,* *respectively. Then they satisfy the following*:

$$r'(\xi,\xi) = \frac{1}{2}g'(d\theta,d\theta), \tag{5.5}$$

$$r'(\xi,Y) = -\frac{1}{2}\mathrm{tr}_{g'}[(\nabla' d\theta)(Y,\cdot)], \tag{5.6}$$

$$r'(Y,Z) = \check{r}'(Y,Z) + \frac{1}{2}\mathrm{tr}_{g'}(d\theta \otimes d\theta)(Y,Z) \tag{5.7}$$

for arbitrary vectors Y and Z orthogonal to the fiber at a point in M, where $\check{r}'(Y,Z) = \check{r}'(\pi_* Y, \pi_* Z)\circ\pi$.

By taking contraction again, we have the relation between the scalar curvatures s and $\check{s}$.

Corollary 5.3

$$s = \check{s}\circ\pi + \frac{1}{2}g'(d\theta,d\theta). \tag{5.8}$$

The traceless Ricci tensors then satisfy the following

Proposition 5.4 *Let r'_0 and $\check{r}'_0$ denote the traceless Ricci tensors of g' and $\check{g}'$, respectively. Then the following hold*:

$$r'_0(\xi,\xi) = \frac{1}{2(n+1)}\Big(2\check{s} + (n+2)g'(d\theta,d\theta)\Big), \tag{5.9}$$

$$r'_0(\xi,Y) = -\frac{1}{2}\mathrm{tr}_{g'}[(\nabla' d\theta)(Y,\cdot)], \tag{5.10}$$

$$\begin{aligned} r'_0(Y,Z) &= \check{r}'_0(Y,Z) + \frac{\check{s}}{n(n+1)}g'(Y,Z) \\ &+\frac{1}{2}\Big(\mathrm{tr}_{g'}(d\theta\otimes d\theta)(Y,Z) - \frac{1}{n+1}g'(d\theta,d\theta)g'(Y,Z)\Big). \end{aligned} \tag{5.11}$$

Also, similar formulas for the Weyl conformal curvature tensors can be derived as follows:

$$\begin{aligned} g'(W'(\xi,Y)Z,\xi) &= \frac{1}{n-1}\check{r}'_0(Y,Z) \\ &+\frac{n+1}{4(n-1)}\Big(\mathrm{tr}_{g'}(d\theta\otimes d\theta)(Y,Z) - \frac{2}{n}g'(d\theta,d\theta)g'(Y,Z)\Big), \end{aligned} \tag{5.12}$$

$$\begin{aligned} g'(W'(X,Y)Z,\xi) &= -\frac{1}{2}(\nabla'_Z d\theta)(X,Y) \\ &+\frac{1}{2(n-1)}\Big(\mathrm{tr}_{g'}[(\nabla' d\theta)(X,\cdot)]g'(Y,Z) \\ &-\mathrm{tr}_{g'}[(\nabla' d\theta)(Y,\cdot)]g'(X,Z)\Big), \end{aligned} \tag{5.13}$$

$$
\begin{aligned}
(5.14) \qquad g'(W'(X,Y)Z,Z') = {} & g'(\check{W}'(X,Y)Z,Z') \\
& + \frac{1}{(n-1)(n-2)} g' \owedge \check{r}'(X,Y,Z,Z') \\
& - \frac{\check{s}}{n(n-1)(n-2)} g' \owedge g'(X,Y,Z,Z') \\
& - \frac{1}{2} d\theta(X,Y) d\theta(Z,Z') \\
& + \frac{1}{4} \Big(d\theta(X,Z') d\theta(Y,Z) - d\theta(X,Z) d\theta(Y,Z') \Big) \\
& + \frac{1}{2(n-1)} g' \owedge \mathrm{tr}_{g'}(d\theta \otimes d\theta)(X,Y,Z,Z') \\
& + \frac{1}{4n(n-1)} g'(d\theta, d\theta) g' \owedge g'(X,Y,Z,Z'),
\end{aligned}
$$

where $\owedge$ denotes the Kulkarni-Nomizu product, which is defined by

$$
\begin{aligned}
(h \owedge k)(X,Y,Z,Z') \ := \ & h(X,Z')k(Y,Z) - h(Y,Z')k(X,Z) \\
& + h(Y,Z)k(X,Z') - h(X,Z)k(Y,Z')
\end{aligned}
$$

for symmetric $(2,0)$-tensor fields h and k.

We now return to the situation in Proposition 2.14. Let $\pi : (M, g') \to (N, \check{g}')$ be a pseudo-Riemannian submersion with totally geodesic fibers, where (M, g') is an oriented pseudo-Riemannian four-manifold with neutral metric g' and $(N, \check{g}')$ is a Lorentzian three-manifold, and ξ the unit time-like vector field tangent to the fibers. Let $\{e^1, e^2, e^3, e^4 := \theta\}$ be a local oriented orthonormal coframe field on (M, g') such that

$$
g' = (e^1)^2 + (e^2)^2 - (e^3)^2 - (e^4)^2,
$$

where $\{e^1, e^2, e^3\}$ is the pull-back of a local oriented orthonormal coframe field $\{\check{e}^1, \check{e}^2, \check{e}^3\}$ on $(N, \check{g}')$ satisfying

$$
\check{g}' = (\check{e}^1)^2 + (\check{e}^2)^2 - (\check{e}^3)^2.
$$

For simplicity, we write $\check{e}^a$ as e^a $(a = 1, 2, 3)$. Since $d\theta$ is a basic two-form, there exists a two-form α on N such that $d\theta = \pi^* \alpha$. We also write α as $d\theta$ for brevity. By the relations

$$
(\check{*}' d\theta)(e_1) = -d\theta(e_2, e_3), \quad (\check{*}' d\theta)(e_2) = d\theta(e_1, e_3), \quad (\check{*}' d\theta)(e_3) = d\theta(e_1, e_2),
$$

it is verified that

$$
\begin{aligned}
(5.15) \qquad \mathrm{tr}_{g'}[d\theta \otimes d\theta] &= (\check{*}' d\theta \otimes \check{*}' d\theta) - \check{g}'(\check{*}' d\theta, \check{*}' d\theta)\check{g}, \\
(5.16) \qquad \check{*}'(\nabla'_X d\theta) &= \nabla'_X \check{*}' d\theta
\end{aligned}
$$

for any vector field X on N, where $\check{*}'$ denotes the Hodge star operator on $(N, \check{g}')$.

For any (4,0)-tensor field T, we denote by T_{ABCD} the components of T with respect to a local frame field $\{e_A\}$. For example, W'_{ABCD} is defined to be

$$W'_{ABCD} := g'(W'(e_C, e_D)e_B, e_A)$$

$(1 \leq A, B, C, D \leq 4)$. Then the components of the Weyl conformal tensor W' of (M, g') are given by

$$\begin{aligned}
W'_{4b4d} &= \frac{1}{2}(r' + \check{*}'d\theta \otimes \check{*}'d\theta)_0(e_b, e_d), \\
W'_{4bcd} &= \frac{1}{2}(\nabla' \check{*}'d\theta)^{\mathrm{sym}}(e_b, e_b), \\
&\quad (b, c, d) = (1, 2, 3), (2, 3, 1), (3, 2, 1), \\
W'_{4121} &= W'_{4323} = \frac{1}{2}(\nabla' \check{*}'d\theta)^{\mathrm{sym}}(e_1, e_3), \\
W'_{4131} &= -W'_{4232} = \frac{1}{2}(\nabla' \check{*}'d\theta)^{\mathrm{sym}}(e_1, e_2), \\
W'_{4212} &= W'_{4313} = -\frac{1}{2}(\nabla' \check{*}'d\theta)^{\mathrm{sym}}(e_2, e_3), \\
W'_{abab} &= \frac{1}{2}(\check{r}' + \check{*}'d\theta \otimes \check{*}'d\theta)_0(e_c, e_c), \\
&\quad (a, b, c) = (1, 2, 3), (2, 3, 1), (3, 1, 2), \\
W'_{1213} &= \frac{1}{2}(\check{r}' + \check{*}'d\theta \otimes \check{*}'d\theta)_0(e_2, e_3), \\
W'_{2123} &= \frac{1}{2}(\check{r}' + \check{*}'d\theta \otimes \check{*}'d\theta)_0(e_1, e_3), \\
W'_{3132} &= \frac{1}{2}(\check{r}' + \check{*}'d\theta \otimes \check{*}'d\theta)_0(e_1, e_2),
\end{aligned}$$

where $(\check{r}' + \check{*}'d\theta \otimes \check{*}'d\theta)_0$ denotes the traceless part of $\check{r}' + \check{*}'d\theta \otimes \check{*}'d\theta$.

A direct computation shows that g' (and thus g) is self-dual if and only if

$$\begin{aligned}
W'_{1212} - W'_{1234} = W'_{1312} - W'_{1334} = W'_{1412} - W'_{1434} = 0, \\
W'_{1213} - W'_{1224} = W'_{1313} - W'_{1324} = W'_{1413} - W'_{1424} = 0, \\
W'_{1214} - W'_{1232} = W'_{1314} - W'_{1332} = W'_{1414} - W'_{1432} = 0,
\end{aligned}$$

which are also equivalent to the following condition:

$$(\check{r}' + \check{*}'d\theta \otimes \check{*}'d\theta)_0 - (\nabla' \check{*}'d\theta)^{\mathrm{sym}} \equiv 0. \tag{5.17}$$

Next, we recall the definition of Einstein-Weyl structures (see Higa [33], Pedersen-Swann [81], cf. [44]). Let N be a smooth three-dimensional manifold equipped with a conformal structure $\check{C}$ of Lorentzian metrics. An affine connection D on N is called a *Weyl connection* on $(N, \check{C})$ if D is torsion-free and preserves the conformal structure $\check{C}$. Then, for a metric representative $\check{g}$ of $\check{C}$, there exists a one-form $\check{\beta}$ such that

$$D\check{g} = -2\check{\beta}\otimes\check{g}.$$

Conversely, for a metric $\check{g}$ of $\check{C}$ and a one-form $\check{\beta}$, there exists a unique Weyl connection D such that $D\check{g} = -2\check{\beta}\otimes\check{g}$. Now, take another metric $\check{g}' = u^2\check{g}$ of $\check{C}$ and a one-form $\check{\beta}'$. Then $(\check{g}, -2\check{\beta})$ and $(\check{g}', -2\check{\beta}')$ define the same Weyl connection D if and only if they satisfy the following gauge relation:

$$\check{\beta}' = -d\log u + \check{\beta}.$$

Let R^D, r^D and $s^D_{\check{g}}$ denote the curvature tensor, the Ricci tensor, and the scalar curvature with respect to $\check{g} \in \check{C}$, respectively:

$$\begin{aligned} R^D(X,Y)Z &:= D_X(D_YZ) - D_Y(D_XZ) - D_{[X,Y]}Z, \\ r^D(Y,Z) &:= \mathrm{tr}(X \mapsto R^D(X,Z)Y), \quad s^D_{\check{g}} := \mathrm{tr}_{\check{g}}(r^D). \end{aligned}$$

A Weyl structure (C, D) on N is said to be Einstein-Weyl if the symmetrized Ricci tensor $r^{D(\mathrm{sym})}$ of D is proportional to a (hence any) metric representative $\check{g}$ of $\check{C}$, that is, $r_0^{D(\mathrm{sym})} \equiv 0$, where the subscription 0 means the traceless part. The relationship between the Ricci curvatures r^D and $\check{r}$ of the Weyl connection D and that of the Levi-Civita connection ∇ of $\check{g}$ is given by

$$r_0^{D(\mathrm{sym})} = (\check{r} + \check{\beta}\otimes\check{\beta})_0 - (\nabla\check{\beta})_0^{\mathrm{sym}}, \quad r^{D(\mathrm{skew})} = \frac{3}{2}d\check{\beta}. \tag{5.18}$$

Therefore, a Weyl structure (C, D) on N is Einstein-Weyl if and only if

$$(\check{r} + \check{\beta}\otimes\check{\beta})_0 - (\nabla\check{\beta})_0^{\mathrm{sym}} \equiv 0. \tag{5.19}$$

In our situation, the one-form $\check{\beta}$, induced from g and ξ, is defined by (2.32) in Section 2.3:

$$\pi^*\check{\beta} = \frac{-dg(\xi,\xi) - 2 *_g (\xi^\flat \wedge d\xi^\flat)}{2g(\xi,\xi)}.$$

For the metric $g' = |g(\xi,\xi)|^{-1}g$, we have $\xi'^\flat = -\theta$ and the corresponding one-form $\check{\beta}'$ is given by

$$\pi^*\check{\beta}' = *_{g'}(\theta \wedge d\theta) = \pi^*(\check{*}' d\theta). \tag{5.20}$$

Here $d\theta$ in the last term is identified with the corresponding two-form on N. If the relation (5.20) holds (i.e., $\check{\beta}' = \check{*}' d\theta$), then we have

$$\mathrm{tr}_{\check{g}'}(\nabla' \check{\beta}') = \pm \check{\delta}' \check{\beta}' = \pm \check{*}' d(d\theta) \equiv 0,$$

that is, $(\nabla' \check{\beta}')_0^{\mathrm{sym}} = (\nabla' \check{\beta}')^{\mathrm{sym}}$. Therefore, (5.19) and (5.17) are equivalent, and hence Proposition 2.14 is proved. $\square$

We next prove Proposition 2.25. Under the situation in Proposition 2.25, we may assume that $g' = -\theta^2 + V^2 g_{S_1^3}$ and $\check{g}' = V^2 g_{S_1^3}$. Then the Levi-Civita connection D of $g_{S_1^3}$ satisfies that $D\check{g}' = 2d\log V \otimes \check{g}'$. Since the de Sitter space S_1^3 is Einstein, we have $r_0^{D(\mathrm{sym})} \equiv 0$. Hence g is self-dual, that is, $W_- \equiv 0$. By (5.18), this is equivalent to $(\check{r}' + \check{\beta})_0 = (\nabla' \check{\beta})_0^{\mathrm{sym}}$, where $\check{\beta} = -d\log V$. (Note that the Levi-Civita connection ∇' of $\check{g}' = V^2 g_{S_1^3}$ was denoted by D' in Section 2.4.) Taking account of (5.17), we see that the self-dual part W_+ is determined by $(\nabla' \check{*}' d\theta)^{\mathrm{sym}}$. Since (V, θ) satisfies (2.37): $\check{*} dV = d\theta$, we have

$$\check{*}' d\theta = V^{-1} \check{*} d\theta = -V^{-1} dV = -d\log V,$$

where $\check{*}$ denotes the Hodge star operator of S_1^3. By the relation between ∇' and D, we obtain

$$\begin{aligned}
\nabla' \check{*}' d\theta &= -\nabla' d\log V \\
&= -(Dd\log V - 2d\log V \otimes d\log V + \|d\log V\|^2 g_{S_1^3}) \\
&= -V^{-2}(VDdV - 3dV \otimes dV + \|dV\|^2 g_{S_1^3}),
\end{aligned}$$

where $\|\cdot\|^2$ denotes the indefinite squared norm with respect to $g_{S_1^3}$. This completes the proof of Proposition 2.25. $\square$

5.2 Hirzebruch signature and Euler characteristic

We recall Hirzebruch signature and Gauss-Bonnet formulas for a compact four-manifold with a neutral metric in terms of its curvature tensor (see, e.g., Avez [5] and Chern [18] for Gauss-Bonnet formula, Matsushita [72] for signature formula, and Law [57] and Matsushita-Law [73] for both formulas).

Let (M, g) be a compact oriented pseudo-Riemannian four-manifold with neutral metric g, and let $\{e_1, e_2, e_3, e_4\}$ be a local oriented orthonormal frame field on (M, g) and $\{e^1, e^2, e^3, e^4\}$ its dual coframe field such that

$$g = (e^1)^2 + (e^2)^2 - (e^3)^2 - (e^4)^2. \tag{5.21}$$

Let ∇ be the Levi-Civita connection of (M,g) and $\Omega^j{}_k$ the components of the curvature form of ∇ with respect to $\{e_1,e_2,e_3,e_4\}$. Then the first Pontrjagin form $p_1(M,\nabla)$ can be expressed as

$$\begin{aligned}
p_1(M,\nabla) &= -\frac{1}{8\pi^2}\mathrm{tr}(\Omega\wedge\Omega) = -\frac{1}{8\pi^2}\sum_{j,k}\Omega^j{}_k\wedge\Omega^k{}_j \\
&= -\frac{1}{8\pi^2}\sum_{j,k,l,m} g_{jm}g_{kl}\Omega^{jl}\wedge\Omega^{km} = \frac{1}{8\pi^2}\sum_{j,k} g_{jj}g_{kk}\Omega^{jk}\wedge\Omega^{jk} \\
&= \frac{1}{4\pi^2}(\Omega^{12}\wedge\Omega^{12} - \Omega^{13}\wedge\Omega^{13} - \Omega^{14}\wedge\Omega^{14} \\
&\qquad -\Omega^{23}\wedge\Omega^{23} - \Omega^{24}\wedge\Omega^{24} + \Omega^{34}\wedge\Omega^{34}).
\end{aligned}$$

Let $\{\lambda_+^1,\lambda_+^2,\lambda_+^3\}$ and $\{\lambda_-^1,\lambda_-^2,\lambda_-^3\}$ be local orthonormal frame fields defined by

$$\begin{aligned}
\lambda_+^1 &= (e^1\wedge e^2 + e^3\wedge e^4)/\sqrt{2}, \quad \lambda_-^1 = (e^1\wedge e^2 - e^3\wedge e^4)/\sqrt{2}, \\
\lambda_+^2 &= (e^1\wedge e^3 + e^2\wedge e^4)/\sqrt{2}, \quad \lambda_-^2 = (e^1\wedge e^3 - e^2\wedge e^4)/\sqrt{2}, \\
\lambda_+^3 &= (e^1\wedge e^4 + e^3\wedge e^2)/\sqrt{2}, \quad \lambda_-^3 = (e^1\wedge e^4 - e^3\wedge e^2)/\sqrt{2},
\end{aligned}$$

which satisfy

$$\begin{gathered}
\lambda_+^1\wedge\lambda_+^1 = -\lambda_+^2\wedge\lambda_+^2 = -\lambda_+^3\wedge\lambda_+^3 = -\lambda_-^1\wedge\lambda_-^1 = \lambda_-^2\wedge\lambda_-^2 = \lambda_-^3\wedge\lambda_-^3 = *1; \\
\lambda_+^p\wedge\lambda_+^q = \lambda_-^p\wedge\lambda_-^q = 0 \ (p\neq q), \quad \lambda_+^p\wedge\lambda_-^q = 0 \ (\text{for any } p,q=1,2,3).
\end{gathered}$$

By using $\{\lambda_\pm^1,\lambda_\pm^2,\lambda_\pm^3\}$, the first Pontrjagin form $p_1(M,\nabla)$ is given as

$$\begin{aligned}
&p_1(M,\nabla) \\
&= \frac{1}{8\pi^2}\{(R(\lambda_+^1)+R(\lambda_-^1))^2 - (R(\lambda_+^2)+R(\lambda_-^2))^2 - (R(\lambda_+^3)+R(\lambda_-^3))^2 \\
&\qquad -(R(\lambda_+^3)-R(\lambda_-^3))^2 - (R(\lambda_+^2)-R(\lambda_-^2))^2 + (R(\lambda_+^1)-R(\lambda_-^1))^2\} \\
&= \frac{1}{4\pi^2}\{R(\lambda_+^1)^2 - R(\lambda_+^2)^2 - R(\lambda_+^3)^2 + R(\lambda_-^1)^2 - R(\lambda_-^2)^2 - R(\lambda_-^3)^2\}.
\end{aligned}$$

where R is the curvature operator. Note that R has the following matrix representation:

$$R\begin{pmatrix}\lambda_+^1\\ \lambda_+^2\\ \lambda_+^3\\ \lambda_-^1\\ \lambda_-^2\\ \lambda_-^3\end{pmatrix} = \left(\begin{array}{ccc|ccc}
a_{11} & a_{12} & a_{13} & b_{11} & b_{12} & b_{13} \\
-a_{12} & a_{22} & a_{23} & b_{21} & b_{22} & b_{23} \\
-a_{13} & a_{23} & a_{33} & b_{31} & b_{32} & b_{33} \\ \hline
b_{11} & -b_{21} & -b_{31} & d_{11} & d_{12} & d_{13} \\
-b_{12} & b_{22} & b_{32} & -d_{12} & d_{22} & d_{23} \\
-b_{13} & b_{23} & b_{33} & -d_{13} & d_{23} & d_{33}
\end{array}\right)\begin{pmatrix}\lambda_+^1\\ \lambda_+^2\\ \lambda_+^3\\ \lambda_-^1\\ \lambda_-^2\\ \lambda_-^3\end{pmatrix}.$$

Hence it follows from the first Bianchi identity that

$$a_{11} + a_{22} + a_{33} = d_{11} + d_{22} + d_{33} = s/4,$$

where s denotes the scalar curvature of (M, g). The components b_{pq} are related to the traceless Ricci tensor Ric_0 of (M, g). Let (Z_{ij}) be the matrix representation of Ric_0 with respect to $\{e_1, e_2, e_3, e_4\}$ (i.e., $Z_{ij} = \mathrm{Ric}_0(e_i, e_j)$) and $(Z^i{}_j) = (\sum_{k=1}^4 g^{ik} Z_{kj})$ denote the corresponding endomorphism on TM. By definition, the components Z^i_j are expressed as

$$\begin{aligned}
Z^1{}_1 &= \frac{1}{2}(R^{12}{}_{12} + R^{13}{}_{13} + R^{14}{}_{14} - R^{32}{}_{32} - R^{24}{}_{24} - R^{34}{}_{34}),\\
Z^1{}_2 &= -R^{13}{}_{32} + R^{14}{}_{24} = Z^2{}_1,\\
Z^1{}_3 &= R^{12}{}_{32} + R^{14}{}_{34} = -Z^3{}_1,\\
Z^1{}_4 &= -R^{12}{}_{24} - R^{13}{}_{34} = -Z^4{}_1,\\
Z^2{}_2 &= \frac{1}{2}(R^{12}{}_{12} - R^{13}{}_{13} - R^{14}{}_{14} + R^{32}{}_{32} + R^{24}{}_{24} - R^{34}{}_{34}),\\
Z^2{}_3 &= R^{12}{}_{13} + R^{24}{}_{34} = -Z^3{}_2,\\
Z^2{}_4 &= R^{12}{}_{14} + R^{32}{}_{34} = -Z^4{}_2,\\
Z^3{}_3 &= \frac{1}{2}(-R^{12}{}_{12} + R^{13}{}_{13} - R^{14}{}_{14} + R^{32}{}_{32} - R^{24}{}_{24} + R^{43}{}_{43}),\\
Z^3{}_4 &= R^{13}{}_{14} - R^{32}{}_{24} = Z^4{}_3,\\
Z^4{}_4 &= \frac{1}{2}(-R^{12}{}_{12} - R^{13}{}_{13} + R^{14}{}_{14} - R^{32}{}_{32} + R^{24}{}_{24} + R^{34}{}_{34})\\
&= -(Z^1{}_1 + Z^2{}_2 + Z^3{}_3).
\end{aligned}$$

Then we have the following relation between the components b_{pq} and $Z^i{}_j$:

$$\begin{aligned}
b_{11} &= (Z^1{}_1 + Z^2{}_2)/2, & b_{12} &= (Z^1{}_4 + Z^2{}_3)/2, & b_{13} &= -(Z^1{}_3 - Z^2{}_4)/2,\\
b_{21} &= (Z^1{}_4 - Z^2{}_3)/2, & b_{22} &= (Z^1{}_1 + Z^3{}_3)/2, & b_{23} &= (Z^1{}_2 + Z^3{}_4)/2,\\
b_{31} &= -(Z^1{}_3 + Z^2{}_4)/2, & b_{32} &= -(Z^1{}_2 - Z^3{}_4)/2, & b_{33} &= -(Z^2{}_2 + Z^3{}_3)/2.
\end{aligned}$$

By using the expression above, we have

$$\begin{aligned}
R(\lambda_+^1)^2 &= \{(a_{11})^2 - (a_{12})^2 - (a_{13})^2 - (b_{11})^2 + (b_{12})^2 + (b_{13})^2\} * 1,\\
R(\lambda_+^2)^2 &= \{(a_{12})^2 - (a_{22})^2 - (a_{23})^2 - (b_{21})^2 + (b_{22})^2 + (b_{23})^2\} * 1,\\
R(\lambda_+^3)^2 &= \{(a_{13})^2 - (a_{23})^2 - (a_{33})^2 - (b_{31})^2 + (b_{32})^2 + (b_{33})^2\} * 1,\\
R(\lambda_-^1)^2 &= \{(b_{11})^2 - (b_{21})^2 - (b_{31})^2 - (d_{11})^2 + (d_{12})^2 + (d_{13})^2\} * 1,\\
R(\lambda_-^2)^2 &= \{(b_{12})^2 - (b_{22})^2 - (b_{32})^2 - (d_{12})^2 + (d_{22})^2 + (d_{23})^2\} * 1,\\
R(\lambda_-^3)^2 &= \{(b_{13})^2 - (b_{23})^2 - (b_{33})^2 - (d_{13})^2 + (d_{23})^2 + (d_{33})^2\} * 1.
\end{aligned}$$

We therefore obtain the following expression of $p_1(M,\nabla)$:

$$\begin{aligned} p_1(M,\nabla) &= \frac{1}{4\pi^2}\{(a_{11})^2-(a_{12})^2-(a_{13})^2-(a_{12})^2+(a_{22})^2+(a_{23})^2 \\ &\quad -(a_{13})^2+(a_{23})^2+(a_{33})^2-(d_{11})^2+(d_{12})^2+(d_{13})^2 \\ &\quad +(d_{12})^2-(d_{22})^2-(d_{23})^2+(d_{13})^2-(d_{23})^2-(d_{33})^2\}*1. \end{aligned}$$

We may write the matrices $A=(a_{pq})$ and $D=(d_{pq})$ as

$$A = W_+ + (s/12)\mathrm{Id}, \quad D = W_- + (s/12)\mathrm{Id},$$

where $W_+=(w^+_{pq})$ (resp. $W_-=(w^-_{pq})$) denotes the self-dual (resp. anti-self-dual) part of the Weyl conformal tensor W. Then we have

$$\begin{aligned} &p_1(M,\nabla) \\ &= \frac{1}{4\pi^2}\Bigg[\left(w^+_{11}+\frac{s}{12}\right)^2-(w^+_{12})^2-(w^+_{13})^2-(w^+_{12})^2+\left(w^+_{22}+\frac{s}{12}\right)^2+(w^+_{23})^2 \\ &\quad -(w^+_{13})^2+(w^+_{23})^2+\left(w^+_{33}+\frac{s}{12}\right)^2-\left(w^-_{11}+\frac{s}{12}\right)^2+(w^-_{12})^2+(w^-_{13})^2 \\ &\quad +(w^-_{12})^2-\left(w^-_{22}+\frac{s}{12}\right)^2-(w^-_{23})^2+(w^-_{13})^2-(w^-_{23})^2-\left(w^-_{33}+\frac{s}{12}\right)^2\Bigg]*1 \\ &= \frac{1}{4\pi^2}\Bigg[(w^+_{11})^2-(w^+_{12})^2-(w^+_{13})^2-(w^+_{12})^2+(w^+_{22})^2+(w^+_{23})^2 \\ &\qquad -(w^+_{13})^2+(w^+_{23})^2+(w^+_{33})^2+\frac{s}{6}(w^+_{11}+w^+_{22}+w^+_{33})+\frac{s^2}{48} \\ &\qquad -(w^-_{11})^2+(w^-_{12})^2+(w^-_{13})^2+(w^-_{12})^2-(w^-_{22})^2-(w^-_{23})^2 \\ &\qquad +(w^-_{13})^2-(w^-_{23})^2-(w^-_{33})^2-\frac{s}{6}(w^-_{11}+w^-_{22}+w^-_{33})-\frac{s^2}{48}\Bigg]*1 \\ &= \frac{1}{4\pi^2}\left\{[(w^+_{11})^2+(w^+_{22})^2+(w^+_{33})^2+2(-(w^+_{12})^2-(w^+_{13})^2+(w^+_{23})^2)]\right. \\ &\qquad \left.-[(w^-_{11})^2+(w^-_{22})^2+(w^-_{33})^2+2(-(w^-_{12})^2-(w^-_{13})^2+(w^-_{23})^2)]\right\}*1. \end{aligned}$$

By Hirzebruch signature theorem, the signature $\tau(M)$ is expressed as $\tau(M)=(1/3)p_1(M)$, so we have the following

Proposition 5.5 *Let (M,g) be a compact oriented four-manifold with a neutral metric g. Then its signature $\tau(M)$ is expressed as*

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2-|W_-|^2\right)*1, \tag{5.22}$$

where the squared norms $|W_\pm|^2$ are given respectively by

$$|W_\pm|^2 := (w^\pm_{11})^2+(w^\pm_{22})^2+(w^\pm_{33})^2+2(-(w^\pm_{12})^2-(w^\pm_{13})^2+(w^\pm_{23})^2).$$

As an application, we obtain the following (cf. [47]):

Proposition 5.6 *Let (M, g, I) be a compact neutral Kähler surface. If g is anti-self-dual with respect to the complex orientation, then the signature $\tau(M)$ is nonpositive, and $\tau(M) = 0$ only if g is conformally-flat.*

We next recall Gauss-Bonnet formula (cf. Avez [5], Chern [18]). Let (M, g) be a compact oriented pseudo-Riemannian manifold with neutral metric g and ∇ its Levi-Civita connection. Let $\{e_1, e_2, e_3, e_4\}$ and $\{e^1, e^2, e^3, e^4\}$ be local orthonormal frame field on (M, g) and its dual coframe field expressed as (5.21), respectively. Then the Euler form $e(M, \nabla)$ of (M, g) is given by

$$\begin{aligned}
e(M,\nabla) &= \frac{-1}{32\pi^2}\sum_{i,j,k,l}\varepsilon_{ijkl}\Omega^{ij}\wedge\Omega^{kl} \\
&= \frac{-1}{4\pi^2}(\Omega^{12}\wedge\Omega^{34} - \Omega^{13}\wedge\Omega^{24} - \Omega^{14}\wedge\Omega^{32}) \\
&= \frac{-1}{8\pi^2}\{(R(\lambda^1_+) + R(\lambda^1_-))\wedge(R(\lambda^1_+) - R(\lambda^1_-)) \\
&\quad -(R(\lambda^2_+) + R(\lambda^2_-))\wedge(R(\lambda^2_+) - R(\lambda^2_-)) \\
&\quad -(R(\lambda^3_+) + R(\lambda^3_-))\wedge(R(\lambda^3_+) - R(\lambda^3_-))\} \\
&= \frac{-1}{8\pi^2}\{R(\lambda^1_+)^2 - R(\lambda^2_+)^2 - R(\lambda^3_+)^2 \\
&\quad -R(\lambda^1_-)^2 + R(\lambda^2_-)^2 + R(\lambda^3_-)^2\}.
\end{aligned}$$

By a computation similar to that done for the signature, we obtain

$$\begin{aligned}
&e(M,\nabla) \\
&= \frac{-1}{8\pi^2}\{[(a_{11})^2 + (a_{22})^2 + (a_{33})^2 + 2(-(a_{12})^2 - (a_{13})^2 + (a_{23})^2)] \\
&\quad +[(d_{11})^2 + (d_{22})^2 + (d_{33})^2 + 2(-(d_{12})^2 - (d_{13})^2 + (d_{23})^2)] \\
&\quad -2[(b_{11})^2 + (b_{22})^2 + (b_{33})^2 - (b_{12})^2 - (b_{21})^2 - (b_{13})^2 \\
&\quad -(b_{31})^2 + (b_{23})^2) + (b_{32})^2]\} * 1 \\
&= \frac{-1}{8\pi^2}\left(|W_+|^2 + |W_-|^2 - \frac{1}{2}|Z|^2 + \frac{s^2}{24}\right) * 1,
\end{aligned}$$

where Z is the traceless Ricci tensor and $|\ \cdot\ |^2$ denotes the indefinite squared norm.

Summing up these formulas, we obtain

Proposition 5.7 *Let (M, g) be a compact oriented four-manifold with a neutral metric g. Then its Euler characteristic $\chi(M)$ is expressed as*

$$\chi(M) = -\frac{1}{8\pi^2}\int_M \left(|W_+|^2 + |W_-|^2 - \frac{1}{2}|Z|^2 + \frac{s^2}{24}\right) * 1. \tag{5.23}$$

The formulas (5.23) and (5.22) together with (2.15) show the following proposition (cf. [42]):

Proposition 5.8 *Let (M, g, I) be a compact neutral Kähler surface. If g is Einstein, then the squared first Chern class $c_1^2(M, I)$ is nonpositive, and $c_1^2(M, I) = 0$ only if g is Ricci-flat.*

Note that Petean [82] observed this result by taking account of $c_1(M, I) = (1/2\pi)[\gamma] = (s_g/8\pi)[\Omega_I]$ and $[\Omega_I]^2 < 0$, and obtained an interesting result for the existence of neutral Kähler Einstein metrics on compact complex surfaces.

5.3 Liouville's theorem

The following result is referred to as Liouville's theorem in Introduction:

Proposition 5.9 *Let $\mathcal{U}, \mathcal{V}$ be open subsets of $\mathbb{R}^n$ $(n \geq 3)$ and*

$$g = \sum_{j,k=1}^{n} g_{jk} dx^j dx^k$$

a pseudo-Riemannian metric on $\mathbb{R}^n$ such that all g_{jk} are constant $(1 \leq j, k \leq n)$. Let $\varphi : \mathcal{U} \to \mathcal{V}$ be a conformal diffeomorphism. Then φ is given as the composition of inversions, similarities and isometries on $(\mathbb{R}^n, g)$.

For the sake of completeness, we give a proof of Proposition 5.9. The following proof is based on the argument, referred to as that due to Haantjes, in Sasaki [85]. We first recall the following

Lemma 5.10 *Let R and R' denote the curvature tensors of g and $g' = e^{2f}g$, respectively. Then the following relation is satisfied:*

$$R' = R + \left(\nabla df - df \otimes df + \frac{1}{2}\|df\|^2 g\right) \owedge g,$$

where $\owedge$ denotes the Kulkarni-Nomizu product.

Proof of Proposition 5.9. Set $g' := \varphi^* g = e^{2f} g$ and

$$\tau := \nabla df - df \otimes df + \frac{1}{2}\|df\|^2 g.$$

Then, since $\varphi : (\mathcal{U}, g') \to (\mathcal{V}, g)$ is an isometry, the corresponding curvature tensors R' and R satisfy

$$\varphi_*(R'(X,Y)Z) = R(\varphi_* X, \varphi_* Y)\varphi_* Z$$

for arbitrary vector fields X, Y, Z on $\mathcal{U}$. Since g_{jk} are constants by assumption, we have $R = 0$, and thus $R' = 0$. Then it follows from Lemma 5.10 that

$$\tau \owedge g = 0.$$

Taking the g-trace of this identity, we have

$$(n-2)\tau = -(\mathrm{tr}_g \tau) g, \quad (n-2)\mathrm{tr}_g \tau = -n \mathrm{tr}_g \tau.$$

The second relation implies that $\mathrm{tr}_g \tau = 0$, and hence by the first we have $\tau = 0$, since $n \geq 3$. By the definition of τ, we obtain

$$\nabla df - df \otimes df + \frac{1}{2}\|df\|^2 g = 0.$$

In terms of the standard coordinates $(x^1, \ldots, x^n)$ of $\mathbb{R}^n$, this relation is equivalent to

$$\frac{\partial^2 f}{\partial x^j \partial x^k} - \frac{\partial f}{\partial x^j}\frac{\partial f}{\partial x^k} + \frac{1}{2} g(df, df) g_{jk} = 0. \tag{5.24}$$

In the case where $df \equiv 0$, that is, when f is constant, we obtain an isometry $\psi := \lambda_{c^{-1}} \circ \varphi : (\mathcal{U}, g) \to (c^{-1}\mathcal{V}, g)$, where $c := e^f$, $\lambda_c : \mathbb{R}^n \to \mathbb{R}^n$ is the homothety defined by $x \mapsto \lambda_c(x) := cx$ and $c^{-1}\mathcal{V} := \{c^{-1}x \in \mathbb{R}^n \mid x \in \mathcal{V}\}$.

In the case where $\|df\|^2 = g(df, df) \neq 0$, it follows from the relation (5.24) that

$$x^i + \frac{2}{g(df, df)} \sum_{j=1}^n g^{ij} \frac{\partial f}{\partial x^j} = b^i$$

for constants b^i $(i = 1, 2, \ldots, n)$. By changing variables $\tilde{x}^i := x^i - b^i$, we have

$$\frac{\partial f}{\partial \tilde{x}^k} = -\frac{1}{2} g(df, df) \sum_{i=1}^n g_{ik} \tilde{x}^i.$$

Then

$$g(df, df) = \sum_{j,k=1}^{n} g^{jk} \frac{\partial f}{\partial \tilde{x}^j} \frac{\partial f}{\partial \tilde{x}^k} = \frac{1}{4} g(df, df)^2 \|\tilde{x}\|^2,$$

that is,

$$g(df, df) = \frac{4}{\|\tilde{x}\|^2},$$

where $\|\tilde{x}\|^2$ is defined by $\|\tilde{x}\|^2 := \sum_{j,k=1}^{n} g_{jk} \tilde{x}^j \tilde{x}^k$. Noting that

$$\frac{\partial(\|\tilde{x}\|^2)}{\partial \tilde{x}^j} = 2 \sum_{k=1}^{n} g_{jk} \tilde{x}^k,$$

we obtain

$$\frac{\partial f}{\partial \tilde{x}^j} = -\frac{1}{\|\tilde{x}\|^2} \frac{\partial(\|\tilde{x}\|^2)}{\partial \tilde{x}^j} = -\frac{\partial(\log|\|\tilde{x}\|^2|)}{\partial \tilde{x}^j},$$

which implies that

$$e^{2f} = \left(\frac{c}{\|\tilde{x}\|^2} \right)^2.$$

Let $T_b : \mathbb{R}^n \to \mathbb{R}^n$ and $I : \mathbb{R}^n \backslash \mathcal{N} \to \mathbb{R}^n \backslash \mathcal{N}$ be maps defined by

$$T_b(x) := x + b, \quad I(x) := \frac{x}{\|x\|^2},$$

where $x = (x^1, \ldots, x^n)$, $b = (b^1, \ldots, b^n)$ and $\mathcal{N} := \{x \in \mathbb{R}^n \mid \|x\|^2 = 0\}$. Then, in our case, $\psi := T_b^{-1} \circ \varphi \circ (\lambda_c \circ I)^{-1} \circ T_b$ is an isometry of $\mathbb{R}^n$.

Finally, we consider the case where $\|df\|^2 = g(df, df) \equiv 0$ but $df \not\equiv 0$ on some domain in $\mathcal{U}$. Then the condition $g(df, df) \equiv 0$ implies that

$$\frac{\partial^2 f}{\partial x^j \partial x^k} = \frac{\partial f}{\partial x^j} \frac{\partial f}{\partial x^k}.$$

This is equivalent to

$$\frac{\partial f_k}{\partial x^j} = f_k \frac{\partial f}{\partial x^j}, \quad \frac{\partial f_j}{\partial x^k} = f_j \frac{\partial f}{\partial x^k},$$

where $f_j := \partial f / \partial x^j$. Then we obtain

$$\frac{\partial f}{\partial x^j} = b_j e^f, \quad \text{or equivalently,} \quad \frac{\partial(e^{-f})}{\partial x^j} = -b_j$$

for some constant b_j $(j = 1, 2, \ldots, n)$. Set $b^j = \sum_{k=1}^{n} g^{jk} b_k$ and $\langle b, x\rangle := \sum_{j,k=1}^{n} g_{jk} b^j x^k$. Then $d(e^{-f}) = -d(\langle b, x\rangle)$. Therefore we obtain $e^{2f} = (c - \langle b, x\rangle)^{-2}$ for some constant c. By translation if necessary, we may assume that $c = 1$. Indeed, by setting $\tilde{x} := x + a$ for a constant vector a satisfying $\langle a, b\rangle = c - 1$, we have $e^{2f} = (1 - \langle b, \tilde{x}\rangle)^{-2} (> 0)$.

Let $\Phi_b : \{1 - \langle b, x\rangle \neq 0\} \to \{1 - \langle b, x\rangle \neq 0\}$ be a map defined by

$$\Phi_b(x) := \frac{1}{1 - \langle b, x\rangle} \Big(x - \frac{1}{2} \|x\|^2 b \Big)$$

for $b := (b^1, \ldots, b^n)$. Then Φ_b restricted to $(\mathbb{R}^n \backslash \mathcal{N}) \bigcap \{\langle b, x\rangle \neq 1\}$ is obtained as

$$\Phi_b(x) = I \circ T_{-\frac{1}{2}b} \circ I(x).$$

Since $\psi := \varphi \circ \Phi_b^{-1}$ is an isometry between subsets in $(\mathbb{R}^n, g)$, the original map φ is obtained as the composition of inversions, translations and isometries of $(\mathbb{R}^n, g)$. $\Box$

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