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Studies on toric Fano varieties

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Abstract

Chapter 1 is intended to give basic tools for the classification of nonsingular toric Fano varieties by means of the notions of primitive collections and primitive relations due to Batyrev. By using them we can easily deal with equivariant blow-ups and blow-downs, and obtain an easy criterion to determine whether a given nonsingular toric variety is a Fano variety or not. As applications of these results, we prove a toric version of a theorem of Mori, and in principle, can classify all nonsingular toric Fano varieties obtained from a given nonsingular toric Fano variety by finite successions of equivariant blow-ups and blow-downs through nonsingular toric Fano varieties. In particular, we obtain a new method for the classification of nonsingular toric Fano varieties of dimension at most four. This method is extended to the case of Gorenstein toric Fano varieties endowed with natural resolutions of singularities, by which we obtain a new method for the classification of Gorenstein toric Fano surfaces.

In Chapter 2, we investigate whether the 124 nonsingular toric Fano 4-folds admit totally nondegenerate embeddings from abelian surfaces or not. In consequence, we determine the possibilities of these embeddings, except for the remaining 21 nonsingular toric Fano 4-folds.

In Chapter 3, we obtain a complete classification of toric weakened Fano 3-folds, that is, smooth toric weak Fano 3-folds which are not Fano but are deformed to smooth Fano 3-folds. There exist exactly 15 toric weakened Fano 3-folds up to isomorphisms.

Finally, we construct one-parameter complex analytic families whose special fibers are complete toric varieties in Chapter 4. Under appropriate assumptions, the general fibers of these families also become toric varieties, and the corresponding fans are explicitly described by the data of the fans associated to the special fibers. Using these families, we construct a deformation family for a certain toric weakened Fano 3-fold. Moreover, we get certain examples of toric weakened Fano 4-folds.

Contents

Chapter 1

Toward the classification of higher-dimensional toric Fano varieties

1.1 Introduction

A *Gorenstein toric Fano variety* is a complete toric variety X with at most Gorenstein singularities such that the anticanonical divisor $-K_X$ is ample. Gorenstein toric Fano varieties are very important as ambient spaces of Calabi-Yau varieties, and, for instance, Batyrev [4] systematically constructed examples of mirror symmetric pairs of Calabi-Yau varieties as hypersurfaces in Gorenstein toric Fano varieties. The set of isomorphism classes of Gorenstein toric Fano d-folds is a finite set for any dimension d (see Batyrev [3]). Nonsingular toric Fano d-folds are classified for $d \leq 4$ and Gorenstein toric Fano d-folds are classified for $d \leq 3$ (see Batyrev [6] and Watanabe-Watanabe [36] in the nonsingular cases, and Koelman [16], Kreuzer-Skarke [17] and [18] in the Gorenstein cases). In this chapter, we study the classification of higher-dimensional nonsingular or Gorenstein toric Fano varieties using the notions of primitive collections and primitive relations introduced by Batyrev [5]. First we consider the nonsingular case.

Definition 1.1.1 Let $\mathcal{F}_d$ be the set of isomorphism classes of toric Fano d-folds. X_1 and X_2 in $\mathcal{F}_d$ are said to be *F-equivalent* if there exists a sequence of equivariant blow-ups and blow-downs from X_1 to X_2 through nonsingular toric *Fano* d-folds, namely there exist nonsingular toric Fano d-folds $Y_0 = X_1, Y_1, \ldots, Y_{2l} = X_2$ together with finite successions $Y_j \to Y_{j-1}$ and $Y_j \to Y_{j+1}$, for each odd $1 \leq j \leq 2l-1$, of equivariant blow-ups through nonsingular toric Fano d-folds. We denote the relation by $X_1 \overset{\mathrm{F}}{\sim} X_2$. Then "$\overset{\mathrm{F}}{\sim}$" is obviously an equivalence relation.

Remark 1.1.2 For equivariant birational maps of complete nonsingular toric varieties which need not be Fano varieties, related factorization conjectures have been proposed by Oda [27]. The weak version analogous to the factorization in Definition 1.1.1 was proved by Włodarczyk [37] and Morelli [22], while the strong version was proved by Morelli [22] and later supplemented by Abramovich-Matsuki-Rashid [1].

As we see in this chapter, if we get a complete system of representatives for $(\mathcal{F}, \overset{\mathrm{F}}{\sim})$, then we get the classification of nonsingular toric Fano d-folds. The following conjecture for nonsingular toric Fano d-folds holds for $d \leq 4$ as a consequence of the known classification.

Conjecture 1.1.3 *Any nonsingular toric Fano d-fold is either pseudo-symmetric or F-equivalent to the d-dimensional projective space $\mathbf{P}^d$.*

In this chapter, we prove this conjecture for $d = 3$ and $d = 4$ without using the classification. As a result, we get a new method for the classification of nonsingular toric Fano 3-folds and 4-folds. Using this method for the classification, we can show that there exist 124 nonsingular toric Fano 4-folds up to isomorphism.

On the other hand, Gorenstein toric Fano d-folds are related to nonsingular toric weak Fano d-folds, where a nonsingular toric *weak* Fano variety is a nonsingular projective toric variety X such that the anticanonical divisor $-K_X$ is nef and big, and the methods for nonsingular toric Fano d-folds are extended to the case of nonsingular weak toric Fano d-folds. As a result, we get a new method for the classification of Gorenstein toric Fano surfaces.

The content of this chapter is as follows: In Section 1.2, we study basic concepts on toric Fano varieties, and recall the correspondence between Gorenstein toric Fano varieties and reflexive polytopes. In Sections 1.3 and 1.4, we introduce primitive collections and primitive relations. We can characterize toric Fano varieties using them, and calculate them before and after an equivariant blow-up. Moreover, we have a criterion for the possibility of an equivariant blow-down in terms of primitive collections and primitive relations. In Section 1.4, we give a new nonsingular toric Fano 4-fold which is missing in the classification of Batyrev [6]. In Section 1.5, we give a toric version of a theorem of Mori as an application of Sections 1.3 and 1.4. In Section 1.6, we give a procedure for the classification which says that we have only to get a complete system of representatives for the F-equivalence relation for the set of isomorphism classes of nonsingular toric Fano d-folds. We also study a correspondence between toric weak Fano varieties and Gorenstein toric Fano varieties. Especially, we get a new method for the classification of Gorenstein toric Fano surfaces. In Sections 1.7 and 1.8, we prove Conjecture 1.1.3 for $d = 3$ and $d = 4$. We give the table of the 124 nonsingular toric Fano 4-folds in terms of primitive relations in Section 1.9. In Section 1.10, as an application of Sections 1.3 and 1.4, we describe all the equivariant blow-up relations among nonsingular toric Fano 4-folds using the classification of Batyrev [6].

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1.2 Reflexive polytopes

In this section, we recall some basic notation and facts about toric Fano varieties (see Batyrev [4], Fulton [9], and Oda [26] for more details). The following notation is used throughout this chapter.

Let N be a free abelian group of rank d and $M := \mathrm{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$ the dual group. The natural pairing $\langle\ ,\ \rangle : M \times N \to \mathbf{Z}$ is extended to a bilinear form $\langle\ ,\ \rangle : M_{\mathbf{R}} \times N_{\mathbf{R}} \to \mathbf{R}$, where $M_{\mathbf{R}} := M \otimes_{\mathbf{Z}} \mathbf{R}$, $N_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R}$.

For a finite complete fan Σ in N and $0 \leq i \leq d$, we put $\Sigma(i) := \{\sigma \in \Sigma \mid \dim \sigma = i\}$. Each $\tau \in \Sigma(1)$ determines a unique element $e(\tau) \in N$ which generates the semigroup $\tau \cap N$. We put

$$\mathrm{G}(\Sigma) := \{e(\tau) \in N \mid \tau \in \Sigma(1)\}$$

and $\mathrm{G}(\sigma) := \sigma \cap \mathrm{G}(\Sigma)$ for $\sigma \in \Sigma$.

Definition 1.2.1 (Batyrev [4]) A d-dimensional convex lattice polytope $\Delta \subset N_{\mathbf{R}}$ is called a *reflexive polytope* if the origin 0 is in the interior of Δ and the polar

$$\Delta^* := \{y \in M_{\mathbf{R}} \mid \langle y, x\rangle \geq -1,\ \forall x \in \Delta\} \subset M_{\mathbf{R}}$$

is also a convex lattice polytope.

For a d-dimensional convex polytope $\Delta \subset N_{\mathbf{R}}$ and $0 \leq i \leq d-1$, we denote by $\Delta(i)$ the set of i-dimensional faces of Δ.

Let $\Delta \subset N_{\mathbf{R}}$ be a convex lattice polytope such that 0 is in the interior of Δ. For any i-dimensional face $\delta \subset \Delta$ $(0 \leq i \leq d-1)$, let

$$\sigma(\delta) := \{rx \in N_{\mathbf{R}} \mid r \in \mathbf{R}_{\geq 0},\ x \in \delta\}.$$

Then $\sigma(\delta)$ is an $(i+1)$-dimensional strongly convex rational polyhedral cone in $N_{\mathbf{R}}$. Moreover,

$$\Sigma(\Delta) := \{\sigma(\delta) \mid \delta \in \Delta(i)\ (0 \leq i \leq d-1)\} \cup \{0\}$$

is a finite complete fan in N.

Proposition 1.2.2 (Batyrev [4]) *If $\Delta \subset N_{\mathbf{R}}$ is a reflexive polytope, then $T_N\mathrm{emb}(\Sigma(\Delta))$ is a Gorenstein toric Fano variety. Conversely, if Σ is a finite complete fan in N such that $T_N\mathrm{emb}(\Sigma)$ is a Gorenstein toric Fano variety, then $\mathrm{Conv}(\mathrm{G}(\Sigma)) \subset N_{\mathbf{R}}$ is a reflexive polytope, where $\mathrm{Conv}(\mathrm{G}(\Sigma))$ is the convex hull of $\mathrm{G}(\Sigma) \subset N_{\mathbf{R}}$. Moreover, any two Gorenstein toric Fano varieties $T_N\mathrm{emb}(\Sigma(\Delta_1))$ and $T_N\mathrm{emb}(\Sigma(\Delta_2))$ corresponding to two reflexive polytopes $\Delta_1 \subset N_{\mathbf{R}}$ and $\Delta_2 \subset N_{\mathbf{R}}$ are isomorphic if and only if Δ_1 and Δ_2 are equivalent up to unimodular transformation of the lattice N.*

Remark 1.2.3 A reflexive polytope Δ is called a *Fano polytope* if $\Sigma(\Delta)$ is nonsingular.

1.3 Primitive collections and primitive relations

Primitive collections and primitive relations, introduced by Batyrev [5], are very convenient in describing higher-dimensional fans. In this section, we recall these concepts and, by making use of them, characterize toric Fano varieties.

Definition 1.3.1 Let Σ be a finite complete *simplicial* fan in N. A nonempty subset $P \subset \mathrm{G}(\Sigma)$ is a *primitive collection* of Σ, if $\mathrm{Cone}(P) \notin \Sigma$, while $\mathrm{Cone}(P \setminus \{x\}) \in \Sigma$ for every $x \in P$, where $\mathrm{Cone}(S) := \sum_{x \in S} \mathbf{R}_{\geq 0} x$ for any subset $S \subset N_{\mathbf{R}}$.

We denote by $\mathrm{PC}(\Sigma)$ the set of primitive collections of Σ.

Remark 1.3.2 By definition, for any subset $S \subset \mathrm{G}(\Sigma)$ which does not generate a cone in Σ, there exists a primitive collection $P \in \mathrm{PC}(\Sigma)$ such that $P \subset S$.

Definition 1.3.3 Let Σ_1 and Σ_2 be finite complete simplicial fans in N. Then $\mathrm{PC}(\Sigma_1)$ and $\mathrm{PC}(\Sigma_2)$ are *isomorphic* if there exists a bijective map $\varphi : \mathrm{G}(\Sigma_1) \to \mathrm{G}(\Sigma_2)$ which induces a well-defined bijective map

$$\varphi_* : \mathrm{PC}(\Sigma_1) \ni P \longmapsto \varphi(P) \in \mathrm{PC}(\Sigma_2).$$

By Definitions 1.3.1 and 1.3.3, we immediately get the following:

Proposition 1.3.4 *Let Σ_1 and Σ_2 be finite complete simplicial fans in N. Then Σ_1 and Σ_2 are combinatorially equivalent if and only if $\mathrm{PC}(\Sigma_1)$ and $\mathrm{PC}(\Sigma_2)$ are isomorphic. Here Σ_1 and Σ_2 are said to be combinatorially equivalent if there exists a bijective map*

$$\psi : \mathrm{G}(\Sigma_1) \longrightarrow \mathrm{G}(\Sigma_2)$$

such that for any nonempty subset $S \subset \mathrm{G}(\Sigma_1)$, we have $\mathrm{Cone}(S) \in \Sigma_1$ when and only when $\mathrm{Cone}(\psi(S)) \in \Sigma_2$.

In the nonsingular case, we have the following additional information:

Definition 1.3.5 Let Σ be a finite complete nonsingular fan in N and $P = \{x_1, \ldots, x_l\} \in \mathrm{PC}(\Sigma)$. Then there is a unique element $\sigma(P) \in \Sigma$ such that

$$x_1 + \cdots + x_l \in \mathrm{Relint}(\sigma(P)),$$

where $\mathrm{Relint}(S)$ stands for the relative interior of S for any subset $S \subset N_{\mathbf{R}}$. Hence we get a linear relation

$$x_1 + \cdots + x_l = a_1 y_1 + \cdots + a_m y_m \ (a_1, \ldots, a_m \in \mathbf{Z}_{>0}),$$

where $\mathrm{G}(\sigma(P)) = \{y_1, \ldots, y_m\}$. We call this relation the *primitive relation* for P.

The integer $\deg P := l - (a_1 + \cdots + a_m)$ is called the *degree* of P.

By this definition and Proposition 1.3.4, we get the following characterization of isomorphism classes of complete nonsingular toric varieties.

Proposition 1.3.6 *Let Σ_1 and Σ_2 be finite complete nonsingular fans in N. Then the complete nonsingular toric varieties $T_N\mathrm{emb}(\Sigma_1)$ and $T_N\mathrm{emb}(\Sigma_2)$ are isomorphic if and only if there exists an isomorphism from $\mathrm{PC}(\Sigma_1)$ to $\mathrm{PC}(\Sigma_2)$ which preserves their primitive relations.*

Let Σ be a finite complete nonsingular fan in N and $X := T_N\mathrm{emb}(\Sigma)$. Then for any $P \in \mathrm{PC}(\Sigma)$, we can define an element $r(P) \in A_1(X)$ in the following way, where $A_1(X)$ is the $\mathbf{Z}$-module of algebraic 1-cycles modulo numerical equivalence.

Proposition 1.3.7 (e.g., Fulton [9], Oda [26]) *Let Σ be a finite complete nonsingular fan in N and $X := T_N\mathrm{emb}(\Sigma)$. Then we have an exact sequence of $\mathbf{Z}$-modules*

$$0 \longrightarrow M \xrightarrow{\varphi} \mathbf{Z}^{\mathrm{G}(\Sigma)} \xrightarrow{\psi} \mathrm{Pic}(X) \longrightarrow 0 \text{ (exact)}.$$

By the exact sequence in Proposition 1.3.7, we have $\mathrm{Pic}(X) \cong \mathbf{Z}^{\mathrm{G}(\Sigma)}/M$ and hence

$$A_1(X) \cong \mathrm{Hom}_{\mathbf{Z}}(\mathrm{Pic}(X), \mathbf{Z}) \cong \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}^{\mathrm{G}(\Sigma)}/M, \mathbf{Z}) \cong M^{\perp} \subset \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}^{\mathrm{G}(\Sigma)}, \mathbf{Z}).$$

Consequently, we have

$$A_1(X) \cong \left\{ (a_x)_{x \in \mathrm{G}(\Sigma)} \in \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}^{\mathrm{G}(\Sigma)}, \mathbf{Z}) \;\middle|\; \sum_{x \in \mathrm{G}(\Sigma)} a_x x = 0 \right\}.$$

Let $P = \{x_1, \ldots, x_l\} \in \mathrm{PC}(\Sigma)$ and let

$$x_1 + \cdots + x_l = a_1 y_1 + \cdots + a_m y_m$$

be the primitive relation for P. Then we get a linear relation

$$x_1 + \cdots + x_l - (a_1 y_1 + \cdots + a_m y_m) = 0.$$

Thus we can define $r(P) = (r(P)_x)_{x \in \mathrm{G}(\Sigma)} \in A_1(X)$ by

$$r(P)_x := \begin{cases} 1 & \text{if } \ x = x_i \ (1 \le i \le l) \\ -a_j & \text{if } \ x = y_j \ (1 \le j \le m) \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, for any wall $\tau \in \Sigma(d-1)$, there is a linear relation

$$b_1 z_1 + \cdots + b_{d-1} z_{d-1} + b_d z_d + b_{d+1} z_{d+1} = 0 \ (b_1, \ldots, b_{d+1} \in \mathbf{Z},\ b_d = b_{d+1} = 1),$$

where $\mathrm{G}(\tau) = \{z_1, \ldots, z_{d-1}\}$, while $\mathrm{Cone}(\mathrm{G}(\tau) \cup \{z_d\})$ and $\mathrm{Cone}(\mathrm{G}(\tau) \cup \{z_{d+1}\})$ are the d-dimensional strongly convex rational polyhedral cones in Σ which contain τ as a face. We define $v(\tau) = (v(\tau)_x)_{x \in \mathrm{G}(\Sigma)} \in A_1(X)$ by

$$v(\tau)_x := \begin{cases} b_i & \text{if} \quad x = z_i \ (1 \leq i \leq d+1) \\ 0 & \quad\ \text{otherwise.} \end{cases}$$

Concerning this definition, the following is very useful.

Theorem 1.3.8 (Batyrev [5], [6], Reid [30]) *Let Σ be a finite complete nonsingular fan in N and $X = T_N\mathrm{emb}(\Sigma)$. Then we have*

$$\mathbf{NE}(X) = \sum_{\tau \in \Sigma(d-1)} \mathbf{R}_{\geq 0} v(\tau) = \sum_{P \in \mathrm{PC}(\Sigma)} \mathbf{R}_{\geq 0} r(P),$$

where $\mathbf{NE}(X) \subset A_1(X) \otimes_{\mathbf{Z}} \mathbf{R}$ is the Mori cone of effective 1-cycles.

The following theorem is the toric Nakai criterion.

Theorem 1.3.9 (Oda [26], Oda-Park [28]) *Let Σ be a finite complete nonsingular fan in N and $X := T_N\mathrm{emb}(\Sigma)$. Then a T_N-invariant divisor $D \in T_N\mathrm{Div}(X)$ is ample if and only if*

$$\left(D.\overline{\mathrm{orb}(\tau)}\right) > 0 \textit{ for all } \tau \in \Sigma(d-1).$$

By Theorems 1.3.8 and 1.3.9, we can characterize nonsingular toric Fano varieties in terms of primitive collections.

Theorem 1.3.10 (Batyrev [6]) *Let Σ be a finite complete nonsingular fan in N and $X := T_N\mathrm{emb}(\Sigma)$. Then X is a nonsingular toric Fano variety (resp. $-K_X$ is a nef divisor) if and only if*

$$\deg P > 0 \ (\textit{resp. } \deg P \geq 0) \textit{ for all } P \in \mathrm{PC}(\Sigma).$$

Proof. ${}^t(1, 1, \ldots, 1) \in \mathbf{Z}^{\mathrm{G}(\Sigma)}$ corresponds to the anticanonical divisor of X. So for $P \in \mathrm{PC}(\Sigma)$,

$$(-K_X.r(P)) = \deg P.$$

Hence by Theorems 1.3.8 and 1.3.9, we are done. q.e.d.

1.4 Equivariant blow-ups and blow-downs

Let Σ be a finite complete simplicial fan in N. In this section, we investigate how the set $\mathrm{PC}(\Sigma)$ of primitive collections change by star subdivisions. In particular, we deal with equivariant blow-ups and blow-downs of nonsingular complete toric varieties in terms of the primitive collections and primitive relations.

Definition 1.4.1 Let Σ be a finite complete simplicial fan in N and $\sigma \in \Sigma$ with $\dim \sigma = l,\ 2 \leq l \leq d$. For $x \in (\mathrm{Relint}(\sigma)) \cap N$ with x primitive in N, we define the star subdivision of Σ along (σ, x) in the following way.

First, we define the strongly convex rational polyhedral cones σ_i $(1 \leq i \leq l)$ by

$$\sigma_i := \mathrm{Cone}\left(\{x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_l\}\right)\ (1 \leq i \leq l),$$

where $\mathrm{G}(\sigma) = \{x_1, \ldots, x_l\}$. Then for $\tau \in \Sigma$ such that $\sigma \prec \tau$, we can write τ uniquely as

$$\tau = \sigma + \tau' \text{ with } \tau' \in \Sigma,\ \sigma \cap \tau' = \{0\}.$$

In this notation, we have a finite complete simplicial fan $\Sigma^*_{(\sigma,x)}$ in N defined by

$$\Sigma^*_{(\sigma,x)} := (\Sigma \setminus \{\tau \in \Sigma \mid \sigma \prec \tau\}) \cup \{\text{the faces of } \sigma_i + \tau' \mid \tau \in \Sigma,\ \sigma \prec \tau,\ 1 \leq i \leq l\}.$$

We call $\Sigma^*_{(\sigma,x)}$ the *star subdivision* of Σ along (σ, x).

Remark 1.4.2 (Fulton [9], Oda [26]) In Definition 1.4.1, if Σ is nonsingular and $x = x_1 + \cdots + x_l$, then the equivariant proper birational morphism $T_N\mathrm{emb}(\Sigma^*_{(\sigma,x)}) \to T_N\mathrm{emb}(\Sigma)$ corresponding to this star subdivision is the equivariant blow-up along $\overline{\mathrm{orb}(\sigma)}$.

The following is one of the main theorems of this chapter.

Theorem 1.4.3 *Let Σ be a finite complete simplicial fan in N, $\sigma \in \Sigma$ and x a primitive element in $(\mathrm{Relint}(\sigma)) \cap N$. Then the primitive collections of $\Sigma^*_{(\sigma,x)}$ are*

(1) $\mathrm{G}(\sigma)$,

(2) $P \in \mathrm{PC}(\Sigma)$ *such that* $\mathrm{G}(\sigma) \not\subset P$, *and*

(3) *the minimal elements in the set* $\{(P \setminus \mathrm{G}(\sigma)) \cup \{x\} \mid P \in \mathrm{PC}(\Sigma),\ P \cap \mathrm{G}(\sigma) \neq \emptyset\}$.

To prove this theorem, we need the following three lemmas.

Lemma 1.4.4 *Let Σ be a finite complete simplicial fan in N, $\sigma \in \Sigma$, and x a primitive element in* $(\mathrm{Relint}(\sigma)) \cap N$. *For any $\tau^* \in \Sigma^*_{(\sigma,x)}$, if $x \in \tau^*$, then there exists $\tau' \in \Sigma$ such that $\tau' \cap \sigma = \{0\}$ and $\tau^* \prec \sigma_i + \tau' \in \Sigma^*_{(\sigma,x)}$ for some i with $1 \leq i \leq l$. Moreover, $\sigma_j + \tau' \in \Sigma^*_{(\sigma,x)}$ for all j $(1 \leq j \leq l)$, where $l = \dim \sigma$.*

The proof is obvious by Definition 1.4.1.

Lemma 1.4.5 *Let Σ be a finite complete simplicial fan in N, $\sigma \in \Sigma$, and x a primitive element in* $(\mathrm{Relint}(\sigma)) \cap N$. *Then $P^* \in \mathrm{PC}\left(\Sigma^*_{(\sigma,x)}\right)$ and $x \in P^*$ imply $\mathrm{G}(\sigma) \cap P^* = \emptyset$.*

Proof. Let $P^* \in \mathrm{PC}(\Sigma^*_{(\sigma,x)})$, $x \in P^*$, and suppose $\mathrm{G}(\sigma) \cap P^* \neq \emptyset$. Then $P^* \setminus \mathrm{G}(\sigma)$ generates a cone in Σ containing x. So, by Lemma 1.4.4, there exists $\tau' \in \Sigma$ such that

$$P^* \setminus \mathrm{G}(\sigma) \subset \mathrm{G}(\sigma_i + \tau')\ (1 \leq \exists i \leq l),\ \sigma \cap \tau' = \{0\},\ \sigma + \tau' \in \Sigma.$$

Since $(P^* \setminus \mathrm{G}(\sigma)) \setminus \{x\} \subset \mathrm{G}(\tau')$, we have an index j $(1 \leq j \leq l)$ such that

$$P^* \subset \mathrm{G}(\sigma_j + \tau'),\ \sigma_j + \tau' \in \Sigma^*_{(\sigma,x)},$$

which contradicts the assumption. q.e.d.

Lemma 1.4.6 *Let Σ be a finite complete simplicial fan in N, $\sigma \in \Sigma$, and x a primitive element in* $(\mathrm{Relint}(\sigma)) \cap N$. *Then for any $P^* \in \mathrm{PC}(\Sigma^*_{(\sigma,x)})$ which contains x, there exists $P \in \mathrm{PC}(\Sigma)$ such that*

$$(P \setminus \mathrm{G}(\sigma)) \cup \{x\} = P^*.$$

Proof. Let $P^* \in \mathrm{PC}(\Sigma^*_{(\sigma,x)})$, $x \in P^*$, and suppose $\mathrm{G}(\sigma) \cup (P^* \setminus \{x\})$ generates a strongly convex rational polyhedral cone in Σ. Then there exists $\tau' \in \Sigma$ such that

$$\mathrm{Cone}\,(\mathrm{G}(\sigma) \cup (P^* \setminus \{x\})) = \sigma + \tau',\ \sigma \cap \tau' = \{0\}.$$

Since $\mathrm{G}(\sigma) \cap P^* = \emptyset$ by Lemma 1.4.5, we have $P^* \subset \mathrm{G}(\sigma_i + \tau')$ for all i $(1 \leq i \leq l)$. This contradicts $P^* \in \mathrm{PC}(\Sigma^*_{(\sigma,x)})$. Therefore $\mathrm{G}(\sigma) \cup (P^* \setminus \{x\})$ contains a primitive collection of Σ.

Let $P \subset \mathrm{G}(\sigma) \cup (P^* \setminus \{x\})$, $P \in \mathrm{PC}(\Sigma)$. For any $y \in P^* \setminus \{x\}$, $P^* \setminus \{y\}$ generates a strongly convex rational polyhedral cone in $\Sigma^*_{(\sigma,x)}$ which contains x. Therefore, by Lemma 1.4.4, there exists $\tau' \in \Sigma$ such that

$$P^* \setminus \{y\} \subset \mathrm{G}(\sigma_i + \tau') \ (1 \leq \exists i \leq l), \ \sigma \cap \tau' = \{0\}.$$

Then $P^* \setminus \{x, y\} \subset \mathrm{G}(\tau')$ because $\mathrm{G}(\sigma) \cap P^* = \emptyset$ by Lemma 1.4.5. So we have

$$\mathrm{Cone}\,(\mathrm{G}(\sigma) \cup (P^* \setminus \{x, y\})) = \sigma + \mathrm{Cone}(P^* \setminus \{x, y\}) \prec \sigma + \tau' \in \Sigma,$$

and consequently $\mathrm{G}(\sigma) \cup (P^* \setminus \{x, y\})$ generates a strongly convex rational polyhedral cone in Σ.

On the other hand, suppose $P^* \setminus \{x\} \not\subset P$. Then there exists $y \in P^* \setminus \{x\}$ such that $P \subset \mathrm{G}(\sigma) \cup (P^* \setminus \{x, y\})$. This contradicts $P \in \mathrm{PC}(\Sigma)$. Therefore $P^* \setminus \{x\} \subset P$, and hence clearly $(P \setminus \mathrm{G}(\sigma)) \cup \{x\} = P^*$. q.e.d.

We are now ready to prove Theorem 1.4.3.

Proof of Theorem 1.4.3. We put

$$\mathcal{P} := \left\{ P^* \in \mathrm{PC}(\Sigma^*_{(\sigma,x)}) \;\middle|\; x \notin P^* \right\}, \ \mathcal{P}' := \mathrm{PC}(\Sigma^*_{(\sigma,x)}) \setminus \mathcal{P},$$

$$\mathcal{S} := \{P \in \mathrm{PC}(\Sigma) \mid \mathrm{G}(\sigma) \not\subset P\} \cup \{\mathrm{G}(\sigma)\},$$

and let $\mathcal{T}$ be the set of minimal elements of

$$\{(P \setminus \mathrm{G}(\sigma)) \cup \{x\} \mid P \in \mathrm{PC}(\Sigma), \ P \cap \mathrm{G}(\sigma) \neq \emptyset\}.$$

Then to prove the theorem, we have only to prove $\mathcal{P} = \mathcal{S}$ and $\mathcal{P}' = \mathcal{T}$.

"$\mathcal{P} = \mathcal{S}$" Clearly, we have $\mathrm{G}(\sigma) \in \mathcal{P}$. Let $P \in \mathrm{PC}(\Sigma)$, $\mathrm{G}(\sigma) \not\subset P$. Then for any $y \in P$, $P \setminus \{y\}$ generates a strongly convex rational polyhedral cone in $\Sigma^*_{(\sigma,x)}$ because $\mathrm{G}(\sigma) \not\subset P \setminus \{y\}$. On the other hand, since $x \notin P$, P does not generate a strongly convex rational polyhedral cone in $\Sigma^*_{(\sigma,x)}$. So we have $P \in \mathcal{P}$. Conversely, let $P^* \in \mathcal{P}$. If $\mathrm{G}(\sigma) \subset P^*$, then $P^* = \mathrm{G}(\sigma) \in \mathcal{S}$, since $\mathrm{G}(\sigma) \in \mathcal{P}$. If $\mathrm{G}(\sigma) \not\subset P^*$, then for any $y \in P^*$, $P^* \setminus \{y\}$ generates a strongly convex rational polyhedral cone in Σ because $x \notin P^*$. Clearly, P^* does not generate a strongly convex rational polyhedral cone in Σ. Therefore $P^* \in \mathrm{PC}(\Sigma)$ and we have $P^* \in \mathcal{S}$.

"$\mathcal{P}' = \mathcal{T}$" Let $(P \setminus \mathrm{G}(\sigma)) \cup \{x\} \in \mathcal{T}$ $(P \in \mathrm{PC}(\Sigma))$ and suppose that $(P \setminus \mathrm{G}(\sigma)) \cup \{x\}$ generates a strongly convex rational polyhedral cone in $\Sigma^*_{(\sigma,x)}$. Then there exists $\tau' \in \Sigma$ such that

$$\mathrm{Cone}\,((P \setminus \mathrm{G}(\sigma)) \cup \{x\}) \prec \sigma_i + \tau' \in \Sigma^*_{(\sigma,x)} \ (1 \leq \forall i \leq l),\ \sigma \cap \tau' = \{0\}.$$

Since $P \setminus \mathrm{G}(\sigma) \subset \mathrm{G}(\tau')$, we have $P \subset \mathrm{G}(\sigma + \tau')$ $(\sigma + \tau' \in \Sigma)$, a contradiction to $P \in \mathrm{PC}(\Sigma)$. Therefore $(P \setminus \mathrm{G}(\sigma)) \cup \{x\}$ contains a primitive collection of $\Sigma^*_{(\sigma,x)}$. So let $P^* \subset (P \setminus \mathrm{G}(\sigma)) \cup \{x\}$, $P^* \in \mathrm{PC}(\Sigma^*_{(\sigma,x)})$. Then $x \in P^*$ because $P \setminus \mathrm{G}(\sigma)$ generates a strongly convex rational polyhedral cone in $\Sigma^*_{(\sigma,x)}$. So by Lemma 1.4.6, there exists $P' \in \mathrm{PC}(\Sigma)$ such that $P^* = (P' \setminus \mathrm{G}(\sigma)) \cup \{x\}$. Since $(P' \setminus \mathrm{G}(\sigma)) \cup \{x\} = P^* \subset (P \setminus \mathrm{G}(\sigma)) \cup \{x\}$, we have $(P' \setminus \mathrm{G}(\sigma)) \cup \{x\} = (P \setminus \mathrm{G}(\sigma)) \cup \{x\}$ by minimality. Therefore $(P \setminus \mathrm{G}(\sigma)) \cup \{x\} = P^* \in \mathrm{PC}(\Sigma^*_{(\sigma,x)})$. Conversely, let $P^* \in \mathrm{PC}(\Sigma^*_{(\sigma,x)})$, $x \in P^*$. Then by Lemma 1.4.6, P^* is clearly expressed in the form as stated. q.e.d.

By using Theorem 1.4.3, we can construct a nonsingular toric Fano 4-fold which is missing in the table of Batyrev [6].

Example 1.4.7 Let $d = 4$, Σ a fan in N corresponding to $\mathbf{P}^2 \times \mathbf{P}^2$ and $\mathrm{G}(\Sigma) = \{x_1, \ldots, x_6\}$. Then the primitive relations of Σ are

$$x_1 + x_2 + x_3 = 0,\ x_4 + x_5 + x_6 = 0.$$

We get a nonsingular toric Fano 4-fold W by equivariant blow-ups of $\mathbf{P}^2 \times \mathbf{P}^2$ along three T_N-invariant 2-dimensional irreducible closed subvarieties

$$\overline{\mathrm{orb}\,(\{x_1, x_4\})},\ \overline{\mathrm{orb}\,(\{x_2, x_5\})},\ \overline{\mathrm{orb}\,(\{x_3, x_6\})}.$$

Let Σ_W be the fan in N corresponding to W and $\mathrm{G}(\Sigma_W) = \mathrm{G}(\Sigma) \cup \{x_7, x_8, x_9\}$. Then the primitive relations of Σ_W are

$$x_1 + x_4 = x_7,\ x_2 + x_5 = x_8,\ x_3 + x_6 = x_9,$$

$$x_1 + x_2 + x_3 = 0,\ x_4 + x_5 + x_6 = 0,\ x_7 + x_8 + x_9 = 0,$$

$$x_1 + x_2 + x_9 = x_6,\ x_4 + x_5 + x_9 = x_3,\ x_1 + x_3 + x_8 = x_5,$$

$$x_4 + x_6 + x_8 = x_2,\ x_2 + x_3 + x_7 = x_4,\ x_5 + x_6 + x_7 = x_1,$$

$$x_1 + x_8 + x_9 = x_5 + x_6,\ x_4 + x_8 + x_9 = x_2 + x_3,\ x_2 + x_7 + x_9 = x_4 + x_6,$$

$$x_5 + x_7 + x_9 = x_1 + x_3,\ x_3 + x_7 + x_8 = x_4 + x_5,\ x_6 + x_7 + x_8 = x_1 + x_2.$$

This is easily confirmed by Theorem 1.4.3. W is missing in the table of Batyrev [6].

By Theorem 1.4.3, we get a way to calculate $\mathrm{PC}(\Sigma^*_{(\sigma,x)})$ from $\mathrm{PC}(\Sigma)$. Conversely, the following easy lemma enables us to calculate $\mathrm{PC}(\Sigma)$ from $\mathrm{PC}(\Sigma^*_{(\sigma,x)})$.

Lemma 1.4.8 *Let Σ be a complete simplicial fan in N, $\sigma \in \Sigma$, and $x \in (\mathrm{Relint}(\sigma)) \cap N$ which generates the semigroup $(\mathbf{R}_{\geq 0}x) \cap N$. If $P \in \mathrm{PC}(\Sigma)$ and $\mathrm{G}(\sigma) \subset P$, then $(P \setminus \mathrm{G}(\sigma)) \cup \{x\} \in \mathrm{PC}(\Sigma^*_{(\sigma,x)})$.*

Proof. We have only to prove that $(P \setminus \mathrm{G}(\sigma)) \cup \{x\}$ is a minimal element in $\{(P' \setminus \mathrm{G}(\sigma)) \cup \{x\} \mid P' \in \mathrm{PC}(\Sigma),\ P' \cap \mathrm{G}(\sigma) \neq \emptyset\}$. Suppose there exists $P' \in \mathrm{PC}(\Sigma)$ such that

$$P' \setminus \mathrm{G}(\sigma) \subset P \setminus \mathrm{G}(\sigma),\ P' \cap \mathrm{G}(\sigma) \neq \emptyset.$$

Since $\mathrm{G}(\sigma) \subset P$, we have $P' \subset P$, hence $P = P'$ because $P,\ P' \in \mathrm{PC}(\Sigma)$. Therefore P is a minimal element. q.e.d.

Corollary 1.4.9 *Let Σ be a finite complete simplicial fan in N, $\sigma \in \Sigma$, and $x \in N \cap (\mathrm{Relint}(\sigma))$ which generates the semigroup $(\mathbf{R}_{\geq 0}x) \cap N$. Then the primitive collections of Σ are*

(1) *$P^* \in \mathrm{PC}(\Sigma^*_{(\sigma,x)})$ such that $P^* \neq \mathrm{G}(\sigma)$, $x \notin P^*$, and*

(2) *$(P^* \setminus \{x\}) \cup \mathrm{G}(\sigma)$, where $P^* \in \mathrm{PC}(\Sigma^*_{(\sigma,x)})$ such that $x \in P^*$ and $(P^* \setminus \{x\}) \cup S \notin \mathrm{PC}(\Sigma^*_{(\sigma,x)})$ for any subset $S \subset \mathrm{G}(\sigma)$.*

This immediately follows from Theorem 1.4.3 and Lemma 1.4.8.

We close this section by giving an easy criterion for the possibility of equivariant blow-down in the nonsingular case.

Theorem 1.4.10 *Let Σ^* be a finite complete* nonsingular *fan in N. Then the following are equivalent.*

(1) *There exist a complete nonsingular toric variety X and an equivariant blow-up $\varphi : T_N\mathrm{emb}(\Sigma^*) \to X$ along a T_N-invariant closed irreducible subvariety of X.*

(2) *There exists* $P^* \in \mathrm{PC}(\Sigma^*)$ *such that the corresponding primitive relation is*

$$x_1 + \cdots + x_l = x, \ P^* = \{x_1, \ldots, x_l\}, \ \textit{for some } x \in \mathrm{G}(\Sigma^*)$$

and for any $\sigma^* \in \Sigma^*$ *which contains* x*, each of*

$$(\mathrm{G}(\sigma^*) \cup P^*) \setminus \{x_i\} \qquad (1 \leq i \leq l)$$

generates a strongly convex rational polyhedral cone in Σ^**.*

(3) *There exists* $P^* \in \mathrm{PC}(\Sigma^*)$ *such that the corresponding primitive relation is*

$$x_1 + \cdots + x_l = x, \ P^* = \{x_1, \ldots, x_l\}, \ \textit{for some } x \in \mathrm{G}(\Sigma^*)$$

and for any $P' \in \mathrm{PC}(\Sigma^*)$ *which satisfies the conditions* $P^* \cap P' \neq \emptyset$ *and* $P^* \neq P'$*,*

$$(P' \setminus P^*) \cup \{x\}$$

contains a primitive collection of Σ^**.*

Proof. We prove (1) $\Longrightarrow$ (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1).

(1) $\Longrightarrow$ (3) is trivial by Theorem 1.4.3.

(3) $\Longrightarrow$ (2). Suppose that there exists $\sigma^* \in \Sigma^*$ such that $x \in \sigma^*$ and

$$(\mathrm{G}(\sigma^*) \cup P^*) \setminus \{x_i\} \text{ for some } i \ (1 \leq i \leq l)$$

does not generate a strongly convex rational polyhedral cone in Σ^*. Then $(\mathrm{G}(\sigma^*) \cup P^*) \setminus \{x_i\}$ contains a primitive collection $P' \in \mathrm{PC}(\Sigma^*)$. Since $P^* \cap P' \neq \emptyset$ and $P^* \neq P'$, by (3),

$$(P' \setminus P^*) \cup \{x\} \subset \mathrm{G}(\sigma^*)$$

contains a primitive collection of Σ^*, a contradiction.

(2) $\Longrightarrow$ (1). For any $\sigma^* \in \Sigma^*$ which contains x, define a strongly convex rational polyhedral cone σ' in $N_{\mathbf{R}}$ by

$$\sigma' := \mathrm{Cone}\left((\mathrm{G}(\sigma^*) \cup P^*) \setminus \{x\}\right).$$

Then the finite complete nonsingular fan Σ in N defined by

$$\Sigma := (\Sigma^* \setminus \{\sigma^* \in \Sigma^* \,|x \in \sigma^*\}) \cup \{\sigma' \text{ and the faces of } \sigma' \,|\sigma^* \in \Sigma^*, \ x \in \sigma^*\}$$

gives a complete nonsingular toric variety $X = T_N\mathrm{emb}(\Sigma)$ and an equivariant blow-up $\varphi : T_N\mathrm{emb}(\Sigma) \to X$. q.e.d.

The equivalence (1) $\Longleftrightarrow$ (3) is a useful criterion for the possibility of equivariant blow-down in the nonsingular case.

1.5 Decomposition of birational morphisms

In this section, we prove a toric version of a theorem of Mori which claims that "a proper birational morphism between nonsingular Fano 3-folds is always decomposed into a composite of blow-ups", and study the higher-dimensional version. In the proof of the theorem, the results of Sections 1.3 and 1.4 are used.

The following proposition is in essential use in the proof of the main theorem of this section.

Proposition 1.5.1 *Let $X := T_N\mathrm{emb}(\Sigma)$ be a nonsingular toric Fano d-fold (resp. $-K_X$ is nef), $x_1 + \cdots + x_l = x$ a primitive relation of Σ and $\varphi : X \to X' := T_N\mathrm{emb}(\Sigma')$ the equivariant blow-down with respect to $x_1 + \cdots + x_l = x$. Then X' is not a nonsingular toric Fano d-fold (resp. $-K_{X'}$ is not nef) if and only if there exists a primitive relation of Σ of the form*

$$y_1 + \cdots + y_m = a_1 z_1 + \cdots + a_n z_n + bx + c_1 x_1 + \cdots + c_{l-1} x_{l-1}$$

up to change of the indices, such that

(1) $a_1, \ldots, a_n, b > 0,\ c_1, \ldots, c_{l-1} \geq 0,$

(2) $m - (a_1 + \cdots + a_n + b + c_1 + \cdots + c_{l-1}) > 0$ *(resp.* ≥ 0*),*

(3) $m - (a_1 + \cdots + a_n + bl + c_1 + \cdots + c_{l-1}) \leq 0$ *(resp.* < 0*) and*

(4) $m + n + l \leq d + 1.$

Proof. The sufficiency is trivial by Theorem 1.3.10.

By Corollary 1.4.9, for any new primitive collection $P' \in \mathrm{PC}(\Sigma')$ added by the equivariant blow-down with respect to $x_1 + \cdots + x_l = x$, there exists

$$P = \{u_1, \ldots, u_r, x\} \in \mathrm{PC}(\Sigma)$$

such that $P' = \{u_1, \ldots, u_r, x_1, \ldots, x_l\}$. Let the primitive relation corresponding to P be

$$u_1 + \cdots + u_r + x = h_1 v_1 + \cdots + h_s v_s.$$

Then $\mathrm{Cone}(\{v_1, \ldots, v_s\}) \in \Sigma'$ because $x \notin \{v_1, \ldots, v_s\}$. So the primitive relation corresponding to P' is

$$u_1 + \cdots + u_r + x_1 + \cdots + x_l = h_1 v_1 + \cdots + h_s v_s.$$

Therefore $\deg P' = r + l - (h_1 + \cdots + h_s) > r + 1 - (h_1 + \cdots + h_s) = \deg P > 0$.

By the above discussion, if X' is not a nonsingular toric Fano d-fold, then there exists a primitive collection P in $\mathrm{PC}(\Sigma)$ such that P is in $\mathrm{PC}(\Sigma')$, its primitive relation contains x on the right-hand side and $r(P)$ is contained in an extremal ray of $\mathbf{NE}(X')$. So we get the conditions (1) and (4). Since X is a Fano variety while X' is not a Fano variety, we get the conditions (2) and (3). q.e.d.

Example 1.5.2 We consider Proposition 1.5.1 in the case of the equivariant blow-down $\varphi : X \to X'$ with respect to the primitive relation of Σ of the form $x_1 + x_2 = x$.

(1) "$d = 2$" X' is always a nonsingular toric Fano surface. On the other hand, if $-K_X$ is nef, then $-K_{X'}$ is always nef.

(2) "$d = 3$" X' is not a nonsingular toric Fano 3-fold if and only if there exists the following primitive relation of Σ.

$$y_1 + y_2 = x \quad (\{y_1, y_2\} \cap \{x_1, x_2\} = \emptyset)\,.$$

(3) "$d = 4$" X' is not a nonsingular toric Fano 4-fold if and only if there exists one of the following primitive relations of Σ.

$$y_1 + y_2 = x,\ y_1 + y_2 + y_3 = 2x,\ y_1 + y_2 + y_3 = x + x_1 \quad (\{y_1, y_2, y_3\} \cap \{x_1, x_2\} = \emptyset)\,.$$

Next, let $d = 3$ and let $\varphi : X \to X'$ be the equivariant blow-down with respect to the primitive relation of Σ, $x_1 + x_2 + x_3 = x$. Then X' is always a nonsingular toric Fano 3-fold by Proposition 1.5.1. We need these facts later.

The following is the toric version of the Mori theory.

Proposition 1.5.3 (Reid [30]) *Let Σ be a finite complete nonsingular fan in N, $X := T_N\mathrm{emb}(\Sigma)$ a projective toric variety, and $P = \{x_1, \ldots, x_l\} \in \mathrm{PC}(\Sigma)$ with the primitive relation corresponding to P being $x_1 + \cdots + x_l = a_1 y_1 + \cdots + a_m y_m$. If $r(P)$ is contained in an extremal ray of $\mathbf{NE}(X)$ and $m \geq 1$, then there exist a nonsingular projective toric d-fold X' and an equivariant morphism*

$$\mathrm{Cont}_P : X \longrightarrow X'$$

such that the following are satisfied:

(1) *For any* $\tau \in \Sigma$, *the image of* $\overline{\mathrm{orb}(\tau)}$ *by* Cont_P *is a point if and only if* $v(\tau) = r(P) \in A_1(X)$.

(2) *Let* Σ' *be a fan in* N *such that* $X' = T_N\mathrm{emb}(\Sigma')$. *If* $m = 1$, *then* Σ' *is simplicial and*

$$\sigma' = \mathrm{Cone}\,(\{x_1, \ldots, x_l\}) \in \Sigma',\ \mathrm{G}(\Sigma') = \mathrm{G}(\Sigma) \setminus \{y_1\}.$$

Moreover, $\Sigma = (\Sigma')^*_{(\sigma',x)}$, *where* $x := (x_1 + \cdots + x_l)/a_1$. *In particular, if* $a_1 = 1$, *then* X' *is nonsingular and* Cont_P *is an equivariant blow-up.*

To prove the main theorem of this section, we suppose $d = 3$. Let $\varphi : Y \longrightarrow X$ be an equivariant morphism between nonsingular toric Fano 3-folds, and Σ and $\widetilde{\Sigma}$ fans in N such that $X = T_N\mathrm{emb}(\Sigma)$ and $Y = T_N\mathrm{emb}(\widetilde{\Sigma})$. To apply Propositions 1.5.1 and 1.5.3, we have to investigate the subdivision of a 3-dimensional strongly convex rational polyhedral cone in Σ. The following lemma is fundamental in classifying subdivisions.

Lemma 1.5.4 *Let* $d = \mathrm{rank}\ N = 3$, Σ *and* $\widetilde{\Sigma}$ *finite complete nonsingular fans in* N *and* $\varphi : T_N\mathrm{emb}(\widetilde{\Sigma}) \longrightarrow T_N\mathrm{emb}(\Sigma)$ *an equivariant morphism. For any* $\sigma \in \Sigma(3)$ *for which* $\mathrm{G}(\sigma) = \{x_1, x_2, x_3\}$, *let* $\widetilde{\sigma}$ *be the unique strongly convex rational polyhedral cone in* $\widetilde{\Sigma} \setminus \{0\}$ *such that* $x_1 + x_2 + x_3 \in \mathrm{Relint}(\widetilde{\sigma})$. *Then we have the following*:

(1) $\dim \widetilde{\sigma} = 3 \Longleftrightarrow \sigma = \widetilde{\sigma} \in \widetilde{\Sigma}$.

(2) $\dim \widetilde{\sigma} = 2 \Longleftrightarrow \mathrm{G}(\widetilde{\sigma}) = \{x, x_3\}$ *where* $x := x_1 + x_2$ *up to change of the indices.*

(3) $\dim \widetilde{\sigma} = 1 \Longleftrightarrow \mathrm{G}(\widetilde{\sigma}) = \{x\}$ *where* $x := x_1 + x_2 + x_3$.

Proof. The sufficiency is trivial. Let $s = \dim \widetilde{\sigma}$ and $\mathrm{G}(\widetilde{\sigma}) = \{y_1, \ldots, y_s\}$. Then $\widetilde{\sigma} \subset \sigma$, since φ is an equivariant morphism, and hence we have

$$y_i = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3\ (1 \leq i \leq s),\ a_{ij} \in \mathbf{Z}_{\geq 0}\ (1 \leq i \leq s,\ 1 \leq j \leq 3).$$

If we put $x_1 + x_2 + x_3 = b_1y_1 + \cdots + b_sy_s$ $(b_1, \ldots, b_s \in \mathbf{Z}_{>0})$, then $b_1 = \cdots = b_s = 1$ because a_{ij} $(1 \leq i \leq s,\ 1 \leq j \leq 3)$ are nonnegative. q.e.d.

Now we are ready to classify the subdivisions of a 3-dimensional strongly convex rational polyhedral cone $\sigma \in \Sigma(3)$. There are five types of subdivisions for σ. Let $\mathrm{G}(\sigma) = \{x_1, x_2, x_3\}$.

(1) "$\dim \widetilde{\sigma} = 3$" $\sigma = \widetilde{\sigma} \in \widetilde{\Sigma}(3)$ by Lemma 1.5.4.

(2) "$\dim \widetilde{\sigma} = 2$" By Lemma 1.5.4, we have $x_1+x_2+x_3 \in \mathrm{Cone}\left(\{x_3, x_4\}\right) \in \widetilde{\Sigma}(2)$, where $x_4 := x_1 + x_2 \in \mathrm{G}(\widetilde{\Sigma})$. Then $\{x_1, x_2\} \in \mathrm{PC}(\widetilde{\Sigma})$ and $r(\{x_1, x_2\})$ is contained in an extremal ray of $\mathbf{NE}(Y)$, since $\deg\left(\{x_1, x_2\}\right) = 1$. So $\sigma_1 := \mathrm{Cone}\left(\{x_1, x_3, x_4\}\right),\ \sigma_2 := \mathrm{Cone}\left(\{x_2, x_3, x_4\}\right)$ are in $\widetilde{\Sigma}(3)$ and $\sigma = \sigma_1 \cup \sigma_2$ by Theorem 1.4.10 (2) and Proposition 1.5.3.

(3) "$\dim \widetilde{\sigma} = 1$ and $\{x_1, x_2\} \in \mathrm{PC}(\widetilde{\Sigma})$" Let $x_4 := x_1 + x_2 + x_3 \in \mathrm{G}(\widetilde{\Sigma})$ and $x_5 := x_1 + x_2$. Then $x_5 \in \mathrm{G}(\widetilde{\Sigma})$ and the primitive relation corresponding to $\{x_1, x_2\}$ is $x_1 + x_2 = x_5$. Since $x_3 + x_5 = x_4$, we have $\{x_3, x_5\} \in \mathrm{PC}(\widetilde{\Sigma})$ and $x_3 + x_5 = x_4$ is the corresponding primitive relation. Hence, since $r(\{x_1, x_2\})$ and $r(\{x_3, x_5\})$ are contained in an extremal ray of $\mathbf{NE}(Y)$, we see that $\sigma_1 := \mathrm{Cone}\left(\{x_1, x_3, x_4\}\right),\ \sigma_2 := \mathrm{Cone}\left(\{x_1, x_4, x_5\}\right),\ \sigma_3 := \mathrm{Cone}\left(\{x_2, x_3, x_4\}\right),\ \sigma_4 := \mathrm{Cone}\left(\{x_2, x_4, x_5\}\right)$ are in $\widetilde{\Sigma}(3)$ and $\sigma = \sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \sigma_4$ for the same reason as above.

(4) "$\dim \widetilde{\sigma} = 1$, $\{x_1, x_2, x_3\} \in \mathrm{PC}(\widetilde{\Sigma})$ and $r\left(\{x_1, x_2, x_3\}\right)$ is contained in an extremal ray of $\mathbf{NE}(Y)$" Let $x_4 := x_1 + x_2 + x_3$. Then by Proposition 1.5.3, we have $\sigma_1 := \mathrm{Cone}\left(\{x_1, x_2, x_4\}\right),\ \sigma_2 := \mathrm{Cone}\left(\{x_2, x_3, x_4\}\right),\ \sigma_3 := \mathrm{Cone}\left(\{x_1, x_3, x_4\}\right)$ are in $\widetilde{\Sigma}(3)$ and $\sigma = \sigma_1 \cup \sigma_2 \cup \sigma_3$.

(5) "$\dim \widetilde{\sigma} = 1$, $\{x_1, x_2, x_3\} \in \mathrm{PC}(\widetilde{\Sigma})$ and $r\left(\{x_1, x_2, x_3\}\right)$ is not contained in an extremal ray of $\mathbf{NE}(Y)$" Let $x_4 := x_1 + x_2 + x_3$. Then the primitive relation corresponding to $\{x_1, x_2, x_3\}$ is $x_1 + x_2 + x_3 = x_4$ and so $\deg\left(\{x_1, x_2, x_3\}\right) = 2$. Therefore there exist two primitive collections $P_1,\ P_2 \in \mathrm{PC}(\widetilde{\Sigma})$ such that $\deg P_1 = \deg P_2 = 1$ and $r\left(\{x_1, x_2, x_3\}\right) = r(P_1) + r(P_2)$. On the other hand, there are two types of primitive relations corresponding to the primitive collection P such that $\deg P = 1$ and $r(P)$ is contained in an extremal ray. The possibilities are

$$(a)\ z_1 + z_2 + z_3 = 2z_4,\ (b)\ w_1 + w_2 = w_3.$$

By easy calculation, the combinations $((a),(a))$ and $((b),(b))$ are impossible. In the case of the combination $((a),(b))$, we have $z_4 = w_1 = x_4,\ w_3 = z_1,\ w_2 = x_1,\ z_2 = x_2$ and $z_3 = x_3$. Then putting $x_5 := z_1$, we have $\sigma_1 := \mathrm{Cone}\left(\{x_1, x_2, x_5\}\right),\ \sigma_2 := \mathrm{Cone}\left(\{x_2, x_4, x_5\}\right),\ \sigma_3 := \mathrm{Cone}\left(\{x_1, x_3, x_5\}\right),\ \sigma_4 := \mathrm{Cone}\left(\{x_3, x_4, x_5\}\right),\ \sigma_5 :=$

Cone $(\{x_2, x_3, x_4\})$ are in $\widetilde{\Sigma}(3)$ and $\sigma = \sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \sigma_4 \cup \sigma_5$ for the same reason as in (2).

By the above classification, we get the following main theorem in this section. This is a toric version of a theorem of Mori.

Theorem 1.5.5 *Let X and Y be nonsingular toric Fano 3-folds, and $\varphi : Y \longrightarrow X$ an equivariant morphism. Then we have a decomposition of φ*

$$Y = X_r \xrightarrow{\varphi_r} X_{r-1} \xrightarrow{\varphi_{r-1}} \cdots\cdots \xrightarrow{\varphi_2} X_1 \xrightarrow{\varphi_1} X_0 = X,$$

where X_i $(0 \leq i \leq r)$ is a nonsingular toric Fano 3-fold, φ_j $(2 \leq j \leq r)$ is an equivariant blow-up along a T_N-invariant 1-dimensional irreducible closed subvariety of X_{j-1} and φ_1 is an equivariant blow-up along some T_N-invariant points of X.

Proof. In the above classification, carry out equivariant blow-downs in the order (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1), (2) $\Longrightarrow$ (1), (5) $\Longrightarrow$ (4) $\Longrightarrow$ (1) and (4) $\Longrightarrow$ (1). Then by Proposition 1.5.1 and Example 1.5.2, we get a decomposition as in the statement. q.e.d.

If $d \geq 4$, the method we employed in the 3-dimensional case is insufficient. For example, in the case of $d = 4$, there is a subdivision of a 4-dimensional strongly convex rational polyhedral cone $\sigma \in \Sigma(4)$ such that the primitive relations corresponding to $\left\{P \in \mathrm{PC}(\widetilde{\Sigma}) \mid P \subset \sigma\right\} \subset \mathrm{PC}(\widetilde{\Sigma})$ are

$$x_1 + x_2 + x_3 = x_5,\ x_2 + x_4 = x_6 \text{ and } x_1 + x_3 + x_6 = x_4 + x_5,$$

where $\mathrm{G}(\sigma) = \{x_1, x_2, x_3, x_4\},\ x_5,\ x_6 \in \mathrm{G}(\widetilde{\Sigma})$. This does not contradict the fact that Y is a nonsingular toric Fano variety, but we cannot decide by Proposition 1.5.1 whether the equivariant blow-down of Y with respect to the primitive relation $x_2 + x_4 = x_6$ is also a nonsingular toric Fano variety or not. However, there is still a possibility of the decomposition similar to that in Theorem 1.5.5 in the case $d \geq 4$.

Conjecture 1.5.6 *Let X and Y be nonsingular toric Fano d-folds, and $\varphi : Y \longrightarrow X$ an equivariant morphism. Then we have a decomposition of φ*

$$Y = X_r \xrightarrow{\varphi_r} X_{r-1} \xrightarrow{\varphi_{r-1}} \cdots\cdots \xrightarrow{\varphi_2} X_1 \xrightarrow{\varphi_1} X_0 = X,$$

where X_i $(0 \leq i \leq r)$ is a nonsingular toric Fano d-fold, and each of φ_j $(1 \leq j \leq r)$ is an equivariant blow-up along a T_N-invariant irreducible closed subvariety of X_{j-1}.

1.6 Program for the classification of toric Fano varieties

In this section, we describe a program for the classification of nonsingular toric Fano varieties. This program can be extended to the case of Gorenstein toric Fano varieties endowed with natural resolution of singularities.

First, we consider the classification of nonsingular toric Fano d-folds. We define the F-equivalence relation again. Let

$$\mathcal{F}_d := \{\text{nonsingular toric Fano } d\text{-folds}\} \,/ \cong .$$

Definition 1.6.1 X_1 and X_2 in $\mathcal{F}_d$ are said to be *F-equivalent* if there exists a sequence of equivariant blow-ups and blow-downs from X_1 to X_2 through toric *Fano* d-folds, namely there exist nonsingular toric Fano d-folds $Y_0 = X_1, Y_1, \ldots, Y_{2l} = X_2$ together with finite successions $Y_j \to Y_{j-1}$ and $Y_j \to Y_{j+1}$, for each odd $1 \leq j \leq 2l-1$, of equivariant blow-ups through nonsingular toric Fano d-folds. We denote the relation by $X_1 \overset{\mathrm{F}}{\sim} X_2$. Then "$\overset{\mathrm{F}}{\sim}$" is obviously an equivalence relation.

In order to classify nonsingular toric Fano d-folds, it follows from Proposition 1.3.6, Theorems 1.3.10, 1.4.3, 1.4.10 and Corollory 1.4.9 that we have only to solve the following problem.

Problem 1.6.2 *Obtain a complete system of representatives for* $(\mathcal{F}_d, \overset{\mathrm{F}}{\sim})$.

In regard to Problem 1.6.2, we propose the following conjecture.

Conjecture 1.6.3 *Any nonsingular toric Fano d-fold is either pseudo-symmetric or F-equivalent to the d-dimensional projective space* $\mathbf{P}^d$*, where a nonsingular toric Fano d-fold* $T_N\mathrm{emb}(\Sigma)$ *is said to be* pseudo-symmetric *if there exist two d-dimensional strongly convex rational polyhedral cones* σ*,* $\sigma' \in \Sigma(d)$ *such that* $\sigma = -\sigma' := \{-x \in N_{\mathbf{R}} \,|\, x \in \sigma'\}$.

If Conjecture 1.6.3 is true, we obtain a complete system of representatives for $(\mathcal{F}_d, \overset{\mathrm{F}}{\sim})$, since pseudo-symmetric ones are already completely classified in the following fashion.

Definition 1.6.4 Let $k \in \mathbf{Z}_{>0}$, $d = 2k$ and $\{e_1, \ldots, e_d\}$ a basis of N. The $2k$-dimensional *del Pezzo variety* V^{2k} is the nonsingular toric Fano $2k$-fold corresponding to the Fano polytope in $N_{\mathbf{R}}$ defined by

$$\mathrm{Conv}\left(\{e_1, \ldots, e_d, -e_1, \ldots, -e_d, e_1 + \cdots + e_d, -(e_1 + \cdots + e_d)\}\right),$$

while the $2k$-dimensional *pseudo del Pezzo variety* $\widetilde{V}^{2k}$ is the nonsingular toric Fano $2k$-fold corresponding to the Fano polytope in $N_{\mathbf{R}}$ defined by

$$\mathrm{Conv}\left(\{e_1, \ldots, e_d, -e_1, \ldots, -e_d, e_1 + \cdots + e_d\}\right).$$

Remark 1.6.5 In the table obtained in Section 1.10, (117) is the 4-dimensional pseudo del Pezzo variety $\widetilde{V}^4$, while (118) is the 4-dimensional del Pezzo variety V^4.

Theorem 1.6.6 (Ewald [8], Voskresenskij-Klyachko [35]) *For any pseudo symmetric toric Fano variety X, there exist $s, m, n \in \mathbf{Z}_{\geq 0}$, $k_1, \ldots, k_m, l_1, \ldots l_n \in \mathbf{Z}_{>0}$ such that*

$$X \cong (\mathbf{P}^1)^s \times V^{2k_1} \times \cdots \times V^{2k_m} \times \widetilde{V}^{2l_1} \times \cdots \times \widetilde{V}^{2l_n},$$

where V^{2k_i} is the $2k_i$-dimensional del Pezzo variety, while $\widetilde{V}^{2l_j}$ is the $2l_j$-dimensional psudo del Pezzo variety for $1 \leq i \leq m$, $1 \leq j \leq n$.

Conjecture 1.6.3 is very hard to deal with in general. So we investigate Conjecture 1.6.3 in a certain special class of nonsingular toric Fano d-folds.

Theorem 1.6.7 *Let $r, a_1, \ldots, a_r$ in $\mathbf{Z}_{>0}$ and $a_1 + \cdots + a_r = d$. Then we have*

$$\mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_r} \overset{\mathrm{F}}{\sim} \mathbf{P}^d.$$

Proof. We are going to prove this by induction on d.

Let Σ be a fan in N corresponding to the d-dimensional projective space and $\mathrm{G}(\Sigma) = \{x_1, \ldots, x_{d+1}\}$. Then the primitive relation is

$$x_1 + \cdots + x_{d+1} = 0.$$

By the equivariant blow-up along $\{x_1, \ldots, x_{a_1+1}\}$ for $1 \leq a_1 < d$, we get a fan Σ_1 in N whose primitive relations are

$$x_1 + \cdots + x_{a_1+1} = x_{d+2},\ x_{a_1+2} + \cdots + x_{d+2} = 0,$$

where $\mathrm{G}(\Sigma_1) = \mathrm{G}(\Sigma) \cup \{x_{d+2}\}$. Moreover, by the equivariant blow-up of Σ_1 along $\{x_1, x_{a_1+2}, \ldots, x_{d+1}\}$, we get a fan Σ_2 in N whose primitive relations are

$$x_1 + x_{a_1+2} + \cdots + x_{d+1} = x_{d+3},\ x_2 + \cdots + x_{a_1+1} + x_{d+3} = 0,\ x_{d+2} + x_{d+3} = x_1,$$

$$x_1 + \cdots + x_{a_1+1} = x_{d+2},\ x_{a_1+2} + \cdots + x_{d+2} = 0,$$

where $\mathrm{G}(\Sigma_2) = \mathrm{G}(\Sigma_1) \cup \{x_{d+3}\}$. Then $T_N\mathrm{emb}(\Sigma_1)$ and $T_N\mathrm{emb}(\Sigma_2)$ are nonsingular toric Fano d-folds by Theorem 1.3.10. By Theorem 1.4.10, Σ_2 can be equivariantly blown-down to a fan Σ' in N with respect to the primitive relation $x_{d+2} + x_{d+3} = x_1$. The primitive relations of Σ' are

$$x_2 + \cdots + x_{a_1+1} + x_{d+3} = 0,\ x_{a_1+2} + \cdots + x_{d+2} = 0,$$

where $\mathrm{G}(\Sigma') = \{x_2, \ldots, x_{d+3}\}$. So the toric variety corresponding to Σ' is isomorphic to $\mathbf{P}^{a_1} \times \mathbf{P}^{d-a_1}$, and we have

$$\mathbf{P}^d \overset{\mathrm{F}}{\sim} \mathbf{P}^{a_1} \times \mathbf{P}^{d-a_1}.$$

Then by the induction assumption, we have

$$\mathbf{P}^{d-a_1} \overset{\mathrm{F}}{\sim} \mathbf{P}^{a_2} \times \cdots \times \mathbf{P}^{a_r}.$$

q.e.d.

Next, we consider more complicated nonsingular toric Fano d-folds.

Definition 1.6.8 (Batyrev [5]) Let Σ be a finite complete nonsingular fan in N. Then Σ is called a *splitting fan* if for any two distinct primitive collections P_1 and P_2 in $\mathrm{PC}(\Sigma)$, we have $P_1 \cap P_2 = \emptyset$.

The following is well-known.

Theorem 1.6.9 (Kleinschmidt [15]) *Let Σ be a finite complete nonsingular fan in N and $X := T_N\mathrm{emb}(\Sigma)$. If the Picard number of X is two or three, then X is projective. Moreover, if the Picard number of X is two, then Σ is a splitting fan.*

The nonsingular toric d-folds corresponding to splitting fans are characterized by the following proposition.

Proposition 1.6.10 (Batyrev [5]) *Let Σ be a finite complete nonsingular fan in N. Then Σ is a splitting fan if and only if there exist toric manifolds $X_0, \dots, X_r$ such that X_0 is a projective space, $X_r = T_N\mathrm{emb}(\Sigma)$ and for $1 \leq i \leq r$, X_i is an equivariant projective space bundle over X_{i-1}.*

For any splitting fan Σ in N, $T_N\mathrm{emb}(\Sigma)$ is projective by Proposition 1.6.10. So the assumption in the following lemma is satisfied.

Lemma 1.6.11 (Batyrev [5]) *Let Σ be a finite complete nonsingular fan in N such that $T_N\mathrm{emb}(\Sigma)$ is projective. Then there exists a primitive collection P in $\mathrm{PC}(\Sigma)$ such that $\sigma(P) = 0$.*

Theorem 1.6.12 *Let Σ be a splitting fan in N and let $P = \{x_1, \dots, x_r\}$ be a primitive collection such that $\sigma(P) = 0$. If, for any primitive collection P' in $\mathrm{PC}(\Sigma)$ such that $\sigma(P') \cap P \neq \emptyset$, there exists y in P' such that y is not in $\sigma(P'')$ for any P'' in $\mathrm{PC}(\Sigma)$, then there exists a nonsingular toric Fano $(d-r+1)$-fold X' in $\mathcal{F}^{d-r+1}$ such that*

$$T_N\mathrm{emb}(\Sigma) \overset{\mathrm{F}}{\sim} \mathbf{P}^{r-1} \times X'.$$

Proof. If $\sigma(P') \cap P = \emptyset$ for any primitive collection P' in $\mathrm{PC}(\Sigma)$, then $T_N\mathrm{emb}(\Sigma)$ is isomorphic to the product as in the statement.

So, let $P' = \{y_1, \dots, y_s\}$ be a primitive collection such that $\sigma(P') \cap P \neq \emptyset$, and x_i in $\sigma(P') \cap P$. Then by assumption, there exists y_j in P' such that y_j is not in $\sigma(P'')$ for any P'' in $\mathrm{PC}(\Sigma)$. The primitive relations of Σ are

$$x_1 + \cdots + x_r = 0,\ y_1 + \cdots + y_s = ax_i + \cdots \ (a > 0),\ \dots.$$

By the equivariant blow-up along $\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r, y_j\}$, we get a fan Σ_1 in N whose primitive relations are

$$x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_r + y_j = z,\ x_i + z = y_j,$$

$$y_1 + \cdots + y_{j-1} + y_{j+1} + \cdots + y_s + z = (a-1)x_i + \cdots,$$

$$x_1 + \cdots + x_r = 0,\ y_1 + \cdots + y_s = ax_i + \cdots,\ \dots,$$

where $\mathrm{G}(\Sigma_1) = \mathrm{G}(\Sigma) \cup \{z\}$ and the first three primitive relations are new. Then $T_N\mathrm{emb}(\Sigma_1)$ is a nonsingular toric Fano d-fold by Theorem 1.3.10. By Theorem 1.4.10, Σ_1

can be equivariantly blown-down to a fan Σ' in N with respect to the primitive relation $x_i + z = y_j$. The primitive relations of Σ' are

$$x_1 + \cdots + x_r = 0,\ y_1 + \cdots + y_{j-1} + y_{j+1} + \cdots + y_s + z = (a-1)x_i + \cdots,\ \ldots,$$

where $\mathrm{G}(\Sigma') = (\mathrm{G}(\Sigma) \setminus \{y_j\}) \cup \{z\}$. Then $T_N\mathrm{emb}(\Sigma')$ is also a nonsingular toric Fano d-fold by Theorem 1.3.10, and Σ' satisfies the assumption of the statement. So we can replace Σ by Σ' and carry out this operation again. This operation terminates in finite steps and $T_N\mathrm{emb}(\Sigma')$ becomes a product as in the statement. q.e.d.

By Theorems 1.6.7 and 1.6.12, we get the following immediately.

Corollary 1.6.13 *Let Σ be a splitting fan in N and let $T_N\mathrm{emb}(\Sigma)$ be a nonsingular toric Fano d-fold. If the Picard number of $T_N\mathrm{emb}(\Sigma)$ is not greater than three, then $T_N\mathrm{emb}(\Sigma)$ is F-equivalent to the d-dimensional projective space.*

Next, we consider the classification of Gorenstein toric Fano varieties.

Let Δ be a reflexive polytope in $N_{\mathbf{R}}$. For any $\delta \in \Delta(d-1)$, subdivide δ as

$$\delta = S_{\delta,1} \cup S_{\delta,2} \cup \cdots \cup S_{\delta,k(\delta)},$$

where $S_{\delta,i}\ \ (1 \le i \le k(\delta))$ are $(d-1)$-dimensional simplices such that

$$S_{\delta,i} \cap N = S_{\delta,i}(0) \subset \delta \cap N\ \ (1 \le i \le k(\delta))\,.$$

Then we can define a finite complete fan $\widetilde{\Sigma(\Delta)}$ in N by

$$\widetilde{\Sigma(\Delta)} := \{\sigma(S_{\delta,i}) \text{ and the faces of } \sigma(S_{\delta,i})\ \mid \delta \in \Delta(d-1),\ 1 \le i \le k(\delta)\} \cup \{0\}.$$

Proposition 1.6.14 (Batyrev [4]) *Let Δ be a reflexive polytope in $N_{\mathbf{R}}$. Then there exists a subdivision of $\Sigma(\Delta)$ as above such that $T_N\mathrm{emb}\left(\widetilde{\Sigma(\Delta)}\right)$ is a projective toric variety with only Gorenstein terminal quotient singularities. Moreover, the equivariant morphism corresponding to this subdivision $\varphi : T_N\mathrm{emb}\left(\widetilde{\Sigma(\Delta)}\right) \longrightarrow T_N\mathrm{emb}(\Sigma(\Delta))$ is crepant.*

Remark 1.6.15 In Proposition 1.6.14, if $T_N\mathrm{emb}\left(\widetilde{\Sigma(\Delta)}\right)$ is nonsingular, then for any $P \in \mathrm{PC}\left(\widetilde{\Sigma(\Delta)}\right)$, we have $\deg P \ge 0$ because $\mathrm{Conv}\left(\mathrm{G}\left(\widetilde{\Sigma(\Delta)}\right)\right) = \Delta$. By Theorem 1.3.10, this means that the anticanonical divisor of $T_N\mathrm{emb}\left(\widetilde{\Sigma(\Delta)}\right)$ is nef.

Definition 1.6.16 Let X be a nonsingular projective algebraic variety. Then X is called a nonsingular *weak Fano* variety if the anticanonical divisor $-K_X$ is nef and big.

By the following proposition, the condition "big" is automatic in the case of toric varieties.

Proposition 1.6.17 *Let Σ be a finite complete nonsingular fan in N such that the corresponding toric d-fold $X := T_N\mathrm{emb}(\Sigma)$ is projective. Then the following are equivalent.*

(1) *X is a nonsingular toric weak Fano variety.*

(2) *The anticanonical divisor $-K_X$ is nef.*

(3) *For any $P \in \mathrm{PC}(\Sigma)$, we have $\deg P \geq 0$.*

Proof. The equivalence (2) $\Longleftrightarrow$ (3) follows from Theorem 1.3.10.

Suppose the anticanonical divisor $-K_X$ is nef. Then $\Delta = \mathrm{Conv}\,(\mathrm{G}(\Sigma))$ is a reflexive polytope. So we have $(-K_X)^d = \mathrm{vol}_d(\Delta^*) > 0$. Therefore $-K_X$ is big. q.e.d.

For the Gorenstein toric Fano varieties endowed with crepant resolutions of singularities as Proposition 1.6.14, we can consider instead the nonsingular toric weak Fano varieties by Propositions 1.6.14, 1.6.17 and Remark 1.6.15. In this case, we can apply the method for nonsingular toric Fano varieties by Theorem 1.3.10 and Proposition 1.6.17. In particular, in the cases of $d = 2$ and $d = 3$, $T_N\mathrm{emb}\left(\widetilde{\Sigma(\Delta)}\right)$ is always nonsingular.

We introduce the same concepts for nonsingular toric weak Fano d-folds as in the case of nonsingular toric Fano d-folds. Let

$$\mathcal{F}_d^{\mathrm{w}} := \{\text{nonsingular toric weak Fano } d\text{-folds}\} / \cong .$$

First, we define the concept, flop, for nonsingular projective toric d-folds.

Definition 1.6.18 Let $X = T_N\mathrm{emb}(\Sigma)$ be a nonsingular projective toric d-fold and P a primitive collection of Σ with primitive relation

$$x_1 + \cdots + x_l = y_1 + \cdots + y_l.$$

If $r(P)$ is contained in an extremal ray of $\mathbf{NE}(X)$, then we can do the following operation. First, blow-up X along $\{y_1, \ldots, y_l\}$ to get the toric variety $X' = T_N\mathrm{emb}(\Sigma')$ and the primitive relation of Σ', $x_1 + \cdots + x_l = z$, where $\mathrm{G}(\Sigma') = \mathrm{G}(\Sigma) \cup \{z\}$. Next, blow-down X' with respect to $x_1 + \cdots + x_l = z$ to get the toric variety $X^+ = T_N\mathrm{emb}(\Sigma^+)$ and the primitive relation of Σ^+,

$$y_1 + \cdots + y_l = x_1 + \cdots + x_l,$$

where $\mathrm{G}(\Sigma^+) = \mathrm{G}(\Sigma)$. We call this operation *flop*.

Definition 1.6.19 X_1 and X_2 in $\mathcal{F}_d^{\mathrm{w}}$ are said to be *weakly-F-equivalent* if there exists a sequence of equivariant blow-ups, blow-downs and flops from X_1 to X_2 through toric *weak Fano* d-folds, namely there exist nonsingular toric weak Fano d-folds $Y_0 = X_1, Y_1, \ldots, Y_{3l} = X_2$ together with finite successions $Y_{3j-2} \to Y_{3j-3}$ and $Y_{3j-2} \to Y_{3j-1}$, for each $1 \leq j \leq l$, of equivariant blow-ups through nonsingular toric Fano d-folds, and finite successions $Y_{3k-1} \leftrightarrow Y_{3k}$, for each $1 \leq k \leq l$, of flop through nonsingular toric Fano d-folds. We denote the relation by $X_1 \overset{\mathrm{wF}}{\sim} X_2$. Then "$\overset{\mathrm{wF}}{\sim}$" is obviously an equivalence relation.

Corresponding to Conjecture 1.6.3, we may propose the following conjecture for nonsingular toric weak Fano d-folds.

Conjecture 1.6.20 *Any nonsingular toric weak Fano d-fold is weakly-F-equivalent to the d-dimensional projective space $\mathbf{P}^d$.*

Remark 1.6.21 Since the 4-dimensional pseudo del Pezzo variety and the 4-dimensional del Pezzo variety can be equivariantly blown-up to nonsingular toric weak Fano 4-folds, we exclude the pseudo-symmetric toric Fano varieties from Conjecture 1.6.20.

We can easily prove Conjectures 1.6.3 and 1.6.20 for $d = 2$.

Theorem 1.6.22 *Any nonsingular toric del Pezzo surface is F-equivalent to the projective plane $\mathbf{P}^2$, while any nonsingular toric weak Fano surface is weakly-F-equivalent to $\mathbf{P}^2$. In particular, Conjectures* 1.6.3 *and* 1.6.20 *are true for $d = 2$, and we obtain a new method for the classification of Gorenstein toric Fano surfaces by the above discussion.*

Proof. We prove Theorem 1.6.22 in the case of nonsingular toric weak Fano surfaces. We can similarly prove Theorem 1.6.22 in the case of nonsingular toric Fano surfaces.

By Proposition 1.5.1 and Example 1.5.2, if a nonsingular toric weak Fano surface X is not minimal in the sense of equivariant blow-ups, then X can be equivariantly blown-down to nonsingular toric weak Fano surface. On the other hand, the minimal complete nonsingular toric surfaces in the sense of equivariant blow-ups are $\mathbf{P}^2$ and $\mathbf{P}_{\mathbf{P}^1}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(a))$ ($a \geq 0$ and $a \neq 1$) (See Oda [26]). So the minimal nonsingular toric weak Fano surfaces in the sense of equivariant blow-ups are

$$\mathbf{P}^2,\ \mathbf{P}^1 \times \mathbf{P}^1 \text{ and } \mathbf{P}_{\mathbf{P}^1}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(2)).$$

These are weakly-F-equivalent to the 2-dimensional projective space $\mathbf{P}^2$ by easy calculation. q.e.d.

1.7 The classification of nonsingular toric Fano 3-folds

We devote this section to proving Conjecture 1.6.3 for $d = 3$. Throughout this section, we assume $d = 3$.

Theorem 1.7.1 *Every nonsingular toric Fano 3-fold is F-equivalent to the 3-dimensional projective space $\mathbf{P}^3$. In particular, Conjecture 1.6.3 is true for $d = 3$, and we obtain a new method for the classification of nonsingular toric Fano 3-folds.*

To prove Theorem 1.7.1, we prove the following lemma. For a toric variety X, we denote by $\rho(X)$ the Picard number of X.

Lemma 1.7.2 *Let $X = T_N\mathrm{emb}(\Sigma)$ be a nonsingular toric Fano 3-fold and $\rho(X) \geq 2$. Then, there exists a primitive collection P in $\mathrm{PC}(\Sigma)$ such that $\#P = 2$.*

Proof. Suppose that there does not exist a primitive collection P in $\mathrm{PC}(\Sigma)$ such that $\#P = 2$. Let $\Delta(\Sigma)$ be the Fano polytope corresponding to X. Then the f-vector of $\Delta(\Sigma)$ is

$$(\rho(X) + 3,\ (\rho(X) + 2)(\rho(X) + 3)/2,\ f_3)$$

by assumption. By the Dehn-Sommerville equalities (see Oda [26]), we have $\rho(X) = 1$ and $f_3 = 4$. q.e.d.

Remark 1.7.3 The method in the proof of Lemma 1.7.2 is not available for $d \geq 4$, because, for any $f_0 > 0$, there always exists a simplicial polytope whose f-vector is

$$(f_0,\ f_0(f_0-1)/2, \ldots\ldots).$$

Proof of Theorem 1.7.1. Let $X = T_N\mathrm{emb}(\Sigma)$ be a nonsingular toric Fano 3-fold.

If $\rho(X) = 2$, then Σ is a splitting fan by Theorem 1.6.9, and hence X is F-equivalent to $\mathbf{P}^3$ by Corollary 1.6.13.

Suppose that $\rho(X) \geq 3$. Then, there exists a primitive collection P in $\mathrm{PC}(\Sigma)$ such that $\#P = 2$ by Lemma 1.7.2. According to Theorem 1.3.10, we have two cases.

(1) " There exists a primitive collection P in $\mathrm{PC}(\Sigma)$ with primitive relation $x_1 + x_2 = x$ $(x_1, x_2, x \in \mathrm{G}(\Sigma))$." Since $\deg P = 1$, $r(P)$ is contained in an extremal ray of $\mathbf{NE}(X)$. So, X can be equivariantly blown-down with respect to $x_1 + x_2 = x$. Let $\varphi : X \to Y$ be the equivariant blow-down with respect to $x_1 + x_2 = x$. By Proposition 1.5.1 and Example 1.5.2, if Y is not a nonsingular toric Fano 3-fold, then there exists a primitive collection P' in $\mathrm{PC}(\Sigma)$ with primitive relation

$$y_1 + y_2 = x \quad (\{x_1, x_2\} \cap \{y_1, y_2\} = \emptyset).$$

Since $\deg P' = 1$, $\{x, x_1, y_1\}$, $\{x, x_1, y_2\}$,$\{x, x_2, y_1\}$ and $\{x, x_2, y_2\}$ generate strongly convex rational polyhedral cones of $\Sigma(3)$ by Theorem 1.4.10. Since $\rho(X) \geq 3$, there exists z in $\mathrm{G}(\Sigma) \setminus \{x, x_1, x_2, y_1, y_2\}$. $\{x, z\}$ is obviously a primitive collection of Σ. If the primitive relation of $\{x, z\}$ is $x + z = z'$ $(z' \in \mathrm{G}(\Sigma))$, then, obviously, X can be equivariantly blown-down to a nonsingular toric Fano 3-fold with respect to $x + z = z'$. If the primitive relation of $\{x, z\}$ is $x + z = 0$ and $\rho(X) \geq 4$, then there exists w in $\mathrm{G}(\Sigma) \setminus \{x, x_1, x_2, y_1, y_2, z\}$, and hence we can replace z by w. If the primitive relation of $\{x, z\}$ is $x + z = 0$ and $\rho(X) = 3$, then the primitive relations of Σ are

$$x_1 + x_2 = x,\ y_1 + y_2 = x \text{ and } x + z = 0.$$

Thus, Σ is a splitting fan, and hence X is F-equivalent to $\mathbf{P}^3$ by Corollary 1.6.13.

(2) "For any primitive collection P in $\mathrm{PC}(\Sigma)$ such that $\#P = 2$, its primitive relation is $x_1 + x_2 = 0$ $(x_1, x_2 \in \mathrm{G}(\Sigma))$." There exists a primitive relation $x_1 + x_2 = 0$ by

Lemma 1.7.2. Let $\{x_1, x_1', x_1''\}$ generate a 3-dimensional strongly convex rational polyhedral cone in Σ, where x_1' and x_1'' are in $\mathrm{G}(\Sigma)$. By assumption, there exist distinct elements y_1 and y_2 in $\mathrm{G}(\Sigma) \setminus \{x_1, x_2, x_1', x_1''\}$. If $\{y_1, y_2\}$ is not a primitive collection, then $\{x_2, y_1, y_2\}$ generates a 3-dimensional cone in Σ. Since $\{x_2, y_1\}$ and $\{x_2, y_2\}$ are also not a primitive collection by assumption, the open set $N_{\mathbf{R}} \setminus (\mathrm{Cone}(\{x_2, y_1\}) \cup \mathrm{Cone}(\{x_2, y_2\}) \cup \mathrm{Cone}(\{y_1, y_2\}))$ has two connected components. If $\{x_2, y_1, y_2\}$ is a primitive collection, then there exist elements of $\mathrm{G}(\Sigma)$ in both connected components, and hence there exists a primitive relation like $u_1 + u_2 = u$. This contradicts the assumption. Therefore, either $\{x_1', y_1\}$ or $\{x_1'', y_1\}$ is a primitive collection, because otherwise, both $\{x_2, y_1, x_1'\}$ and $\{x_2, y_1, x_1''\}$ generate 3-dimensional cones in Σ. So, we have two primitive relations $y_1 + x_1' = 0$ and $y_2 + x_1'' = 0$ up to change of the indices. Therefore,

$$\mathrm{Cone}(\{x_1, x_1', x_1''\}) = -\mathrm{Cone}(\{x_2, y_1, y_2\}),$$

and hence $T_N\mathrm{emb}(\Sigma)$ is a pseudo-symmetric toric Fano 3-fold. Conversely, let $\{y_1, y_2\}$ be a primitive collection. Then the corresponding primitive relation is $y_1 + y_2 = 0$ by assumption, and x_1, x_2, y_1 and y_2 are contained in a plane. So, there exists z in $\mathrm{G}(\Sigma) \setminus \{x_1, x_2, x_1', x_1'', y_1, y_2\}$, and both $\{x_1', z\}$ and $\{x_1'', z\}$ are primitive collections. This contradicts the assumption. On the other hand, by Theorem 1.6.6, the psudo-symmetric toric Fano 3-folds are

$$\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1,\ \mathbf{P}^1 \times V^2 \text{ and } \mathbf{P}^1 \times \widetilde{V}^2.$$

By Definition 1.6.4 and Theorem 1.6.7, these are F-equivalent to $\mathbf{P}^3$. q.e.d.

1.8 The classification of nonsingular toric Fano 4-folds

In this section, we prove Conjecture 1.6.3 for $d = 4$. As a result, we obtain a new method for the classification of nonsingular toric Fano 4-folds. Using this method for the classification, we obtain the 124 nonsingular toric Fano 4-folds.

Theorem 1.8.1 *Every nonsingular toric Fano 4-fold other than the 4-dimensional del Pezzo variety V^4 and the 4-dimensional pseudo del Pezzo variety $\widetilde{V}^4$ is F-equivalent to* *the 4-dimensional projective space* $\mathbf{P}^4$*. In particular, Conjecture* 1.6.3 *is true for* $d = 4$*, and hence we obtain a new method for the classification of nonsingular toric Fano* 4*-folds.*

We devote the rest of this section to proving Theorem 1.8.1. So, let $X = T_N\mathrm{emb}(\Sigma)$ be a nonsingular toric Fano 4-fold and $\rho = \rho(X)$ the Picard number of X.

If $\rho(X) = 2$, then Σ is a splitting fan by Theorem 1.6.9, and hence X is F-equivalent to $\mathbf{P}^4$ by Corollary 1.6.13.

The following theorem holds for any nonsingular projective toric d-folds of Picard number 3.

Theorem 1.8.2 (Batyrev [5]) *Let* $X = T_N\mathrm{emb}(\Sigma)$ *be a nonsingular projective toric* d*-fold of Picard number* 3*. Then, one of the following holds.*

(1) Σ *is a splitting fan.*

(2) $\#\,\mathrm{PC}(\Sigma) = 5$.

Moreover, in the case of (2)*, there exists* $(p_0, p_1, p_2, p_3, p_4) \in (\mathbf{Z}_{>0})^5$ *such that the primitive relations of* Σ *are*

$$v_1 + \cdots + v_{p_0} + y_1 + \cdots + y_{p_1} = c_2 z_2 + \cdots + c_{p_2} z_{p_2} + (b_1 + 1)t_1 + \cdots + (b_{p_3} + 1)t_{p_3},$$

$$y_1 + \cdots + y_{p_1} + z_1 + \cdots + z_{p_2} = u_1 + \cdots + u_{p_4}, \ \ z_1 + \cdots + z_{p_2} + t_1 + \cdots + t_{p_3} = 0,$$

$$t_1 + \cdots + t_{p_3} + u_1 + \cdots + u_{p_4} = y_1 + \cdots + y_{p_1} \ \textit{and}$$

$$u_1 + \cdots + u_{p_4} + v_1 + \cdots + v_{p_0} = c_2 z_2 + \cdots + c_{p_2} z_{p_2} + b_1 t_1 + \cdots + b_{p_3} t_{p_3},$$

where

$$\mathrm{G}(\Sigma) = \{v_1, \ldots, v_{p_0}, y_1, \ldots, y_{p_1}, z_1, \ldots, z_{p_2}, t_1, \ldots, t_{p_3}, u_1, \cdots, u_{p_4}\},$$

and $c_2, \ldots, c_{p_2}, b_1, \ldots, b_{p_3} \in \mathbf{Z}_{>0}$.

The following holds.

Proposition 1.8.3 *In Theorem* 1.8.2*, suppose that* X *is a nonsingular toric Fano* d*-fold. If* $p_1 = 1$ *or* $p_4 = 1$*, then* X *can be equivariantly blown-down to a nonsingular toric Fano* d*-fold.*

Proof. We prove Proposition 1.8.3 for the case of $p_1 = 1$. We can prove the case of $p_4 = 1$ similarly.

By assumption, we have the primitive relation

$$t_1 + \cdots + t_{p_3} + u_1 + \cdots + u_{p_4} = y_1.$$

The primitive collections which have common elements with $\{t_1, \ldots, t_{p3}, u_1, \ldots, u_{p_4}\}$ are

$$\{z_1, \ldots, z_{p_2}, t_1, \ldots, t_{p_3}\} \text{ and } \{u_1, \cdots, u_{p_4}, v_1, \ldots, v_{p_0}\}.$$

Since $\{z_1, \ldots, z_{p_2}, y_1\}$ and $\{v_1, \ldots, v_{p_0}, y_1\}$ are in $\mathrm{PC}(\Sigma)$, X can be equivariantly blown-down to a toric variety X' by Theorem 1.4.10. X' is obviously a nonsingular toric Fano variety by Proposition 1.5.1. q.e.d.

Let $\rho = 3$. Since $\#\,\mathrm{G}(\Sigma) = 7$, we have $(p_0, p_1, p_2, p_3, p_4) = (1, 1, 1, 1, 3)$, $(1, 1, 1, 2, 2)$ or their permutations. By Proposition 1.8.3, if $(p_0, p_1, p_2, p_3, p_4) \neq (1, 2, 1, 1, 2)$, then X can be equivariantly blown-down to a nonsingular toric Fano 4-fold. So, let $(p_0, p_1, p_2, p_3, p_4) = (1, 2, 1, 1, 2)$. Then the primitive relations of Σ are

$$v_1 + y_1 + y_2 = (b_1 + 1)t_1,\ y_1 + y_2 + z_1 = u_1 + u_2,\ z_1 + t_1 = 0,$$

$$t_1 + u_1 + u_2 = y_1 + y_2 \text{ and } u_1 + u_2 + v_1 = b_1 t_1,$$

where $b_1 = 0$ or 1. If $b_1 = 0$, then X can be equivariantly blown-down to a nonsingular toric Fano 4-fold with respect to $v_1 + y_1 + y_2 = t_1$ by Theorem 1.4.10 and Proposition 1.5.1. On the other hand, if $b_1 = 1$, we can show easily that X is F-equivalent to $\mathbf{P}^4$ (see G_1 in the table of Section 1.10).

Next, we consider the case of $\rho \geq 4$. We need the following proposition.

Proposition 1.8.4 *Let $X = T_N\mathrm{emb}(\Sigma)$ be a nonsingular toric Fano 4-fold and $\rho(X) \geq 3$. Then, there exists a primitive collection P in $\mathrm{PC}(\Sigma)$ such that $\#P = 2$.*

To prove Proposition 1.8.4, we have to prove the following three lemmas.

Lemma 1.8.5 *Let $X = T_N\mathrm{emb}(\Sigma)$ be a nonsingular toric Fano 4-fold and $\rho(X) \geq 3$. If there does not exist a primitive collection P in $\mathrm{PC}(\Sigma)$ such that $\#P = 2$, then there does not exist a primitive collection P' in $\mathrm{PC}(\Sigma)$ such that $\#P' = 4$.*

Proof. Suppose that there exists a primitive collection $P' = \{x_1, x_2, x_3, x_4\}$ in $\mathrm{PC}(\Sigma)$. Then the open set

$$N_{\mathbf{R}} \setminus (\mathrm{Cone}\,(\{x_2, x_3, x_4\}) \cup \mathrm{Cone}\,(\{x_1, x_3, x_4\}) \cup \mathrm{Cone}\,(\{x_1, x_2, x_4\}) \cup \mathrm{Cone}\,(\{x_1, x_2, x_3\}))$$

has two connected components. Therefore, since there exist at least two other elements by the assumption $\rho(X) \geq 3$, there exists a primitive relation P in $\mathrm{PC}(\Sigma)$ such that $\#P = 2$. This contradicts the assumption. q.e.d.

Lemma 1.8.6 *Let $X = T_N\mathrm{emb}(\Sigma)$ be a nonsingular toric Fano 4-fold and $\rho(X) \geq 3$. If there does not exist a primitive collection P in $\mathrm{PC}(\Sigma)$ such that $\#P = 2$, then there does not exist a primitive relation of the form*

$$x_1 + x_2 + x_3 = ax_4 \quad (a = 1, 2).$$

Proof. Suppose that there exists a primitive collection P in $\mathrm{PC}(\Sigma)$ with primitive relation $x_1 + x_2 + x_3 = ax_4$ $(a = 1, 2)$. If $r(P)$ is contained in an extremal ray of $\mathbf{NE}(X)$, then there exist $z_1, z_2 \in \mathrm{G}(\Sigma) \setminus \{x_1, x_2, x_3, x_4\}$ such that $\{x_i, x_j, x_4, z_k\}$ generate 4-dimensional strongly convex rational polyhedral cones in Σ for $1 \leq i < j \leq 3,\ 1 \leq k \leq 2$. Since $\#\,\mathrm{G}(\Sigma) = \rho+4 \geq 7$, there exists $w \in \mathrm{G}(\Sigma) \setminus \{x_1, x_2, x_3, x_4, z_1, z_2\}$, and hence $\{x_4, w\}$ is a primitive collection of Σ. This contradicts the assumption. So, there does not exists a primitive relation of the form $x_1 + x_2 + x_3 = 2x_4$, since its degree is one. On the other hand, suppose that the primitive relation $x_1 + x_2 + x_3 = x_4$ is represented as the sum of two primitive relations of degree one. By Lemma 1.8.5 and assumption, for any primitive collection P' such that $\deg P' = 1$, its primitive relation is of the form $y_1+y_2+y_3 = y_4+y_5$. Therefore, there exist two primitive relations

$$t_1 + t_2 + x_1 = x_4 + s \text{ and } s + x_2 + x_3 = t_1 + t_2$$

such that

$$\{t_1, t_2, x_4, s\},\ \{t_1, x_1, x_4, s\},\ \{t_2, x_1, x_4, s\},\ \{s, x_2, t_1, t_2\},\ \{s, x_3, t_1, t_2\} \text{ and } \{x_2, x_3, t_1, t_2\}$$

generate 4-dimensional strongly convex rational polyhedral cones in Σ. This is a contradiction, because there exist three 4-dimensional strongly convex rational polyhedral cones generated by $\{t_1, t_2, x_4, s\}$, $\{s, x_2, t_1, t_2\}$ and $\{s, x_3, t_1, t_2\}$, and they contain the 3-dimensional strongly convex rational polyhedral cone generated by $\{t_1, t_2, s\}$. q.e.d.

Lemma 1.8.7 *Let $X = T_N\mathrm{emb}(\Sigma)$ be a nonsingular toric Fano 4-fold and $\rho(X) \geq 3$. If there does not exist a primitive collection P in $\mathrm{PC}(\Sigma)$ such that $\#P = 2$, then there exists a primitive collection $P' = \{x_1, x_2, x_3\}$ in $\mathrm{PC}(\Sigma)$ such that $x_1 + x_2 + x_3 \neq 0$.*

Proof. Suppose that $x_1 + x_2 + x_3 = 0$ for any primitive collection $P' = \{x_1, x_2, x_3\}$ in $\mathrm{PC}(\Sigma)$. By Lemmas 1.8.5, 1.8.6, and assumption, for any primitive collection P in $\mathrm{PC}(\Sigma)$, we have $\#P = 3$. If Σ is a splitting fan, then X is isomorphic to $\mathbf{P}^2 \times \mathbf{P}^2$, and hence $\rho(X) = 2$. So, there exist two primitive collections P_1, P_2 in $\mathrm{PC}(\Sigma)$ such that $P_1 \cap P_2 = \emptyset$. If $P_1 = \{x_1, x_2, x_3\}$ and $P_2 = \{x_1, x_4, x_5\}$, that is, $\#(P_1 \cap P_2) = 1$, then we have $x_2 + x_3 = x_4 + x_5$, and hence $\{x_2, x_3\}$ or $\{x_4, x_5\}$ in $\mathrm{PC}(\Sigma)$. This contradicts the assumption. The case $P_1 = \{x_1, x_2, x_3\}$ and $P_2 = \{x_1, x_2, x_4\}$, that is, $\#(P_1 \cap P_2) = 2$, is also impossible, because $x_3 = x_4$. q.e.d.

Proof of Proposition 1.8.4. By Lemmas 1.8.6 and 1.8.7, there exists a primitive collection P in $\mathrm{PC}(\Sigma)$ with primitive relation $x_1 + x_2 + x_3 = x_4 + x_5$. Since $\deg P = 1$, we have three 4-dimensional strongly convex rational polyhedral cones generated by $\{x_i, x_j, x_4, x_5\}$, where $1 \leq i < j \leq 3$. There exist distinct elements y_1 and y_2 in $\mathrm{G}(\Sigma) \setminus \{x_1, x_2, x_3, x_4, x_5\}$ by the assumption $\rho \geq 3$, and hence we have $\{x_4, x_5, y_1\}$ and $\{x_4, x_5, y_2\}$ in $\mathrm{PC}(\Sigma)$. If $y_1 + x_4 + x_5 = 0$, then we have $y_2 + x_4 + x_5 \neq 0$. Therefore, we have $y_2 + x_4 + x_5 = x_1 + x_2$ up to change of indices. This is a contradiction, because we have $x_3 + y_2 = 0$, and hence $\{x_3, y_2\}$ is in $\mathrm{PC}(\Sigma)$. The case $y_1 + x_4 + x_5 \neq 0$ is similar. q.e.d.

Let $\rho \geq 4$. Then, there exists a primitive collection of Σ whose cardinality is two by Proposition 1.8.4. We divide the proof of Theorem 1.8.1 for $\rho \geq 4$ into two cases.

(1) "There exists a primitive relation $x_1 + x_2 = x$, where $x_1, x_2, x \in \mathrm{G}(\Sigma)$."

Let $\varphi : X \to X'$ be the equivariant blow-down with respect to $x_1 + x_2 = x$. If X' is not a nonsingular toric Fano 4-fold, then, by Proposition 1.5.1 and Example 1.5.2, there exist one of the following primitive relations:

$$y_1 + y_2 + y_3 = 2x,\ y_1 + y_2 + y_3 = x + x_1 \text{ and } y_1 + y_2 = x,$$

where y_1, y_2, y_3 in $\mathrm{G}(\Sigma)$.

(1.1) "$y_1 + y_2 + y_3 = 2x$ or $y_1 + y_2 + y_3 = x + x_1$." Since the degree is one, we have six 4-dimensional strongly convex rational polyhedral cones generated by $\{x_i, x, y_j, y_k\}$, where $1 \leq i \leq 2$, $1 \leq j < k \leq 3$. There exist distinct elements z_1 and z_2 in $\mathrm{G}(\Sigma) \setminus \{x_1, x_2, x, y_1, y_2, y_3\}$, because $\#\mathrm{G}(\Sigma) = \rho + 4 \geq 8$, and hence we have two primitive collections $\{x, z_1\}$ and $\{x, z_2\}$ in $\mathrm{PC}(\Sigma)$. Therefore, we obtain a primitive relation of Σ of the form

$$x + z_1 = w \quad (w \in \{x_1, x_2, y_1, y_2, y_3\})$$

up to change of indices. Let $\varphi : X \to X''$ be the equivariant blow-down with respect to $x + z_1 = w$.

(1.1.1) "$w \neq x_1$ or $y_1+y_2+y_3 = 2x$ is a primitive relation of Σ." Then X'' is obviously a nonsingular toric Fano 4-fold.

(1.1.2) "$w = x_1$ and $y_1 + y_2 + y_3 = x + x_1$ is a primitive relation of Σ." In this case, X'' is not a nonsingular toric Fano 4-fold by Proposition 1.5.1 and Example 1.5.2. Since $\rho \geq 4$, there exists $t \in \mathrm{G}(\Sigma) \setminus \{x_1, x_2, x, y_1, y_2, y_3, z_1\}$. So, we have one of the following primitive relations of Σ up to change of indices:

$$t + x_1 = y_1,\ t + x_1 = x_2 \text{ and } t + x_1 = z_1.$$

Let $\varphi' : X \to X'''$ be the equivariant blow-down with respect to this primitive relation. Then, X''' is obviously a nonsingular toric Fano 4-fold.

(1.2) "$y_1 + y_2 = x$." Since the degree is one, there exist two elements z_1 and z_2 in $\mathrm{G}(\Sigma) \setminus \{x_1, x_2, x, y_1, y_2\}$, and we have eight 4-dimensional strongly convex rational polyhedral cones generated by $\{x_i, x, y_j, z_k\}$, where $1 \leq i, j, k \leq 2$. There exist w in $\mathrm{G}(\Sigma) \setminus \{x_1, x_2, x, y_1, y_2, z_1, z_2\}$, because $\#\mathrm{G}(\Sigma) = \rho + 4 \geq 8$, and hence $P = \{x, w\}$ is a primitive collection of Σ.

(1.2.1) "The primitive relation of P is $x+w = t$, where t in $\{z_1, z_2\}$." Let $\varphi : X \to X''$ be the equivariant blow-down with respect to $x + w = t$. Then, X'' is obviously a nonsingular toric Fano 4-fold.

(1.2.2) "The primitive relation of P is $x + w = t$, where t in $\{x_1, x_2, y_1, y_2\}$." Let $\varphi : X \to X''$ be the equivariant blow-down with respect to $x + w = t$. If X'' is not a nonsingular toric Fano 4-fold, then we obviously have a primitive relation $z_1 + z_2 = t$ by Proposition 1.5.1. We may assume $t = x_2$ without loss of generality. Then, we have four 4-dimensional strongly convex rational polyhedral cones generated by $\{x_2, y_i, y_j, w\}$, where $1 \leq i, j \leq 2$. $\{x_1, x, y_1, z_1\}$ is a $\mathbf{Z}$-basis of N. Using this basis, we have

$$x_2 = -x_1 + x,\ y_2 = x - y_1,\ z_2 = -x_1 + x - z_1 \text{ and } w = -x_1.$$

Since the coefficient of x in none of these relation is negative, there exist u in $\mathrm{G}(\Sigma) \setminus \{x_1, x_2, x, y_1, y_2, z_1, z_2, w\}$ by the completeness of Σ, and hence we have two primitive collections $\{x, u\}$ and $\{x_2, u\}$ in $\mathrm{PC}(\Sigma)$. Therefore, we have a primitive relation either $x + u = s$ or $x_2 + u = s$, where s is in $\{x_1, y_1, y_2, z_1, z_2, w\}$. Let $\varphi : X \to X''$ be the equivariant blow-down with respect to $x + u = s$. Then, X'' is obviously a nonsingular toric Fano 4-fold. The case of the blow-down with respect to $x_2 + u = s$ is similar.

(1.2.3) "The primitive relation of P is $x + w = 0$." If $\rho \geq 5$, then there exist v in $\mathrm{G}(\Sigma) \setminus \{x_1, x_2, x, y_1, y_2, z_1, z_2, w, v\}$, and hence we have the primitive relation $x + v \neq 0$. In this case, we can use the same method as in (1.2.1) or (1.2.2).

So let $\rho = 4$ and $\mathrm{G}(\Sigma) = \{x_1, x_2, x, y_1, y_2, z_1, z_2, w\}$. Then, either $\{z_1, z_2\}$ or $\{x, z_1, z_2\}$ is a primitive collection of Σ.

(1.2.3.1) "$z_1 + z_2 = 0$ is a primitive relation of Σ." X is obviously a nonsingular toric Fano 4-fold in this case. The primitive relations of Σ are

$$x_1 + x_2 = x,\ y_1 + y_2 = x,\ x + w = 0 \text{ and } z_1 + z_2 = 0.$$

Therefore, Σ is a splitting fan, and hence X is F-equivalent to $\mathbf{P}^4$ by Theorems 1.6.7 and 1.6.12.

(1.2.3.2) "$z_1 + z_2 = x$ is a primitive relation of Σ." X is obviously a nonsingular toric Fano 4-fold in this case. The primitive relations of Σ are

$$x_1 + x_2 = x,\ y_1 + y_2 = x,\ x + w = 0 \text{ and } z_1 + z_2 = x.$$

Therefore, Σ is a splitting fan, and hence X is F-equivalent to $\mathbf{P}^4$ by Theorems 1.6.7 and 1.6.12.

(1.2.3.3) "$z_1 + z_2 = t$ is a primitive relation of Σ, where t in $\{x_1, x_2, y_1, y_2, w\}$." Let $\varphi : X \to X''$ be the equivariant blow-down with respect to $z_1 + z_2 = t$. Then X'' is obviously a nonsingular toric Fano 4-fold by Proposition 1.5.1 and Example 1.5.2.

(1.2.3.4) "$z_1 + z_2 + x = 0$ is a primitive relation of Σ." This is impossible, because $z_1 + z_2 = -x = w$, and hence $\{z_1, z_2\}$ is a primitive collection of Σ.

(1.2.3.5) "$z_1 + z_2 + x = ax_1$ is a primitive relation of Σ, where $a = 1$ or 2." Since $ax_1 + w = z_1 + z_2$, $\{t, w\}$ is a primitive collection of Σ. There exists u in $\{x_2, y_1, y_2, z_1, z_2\}$ such that the primitive relation of $\{x_1, w\}$ is $x_1 + w = u$, because $x + w = 0$. Since $x_1 - x - u = 0$, we have $u = x_2$. Because, otherwise, $\{x_1, x, u\}$ is a part of a $\mathbf{Z}$-basis of

N. However, this contradicts the fact $x_1+x_2=x$. We can replace x_1 by x_2, y_1 or y_2, and repeat the same argument.

(1.2.3.6) "$z_1+z_2+x=aw$ is a primitive relation of Σ, where $a=1$ or 2." We have $z_1+z_2=aw-x=(a+1)w$. This is a contradiction.

(1.2.3.7) "$z_1+z_2+x=x_i+y_j$ is a primitive relation of Σ, where $1\le i,j\le 2$." X is obviously a nonsingular toric Fano 4-fold. We can show easily that X is F-equivalent to $\mathbf{P}^4$ (See M_2 in the table of Section 1.10).

(2) "There does not exist a primitive collection $P=\{x_1,x_2\}$ in $\mathrm{PC}(\Sigma)$ with primitive relation $x_1+x_2\neq 0$."

In this case, we need the following lemma. This lemma can be proved in the same way as Lemmas 1.8.5 and 1.8.6.

Lemma 1.8.8 *Let $X=T_N\mathrm{emb}(\Sigma)$ be a nonsingular toric Fano 4-fold and $\rho(X)\ge 4$. If there does not exist a primitive collection $P=\{x_1,x_2\}$ in $\mathrm{PC}(\Sigma)$ with primitive relation $x_1+x_2\neq 0$, then the following hold.*

(1) *There does not exist a primitive collection P in $\mathrm{PC}(\Sigma)$ such that $\#P=4$.*

(2) *There does not exist a primitive relation of Σ of the form*

$$x_1+x_2+x_3=ax_4\quad (a=1,2).$$

Since $\rho\ge 4$, there exists a primitive collection $P=\{x_1,x_2\}$ in $\mathrm{PC}(\Sigma)$ with primitive relation $x_1+x_2=0$, by Proposition 1.8.4. We fix this P.

(2.1) "$r(P)$ is contained in an extremal ray of $\mathbf{NE}(X)$." By toric Mori theory, there exists a nonsingular projective toric 3-fold $Y=T_N\mathrm{emb}(\Sigma^*)$ such that X is an equivariant $\mathbf{P}^1$-bundle over Y, $\mathrm{G}(\Sigma^*)\subset\mathrm{G}(\Sigma)$, and if P^* is a primitive collection of Σ^*, then P^* is also a primitive collection of Σ. Let $\#\,\mathrm{G}(\Sigma^*)=n$ and n_0 the number of the primitive collections of Σ^* whose cardinality is two. Then the f-vector of the 3-dimensional simplicial convex polytope corresponding to Σ^* is $(n,n(n-1)/2-n_0,f_2)$. By the Dehn-Sommerville equalities (see Oda [26]), we have $n_0=(n-3)(n-4)/2$. So by assumption, we have $n_0=(n-3)(n-4)/2\le n/2$. Since $\rho\ge 4$, we have $n=6$, and hence the primitive relations of Σ are

$$x_1+x_2=0,\ x_3+x_4=0,\ x_5+x_6=0 \text{ and } x_7+x_8=0.$$

Thus, X is $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$, and hence X is F-equivalent to $\mathbf{P}^4$ by Theorem 1.6.7.

(2.2) "$r(P)$ is not contained in an extremal ray of $\mathbf{NE}(X)$." By Lemma 1.8.8, there exist two primitive relations of Σ of the form

$$x_1 + y_1 + y_2 = z_1 + z_2 \text{ and } x_2 + z_1 + z_2 = y_1 + y_2,$$

with y_1, y_2, z_1, z_2 in $\mathrm{G}(\Sigma)$. We have five 4-dimensional strongly convex rational polyhedral cones of Σ generated by

$$\{x_1, y_1, z_1, z_2\},\ \{x_1, y_2, z_1, z_2\},\ \{y_1, y_2, z_1, z_2\},\ \{x_2, y_1, y_2, z_1\} \text{ and } \{x_2, y_1, y_1, z_2\}.$$

By the assumption $\rho \geq 4$, there exists w in $\mathrm{G}(\Sigma) \setminus \{x_1, x_2, y_1, y_2, z_1, z_2\}$ such that either $\{z_1, z_2, w\}$ or $\{z_1, z_2, w\}$ is a primitive collection of Σ, because there exists at most one primitive collection among $\{z_1, w\}$, $\{z_2, w\}$, $\{y_1, w\}$ and $\{y_1, w\}$, and the others generate 2-dimensional strongly convex rational polyhedral cones of Σ. If $w + z_1 + z_2 = 0$ is a primitive relation, then we have $y_1 + y_2 + w = x_2$. So, by assumption, $\{y_1, y_2\}$, $\{y_1, w\}$ and $\{y_2, w\}$ are not primitive collections. Therefore, $\{y_1, y_2, w\}$ is a primitive collection of Σ. This contradicts Lemma 1.8.8.

By the above discussion, we have the primitive relations $w + z_1 + z_2 = t_1 + t_2$ and $w + y_1 + y_2 = s_1 + s_2$, where the possibilities for $\{t_1, t_2\}$ are $\{x_1, y_1\}$ and $\{x_1, y_2\}$, while the possibilities for $\{s_1, s_2\}$ are $\{x_2, z_1\}$ and $\{x_2, z_2\}$. So, we have $4 \leq \rho \leq 6$.

(2.2.1) "$\rho = 4$" X is obviously a nonsingular toric Fano 4-fold. We can show easily that X is F-equivalent to $\mathbf{P}^4$ (See M_1 in the table of Section 1.10).

(2.2.2) "$\rho = 5$" X is the 4-dimensional pseudo del Pezzo variety. Moreover, X is not F-equivalent to $\mathbf{P}^4$ (See (117) in the table of Section 1.10). The primitive relations of Σ are

$$x_0 + x_4 = 0,\ x_1 + x_5 = 0,\ x_2 + x_6 = 0,\ x_3 + x_7 = 0,$$

$$x_0+x_1+x_2 = x_7+x_8,\ x_0+x_1+x_3 = x_6+x_8,\ x_0+x_2+x_3 = x_5+x_8,\ x_1+x_2+x_3 = x_4+x_8,$$

$$x_4+x_5+x_8 = x_2+x_3,\ x_4+x_6+x_8 = x_1+x_3,\ x_4+x_7+x_8 = x_1+x_2,\ x_5+x_6+x_8 = x_0+x_3,$$

$$x_5 + x_7 + x_8 = x_0 + x_2,\ x_6 + x_7 + x_8 = x_0 + x_1,$$

where $\mathrm{G}(\Sigma) = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$.

(2.2.3) "$\rho = 6$" X is the 4-dimensional del Pezzo variety. Moreover, X is not F-equivalent to $\mathbf{P}^4$ (See (118) in the table of Section 1.10). The primitive relations of Σ

are

$$x_0 + x_4 = 0,\ x_1 + x_5 = 0,\ x_2 + x_6 = 0,\ x_3 + x_7 = 0,\ x_8 + x_9 = 0,$$

$$x_0+x_1+x_2 = x_7+x_8,\ x_0+x_1+x_3 = x_6+x_8,\ x_0+x_2+x_3 = x_5+x_8,\ x_1+x_2+x_3 = x_4+x_8,$$

$$x_0+x_1+x_9 = x_6+x_7,\ x_0+x_2+x_9 = x_5+x_7,\ x_0+x_3+x_9 = x_5+x_6,\ x_1+x_2+x_9 = x_4+x_7,$$

$$x_1+x_3+x_9 = x_4+x_6,\ x_2+x_3+x_9 = x_4+x_5,\ x_4+x_5+x_6 = x_3+x_9,\ x_4+x_5+x_7 = x_2+x_9,$$

$$x_4+x_6+x_7 = x_1+x_9,\ x_5+x_6+x_7 = x_0+x_9,\ x_4+x_5+x_8 = x_2+x_3,\ x_4+x_6+x_8 = x_1+x_3,$$

$$x_4+x_7+x_8 = x_1+x_2,\ x_5+x_6+x_8 = x_0+x_3,\ x_5+x_7+x_8 = x_0+x_2,\ x_6+x_7+x_8 = x_0+x_1,$$

where $\mathrm{G}(\Sigma) = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$.

1.9 124 nonsingular toric Fano 4-folds

In this section, we describe the 124 nonsingular toric Fano 4-folds in terms of primitive relations. We use the same notation as in Batyrev [6] (see also Section 1.10). Let $\mathrm{G}(\Sigma) = \{x_1, x_2, \ldots\}$.

(1) 4-dimensional projective space $\mathbf{P}^4$. The primitive relation is $x_1+x_2+x_3+x_4+x_5 = 0$.

"type B" The primitive relations are as follows:

Case	2	3	4	5	6
$x_1 + x_2 + x_3 + x_4 =$	$3x_5$	$2x_5$	x_5	0	0
$x_5 + x_6 =$	0	0	0	0	x_1
Notation	B_1	B_2	B_3	B_4	B_5

"type C" The primitive relations are $x_1 + x_2 + x_3 = 0$ and

Case	7	8	9	10
$x_4 + x_5 + x_6 =$	$2x_1$	x_1	$x_1 + x_2$	0
Notation	C_1	C_2	C_3	C_4

"type E" The primitive relations are $x_1 + x_7 = 0$, $x_1 + x_2 = x_6$, $x_6 + x_7 = x_2$ and

Case	11	12	13
$x_2+x_3+x_4+x_5=$	$2x_1$	x_1	0
$x_3+x_4+x_5+x_6=$	$3x_1$	$2x_1$	x_1
Notation	E_1	E_2	E_3

"type D" The primitive relations are as follows:

Case	14	15	16	17	18	19	20	21	22	23	24
$x_1+x_2+x_3=$	$2x_6$	$2x_4$	x_4+x_6	$2x_6$	$2x_4$	x_6	0	x_4	x_4+x_6	x_6	0
$x_4+x_5=$	x_6	x_6	x_6	x_1	0	x_6	x_1	x_6	0	x_1	x_1
$x_6+x_7=$	0	0	0	0	0	0	x_1	0	0	0	x_4
Notation	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}

Case	25	26	27	28	29	30	31	32
$x_1+x_2+x_3=$	x_4	0	0	0	x_4+x_7	0	$2x_7$	x_7
$x_4+x_5=$	0	0	0	0	x_6	x_1	x_6	x_6
$x_6+x_7=$	0	0	x_1	x_4	0	x_2	0	0
Notation	D_{12}	D_{13}	D_{14}	D_{15}	D_{16}	D_{17}	D_{18}	D_{19}

"type G" The primitive relations are as follows:

Case	33	34	35	36	37	38
$x_1+x_7=$	0	x_4	0	x_4	x_4	x_4
$x_2+x_3+x_4=$	x_1	x_7	0	x_7	x_7	x_7
$x_4+x_5+x_6=$	$2x_1$	$2x_1$	x_1	x_1+x_2	0	x_1
$x_5+x_6+x_7=$	x_2+x_3	x_1	x_2+x_3	x_2	x_2+x_3	0
$x_1+x_2+x_3=$	x_5+x_6	0	x_5+x_6	0	0	0
Notation	G_1	G_2	G_3	G_4	G_5	G_6

"type H" The primitive relations are $x_1+x_2=x_8$, $x_7+v_8=x_1$, $x_1+x_6=x_7$, $x_2+x_7=0$, $x_6+x_8=0$ and

Case	39	40	41	42	43	44	45	46	47	48
$x_3+x_4+x_5=$	$2x_1$	x_1+x_8	$2x_8$	x_1	x_8	x_2+x_8	$2x_2$	0	x_2	x_2+x_6
Notation	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9	H_{10}

"type L" The primitive relations are $x_1+x_8=0$ and

Case	49	50	51	52	53	54	55	56	57	58	59	60	61
$x_2+x_3=$	x_1	x_1	x_1	x_1	0	0	x_1	0	0	x_1	0	x_1	x_1
$x_4+x_5=$	x_1	x_3	x_1	x_3	x_3	x_3	0	0	0	x_3	x_3	x_8	x_1
$x_6+x_7=$	x_1	x_3	x_4	x_4	x_3	x_4	x_4	0	x_4	x_2	x_2	x_4	x_8
Notation	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}	L_{11}	L_{12}	L_{13}

"type I" The primitive relations are $x_7+x_8=x_3$, $x_3+x_6=x_7$, $x_6+x_8=0$, and

Case	62	63	64	65	66	67	68	69	70	71
$x_1+x_2=$	x_3	x_8	x_8	x_7	x_4	x_3	0	x_6	x_7	x_8
$x_3+x_4+x_5=$	$2x_8$	$2x_8$	x_1+x_8	$2x_8$	$2x_8$	x_8	$2x_8$	x_1+x_8	x_8	x_8
$x_4+x_5+x_7=$	x_8	x_8	x_1	x_8	x_8	0	x_8	x_1	0	0
Notation	I_1	I_2	I_3	I_4	I_5	I_6	I_7	I_8	I_9	I_{10}

Case	72	73	74	75	76
$x_1+x_2=$	0	x_6	0	x_4	x_6
$x_3+x_4+x_5=$	x_1+x_8	x_8	x_8	x_8	$2x_8$
$x_4+x_5+x_7=$	x_1	0	0	0	x_8
Notation	I_{11}	I_{12}	I_{13}	I_{14}	I_{15}

"type M" The primitive relations are as follows:

Case	77	78	79	80	81
$x_1+x_8=$	0	0	0	0	x_5
$x_4+x_5=$	0	x_1	x_1	x_1	x_7
$x_6+x_7=$	0	x_1	x_5	0	x_1
$x_1+x_2+x_3=$	x_4+x_6	x_4+x_6	x_4+x_6	x_4+x_6	x_6
$x_4+x_6+x_8=$	x_2+x_3	x_2+x_3	x_2+x_3	x_2+x_3	0
$x_2+x_3+x_5=$	x_6+x_8	x_6	x_6	x_6	x_6+x_8
$x_2+x_3+x_7=$	x_4+x_8	x_4	0	x_4+x_8	0
Notation	M_1	M_2	M_3	M_4	M_5

"type J" The primitive relations are $x_3+x_6=x_7$, $x_1+x_2+x_8=x_4+x_5$, $x_4+x_5+x_6=x_1+x_2$, $x_7+x_8=x_3$, $x_6+x_8=0$ and

Case	82	83
$x_3+x_4+x_5=$	0	x_8
$x_4+x_5+x_7=$	x_6	0
$x_1+x_2+x_3=$	x_6	0
$x_1+x_2+x_7=$	$2x_6$	x_6
Notation	J_1	J_2

"type Q" The primitive relations are $x_8+x_9=0, x_7+x_9=x_1$, $x_1+x_2=x_9$, $x_1+x_8=x_7$, $x_2+x_7=0$ and

Case	84	85	86	87	88	89	90	91	92	93	94	95
$x_3+x_5=$	x_1	x_1	x_9	x_1	x_9	0	x_9	0	x_9	0	0	x_1
$x_4+x_6=$	x_1	x_3	x_9	x_9	x_3	x_1	x_7	x_9	x_2	x_3	0	x_2
Notation	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8	Q_9	Q_{10}	Q_{11}	Q_{12}

Case	96	97	98	99	100
$x_3+x_5=$	x_2	x_2	0	x_9	x_2
$x_4+x_6=$	x_2	x_3	x_2	x_8	x_8
Notation	Q_{13}	Q_{14}	Q_{15}	Q_{16}	Q_{17}

"type K" The primitive relations are $x_7 + x_9 = x_1$, $x_1 + x_8 = x_7$, $x_8 + x_9 = 0$, $x_2 + x_8 = x_6$, $x_6 + x_7 = x_8$, $x_1 + x_6 = 0$, $x_6 + x_9 = x_2$, $x_1 + x_2 = x_9$, $x_2 + x_7 = 0$ and

Case	101	102	103	104
$x_3 + x_4 + x_5 =$	$2x_9$	$x_1 + x_9$	x_9	0
Notation	K_1	K_2	K_3	K_4

"type R" The primitive relations are $x_5 + x_7 = 0$, $x_1 + x_9 = 0$, $x_5 + x_9 = x_8$, $x_1+x_8 = x_5$, $x_7+x_8 = x_9$, $x_2+x_3+x_9 = x_4+x_7$, $x_2+x_3+x_5 = x_1+x_4$, $x_2+x_3+x_8 = x_4$, $x_1 + x_4 + x_7 = x_2 + x_3$ and

Case	105	106	107
$x_4 + x_6 =$	x_8	x_5	0
$x_2 + x_3 + x_6 =$	0	x_1	$x_1 + x_7$
Notation	R_1	R_2	R_3

(108) The primitive relations are $x_7+x_9 = 0$, $x_8+x_9 = x_1$, $x_3+x_5 = x_4$, $x_4+x_6 = x_5$, $x_1 + x_7 = x_8$, $x_3 + x_6 = 0$, $x_1 + x_2 + x_5 = x_6 + x_9$, $x_1 + x_2 + x_4 = x_9$, $x_2 + x_5 + x_8 = x_6$ and $x_2 + x_4 + x_8 = 0$.

"type U" The primitive relations are $x_1 + x_3 = x_2$, $x_2 + x_4 = x_3$, $x_1 + x_4 = 0$, $x_3 + x_5 = x_4$, $x_4 + x_6 = x_5$, $x_2 + x_5 = 0$, $x_1 + x_5 = x_6$, $x_2 + x_6 = x_1$, $x_3 + x_6 = 0$ and

Case	109	110	111	112	113	114	115	116
$x_8 + x_7 =$	x_1	x_1	x_1	0	0	x_1	x_1	x_1
$x_9 + x_{10} =$	x_1	x_8	x_2	x_8	0	0	x_3	x_4
Notation	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8

(117) 4-dimensional pseudo del Pezzo variety (see Definition 1.6.4 and Remark 1.6.5). The primitive relations are $x_4+x_9 = 0$, $x_1+x_5 = 0$, $x_2+x_6 = 0$, $x_3+x_7 = 0$, $x_1+x_2+x_9 = x_7 + x_8$, $x_1 + x_3 + x_9 = x_6 + x_8$, $x_2 + x_3 + x_9 = x_5 + x_8$, $x_1 + x_2 + x_3 = x_4 + x_8$, $x_4+x_5+x_8 = x_2+x_3$, $x_4+x_6+x_8 = x_1+x_3$, $x_4+x_7+x_8 = x_1+x_2$, $x_5+x_6+x_8 = x_3+x_9$, $x_5 + x_7 + x_8 = x_2 + x_9$ and $x_6 + x_7 + x_8 = x_1 + x_9$.

(118) 4-dimensional del Pezzo variety (see Definition 1.6.4 and Remark 1.6.5). The primitive relations are $x_4+x_{10}=0$, $x_1+x_5=0$, $x_2+x_6=0$, $x_3+x_7=0$, $x_8+x_9=0$, $x_1+x_2+x_{10}=x_7+x_8$, $x_1+x_3+x_{10}=x_6+x_8$, $x_2+x_3+x_{10}=x_5+x_8$, $x_1+x_2+x_3=x_4+x_8$, $x_1+x_9+x_{10}=x_6+x_7$, $x_2+x_9+x_{10}=x_5+x_7$, $x_3+x_9+x_{10}=x_5+x_6$, $x_1+x_2+x_9=x_4+x_7$, $x_1+x_3+x_9=x_4+x_6$, $x_2+x_3+x_9=x_4+x_5$, $x_4+x_5+x_6=x_3+x_9$, $x_4+x_5+x_7=x_2+x_9$, $x_4+x_6+x_7=x_1+x_9$, $x_5+x_6+x_7=x_9+x_{10}$, $x_4+x_5+x_8=x_2+x_3$, $x_4+x_6+x_8=x_1+x_3$, $x_4+x_7+x_8=x_1+x_2$, $x_5+x_6+x_8=x_3+x_{10}$, $x_5+x_7+x_8=x_2+x_{10}$ and $x_6+x_7+x_8=x_1+x_{10}$.

(119) $S_2\times S_2$. The primitive relations are $x_1+x_3=0$, $x_1+x_4=x_5$, $x_2+x_4=x_3$, $x_2+x_5=0$, $x_3+x_5=x_4$, $x_6+x_8=0$, $x_6+x_9=x_{10}$, $x_7+x_9=x_8$, $x_7+x_{10}=0$ and $x_8+x_{10}=x_9$.

(120) $S_2\times S_3$. The primitive relations are $x_1+x_3=0$, $x_1+x_4=x_5$, $x_2+x_4=x_3$, $x_2+x_5=0$, $x_3+x_5=x_4$, $x_6+x_8=x_7$, $x_6+x_9=0$, $x_6+x_{10}=x_{11}$, $x_7+x_9=x_8$, $x_7+x_{10}=0$, $x_7+x_{11}=x_6$, $x_8+x_{10}=x_9$, $x_8+x_{11}=0$ and $x_9+x_{11}=x_{10}$.

(121) $S_3\times S_3$. The primitive relations are $x_1+x_3=x_2$, $x_1+x_4=0$, $x_1+x_5=x_6$, $x_2+x_4=x_3$, $x_2+x_5=0$, $x_2+x_6=x_1$, $x_3+x_5=x_4$, $x_3+x_6=0$, $x_4+x_6=x_5$, $x_7+x_9=x_8$, $x_7+x_{10}=0$, $x_7+x_{11}=x_{12}$, $x_8+x_{10}=x_9$, $x_8+x_{11}=0$, $x_8+x_{12}=x_7$, $x_9+x_{11}=x_{10}$, $x_9+x_{12}=0$ and $x_{10}+x_{12}=x_{11}$.

"type Z" The primitive relations are $x_1+x_2+x_5=0$, $x_1+x_2+x_6=x_7$, $x_2+x_4+x_5=x_8$, $x_2+x_4+x_6=x_7+x_8$ and

Case	122	123
$x_3+x_8+x_7=$	0	x_2
$x_3+x_4+x_6=$	x_1+x_5	0
$x_3+x_4+x_7=$	x_1	x_1+x_2
$x_3+x_6+x_8=$	x_5	x_2+x_5
Notation	Z_1	Z_2

(124) The primitive relations are $x_1+x_4=x_7$, $x_2+x_5=x_8$, $x_3+x_6=x_9$, $x_1+x_2+x_3=0$, $x_4+x_5+x_6=0$, $x_7+x_8+x_9=0$, $x_1+x_2+x_9=x_6$, $x_4+x_5+x_9=x_3$, $x_1+x_3+x_8=x_5$, $x_4+x_6+x_8=x_2$, $x_2+x_3+x_7=x_4$, $x_5+x_6+x_7=x_1$, $x_1+x_8+x_9=x_5+x_6$, $x_4+x_8+x_9=x_2+x_3$, $x_2+x_7+x_9=x_4+x_6$, $x_5+x_7+x_9=x_1+x_3$, $x_3+x_7+x_8=x_4+x_5$ and $x_6+x_7+x_8=x_1+x_2$ (see Example 1.4.7).

1.10 Equivariant blow-up relations among nonsingular toric Fano 4-folds

In this section, we describe all the equivariant blow-up relations among nonsingular toric Fano 4-folds using the results of Sections 1.3, 1.4, 1.6 and 1.8. In Table 1, we use the same notation as in Batyrev [6], and i-blow-up means the equivariant blow-up along a T_N-invariant irreducible closed subvariety of codimension i.

Table 1: equivariant blow-up relations among nonsingular toric Fano 4-folds

	equivariant blow-up	notation
(1)	none	$\mathbf{P}^4$
(2)	none	B_1
(3)	none	B_2
(4)	4-blow-up of $\mathbf{P}^4$	B_3
(5)	none	B_4
(6)	2-blow-up of $\mathbf{P}^4$	B_5
(7)	none	C_1
(8)	3-blow-up of $\mathbf{P}^4$	C_2
(9)	none	C_3
(10)	none	C_4
(11)	2-blow-up of B_1, B_2	E_1
(12)	2-blow-up of B_2, B_3	E_2
(13)	2-blow-up of B_3, B_4, 4-blow-up of B_5	E_3
(14)	none	D_1
(15)	2-blow-up of C_1	D_2
(16)	none	D_3
(17)	2-blow-up of B_2	D_4
(18)	none	D_5
(19)	2-blow-up of C_3	D_6
(20)	none	D_7

(21)	2-blow-up of C_2, 3-blow-up of B_3	D_8
(22)	none	D_9
(23)	2-blow-up of B_5, 3-blow-up of B_3	D_{10}
(24)	2-blow-up of B_5, C_2	D_{11}
(25)	3-blow-up of B_4	D_{12}
(26)	none	D_{13}
(27)	2-blow-up of B_4	D_{14}
(28)	2-blow-up of C_4	D_{15}
(29)	2-blow-up of C_3	D_{16}
(30)	2-blow-up of B_5	D_{17}
(31)	2-blow-up of C_1	D_{18}
(32)	2-blow-up of C_2, 3-blow-up of B_5	D_{19}
(33)	none	G_1
(34)	2-blow-up of C_2, 3-blow-up of C_1	G_2
(35)	3-blow-up of C_3	G_3
(36)	2-blow-up of C_2, 3-blow-up of C_3	G_4
(37)	2-blow-up of C_3, 3-blow-up of C_4	G_5
(38)	2-blow-up of C_4, 3-blow-up of C_2	G_6
(39)	2-blow-up of D_2	H_1
(40)	2-blow-up of D_3	H_2
(41)	2-blow-up of D_1, D_5	H_3
(42)	2-blow-up of D_8, D_9	H_4
(43)	2-blow-up of D_6, D_{12}, D_{16}	H_5
(44)	2-blow-up of D_3, D_9	H_6
(45)	2-blow-up of D_2, D_5, D_{18}	H_7
(46)	2-blow-up of D_{13}, D_{15}	H_8
(47)	2-blow-up of D_8, D_{12}, D_{19}, 3-blow-up of E_3	H_9
(48)	2-blow-up of D_9, D_{16}	H_{10}
(49)	none	L_1
(50)	2-blow-up of D_7	L_2
(51)	2-blow-up of D_6	L_3

(52)	2-blow-up of $D_8,\ D_{10},\ D_{11}$	L_4
(53)	none	L_5
(54)	2-blow-up of $D_{12},\ D_{14}$	L_6
(55)	2-blow-up of D_{15}	L_7
(56)	none	L_8
(57)	2-blow-up of D_{13}	L_9
(58)	2-blow-up of $D_{10},\ D_{17}$	L_{10}
(59)	2-blow-up of D_{14}	L_{11}
(60)	2-blow-up of $D_{11},\ D_{17},\ D_{19}$	L_{12}
(61)	2-blow-up of D_7	L_{13}
(62)	2-blow-up of D_4	I_1
(63)	2-blow-up of $D_1,\ D_6$	I_2
(64)	2-blow-up of $D_3,\ D_8$	I_3
(65)	2-blow-up of D_{10}	I_4
(66)	2-blow-up of $E_2,\ D_4,\ D_{10}$	I_5
(67)	2-blow-up of D_{10}, 3-blow-up of D_{11}	I_6
(68)	2-blow-up of $D_5,\ D_{12}$	I_7
(69)	2-blow-up of $D_8,\ D_{16}, G_4$	I_8
(70)	2-blow-up of D_{14}, 3-blow-up of D_7	I_9
(71)	2-blow-up of $D_6,\ D_{15},\ G_5$	I_{10}
(72)	2-blow-up of $D_9,\ D_{12}$	I_{11}
(73)	2-blow-up of $D_{15},\ D_{19},\ G_6$, 3-blow-up of D_{11}	I_{12}
(74)	2-blow-up of $D_{12},\ D_{13}$, 3-blow-up of D_{14}	I_{13}
(75)	2-blow-up of $E_3,\ D_{10},\ D_{14}$, 3-blow-up of D_{17}	I_{14}
(76)	2-blow-up of $D_{18},\ D_{19},\ G_2$	I_{15}
(77)	none	M_1
(78)	none	M_2
(79)	2-blow-up of $G_3,\ G_5$	M_3
(80)	2-blow-up of G_3	M_4
(81)	2-blow-up of $G_4,\ G_6$	M_5
(82)	2-blow-up of $G_1,\ G_3$	J_1

(83)	2-blow-up of G_3, 3-blow-up of G_5	J_2
(84)	2-blow-up of L_2	Q_1
(85)	2-blow-up of $H_4,\ L_4$	Q_2
(86)	2-blow-up of $L_1,\ L_5$	Q_3
(87)	2-blow-up of L_3	Q_4
(88)	2-blow-up of $H_5,\ L_3,\ L_6$	Q_5
(89)	2-blow-up of L_6	Q_6
(90)	2-blow-up of L_7	Q_7
(91)	2-blow-up of $L_5,\ L_9$	Q_8
(92)	2-blow-up of $L_3,\ L_7,\ I_{10}$	Q_9
(93)	2-blow-up of $H_8,\ L_7,\ L_9$	Q_{10}
(94)	2-blow-up of $L_8,\ L_9$	Q_{11}
(95)	2-blow-up of $L_{10},\ L_{12},\ I_6$	Q_{12}
(96)	2-blow-up of $L_2,\ L_5,\ L_{13}$	Q_{13}
(97)	2-blow-up of $H_9,\ L_4,\ L_6,\ L_{12},\ I_{14}$	Q_{14}
(98)	2-blow-up of $L_6,\ L_9,\ L_{11},\ I_{13}$	Q_{15}
(99)	2-blow-up of $L_{11},\ L_{13},\ I_9$	Q_{16}
(100)	2-blow-up of $L_7,\ L_{12},\ I_{12}$	Q_{17}
(101)	2-blow-up of $H_1,\ H_3,\ H_7$	K_1
(102)	2-blow-up of $H_2,\ H_6,\ H_{10}$	K_2
(103)	2-blow-up of $H_4,\ H_5,\ H_9$	K_3
(104)	2-blow-up of H_8	K_4
(105)	2-blow-up of M_3	R_1
(106)	2-blow-up of $M_2,\ M_4$	R_2
(107)	2-blow-up of $M_1,\ M_4$	R_3
(108)	2-blow-up of $I_{11},\ I_{13}$	
(109)	2-blow-up of $Q_1,\ Q_3,\ Q_{13}$	U_1
(110)	2-blow-up of $Q_2,\ Q_5,\ Q_{14},\ K_3$	U_2
(111)	2-blow-up of $Q_4,\ Q_9$	U_3
(112)	2-blow-up of $Q_{10},\ K_4$	U_4
(113)	2-blow-up of Q_{11}	U_5

(114)	2-blow-up of $Q_6,\ Q_8,\ Q_{15}$	U_6
(115)	2-blow-up of $Q_7,\ Q_{12},\ Q_{17}$	U_7
(116)	2-blow-up of Q_{16}	U_8
(117)	none (See Definition 1.6.4 and Remark 1.6.5)	$\widetilde{V}^4$
(118)	none (See Definition 1.6.4 and Remark 1.6.5)	V^4
(119)	2-blow-up of $Q_{10},\ Q_{11}$	$S_2 \times S_2$
(120)	2-blow-up of $U_4,\ U_5,\ S_2 \times S_2$	$S_2 \times S_3$
(121)	2-blow-up of $S_2 \times S_3$	$S_3 \times S_3$
(122)	2-blow-up of G_6	Z_1
(123)	2-blow-up of G_4	Z_2
(124)	2-blow-up of Z_1 (See Example 1.4.7)	W

Chapter 2

Remarks on abelian surfaces in nonsingular toric Fano 4-folds

2.1 Introduction

There exist no embeddings from abelian surfaces into nonsingular projective toric 3-folds over **C** (see, e.g., Kajiwara [13] and [14]). So, the next problem is to study which nonsingular projective toric 4-folds admit embeddings from abelian surfaces. This problem was considered by many people (see Horrocks-Mumford [11], Hulek [12], Kajiwara [13], [14], Lange [19] and Sankaran [31]). In this chapter, we consider the following problem.

Problem 2.1.1 Which nonsingular toric Fano 4-fold admits a *totally nondegenerate embedding* from an abelian surface (see Definition 2.2.1)?

There exist exactly 124 nonsingular toric Fano 4-folds up to isomorphism (see Batyrev [6] and Sato [33]). We give a partial answer to Problem 2.1.1 (see Theorem 2.6.4).

The content of this chapter is as follows: In Section 2.2, we recall the definition of a *totally nondegenerate embedding*. In Section 2.3, we describe criteria for the non-existence of totally nondegenerate finite morphisms, and using these criteria, we show the non-existence for some nonsingular projective toric 4-folds. In Section 2.4, we consider the relationship between 2-blow-ups of toric 4-folds and totally nondegenerate finite morphisms. As a result, we can derive the main result in Section 2.6. In Section 2.5, we show the non-existence of totally nondegenerate finite morphisms for some nonsingular toric Fano 4-folds. In Section 2.6, we obtain the main result.

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2.2 totally nondegenerate embedding

The following notation is used throughout this chapter. For fundamental properties of the toric geometry, see Oda [26].

Let $N := \mathbf{Z}^4$ and $M := \mathrm{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$ the dual group. For a finite complete nonsingular fan Σ in N and $0 \le i \le 4$, we put $\Sigma(i) := \{\sigma \in \Sigma \mid \dim \sigma = i\}$. Each $\tau \in \Sigma(1)$ determines a unique element $e(\tau) \in N$ which generates the semigroup $\tau \cap N$. We put

$$\mathrm{G}(\Sigma) := \{e(\tau) \in N \mid \tau \in \Sigma(1)\}\,.$$

Let X be the complete nonsingular toric 4-fold corresponding to the fan Σ. Let $\mathrm{G}(\Sigma) = \{x_1, \ldots, x_n\}$ and let $\{D_1, \ldots, D_n\}$ be the corresponding T_N-invariant prime divisors on X. In particular, the Picard number of X is $n - 4$.

In this chapter, we study the finite morphisms from abelian surfaces to nonsingular complete toric 4-folds satisfying the following condition.

Definition 2.2.1 Let X be a 4-dimensional complete nonsingular toric variety and A an abelian surface. A finite morphism $\varphi : A \to X$ is called a *totally nondegenerate finite morphism* if $D_i \cap \varphi(A)$ is non-empty on $\varphi(A)$ for any T_N-invariant prime divisor D_i $(1 \le i \le n)$. If φ is an embedding, we call φ a *totally nondegenerate embedding.*

Notation 2.2.2 We use the following notation throughout this chapter.

(1) Let $\varphi : A \to X$ be a totally nondegenerate finite morphism. For a T_N-invariant prime divisor D_i on X, we put $C_i := \varphi^* D_i$. C_i is an effective divisor on A.

(2) The types of nonsingular toric Fano 4-folds are in the sense of Batyrev [6] and Sato [33]. We use characters $\mathcal{B}, \mathcal{C}, \mathcal{D} \ldots$ instead of $B, C, D \ldots$. See Table 1 in Section 2.7.

For a basis $\{x_1, x_2, x_3, x_4\}$ of N, by computing the divisors of the rational functions $\mathbf{e}(x_1^*)$, $\mathbf{e}(x_2^*)$, $\mathbf{e}(x_3^*)$, $\mathbf{e}(x_4^*) \in \mathbf{C}(X)$, where $\{x_1^*, x_2^*, x_3^*, x_4^*\} \subset M$ is the dual basis of $\{x_1, x_2, x_3, x_4\}$, we obtain four linear relations

$$\mathrm{div}\,(\mathbf{e}(x_1^*)) = 0,\ \mathrm{div}\,(\mathbf{e}(x_2^*)) = 0,\ \mathrm{div}\,(\mathbf{e}(x_3^*)) = 0 \text{ and } \mathrm{div}\,(\mathbf{e}(x_4^*)) = 0$$

among T_N-invariant prime divisors in $\mathrm{Pic}(X)$. We often use this argument in the following sections.

2.3 Criteria for non-existence

In this section, we present criteria for the non-existence of totally nondegenerate finite morphism from abelian surface A to a projective nonsingular toric 4-fold. We reduce Kajiwara's method in [13] and [14] to a more convenient form. For fundamental properties of primitive collections and primitive relations, see Batyrev [5], [6] and Sato [33].

Lemma 2.3.1 *Let A be an abelian surface and D an effective divisor on A. Then we have $D^2 \geq 0$.*

Proof. We may assume that D is an irreducible curve on A. For some point $x \in A$, we have $D(D+x) \geq 0$, where $D+x$ is the translation of D by x. Since D and $D+x$ are algebraically equivalent, we have $D^2 = D(D+x) \geq 0$. q.e.d.

Lemma 2.3.2 *Let X be a complete nonsingular toric 4-fold and $\varphi : A \to X$ a totally nondegenerate finite morphism. If $(\varphi^* D_i)(\varphi^* D_j) = 0$ and $(\varphi^* D_j)(\varphi^* D_k) = 0$, where $1 \leq i, j, k \leq n$, then we have $(\varphi^* D_i)(\varphi^* D_k) = 0$.*

Proof. Suppose that $(\varphi^* D_i)(\varphi^* D_k) > 0$. Since $(\varphi^* D_i + \varphi^* D_k)^2 = (\varphi^* D_i)^2 + (\varphi^* D_k)^2 + 2(\varphi^* D_i)(\varphi^* D_k) \geq 2(\varphi^* D_i)(\varphi^* D_k) > 0$ by Lemma 2.3.1, $\varphi^* D_i + \varphi^* D_k$ is an ample divisor on A. On the other hand, $(\varphi^* D_i + \varphi^* D_k)(\varphi^* D_j) = 0$ by assumption. This contradicts the fact that $\varphi^* D_i + \varphi^* D_k$ is ample. Therefore, $(\varphi^* D_i)(\varphi^* D_k) = 0$. q.e.d.

For a totally nondegenerate finite morphism $\varphi : A \to X$, we define a graph Γ_φ as follows: The vertex set of Γ_φ is $\{1, \ldots, n\}$, and $\{i, j\}$ is an edge of Γ_φ if $i \neq j$ and $(\varphi^* D_i)(\varphi^* D_j) = C_i C_j = 0$.

Remark 2.3.3 Lemma 2.3.2 implies that every connected component of Γ_φ is complete, that is, any pair of distinct vertices is connected by an edge. In particular, if Γ_φ is connected, then Γ_φ is a complete graph.

Lemma 2.3.4 *Let X be a* projective *nonsingular toric* 4*-fold. If there exists a totally nondegenerate finite morphism $\varphi : A \to X$, then $(\varphi^* D_i)(\varphi^* D_j) > 0$ for some $1 \leq i < j \leq n$.*

Proof. Since X is projective, there exists an ample effective divisor $\sum_{k=1}^{n} a_k D_k$ on X. Since $\sum_{k=1}^{n} a_k \varphi^* D_k$ is also ample on A, we have $(\sum_{k=1}^{n} a_k \varphi^* D_k)^2 > 0$. Therefore, we have $(\varphi^* D_i)(\varphi^* D_j) > 0$ for some $1 \leq i \leq j \leq n$. If $i = j$, then there exists $1 \leq l \leq n$ such that $l \neq i$ and $(\varphi^* D_i)(\varphi^* D_l) > 0$ by Lemma 2.3.2. q.e.d.

Remark 2.3.5 Lemma 2.3.4 implies that Γ_φ is not complete. In particular, by Remark 2.3.3, Γ_φ is not connected.

Remark 2.3.6 For $n > 5$, the assertion in Remark 2.3.5 is also true if we replace the vertex set of Γ_φ by $S \subset \{1, \ldots, n\}$ such that $\{D_i\}_{i \in S} \subset \mathrm{Pic}(X)$ generates $\mathrm{Pic}(X)$.

By using this incompleteness of Γ_φ, we can show the non-existence of totally nondegenerate finite morphisms for some projective nonsingular toric 4-folds. For example, the following holds.

Example 2.3.7 Let X be the nonsingular projective toric 4-fold corresponding to the fan Σ with primitive relations

$$x_1 + x_7 = 0,\ x_2 + x_3 + x_4 = a x_1,\ x_4 + x_5 + x_6 = (a+1) x_1,$$

$$x_5 + x_6 + x_7 = x_2 + x_3 \text{ and } x_1 + x_2 + x_3 = x_5 + x_6,$$

where $\mathrm{G}(\Sigma) = \{x_1, \ldots, x_7\}$ and a is a positive integer. $D_1 D_7 = 0$ on X because $\{x_1, x_7\}$ is a primitive collection of Σ, and by a basis $\{x_1, x_2, x_4, x_5\}$ of N, we have

$$(1)\ D_1 + aD_3 + (a+1)D_6 - D_7 = 0,\ (2)\ D_2 - D_3 = 0,$$

$$(3)\ -D_3 + D_4 - D_6 = 0 \text{ and } (4)\ D_5 - D_6 = 0$$

in $\mathrm{Pic}(X)$, respectively. Suppose that there exists a totally nondegenerate finite morphism $\varphi : A \to X$. By intersecting D_1 with both sides of (1) and restricting the result to A, we have $C_1^2 + aC_1C_3 + (a+1)C_1C_6 - C_1C_7 = 0$. So, we have $C_1C_3 = C_1C_6 = 0$ and $C_1C_2 = C_1C_4 = C_1C_5 = 0$ by (2), (3) and (4). Hence Γ_φ is connected, a contradiction to Remark 2.3.5. Therefore, X admits no totally nondegenerate finite morphism.

Remark 2.3.8 In Example 2.3.7, if $a = 1$, then X is the nonsingular toric Fano 4-fold of type $\mathcal{G}_1$. So there exists no totally nondegenerate finite morphism to the nonsingular toric Fano 4-fold of type $\mathcal{G}_1$.

Proposition 2.3.9 *If a nonsingular projective toric* 4*-fold* X *has an equivariant projective birational divisorial contraction to a possibly singular point, then there exist no totally nondegenerate finite morphism to* X*.*

Proof. Suppose that there exists a totally nondegenerate finite morphsim $\varphi : A \to X$. By the Mori theory for projective toric varieties (see Reid [30]), we may assume, without loss of generality, that we have a primitive relation $x_1+x_2+x_3+x_4 = ax_5$, where a is a positive integer. Obviously, $D_5D_i = 0$ for $6 \le i \le n$. By a basis $\{x_1, x_2, x_3, x_5\}$ of N, we have $aD_4 + D_5 + b_6D_6 + \cdots + b_nD_n = 0$ in $\mathrm{Pic}(X)$. So $aC_4C_5 + C_5^2 = 0$ on A. Therefore, $C_4C_5 = 0$, and similarly $C_iC_5 = 0$ for $1 \le i \le 4$. This means that Γ_φ is connected, a contradiction to Remark 2.3.5. q.e.d.

Remark 2.3.10 By Proposition 2.3.9, there exist no totally nondegenerate finite morphism to the nonsingular toric Fano 4-folds of types $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathcal{E}_3$.

We now consider the case where X is decomposed into the product of $\mathbf{P}^1$ and a projective nonsingular toric 3-fold. Let $X = \mathbf{P}^1 \times X'$ be a projective nonsingular toric 4-fold, where X' is a projective nonsingular toric 3-fold. Suppose that x_1 and x_2 correspond to the class of fibers of the first projection $X \to \mathbf{P}^1$, where $\mathrm{G}(\Sigma) = \{x_1, x_2, \ldots, x_n\}$. Then the following holds.

Proposition 2.3.11 *Suppose that there exists a totally nondegenerate finite morphism* $\varphi : A \to X$*. We define a subgraph* Γ' *of* Γ_φ *as follows: The vertex set of* Γ' *is* $\{3, \ldots, n\}$*, and* $\{i, j\}$ $(3 \le i < j \le n)$ *is an edge of* Γ' *if* $\{i, j\}$ *is an edge of* Γ_φ*. Then* Γ' *is not complete.*

Proof. Since each fiber of the second projection $p_2 : X \to X'$ is $\mathbf{P}^1$, each fiber of p_2 is not contained in the abelian surface A. So $p_2 \circ \varphi$ is a finite morphism. Therefore, for an ample divisor $E = \sum_{i=3}^n a_iE_i$ on the projective variety X', where $E_3, \ldots, E_n$ are the toric divisors corresponding to $x_3, \ldots, x_n$, respectively, $(p_2 \circ \varphi)^*(E) = \sum_{i=3}^n a_i(p_2 \circ \varphi)^*D_i$ is also an ample divisor on A. So, there exist $3 \le i < j \le n$ such that $((p_2 \circ \varphi)^*D_i)((p_2 \circ \varphi)^*D_j) \neq 0$. Since $\{i, j\}$ is not an edge of Γ', the graph Γ' is not complete. q.e.d.

Example 2.3.12 Let X be the nonsingular projective toric 4-fold corresponding to the fan Σ with primitive relations

$$x_1 + x_8 = 0,\ x_2 + x_3 = 0,\ x_4 + x_5 = ax_3 \text{ and } x_6 + x_7 = ax_3,$$

where $\mathrm{G}(\Sigma) = \{x_1, \ldots, x_8\}$ and a is a positive integer. $D_1D_8 = D_2D_3 = D_4D_5 = D_6D_7 = 0$ on X, and by a basis $\{x_1, x_2, x_4, x_6\}$ of N, we have

$$(1)\ D_1 - D_8 = 0,\ (2)\ D_2 - D_3 - aD_5 - aD_7 = 0,\ (3)\ D_4 - D_5 = 0 \text{ and } (4)\ D_6 - D_7 = 0$$

in $\mathrm{Pic}(X)$, respectively. X is isomorphic to $\mathbf{P}^1 \times X'$, where X' is a toric 3-fold, and D_1 and D_8 are fibers of the first projection $X \to \mathbf{P}^1$. Suppose that there exists a totally nondegenerate finite morphism $\varphi : A \to X$. By (2), we have $C_3^2 + aC_3C_5 + aC_3C_7 = C_2C_3 = 0$, and hence $C_3C_5 = C_3C_7 = 0$. Consequently, $C_3C_4 = C_3C_6 = 0$ by (3) and (4). Thus, the graph Γ' as in Proposition 2.3.11 is connected, a contradiction to Proposition 2.3.11. Therefore, X admits no totally nondegenerate finite morphism.

Remark 2.3.13 In Example 2.3.12, if $a = 1$, then X is the nonsingular toric Fano 4-fold of type $\mathcal{L}_5$. So there exist no totally nondegenerate finite morphism to the nonsingular toric Fano 4-fold of type $\mathcal{L}_5$.

For the main theorem of this chapter, we show some results for the non-existence of totally nondegenerate finite morphisms using Remark 2.3.5 and Proposition 2.3.11.

Proposition 2.3.14 *Let X be an F_a-bundle over $\mathbf{P}^2$, where F_a is the Hirzebruch surface of degree a $(a \geq 0)$, and $\mathrm{G}(\Sigma) = \{x_1, \ldots x_7\}$. We introduce a coordinate in N so that the coordinates of $x_1, x_2, x_3, x_4, x_5, x_6$ and x_7 are*

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ s \\ t \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ a \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix},$$

respectively, where s and t are integers. In this situation, the following hold:

(1) *If $s = t = 0$, then X is isomorphic to $\mathbf{P}^2 \times \mathrm{F}_a$.*

(2) *In the case $s \neq 0$ or $t \neq 0$, if one of the following conditions is satisfied, then X admits no totally nondegenerate finite morphism.*

(a) $a = 0$.

(b) $a > 0$ *and* $t \geq 0$.

(c) $a > 0$, $s > 0$, $t < 0$ *and* $as + t \geq 0$.

Proof. Suppose that there exists a totally nondegenerate finite morphism $\varphi : A \to X$. (i) is obvious. So, let $s \neq 0$ or $t \neq 0$. By a basis $\{x_1, x_2, x_4, x_6\}$ of N, we have

(1) $D_1 - D_3 = 0$, (2) $D_2 - D_3 = 0$, (3) $sD_3 + D_4 - D_5 = 0$ and (4) $tD_3 + aD_5 + D_6 - D_7 = 0$

in $\mathrm{Pic}(X)$, respectively. Moreover, we have $D_4D_5 = D_6D_7 = 0$ on X, and $C_4^2 = C_5^2 = C_6^2 = C_7^2 = 0$ on A.

(a) Let $a = 0$. In this case, X is isomorphic to $\mathbf{P}^1 \times X'$, where X' is a toric 3-fold.

If $s = 0$ and $t \neq 0$, then D_4 and D_5 are in the class of fibers of the first projection $X \to \mathbf{P}^1$. Since $tD_3 + D_6 - D_7 = 0$ by (4), we have $tC_3C_6 = -C_6^2 + C_6C_7 = 0$. Therefore, $C_3C_6 = 0$, and hence $C_1C_6 = C_2C_6 = 0$ by (1) and (2). This contradicts Proposition 2.3.11.

If $s \neq 0$ and $t = 0$, then D_6 and D_7 are in the class of fibers of the first projection $X \to \mathbf{P}^1$. Since $sC_3C_4 = -C_4^2 + C_4C_5 = 0$ by (3), we have $C_3C_4 = 0$. On the other hand, $C_1C_4 = C_2C_4 = 0$ by (1) and (2). This contradicts Proposition 2.3.11.

(b) Let $a > 0$ and $t \geq 0$. Since

$$(5)\ tC_3C_6 + aC_5C_6 = -C_6^2 + C_6C_7 = 0$$

by (4), we have $C_5C_6 = 0$.

If $t > 0$, then $C_3C_6 = 0$ by (5). Moreover $C_1C_6 = C_2C_6 = 0$ by (1) and (2). So Γ_φ is connected, a contradiction to Remark 2.3.5.

Let $t = 0$. Then $s \neq 0$ by assumption. So, we have $C_3C_4 = 0$ as above, and hence $C_1C_4 = C_2C_4 = 0$ by (1) and (2). So, Γ_φ is connected, a contradiction to Remark 2.3.5.

(c) Let $a > 0$, $s > 0$, $t < 0$ and $as + t \geq 0$. Then we have $C_3C_4 = C_1C_4 = C_2C_4 = 0$ as above. On the other hand, by (3) and (4), we have

$$(6)\ -tD_4 + (as + t)D_5 + sD_6 - sD_7 = 0$$

in $\mathrm{Pic}(X)$. So, $-tC_4C_6 + (as+t)C_5C_6 = -sC_6^2 + sC_6C_7 = 0$, and hence we have $C_4C_6 = 0$ by the assumptions $t < 0$ and $as + t \geq 0$. Therefore, Γ_φ is connected, a contradiction to Remark 2.3.5. q.e.d.

Remark 2.3.15 By Proposition 2.3.14, there exist no totally nondegenerate finite morphism to the nonsingular toric Fano 4-folds of types $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, $\mathcal{D}_5$, $\mathcal{D}_6$, $\mathcal{D}_8$, $\mathcal{D}_9$, $\mathcal{D}_{12}$ and $\mathcal{D}_{16}$. The corresponding a, s and t are as follows:

	$\mathcal{D}_1$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{D}_5$	$\mathcal{D}_6$	$\mathcal{D}_8$	$\mathcal{D}_9$	$\mathcal{D}_{12}$	$\mathcal{D}_{16}$
a	1	1	1	0	1	1	0	0	1
s	0	2	1	2	0	1	1	1	1
t	2	0	1	0	1	0	1	0	-1

2.4 2-blow-up

The following is useful for deriving the main result in Section 2.6.

Proposition 2.4.1 *Let X and $\widetilde{X}$ be nonsingular projective toric 4-folds and $\psi : \widetilde{X} \to X$ a 2-blow-up, where a "2-blow-up" means an equivariant blow-up along a T_N-invariant subvariety of codimension 2. If X admits no totally nondegenerate finite morphism, then $\widetilde{X}$ admits no totally nondegenerate finite morphism either.*

Proof. If there exists a totally nondegenerate finite morphism $\varphi : A \to \widetilde{X}$, then $\psi \circ \varphi : A \to X$ is also a totally nondegenerate finite morphism (see Mumford [24], p. 88). This is a contradiction. q.e.d.

In particular, we have the following.

Corollary 2.4.2 *Let $X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_{n-1} \leftarrow X_n$ be a sequence of 2-blow-ups among nonsingular projective toric 4-folds. If X_1 admits no totally nondegenerate finite morphism, then X_n admits no totally nondegenerate finite morphism either.*

We close this section by proposing the following conjecture.

Conjecture 2.4.3 *Let X and $\widetilde{X}$ be nonsingular projective toric 4-folds and $\psi : \widetilde{X} \to X$ a 2-blow-up. If X admits no totally nondegenerate embedding, then $\widetilde{X}$ admits no totally nondegenerate embedding either.*

2.5 Certain examples

In this section, to describe the main result in Section 2.6, we show the non-existence of totally nondegenerate embeddings for certain nonsingular toric Fano 4-folds.

(a) "type $\mathcal{I}$'s" Let X be the nonsingular projective toric 4-fold corresponding to the fan Σ defined as follows: Let $\mathrm{G}(\Sigma) = \{x_1, \ldots, x_8\} \subset N$ such that the coordinates of $x_1, \ldots, x_8$ are

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\b\\0\\c\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\-1\\a+1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix},$$

respectively, and that the primitive collections of Σ are $\{x_3, x_4, x_5\}$, $\{x_4, x_5, x_7\}$, $\{x_7, x_8\}$, $\{x_3, x_6\}$, $\{x_6, x_8\}$ and $\{x_1, x_2\}$. For certain values of a, b and c, X becomes the nonsingular toric Fano 4-fold of type $\mathcal{I}$. The corresponding a, b and c are as follows:

	$\mathcal{I}_4$	$\mathcal{I}_6$	$\mathcal{I}_{12}$	$\mathcal{I}_{15}$
a	1	0	0	1
b	1	1	0	0
c	-1	0	-1	-1

$D_7D_8 = D_3D_6 = D_6D_8 = D_1D_2 = 0$ on X, and by a basis $\{x_1, x_3, x_4, x_8\}$ of N, we have

$$(1)\ D_1 - D_2 = 0,\ (2)\ bD_2 + D_3 - D_5 + D_7 = 0,$$

$$(3)\ D_4 - D_5 = 0 \text{ and } (4)\ cD_2 + (a+1)D_5 - D_6 - D_7 + D_8 = 0$$

in $\mathrm{Pic}(X)$, respectively. Moreover, we have

$$(5)\ (b+c)D_2 + D_3 + aD_5 - D_6 + D_8 = 0$$

by (2) and (4). Suppose that there exists a totally nondegenerate finite morphism $\varphi : A \to X$.

If $b = 0$ and $c = -1$, then $C_5C_6 = C_3C_6 + C_6C_7 = 0$ by (2). On the other hand, by (4), we have $C_2C_5 = C_5C_8 + (a+1)C_5^2 - C_5C_6 - C_5C_7 = 0$. Hence Γ_φ is connected, a contradiction to Remark 2.3.5. Therefore, X admits no totally nondegenerate finite morphism. In particular, the nonsingular toric Fano 4-folds of types $\mathcal{I}_{12}$ and $\mathcal{I}_{15}$ admit no totally nondegenerate finite morphism.

In the case $b=1$, if $a=1$ and $b+c=0$, then $C_3C_5=-C_3^2+C_3C_6-C_3C_8=0$ by (5), and $C_2C_3=-C_3^2+C_3C_5-C_3C_7=0$ by (2). On the other hand, if $a=0$ and $b+c=1$, then $C_2C_3=0$ by (5), and $C_3C_5=0$ by (2). In any case, Γ_φ is connected, a contradiction to Remark 2.3.5. Therefore, X admits no totally nondegenerate finite morphism. In particular, the nonsingular toric Fano 4-folds of types $\mathcal{I}_4$ and $\mathcal{I}_6$ admit no totally nondegenerate finite morphism.

(b) "type $\mathcal{J}_2$" Let X be the nonsingular projective toric 4-fold corresponding to the fan Σ with primitive relations

$$x_3+x_6=x_7,\ x_1+x_2+x_8=x_4+x_5,\ x_4+x_5+x_6=x_1+x_2,\ x_7+x_8=x_3,\ x_6+x_8=0,$$

$$x_3+x_4+x_5=x_8,\ x_4+x_5+x_7=0,\ x_1+x_2+x_3=0 \text{ and } x_1+x_2+x_7=x_6,$$

where $\mathrm{G}(\Sigma)=\{x_1,\ldots,x_8\}$. X is the nonsingular toric Fano 4-fold of type $\mathcal{J}_2$. $D_3D_6=D_7D_8=D_6D_8=0$ on X, and by a basis $\{x_1,x_2,x_4,x_5\}$ of N, we have

$$(1)\ D_1-D_3+D_6-D_8=0,\ (2)\ D_2-D_3+D_6-D_8=0,$$

$$(3)\ D_4-D_6-D_7+D_8=0 \text{ and } (4)\ D_5-D_6-D_7+D_8=0$$

in $\mathrm{Pic}(X)$, respectively. So, we have $D_1=D_2$ and $D_4=D_5$. Suppose that there exists a totally nondegenerate finite morphism $\varphi: A\to X$. By (1), we have $C_1C_3=C_3^2-C_3C_6+C_3C_8=0$. On the other hand, by (3), we have $C_3C_4=C_3C_6+C_3C_7-C_3C_8=0$. Hence Γ_φ is connected, a contradiction to Remark 2.3.5. Therefore, X admits no totally nondegenerate finite morphism.

(c) "type $\mathcal{L}$'s" Let X be the nonsingular projective toric 4-fold corresponding to the fan Σ defined as follows: Let $\mathrm{G}(\Sigma)=\{x_1,\ldots,x_8\}\subset N$ such that the coordinates of $x_1,\ldots,x_8$ are

$$\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ \begin{pmatrix}-1\\1\\0\\0\end{pmatrix},\ \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\1\\0\end{pmatrix},\ \begin{pmatrix}a\\b\\-1\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\0\\1\end{pmatrix},\ \begin{pmatrix}c\\d\\0\\-1\end{pmatrix},\ \begin{pmatrix}0\\-1\\0\\0\end{pmatrix},$$

respectively, and that the primitive collections of Σ are $\{x_1, x_8\}$, $\{x_2, x_3\}$, $\{x_4, x_5\}$ and $\{x_6, x_7\}$. For certain values of a, b, c and d, X becomes the nonsingular toric Fano 4-fold of type $\mathcal{L}$. The corresponding a, b, c and d are as follows:

	$\mathcal{L}_1$	$\mathcal{L}_2$	$\mathcal{L}_{10}$
a	0	1	1
b	1	0	0
c	0	1	-1
d	1	0	1

$D_1D_8 = D_2D_3 = D_4D_5 = D_6D_7 = 0$ on X, and by a basis $\{x_1, x_3, x_4, x_6\}$ of N, we have

$$(1)\ D_1 + D_2 + bD_5 + dD_7 - D_8 = 0,\ (2)\ - D_2 + D_3 + aD_5 + cD_7 = 0,$$

$$(3)\ D_4 - D_5 = 0 \text{ and } (4)\ D_6 - D_7 = 0$$

in $\mathrm{Pic}(X)$, respectively. Suppose that there exists a totally nondegenerate finite morphism $\varphi : A \to X$. By (1) and (2), we have $C_1C_2 + bC_1C_5 + dC_1C_7 = -C_1^2 + C_1C_8 = 0$ and $aC_3C_5 + cC_3C_7 = -C_3^2 + C_2C_3 = 0$, respectively, and hence we have $C_1C_2 = 0$.

If either $a > 0$ and $c > 0$ or $b > 0$ and $d > 0$, then either $C_3C_5 = C_3C_5 = 0$ or $C_1C_5 = C_1C_7 = 0$, respectively. In any case, Γ_φ is connected, a contradiction to Remark 2.3.5. Therefore, X admits no totally nondegenerate finite morphism. In particular, the nonsingular toric Fano 4-folds of types $\mathcal{L}_1$ and $\mathcal{L}_2$ admit no totally nondegenerate finite morphism.

Let X be of type $\mathcal{L}_{10}$. Since $b \geq 0$ and $d > 1$, we have $C_1C_7 = 0$. Moreover, by (2), we have $C_1C_5 = -C_1C_3 + C_2C_3 + C_1C_7 = 0$. Hence Γ_φ is connected, a contradiction to Remark 2.3.5. Therefore, the nonsingular toric Fano 4-fold of type $\mathcal{L}_{10}$ admits no totally nondegenerate finite morphism.

(d) "type $\mathcal{L}_{12}$" This case is special. Let X be the nonsingular projective toric 4-fold corresponding to the fan Σ with primitive relations

$$x_1 + x_8 = 0,\ x_2 + x_3 = x_1,\ x_4 + x_5 = x_8 \text{ and } x_6 + x_7 = x_4,$$

where $\mathrm{G}(\Sigma) = \{x_1, \ldots, x_8\}$. X is the nonsingular toric Fano 4-fold of type $\mathcal{L}_{12}$. $D_1D_8 = D_2D_3 = D_4D_5 = D_6D_7 = 0$ on X, and by a basis $\{x_1, x_2, x_4, x_6\}$ of N, we have

$$(1)\ D_1 + D_3 - D_5 - D_8 = 0,\ (2)\ D_2 - D_3 = 0,$$

$$(3)\ D_4 - D_5 + D_7 = 0 \text{ and } (4)\ D_6 - D_7 = 0$$

in $\mathrm{Pic}(X)$, respectively. So we have $D_2^2 = D_3^2 = D_6^2 = D_7^2 = 0$,

$$(5)\ D_8^2 = D_3D_8 - D_5D_8 \text{ and } (6)\ D_5^2 = D_5D_7$$

on X. Suppose that there exists a totally nondegenerate embedding $\varphi : A \hookrightarrow X$. Then by (3), we have $D_4D_7A = -(D_4A)^2 + D_4D_5A = 0$. Moreover, since D_1A is an effective divisor on A and $D_3D_8A - D_5D_8A = 0$ by (5), we have $(D_1A)^2 = (-D_3A + D_5A + D_8A)^2 = -2D_3D_5A - 2D_3D_8A + 2D_5D_8A = -2D_3D_5A \geq 0$. Therefore, we have $D_3D_5A = 0$. On the other hand, $\{D_3D_5, D_3D_7, D_3D_8, D_5D_7, D_5D_8, D_7D_8\}$ generates $\mathrm{A}^2(X)$ by the equalities (5) and (6). So, we can express the class of A in $\mathrm{A}^2(X)$ as

$$A = a_1D_3D_5 + a_2D_3D_7 + a_3D_3D_8 + a_4D_5D_7 + a_5D_5D_8 + a_6D_7D_8 \in \mathrm{A}^2(X).$$

Since $D_3D_5^2D_7 = D_3(D_4 + D_7)D_5D_7 = 0$, $D_3D_5^2D_8 = D_3(D_4 + D_7)D_5D_8 = 1$ and $D_5^2D_7D_8 = (D_4 + D_7)D_5D_7D_8 = 0$, we have the following:

$$(7) \begin{cases} D_3D_5A = a_4D_3D_5^2D_7 + a_5D_3D_5^2D_8 + a_6D_3D_5D_7D_8 = a_5 + a_6 = 0, \\ D_3D_7A = a_5D_3D_5D_7D_8 = a_5 = 0 \text{ and} \\ D_5D_7A = a_1D_3D_5^2D_7 + a_3D_3D_5D_7D_8 + a_5D_5^2D_7D_8 = a_3 = 0. \end{cases}$$

By these equalities, we have $a_3 = a_5 = a_6 = 0$. So, $A^2 = a_1D_3D_5A + a_2D_3D_7A + a_4D_5D_7A = 0$. Therefore, by the following, we have $c_2(X)A = 0$.

Lemma 2.5.1 (Van de Ven [34], Proposition 3) *Let $A \hookrightarrow X$ be an embedding from an abelian surface A to a 4-dimensional nonsingular projective toric variety X. Then, we have $c_2(X)A = A^2$ in the group $\mathrm{A}^4(X)$ of codimension four cycles on X modulo rational equivalence.*

Since

$$c_2(X) = \sum_{1 \leq i < j \leq n} D_iD_j,$$

Γ_φ is connected. This contradicts Remark 2.3.5. So, X admits no totally nondegenerate embedding.

(e) "type $\mathcal{M}$'s" Let X be the nonsingular projective toric 4-fold corresponding to the fan Σ defined as follows: Let $\mathrm{G}(\Sigma) = \{x_1, \ldots, x_7\} \subset N$ such that the coordinates of

$x_1, \dots, x_7$ are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} a \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} ac+b \\ 0 \\ -c \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

respectively, and that the primitive collections of Σ are $\{x_1, x_8\}$, $\{x_1, x_2, x_3\}$, $\{x_4, x_6, x_8\}$, $\{x_4, x_5\}$, $\{x_6, x_7\}$, $\{x_2, x_3, x_5\}$ and $\{x_2, x_3, x_7\}$. For certain values of a, b and c, X becomes the nonsingular toric Fano 4-fold of type $\mathcal{M}$. The corresponding a, b and c are as follows:

	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$
a	0	1	1	1
b	0	1	0	0
c	0	0	1	0

$D_1D_8 = D_4D_5 = D_6D_7 = 0$ on X, and by a basis $\{x_1, x_2, x_4, x_6\}$ of N, we have

$$(1)\ D_1 - D_3 + aD_5 + (ac+b)D_7 - D_8 = 0,\ (2)\ D_2 - D_3 = 0,$$

$$(3)\ D_3 + D_4 - D_5 - cD_7 = 0 \text{ and } (4)\ D_3 + D_6 - D_7 = 0$$

in $\mathrm{Pic}(X)$, respectively. Suppose that there exists a totally nondegenerate finite morphism $\varphi : A \to X$. By (4), we have $C_3C_6 = -C_6^2 + C_6C_7 = 0$. By (1), (3) and (4), we have

$$(5)\ D_1 + (a-1)D_3 + aD_4 + bD_7 - D_8 = 0 \text{ and } (6)\ D_1 + aD_5 + D_6 + (ac+b-1)D_7 - D_8 = 0.$$

Let $a \geq 0$, $b \geq 0$ and $c \geq 0$.

If $a = 1$ and $c = 0$, then, by (3), we have $C_3C_4 = -C_4^2 + C_4C_5 = 0$. On the other hand, by (5), we have $C_1C_4 + bC_1C_7 = -C_1^2 + C_1C_8 = 0$, and hence $C_1C_4 = 0$. Hence Γ_φ is connected, a contradiction to Remark 2.3.5. Therefore, X admits no totally nondegenerate finite morphism. In particular, the nonsingular toric Fano 4-folds of types $\mathcal{M}_2$ and $\mathcal{M}_4$ admit no totally nondegenerate finite morphism.

If $a = b = c = 0$, then $C_3C_4 = 0$ as above. On the other hand, by (5), we have $C_1C_3 = C_1^2 - C_1C_8 = 0$, and hence $C_1C_3 = 0$. Hence Γ_φ is connected, a contradiction to Remark 2.3.5. Therefore, X admits no totally nondegenerate finite morphism. In particular, the nonsingular toric Fano 4-fold of type $\mathcal{M}_1$ admits no totally nondegenerate finite morphism.

If $a = c = 1$, then $C_1C_4 = 0$ as above. On the other hand, by (6), we have $C_1C_5 + C_1C_6 + bC_1C_7 = -C_1^2 + C_1C_8 = 0$, and hence $C_1C_6 = 0$. Hence Γ_φ is connected, a contradiction to Remark 2.3.5. Therefore, X admits no totally nondegenerate finite morphism. In particular, the nonsingular toric Fano 4-fold of type $\mathcal{M}_3$ admits no totally nondegenerate finite morphism.

(f) "type $\mathcal{M}_5$" Let X be the nonsingular projective toric 4-fold corresponding to the fan Σ with primitive relations

$$x_1 + x_8 = x_5,\ x_4 + x_5 = x_7,\ x_6 + x_7 = x_1,\ x_1 + x_2 + x_3 = x_6,\ x_2 + x_3 + x_5 = x_6 + x_8$$

$$x_2 + x_3 + x_7 = 0 \text{ and } x_4 + x_6 + x_8 = 0,$$

where $\mathrm{G}(\Sigma) = \{x_1, \ldots, x_8\}$. X is the nonsingular toric Fano 4-fold of type $\mathcal{M}_5$. $D_1D_8 = D_4D_5 = D_6D_7 = 0$ on X, and by a basis $\{x_1, x_2, x_4, x_6\}$ of N, we have

$$(1)\ D_1 - D_3 + D_5 + D_7 = 0,\ (2)\ D_2 - D_3 = 0,$$

$$(3)\ D_4 - D_5 - D_8 = 0 \text{ and } (4)\ D_3 - D_5 + D_6 - D_7 - D_8 = 0$$

in $\mathrm{Pic}(X)$, respectively. Suppose that there exists a totally nondegenerate finite morphism $\varphi : A \to X$. By (1), (3) and (4), we have $C_5C_8 = -C_4C_5 + C_5^2 = 0$ and $C_6C_8 = -C_1C_8 + C_8^2 = 0$. On the other hand, by Lemma 2.3.2, we have $C_1C_5 = C_1C_7 = 0$. So, by (1), we have $C_1C_3 = C_1^2 + C_1C_5 + C_1C_7 = 0$. Hence Γ_φ is connected, a contradiction to Remark 2.3.5. Therefore, X admits no totally nondegenerate finite morphism.

Remark 2.5.2 Kajiwara [13], [14] showed that the pseudo del Pezzo 4-fold $\widetilde{V}^4$ admits no totally nondegenerate finite morphism similarly as above.

2.6 The main results

By Examples 2.3.7 and 2.3.12, Propositions 2.3.9 and 2.3.14, the results (a), (b), (c), (d), (e) and (f) in Section 2.5, and Remark 2.5.2, the nonsingular toric Fano 4-folds of types $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, $\mathcal{D}_5$, $\mathcal{D}_6$, $\mathcal{D}_8$, $\mathcal{D}_9$, $\mathcal{D}_{12}$, $\mathcal{D}_{16}$, $\mathcal{G}_1$, $\mathcal{I}_4$, $\mathcal{I}_6$, $\mathcal{I}_{12}$, $\mathcal{I}_{15}$, $\mathcal{J}_2$, $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_5$, $\mathcal{L}_{10}$, $\mathcal{L}_{12}$, $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, $\mathcal{M}_4$, $\mathcal{M}_5$ and $\widetilde{V}^4$ admit no totally nondegenerate embedding. Moreover, by Corollary 2.4.2, any nonsingular projective toric 4-fold admits no totally nondegenerate embedding, if it is obtained by finite successions of 2-blow-ups from one of them.

Remark 2.6.1 Kajiwara [14] and Sankaran [31] showed that the nonsingular toric Fano 4-folds of types $\mathcal{B}_5$, $\mathcal{D}_{19}$, $\mathcal{G}_2$ and $\mathcal{G}_6$ admit no totally nondegenerate embedding using more complicated methods (see Kajiwara [14] for types $\mathcal{D}_{19}$, $\mathcal{G}_2$ and $\mathcal{G}_6$, Sankaran [31] for type $\mathcal{B}_5$). Since their method differs from ours, we cannot determine whether nonsingular projective toric 4-folds obtained by finite successions of 2-blow-ups from one of these types admit a totally nondegenerate embedding or not.

To describe the main result, we need the following proposition.

Proposition 2.6.2 *If X is a nonsingular toric Fano 4-fold such that $X \cong X_1 \times X_2$, where X_1 and X_2 are nonsingular toric del Pezzo surfaces, then there exists a totally nondegenerate embedding.*

Proof. A smooth element E_1 in $|-K_{X_1}|$ (resp. E_2 in $|-K_{X_2}|$) is an elliptic curve. By an easy calculation of intersection numbers, $E_1 \times E_2 \hookrightarrow X$ is obviously a totally nondegenerate embedding. q.e.d.

Remark 2.6.3 In Proposition 2.6.2, if there exists an abelian surface embedding $A \hookrightarrow X$, then A is isomorphic to the direct product of two elliptic curves as stated in the proof of Proposition 2.6.2, by the results of Kajiwara [13] and [14].

By these results and Table 1 in Sato [33], we get the following:

Theorem 2.6.4 *Let X be a nonsingular toric Fano 4-fold. Then, one of the following holds.*

(1) *X admits no totally nondegenerate embedding.*

(2) *$X \cong \mathbf{P}^4$ or $X \cong \mathbf{P}^1 \times \mathbf{P}^3$. There exists a totally nondegenerate embedding in this case* (*see* Horrocks-Mumford [11] *and* Lange [19]).

(3) *$X \cong X_1 \times X_2$, where X_1 and X_2 are nonsingular toric del Pezzo surfaces. There exists a totally nondegenerate embedding in this case* (*see Proposition* 2.6.2).

(4) *X is of one of the types $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{D}_7$, $\mathcal{D}_{10}$, $\mathcal{D}_{11}$, $\mathcal{D}_{14}$, $\mathcal{D}_{17}$, $\mathcal{D}_{18}$, $\mathcal{G}_3$, $\mathcal{G}_4$, $\mathcal{G}_5$, $\mathcal{L}_{11}$, $\mathcal{L}_{13}$, $\mathcal{I}_9$, $\mathcal{Q}_{16}$, $\mathcal{U}_8$, V^4, $\mathcal{Z}_1$, $\mathcal{Z}_2$ and $\mathcal{W}$.*

Remark 2.6.5 For the nonsingular toric Fano 4-fold X of type $\mathcal{C}_1$, Sankaran [31] showed that there exists a totally nondegenerate embedding $A \hookrightarrow X$. However, his paper seems to contain gaps unfortunately. So, we do not yet know whether X admits a totally nondegenerate embedding or not.

2.7 Table of nonsingular toric Fano 4-folds

In this section, we give the table of nonsingular toric Fano 4-folds classified in Batyrev [6] and Sato [33] with 2-blow-up relations among them. We describe the results about totally nondegenerate embeddings obtained in the previous sections. In the third column, we show whether the nonsingular toric Fano 4-fold admits a totally nondegenerate embedding or not. The symbol "∃" means that there exists a totally nondegenerate embedding, while the symbol "×" means that there does not exist a totally nondegenerate embedding. We omit a reference in the case where the noningular toric Fano 4-fold is obtained by finite successions of 2-blow-ups from one of the nonsingular toric Fano 4-folds of types $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, $\mathcal{D}_5$, $\mathcal{D}_6$, $\mathcal{D}_8$, $\mathcal{D}_9$, $\mathcal{D}_{12}$, $\mathcal{D}_{16}$, $\mathcal{G}_1$, $\mathcal{I}_4$, $\mathcal{I}_6$, $\mathcal{I}_{12}$, $\mathcal{I}_{15}$, $\mathcal{J}_2$, $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_5$, $\mathcal{L}_{10}$, $\mathcal{L}_{12}$, $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, $\mathcal{M}_4$, $\mathcal{M}_5$ and $\widetilde{V}^4$ (see Corollary 2.4.2).

Table 1: nonsingular toric Fano 4-folds

	2-blow-up of	embedding	notation
(1)	none	∃ (See Horrocks-Mumford [11])	$\mathbf{P}^4$
(2)	none	× (See Proposition 2.3.9)	$\mathcal{B}_1$
(3)	none	× (See Proposition 2.3.9)	$\mathcal{B}_2$
(4)	none	× (See Proposition 2.3.9)	$\mathcal{B}_3$
(5)	none	∃ (See Lange [19])	$\mathcal{B}_4$
(6)	$\mathbf{P}^4$	× (See Sankaran [31])	$\mathcal{B}_5$
(7)	none	unknown	$\mathcal{C}_1$
(8)	none	unknown	$\mathcal{C}_2$
(9)	none	unknown	$\mathcal{C}_3$
(10)	none	∃ (See Proposition 2.6.2)	$\mathcal{C}_4$

(11)	$\mathcal{B}_1,\ \mathcal{B}_2$	$\times$	$\mathcal{E}_1$
(12)	$\mathcal{B}_2,\ \mathcal{B}_3$	$\times$	$\mathcal{E}_2$
(13)	$\mathcal{B}_3,\ \mathcal{B}_4$	$\times$	$\mathcal{E}_3$
(14)	none	$\times$ (See Proposition 2.3.14)	$\mathcal{D}_1$
(15)	$\mathcal{C}_1$	$\times$ (See Proposition 2.3.14)	$\mathcal{D}_2$
(16)	none	$\times$ (See Proposition 2.3.14)	$\mathcal{D}_3$
(17)	$\mathcal{B}_2$	$\times$	$\mathcal{D}_4$
(18)	none	$\times$ (See Proposition 2.3.14)	$\mathcal{D}_5$
(19)	$\mathcal{C}_3$	$\times$ (See Proposition 2.3.14)	$\mathcal{D}_6$
(20)	none	unknown	$\mathcal{D}_7$
(21)	$\mathcal{C}_2$	$\times$ (See Proposition 2.3.14)	$\mathcal{D}_8$
(22)	none	$\times$ (See Proposition 2.3.14)	$\mathcal{D}_9$
(23)	$\mathcal{B}_5$	unknown	$\mathcal{D}_{10}$
(24)	$\mathcal{B}_5,\ \mathcal{C}_2$	unknown	$\mathcal{D}_{11}$
(25)	none	$\times$ (See Proposition 2.3.14)	$\mathcal{D}_{12}$
(26)	none	$\exists$ (See Proposition 2.6.2)	$\mathcal{D}_{13}$
(27)	$\mathcal{B}_4$	unknown	$\mathcal{D}_{14}$
(28)	$\mathcal{C}_4$	$\exists$ (See Proposition 2.6.2)	$\mathcal{D}_{15}$
(29)	$\mathcal{C}_3$	$\times$ (See Proposition 2.3.14)	$\mathcal{D}_{16}$
(30)	$\mathcal{B}_5$	unknown	$\mathcal{D}_{17}$
(31)	$\mathcal{C}_1$	unknown	$\mathcal{D}_{18}$
(32)	$\mathcal{C}_2$	$\times$ (See Kajiwara [14])	$\mathcal{D}_{19}$
(33)	none	$\times$ (See Example 2.3.7)	$\mathcal{G}_1$
(34)	$\mathcal{C}_2$	$\times$ (See Kajiwara [14])	$\mathcal{G}_2$
(35)	none	unknown	$\mathcal{G}_3$
(36)	$\mathcal{C}_2$	unknown	$\mathcal{G}_4$
(37)	$\mathcal{C}_3$	unknown	$\mathcal{G}_5$
(38)	$\mathcal{C}_4$	$\times$ (See Kajiwara [14])	$\mathcal{G}_6$
(39)	$\mathcal{D}_2$	$\times$	$\mathcal{H}_1$
(40)	$\mathcal{D}_3$	$\times$	$\mathcal{H}_2$
(41)	$\mathcal{D}_1,\ \mathcal{D}_5$	$\times$	$\mathcal{H}_3$

(42)	$\mathcal{D}_8,\ \mathcal{D}_9$	$\times$	$\mathcal{H}_4$
(43)	$\mathcal{D}_6,\ \mathcal{D}_{12},\ \mathcal{D}_{16}$	$\times$	$\mathcal{H}_5$
(44)	$\mathcal{D}_3,\ \mathcal{D}_9$	$\times$	$\mathcal{H}_6$
(45)	$\mathcal{D}_2,\ \mathcal{D}_5,\ \mathcal{D}_{18}$	$\times$	$\mathcal{H}_7$
(46)	$\mathcal{D}_{13},\ \mathcal{D}_{15}$	$\exists$ (See Proposition 2.6.2)	$\mathcal{H}_8$
(47)	$\mathcal{D}_8,\ \mathcal{D}_{12},\ \mathcal{D}_{19}$	$\times$	$\mathcal{H}_9$
(48)	$\mathcal{D}_9,\ \mathcal{D}_{16}$	$\times$	$\mathcal{H}_{10}$
(49)	none	$\times$ (See (c) in Section 2.5)	$\mathcal{L}_1$
(50)	$\mathcal{D}_7$	$\times$ (See (c) in Section 2.5)	$\mathcal{L}_2$
(51)	$\mathcal{D}_6$	$\times$	$\mathcal{L}_3$
(52)	$\mathcal{D}_8,\ \mathcal{D}_{10},\ \mathcal{D}_{11}$	$\times$	$\mathcal{L}_4$
(53)	none	$\times$ (See Example 2.3.12)	$\mathcal{L}_5$
(54)	$\mathcal{D}_{12},\ \mathcal{D}_{14}$	$\times$	$\mathcal{L}_6$
(55)	$\mathcal{D}_{15}$	$\exists$ (See Proposition 2.6.2)	$\mathcal{L}_7$
(56)	none	$\exists$ (See Proposition 2.6.2)	$\mathcal{L}_8$
(57)	$\mathcal{D}_{13}$	$\exists$ (See Proposition 2.6.2)	$\mathcal{L}_9$
(58)	$\mathcal{D}_{10},\ \mathcal{D}_{17}$	$\times$ (See (c) in Section 2.5)	$\mathcal{L}_{10}$
(59)	$\mathcal{D}_{14}$	unknown	$\mathcal{L}_{11}$
(60)	$\mathcal{D}_{11},\ \mathcal{D}_{17},\ \mathcal{D}_{19}$	$\times$ (See (d) in Section 2.5)	$\mathcal{L}_{12}$
(61)	$\mathcal{D}_7$	unknown	$\mathcal{L}_{13}$
(62)	$\mathcal{D}_4$	$\times$	$\mathcal{I}_1$
(63)	$\mathcal{D}_1,\ \mathcal{D}_6$	$\times$	$\mathcal{I}_2$
(64)	$\mathcal{D}_3,\ \mathcal{D}_8$	$\times$	$\mathcal{I}_3$
(65)	$\mathcal{D}_{10}$	$\times$ (See (a) in Section 2.5)	$\mathcal{I}_4$
(66)	$\mathcal{E}_2,\ \mathcal{D}_4,\ \mathcal{D}_{10}$	$\times$	$\mathcal{I}_5$
(67)	$\mathcal{D}_{10}$	$\times$ (See (a) in Section 2.5)	$\mathcal{I}_6$
(68)	$\mathcal{D}_5,\ \mathcal{D}_{12}$	$\times$	$\mathcal{I}_7$
(69)	$\mathcal{D}_8,\ \mathcal{D}_{16},\ \mathcal{G}_4$	$\times$	$\mathcal{I}_8$
(70)	$\mathcal{D}_{14}$	unknown	$\mathcal{I}_9$
(71)	$\mathcal{D}_6,\ \mathcal{D}_{15},\ \mathcal{G}_5$	$\times$	$\mathcal{I}_{10}$
(72)	$\mathcal{D}_9,\ \mathcal{D}_{12}$	$\times$	$\mathcal{I}_{11}$

(73)	$\mathcal{D}_{15},\ \mathcal{D}_{19},\ \mathcal{G}_6$	$\times$ (See (a) in Section 2.5)	$\mathcal{I}_{12}$
(74)	$\mathcal{D}_{12},\ \mathcal{D}_{13}$	$\times$	$\mathcal{I}_{13}$
(75)	$\mathcal{E}_3,\ \mathcal{D}_{10},\ \mathcal{D}_{14}$	$\times$	$\mathcal{I}_{14}$
(76)	$\mathcal{D}_{18},\ \mathcal{D}_{19},\ \mathcal{G}_2$	$\times$ (See (a) in Section 2.5)	$\mathcal{I}_{15}$
(77)	none	$\times$ (See (e) in Section 2.5)	$\mathcal{M}_1$
(78)	none	$\times$ (See (e) in Section 2.5)	$\mathcal{M}_2$
(79)	$\mathcal{G}_3,\ \mathcal{G}_5$	$\times$ (See (e) in Section 2.5)	$\mathcal{M}_3$
(80)	$\mathcal{G}_3$	$\times$ (See (e) in Section 2.5)	$\mathcal{M}_4$
(81)	$\mathcal{G}_4,\ \mathcal{G}_6$	$\times$ (See (f) in Section 2.5)	$\mathcal{M}_5$
(82)	$\mathcal{G}_1,\ \mathcal{G}_3$	$\times$	$\mathcal{J}_1$
(83)	$\mathcal{G}_3$	$\times$ (See (b) in Section 2.5)	$\mathcal{J}_2$
(84)	$\mathcal{L}_2$	$\times$	$\mathcal{Q}_1$
(85)	$\mathcal{H}_4,\ \mathcal{L}_4$	$\times$	$\mathcal{Q}_2$
(86)	$\mathcal{L}_1,\ \mathcal{L}_5$	$\times$	$\mathcal{Q}_3$
(87)	$\mathcal{L}_3$	$\times$	$\mathcal{Q}_4$
(88)	$\mathcal{H}_5,\ \mathcal{L}_3,\ \mathcal{L}_6$	$\times$	$\mathcal{Q}_5$
(89)	$\mathcal{L}_6$	$\times$	$\mathcal{Q}_6$
(90)	$\mathcal{L}_7$	$\times$	$\mathcal{Q}_7$
(91)	$\mathcal{L}_5,\ \mathcal{L}_9$	$\times$	$\mathcal{Q}_8$
(92)	$\mathcal{L}_3,\ \mathcal{L}_7,\ \mathcal{I}_{10}$	$\times$	$\mathcal{Q}_9$
(93)	$\mathcal{H}_8,\ \mathcal{L}_7,\ \mathcal{L}_9$	$\exists$ (See Proposition 2.6.2)	$\mathcal{Q}_{10}$
(94)	$\mathcal{L}_8,\ \mathcal{L}_9$	$\exists$ (See Proposition 2.6.2)	$\mathcal{Q}_{11}$
(95)	$\mathcal{L}_{10},\ \mathcal{L}_{12},\ \mathcal{I}_6$	$\times$	$\mathcal{Q}_{12}$
(96)	$\mathcal{L}_2,\ \mathcal{L}_5,\ \mathcal{L}_{13}$	$\times$	$\mathcal{Q}_{13}$
(97)	$\mathcal{H}_9,\ \mathcal{L}_4,\ \mathcal{L}_6,\ \mathcal{L}_{12},\ \mathcal{I}_{14}$	$\times$	$\mathcal{Q}_{14}$
(98)	$\mathcal{L}_6,\ \mathcal{L}_9,\ \mathcal{L}_{11},\ \mathcal{I}_{13}$	$\times$	$\mathcal{Q}_{15}$
(99)	$\mathcal{L}_{11},\ \mathcal{L}_{13},\ \mathcal{I}_9$	unknown	$\mathcal{Q}_{16}$
(100)	$\mathcal{L}_7,\ \mathcal{L}_{12},\ \mathcal{I}_{12}$	$\times$	$\mathcal{Q}_{17}$
(101)	$\mathcal{H}_1,\ \mathcal{H}_3,\ \mathcal{H}_7$	$\times$	$\mathcal{K}_1$
(102)	$\mathcal{H}_2,\ \mathcal{H}_6,\ \mathcal{H}_{10}$	$\times$	$\mathcal{K}_2$
(103)	$\mathcal{H}_4,\ \mathcal{H}_5,\ \mathcal{H}_9$	$\times$	$\mathcal{K}_3$

(104)	$\mathcal{H}_8$	$\exists$ (See Proposition 2.6.2)	$\mathcal{K}_4$
(105)	$\mathcal{M}_3$	$\times$	$\mathcal{R}_1$
(106)	$\mathcal{M}_2,\ \mathcal{M}_4$	$\times$	$\mathcal{R}_2$
(107)	$\mathcal{M}_1,\ \mathcal{M}_4$	$\times$	$\mathcal{R}_3$
(108)	$\mathcal{I}_{11},\ \mathcal{I}_{13}$	$\times$	$\mathcal{P}$
(109)	$\mathcal{Q}_1,\ \mathcal{Q}_3,\ \mathcal{Q}_{13}$	$\times$	$\mathcal{U}_1$
(110)	$\mathcal{Q}_2,\ \mathcal{Q}_5,\ \mathcal{Q}_{14},\ \mathcal{K}_3$	$\times$	$\mathcal{U}_2$
(111)	$\mathcal{Q}_4,\ \mathcal{Q}_9$	$\times$	$\mathcal{U}_3$
(112)	$\mathcal{Q}_{10},\ \mathcal{K}_4$	$\times$	$\mathcal{U}_4$
(113)	$\mathcal{Q}_{11}$	$\exists$ (See Proposition 2.6.2)	$\mathcal{U}_5$
(114)	$\mathcal{Q}_6,\ \mathcal{Q}_8,\ \mathcal{Q}_{15}$	$\times$	$\mathcal{U}_6$
(115)	$\mathcal{Q}_7,\ \mathcal{Q}_{12},\ \mathcal{Q}_{17}$	$\times$	$\mathcal{U}_7$
(116)	$\mathcal{Q}_{16}$	unknown	$\mathcal{U}_8$
(117)	none	$\times$ (See Kajiwara [13] and [14])	$\widetilde{V}^4$
(118)	none	unknown	V^4
(119)	$\mathcal{Q}_{10},\ \mathcal{Q}_{11}$	$\exists$ (See Proposition 2.6.2)	$S_2 \times S_2$
(120)	$\mathcal{U}_4,\ \mathcal{U}_5,\ S_2 \times S_2$	$\exists$ (See Proposition 2.6.2)	$S_2 \times S_3$
(121)	$S_2 \times S_3$	$\exists$ (See Proposition 2.6.2)	$S_3 \times S_3$
(122)	$\mathcal{G}_6$	unknown	$\mathcal{Z}_1$
(123)	$\mathcal{G}_4$	unknown	$\mathcal{Z}_2$
(124)	$\mathcal{Z}_1$	unknown	$\mathcal{W}$

Chapter 3

The classification of smooth toric weakened Fano 3-folds

3.1 Introduction

A *Fano* (resp. *weak Fano*) variety X is a smooth projective variety whose anti-canonical divisor $-K_X$ is ample (resp. nef and big). Minagawa [20] introduce the concept of a *weakened Fano* variety in connection with "Reid's Fantasy" for weak Fano 3-folds. A weak Fano variety X is called a *weakened Fano* variety if it is not Fano but is deformed to Fano under a small deformation (see Definition 3.4.1). In this chapter, we consider the classification problem of weakened Fano 3-folds for the case of toric varieties. As a result, we can determine the structures of toric weakened Fano 3-folds using a result of Minagawa [20], [21]. There exist exactly 15 toric weakened Fano 3-folds up to isomorphisms (see Theorem 3.4.17). There are three cases: (1) $\mathbf{P}^1 \times X'$, where X' is a toric weak del Pezzo surface but not a del Pezzo surface, (2) toric del Pezzo surface bundles over $\mathbf{P}^1$ and (3) toric weakened del Pezzo surface bundles over $\mathbf{P}^1$.

The content of this chapter is as follows: Section 3.2 is a section for preparation. We review the basic concepts such as the toric Mori theory, primitive collections and primitive relations. In Section 3.3, we give the classification of toric weak del Pezzo surfaces. This is necessary for the classification of smooth toric weakened Fano 3-folds. In Section 3.4, we give the classification of smooth toric weakened Fano 3-folds.

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3.2 Primitive collections and primitive relations

In this section, we review the concepts of smooth complete toric varieties using the notion of primitive collections and primitive relations. See Batyrev [5], [6] and Sato [33] more precisely. For fundamental properties of the toric geometry, see Fulton [9] and Oda [26]. We work over the complex field $\mathbf{C}$ throughout this chapter.

Definition 3.2.1 Let X be a smooth complete toric d-fold, Σ the corresponding fan in $N := \mathbf{Z}^d$ and $\mathrm{G}(\Sigma) \subset N$ the set of primitive generators of 1-dimensional cones in Σ. A subset $P \subset \mathrm{G}(\Sigma)$ is called a *primitive collection* of Σ if P does not generate a cone in Σ, while any proper subset of P generates a cone in Σ. We denote by $\mathrm{PC}(\Sigma)$ the set of primitive collections of Σ.

Let $P = \{x_1, \ldots, x_m\}$ be a primitive collection of Σ. Then, there exists a unique cone $\sigma(P)$ in Σ such that $x_1 + \cdots + x_m$ is contained in the relative interior of $\sigma(P)$, because X is complete. So, we get an equality

$$x_1 + \cdots + x_m = a_1 y_1 + \cdots + a_n y_n,$$

where $y_1, \ldots, y_n$ are the generators of $\sigma(P)$, that is, $\sigma(P) \cap \mathrm{G}(\Sigma) = \{y_1, \ldots, y_n\}$ and $a_1, \ldots, a_n$ are positive integers. We call this equality the *primitive relation* of P. Thus, we get an element $r(P)$ in $A_1(X)$ for any primitive collection $P \in \mathrm{PC}(\Sigma)$, where $A_1(X)$ is the group of 1-cycles on X modulo rational equivalences. We define the *degree* of P as $\deg P := (-K_X \cdot r(P)) = m - (a_1 + \cdots + a_n)$. The following is important.

Proposition 3.2.2 (Batyrev [5], Reid [30]) *Let X be a smooth projective toric variety and Σ the corresponding fan. Then*

$$\mathbf{NE}(X) = \sum_{P \in \mathrm{PC}(\Sigma)} \mathbf{R}_{\geq 0} r(P),$$

where $\mathbf{NE}(X)$ is the Mori cone of X.

A primitive collection P is called an *extremal* primitive collection when $r(P)$ is contained in an extremal ray of $\mathbf{NE}(X)$.

3.3 Toric weak Fano varieties

In this section, we review the concepts of toric Fano varieties and toric weak Fano varieties. In particular, we give the classification of toric weak del Pezzo surfaces.

Definition 3.3.1 Let X be a smooth projective algebraic variety. Then, X is called a *Fano* variety (resp. *weak Fano* variety), if its anti-canonical divisor $-K_X$ is ample (resp. nef and big).

In the toric case, Fano varieties and weak Fano varieties are characterized as follows.

Proposition 3.3.2 (Batyrev [6], Sato [33]) *Let X be a smooth projective toric variety and Σ the corresponding fan. Then, X is Fano* (*resp. weak Fano*) *if and only if* $\deg P > 0$ (*resp.* $\deg P \geq 0$) *for any primitive collection* $P \in \mathrm{PC}(\Sigma)$.

By Propositions 3.2.2 and 3.3.2, the following holds. This is very useful in Section 3.4.

Corollary 3.3.3 *Let X be a toric weak Fano variety and Σ the corresponding fan. Then, for any primitive collection $P = \{x_1, x_2\} \in \mathrm{PC}(\Sigma)$, the corresponding primitive relation is one of the following.*

(1) $x_1 + x_2 = 0$.

(2) $x_1 + x_2 = ay$ ($y \in \mathrm{G}(\Sigma)$, *and either* $a = 1$ *or* 2).

(3) $x_1 + x_2 = y_1 + y_2$ ($y_1,\ y_2 \in \mathrm{G}(\Sigma)$).

There exists a one-to-one correspondence between smooth toric weak del Pezzo surfaces, that is, 2-dimensional smooth toric weak Fano varieties and Gorenstein toric del Pezzo surfaces (see Section 6 in Sato [33]). Since Gorenstein toric del Pezzo surfaces are classified (see Koelman [16]), we can completely classify smooth toric weak del Pezzo surfaces. We give all the smooth toric weak del Pezzo surfaces by giving the elements of $\mathrm{G}(\Sigma)$ (see Table 1). This classification is necessary in the following section.

Table 1: Smooth toric weak del Pezzo surfaces

	$\mathrm{G}(\Sigma)$	notation
(1)	$\begin{pmatrix}1\\0\end{pmatrix},\ \begin{pmatrix}0\\1\end{pmatrix},\ \begin{pmatrix}-1\\-1\end{pmatrix}.$	$\mathbf{P}^2$
(2)	$\pm\begin{pmatrix}1\\0\end{pmatrix},\ \pm\begin{pmatrix}0\\1\end{pmatrix}.$	$\mathbf{P}^1\times\mathbf{P}^1$
(3)	$\pm\begin{pmatrix}1\\0\end{pmatrix},\ \begin{pmatrix}0\\1\end{pmatrix},\ \begin{pmatrix}1\\-1\end{pmatrix}.$	F_1
(4)	$\pm\begin{pmatrix}1\\0\end{pmatrix},\ \begin{pmatrix}1\\\pm1\end{pmatrix}.$	F_2
(5)	$\pm\begin{pmatrix}1\\0\end{pmatrix},\ \pm\begin{pmatrix}0\\1\end{pmatrix},\ \begin{pmatrix}1\\1\end{pmatrix}.$	S_7
(6)	$\pm\begin{pmatrix}1\\0\end{pmatrix},\ \begin{pmatrix}1\\\pm1\end{pmatrix},\ \begin{pmatrix}0\\1\end{pmatrix}.$	W_3
(7)	$\pm\begin{pmatrix}1\\0\end{pmatrix},\ \pm\begin{pmatrix}0\\1\end{pmatrix},\ \pm\begin{pmatrix}1\\1\end{pmatrix}.$	S_6
(8)	$\pm\begin{pmatrix}1\\0\end{pmatrix},\ \pm\begin{pmatrix}0\\1\end{pmatrix},\ \begin{pmatrix}1\\\pm1\end{pmatrix}.$	W_4^1
(9)	$\pm\begin{pmatrix}1\\0\end{pmatrix},\ \begin{pmatrix}1\\\pm1\end{pmatrix},\ \begin{pmatrix}0\\1\end{pmatrix},\ \begin{pmatrix}-1\\1\end{pmatrix}.$	W_4^2
(10)	$\pm\begin{pmatrix}1\\0\end{pmatrix},\ \begin{pmatrix}1\\\pm1\end{pmatrix},\ \begin{pmatrix}0\\1\end{pmatrix},\ \begin{pmatrix}1\\2\end{pmatrix}.$	W_4^3
(11)	$\pm\begin{pmatrix}1\\0\end{pmatrix},\ \pm\begin{pmatrix}0\\1\end{pmatrix},\ \begin{pmatrix}1\\\pm1\end{pmatrix},\ \begin{pmatrix}-1\\1\end{pmatrix}.$	W_5^1
(12)	$\pm\begin{pmatrix}1\\0\end{pmatrix},\ \pm\begin{pmatrix}0\\1\end{pmatrix},\ \begin{pmatrix}1\\\pm1\end{pmatrix},\ \begin{pmatrix}1\\-2\end{pmatrix}.$	W_5^2
(13)	$\pm\begin{pmatrix}1\\0\end{pmatrix},\ \pm\begin{pmatrix}0\\1\end{pmatrix},\ \begin{pmatrix}1\\\pm1\end{pmatrix},\ \begin{pmatrix}-1\\\pm1\end{pmatrix}.$	W_6^1
(14)	$\pm\begin{pmatrix}1\\0\end{pmatrix},\ \pm\begin{pmatrix}0\\1\end{pmatrix},\ \begin{pmatrix}1\\\pm1\end{pmatrix},\ \begin{pmatrix}1\\-2\end{pmatrix},\ \begin{pmatrix}-1\\1\end{pmatrix}.$	W_6^2
(15)	$\pm\begin{pmatrix}1\\0\end{pmatrix},\ \pm\begin{pmatrix}0\\1\end{pmatrix},\ \begin{pmatrix}1\\\pm1\end{pmatrix},\ \begin{pmatrix}1\\\pm2\end{pmatrix}.$	W_6^3
(16)	$\pm\begin{pmatrix}1\\0\end{pmatrix},\ \pm\begin{pmatrix}0\\1\end{pmatrix},\ \begin{pmatrix}1\\\pm1\end{pmatrix},\ \begin{pmatrix}1\\-2\end{pmatrix},\ \begin{pmatrix}-1\\1\end{pmatrix},\ \begin{pmatrix}-2\\1\end{pmatrix}.$	W_7

In Table 1, we denote by F_a the Hirzebruch surface of degree a, while we denote by S_n the del Pezzo surface of degree n.

3.4 Classification

Minagawa defined the following concept in connection with "Reid's fantasy" for weak Fano 3-folds.

Definition 3.4.1 (Minagawa [20]) Let X be a weak Fano variety. Then, X is called a *weakened Fano* variety if

(1) X is not a Fano variety and

(2) there exists a small deformation $\varphi : \mathcal{X} \to \Delta_\epsilon := \{t \in \mathbf{C} \mid |t| < \epsilon << 1\}$ such that $\mathcal{X}_0 := \varphi^{-1}(0) \cong X$, while $\mathcal{X}_t := \varphi^{-1}(t)$ is a Fano variety for any $t \in \Delta_\epsilon \setminus \{0\}$.

Remark 3.4.2 Let X be a weak del Pezzo surface. If X is not a del Pezzo surface, then X is a weakened del Pezzo surface.

Remark 3.4.3 Weakened Fano 3-folds of Picard number two are studied in Minagawa [20].

The main purpose of this chapter is to classify toric weakened Fano 3-folds.

Minagawa characterized weakened Fano 3-folds using the notion of primitive contractions.

Theorem 3.4.4 (Minagawa [20], [21]) *Let X be a weak Fano 3-fold and not a Fano 3-fold. Then, X is a weakened Fano 3-fold if and only if every primitive crepant contraction $f : X \to \overline{X}$ is a divisorial contraction which contracts a divisor $E \subset X$ to a curve $\overline{C} \subset \overline{X}$ such that*

(1) *$f|_E : E \to \overline{C}$ is a $\mathbf{P}^1$-bundle structure,*

(2) *$\overline{C} \cong \mathbf{P}^1$ and*

(3) *$(-K_{\overline{X}} \cdot \overline{C}) = 2$.*

Such contractions are called $(0, 2)$-type *contractions.*

We study contractions of $(0, 2)$-type for the case of toric varieties. Let X be a toric weakened Fano 3-fold and Σ the corresponding fan in $N \cong \mathbf{Z}^3$. By the assumptions f is crepant and f contracts a divisor to a curve, without loss of generalities, we can assume that Σ contains four 3-dimensional cones

$$\sigma_1 = \mathbf{R}_{\geq 0}x_0 + \mathbf{R}_{\geq 0}x_+ + \mathbf{R}_{\geq 0}y_+,\ \sigma_2 = \mathbf{R}_{\geq 0}x_0 + \mathbf{R}_{\geq 0}x_+ + \mathbf{R}_{\geq 0}y_-,$$

$$\sigma_3 = \mathbf{R}_{\geq 0}x_0 + \mathbf{R}_{\geq 0}x_- + \mathbf{R}_{\geq 0}y_+ \text{ and } \sigma_4 = \mathbf{R}_{\geq 0}x_0 + \mathbf{R}_{\geq 0}x_- + \mathbf{R}_{\geq 0}y_-,$$

where

$$x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ x_+ = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ x_- = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},\ y_+ = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } y_- = \begin{pmatrix} \alpha \\ \beta \\ -1 \end{pmatrix}$$

for some integers α and β. In this case, E is the toric divisor corresponding to the 1-dimensional cone $\mathbf{R}_{\geq 0}x_0$ in Σ. The conditions (1) and (2) are automatically satisfied. On the other hand, the set of maximal cones of the fan $\overline{\Sigma}$ corresponding to $\overline{X}$ is

$$(\{\text{the maximal cones of } \Sigma\} \setminus \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}) \cup \{\overline{\sigma_1} := \sigma_1 \cup \sigma_3,\ \overline{\sigma_2} := \sigma_2 \cup \sigma_4\}.$$

$\overline{C}$ is the torus invariant curve corresponding to the 2-dimensional cone $\mathbf{R}_{\geq 0}x_+ + \mathbf{R}_{\geq 0}x_-$ in $\overline{\Sigma}$. Put C be the torus invariant curve corresponding to the cone $\mathbf{R}_{\geq 0}x_0 + \mathbf{R}_{\geq 0}x_+$ on X. Then, we have $\left(-K_{\overline{X}} \cdot \overline{C}\right) = (-K_{\overline{X}} \cdot f_*C) = (f^*(-K_{\overline{X}}) \cdot C) = (-K_X \cdot C)$, because $\overline{C} = f_*C$ in $A_1(\overline{X})$, while $f^*K_{\overline{X}} = K_X$ in $\mathrm{Pic}(X)$. Let D_0, D_+, D_-, H_+ and H_- be the toric divisors on X corresponding to x_0, x_+, x_-, y_+ and y_-, respectively. Then, the equalities

$$D_0 + D_+ + D_- + \alpha H_- + \mathcal{D}_1 = 0,\ D_+ - D_- + \beta H_- + \mathcal{D}_2 = 0 \text{ and}$$

$$H_+ - H_- + \mathcal{D}_3 = 0$$

hold in $\mathrm{Pic}(X)$, where $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ are linear combinations of prime toric devisors on X other than D_0, D_+, D_-, H_+ and H_-. Note $(H_+ \cdot C) = (H_- \cdot C) = 1$ and $(D_- \cdot C) = (\mathcal{D}_1 \cdot C) = (\mathcal{D}_2 \cdot C) = (\mathcal{D}_3 \cdot C) = 0$. By these equalities, we have $(D_+ \cdot C) = -\beta$ and $(D_0 \cdot C) = \beta - \alpha$. Thus, we have $\left(-K_{\overline{X}} \cdot \overline{C}\right) = (-K_X \cdot C) = (D_0 \cdot C) + (D_+ \cdot C) + (D_- \cdot C) + (H_+ \cdot C) + (H_- \cdot C) = (\beta - \alpha) - \beta + 2 = 2 - \alpha$. So, by the assumption $\left(-K_{\overline{X}} \cdot \overline{C}\right) = 2$, we have $\alpha = 0$ and E is isomorphic to the Hirzebruch surface F_β of degree β. Moreover, since X is a weak Fano 3-fold, we have $-2 \leq \beta \leq 2$. We may assume $0 \leq \beta \leq 2$. As a result, we get the following.

Proposition 3.4.5 *Let X be a toric weak Fano 3-fold and Σ the corresponding fan. There exists a $(0,2)$-type contraction $f : X \to \overline{X}$ if and only if, by some automorphism of $N = \mathbf{Z}^3$, Σ contains four 3-dimensional cones*

$$\sigma_1 = \mathbf{R}_{\geq 0}x_0 + \mathbf{R}_{\geq 0}x_+ + \mathbf{R}_{\geq 0}y_+,\ \sigma_2 = \mathbf{R}_{\geq 0}x_0 + \mathbf{R}_{\geq 0}x_+ + \mathbf{R}_{\geq 0}y_-,$$

$$\sigma_3 = \mathbf{R}_{\geq 0}x_0 + \mathbf{R}_{\geq 0}x_- + \mathbf{R}_{\geq 0}y_+ \text{ and } \sigma_4 = \mathbf{R}_{\geq 0}x_0 + \mathbf{R}_{\geq 0}x_- + \mathbf{R}_{\geq 0}y_-,$$

where

$$x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ x_+ = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ x_- = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},\ y_+ = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } y_- = \begin{pmatrix} 0 \\ a \\ -1 \end{pmatrix},$$

and $0 \leq a \leq 2$. In particular, the exceptional divisor E of f corresponds to $\mathbf{R}_{\geq 0}x_0 \in \Sigma$ and $E \cong F_a$.

The following is fundamental to study the classification of toric weakened Fano 3-folds.

Lemma 3.4.6 *Let X be a toric weakened Fano 3-fold and Σ the corresponding fan. Then, for any primitive collection $P = \{x_1, x_2\} \in \mathrm{PC}(\Sigma)$ with primitive relation $x_1+x_2 = y_1+y_2$, where $\sigma(P) \cap \mathrm{G}(\Sigma) = \{y_1, y_2\}$, we have $\mathbf{R}_{\geq 0}x_1 + \mathbf{R}_{\geq 0}x_2 \supset \mathbf{R}_{\geq 0}y_1 + \mathbf{R}_{\geq 0}y_2$.*

Proof. Suppose that x_1, x_2, y_1, y_2 generates a 3-dimensional cone. If P is extremal, then the corresponding primitive contraction is a small contraction. So, by Theorem 3.4.4, P is not extremal. Therefore, there exist extremal primitive collections $P_1, \ldots, P_n$ such that

$$r(P) = \sum_{i=1}^{n} a_i r(P_i),$$

where $a_1, \ldots, a_n$ are positive integers and $n \geq 2$. Since $\deg P = (-K_X \cdot r(P)) = 0$, for any $1 \leq i \leq n$, we have $\deg P_i = (-K_X \cdot r(P_i)) = 0$ and the corresponding primitive crepant contractions are $(0,2)$-type by Theorem 3.4.4. Let $x_i' + x_i'' = 2y_i'$ be the corresponding primitive relation of P_i for any $1 \leq i \leq n$. Then, there exist $1 \leq j, k \leq n$ such that $y_1 = y_j'$ and $y_2 = y_k'$. This is impossible because P_j and P_k are extremal. Thus, we have $\mathbf{R}_{\geq 0}x_1 + \mathbf{R}_{\geq 0}x_2 \supset \mathbf{R}_{\geq 0}y_1 + \mathbf{R}_{\geq 0}y_2$. q.e.d.

Corollary 3.4.7 *Let X be a toric weakened Fano 3-fold and Σ the corresponding fan. For any primitive collecion $P = \{x_1, x_2\} \in \mathrm{PC}(\Sigma)$ such that $x_1 + x_2 \neq 0$, there exists an element $z \in \mathrm{G}(\Sigma)$ such that z is contained in the relative interior of $\mathbf{R}_{\geq 0}x_1 + \mathbf{R}_{\geq 0}x_2$.*

Proof. This is obvious by Corollary 3.3.3 and Lemma 3.4.6. q.e.d.

Now, we can start the classification of toric weakened Fano 3-folds.

Let X be a toric weakened Fano 3-fold and Σ the corresponding fan. Since X is not Fano, there exists a $(0,2)$-type contraction. We use the notation as in Proposition 3.4.5. Put

$$\mathcal{S}_+ = \left\{ \alpha x_0 + \beta x_+ \in N_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R} \cong \mathbf{R}^3 \;\middle|\; \alpha, \beta \in \mathbf{R}, \beta \geq 0 \right\},$$

$$\mathcal{S}_- = \{ \alpha x_0 + \beta x_- \in N_{\mathbf{R}} \mid \alpha, \beta \in \mathbf{R}, \beta \geq 0 \},$$

$$\mathcal{T}_+ = \{ \alpha x_0 + \beta y_+ \in N_{\mathbf{R}} \mid \alpha, \beta \in \mathbf{R}, \beta \geq 0 \},$$

$$\mathcal{T}_- = \{ \alpha x_0 + \beta y_- \in N_{\mathbf{R}} \mid \alpha, \beta \in \mathbf{R}, \beta \geq 0 \},$$

$$\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_- \text{ and } \mathcal{T} = \mathcal{T}_+ \cup \mathcal{T}_-.$$

$\mathcal{S}_+$, $\mathcal{S}_-$, $\mathcal{T}_+$, $\mathcal{T}_-$, $\mathcal{S}$ and $\mathcal{T}$ are connected subsets in $N_{\mathbf{R}}$. The following holds.

Lemma 3.4.8 *$I := \mathrm{G}(\Sigma) \setminus \{x_0, x_+, x_-, y_+, y_-\}$ is contained in either $\mathcal{S}$ or $\mathcal{T}$.*

Proof. First, we show $\mathrm{G}(\Sigma) \subset \mathcal{S} \cup \mathcal{T}$. Suppose that there exists $z \in I$ such that $z \notin \mathcal{S} \cup \mathcal{T}$. Since $\{x_0, z\}$ is a primitive collection, there exists $z' \in \mathrm{G}(\Sigma)$ such that z' is contained in the relative interior of $\mathbf{R}_{\geq 0} x_0 + \mathbf{R}_{\geq 0} z$ by Corollary 3.4.7. We can replace z by z' and do this discussion again. This contradicts the fact $\mathrm{G}(\Sigma)$ is a finite set. So, we have $\mathrm{G}(\Sigma) \subset \mathcal{S} \cup \mathcal{T}$.

Next, suppose that there exist z_1 and $z_2 \in I$ such that $x_0 + z_1 \neq 0$, $x_0 + z_2 \neq 0$, $z_1 \in \mathcal{S}_+$ and $z_2 \in \mathcal{T}_+$ (the other cases are similar). By the similar discussion as above, we can choose such z_1 and z_2 as $\{x_+, z_1\}$ and $\{y_+, z_2\}$ generate 2-dimensional cones in Σ. On the other hand, $\{z_1, z_2\}$ generates a 2-dimensional cone in Σ by Corollary 3.4.7 and the above discussion. Therefore, either $\{x_+, z_2\}$ or $\{y_+, z_1\}$ is a primitive collection. By Corollary 3.4.7, this contradicts the fact $\mathrm{G}(\Sigma) \subset \mathcal{S} \cup \mathcal{T}$. q.e.d.

Lemma 3.4.9 *If $\{y_+, y_-\}$ (resp. $\{x_+, x_-\}$) is a primitive collection and $I \subset \mathcal{S}$ (resp. $I \subset \mathcal{T}$), then, for any primitive collection $P \in \mathrm{PC}(\Sigma)$ such that $P \neq \{y_+, y_-\}$ (resp. $P \neq \{x_+, x_-\}$), we have $P \cap \{y_+, y_-\} = \emptyset$ (resp. $P \cap \{x_+, x_-\} = \emptyset$).*

Proof. We show this lemma for the case $I \subset \mathcal{S}$. Another case is similar.

Let $P \in \mathrm{PC}(\Sigma)$ be a primitive collection such that $P \neq \{y_+, y_-\}$. By Corollary 3.4.7 and Lemma 3.4.8, we have $\#P = 3$. So, without loss of generalities, we may assume $P = \{y_+, z_1, z_2\}$, where $z_1, z_2 \in I$. Since y_+, z_1 and z_2 are linearly independent over $\mathbf{R}$, there exists a element of $\mathrm{G}(\Sigma)$ in the interior of $\mathbf{R}_{\geq 0}y_+ + \mathbf{R}_{\geq 0}z_1 + \mathbf{R}_{\geq 0}z_2$. This contradicts Lemma 3.4.8. q.e.d.

Remark 3.4.10 In Lemma 3.4.9, the condition $\{x_+, x_-\}$ is a primitive collection holds automatically. Moreover, if $E \cong \mathbf{P}^1 \times \mathbf{P}^1$ or $E \cong F_2$, that is, $a = 0$ or $a = 2$, then the condition $\{y_+, y_-\}$ is a primitive collection also holds automatically. In these cases, $y_+ + y_- = 0$ and $y_+ + y_- = 2z$ ($z \in \mathrm{G}(\Sigma)$) are the corresponding primitive relations, respectively.

Corollary 3.4.11 *Under the assumption in Lemma* 3.4.9*, if the Picard number of X is not three, then one of the following holds.*

(1) *$I \subset \mathcal{S}$. Moreover, X is a toric surface bundle over $\mathbf{P}^1$ such that the fan corresponding to a fiber is in $\mathcal{S}$.*

(2) *$I \subset \mathcal{T}$ and $E \cong \mathbf{P}^1 \times \mathbf{P}^1$. Moreover, X is a toric surface bundle over $\mathbf{P}^1$ such that tha fan corresponding to a fiber is in $\mathcal{T}$.*

Proof. See Proposition 4.1, Theorem 4.3 and Corollary 4.4 in Batyrev [5]. q.e.d.

By these results, we can complete the classification of toric weakened Fano 3-folds. We split the classification into three cases, that is, (I) $E \cong \mathbf{P}^1 \times \mathbf{P}^1$, (II) $E \cong F_1$ and (III) $E \cong F_2$.

(I) $E \cong \mathbf{P}^1 \times \mathbf{P}^1$ ($a = 0$).

If $I \subset \mathcal{S}$, then X is isomorphic to $\mathbf{P}^1 \times X'$ by (1) in Corollary 3.4.11, where X' is a toric weak del Pezzo surface but not a toric del Pezzo surface. In this case, by the classification of toric weak del Pezzo surfaces in Section 3.3, there exist exactly 11 toric weakened Fano 3-folds up to isomorphisms: $\mathbf{P}^1 \times F_2$, $\mathbf{P}^1 \times W_3$, $\mathbf{P}^1 \times W_4^1$, $\mathbf{P}^1 \times W_4^2$, $\mathbf{P}^1 \times W_4^3$, $\mathbf{P}^1 \times W_5^1$, $\mathbf{P}^1 \times W_5^2$, $\mathbf{P}^1 \times W_6^1$, $\mathbf{P}^1 \times W_6^2$, $\mathbf{P}^1 \times W_6^3$ and $\mathbf{P}^1 \times W_7$ (see Table 1 in Section 3.3).

If $I \subset \mathcal{T}$, then X is a toric surface bundle over $\mathbf{P}^1$ by (2) in Corollary 3.4.11. A fiber X' of this bundle structure corresponds to the 2-dimensional fan in $\mathcal{T}$ and X' is a toric weak del Pezzo surface. Therefore, by the classification of toric weak del Pezzo surfaces in Section 3.3, in this case, we obtain two new toric weakened Fano 3-folds X_3^0 and X_4^0 whose Picard numbers are 3 and 4, respectively. The other cases are impossible because there exists a crepant contraction which is not of $(0,2)$-type. Put

$$z_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},\ z_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix},$$

Σ_3^0 the fan corresponding to X_3^0 and Σ_4^0 the fan corresponding to X_4^0. Then $\mathrm{G}(\Sigma_3^0) = \{x_0, x_+, x_-, y_+, y_-, z_1\}$, while $\mathrm{G}(\Sigma_4^0) = \{x_0, x_+, x_-, y_+, y_-, z_1, z_2\}$. In particular, X_3^0 is a F_1-bundle over $\mathbf{P}^1$, while X_4^0 is a S_7-bundle over $\mathbf{P}^1$. Moreover, we have

$$\left(-K_{X_3^0}\right)^3 = 52 \text{ and } \left(-K_{X_4^0}\right)^3 = 38.$$

There exists a sequence of equivariant blow-ups along curves

$$X_4^0 \longrightarrow X_3^0 \longrightarrow \mathbf{P}_{\mathbf{P}^1}\left(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(2)\right).$$

On the other hand, there exists a sequence of blow-ups along curves

$$Y_4^0 \longrightarrow Y_3^0 \longrightarrow \mathbf{P}_{\mathbf{P}^1}\left(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)\right),$$

where Y_3^0 is a toric Fano 3-fold of Picard number 3 and of type no.31 on the table in Mori-Mukai [23], while Y_4^0 is a Fano 3-fold, which is not toric, of Picard number 4 and of type no.8 on the table in Mori-Mukai [23]. X_3^0 and X_4^0 are deformed to Y_3^0 and Y_4^0 under small deformations, respectively. Moreover, $\mathbf{P}_{\mathbf{P}^1}\left(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(2)\right)$ is deformed to $\mathbf{P}_{\mathbf{P}^1}\left(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)\right)$, though $\mathbf{P}_{\mathbf{P}^1}\left(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)\right)$ is not Fano (see Ashikaga-Konno [2], Harris [10] and Nakamura [25]).

Corollary 3.4.12 *Let X be a toric weakened Fano 3-fold. If there exists a $(0,2)$-type contraction whose exceptional divisor is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$, then the exceptional divisors of other $(0,2)$-type contractions are also isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$.*

(II) $E \cong F_1$ $(a = 1)$.

In this case, we have to consider the following (see Remark 3.4.10).

Lemma 3.4.13 *$\{y_+, y_-\}$ is a primitive collection. Moreover, $z := y_+ + y_-$ is contained in $\mathrm{G}(\Sigma)$ and the primitive relation of $\{y_+, y_-\}$ is $y_+ + y_- = z$.*

To prove Lemma 3.4.13, we need the following.

Lemma 3.4.14 *Let X be a toric weakened Fano 3-fold and Σ the corresponding fan. Any primitive collection $P = \{x_1, x_2\}$ such that its primitive relation is $x_1 + x_2 = 2y$ ($y \in \mathrm{G}(\Sigma)$) and $\{x_1, y\}$ is a part of a $\mathbf{Z}$-basis of N is extremal.*

Proof. If P is not extremal, by the same argument as in the proof of Lemma 3.4.6, there exists an extremal primitive relation $x' + x'' = 2y$ whose corresponding contraction is $(0,2)$-type, where $x', x'' \in \mathrm{G}(\Sigma)$. By the assumption $\{x_1, y\}$ is a part of a $\mathbf{Z}$-basis of N, $\mathbf{R}_{\geq 0}x_1 + \mathbf{R}_{\geq 0}x_2$ does not contain $\mathbf{R}_{\geq 0}x' + \mathbf{R}_{\geq 0}x''$. This is impossible, because X is a weak Fano variety and $x' + x'' = 2y$ corresponds to a $(0,2)$-type contraction. q.e.d.

Proof of Lemma 3.4.13. Suppose that $\{y_+, y_-\}$ is not a primitive collection. Then, we have two primitive relations

$$x_- + y_+ + y_- = x_0 \text{ and } x_0 + y_+ + y_- = x_+.$$

By the completeness of X, there exists

$$z = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \in I$$

such that $\alpha < 0$. Obviously, $\{x_+, z\}$ is a primitive collection. If $I \subset \mathcal{S}$, then, by the same argument as in the proof of Lemma 3.4.8, the corresponding primitive relation have to be $x_+ + z = 0$. So, we have

$$x_- + z = 2\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \in \mathrm{G}(\Sigma).$$

By Lemma 3.4.14, this primitive collection is extremal. However, the corresponding contraction is not of $(0,2)$-type. So, suppose that $I \subset \mathcal{T}$. This case is also impossible by the same argument as in the proof of Lemma 3.4.8. Therefore, $\{y_+, y_-\}$ is a primitive collection. q.e.d.

Since the Picard number of X is not three, we have $I \subset \mathcal{S}$, and hence X is a toric surface bundle over $\mathbf{P}^1$ by (1) in Corollary 3.4.11 and Lemma 3.4.13. The fan corresponding to a fiber of this bundle structure is in $\mathcal{S}$. By the classification of toric weak del Pezzo surfaces in Section 3.3, in this case, we obtain two new toric weakened Fano 3-folds X_4^1 and X_5^1 whose Picard numbers are 4 and 5, respectively. The other cases are impossible since there exists a crepant contraction which is not of $(0,2)$-type. Put

$$z_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ z_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \ z_3 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix},$$

Σ_4^1 the fan corresponding to X_4^1 and Σ_5^1 the fan corresponding to X_5^1. Then, $\mathrm{G}(\Sigma_4^1) = \{x_0, x_+, x_-, y_+, y_-, z_1, z_2\}$, while $\mathrm{G}(\Sigma_5^1) = \{x_0, x_+, x_-, y_+, y_-, z_1, z_2, z_3\}$. In particular, X_4^1 is a W_3-bundle over $\mathbf{P}^1$, while X_5^1 is a W_4^1-bundle over $\mathbf{P}^1$ (see Table 1 in Section 3.3). Moreover, we have

$$\left(-K_{X_4^1}\right)^3 = 46 \text{ and } \left(-K_{X_5^1}\right)^3 = 36.$$

There exists an equivariant blow-up along a curve

$$X_5^1 \longrightarrow X_4^1.$$

On the other hand, there exists a blow-up along a curve

$$Y_5^1 \longrightarrow Y_4^1,$$

where Y_4^1 is a toric Fano 3-fold of Picard number 4 and of type no.12 on the table in Mori-Mukai [23], while Y_5^1 is a Fano 3-fold, which is not toric, of Picard number 5 and of type no.2 on the table in Mori-Mukai [23]. X_4^1 and X_5^1 are deformed to Y_4^1 and Y_5^1 under small deformations, respectively.

Corollary 3.4.15 *Let X be a toric weakened Fano 3-fold. If there exists a $(0,2)$-type contraction whose exceptional divisor is isomorphic to F_1, then X is isomorphic to either X_4^1 or X_5^1.*

(III) $E \cong F_2$ $(a = 2)$.

We have a primitive relation $y_+ + y_- = 2z$, where

$$z = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in \mathrm{G}(\Sigma).$$

So, the Picard number of X is not three. Therefore, $I \subset \mathcal{S}$, and hence X is a toric surface bundle over $\mathbf{P}^1$ by (1) in Corollary 3.4.11. The fan corresponding to a fiber of this bundle structure is in $\mathcal{S}$. On the other hand, by Lemma 3.4.14, the primitive relation $y_+ + y_- = 2z$ is extremal. Therefore, the corresponding contraction is of $(0,2)$-type. Moreover, its exceptional divisor have to be isomorphic to F_2 by Corollaries 3.4.12 and 3.4.15. This is impossible. Thus, there exists no $(0,2)$-type contraction whose exceptional divisor is isomorphic to F_2 on any toric weakened Fano 3-fold.

Remark 3.4.16 There is an example of a general weakened Fano 3-fold which has a $(0,2)$-type contraction whose exceptional divisor is isomorphic to F_2 (see Minagawa [20]).

Thus, we obtain the classification of toric weakened Fano 3-folds.

Theorem 3.4.17 *There exist exactly* 15 *smooth toric weakened Fano* 3*-folds up to isomorphisms. There are following three cases*:

(1) $\mathbf{P}^1 \times X'$, *where* X' *is a toric weak del Pezzo surface but not a del Pezzo surface, that is, toric weakened del Pezzo surface*: $\mathbf{P}^1 \times F_2$, $\mathbf{P}^1 \times W_3$, $\mathbf{P}^1 \times W_4^1$, $\mathbf{P}^1 \times W_4^2$, $\mathbf{P}^1 \times W_4^3$, $\mathbf{P}^1 \times W_5^1$, $\mathbf{P}^1 \times W_5^2$, $\mathbf{P}^1 \times W_6^1$, $\mathbf{P}^1 \times W_6^2$, $\mathbf{P}^1 \times W_6^3$ *and* $\mathbf{P}^1 \times W_7$.

(2) *Toric del Pezzo surface bundles over* $\mathbf{P}^1$: X_3^0 *and* X_4^0.

(3) *Toric weakened del Pezzo surface bundles over* $\mathbf{P}^1$ *which are not decomposed into a direct product of* $\mathbf{P}^1$ *and a toric weakened del Pezzo surface*: X_4^1 *and* X_5^1.

Chapter 4

Jumping deformations of complete toric varieties

4.1 Introduction

It is well-known that the Hirzebruch surface F_a ($a \geq 0$) of degree a is deformed in a one-parameter family to F_{a-2k}, where k is a positive integer such that $a-2k \geq 0$. In particular, if $a \equiv a'$ (mod 2), then F_a and $F_{a'}$ are homeomorphic. In this chapter, we generalize this classical result to certain nonsingular complete toric varieties. Namely, for a nonsingular complete toric d-fold V which have a toric fibration onto $\mathbf{P}^1$ such that its general fiber is isomorphic to either $\mathbf{P}^{d-1}$ or a toric bundle over $\mathbf{P}^1$, we construct a complex analytic family $\{V_t\}_{t\in\mathbf{C}}$ such that $V_0 \cong V$ and that $\{V_t\}_{t\neq 0}$ are mutually isomorphic. Moreover, under appropriate assumptions, the general fiber of this family is explicitly described by the data of the fan corresponding to V.

As an application of this construction of families, we construct a deformation family for a certain toric weakened Fano variety, that is, a nonsingular toric weak Fano varieties which is not Fano but is deformed to Fano varieties. Toric weakened Fano d-folds are classified for $d \leq 3$ (see Sato [32]). Moreover, we obtain certain examples of toric weakened Fano 4-folds.

The content of this chapter is as follows: In Section 4.2, we review the homogeneous coordinate of a toric varietiy, which is a key to our main result. In Section 4.3, we construct complex analytic families of nonsingular complete varieties over $\mathbf{C}$ as stated above. In

Section 4.4, as an application of the construction, we study deformations among $\mathbf{P}^{d-1}$-bundles over $\mathbf{P}^1$. In Section 4.5, we give certain examples of toric weakened Fano 3-folds and 4-folds using the families constructed in Section 4.3.

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4.2 Homogeneous coordinates of toric varieties

In this section, we recall homogeneous coordinates of toric varieties (see Cox [7] and Oda [26]).

Let $N = \mathbf{Z}^d$ with elements regarded as column vectors, $M := \mathrm{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$, $N_{\mathbf{R}} := N \otimes \mathbf{R}$, $M_{\mathbf{R}} := M \otimes \mathbf{R}$ and Σ a fan in N. Throughout this chapter, we mean by a cone a nonsingular rational cone, and by a fan in N a nonsingular fan which contains at least one d-dimensional cone. For $0 \leq i \leq d$, we put $\Sigma(i) := \{\sigma \in \Sigma \mid \dim \sigma = i\}$. Each $\tau \in \Sigma(1)$ determines a unique element $e(\tau) \in N$ which generates the semigroup $\tau \cap N$. We call

$$\mathrm{G}(\Sigma) := \{e(\tau) \in N \mid \tau \in \Sigma(1)\}$$

the set of primitive generators of Σ. Put $\mathrm{G}(\sigma) := \sigma \cap \mathrm{G}(\Sigma)$. We introduce variables $\{\mathcal{X}_\rho \mid \rho \in \mathrm{G}(\Sigma)\}$ and consider the polynomial ring $S := \mathbf{C}\left[\mathcal{X}_\rho \mid \rho \in \mathrm{G}(\Sigma)\right]$, which we call the *homogeneous coordinate ring* of the nonsingular toric d-fold V corresponding to Σ. Put

$$Z := \left\{ X = (X_\rho)_{\rho \in \mathrm{G}(\Sigma)} \in \mathbf{C}^{\mathrm{G}(\Sigma)} \;\middle|\; \prod_{\rho \in \mathrm{G}(\Sigma) \setminus \mathrm{G}(\sigma)} X_\rho = 0 \text{ for any } \sigma \in \Sigma \right\} \subset \mathbf{C}^{\mathrm{G}(\Sigma)}.$$

On the other hand, by the exact sequence

$$0 \to M \to \mathbf{Z}^{\mathrm{G}(\Sigma)} \to \mathrm{Pic}(V) \to 0,$$

we have an exact sequence

$$1 \to G := \mathrm{Hom}_{\mathbf{Z}}\left(\mathrm{Pic}(V), \mathbf{C}^\times\right) \to \left(\mathbf{C}^\times\right)^{\mathrm{G}(\Sigma)} \to T_N \to 1.$$

Since $\left(\mathbf{C}^\times\right)^{\mathrm{G}(\Sigma)}$ acts naturally on $\mathbf{C}^{\mathrm{G}(\Sigma)}$, the subgroup $G \subset \left(\mathbf{C}^\times\right)^{\mathrm{G}(\Sigma)}$ acts on $\mathbf{C}^{\mathrm{G}(\Sigma)}$ as

$$gt := \left(g\left([D_\rho]\right) t_\rho\right)_{\rho \in \mathrm{G}(\Sigma)},$$

where $g \in G$, $t = (t_\rho)_{\rho \in \mathrm{G}(\Sigma)} \in \mathbf{C}^{\mathrm{G}(\Sigma)}$ and $[D_\rho] \in \mathrm{Pic}(V)$ is the class of the T_N-invariant prime divisor D_ρ corresponding to ρ. In this setting, the following holds.

Proposition 4.2.1 (Cox [7], Theorem 2.1) *The subset* $\mathbf{C}^{\mathrm{G}(\Sigma)} \setminus Z \subset \mathbf{C}^{\mathrm{G}(\Sigma)}$ *is invariant under the action of* G*, and* V *is the geometric quotient of* $\mathbf{C}^{\mathrm{G}(\Sigma)} \setminus Z$ *by* G*. We denote* $\mathbf{C}^{\mathrm{G}(\Sigma)} \setminus Z$ *by* $\mathbf{U}(\Sigma)$*.*

We need the following proposition for this description.

Proposition 4.2.2 (Cox [7], Theorem 2.1) *For any* $\sigma \in \Sigma$*, we have*

$$U_\sigma \cong \left(\mathbf{U}(\Sigma)_\sigma := \left\{ X = (X_\rho)_{\rho \in \mathrm{G}(\Sigma)} \in \mathbf{U}(\Sigma) \;\middle|\; \prod_{\rho \in \mathrm{G}(\Sigma) \setminus \mathrm{G}(\sigma)} X_\rho \neq 0 \right\} \right) \Big/ G,$$

where $U_\sigma \subset V$ *is the affine toric subvariety corresponding to* σ*.*

4.3 Constructions of families

In this section, we construct one-parameter complex analytic families whose fibers are nonsingular complete varieties. Especially, the special fibers are nonsingular complete toric varieties. This is a generalization of the classical results on deformations among Hirzebruch surfaces.

Let $\widetilde{N} := \{\mathbf{n} \in N \,|\, \text{the } d\text{-th coordinate of } \mathbf{n} \text{ is } 0\}$ and $\widetilde{\Sigma}$ a complete fan in $\widetilde{N}$. For a complete fan Σ in N containing $\widetilde{\Sigma}$ as a subfan, we define subfans of Σ as follows:

$$\Sigma^+ := \{\sigma \in \Sigma \,|\, \text{the } d\text{-th coordinate of } \mathbf{n} \text{ is nonnegative for any } \mathbf{n} \in \sigma\}$$

$$\Sigma^- := \{\sigma \in \Sigma \,|\, \text{the } d\text{-th coordinate of } \mathbf{n} \text{ is nonpositive for any } \mathbf{n} \in \sigma\}$$

Then, we have $\widetilde{\Sigma} = \Sigma^+ \cap \Sigma^-$. We denote by V (resp. V^+, V^-, $\widetilde{V}$) the nonsingular toric variety corresponding to the fan Σ (resp. Σ^+, Σ^-, $\widetilde{\Sigma}$).

Remark 4.3.1 V has a toric fibration $V \to \mathbf{P}^1$ whose general fiber is isomorphic to $\widetilde{V}$.

In the above situation, let

$$\mathrm{G}(\widetilde{\Sigma}) = \{\mathbf{e}_1, \ldots, \mathbf{e}_{d-1}, \mathbf{a}_1, \ldots, \mathbf{a}_\rho\}, \ \mathrm{G}(\Sigma^+) = \{\mathbf{b}_1, \ldots, \mathbf{b}_m\} \cup \mathrm{G}(\widetilde{\Sigma}),$$

$$\mathrm{G}(\Sigma^-) = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\} \cup \mathrm{G}(\widetilde{\Sigma}),$$

$\{\mathbf{e}_1, \ldots, \mathbf{e}_{d-1}, \mathbf{b}_1\}$ the standard basis for N, and

$$(\mathbf{a}_1, \ldots, \mathbf{a}_\rho, \mathbf{b}_2, \ldots, \mathbf{b}_m, \mathbf{c}_1, \ldots, \mathbf{c}_n) =$$

$$\begin{pmatrix} a_{1,1} & \cdots & a_{\rho,1} & b_{2,1} & \cdots & b_{m,1} & c_{1,1} & \cdots & c_{n,1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{1,d} & \cdots & a_{\rho,d} & b_{2,d} & \cdots & b_{m,d} & c_{1,d} & \cdots & c_{n,d} \end{pmatrix}.$$

Suppose that $\widetilde{V}$ is isomorphic to either $\mathbf{P}^{d-1}$ or a toric bundle over $\mathbf{P}^1$. If $\widetilde{V}$ is isomorphic to a toric bundle over $\mathbf{P}^1$, suppose that the T_N-invariant prime divisors on $\widetilde{V}$ corresponding to $\mathbf{e}_1$ and $\mathbf{a}_1$ correspond to fibers. Suppose further that $\{\mathbf{e}_1, \ldots, \mathbf{e}_{d-1}, \mathbf{b}_1\}$ generates a d-dimensional cone in Σ^+, while $\{\mathbf{e}_1, \ldots, \mathbf{e}_{d-1}, \mathbf{c}_1\}$ generates a d-dimensional cone in Σ^-. For a nonnegative integer k, we construct a complex analytic family.

Since $\{\mathbf{a}_1, \ldots, \mathbf{a}_\rho\} \subset \mathrm{G}(\widetilde{\Sigma})$, we have $a_{1,d} = \cdots = a_{\rho,d} = 0$. We have $c_{1,d} = -1$, by the assumption that $\{\mathbf{e}_1, \ldots, \mathbf{e}_{d-1}, \mathbf{c}_1\}$ generates a d-dimensional cone in Σ^-.

Let $D_1, \ldots, D_{d-1}, A_1, \ldots, A_\rho, B_1, \ldots, B_m, C_1, \ldots, C_n$ be the T_N-invariant prime divisors corresponding to $\mathbf{e}_1, \ldots, \mathbf{e}_{d-1}, \mathbf{a}_1, \ldots, \mathbf{a}_\rho, \mathbf{b}_1, \ldots, \mathbf{b}_m, \mathbf{c}_1, \ldots, \mathbf{c}_n$, respectively. Then, by computing the divisors of the rational functions $\mathbf{e}(\mathbf{e}_1^*), \ldots, \mathbf{e}(\mathbf{e}_{d-1}^*), \mathbf{e}(\mathbf{b}_1^*) \in \mathbf{C}(V)$, where $\{\mathbf{e}_1^*, \ldots, \mathbf{e}_{d-1}^*, \mathbf{b}_1^*\} \subset M$ is the dual basis of $\{\mathbf{e}_1, \ldots, \mathbf{e}_{d-1}, \mathbf{b}_1\}$, we have

$$D_1 + a_{1,1}A_1 + \cdots + a_{\rho,1}A_\rho + b_{2,1}B_2 + \cdots + b_{m,1}B_m + c_{1,1}C_1 + \cdots + c_{n,1}C_n = 0,$$

$$D_2 + a_{1,2}A_1 + \cdots + a_{\rho,2}A_\rho + b_{2,2}B_2 + \cdots + b_{m,2}B_m + c_{1,2}C_1 + \cdots + c_{n,2}C_n = 0,$$

$$\vdots$$

$$D_{d-1} + a_{1,d-1}A_1 + \cdots + a_{\rho,d-1}A_\rho + b_{2,d-1}B_2 + \cdots + b_{m,d-1}B_m + c_{1,d-1}C_1 + \cdots + c_{n,d-1}C_n = 0,$$

$$B_1 + b_{2,d}B_2 + \cdots + b_{m,d}B_m - C_1 + c_{2,d}C_2 + \cdots + c_{n,d}C_n = 0$$

in $\mathrm{Pic}(V)$, respectively. Using these equalities, we calculate the homogeneous coordinates of V, V^+, V^- and $\widetilde{V}$.

Let $(X_1, \ldots, X_{d-1}, Y_1, \ldots, Y_\rho, Z_1, \ldots, Z_m, W_1, \ldots, W_n) \in \mathbf{U}(\Sigma)$ be the homogeneous coordinate of V corresponding to $\mathbf{e}_1, \ldots, \mathbf{e}_{d-1}, \mathbf{a}_1, \ldots, \mathbf{a}_\rho, \mathbf{b}_1, \ldots, \mathbf{b}_m, \mathbf{c}_1, \ldots, \mathbf{c}_n$, respectively. Then the action of $G := \mathrm{Hom}_{\mathbf{Z}}(\mathrm{Pic}(V), \mathbf{C}^\times)$ on $\mathbf{U}(\Sigma)$ is as follows: $g \in G$ acts as

$$(4.1)\ (g\left(-\left(a_{1,1}A_1 + \cdots + a_{\rho,1}A_\rho + b_{2,1}B_2 + \cdots + b_{m,1}B_m + c_{1,1}C_1 + \cdots + c_{n,1}C_n\right)\right) X_1,$$

$$\ldots, g\left(-\left(a_{1,d-1}A_1 + \cdots + a_{\rho,d-1}A_\rho + b_{2,d-1}B_2 + \cdots + b_{m,d-1}B_m +\right.\right.$$

$$\left.\left. c_{1,d-1}C_1 + \cdots + c_{n,d-1}C_n\right)\right) X_{d-1},$$

$$g(A_1)Y_1, \ldots, g(A_\rho)Y_\rho,$$

$$g\left(-\left(b_{2,d}B_2+\cdots+b_{m,d}B_m-C_1+c_{2,d}C_2+\cdots+c_{n,d}C_n\right)\right)Z_1,$$

$$g(B_2)Z_2,\ldots,g(B_m)Z_m,g(C_1)W_1,\cdots,g(C_n)W_n).$$

Let $\varphi:(X_1,\ldots,X_{d-1},Y_1,\ldots,Y_\rho,Z_1,\ldots,Z_m,W_1,\ldots,W_n)\mapsto(X_1^+,\ldots,X_{d-1}^+,Y_1^+,\ldots,Y_\rho^+,$ $Z_1^+,\ldots,Z_m^+)$ be a surjective morphism from $\bigcup_{\sigma\in\Sigma^+}\mathbf{U}(\Sigma)_\sigma\subset\mathbf{U}(\Sigma)$ to $\mathbf{U}(\Sigma^+)$ given by

$$\left(X_1^+,\ldots,X_{d-1}^+,Y_1^+,\ldots,Y_\rho^+,Z_1^+,\ldots,Z_m^+\right)= \tag{4.2}$$

$$\Big(X_1W_1^{c_{1,1}}\cdots W_n^{c_{n,1}},\ldots,X_{d-1}W_1^{c_{1,d-1}}\cdots W_n^{c_{n,d-1}},$$

$$Y_1,\ldots,Y_\rho,Z_1W_1^{c_{1,d}}\cdots W_n^{c_{n,d}},Z_2,\ldots,Z_m\Big),$$

where $\left(X_1^+,\ldots,X_{d-1}^+,Y_1^+,\ldots,Y_\rho^+,Z_1^+,\ldots,Z_m^+\right)$ is the homogeneous coordinate of $V^+\cong\mathbf{U}(\Sigma^+)/G^+$ with $G^+:=\mathrm{Hom}_{\mathbf{Z}}(\mathrm{Pic}(V^+),\mathbf{C}^\times)$ corresponding to $\mathbf{e}_1,\ldots,\mathbf{e}_{d-1},\mathbf{a}_1,\ldots,\mathbf{a}_\rho,\mathbf{b}_1,$ $\ldots,\mathbf{b}_m$, respectively. φ is well-defined, since $W_1,\ldots,W_n\neq0$ on $\bigcup_{\sigma\in\Sigma^+}\mathbf{U}(\Sigma)_\sigma$. Moreover, since φ is compartible with the action of G and G^+ by (4.1), φ induces the isomorphism $\widetilde{\varphi}:$ $\left(\bigcup_{\sigma\in\Sigma^+}\mathbf{U}(\Sigma)_\sigma\right)/G\subset V\to V^+$. Similarly, the morphism $\psi:(X_1,\ldots,X_{d-1},Y_1,\ldots,Y_\rho,Z_1,$ $\ldots,Z_m,W_1,\ldots,W_n)\mapsto(X_1^-,\ldots,X_{d-1}^-,Y_1^-,\ldots,Y_\rho^-,W_1^-,\ldots,W_n^-)$ from $\bigcup_{\sigma\in\Sigma^-}\mathbf{U}(\Sigma)_\sigma\subset$ $\mathbf{U}(\Sigma)$ to $\mathbf{U}(\Sigma^-)$ given by

$$\left(X_1^-,\ldots,X_{d-1}^-,Y_1^-,\ldots,Y_\rho^-,W_1^-,\ldots,W_n^-\right)= \tag{4.3}$$

$$\Big(X_1Z_1^{c_{1,1}}Z_2^{b_{2,d}c_{1,1}+b_{2,1}}\cdots Z_m^{b_{m,d}c_{1,1}+b_{m,1}},\ldots,$$

$$X_{d-1}Z_1^{c_{1,d-1}}Z_2^{b_{2,d}c_{1,d-1}+b_{2,d-1}}\cdots Z_m^{b_{m,d}c_{1,d-1}+b_{m,d-1}},Y_1,\ldots,Y_\rho,$$

$$Z_1^{-1}Z_2^{-b_{2,d}}\cdots Z_m^{-b_{m,d}}W_1,W_2,\ldots,W_n\Big)$$

induces the isomorphism $\widetilde{\psi}:\left(\bigcup_{\sigma\in\Sigma^-}\mathbf{U}(\Sigma)_\sigma\right)/G\subset V\to V^-\cong\mathbf{U}(\Sigma^-)/G^-$, where $G^-:=$ $\mathrm{Hom}_{\mathbf{Z}}(\mathrm{Pic}(V^-),\mathbf{C}^\times)$ and $\left(X_1^-,\ldots,X_{d-1}^-,Y_1^-,\ldots,Y_\rho^-,W_1^-,\ldots,W_n^-\right)$ is the homogeneous coordinate of V^- corresponding to $\mathbf{e}_1,\ldots,\mathbf{e}_{d-1},\mathbf{a}_1,\ldots,\mathbf{a}_\rho,\mathbf{c}_1,\ldots,\mathbf{c}_n$, respectively.

$g^+\in G^+$ and $g^-\in G^-$ act on $\mathbf{U}(\Sigma^+)$ and $\mathbf{U}(\Sigma^-)$ as

$$\Big(g^+\left(-\left(a_{1,1}A_1+\cdots+a_{\rho,1}A_\rho+b_{2,1}B_2+\cdots+b_{m,1}B_m\right)\right)X_1^+,\ldots, \tag{4.4}$$

$$g^+\left(-\left(a_{1,d-1}A_1+\cdots+a_{\rho,d-1}A_\rho+b_{2,d-1}B_2+\cdots+b_{m,d-1}B_m\right)\right)X_{d-1}^+,$$

$$g^+(A_1)Y_1^+,\ldots,g^+(A_\rho)Y_\rho^+,g^+\left(-\left(b_{2,d}B_2+\cdots+b_{m,d}B_m\right)\right)Z_1^+,$$

$$g^+(B_2)Z_2^+,\ldots,g^+(B_m)Z_m^+\Big)$$

and

$$\Big(g^-\left(-\left(a_{1,1}A_1+\cdots+a_{\rho,1}A_\rho+(c_{2,1}+c_{1,1}c_{2,d})C_2+\cdots+(c_{n,1}+c_{1,1}c_{n,d})C_n\right)\right)X_1^-,\ldots,$$

$$g^-\left(-\left(a_{1,d-1}A_1+\cdots+a_{\rho,d-1}A_\rho+(c_{2,d-1}+c_{1,d-1}c_{2,d})C_2+\cdots+\right.\right.$$

$$\left.\left.(c_{n,d-1}+c_{1,d-1}c_{n,d})C_n\right)\right)X_{d-1}^-,g^-(A_1)Y_1^-,\ldots,g^-(A_\rho)Y_\rho^-,$$

$$g^-\left(-(c_{2,d}C_2+\cdots+c_{n,d}C_n)\right)W_1^-,g^-(C_2)W_2^-,\cdots,g^-(C_n)W_n^-\Big),$$

respectively. So, similarly as (4.2) and (4.3), we have isomorphisms $(X_1^+,\ldots,X_{d-1}^+,Y_1^+,\ldots,Y_\rho^+,Z_1^+,\ldots,Z_m^+)\mapsto(x_1^+,\ldots,x_{d-1}^+,y_1^+,\ldots,y_\rho^+,z)$ from $\left(\bigcup_{\sigma\in\widetilde{\Sigma}}\mathbf{U}(\Sigma^+)_\sigma\right)/G^+\subset V^+$ to $\widetilde{V}\times\mathbf{C}^\times$ and $(X_1^-,\ldots,X_{d-1}^-,Y_1^-,\ldots,Y_\rho^-,\ W_1^-,\ldots,W_n^-)\mapsto(x_1^-,\ldots,x_{d-1}^-,y_1^-,\ldots,y_\rho^-,w)$ from $\left(\bigcup_{\sigma\in\widetilde{\Sigma}}\mathbf{U}(\Sigma^-)_\sigma\right)/G^-\subset V^-$ to $\widetilde{V}\times\mathbf{C}^\times$ given by

$$\left(x_1^+,\ldots,x_{d-1}^+,y_1^+,\ldots,y_\rho^+,z\right)= \tag{4.5}$$

$$\Big(X_1^+(Z_2^+)^{b_{2,1}}\cdots(Z_m^+)^{b_{m,1}},\ldots,X_{d-1}^+(Z_2^+)^{b_{2,d-1}}\cdots(Z_m^+)^{b_{m,d-1}},$$

$$Y_1^+,\ldots,Y_\rho^+,Z_1^+(Z_2^+)^{b_{2,d}}\cdots(Z_m^+)^{b_{m,d}}\Big)$$

and

$$\left(x_1^-,\ldots,x_{d-1}^-,y_1^-,\ldots,y_\rho^-,w\right)= \tag{4.6}$$

$$\Big(X_1^-(W_2^-)^{c_{2,1}+c_{1,1}c_{2,d}}\cdots(W_n^-)^{c_{n,1}+c_{1,1}c_{n,d}},\ldots,$$

$$X_{d-1}^-(W_2^-)^{c_{2,d-1}+c_{1,d-1}c_{2,d}}\cdots(W_n^-)^{c_{n,d-1}+c_{1,d-1}c_{n,d}},Y_1^-,\ldots,Y_\rho^-,$$

$$W_1^-(W_2^-)^{-c_{2,d}}\cdots(W_n^-)^{-c_{n,d}}\Big),$$

respectively, where $(x_1^+,\ldots,x_{d-1}^+,y_1^+,\ldots,y_\rho^+)$ and $(x_1^-,\ldots,x_{d-1}^-,y_1^-,\ldots,y_\rho^-)$ are homogeneous coordinates of $\widetilde{V}$, while $z,w\in\mathbf{C}^\times$. These two coordinates are related as follows:

$$x_1^+=x_1^-w^{c_{1,1}},\ldots,x_{d-1}^+=x_{d-1}^-w^{c_{1,d-1}},y_1^+=y_1^-,\ldots,y_\rho^+=y_\rho^-,z=1/w.$$

We construct a one-parameter family of complete varieties parameterized by $t\in\mathbf{C}$ by changing this relation: Let $\{V_t\}_{t\in\mathbf{C}}$ be the family we obtain by patching V^+ and V^- along $\widetilde{V}$ by the automorphism $(x_1^-,\ldots,x_{d-1}^-,y_1^-,\ldots,y_\rho^-,w)\mapsto(x_1^+,\ldots,x_{d-1}^+,y_1^+,\ldots,y_\rho^+,z)$ defined by

$$x_1^+=x_1^-w^{c_{1,1}}+ty_1^-w^k,x_2^+=x_2^-w^{c_{1,2}},\ldots,x_{d-1}^+=x_{d-1}^-w^{c_{1,d-1}}, \tag{4.7}$$

$$y_1^+=y_1^-,\ldots,y_\rho^+=y_\rho^-,z=1/w.$$

This is well-defined, since $D_1 = A_1$ in $\mathrm{Pic}(\widetilde{V})$ and the combinatorial structures of the neighborhoods of $\mathbf{e}_1$ and $\mathbf{a}_1$ in $\widetilde{\Sigma}$ are equivalent by the assumption $\widetilde{V}$ is isomorphic to either $\mathbf{P}^{d-1}$ or a toric bundle over $\mathbf{P}^1$. Thus, we have the following.

Theorem 4.3.2 *$\{V_t\}_{t\in\mathbf{C}}$ is a complex analytic family whose special fiber V_0 is isomorphic to V.*

Next, we calculate the general fibers of this family under appropriate assumptions. We introduce some notation.

For any $\mathbf{q} = (q_1, \ldots, q_{d-1}) \in \mathbf{Z}^{d-1}$, we define a complete fan $\mathbf{q}^-\Sigma$ in N as follows:

$$\mathbf{q}^-\Sigma := \Sigma^+ \cup \left\{\mathbf{q}^-\sigma \mid \sigma \in \Sigma^-\right\},$$

where $\mathbf{q}^-\sigma$ is the image of σ under the automorphism of $N_\mathbf{R}$ corresponding to the matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & q_1 \\ 0 & 1 & \cdots & 0 & q_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & q_{d-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

acting from the left on the elements of $N = \mathbf{Z}^d$ regarded as column vectors. We denote by q^-V the nonsingular toric d-fold corresponding to the fan $q^-\Sigma$.

Theorem 4.3.3 *For any t in $\mathbf{C}^\times := \mathbf{C} \setminus \{0\}$, we have*

$$V_t \cong (2k, -ka_{1,2}, \ldots, -ka_{1,d-1})^- V,$$

if the following conditions are satisfied:

(1) $b_{2,1} = \cdots = b_{m,1} = 0$ *and*

(2) $kc_{1,d} + c_{1,1} \geq 0, \ldots, kc_{n,d} + c_{n,1} \geq 0.$

Proof. Let $t \neq 0$.

We can define an automorphism $(x_1^+, \ldots, x_{d-1}^+, y_1^+, \ldots, y_\rho^+, z) \mapsto (\hat{x}_1^+, \ldots, \hat{x}_{d-1}^+, \hat{y}_1^+, \ldots, \hat{y}_\rho^+, \hat{z})$ of $\widetilde{V} \times \mathbf{C}^\times$ by

$$\hat{x}_1^+ := x_1^+ z^k - t y_1^+, \hat{x}_2^+ := x_2^+, \ldots, \hat{x}_{d-1}^+ := x_{d-1}^+, \tag{4.8}$$

$$\hat{y}_1^+ := tx_1^+, \hat{y}_2^+ := y_2^+, \ldots, \hat{y}_\rho^+ := y_\rho^+, \hat{z} := z.$$

In fact, since $\widetilde{V}$ is isomorphic to either $\mathbf{P}^{d-1}$ or a toric bundle over $\mathbf{P}^1$, this is a morphism, and we can easily construct the inverse of this morphism. By the automorphism $(x_1^-, \ldots, x_{d-1}^-, y_1^-, \ldots, y_\rho^-, w) \mapsto (x_1^+, \ldots, x_{d-1}^+, y_1^+, \ldots, y_\rho^+, z)$ given by the relations (4.7), we have

$$\hat{x}_1^+ = (x_1^- w^{c_{1,1}} + ty_1^- w^k)w^{-k} - ty_1^- = x_1^- w^{c_{1,1}-k},$$

$$\hat{x}_2^+ = x_2^- w^{c_{1,2}}, \ldots, \hat{x}_{d-1}^+ = x_{d-1}^- w^{c_{1,d-1}},$$

$$\hat{y}_1^+ = t(x_1^- w^{c_{1,1}} + ty_1^- w^k),\ \hat{y}_2^+ = y_2^-, \ldots, \hat{y}_\rho^+ = y_\rho^-,\ \hat{z} = 1/w.$$

By considering the action of $\widetilde{G}$ on $x_1^+, \ldots, x_{d-1}^+, y_1^+$, these relations are equivalent to

$$\hat{x}_1^+ = x_1^- w^{c_{1,1}-2k},\ \hat{x}_2^+ = x_2^- w^{c_{1,2}+ka_{1,2}}, \ldots, \hat{x}_{d-1}^+ = x_{d-1}^- w^{c_{1,d-1}+ka_{1,d-1}}, \tag{4.9}$$

$$\hat{y}_1^+ = t(x_1^- w^{c_{1,1}-k} + ty_1^-),\ \hat{y}_2^+ = y_2^-, \ldots, \hat{y}_\rho^+ = y_\rho^-,\ \hat{z} = 1/w.$$

Let

$$\hat{x}_1^- := x_1^-, \ldots, \hat{x}_{d-1}^- := x_{d-1}^-, \tag{4.10}$$

$$\hat{y}_1^- := t(x_1^- w^{c_{1,1}-k} + ty_1^-),\ \hat{y}_2^- := y_2^-, \ldots, \hat{y}_\rho^- := y_\rho^-,\ \hat{w} := w.$$

Then $(x_1^-, \ldots, x_{d-1}^-, y_1^-, \ldots, y_\rho^-, w) \mapsto (\hat{x}_1^-, \ldots, \hat{x}_{d-1}^-, \hat{y}_1^-, \ldots, \hat{y}_\rho^-, \hat{w})$ determines an automorphism of $\widetilde{V} \times \mathbf{C}^\times$, and in terms of this new coordinate, the automorphism $(x_1^-, \ldots, x_{d-1}^-, y_1^-, \ldots, y_\rho^-, w) \mapsto (\hat{x}_1^+, \ldots, \hat{x}_{d-1}^+, \hat{y}_1^+, \ldots, \hat{y}_\rho^+, \hat{z})$ given by the relations (4.9) is described as the automorphism $(\hat{x}_1^-, \ldots, \hat{x}_{d-1}^-, \hat{y}_1^-, \ldots, \hat{y}_\rho^-, \hat{w}) \mapsto (\hat{x}_1^+, \ldots, \hat{x}_{d-1}^+, \hat{y}_1^+, \ldots, \hat{y}_\rho^+, \hat{z})$ given by

$$\hat{x}_1^+ = \hat{x}_1^- \hat{w}^{c_{1,1}-2k},\ \hat{x}_2^+ = \hat{x}_2^- \hat{w}^{c_{1,2}+ka_{1,2}}, \ldots, \hat{x}_{d-1}^+ = \hat{x}_{d-1}^- \hat{w}^{c_{1,d-1}+ka_{1,d-1}}, \tag{4.11}$$

$$\hat{y}_1^+ = \hat{y}_1^-, \ldots, \hat{y}_\rho^+ = \hat{y}_\rho^-,\ \hat{z} = 1/\hat{w}.$$

We can show that the automorphisms $(x_1^+, \ldots, x_{d-1}^+, y_1^+, \ldots, y_\rho^+, z) \mapsto (\hat{x}_1^+, \ldots, \hat{x}_{d-1}^+, \hat{y}_1^+, \ldots, \hat{y}_\rho^+, \hat{z})$ in (4.8) and $(x_1^-, \ldots, x_{d-1}^-, y_1^-, \ldots, y_\rho^-, w) \mapsto (\hat{x}_1^-, \ldots, \hat{x}_{d-1}^-, \hat{y}_1^-, \ldots, \hat{y}_\rho^-, \hat{w})$ in (4.10) of $\widetilde{V} \times \mathbf{C}^\times$ are extended to automorphisms of V^+ and V^-, respectively as follows: Put

$$\hat{X}_1^+ := X_1^+(Z_1^+)^k(Z_2^+)^{kb_{2,d}-b_{2,1}} \cdots (Z_m^+)^{kb_{m,d}-b_{m,1}} - tY_1^+(Z_2^+)^{-b_{2,1}} \cdots (Z_m^+)^{-b_{m,1}},$$

$$\hat{X}_2^+ := X_2^+, \ldots, \hat{X}_{d-1}^+ := X_{d-1}^+,\ \hat{Y}_1^+ := tX_1^+(Z_2^+)^{b_{2,1}} \cdots (Z_m^+)^{b_{m,1}},$$

$$\hat{Y}_2^+ := Y_2^+, \dots, \hat{Y}_\rho^+ := Y_\rho^+,\ \hat{Z}_1^+ := Z_1^+, \dots, \hat{Z}_m^+ := Z_m^+.$$

By the assumption $b_{2,1} = \cdots = b_{m,1} = 0$, this defines an automorphism $(X_1^+, \dots, X_{d-1}^+, Y_1^+, \dots, Y_\rho^+, Z_1^+, \dots, Z_m^+) \mapsto (\hat{X}_1^+, \dots, \hat{X}_{d-1}^+, \hat{Y}_1^+, \dots, \hat{Y}_\rho^+, \hat{Z}_1^+, \dots, \hat{Z}_m^+)$ of V^+, and, obviously, the restriction of this automorphism through the isomorphisms $(X_1^+, \dots, X_{d-1}^+, Y_1^+, \dots, Y_\rho^+, Z_1^+, \dots, Z_m^+) \mapsto (x_1^+, \dots, x_{d-1}^+, y_1^+, \dots, y_\rho^+, z)$ and $(\hat{X}_1^+, \dots, \hat{X}_{d-1}^+, \hat{Y}_1^+, \dots, \hat{Y}_\rho^+, \hat{Z}_1^+, \dots, \hat{Z}_m^+) \mapsto (\hat{x}_1^+, \dots, \hat{x}_{d-1}^+, \hat{y}_1^+, \dots, \hat{y}_\rho^+, \hat{z})$ defined by the equalities (4.5) from $\mathbf{U}(\widetilde{\Sigma})/G^+ \subset V^+$ to $\widetilde{V} \times \mathbf{C}^\times$ is the automorphism $(x_1^+, \dots, x_{d-1}^+, y_1^+, \dots, y_\rho^+, z) \mapsto (\hat{x}_1^+, \dots, \hat{x}_{d-1}^+, \hat{y}_1^+, \dots, \hat{y}_\rho^+,\ \hat{z})$ corresponding to the equalities (4.8). Similarly, by the assumption $kc_{1,d} + c_{1,1} \geq 0, \dots, kc_{n,d} + c_{n,1} \geq 0$, by putting

$$\hat{X}_1^- := X_1^-, \dots, \hat{X}_{d-1}^- := X_{d-1}^-,$$

$$\hat{Y}_1^- := tX_1^-(W_1^-)^{kc_{1,d}+c_{1,1}} \cdots (W_n^-)^{kc_{n,d}+c_{n,1}},$$

$$\hat{Y}_2^- := Y_2^-, \dots, \hat{Y}_\rho^- := Y_\rho^-,\ \hat{W}_1^- := W_1^-, \dots, \hat{W}_n^- := W_n^-,$$

we get an automorphism $(X_1^-, \dots, X_{d-1}^-, Y_1^-, \dots, Y_\rho^-, W_1^-, \dots, W_n^-) \mapsto (\hat{X}_1^-, \dots, \hat{X}_{d-1}^-, \hat{Y}_1^-, \dots, \hat{Y}_\rho^-, \hat{W}_1^-, \dots, \hat{W}_n^-)$ of V^- whose restriction through the isomorphisms $(X_1^-, \dots, X_{d-1}^-, Y_1^-, \dots, Y_\rho^-, W_1^-, \dots, W_n^-) \mapsto (x_1^-, \dots, x_{d-1}^-, y_1^-, \dots, y_\rho^-, w)$ and $(\hat{X}_1^-, \dots, \hat{X}_{d-1}^-, \hat{Y}_1^-, \dots, \hat{Y}_\rho^-, \hat{W}_1^-, \dots, \hat{W}_n^-) \mapsto (\hat{x}_1^-, \dots, \hat{x}_{d-1}^-, \hat{y}_1^-, \dots, \hat{y}_\rho^-, \hat{w})$ defined by the equalities (4.6) from $\mathbf{U}(\widetilde{\Sigma})/G^- \subset V^-$ to $\widetilde{V} \times \mathbf{C}^\times$ is the automorphism $(x_1^-, \dots, x_{d-1}^-, y_1^-, \dots, y_\rho^-, w) \mapsto (\hat{x}_1^-, \dots, \hat{x}_{d-1}^-, \hat{y}_1^-, \dots, \hat{y}_\rho^-, \hat{w})$ corresponding to the equalities (4.10).

Next, to show $V_t \cong (2k, -ka_{1,2}, \dots, -ka_{1,d-1})^- V$ for any $t \neq 0$, we have to investigate the action of G^- on $\mathbf{U}(\Sigma^-)$. However, the action (4.4) is obviously equivalent to the following:

$$\begin{aligned}(4.12)\qquad &\Big(g^-\left(-\left(a_{1,1}A_1 + \cdots + a_{\rho,1}A_\rho + (-c_{2,1} - c_{1,1}c_{2,d})C_2 + \cdots\right.\right.\\
&\left.\left.+(-c_{n,1} - c_{1,1}c_{n,d})C_n\right)\right) X_1^-, \dots,\\
&g^-\left(-\left(a_{1,d-1}A_1 + \cdots + a_{\rho,d-1}A_\rho + (-c_{2,d-1} - c_{1,d-1}c_{2,d})C_2 + \cdots +\right.\right.\\
&\left.\left.(-c_{n,d-1} - c_{1,d-1}c_{n,d})C_n\right)\right) X_{d-1}^-, g^-(A_1)Y_1^-, \dots, g^-(A_\rho)Y_\rho^-,\\
&g^-\left(-(c_{2,d}C_2 + \cdots + c_{n,d}C_n)\right) W_1^-, g^-(C_2)W_2^-, \cdots, g^-(C_n)W_n^-\Big),\end{aligned}$$

because $B_1 + b_{2,d}B_2 + \cdots + b_{m,d}B_m - C_1 + c_{2,d}C_2 + \cdots + c_{n,d}C_n = 0$ on V. So, by the automorphism $(\hat{x}_1^-, \dots, \hat{x}_{d-1}^-, \hat{y}_1^-, \dots, \hat{y}_\rho^-, \hat{w}) \mapsto (\hat{x}_1^+, \dots, \hat{x}_{d-1}^+, \hat{y}_1^+, \dots, \hat{y}_\rho^+, \hat{z})$ given by the relations (4.11) and the action (4.12), we have $V_t \cong (2k, -ka_{1,2}, \dots, -ka_{1,d-1})^- V$ for any $t \neq 0$. q.e.d.

4.4 Projective space bundles over the projective line

The classical results for deformations among Hirzebruch surfaces are well-known. As a generalization of this results, for $\mathbf{P}^2$-bundles over $\mathbf{P}^1$, Nakamura [25] showed the following.

Proposition 4.4.1 (Nakamura [25]) *For integers a, b, c, a', b', c', let*

$$V = \mathbf{P}_{\mathbf{P}^1}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c)) \text{ and } V' = \mathbf{P}_{\mathbf{P}^1}(\mathcal{O}(a') \oplus \mathcal{O}(b') \oplus \mathcal{O}(c')).$$

Then, the following are equivalent.

(1) $a + b + c \equiv a' + b' + c' \pmod 3$.

(2) *There exist $\mathbf{P}^2$-bundles over $\mathbf{P}^1$ $V_0, \ldots, V_m$ such that $V_0 \cong V$, $V_m \cong V'$, and that V_{i-1} is deformed to V_i for any $1 \leq i \leq m$.*

(3) *V and V' are homeomorphic.*

We generalize the case (1) $\Longrightarrow$ (2) of Proposition 4.4.1 for $\mathbf{P}^{d-1}$-bundles over $\mathbf{P}^1$ using the one-parameter families constructed in Theorem 4.3.2. Harris [10] studied this case. For fundamental properties of primitive collections and primitive relations, see Batyrev [5], [6] and Sato [33]. We use the notation as in Section 4.3.

Let V be a $\mathbf{P}^{d-1}$-bundle over $\mathbf{P}^1$, that is,

$$V = V(p_1, \ldots, p_{d-1}) := \mathbf{P}_{\mathbf{P}^1}\left(\mathcal{O} \oplus \mathcal{O}(p_1) \oplus \cdots \oplus \mathcal{O}(p_{d-1})\right),$$

where $p_1, \ldots, p_{d-1}$ are nonnegative integers. Then, the primitive relations of the corresponding fan Σ are

$$\mathbf{e}_1 + \cdots + \mathbf{e}_{d-1} + \mathbf{a}_1 = 0 \text{ and } \mathbf{b}_1 + \mathbf{c}_1 = p_1\mathbf{e}_1 + \cdots + p_{d-1}\mathbf{e}_{d-1},$$

where $\mathrm{G}(\Sigma) = \{\mathbf{e}_1, \ldots, \mathbf{e}_{d-1}, \mathbf{a}_1, \mathbf{b}_1, \mathbf{c}_1\}$. For a nonnegative integer k such that $a_1 - k \geq 0$, the conditions in Theorem 4.3.3 are satisfied. Therefore, there exists a one-parameter complex analytic family $\{V_t\}_{t\in\mathbf{C}}$ such that

$$(1)\ V_t \cong \begin{cases} V & \text{if } \ t = 0 \\ (2k, k, \ldots, k)^- V & \text{if } \ t \neq 0. \end{cases}$$

We show that, for $V(p_1, \ldots, p_{d-1})$ and $V(p'_1, \ldots, p'_{d-1})$, if $p_1 + \cdots + p_{d-1} \equiv p'_1 + \cdots + p'_{d-1} \pmod d$, then there exist nonsingular toric d-folds $V_0, \ldots, V_m$ such that each V_i is a

$\mathbf{P}^{d-1}$-bundle over $\mathbf{P}^1$, $V_0 \cong V(p_1, \dots, p_{d-1})$, $V_m \cong V(p'_1, \dots, p'_{d-1})$, and V_{i-1} is deformed by a one-parameter family to V_i for any $1 \leq i \leq m$.

Let $k = 1$. Suppose that there exists $1 \leq i \leq d-1$ such that $p_i \geq 2$. So we may assume $p_1 \geq p_2 \geq \cdots \geq p_l > p_{l+1} = \cdots = p_{d-1} = 0$ by changing the order of the indices, where $p_1 \geq 2$. Then by the family (1), V is deformed to $(2, 1, \dots, 1)^{-}V$. The primitive relations of $(2, 1, \dots, 1)^{-}\Sigma$ are

$$\mathbf{e}_1 + \cdots + \mathbf{e}_{d-1} + \mathbf{a}_1 = 0 \text{ and}$$

$$\mathbf{b}_1 + \mathbf{c}'_1 = \begin{pmatrix} p_1 - 2 \\ p_2 - 1 \\ \vdots \\ p_l - 1 \\ p_{l+1} - 1 \\ \vdots \\ p_{d-1} - 1 \\ 0 \end{pmatrix} = \begin{cases} (p_1 - 1)\mathbf{e}_1 + p_2\mathbf{e}_2 + \cdots + p_l\mathbf{e}_l + \mathbf{a}_1 & \text{if} \quad l < d-1 \\ (p_1 - 2)\mathbf{e}_1 + (p_2 - 1)\mathbf{e}_2 + \cdots + (p_l - 1)\mathbf{e}_l & \text{if} \quad l = d-1, \end{cases}$$

where $\mathrm{G}\left((2, 1, \dots, 1)^{-}\Sigma\right) = \{\mathbf{e}_1, \dots, \mathbf{e}_{d-1}, \mathbf{a}_1, \mathbf{b}_1, \mathbf{c}'_1\}$. We can replace V by $(2, 1, \dots, 1)^{-}V$ and carry out this operation again. This operation terminates in finite steps and V becomes $V(p_1, \dots, p_{d-1})$ such that $p_1 \leq 1, \dots, p_{d-1} \leq 1$. In each step, $p_1 + \cdots + p_{d-1} \in \mathbf{Z}/d\mathbf{Z}$ does not change. Thus, we have the following.

Proposition 4.4.2 *For integers $a_1, \dots, a_d, a'_1, \dots, a'_d$, let $V = \mathbf{P}_{\mathbf{P}^1}(\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_d))$ and $V' = \mathbf{P}_{\mathbf{P}^1}(\mathcal{O}(a'_1) \oplus \cdots \oplus \mathcal{O}(a'_d))$. If $a_1 + \cdots + a_d \equiv a'_1 + \cdots + a'_d \pmod d$, then there exist $\mathbf{P}^{d-1}$-bundles over $\mathbf{P}^1$ $V_0, \dots, V_m$ such that $V_0 \cong V$, $V_m \cong V'$ and V_{i-1} is deformed to V_i for any $1 \leq i \leq m$. In particular, V and V' are homeomorphic.*

4.5 Weakened Fano varieties

The following definition is important for the birational geometry.

Definition 4.5.1 Let V be a nonsingular projective variety. V is called a *Fano* (resp. *weak Fano*) variety if its anti-canonical divisor $-K_V$ is ample (resp. nef and big).

The following definition was proposed by Minagawa [20] in connection with "Reid's fantasy" for weak Fano 3-folds.

Definition 4.5.2 (Minagawa [20]) Let V be a nonsingular weak Fano d-fold over $\mathbf{C}$ and $\Delta_\epsilon := \{t \in \mathbf{C} \,|\, |t| < \epsilon\}$ for a sufficiently small real number $\epsilon > 0$. Then, V is called a *weakened Fano* d-fold if V is not a nonsingular Fano d-fold, and there exists a small deformation $\varphi : \mathcal{V} \to \Delta_\epsilon$ such that $\mathcal{V}_0 := \varphi^{-1}(0) \cong V$, while $\mathcal{V}_t := \varphi^{-1}(t)$ is a nonsingular Fano d-fold for any $t \in \Delta_\epsilon \setminus \{0\}$.

In this section, we construct a deformation family for a certain toric weakened Fano 3-fold using the families constructed in Section 4.3. Toric weakened Fano 3-folds are completely classified by Sato [32]. Moreover, we give certain examples of toric weakened Fano 4-folds. We use the notation as in Section 4.3.

Example 4.5.3 Let V be the nonsingular toric weakened Fano 3-fold of type X_0^3 in the sense of Sato [32], that is, the primitive relations of Σ are

$$\mathbf{e}_1 + \mathbf{a}_1 = \mathbf{e}_2,\ \mathbf{e}_2 + \mathbf{a}_2 = 0 \text{ and } \mathbf{b}_1 + \mathbf{c}_1 = 2\mathbf{e}_1,$$

where $\mathrm{G}(\Sigma) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1, \mathbf{c}_1\}$. V is a F_1-bundle over $\mathbf{P}^1$, where F_1 is the Hirzebruch surface of degree 1. Therefore, by Theorems 4.3.2 and 4.3.3, there exists a complex analytic family $\{V_t\}_{t\in\mathbf{C}}$ such that

$$\begin{cases} V_0 \cong V, \text{ while} \\ V_t \cong (2,-1)^- V \ (t \neq 0). \end{cases}$$

The primitive relations of $(2,-1)^-\Sigma$ are

$$\mathbf{e}_1 + \mathbf{a}_1 = \mathbf{e}_2,\ \mathbf{e}_2 + \mathbf{a}_2 = 0 \text{ and } \mathbf{b}_1 + \mathbf{c}'_1 = \mathbf{e}_2,$$

where $\mathrm{G}(\Sigma) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1, \mathbf{c}'_1\}$. $(2,-1)^-V$ is the toric Fano 3-fold we want (see Section 4 in Sato [32]).

In the same way as in Example 4.5.3, we obtain certain examples of toric weakened Fano 4-folds which does not decomposed into direct products of lower-dimensional varieties. In the following, put $\mathrm{G}(\Sigma) = \{x_1, x_2, \ldots\}$, and the fans corresponding to toric weakened Fano 4-folds are described in terms of primitive relations. We also give the types of general fibers. The types of nonsingular toric Fano 4-folds are in the sense of Batyrev [6] and Sato [33].

(1) $x_1 + x_4 = x_2,\ x_2 + x_3 + x_5 = 0$ and $x_6 + x_7 = 2x_1$ (type D_7).

(2) $x_1 + x_4 = x_2,\ x_2 + x_6 = 0,\ x_3 + x_5 = x_2$ and $x_7 + x_8 = 2x_1$ (type L_1).

(3) $x_1 + x_4 = x_2,\ x_2 + x_6 = 0,\ x_3 + x_5 = x_6$ and $x_7 + x_8 = 2x_1$ (type L_{13}).

(4) $x_1 + x_4 = x_2,\ x_2 + x_5 = x_3,\ x_3 + x_6 = 0$ and $x_7 + x_8 = 2x_1$ (type L_2).

(5) $x_5 + x_6 = 0,\ x_3 + x_7 = 0,\ x_2 + x_3 = x_5,\ x_5 + x_7 = x_2,\ x_2 + x_6 = x_7,\ x_1 + x_4 = x_2$ and $x_8 + x_9 = 2x_1$ (type Q_1).

(6) $x_5 + x_6 = 0,\ x_3 + x_7 = 0,\ x_2 + x_3 = x_5,\ x_5 + x_7 = x_2,\ x_2 + x_6 = x_7,\ x_1 + x_4 = x_3$ and $x_8 + x_9 = 2x_1$ (type Q_{13}).

(7) $x_5 + x_6 = 0,\ x_3 + x_7 = 0,\ x_2 + x_3 = x_5,\ x_5 + x_7 = x_2,\ x_2 + x_6 = x_7,\ x_1 + x_4 = x_5$ and $x_8 + x_9 = 2x_1$ (type Q_8).

(8) $x_5 + x_6 = 0,\ x_3 + x_7 = 0,\ x_2 + x_3 = x_5,\ x_5 + x_7 = x_2,\ x_2 + x_6 = x_7,\ x_1 + x_4 = 0$ and $x_8 + x_9 = 2x_1$ (type Q_{11}).

(9) $x_5 + x_8 = 0,\ x_2 + x_5 = x_3,\ x_3 + x_8 = x_2,\ x_3 + x_6 = x_5,\ x_3 + x_7 = 0,\ x_2 + x_6 = 0,\ x_6 + x_8 = x_7,\ x_2 + x_7 = x_8,\ x_5 + x_7 = x_6,\ x_1 + x_4 = x_2$ and $x_9 + x_{10} = 2x_1$ (type U_1).

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