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Studies on commuting difference systems arising from solvable lattice models

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Abstract

We study a pair of commuting difference operators arising from the elliptic solution of the dynamical Yang-Baxter equation of type C_2. The operators act on the space of meromorphic functions on the weight space of $\mathfrak{sp}_4(\mathbb{C})$. We show that these operators can be identified with the system obtained by van Diejen and by Komori-Hikami with special parameters. It turns out that our case can be related to a pair of difference Lamé operators (two-body Ruijsenaars operators) and thereby we diagonalize the system on the finite dimensional space spanned by the level one characters of the $C_2^{(1)}$-affine Lie algebra.

Contents

Introduction

0.1 Calogero-Moser systems and Lax pair

The Calogero-Moser dynamical systems are widely studied Hamiltonian systems with amazingly rich structure. In the simplest case, it is the classical one-dimensional n-body system given by the Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + g^2 \sum_{j<k} v(x_j - x_k), \tag{0.1}$$

where x_j are coordinates, p_j are momenta and g is the coupling constant. Here the mass is set to be 1. The potential $v(x)$ is given by the following:

$$\begin{aligned} &\mathbf{I}.\ v(x) = \frac{1}{x^2}, \quad \mathbf{II}.\ v(x) = \frac{a^2}{\sinh^2(ax)}, \quad \mathbf{III}.\ v(x) = \frac{a^2}{\sin^2(ax)}, \\ &\mathbf{IV}.\ v(x) = \wp(x), \end{aligned} \tag{0.2}$$

where $\wp(x)$ denotes the Weierstrass $\wp$-function. The $\wp$-function potential **IV** is the most general case of these potentials and the other three can be obtained by letting one or two of the double periods of $\wp$-function tend to infinity. Incidentally, the systems with potential **I** is called the Calogero model and **III** the Sutherland model. Putting $p_j = \dot{x}_j$, the equations of motion are given by

$$\dot{x_j} = \{x_j, H\}, \quad \dot{p_j} = \{p_j, H\}, \tag{0.3}$$

where $\{\ ,\ \}$ is the Poisson bracket.

A Hamiltonian system with a $2n$-dimensional phase space is said to be Liouville integrable if (i) it has n independent Poisson-commuting integrals of motion $\{I_j\}$ (One of the I_j is the Hamiltonian H) and (ii) they are in involution, that is,

$$\text{(i) } \{H, I_j\} = 0, \quad \text{(ii) } \{I_j, I_k\} = 0. \tag{0.4}$$

In 1975, Moser considered the systems (0.1) with potentials **I** and **III** of (0.2) [Mo]. Using the Lax method, he has found n independent integrals of motion explicitly, and for the case **I**, he has also shown that these integrals are in involution. Moser's method was used to investigate analogous systems with potentials **II** [CMR] and **IV** [Ca] by Calogero et al.

We review the Lax method briefly. Suppose that we find a pair of matrices L and M (called the Lax pair), whose elements depend on x_j's and p_j's, so that the matrix equation (called the Lax equation)

$$i\frac{dL}{dt} = i\{L, H\} = [M, L] \tag{0.5}$$

is equivalent to the Hamiltonian equation (0.3). Then,

$$I_k := \mathrm{Tr} L^k = \sum_j (L^k)_{jj}, \quad (k = 1, 2, \ldots, n) \tag{0.6}$$

are integrals of motion, since we obtain

$$\begin{aligned} \frac{d}{dt}\mathrm{Tr}L^k &= -i\mathrm{Tr}[M, L^k] \\ &= -i\mathrm{Tr}(ML^k - L^k M) \;=\; 0. \end{aligned}$$

Here we have used $(d/dt)L^k = [M, L^k]$ (this can be shown by induction) and $\mathrm{Tr}AB = \mathrm{Tr}BA$. For the Hamiltonian (0.1), Calogero and Moser considered the following matrices:

$$L_{jk} = p_j \delta_{jk} + ig(1 - \delta_{jk}) w(x_j - x_k), \tag{0.7}$$

$$M_{jk} = g(1 - \delta_{jk}) y(x_j - x_k) + g\delta_{jk} \sum_{l \neq j} z(x_j - x_l). \tag{0.8}$$

Here the condition that the function $w(x)$ is odd and $y(x), z(x)$ are even, are assumed. Substituting L and M into the Lax equation (0.5) together with (0.3), we find certain functional equations for $w(x), y(x), z(x)$ and $v(x)$. They solved these functional equations and get the potentials (0.2).

It was pointed out by Olshanetsky and Perelomev that one can quantize the Calogero-Moser system in such a way that the integrability is preserved, namely, such that the integrals of motion go over into commuting operators. We put

$$p_j = -i\frac{\partial}{\partial x_j}.$$

Here the Plank constant is set to be 2π. Then the Hamiltonian of the resulting quantum Calogero-Moser system is given by the Schrödinger operator

$$\hat{H} = -\frac{1}{2}\sum_{j=1}^{n} \partial_j^2 + g(g-1)\sum_{j<k} v(x_j - x_k), \quad \left(\partial_j := \frac{\partial}{\partial x_j}\right) \tag{0.9}$$

where the potential $v(x)$ is equal to the classical case (0.2). Note that the parameter g^2 in the classical Hamiltonian (0.1) is replaced by $g(g-1)$. This is due to contributions which arise when the partial derivations in (0.9) act on the potentials. In the quantum mechanics, the equations of motion are given by

$$\frac{dx_j}{dt} = i[x_j, H] \quad \frac{dp_j}{dt} = i[p_j, H],$$

and a Hamiltonian system is called (Liouville) integrable if there exist conserved operators $\{I_k\}$ $(1 \le k \le n)$ such that

$$\text{(i) } [H, I_k] = 0, \quad \text{(ii) } [I_k, I_l] = 0. \tag{0.10}$$

The Lax method is also useful in quantum cases. We choose the quantum Lax matrices L and M, which have quite the same form as (0.7) and (0.8), and consider the quantum Lax equation

$$-i\frac{dL}{dt} = [H, L] = [L, M].$$

For the elliptic potential **IV** in (0.2), the Lax matrix L was introduced by Krichever [Kr] as follows:

$$L_{jk} = p_j \delta_{jk} + ig(1-\delta_{jk}) \frac{\sigma(x_j - x_k + \mu)}{\sigma(\mu)\sigma(x_j - x_k)}, \tag{0.11}$$

where σ is the Weierstrass σ-function and $\mu \in \mathbb{C}$ an auxiliary parameter.

In 1987, Ruijsenaars [R] considered the difference analogue (the so-called "relativistic analogue") of the Calogero-Moser systems. In the quantum elliptic case (potential (0.2) of type **IV**), the system is given by the following difference operators:

$$M_k = \sum_{\substack{I \subset \{1,\dots,n\} \\ |I| = k}} \prod_{\substack{i \in I \\ j \notin I}} \frac{\sigma(x_i - x_j + \beta\hbar)}{\sigma(x_i - x_j)} T_I, \tag{0.12}$$

where $T_I := \prod_{i \in I} T_i$ and T_i are the shift operators defined by

$$T_i f(x_1, \dots, x_n) = f(x_1, \dots, x_i + \hbar, \dots, x_n),$$

and $\beta \in \mathbb{C}$ is a parameter. The rational (**I**), hyperbolic (**II**), trigonometric (**III**) cases are obtained by appropriate degeneration of (0.12). Ruijsenaars has shown the commutativity of the operators $\{M_k\}$ directly, and proposed the corresponding Lax matrix. By sending the step size $\hbar$ to zero, the difference operators M_k go over into the system of differential operators containing $\hat{H}$ (0.9). In the trigonometric case, the eigenvalue problem of the system $\{M_k\}$ has been solved by Macdonald [M]. Their eigenfunctions, called Macdonald polynomials, are deeply investigated as q-orthogonal polynomial theory.

These models are in fact related to the type A simple Lie algebra $\mathfrak{sl}_n(\mathbb{C})$. In the present thesis, we are interested in the Ruijsenaars-type elliptic difference extension of the Calogero-Moser system corresponding to the root system other than type A.

For the trigonometric case, the Macdonald difference system allows at least two theoretical understanding so that we can regard the model as certainly of type A. One is the work by Etingof and Kirillov [EK], who obtained these operators as the image of central elements of the quantum enveloping algebra $U_q(\mathfrak{sl}_n)$ ($q = e^{\pi i \hbar}$) acting on vector valued characters. The other is the work by Cherednik [Ch2], who used the affine Hecke algebra of type A, its representation via q-difference operators, and the center of the algebra. Among these approaches, the former is more close and important for us. However, we will utilize yet another understanding of the system, namely, the idea of the transfer matrix which we will explain in the next section. In fact, as is shown by Hasegawa's work on Ruijsenaars model, the transfer matrix will give the proper difference analogue of the Lax matrix of Krichever's type (0.11), and we can recover the commuting operators $\{M_k\}$ as commuting transfer matrices.

0.2 Yang-Baxter equation and transfer matrix

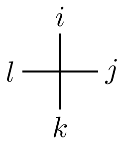
Originally, in Baxter's study of two-dimensional lattice statistical models, the Yang-Baxter equation arose as a condition to provide sufficiently many commuting operators. This is done by taking the trace of the so-called transfer matrix T, the operator which satisfies the "$RTT = TTR$" relation (0.15). Here, in the lattice model situation, the matrix R ("the R-matrix") stands for the local Boltzmann weight of the model.

In the "vertex" type 2-dimensional lattice model, the Boltzmann weight is given as a set of quantities $\{R^{ij}_{kl}\}$, where i, j, k, l take value in $\{1, 2, \ldots, n\}$ and n is fixed once for all (Fig. 0.1) [JM]. Let us explain the notion of the transfer matrix, which plays a key role in this thesis. We introduce an n-dimensional vector space V with basis $\{v_1, \ldots, v_n\}$ and define the matrix $R \in \mathrm{End}(V \otimes V)$ by

$$R(v_i \otimes v_j) = \sum_{k,l} R^{ij}_{kl} v_k \otimes v_l. \tag{0.13}$$

Assume that R depends on a parameter $u \in \mathbb{C}$, (u is called the "spectral parameter" in the context of the inverse scattering method [FT]), and define $\mathcal{T}(\vec{u}, u') \in \mathrm{End}(V^{\otimes N} \otimes V)$ by

$$\begin{aligned}
&\mathcal{T}(\vec{u}, u')(v_{i_1} \otimes \cdots \otimes v_{i_N} \otimes v_{j_1}) \\
&:= \sum_{k_1,\ldots,k_N,j_1'} T^{i_1,\ldots,i_N,j_1}_{k_1,\ldots,k_N,j_1'}(u_1, \ldots, u_N, u') v_{k_1} \otimes \cdots \otimes v_{k_N} \otimes v_{j_1'},
\end{aligned}$$



$= R^{ij}_{kl}$

Figure 0.1: R-matrix or a Boltzmann weight of a vertex model. The indices i, j, k, l run from 1 to n, where n is the number of freedom at each vertices.

where $\vec{u} = (u_1, \dots, u_N)$ and

$$\begin{aligned} &T^{i_1,\dots,i_N,j_1}_{k_1,\dots,k_N,j_1'}(u_1, \dots, u_N, u') \\ &:= \sum_{l_1,\dots,l_{N-1}} R^{i_1,l_1}_{k_1,j_1'}(u_1 - u') R^{i_2,l_2}_{k_2,l_1}(u_2 - u') \dots R^{i_N,j_1}_{k_N,l_{N-1}}(u_N - u'). \end{aligned}$$

This is the naive version of the transfer matrix of the model. In the physics viewpoint, it is natural to assume the "periodic boundary condition", that is, $j_1 = j_1'$. This means to take the summation of $T^{i_1,\dots,i_N,j_1}_{k_1,\dots,k_N,j_1'}(u_1, \dots, u_N, u')$ over $j_1 = j_1'$. The resulting endomorphism $\mathrm{trace}_V(\mathcal{T}(\vec{u}, u')) \in \mathrm{End}(V^{\otimes n})$ is called the row-to-row transfer matrix:

$$\begin{aligned} &\mathrm{trace}_V(\mathcal{T}(\vec{u}, u'))(v_{i_1} \otimes \dots \otimes v_{i_N}) \\ &:= \sum_{k_1,\dots,k_N,j_1} T^{i_1,\dots,i_N,j_1}_{k_1,\dots,k_N,j_1}(u_1, \dots, u_N, u') v_{k_1} \otimes \dots \otimes v_{k_N} \otimes v_{j_1}. \end{aligned}$$

Consider the following condition for the matrix $R(u) \in \mathrm{End}(\mathrm{V} \otimes \mathrm{V})$:

$$\begin{aligned} R_{12}(u - u') R_{13}(u - u'') R_{23}(u' - u'') &= R_{23}(u' - u'') R_{13}(u - u'') R_{12}(u - u') \\ &: V \otimes V \otimes V \to V \otimes V \otimes V, \end{aligned} \tag{0.14}$$

where R_{ij} means R acting in the ith and jth factors of the tensor product $V \otimes V \otimes V$ and u, u', u'' are arbitrary complex numbers. This is called the Yang-Baxter equation.

Because of the construction of the transfer matrix, if $R(u)$ is invertible for all u, we have

$$R(u - u') \mathcal{T}_1(u) \mathcal{T}_2(u') R(u - u')^{-1} = \mathcal{T}_2(u') \mathcal{T}_1(u). \tag{0.15}$$

Then taking the trace on $V \otimes V$ on both sides gives

$$\mathrm{trace}_V \mathcal{T}(u) \mathrm{trace}_V \mathcal{T}(u') = \mathrm{trace}_V \mathcal{T}(u') \mathrm{trace}_V \mathcal{T}(u), \tag{0.16}$$

that is, $[T(u), T(u')] = 0$. This resembles (0.10). To sum up, the Yang-Baxter equation (0.14) will give a one-parameter family of commuting operators.

Solutions of the equation (0.14) were classified by Belavin-Drinfeld [BD] under a certain non-degeneracy condition, and yield three families: rational, trigonometric and elliptic. Let us recall the elliptic solution (Belavin's R-matrix [Bel]) here, which is actually relevant for the Ruijsenaars model (0.12) due to Hasegawa. Fix $\tau, \hbar \in \mathbb{C}$ and assume $\mathrm{Im}\tau > 0$. Put

$$\theta_{m,l}(u,\tau) := \sum_{\mu \in m + l\mathbb{Z}} \exp\left[2\pi i \left(\mu u + \frac{\mu^2}{2l}\tau\right)\right]$$

and $\theta^{(j)}(u) := \theta_{1/2-j/n,1}(u + 1/2, n\tau)$. The zeros of $\theta^{(j)}(u)$ are given by $\mathbb{Z} + (j + n\mathbb{Z})\tau$. Then Belavin's solution $R(u)$ to (0.13) is given by

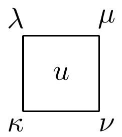
$$R(u)^{ij}_{kl} = \delta_{i+j,k+l \bmod n} \frac{\theta^{(k-l)}(u+\hbar)}{\theta^{(k-i)}(\hbar)\theta^{(i-l)}(u)} \frac{\prod_{k=0}^{n-1} \theta^{(k)}(u)}{\prod_{k=1}^{n-1} \theta^{(k)}(0)}. \tag{0.17}$$

Generally speaking, each solution of the Yang-Baxter equation corresponds to a certain bialgebra (a quantum group) and its representation. As for Belavin's R-matrix (0.17), the relevant algebraic structure is known as the Sklyanin algebra [Sk1], and the n dimensional vector space V should be regarded as its vector representation, which is an analogue of the vector representation of the type A simple Lie algebra $\mathfrak{sl}_n(\mathbb{C})$. Similarly, in the trigonometric case, the solution of the Yang-Baxter equation corresponds to the affine quantum universal enveloping algebra $U_q(\widehat{\mathfrak{sl}}_n(\mathbb{C}))$ and its vector representation. According to the Belavin-Drinfeld classification, the elliptic solutions of the vertex-type Yang-Baxter equation (0.14) exist only for $\mathfrak{sl}_n(\mathbb{C})$. This is in contrast with the situation of elliptic Calogero-type models, which can be generalized to any type of root systems (0.27).

However, there is another type of the Yang-Baxter equation, known as the face type Yang-Baxter equation, which admits more elliptic solutions. In the face-type statistical model, or the interaction-round-a-face model (abbrev. IRF or face model) [Ba2], the Boltzmann weight with spectral parameter u is assigned to each $\lambda, \mu, \nu, \kappa \in \mathcal{S}$, where $\mathcal{S}$ is the set of freedom of the model (Fig. 0.2): Let us denote the weight by

$$W\left(\begin{array}{cc} \lambda & \mu \\ \kappa & \nu \end{array} \middle| u \right). \tag{0.18}$$

To obtain the commutativity of the transfer matrices of this model in the same way as in the vertex model, the following equation should be assumed instead of the vertex-type



$$= W\left(\begin{array}{cc} \lambda & \mu \\ \kappa & \nu \end{array}\middle| u\right).$$

Figure 0.2: A Boltzmann weight of a face model. For each square with states $\lambda, \mu, \nu, \kappa$ at each corners, the assigned quantity have the meaning as the local energy of the configuration.

equation (0.14).

$$\begin{aligned}
&\sum_{\eta} W\left(\begin{array}{cc} \rho & \eta \\ \sigma & \kappa \end{array}\middle| u-v\right) W\left(\begin{array}{cc} \lambda & \mu \\ \rho & \eta \end{array}\middle| u-w\right) W\left(\begin{array}{cc} \mu & \nu \\ \eta & \kappa \end{array}\middle| v-w\right) \\
&= \sum_{\eta} W\left(\begin{array}{cc} \lambda & \eta \\ \rho & \sigma \end{array}\middle| v-w\right) W\left(\begin{array}{cc} \eta & \nu \\ \sigma & \kappa \end{array}\middle| u-w\right) W\left(\begin{array}{cc} \lambda & \mu \\ \eta & \nu \end{array}\middle| u-v\right). \qquad (0.19)
\end{aligned}$$

This is the face-type Yang-Baxter equation.

Actually, as we will see in the body of this thesis, the variables $\lambda, \mu, \nu, \ldots$ in the Boltzmann weight acquire the meaning as dynamical variables in the context of Calogero type systems. That is, λ corresponds to $x = (x_1, \ldots, x_n)$ in the previous formulas. Accordingly, this equation is also called the "dynamical" Yang-Baxter equation [Fe1],[Fe2].

To be more precise, let us recall an elliptic solution of this equation corresponding to $\mathfrak{sl}_n(\mathbb{C})$ after Jimbo-Miwa-Okado [JMO1]. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{sl}_n(\mathbb{C})$ and $\mathfrak{h}^*$ its dual. We realize $\mathfrak{h}^*$ in the n-dimensional vector space $\mathbb{C}^n$ as follows. Let $\{e_i\}_{i=1}^n$ be the orthonormal basis with respect to the inner product $\langle\ ,\ \rangle$ of $\mathbb{C}$, and put $\bar{\varepsilon}_i := e_i - (e_1 + \cdots + e_n)/n$ and $\mathfrak{h}^* := \sum_{i=1}^n \mathbb{C}\bar{\varepsilon}_i$. For an element $\lambda \in \mathfrak{h}^*$ we define

$$W\left(\begin{array}{cc} \lambda & \lambda + \hbar\bar{\varepsilon}_i \\ \lambda + \hbar\bar{\varepsilon}_i & \lambda + 2\hbar\bar{\varepsilon}_i \end{array}\middle| u\right) = \frac{[u+\hbar]}{[\hbar]}, \qquad (0.20)$$

$$W\left(\begin{array}{cc} \lambda & \lambda + \hbar\bar{\varepsilon}_i \\ \lambda + \hbar\bar{\varepsilon}_i & \lambda + \hbar(\bar{\varepsilon}_i + \bar{\varepsilon}_j) \end{array}\middle| u\right) = \frac{[\lambda_{ij} - u]}{[\lambda_{ij}]} \quad (i \neq j), \qquad (0.21)$$

$$W\left(\begin{array}{cc} \lambda & \lambda + \hbar\bar{\varepsilon}_i \\ \lambda + \hbar\bar{\varepsilon}_j & \lambda + \hbar(\bar{\varepsilon}_i + \bar{\varepsilon}_j) \end{array}\middle| u\right) = \frac{[u]}{[\hbar]}\frac{[\lambda_{ij} + \hbar]}{[\lambda_{ij}]} \quad (i \neq j). \qquad (0.22)$$

Here $\lambda_{ij} := \langle \lambda, \bar{\varepsilon}_i - \bar{\varepsilon}_j \rangle$, and

$$[u] := \theta_{1/2,1}\left(u + \frac{1}{2}\right) = i\mathrm{p}^{1/8} \sin \pi u \prod_{m=1}^{\infty} (1 - 2\mathrm{p}^m \cos 2\pi u + \mathrm{p}^{2m})(1 - \mathrm{p}^m) \tag{0.23}$$

($\mathrm{p} := e^{2\pi i \tau}$) denotes Jacobi's first theta function. This is an odd function and has the following quasi-periodicity:

$$[u + m] = (-1)^m [u], \quad [u + m\tau] = (-1)^m e^{-\pi i m^2 \tau - 2\pi i m u} [u] \quad (m \in \mathbb{Z}). \tag{0.24}$$

For the other configurations of $\lambda, \mu, \nu, \kappa \in \mathfrak{h}^*$, we set

$$W\left(\begin{array}{cc} \lambda & \mu \\ \kappa & \nu \end{array} \middle| u \right) = 0. \tag{0.25}$$

Then (0.20), (0.21), (0.22) and (0.25) solve the equation (0.19). It is known that the above face Boltzmann weight and the Belavin's R-matrix are related in the following way. Define the outgoing intertwining vector $\phi(u)^{\mu}_{\lambda,j}$ as

$$\phi(u)^{\mu}_{\lambda,j} := \begin{cases} \vartheta_j(u/n - \langle \lambda, \bar{\varepsilon}_k \rangle)/i\eta(\tau) & (\text{if there exists } k \text{ such that } \mu = \lambda + \hbar \bar{\varepsilon}_k) \\ 0 & (\text{otherwise}) \end{cases} \tag{0.26}$$

where

$$\vartheta_j = \theta_{n/2-j,n}\left(u + \frac{1}{2}, \tau\right) = \sum_{\mu \in n/2 - j + n\mathbb{Z}} \exp\left[2\pi i \left(\mu\left(u + \frac{1}{2}\right) + \frac{\mu^2}{2n}\tau\right)\right],$$

and $\eta(\tau) := \mathrm{p}^{1/24} \prod_{m=1}^{\infty} (1 - \mathrm{p}^m)$ denotes the Dedekind eta function with $\mathrm{p} = e^{2\pi i \tau}$. Then we have

$$\sum_{i,j=1}^{n} R(u - v)^{i,j}_{k,l} \phi(u)^{\mu}_{\lambda,i} \otimes \phi(v)^{\nu}_{\mu,j} = \sum_{\kappa} \phi(v)^{\kappa}_{\lambda,l} \otimes \phi(u)^{\nu}_{\kappa,k} W\left(\begin{array}{cc} \lambda & \mu \\ \kappa & \nu \end{array} \middle| u - v \right).$$

This is the vertex-IRF correspondence we mentioned in Section 0.1, which is a special feature of type A models. Hasegawa used this relation to obtain the difference extension of Krichever's Lax matrix, or an L-operator, that satisfies the equation (0.15).

0.3 The aim and the plan of this paper

In the Hamiltonian system (0.1) or (0.9), the coordinate space of dynamical variables $\{(x_1, x_2, \ldots, x_n)\}$ has an obvious action of the symmetric group S_n, and the Hamiltonian is S_n-invariant. We can regard the symmetric group S_n as the Weyl group associated to the root system of type A_{n-1}. From this point of view, Olshanetsky and Perelomov generalized the system with Hamiltonian (0.1) or (0.9) to arbitrary root systems [OP2], [OP3]. For the root system BC_n, their Hamiltonian takes the form

$$H = \sum_{j=1}^{n} p_j^2 + g \sum_{j<k} \left(v(x_j - x_k) + v(x_j + x_k) \right) \\ + \sum_{j=1}^{n} \left(g_1 v(x_j) + g_2 v(2x_j) \right), \tag{0.27}$$

where the potential $v(x)$ is the same as (0.2). Setting the parameter $g_1 = 0$ (resp. $g_2 = 0$, $g_1 = g_2 = 0$), the Hamiltonian corresponds to the root system B_n (resp. C_n, D_n). These systems also have been studied broadly. The classical Lax pair for the system (0.27) is constructed by Inozemtsev et al.[IM], [I], and the quantum conserved quantities or commuting differential operators are presented by Ochiai, Oshima and Sekiguchi [OOS]. The aim of this thesis is to take a step toward the difference extension of the Hamiltonian of this type and the understanding thereof.

A generalization of the quantum system with Hamiltonian (0.27) to a difference operator, or equivalently, a generalization of Ruijsenaars difference system to BC_n case, is studied by van Diejen [vD2] and Komori-Hikami [KH1], [KH2]. A brief history of this system is as follows. First, van Diejen constructed two elliptic commuting difference operators, one is of the first order and the other is of the n-th order. In particular, he obtained an elliptic extension of the difference Calogero-Moser system of type BC_2 [vD1]. Extending this work by van Diejen, Komori and Hikami obtained a general family of n commuting difference operators with elliptic function coefficients. Besides the step parameter of difference operators and the modulus of elliptic functions, the family contains nine arbitrary parameters. Their construction uses Shibukawa-Ueno's elliptic R-operator [SU] together with the elliptic K-operators [KH3], [KH4], that is, the elliptic solution to the reflection equation [Sk2], [Ch1]. This method can be regarded as an elliptic generalization of the affine Hecke algebra approach to Macdonald systems, which have been extensively used by Cherednik (see [N] for BC_n case).

Comparing to these works, what is characteristic in the present thesis is in the method to construct the commuting operators: We will generalize Hasegawa's construction of Ruijsenaars system as transfer matrix to the root systems other than the type A. As we mentioned before and will review in Appendix B, his transfer matrix $\tilde{L}$ can be regarded as the difference analogue of Krichever's Lax matrix, so that his construction of commuting operators can be regarded as the direct generalization of the classical construction (0.6). However, in the construction used in [H1] and [H3], a relation between Belavin's elliptic quantum R-matrix (0.17) and the face-type solution of the Yang-Baxter equation (0.20), (0.21), (0.22), especially the intertwining vector (0.26), played a central role. For the root systems other than type A, it is known that there are no vertex-type R-matrices nor the intertwining vectors. Nevertheless, the face-type solutions of the Yang-Baxter equation are known for all classical Lie algebras and their vector representations [JMO2]. We will utilize this type of solution to introduce our difference operators. For type A case this approach was pursued by Felder and Varchenko [FV2], but it is nontrivial for the other cases. It turns out that for type C_2 the method works quite satisfactory.

We define a C_2 analogue of Hasegawa's operator $\tilde{L}$ (Appendix B) by using the fused Boltzmann weights for the anti-symmetric representation of C_2 type Lie algebra to obtain a pair of difference operators. Then an argument similar to derive (0.16) works.

Theorem. The following two difference operators commute:

$$\widetilde{M_1} = \sum_{i \in \{\pm 1, \pm 2\}} \prod_{\substack{j \in \{\pm 1, \pm 2\} \\ j \neq \pm i}} \frac{[\lambda_i + \lambda_j - \hbar]}{[\lambda_i + \lambda_j]} T_i^{\hbar}, \tag{0.28}$$

$$\widetilde{M_2} = \sum_{\substack{i=\pm 1 \\ j=\pm 2}} \left(\frac{[\lambda_i + \lambda_j - \hbar]}{[\lambda_i + \lambda_j + \hbar]} T_i^{\hbar} T_j^{\hbar} + \frac{[\lambda_i + \lambda_j - \hbar]\,[\lambda_i + \lambda_j + 2\hbar]}{[\lambda_i + \lambda_j]\,[\lambda_i + \lambda_j + \hbar]} \right). \tag{0.29}$$

We also show that the space spanned by the level one characters of the affine Lie algebra $\widehat{\mathfrak{sp}}_4(\mathbb{C})$ is invariant under the action of the difference operators. Moreover, we will give a simultaneous diagonalization of the system on this space.

Now we explain the plan of this thesis. This thesis consists of four chapters.

In Chapter 1, we present explicit expressions for the system of Boltzmann weights for the vector representation of B_n, C_n and D_n type Lie algebras. Originally, the Boltzmann weight of this type was given by Jimbo, Miwa and Okado [JMO2], but their solution was expressed in terms of square roots of theta functions. Our Boltzmann weights and the original ones can be conjugated by a similarity transformation, which we will give in Appendix A. Since this transformation does not affect the validity of the Yang-Baxter equation, one can see that our Boltzmann weights solve the Yang-Baxter equation. However, a problem arises when we choose the branch of the square roots. In fact, there are at least two ways to choose the sign of products of the square roots which solve the Yang-Baxter equation. Consequently, in this thesis, we will give a proof of the Yang-Baxter equation for our Boltzmann weights directly without using the similarity transformation (Theorem 1). In fact, the properties of our Boltzmann weight such as crossing symmetry, take a slightly different form from the original ones (Proposition 1). Nevertheless, the functional identities we must prove are mostly the same as those in the original paper. Therefore, we show the sketch of the proof, and describe in detail the less written part of the proof given in [JMO2].

In Chapter 2, we study the fusion procedure for the Boltzmann weight of type C_2. First, we introduce a vector space called the path space on which the Boltzmann weights act naturally as a linear operator. This operator should be regarded as a building block, and we will use these blocks to compose more general face operator satisfying the Yang-Baxter equation. We call the matrix coefficients of these composite operators as the fused Boltzmann weight. Their explicit formulas are given in Section 2.2

In Chapter 3, we construct the difference operators by using the fused Boltzmann weights. This is the first main result (Theorem 2) of this thesis. The commutativity of these operators relies on the Yang-Baxter equation of the face type (0.19). In Section 3.2, we will explain how our system can be identified with van Diejen-Komori-Hikami system with special choice of parameters (Theorem 3). We also calculate the differential operators in the limit in Section 3.3. The way to construct the difference operators is applicable to the A_n and B_n cases as well. We deal with these cases in appendix B.

In Chapter 4, we will give the simultaneous diagonalization of our C_2 type difference operators. In Section 4.1, we introduce a finite dimensional space of theta functions invariant under the action of the (affine) Weyl group and its basis after Kac-Peterson [KP]. We show that the difference operators constructed in Section 3.1 preserve this space (Theorem 4). Then our second main theorem states the diagonalization of our operators on this space (Theorem 5). This is an elliptic analogue of the eigenvalue problem of Macdonald operators on the space of symmetric polynomials. It turns out that our operators split into two A_1 operators (difference Lamé operators), and the eigenvalue problem can be reduced to the eigenvalue problem of A_1 type operators.

Chapter 1

Solution of the face-type Yang-Baxter equation of types B_n, C_n, and D_n

1.1 The elliptic solutions of the face-type Yang-Baxter equation

Let $\mathfrak{g}$ be the finite dimensional simple Lie algebra of type X_n, where X_n denotes one of the B_n, C_n, or D_n, $\mathfrak{h}$ its Cartan subalgebra and $\mathfrak{h}^*$ the dual space of $\mathfrak{h}$. We denote by ϖ_j $(0 \leq j \leq n)$ the fundamental weights, and $\mathcal{P}$ the set of weights that belongs to the vector representation $L(\varpi_1)$ of $\mathfrak{g}$. To express these objects, we introduce the n-dimensional Euclidean space with basis $\varepsilon_1, \ldots, \varepsilon_n$ and bilinear form $(\ ,\ )$ as listed below. We give also the root system $R \subset \mathfrak{h}^*$ of $\mathfrak{g}$ (Fix the square length of the long roots is two).

$B_n\,(n \geq 1) : \mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C})$.

- $(\varepsilon_j, \varepsilon_k) := \delta_{jk}$,
- $R := \{\pm(\varepsilon_j \pm \varepsilon_k), \pm\varepsilon_l \,|\, 1 \leq j < k \leq n, 1 \leq l \leq n\}$,
- $\varpi_j = \begin{cases} \varepsilon_1 + \cdots + \varepsilon_j & (1 \leq j \leq n-1), \\ (\varepsilon_1 + \cdots + \varepsilon_n)/2 & (j = n), \end{cases}$
- $\mathcal{P} = \{\pm\varepsilon_1, \pm\varepsilon_2, \ldots, \pm\varepsilon_n, 0\}$.

$C_n\,(n \geq 2) : \mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$.

- $(\varepsilon_j, \varepsilon_k) := \dfrac{1}{2}\delta_{jk}$,

- $R := \{\pm(\varepsilon_j \pm \varepsilon_k), \pm 2\varepsilon_l \,|\, 1 \le j < k \le n, 1 \le l \le n\}$,
- $\varpi_j = \varepsilon_1 + \cdots + \varepsilon_j$,
- $\mathcal{P} = \{\pm\varepsilon_1, \pm\varepsilon_2, \dots, \pm\varepsilon_n\}$.

D_n $(n \ge 3)$: $\mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C})$,

- $(\varepsilon_j, \varepsilon_k) := \delta_{jk}$,
- $R := \{\pm(\varepsilon_j \pm \varepsilon_k) \,|\, 1 \le l \le n\}$,
- $\varpi_j = \begin{cases} \varepsilon_1 + \cdots + \varepsilon_j & (1 \le j \le n-2), \\ (\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n)/2 & (j = n-1), \\ (\varepsilon_1 + \dots \varepsilon_{n-1} + \varepsilon_n)/2 & (j = n), \end{cases}$
- $\mathcal{P} = \{\pm\varepsilon_1, \pm\varepsilon_2, \dots, \pm\varepsilon_n\}$.

Fix a complex parameter $\hbar$. For $p \in \mathcal{P}$, we shall use the following notation frequently,

$$\begin{aligned} \widehat{p} &= 2\hbar p \quad \text{for } C_n, \\ &= \hbar p \quad \text{for } B_n, D_n, \end{aligned}$$

and we denote $\widehat{\mathcal{P}} := \{\widehat{p} \,|\, p \in \mathcal{P}\}$.

The system of Boltzmann weights for the face model is given by a set of functions for any quadruple $(\lambda, \mu, \nu, \kappa)$ of elements of $\mathfrak{h}^*$. Let us denote the functions by

$$W\left(\begin{array}{cc|c} \lambda & \mu & \\ \kappa & \nu & u \end{array} \right),$$

that depend on the spectral parameter $u \in \mathbb{C}$. They satisfy the condition

$$W\left(\begin{array}{cc|c} \lambda & \mu & \\ \kappa & \nu & u \end{array} \right) = 0 \quad \text{unless} \quad \mu - \lambda, \nu - \mu, \kappa - \lambda, \nu - \kappa \in \widehat{\mathcal{P}}.$$

Under the setting above, Jimbo-Miwa-Okado found a solution of the Yang-Baxter equation ((0.19) in Introduction) [JMO2]. The solutions are parameterized in terms of the elliptic theta function $[u]$ (0.23) just like the A_n-type solution (See Section 0.2). Explicitly the solutions are given by the following formulas. For $\lambda \in \mathfrak{h}^*$ and $p \in \mathcal{P}$, we put

$$\lambda_p := (\lambda, p) \quad (p \neq 0) \quad \text{and} \quad \lambda_0 = -\frac{\hbar}{2}.$$

We will write

$$\begin{array}{c} p \\ s\,\boxed{u}\,q \\ r \end{array} = W\left(\begin{array}{cc} \lambda & \lambda+\widehat{p} \\ \lambda+\widehat{s} & \lambda+\widehat{p}+\widehat{q} \end{array} \middle| \, u \right)$$

for $p, q, r, s \in \mathcal{P}$ such that $p+q=r+s$.

$$\begin{array}{c} p \\ p\,\boxed{u}\,p \\ p \end{array} = \frac{[c-u]\,[u+\hbar]}{[c]\,[\hbar]} \quad (p \neq 0), \tag{1.1}$$

$$\begin{array}{c} p \\ p\,\boxed{u}\,q \\ q \end{array} = \frac{[c-u]\,[\lambda_p-\lambda_q-u]}{[c]\,[\lambda_p-\lambda_q]} \quad (p \neq \pm q), \tag{1.2}$$

$$\begin{array}{c} q \\ p\,\boxed{u}\,p \\ q \end{array} = \frac{[c-u]\,[u]\,[\lambda_p-\lambda_q+\hbar]}{[c]\,[\hbar]\,[\lambda_p-\lambda_q]} \quad (p \neq \pm q), \tag{1.3}$$

$$\begin{array}{c} q \\ p\,\boxed{u}\!-\!q \\ -p \end{array} = \frac{[u]\,[\lambda_p+\lambda_q+\hbar+c-u]}{[c]\,[\lambda_p+\lambda_q+\hbar]}\,\frac{g(\lambda+\widehat{p},\lambda)}{g(\lambda,\lambda+\widehat{q})} \quad (p, q \neq 0,\, p \neq q), \tag{1.4}$$

$$\begin{array}{c} p \\ p\,\boxed{u}\!-\!p \\ -p \end{array} = \frac{[c-u]\,[2\lambda_p+\hbar-u]}{[c]\,[2\lambda_p+\hbar]} + \frac{[u]\,[2\lambda_p+\hbar+c-u]}{[c]\,[2\lambda_p+\hbar]} G_{\lambda p} \quad (p \neq 0). \tag{1.5}$$

Only for the B_n type, the next patterns appear:

$$\begin{array}{c} 0 \\ p\,\boxed{u}\,0 \\ -p \end{array} = \frac{[u]\,[\lambda_p+\lambda_0+\hbar+c-u]}{[c]\,[\lambda_p+\lambda_0+\hbar]}\,\frac{g(\lambda+\widehat{p},\lambda)}{g(\lambda,\lambda)} \quad (p \neq 0), \tag{1.6}$$

$$\begin{array}{c} q \\ 0\,\boxed{u}\!-\!q \\ 0 \end{array} = \frac{[u]\,[\lambda_0+\lambda_q+\hbar+c-u]}{[c]\,[\lambda_0+\lambda_q+\hbar]}\,\frac{g(\lambda,\lambda)}{g(\lambda,\lambda+\widehat{q})} \quad (q \neq 0), \tag{1.7}$$

$$\begin{array}{c} 0 \\ 0\,\boxed{u}\,0 \\ 0 \end{array} = \frac{[c+u]\,[2c-u]}{[c]\,[2c]} - \frac{[u]\,[c-u]}{[c]\,[2c]} \left(\sum_{q \neq 0} \frac{[\lambda_q+\hbar/2+2c]}{[\lambda_q+\hbar/2]} G_{\lambda q} \right). \tag{1.8}$$

type	B_n	C_n	D_n
c	$-(2n-1)\hbar/2$	$-(n+1)\hbar$	$-(n-1)\hbar$
$h(\lambda)$	$[\lambda]$	$[2\lambda]$	1

Table 1.1: The crossing parameter c, the function $h(\lambda)$.

In the above formulas, we used the following notations. The crossing parameter c is fixed as in Table 1.1. The function $g(\lambda, \mu)$ is given by

$$g(\lambda, \mu) = a_p h(\mu_p) \prod_{\substack{r \in \mathcal{P} \\ r \neq \pm p, 0}} [\mu_p - \mu_r] \quad (\mu = \lambda + \widehat{p},\ p \in \mathcal{P} - \{0\}), \tag{1.9}$$

$$g(\lambda, \lambda) = (-1)^n \prod_{r \neq 0} [\lambda_0 - \lambda_r]. \tag{1.10}$$

Here $h(\lambda)$ is given in Table 1.1, and $a_p = 1$ $(p = \varepsilon_1, \ldots, \varepsilon_n)$, $= -1$ $(p = -\varepsilon_1, \ldots, -\varepsilon_n)$ for type B_n and $a_p = 1$ in the remaining cases. We denote further

$$G_{\lambda q} = \frac{g(\lambda + \widehat{q}, \lambda)}{g(\lambda, \lambda + \widehat{q})} = \epsilon \frac{h(\lambda_q + \hbar)}{h(\lambda_q)} \prod_{r \neq \pm q, 0} \frac{[\lambda_q + \lambda_r + \hbar]}{[\lambda_q + \lambda_r]}, \tag{1.11}$$

where $\epsilon = -1$ for C_n and $\epsilon = 1$ otherwise.

We adopted a slightly different formulas (1.3), (1.4), (1.6), and (1.7) from the original ones (see (1.42),(1.43) in Appendix A). In Appendix A, we will give a similarity transformation (1.44) which transforms our Boltzmann weights into the original ones.

Theorem 1 *The Boltzmann weights* $W\left(\begin{array}{cc|c} \lambda & \mu & \\ \kappa & \nu & u \end{array}\right)$ (1.1)-(1.8) *solve the face-type Yang-Baxter equation* (0.19, Fig.1.1).

1.2 Proof of the face-type Yang-Baxter equation

This section is devoted to the proof of Theorem 1.

Proposition 1 *The Boltzmann weights* (1.1)-(1.8) *enjoy the following properties.*

- *Initial condition* :

$$W\left(\begin{array}{cc|c} \lambda & \mu & \\ \kappa & \nu & 0 \end{array}\right) = \delta_{\mu\kappa}. \tag{1.12}$$

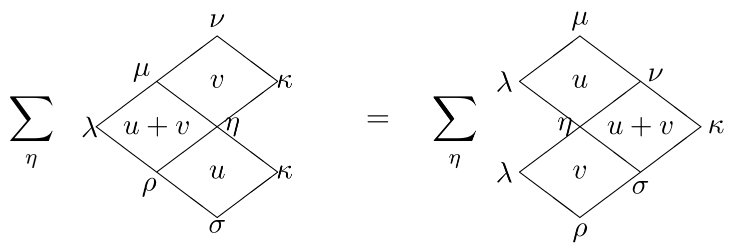


Figure 1.1: The face-type Yang-Baxter equation.

- *Inversion relation* :

$$\sum_\eta W\left(\begin{array}{cc}\lambda & \eta \\ \kappa & \nu\end{array}\Bigg| u\right) W\left(\begin{array}{cc}\lambda & \mu \\ \eta & \nu\end{array}\Bigg| -u\right) = \delta_{\mu\kappa}\, \frac{[c+u]\,[c-u]\,[\hbar+u]\,[\hbar-u]}{[c]^2\,[\hbar]^2}. \tag{1.13}$$

- *Crossing symmetry* :

$$W\left(\begin{array}{cc}\lambda & \mu \\ \kappa & \nu\end{array}\Bigg| u\right) = \frac{g(\lambda,\kappa)}{g(\mu,\nu)}\, W\left(\begin{array}{cc}\kappa & \lambda \\ \nu & \mu\end{array}\Bigg| c-u\right). \tag{1.14}$$

Here $g(\lambda,\mu)$ *is given in* (1.9),(1.10).

- *Reflection symmetry* :

$$W\left(\begin{array}{cc}\lambda & \mu \\ \kappa & \nu\end{array}\Bigg| u\right) = \frac{g(\lambda,\kappa)g(\kappa,\nu)}{g(\lambda,\mu)g(\mu,\nu)}\, W\left(\begin{array}{cc}\lambda & \kappa \\ \mu & \nu\end{array}\Bigg| u\right). \tag{1.15}$$

Before going to the proof we need to prepare three lemmas.

Lemma 1 *If* $f(u)$ *is entire, not identically zero and satisfies*

$$f(u+1) = e^{-2\pi i B} f(u), \quad f(u+\tau) = e^{-2\pi i (A_1 + A_2 u)} f(u),$$

then A_2 *is a non-negative integer,* $f(u)$ *has* A_2 *zeros mod* $\mathbb{Z}+\mathbb{Z}\tau$ *and* $\sum(zeros) = B\tau + A_2/2 - A_1$.

Lemma 2 *For any $\lambda, \mu, \nu, u, v, w, c$ with $u+v+w=c$, we have the identity*

$$
\begin{aligned}
0 = &- [c-u]\,[c-v]\,[c-w]\frac{[\mu+\nu+u]\,[\nu+\lambda+v]\,[\lambda+\mu+w]}{[\mu+\nu]\,[\nu+\lambda]\,[\lambda+\mu]} \\
&+ [c-u]\,[v]\,[w]\frac{[2\lambda-u]\,[\lambda+\mu+c-v]\,[\lambda+\nu+c-w]}{[2\lambda]\,[\lambda+\mu]\,[\lambda+\nu]} \\
&+ [u]\,[c-v]\,[w]\frac{[\mu+\lambda+c-u]\,[2\mu-v]\,[\mu+\nu+c-w]}{[\mu+\lambda]\,[2\mu]\,[\mu+\nu]} \\
&+ [u]\,[v]\,[c-w]\frac{[\nu+\lambda+c-u]\,[\nu+\mu+c-v]\,[2\nu-w]}{[\nu+\lambda]\,[\nu+\mu]\,[2\nu]} \\
&+ [u]\,[v]\,[w]\sum_{\omega}\frac{1}{2}\frac{[\lambda+c-u+\omega]\,[\mu+c-v+\omega]\,[\nu+c-w+\omega]}{[\lambda+\omega]\,[\mu+\omega]\,[\nu+\omega]}e^{2\pi i\varphi(\omega)}
\end{aligned}
\tag{1.16}
$$

Here the summation $\sum_{\omega}$ is over the half periods $\omega = 0, 1/2, \tau/2, (1+\tau)/2$, and

$$
\varphi(\omega) = \begin{cases} 0, & \omega = 0,\ 1/2, \\ c, & \omega = \tau/2,\ (1+\tau)/2. \end{cases} \tag{1.17}
$$

Proof. Let $f(\lambda)$ be the right hand side of (1.16), regarded as a function in λ. This satisfies

$$
f(\lambda+1) = f(\lambda), \quad f(\lambda+\tau) = e^{-2\pi i(v+w)}f(\lambda).
$$

Its apparent poles are located at $\lambda \equiv -\mu, -\nu, 0, -\omega$ modulo 1 and τ. But the residue at $\lambda = -\mu$,

$$
\begin{aligned}
\operatorname*{Res}_{\lambda=-\mu} f(\lambda)d\lambda = &- [c-u]\,[c-v]\,[w]\Bigg(\frac{[c-w]\,[\mu+\nu+u]\,[\nu-\mu+v]}{[\mu+\nu]\,[\nu-\mu]} \\
&+ \frac{[v]\,[-2\mu-u]\,[-\mu+\nu+c-w]}{[-2\mu]\,[-\mu+\nu]} + \frac{[u]\,[2\mu-v]\,[\mu+\nu+c-w]}{[2\mu]\,[\mu+\nu]}\Bigg)
\end{aligned}
$$

is vanishing by the following three-term identity:

$$
\begin{aligned}
&[u+x]\,[u-x]\,[v+y]\,[v-y] - [u+y]\,[u-y]\,[v+x]\,[v-x] \\
&\qquad = [x+y]\,[x-y]\,[u+v]\,[u-v]
\end{aligned}
\tag{1.18}
$$

$(u, v, x, y \in \mathbb{C})$. The identity $\operatorname*{Res}_{\lambda=-\nu} f(\lambda)d\lambda = 0$ holds in the same way. Also we have

$$
\begin{aligned}
\operatorname*{Res}_{\lambda=0} f(\lambda)d\lambda = &[c-u]\,[v]\,[w]\,\frac{[-u]\,[\mu+c-v]\,[\nu+c-v]}{[\mu]\,[\nu]}\lim_{\lambda\to 0}\frac{\lambda}{[2\lambda]} \\
&+ [u]\,[v]\,[w]\,\frac{1}{2}\frac{[c-u]\,[\mu+c-v]\,[\nu+c-w]}{[\mu]\,[\nu]}\lim_{a\to 0}\frac{\lambda}{[\lambda]} = 0,
\end{aligned}
$$

and

$$\operatorname*{Res}_{\lambda=\tau/2} f(\lambda)d\lambda = [c-u]\,[v]\,[w]\,\frac{[\tau-u]\,[\tau/2+\mu+c-v]\,[\tau/2+\nu+c-v]}{2\,[\tau/2+\mu]\,[\tau/2+\nu]}$$
$$+\,[u]\,[v]\,[w]\,\frac{1}{2}\frac{[\tau+c-u]\,[\mu+c-v+\tau/2]\,[\nu+c-w+\tau/2]}{[\tau/2+\mu]\,[\tau/2+\nu]}e^{2\pi ic} = 0.$$

Therefore $f(\lambda)$ does not have any pole. Next we apply Lemma 1, taking $A_1 = v + w, A_2 = B = 0$. $\square$

Lemma 3 *Set*

$$\phi(u) = \frac{d}{du}\log[u],$$

we have then

$$\phi(u+\hbar)+\phi(u-\hbar)-2\phi(u) = \frac{[\hbar]^2[2u][0]'}{[u]^2[u-\hbar][u+\hbar]}. \tag{1.19}$$

Proof of Proposition 1. The initial condition (1.12) is trivial. The crossing symmetry (1.14) and the reflection symmetry (1.15) are easily checked by the explicit form.

Let us prove the inversion relation (1.13). The case of $\lambda = \nu$ is the following identity

$$\sum_{r\in\mathcal{P}}\left(p \overset{r}{\underset{-p}{\boxed{u}}} {-r}\ \ r \overset{q}{\underset{-r}{\boxed{-u}}} {-q} \right) = \delta_{p,q}\,\frac{[c+u]\,[c-u]\,[\hbar+u]\,[\hbar-u]}{[c]^2\,[\hbar]^2}. \tag{1.20}$$

If $p, q \neq 0$, then the equation (1.20) is reduced to

$$\sum_{r\in\mathcal{P}}\frac{[\lambda_p+\lambda_r+\hbar+c-u]\,[\lambda_q+\lambda_r+\hbar+c+u]}{[\lambda_p+\lambda_r+\hbar]\,[\lambda_q+\lambda_r+\hbar]}\,G_{\lambda r}$$
$$+\frac{[c-u]\,[2\lambda_p+\hbar-u]\,[\lambda_p+\lambda_q+\hbar+c+u]}{[u]\,[2\lambda_p+\hbar]\,[\lambda_p+\lambda_q+\hbar]}$$
$$-\frac{[c+u]\,[2\lambda_q+\hbar+u]\,[\lambda_p+\lambda_q+\hbar+c-u]}{[u]\,[2\lambda_q+\hbar]\,[\lambda_p+\lambda_q+\hbar]}$$
$$=\delta_{p,q}\frac{[c-u]\,[c+u]\,[2\lambda_p]\,[2\lambda_q+2\hbar]}{[\hbar]^2\,[2\lambda_p+\hbar]^2}\,G^{-1}_{\lambda p}. \tag{1.21}$$

Set

$$F_{pq}(z) := \frac{[\lambda_p+z+\hbar+c-u]\,[\lambda_q+z+\hbar+c+u]}{[\lambda_p+z+\hbar]\,[\lambda_q+z+\hbar]}\,\frac{\epsilon h(z+\hbar)\,[2z]}{[\hbar]\,h(z)\,[2z+\hbar]}\prod_{r\neq 0}\frac{[z-\lambda_r+\hbar]}{[z-\lambda_r]}$$
$$= F^{(1)}_{pq}(z)F^{(2)}_{pq}(z)F^{(3)}_{pq}(z),$$

where

$$F^{(1)}_{pq}(z) := \frac{[\lambda_p + z + \hbar + c - u]\,[\lambda_q + z + \hbar + c + u]}{[\lambda_p + z + \hbar]\,[\lambda_q + z + \hbar]},$$

$$F^{(2)}_{pq}(z) := \frac{\epsilon h(z+\hbar)\,[2z]}{[\hbar]\,h(z)\,[2z+\hbar]} \quad \text{and} \quad F^{(3)}_{pq}(z) := \prod_{r\neq 0} \frac{[z - \lambda_r + \hbar]}{[z-\lambda_r]}.$$

$F_{pq}(z)$ is a doubly periodic function

$$F_{pq}(z+1) = F_{pq}(z+\tau) = F_{pq}(z),$$

because $F^{(1)}_{pq}(z+\tau) = -e^{-2\pi i(2c)}F_{pq}(z)$, $\quad F^{(3)}_{pq}(z+\tau) = -e^{-2\pi i(2n)}F_{pq}(z)$ and $F^{(2)}_{pq}(z+\tau) = -e^{-2\pi i X}F_{pq}(z)$, where c is given in Table 1.1 and

$$\begin{aligned} X &= -\hbar \quad && \text{for } B_n, \\ &= 2\hbar && \text{for } C_n, \\ &= -2\hbar && \text{for } D_n. \end{aligned}$$

If $p \neq q$, then the poles of $F_{pq}(z)$ are located at

$$z = \lambda_r \ (r \neq 0), \quad -\frac{\hbar}{2} + \omega \ \left(\omega = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\right).$$

If $p = q$, then $F_{pq}(z)$ has an additional pole at $z = -\lambda_p - \hbar$ because of the first factor $F^{(1)}_{pq}(z)$. The residue can be calculated as follows.

$$\begin{aligned} \operatorname*{Res}_{z=-\lambda_r} F_{pq}(z)dz =& \frac{[\lambda_p + \lambda_r + \hbar + c - u]\,[\lambda_q + \lambda_r + \hbar + c + u]}{[\lambda_p + \lambda_r + \hbar]\,[\lambda_q + \lambda_r + \hbar]} \\ & \times \frac{\epsilon h(\lambda_r + \hbar)\,[2\lambda_r]}{[\hbar]\,h(\lambda_r)\,[2\lambda_r + \hbar]} \frac{[\hbar]\,[2\lambda_r + \hbar]}{[0]'\,[2\lambda_r]} \prod_{s \neq 0, r} \frac{[\lambda_r - \lambda_s + \hbar]}{[\lambda_r - \lambda_s]} \\ =& \frac{[\lambda_p + \lambda_r + \hbar + c - u]\,[\lambda_q + \lambda_r + \hbar + c + u]}{[0]'\,[\lambda_p + \lambda_r + \hbar]\,[\lambda_q + \lambda_r + \hbar]} G_{\lambda r}. \end{aligned}$$

We also have

$$\begin{aligned} \operatorname*{Res}_{z=-\hbar/2+\omega} F_{pq}(z)dz =& \frac{[\lambda_p + \omega + \hbar/2 + c - u]\,[\lambda_q + \omega + \hbar/2 + c + u]}{[\lambda_p + \omega + \hbar/2]\,[\lambda_q + \omega + \hbar/2]} \\ & \times \frac{\epsilon h(\omega + \hbar/2)\,[2\omega - \hbar]}{[\hbar]\,h(\omega - \hbar/2)} \prod_{r \neq 0} \frac{[\omega - \lambda_r + \hbar/2]}{[\omega - \lambda_r - \hbar/2]} \lim_{z \to \omega - \hbar/2} \frac{(z - \omega + \hbar/2)}{[2z + \hbar]}, \end{aligned}$$

and the values of the factors herein is listed in Table 1.2. From this table, we conclude

ω	0	$1/2$	$\tau/2$	$(1+\tau)/2$
$\lim\limits_{z\to\omega-\hbar/2}\dfrac{(z-\omega+\hbar/2)}{[2z+\hbar]}$	$\dfrac{1}{2[0]'}$	$-\dfrac{1}{2[0]'}$	$-\dfrac{e^{\pi i\tau}}{2[0]'}$	$\dfrac{e^{\pi i\tau}}{2[0]'}$
$[2\omega-\hbar]$	$[-\hbar]$	$-[-\hbar]$	$-e^{-\pi i\tau+2\pi i\hbar}[-\hbar]$	$e^{-\pi i\tau+2\pi i\hbar}[-\hbar]$
$F^{(3)}_{pq}(\omega-\frac{\hbar}{2})$	1	1	$e^{-2\pi in\hbar}$	$e^{-2\pi in\hbar}$
$\dfrac{h(\omega+\hbar/2)}{h(\omega-\hbar/2)}$ (for B_n)	-1	1	$-e^{-\pi i\hbar}$	$e^{-\pi i\hbar}$
$\dfrac{h(\omega+\hbar/2)}{h(\omega-\hbar/2)}$ (for C_n)	-1	-1	$-e^{-4\pi i\hbar}$	$-e^{-4\pi i\hbar}$

Table 1.2: The values of the factors of $\mathrm{Res}_{z=-\hbar/2+\omega}F_{pq}(z)dz$.

$$\operatorname*{Res}_{z=-\hbar/2+\omega}F_{pq}(z)dz=-\frac{1}{2}\frac{[\lambda_p+\omega+\hbar/2+c-u]\,[\lambda_q+\omega+\hbar/2+c+u]}{[0]'\,[\lambda_p+\omega+\hbar/2]\,[\lambda_q+\omega+\hbar/2]}e^{2\pi i\varphi(\omega)},$$

where $\varphi(\omega)$ is given in (1.17). For $p=q$ case we have the additional residue

$$\begin{aligned}\operatorname*{Res}_{z=-\lambda_p-\hbar}F_{pp}(z)dz=&[c-u]\,[c+u]\,\frac{\epsilon h(-\lambda_p)\,[-2\lambda_p-2\hbar]}{[\hbar]\,h(-\lambda_p-\hbar)\,[-2\lambda_p-\hbar]}\frac{[-2\lambda_p]}{[0]'\,[-2\lambda_p-\hbar]\,[-\hbar]}\\&\times\prod_{r\neq0,\pm p}\frac{[-\lambda_p-\lambda_r]}{[-\lambda_p-\lambda_r-\hbar]}\\&=-\frac{[c-u]\,[c+u]}{[0]'\,[\hbar]^2}\frac{[2\lambda_p]\,[2\lambda_p+2\hbar]}{[2\lambda_p+\hbar]^2}G^{-1}_{\lambda p}.\end{aligned}$$

Then the relation $\sum\mathrm{Res}F_{pq}(z)dz=0$ gives rise to

$$\begin{aligned}&\sum_{r\neq0}\frac{[\lambda_p+\lambda_r+\hbar+c-u]\,[\lambda_q+\lambda_r+\hbar+c+u]}{[0]'\,[\lambda_p+\lambda_r+\hbar]\,[\lambda_q+\lambda_r+\hbar]}G_{\lambda r}\\&\quad-\frac{1}{2}\sum_{\omega}\frac{[\lambda_p+\omega+\hbar/2+c-u]\,[\lambda_q+\omega+\hbar/2+c+u]}{[\lambda_p+\omega+\hbar/2]\,[\lambda_q+\omega+\hbar/2]}e^{2\pi i\varphi(\omega)}\\&\quad-\delta_{p,q}\frac{[c-u]\,[c+u]}{[\hbar]^2}\frac{[2\lambda_p]\,[2\lambda_p+2\hbar]}{[2\lambda_p+\hbar]^2}G^{-1}_{\lambda p}=0\end{aligned}$$

Now we can use Lemma 2 specialized as $\lambda=\lambda_p+\hbar/2$, $\mu=\lambda_q+\hbar/2$, $v=-u$, and $w=c$ to rewrite this formula into (1.21). This completes the case $p,q\neq0$.

The case of $p=q=0$. Put

$$f(u):=\sum_{r\in\mathcal{P}}\left(\begin{array}{ccc}&r&\\0&\boxed{u}&-r\\&0&\end{array}\;\begin{array}{ccc}&0&\\r&\boxed{-u}&0\\&-r&\end{array}\right)-\frac{[c+u]\,[c-u]\,[\hbar+u]\,[\hbar-u]}{[c]^2\,[\hbar]^2},\qquad(1.22)$$

and we can check $f(0) = f(\pm c) = f(\pm\hbar) = 0$. To show $f(\hbar) = 0$ needs lengthy computation, which we will omit here. On the other hand, we can deduce that $f(u)$ should have four zeros because of its transformation property. This is a contradiction, hence $f(u) = 0$.

The case of $p = 0$ and $q \neq 0$ is quite similar. Putting

$$f(u) = \sum_{r\in\mathcal{P}} \left(\begin{array}{ccc} & r & \\ 0 & \boxed{u} & -r \\ & 0 & \end{array} \; \begin{array}{ccc} & q & \\ r & \boxed{-u} & -q \\ & -r & \end{array} \right), \tag{1.23}$$

we have $f(0) = f(\pm c) = 0$. This completes the case $\lambda = \nu$.

The cases $\nu = \lambda + 2\widehat{p}$ $(p \in \mathcal{P})$ are trivial. The remaining cases of the inversion relation (1.13) are easily checked by using the three-term identity (1.18). $\square$

Proof of Theorem 1. Set

$$\begin{aligned} &X(\lambda,\mu,\nu,\kappa,\sigma,\rho\,|\,u,v) \\ &:= \sum_{\eta} W\left(\begin{array}{cc} \rho & \eta \\ \sigma & \kappa \end{array}\middle|\, u\right) W\left(\begin{array}{cc} \lambda & \mu \\ \rho & \eta \end{array}\middle|\, u+v\right) W\left(\begin{array}{cc} \mu & \nu \\ \eta & \kappa \end{array}\middle|\, v\right), \end{aligned} \tag{1.24}$$

$$\begin{aligned} &Y(\lambda,\mu,\nu,\kappa,\sigma,\rho\,|\,u,v) \\ &:= \sum_{\eta} W\left(\begin{array}{cc} \lambda & \eta \\ \rho & \sigma \end{array}\middle|\, v\right) W\left(\begin{array}{cc} \eta & \nu \\ \sigma & \kappa \end{array}\middle|\, u+v\right) W\left(\begin{array}{cc} \lambda & \mu \\ \eta & \nu \end{array}\middle|\, u\right), \end{aligned} \tag{1.25}$$

and

$$Z(\lambda,\mu,\nu,\kappa,\sigma,\rho\,|\,u,v) := X(\lambda,\mu,\nu,\kappa,\sigma,\rho\,|\,u,v) - Y(\lambda,\mu,\nu,\kappa,\sigma,\rho\,|\,u,v). \tag{1.26}$$

Regarding $Z(\lambda,\mu,\nu,\kappa,\sigma,\rho\,|\,u,v)$ as a function of u, we denote it by $Z(u)$. The initial condition (1.12) and the inversion relation (1.13) implies $Z(0) = Z(-v) = 0$. We have

$$Z(\lambda,\mu,\nu,\kappa,\sigma,\rho\,|\,u,v) = -\frac{g(\lambda,\rho)}{g(\nu,\kappa)} Z(\rho,\lambda,\mu,\nu,\kappa,\sigma\,|\,c-u-v,u) \tag{1.27}$$

by the crossing symmetry (1.14). This shows $Z(c-v) = Z(c) = 0$ also. Thus we have found the four zeros at $u = 0, -v, c, c-v$ of $Z(u)$.

We use the quasi-periodicity property:

$$W\left(\begin{array}{cc} \lambda & \mu \\ \kappa & \nu \end{array}\middle|\, u+1\right) = W\left(\begin{array}{cc} \lambda & \mu \\ \kappa & \nu \end{array}\middle|\, u\right),$$

$$W\left(\begin{array}{cc} \lambda & \mu \\ \kappa & \nu \end{array}\middle|\, u+\tau\right) = e^{-2\pi i\tau-2\pi i(2u-c+\xi)} W\left(\begin{array}{cc} \lambda & \mu \\ \kappa & \nu \end{array}\middle|\, u\right),$$

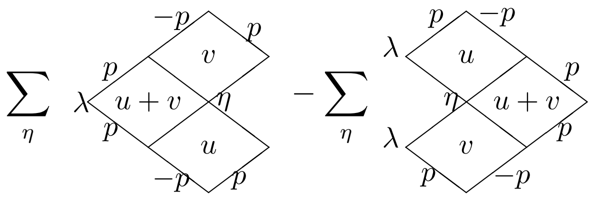


Figure 1.2: The function $Z(\lambda, \lambda + \widehat{p}, \lambda, \lambda + \widehat{p}, \lambda, \lambda + \widehat{p} | u, v)$.

where $\xi = \hbar$ for (1.1), $-\lambda_p + \lambda_q$ for (1.2), 0 for (1.3), $-\lambda_p - \lambda_q - \hbar$ for (1.4, 1.6, 1.7) and $-2\lambda_p - \hbar$ for (1.5, 1.8). From these we have for $Z(u)$:

$$Z(u+1) = Z(u), \quad Z(u+\tau) = e^{-2\pi i\tau - 2\pi i(2\tau + 4u + 2v - 2c + \zeta)} Z(u),$$

where ζ is given by a summation of three ξ's. From Lemma 1, we have $\sum$(zeros) $= 2 - 2\tau - 2v + 2c - \zeta$. On the other hand, we have $\sum$(zeros) $= 2c - 2v$. Therefore, if $\zeta \not\equiv 0 \bmod \mathbb{Z} + \mathbb{Z}\tau$, then $Z(u) = 0$.

Let us verify $\zeta = 0$ cases. Thanks to the symmetry (1.27) and the following symmetry (this follows from the reflection symmetry (1.15))

$$Z(\lambda, \mu, \nu, \kappa, \sigma, \rho \,|\, u, v) = \frac{g(\lambda, \rho) g(\rho, \sigma) g(\sigma, \kappa)}{g(\lambda, \mu) g(\mu, \nu) g(\nu, \kappa)} Z(\lambda, \rho, \sigma, \kappa, \nu, \mu \,|\, v, u),$$

we can reduce the remaining cases ($\zeta \not\equiv 0$) to the following four special cases:

$$Z(\lambda, \lambda + \widehat{p}, \lambda + \widehat{p} + \widehat{q}, \lambda + \widehat{p} + \widehat{q} + \widehat{r}, \lambda + \widehat{q} + \widehat{r}, \lambda + \widehat{r} \,|\, u, v) = 0, \tag{1.28}$$

$(r \neq \pm p, \pm q, p \neq \pm q)$,

$$Z(\lambda, \lambda + \widehat{p}, \lambda, \lambda + \widehat{p}, \lambda, \lambda + \widehat{p} \,|\, u, v) = 0 \quad (p \neq 0), \tag{1.29}$$

$$Z(\lambda, \lambda, \lambda, \lambda, \lambda, \lambda \,|\, u, v) = 0, \tag{1.30}$$

$$Z(\lambda, \lambda, \lambda + \widehat{p}, \lambda + \widehat{p}, \lambda + \widehat{p}, \lambda \,|\, u, v) = 0 \quad (p \neq 0). \tag{1.31}$$

In (1.28), each side of the Yang-Baxter equation contains only one term, and they are manifestly the same. Equation (1.30) follows straightforwardly.

A proof of the case (1.29) can be found in the original literature [JMO2]. However, since the proof is sketchy and seems to contain some typographical errors, we will describe its details here for readers' convenience.

Regarding $Y(\lambda, \lambda+\widehat{p}, \lambda, \lambda+\widehat{p}, \lambda, \lambda+\widehat{p}\,|u,v)$ as a function of λ_p, let us denote it by $f(\lambda_p)$. It reads as

$$\begin{aligned}
f(\lambda_p) =& G_{\lambda p}\frac{[u][v][w]}{[c]^3}\sum_{q\in\mathcal{P}}\frac{[\lambda_q+\lambda_p+\hbar+\widetilde{u}][\lambda_q+\lambda_p+\hbar+\widetilde{v}][\lambda_q+\lambda_p+\hbar+\widetilde{w}]}{[\lambda_q+\lambda_p+\hbar]^3}G_{\lambda q} \\
&+ G_{\lambda p}^{-1}\frac{[\widetilde{u}][\widetilde{v}][\widetilde{w}]}{[c]^3}\frac{[2\lambda_p+\hbar-u][2\lambda_p+\hbar-v][2\lambda_p+\hbar-w]}{[2\lambda_p+\hbar]^3} \\
&+ \sum_{\text{cyclic}}\frac{[u][\widetilde{v}][\widetilde{w}]}{[c]^3}\frac{[2\lambda_p+\hbar+\widetilde{u}][2\lambda_p+\hbar-v][2\lambda_p+\hbar-w]}{[2\lambda_p+\hbar]^3} \\
&+ G_{\lambda p}\sum_{\text{cyclic}}\frac{[\widetilde{u}][v][w]}{[c]^3}\frac{[2\lambda_p+\hbar-u][2\lambda_p+\hbar+\widetilde{v}][2\lambda_p+\hbar+\widetilde{w}]}{[2\lambda_p+\hbar]^3} \\
=:& f_1+f_2+f_3+f_4,
\end{aligned} \tag{1.32}$$

where we put $w=c-u-v, \widetilde{u}=c-u, \widetilde{v}=c-v, \widetilde{w}=c-w$ and the summation $\sum_{\text{cyclic}}$ is over the cyclic permutations of the three variables (u,v,w). From the explicit form, one can see that $X(\lambda, \lambda+\widehat{p}, \lambda, \lambda+\widehat{p}, \lambda, \lambda+\widehat{p}\,|u,v) = f(-\lambda_p-\hbar)$. Therefore we will show $f(\lambda_p)=f(-\lambda_p-\hbar)$ to prove (1.29).

Consider the function

$$\begin{aligned}
\Phi(z) :=& \frac{[z+\lambda_p+\hbar+\widetilde{u}]\,[z+\lambda_p+\hbar+\widetilde{v}]\,[z+\lambda_p+\hbar+\widetilde{w}]}{[z+\lambda_p+\hbar]^3} \\
&\times\frac{[0]'}{[\hbar]}\frac{\varepsilon h(z+\hbar)\,[2z]}{h(z)\,[2z+\hbar]}\prod_{r\neq 0}\frac{[z+\lambda_r+\hbar]}{[z+\lambda_r]}.
\end{aligned}$$

One sees that $\Phi(z)$ is doubly periodic with the periods 1 and τ. Its poles are located at $z\equiv-\lambda_p-\hbar$, $\lambda_q\,(q\in\mathcal{P})$, $-\hbar/2+\omega\,(\omega=0,1/2,\tau/2,(1+\tau)/2)$ modulo 1 and τ. The pole at $z=-\lambda_p-\hbar$ is of the second order, and the others are simple.

Let $f_i(\lambda_p)$ $(i=1,2,3,4)$ be the i-th term of the above function $f(\lambda_p)$. The residue theorem $\sum\mathrm{Res}\Phi(z)dz=0$ together with the formula

$$\operatorname*{Res}_{z=\lambda_q}\Phi(z)dz = -\frac{[\lambda_q+\lambda_p+\hbar+\widetilde{u}][\lambda_q+\lambda_p+\hbar+\widetilde{v}][\lambda_q+\lambda_p+\hbar+\widetilde{w}]}{[\lambda_q+\lambda_p+\hbar]^3}G_{\lambda q}$$

will give the following expression for $f_1(\lambda_p)$:

$$f_1(\lambda_p)=a(\lambda_p)+b(\lambda_p),$$

where

$$a(\lambda_p) := G_{\lambda p}\frac{[u][v][w]}{[c]^3}\sum_{\omega}\operatorname*{Res}_{z=-\hbar/2+\omega}\Phi(z)dz, \tag{1.33}$$

$$b(\lambda_p) := G_{\lambda p}\frac{[u][v][w]}{[c]^3}\operatorname*{Res}_{z=-\lambda_p-\hbar}\Phi(z)dz. \tag{1.34}$$

Here the summation $\sum_\omega$ is over the half periods $\omega = 0, 1/2, \tau/2, (1+\tau)/2$.

From the quasi-periodicity of the theta function (0.24) and the values of the crossing parameters (Table 1.1), we have for $\omega = 0, 1/2, \tau/2, (1+\tau)/2$

$$\operatorname*{Res}_{z=-\hbar/2+\omega} \Phi(z)dz = \frac{1}{2}\frac{[\lambda_p+\hbar/2+\omega+\widetilde{u}][\lambda_p+\hbar/2+\omega+\widetilde{v}][\lambda_p+\hbar/2+\omega+\widetilde{w}]}{[\lambda_p+\hbar/2+\omega]^3}e^{2\pi i\varphi(\omega)}, \tag{1.35}$$

where φ is given in (1.17). Combining (1.33), (1.35) and Lemma 2, we can verify

$$a(\lambda_p) + f_4(\lambda_p) - f_2(-\lambda_p-\hbar) = -a(-\lambda_p-\hbar) + f_2(\lambda_p) - f_4(-\lambda_p-\hbar) = 0. \tag{1.36}$$

Set $\phi(u) = (d/du)\log[u]$, then the residue $\operatorname*{Res}_{z=-\lambda_p-\hbar}\Phi(z)dz$ can be expressed as

$$\begin{aligned}\operatorname*{Res}_{z=-\lambda_p-\hbar} \Phi(z)dz = G_{\lambda p}^{-1}\frac{[\widetilde{u}][\widetilde{v}][\widetilde{w}]}{[0]'[\hbar]^2}\frac{[2\lambda_p+2\hbar][2\lambda_p]}{[2\lambda_p+\hbar]^2}\Bigg(\sum_{\text{cyclic}}\phi(\widetilde{u}) - 3\phi(2\lambda_p) + 3\phi(2\lambda_p+\hbar)\\ + \phi(\hbar) + \sum_{\substack{q\in\mathcal{P}\\ q\neq\pm p}}\{\phi(-\lambda_p+\lambda_q) - \phi(-\lambda_p+\lambda_q-\hbar)\}\Bigg).\end{aligned} \tag{1.37}$$

Since $\phi(u)$ is an odd function, we have from (1.34) and (1.37)

$$\begin{aligned}b(\lambda_p) - b(-\lambda_p-\hbar) = &-3\frac{[u][v][w]}{[c]^3}\frac{[\widetilde{u}][\widetilde{v}][\widetilde{w}]}{[0]'[\hbar]^2}\frac{[2\lambda_p+2\hbar][2\lambda_p]}{[2\lambda_p+\hbar]^2}\\ &\times\{\phi(2\lambda_p)+\phi(2\lambda_p+2\hbar)-2\phi(2\lambda_p+\hbar)\}.\end{aligned} \tag{1.38}$$

On the other hand, using the identity (see (1.18))

$$\begin{aligned}&[2\lambda_p+\hbar+\widetilde{u}][2\lambda_p+\hbar-v][2\lambda_p+\hbar-w] - [2\lambda_p+\hbar-\widetilde{u}][2\lambda_p+\hbar+v][2\lambda_p+\hbar+w]\\ &\quad= [\widetilde{u}][v][w]\frac{[4\lambda_p+2\hbar]}{[2\lambda_p+\hbar]}\end{aligned}$$

and its cyclic permutations of (u, v, w), we have

$$f_3(\lambda_p) - f_3(-\lambda_p-\hbar) = 3\frac{[u][v][w][\widetilde{u}][\widetilde{v}][\widetilde{w}]}{[c]^3}\frac{[4\lambda_p+2\hbar]}{[2\lambda_p+\hbar]^4}. \tag{1.39}$$

Now from (1.38) and (1.39), we have

$$b(\lambda_p) + f_3(\lambda_p) = b(-\lambda_p-\hbar) + f_3(-\lambda_p-\hbar), \tag{1.40}$$

where we used Lemma 3. Combining (1.36) and (1.40), we have obtained $f(\lambda_p) = f(-\lambda_p-\hbar)$. This completes the proof of (1.28).

As for (1.31), the proof reduces to the identity

$$H_{\lambda 0} - H_{\lambda+\hat{p}\,0} = \frac{[\hbar]^2\,[2c]\,[2\lambda_p+\hbar]}{[\lambda_p-\hbar/2]\,[\lambda_p+\hbar/2]^2\,[\lambda_p+3\hbar/2]}, \tag{1.41}$$

where

$$H_{\lambda 0} := \sum_{q\neq 0} \frac{[\lambda_q+\hbar/2+2c]}{[\lambda_q+\hbar/2]} G_{\lambda q}.$$

The function $H_{\lambda 0}$ is doubly periodic in each variable $\lambda_{\varepsilon_1},\ldots,\lambda_{\varepsilon_n}$. Therefore we can evaluate the left hand side of (1.41) at some special point. $\square$

Appendix A: The similarity transformation

The formulas for our Boltzmann weights in Section 1.1 are slightly different from the original formula in Ref.[JMO2]. The original form of type (1.3) and (1.4) are given as follows:

$$\begin{array}{c} q \\ p\,\boxed{u}\,p \\ q \end{array} = \frac{[c-u]\,[u]}{[c]\,[\hbar]} \left(\frac{[\lambda_p-\lambda_q+\hbar]\,[\lambda_p-\lambda_q-\hbar]}{[\lambda_p-\lambda_q]^2} \right)^{1/2} \qquad (p\neq \pm q), \tag{1.42}$$

$$\begin{array}{c} q \\ p\,\boxed{u}\,{-q} \\ -p \end{array} = \frac{[u]\,[\lambda_{p+q}+\hbar+c-u]}{[c]\,[\lambda_{p+q}+\hbar]} \left(G_{\lambda p}G_{\lambda q}\right)^{1/2} \qquad (p\neq q). \tag{1.43}$$

We note that in the B_n case, types (1.6) and (1.7) are included in type (1.43). All the other Boltzmann weights (namely, (1.1),(1.2) and (1.5)) are the same as the ones we adopted in Section 1.1. We denote these weights by $W_{JMO}\left(\begin{array}{cc} \lambda & \mu \\ \kappa & \nu \end{array} \middle| u \right)$. Our Boltzmann weights are obtained from these in the following way. We introduce an ordering on the set $\mathcal{P}$ as

$$\varepsilon_1 \prec \cdots \prec \varepsilon_n \prec 0 \prec -\varepsilon_n \prec \cdots \prec -\varepsilon_1 \quad \text{for } B_n,$$

$$\varepsilon_1 \prec \cdots \prec \varepsilon_n \prec -\varepsilon_n \prec \cdots \prec -\varepsilon_1 \quad \text{for } C_n,\ D_n.$$

For $\lambda,\mu\in\mathfrak{h}^*$, such that $\mu-\lambda=\widehat{p}\in\widehat{\mathcal{P}}$, we define the function $s(\lambda,\mu)$ as follows:
B_n:

$$\begin{aligned} s(\lambda,\mu) &= \prod_{r\prec p} [\lambda_r-\lambda_p]\,[\mu_r-\mu_p] \quad (p\prec 0) \\ &= [-\lambda_p]\,[-\mu_p] \prod_{\substack{r\prec p \\ r\neq 0,-p}} [\lambda_r-\lambda_p]\,[\mu_r-\mu_p] \quad (p\succ 0) \\ &= \prod_{r\prec 0} [\lambda_r-\lambda_0]\,[\lambda_r+\lambda_0] \quad (p=0), \end{aligned}$$

C_n:

$$s(\lambda, \mu) := \prod_{\substack{r \in \mathcal{P} \\ r \prec p}} [\lambda_r - \lambda_p]\,[\mu_r - \mu_p],$$

D_n:

$$s(\lambda, \mu) = \prod_{\substack{p \prec p \\ r \neq -p}} [\lambda_r - \lambda_p]\,[\mu_r - \mu_p].$$

Then the relation between the Boltzmann weights W in Section 1.1 and the ones in [JMO2] is given by

$$W\left(\begin{array}{cc} \lambda & \mu \\ \kappa & \nu \end{array} \middle| u \right) = \left\{ \frac{s(\lambda,\kappa)s(\kappa,\nu)}{s(\lambda,\mu)s(\mu,\nu)} \right\}^{1/2} W_{JMO}\left(\begin{array}{cc} \lambda & \mu \\ \kappa & \nu \end{array} \middle| u \right). \tag{1.44}$$

Chapter 2

Fusion procedure for the face model of type C_2

2.1 Path space and face operator

In the previous chapter, we introduced the Boltzmann weights and proved that they satisfy the YBE. In what follows, we treat only the case of type C_2. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sp}_4(\mathbb{C})$, $\mathfrak{h}$ its Cartan subalgebra and $\mathfrak{h}^*$ the dual space of $\mathfrak{h}$. We recall that the root system R for $(\mathfrak{g}, \mathfrak{h})$ and nondegenerate bilinear form $(\, , \,)$ are given by $R := \{\pm(\varepsilon_1 \pm \varepsilon_2), \pm 2\varepsilon_1, \pm 2\varepsilon_2\} \subset \mathfrak{h}^*$ and

$$(\varepsilon_j, \varepsilon_k) = \frac{1}{2}\delta_{jk}, \tag{2.1}$$

respectively. Note that the square length of the long roots $\pm 2\varepsilon_i$ is two. We will identify the space $\mathfrak{h}$ and its dual $\mathfrak{h}^*$ via the form $(\, , \,)$. The fundamental weights are given by $\varpi_1 = \varepsilon_1, \varpi_2 = \varepsilon_1 + \varepsilon_2$. Let $\mathcal{P}_d$ $(d = 1, 2)$ be the set of weights for the fundamental representation $L(\varpi_d)$. We have

$$\mathcal{P}_1 = \{\pm\varepsilon_1, \pm\varepsilon_2\}, \quad \mathcal{P}_2 = \{\pm(\varepsilon_1 \pm \varepsilon_2), 0\}. \tag{2.2}$$

To construct commuting difference operators, we need the general types of the Boltzmann weights, which we call the fused Boltzmann weights.

First let us introduce the notion of the path space. For any $u \in \mathbb{C}$ and $\lambda, \mu \in \mathfrak{h}^*$ such that $\mu - \lambda \in 2\hbar\mathcal{P}_1$, we introduce a formal symbol $e^{\mu}_{\lambda}(u)$ and the complex vector space

$$\widehat{\mathcal{P}}(\varpi_1^u)^{\mu}_{\lambda} := \begin{cases} \mathbb{C}\, e^{\mu}_{\lambda}(u) & : \mu - \lambda \in 2\hbar\mathcal{P}_1 \\ 0 & : \text{otherwise.} \end{cases}$$

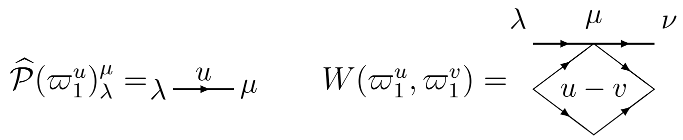


Figure 2.1: Path space and face operator.

We define the path space from λ to μ of type $(u_1, \dots, u_k)$ by

$$\begin{aligned}&\widehat{\mathcal{P}}(\varpi_1^{u_1} \otimes \cdots \otimes \varpi_1^{u_k})_\lambda^\nu \\ &:= \bigoplus_{\mu_1, \cdots, \mu_{k-1} \in \mathfrak{h}^*} \widehat{\mathcal{P}}(\varpi_1^{u_1})_\lambda^{\mu_1} \otimes \widehat{\mathcal{P}}(\varpi_1^{u_2})_{\mu_1}^{\mu_2} \otimes \cdots \otimes \widehat{\mathcal{P}}(\varpi_1^{u_k})_{\mu_{k-1}}^\nu. \end{aligned} \tag{2.3}$$

The following set of paths

$$\{e_\lambda^{\mu_1}(u_1) \otimes e_{\mu_1}^{\mu_2}(u_2) \otimes \cdots \otimes e_{\mu_{k-1}}^\nu(u_k) \mid \mu_i - \mu_{i-1} \in 2\hbar\mathcal{P}_1 (1 \le i \le k), \mu_0 = \lambda, \mu_k = \nu\}$$

forms a basis of the space (2.3). Set also

$$\widehat{\mathcal{P}}(\varpi_1^{u_1} \otimes \cdots \otimes \varpi_1^{u_k})_\lambda := \bigoplus_{\nu \in \mathfrak{h}^*} \widehat{\mathcal{P}}(\varpi_1^{u_1} \otimes \cdots \otimes \varpi_1^{u_k})_\lambda^\nu$$

and

$$\widehat{\mathcal{P}}(\varpi_1^{u_1} \otimes \cdots \otimes \varpi_1^{u_k}) := \bigoplus_{\lambda \in \mathfrak{h}^*} \widehat{\mathcal{P}}(\varpi_1^{u_1} \otimes \cdots \otimes \varpi_1^{u_k})_\lambda.$$

We define the face operator $W(\varpi_1^u, \varpi_1^v) : \widehat{\mathcal{P}}(\varpi_1^u \otimes \varpi_1^v) \to \widehat{\mathcal{P}}(\varpi_1^v \otimes \varpi_1^u)$ by

$$W(\varpi_1^u, \varpi_1^v)\, e_\lambda^\mu(u) \otimes e_\mu^\nu(v) := \sum_{\kappa \in \mathfrak{h}^*} W\left(\begin{array}{cc|c} \lambda & \mu & \\ \kappa & \nu & \end{array} u - v\right) e_\lambda^\kappa(v) \otimes e_\kappa^\nu(u)$$

(Fig.2.1). With these definitions, the Yang-Baxter equation (0.19) reads as follows:

$$\begin{aligned}&(\mathrm{id} \otimes W(\varpi_1^u, \varpi_1^v))\,(W(\varpi_1^u, \varpi_1^w) \otimes \mathrm{id})\,(\mathrm{id} \otimes W(\varpi_1^v, \varpi_1^w)) \\ &= (W(\varpi_1^v, \varpi_1^w) \otimes \mathrm{id})\,(\mathrm{id} \otimes W(\varpi_1^u, \varpi_1^w))\,(W(\varpi_1^u, \varpi_1^v) \otimes \mathrm{id}) \\ &: \widehat{\mathcal{P}}(\varpi_1^u \otimes \varpi_1^v \otimes \varpi_1^w) \to \widehat{\mathcal{P}}(\varpi_1^w \otimes \varpi_1^v \otimes \varpi_1^u). \end{aligned} \tag{2.4}$$

We will recall the fusion procedure. First we consider the composition of the face operators:

$$\begin{aligned}&W\left(\varpi_1^{u_1} \otimes \varpi_1^{u_2} \otimes \cdots \otimes \varpi_1^{u_k}, \varpi_1^v\right) \\ &:= W^{1,2}\left(\varpi_1^{u_1}, \varpi_1^v\right) W^{2,3}\left(\varpi_1^{u_2}, \varpi_1^v\right) \cdots W^{k,k+1}\left(\varpi_1^{u_k}, \varpi_1^v\right) \\ &: \widehat{\mathcal{P}}(\varpi_1^{u_1} \otimes \varpi_1^{u_2} \otimes \cdots \otimes \varpi_1^{u_k} \otimes \varpi_1^v) \to \widehat{\mathcal{P}}(\varpi_1^v \otimes \varpi_1^{u_1} \otimes \varpi_1^{u_2} \cdots \otimes \varpi_1^{u_k}),\end{aligned}$$

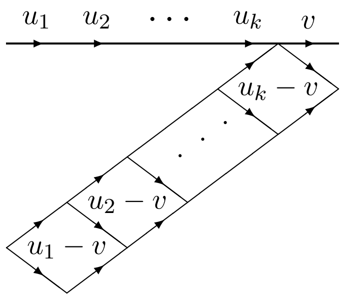


Figure 2.2: The face operator $W\left(\varpi_1^{u_1} \otimes \varpi_1^{u_2} \otimes \cdots \otimes \varpi_1^{u_k}, \varpi_1^{v}\right)$.

where the operators $W\left(\varpi_1^{u_j}, \varpi_1^{v}\right)^{j,j+1}$ are defined by the formula

$$\begin{aligned}
&W\left(\varpi_1^{u_j}, \varpi_1^{v}\right)^{j,j+1} := \mathrm{id}^{\otimes(j-1)} \otimes W\left(\varpi_1^{u_j}, \varpi_1^{v}\right) \otimes \mathrm{id}^{\otimes(k-j)}\\
&: \widehat{\mathcal{P}}(\varpi_1^{u_1} \otimes \cdots \otimes \varpi_1^{u_{j-1}} \otimes \underbrace{\varpi_1^{u_j} \otimes \varpi_1^{v}} \otimes \varpi_1^{u_{j+1}} \otimes \cdots \otimes \varpi_1^{u_k})\\
&\to \widehat{\mathcal{P}}(\varpi_1^{u_1} \otimes \cdots \otimes \varpi_1^{u_{j-1}} \otimes \underbrace{\varpi_1^{v} \otimes \varpi_1^{u_j}} \otimes \varpi_1^{u_{j+1}} \otimes \cdots \otimes \varpi_1^{u_k})
\end{aligned}$$

(Fig. 2.2). Also we set

$$\begin{aligned}
&W\left(\varpi_1^{u_1} \otimes \varpi_1^{u_2} \otimes \cdots \otimes \varpi_1^{u_k}, \varpi_1^{v_1} \otimes \varpi_1^{v_2} \otimes \cdots \otimes \varpi_1^{v_l}\right)\\
&:= \overleftarrow{\prod_{1\le j\le l}} W\left(\varpi_1^{u_1} \otimes \varpi_1^{u_2} \otimes \cdots \otimes \varpi_1^{u_k}, \varpi_1^{v_j}\right)^{[j,k+j]}\\
&: \widehat{\mathcal{P}}(\varpi_1^{u_1} \otimes \cdots \otimes \varpi_1^{u_k} \otimes \varpi_1^{v_1} \otimes \cdots \otimes \varpi_1^{v_l}) \to \widehat{\mathcal{P}}(\varpi_1^{v_1} \otimes \cdots \otimes \varpi_1^{v_l} \otimes \varpi_1^{u_1} \otimes \cdots \otimes \varpi_1^{u_k}),
\end{aligned}$$

where

$$\begin{aligned}
&W(\varpi_1^{u_1} \otimes \cdots \otimes \varpi_1^{u_k}, \varpi_1^{v_j})^{[j,k+j]} := \mathrm{id}^{\otimes(j-1)} \otimes W(\varpi_1^{u_1} \otimes \cdots \otimes \varpi_1^{u_k}, \varpi_1^{v_j}) \otimes \mathrm{id}^{\otimes(l-j)}\\
&: \widehat{\mathcal{P}}(\varpi_1^{v_1} \otimes \cdots \otimes \varpi_1^{v_{j-1}} \otimes \underbrace{\varpi_1^{u_1} \otimes \cdots \otimes \varpi_1^{u_k} \otimes \varpi_1^{v_j}} \otimes \varpi_1^{v_{j+1}} \otimes \cdots \otimes \varpi_1^{v_l})\\
&\to \widehat{\mathcal{P}}(\varpi_1^{v_1} \otimes \cdots \otimes \varpi_1^{v_{j-1}} \otimes \underbrace{\varpi_1^{v_j} \otimes \varpi_1^{u_1} \otimes \cdots \otimes \varpi_1^{u_k}} \otimes \varpi_1^{v_{j+1}} \otimes \cdots \otimes \varpi_1^{v_l})
\end{aligned}$$

(Fig. 2.3).

Let us introduce the fusion projector $\pi_{\varpi_2^u}$ by specializing the parameter in $W\left(\varpi_1^{u}, \varpi_1^{v}\right)$:

$$\pi_{\varpi_2^u} := W(\varpi_1^{u-\hbar}, \varpi_1^{u}) : \widehat{\mathcal{P}}(\varpi_1^{u-\hbar} \otimes \varpi_1^{u}) \to \widehat{\mathcal{P}}(\varpi_1^{u} \otimes \varpi_1^{u-\hbar}). \tag{2.5}$$

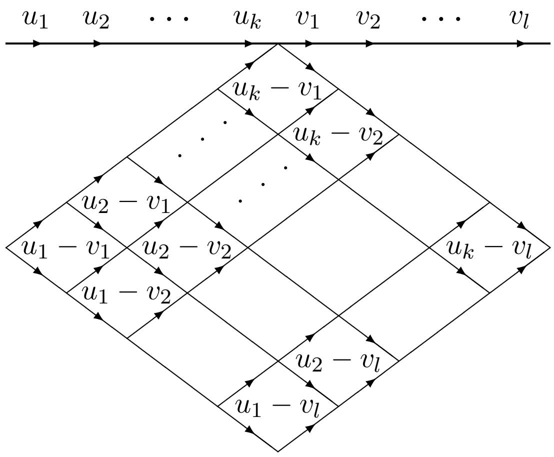


Figure 2.3: The face operator $W\left(\varpi_1^{u_1}\otimes\varpi_1^{u_2}\otimes\cdots\otimes\varpi_1^{u_k},\varpi_1^{v_1}\otimes\varpi_1^{v_2}\otimes\cdots\otimes\varpi_1^{v_l}\right)$.

Lemma 4 *The space $\pi_{\varpi_2^u}(\widehat{\mathcal{P}}(\varpi_1^{u-\hbar}\otimes\varpi_1^u)_\lambda)$ has a basis $\{f_\lambda^{\lambda+\widehat{r}}(u)\}_{r\in\mathcal{P}_2}$ given by*

$$\begin{aligned} f_\lambda^{\lambda+\widehat{p}+\widehat{q}}(u) :=&[\lambda_{p-q}+\hbar]e_\lambda^{\lambda+\widehat{p}}(u)\otimes e_{\lambda+\widehat{p}}^{\lambda+\widehat{p}+\widehat{q}}(u-\hbar)\\ &+[\lambda_{q-p}+\hbar]e_\lambda^{\lambda+\widehat{q}}(u)\otimes e_{\lambda+\widehat{q}}^{\lambda+\widehat{p}+\widehat{q}}(u-\hbar), \end{aligned}\tag{2.6}$$

where $p=\pm\varepsilon_1$, $q=\pm\varepsilon_2$, and

$$f_\lambda^\lambda(u) := \sum_{p\in\mathcal{P}_1}[2\lambda_p+2\hbar]e_\lambda^{\lambda+\widehat{p}}(u)\otimes e_{\lambda+\widehat{p}}^{\lambda}(u-\hbar).\tag{2.7}$$

Proof. For $p,q\in\mathcal{P}_1$, $q\neq\pm p$, we have

$$\pi_{\varpi_2^u}\left(e_\lambda^{\lambda+\widehat{p}}(u-\hbar)\otimes e_{\lambda+\widehat{p}}^{\lambda+2\widehat{p}}(u)\right)=\left(\begin{array}{ccc} & p & \\ p & \boxed{-\hbar} & p\\ & p & \end{array}\right)e_\lambda^{\lambda+\widehat{p}}(u)\otimes e_{\lambda+\widehat{p}}^{\lambda+2\widehat{p}}(u-\hbar)=0,$$

$$\begin{aligned} &\pi_{\varpi_2^u}\left(e_\lambda^{\lambda+\widehat{p}}(u-\hbar)\otimes e_{\lambda+\widehat{p}}^{\lambda+\widehat{p}+\widehat{q}}(u)\right)\\ =&\left(\begin{array}{ccc} & p & \\ p & \boxed{-\hbar} & q\\ & q & \end{array}\right)e_\lambda^{\lambda+\widehat{p}}(u)\otimes e_{\lambda+\widehat{p}}^{\lambda+\widehat{p}+\widehat{q}}(u-\hbar)+\left(\begin{array}{ccc} & p & \\ q & \boxed{-\hbar} & q\\ & p & \end{array}\right)e_\lambda^{\lambda+\widehat{q}}(u)\otimes e_{\lambda+\widehat{q}}^{\lambda+\widehat{p}+\widehat{q}}(u-\hbar)\\ =&\frac{[-2\hbar]}{[-3\hbar]\,[\lambda_{p-q}]}\Big([\lambda_{p-q}+\hbar]e_\lambda^{\lambda+\widehat{p}}(u)\otimes e_{\lambda+\widehat{p}}^{\lambda+\widehat{p}+\widehat{q}}(u-\hbar)\\ &\quad+[\lambda_{q-p}+\hbar]e_\lambda^{\lambda+\widehat{q}}(u)\otimes e_{\lambda+\widehat{q}}^{\lambda+\widehat{p}+\widehat{q}}(u-\hbar)\Big), \end{aligned}$$

and

$$
\begin{aligned}
&\pi_{\varpi_2^u}\left(e_\lambda^{\lambda+\widehat{p}}(u-\hbar)\otimes e_{\lambda+\widehat{p}}^\lambda(u)\right)\\
&=\sum_{r\in\mathcal{P}_1}\left(r\ \overset{p}{\underset{-r}{\boxed{-\hbar}}}\, {-p}\right) e_\lambda^{\lambda+\widehat{r}}(u)\otimes e_{\lambda+\widehat{r}}^\lambda(u-\hbar)\\
&=\frac{[-\hbar]\,[\lambda_{p+q}-\hbar]\,[\lambda_{p-q}-\hbar]}{[-3\hbar]\,[\lambda_{p+q}]\,[\lambda_{p-q}]\,[2\lambda_p]}\left(\sum_{r\in\mathcal{P}_1}[2\lambda_r+2\hbar]e_\lambda^{\lambda+\widehat{r}}(u)\otimes e_{\lambda+\widehat{r}}^\lambda(u-\hbar)\right).
\end{aligned}
$$

Here we have used the identity

$$
\begin{aligned}
&p\ \overset{p}{\underset{-p}{\boxed{-\hbar}}}\, {-p}\\
&=\frac{[-2\hbar]\,[2\lambda_p+2\hbar]}{[-3\hbar]\,[2\lambda_p+\hbar]}-\frac{[-\hbar]\,[2\lambda_p-\hbar]}{[-3\hbar]\,[2\lambda_p+\hbar]}\frac{[2\lambda_p+2\hbar]}{[2\lambda_p]}\frac{[\lambda_{p+q}+\hbar]\,[\lambda_{p-q}+\hbar]}{[\lambda_{p+q}]\,[\lambda_{p-q}]}\\
&=\frac{[2\lambda_p+2\hbar]}{[-3\hbar]\,[2\lambda_p+\hbar]}\frac{[-2\hbar]\,[2\lambda_p]\,[\lambda_{p+q}]\,[\lambda_{p-q}]-[-\hbar]\,[2\lambda_p-\hbar]\,[\lambda_{p+q}+\hbar]\,[\lambda_{p-q}+\hbar]}{[2\lambda_p]\,[\lambda_{p+q}]\,[\lambda_{p-q}]}\\
&=\frac{[-\hbar]}{[-3\hbar]}\frac{[2\lambda_p+2\hbar]}{[2\lambda_p]}\frac{[\lambda_{p+q}-\hbar]\,[\lambda_{p-q}-\hbar]}{[\lambda_{p+q}]\,[\lambda_{p-q}]}
\end{aligned}
$$

which follows from the three-term identity (1.18). □

We define

$$\widehat{\mathcal{P}}(\varpi_2^u)_\lambda := \pi_{\varpi_2^u}(\widehat{\mathcal{P}}(\varpi_1^{u-\hbar}\otimes\varpi_1^u)_\lambda)$$

and $\widehat{\mathcal{P}}(\varpi_2^u)_\lambda^\mu := \widehat{\mathcal{P}}(\varpi_1^{u-\hbar}\otimes\varpi_1^u)_\lambda^\mu \cap \widehat{\mathcal{P}}(\varpi_2^u)_\lambda$. Note that $\dim\widehat{\mathcal{P}}(\varpi_2^u)_\lambda^\mu \le 1$ and the equality holds when $\mu-\lambda\in 2\hbar\mathcal{P}_2$. We can also define $\widehat{\mathcal{P}}(\varpi_{d_1}^{u_1}\otimes\cdots\otimes\varpi_{d_k}^{u_k})_\lambda^\nu$ similarly as in (2.3) ($d_1,\ldots,d_k=1$ or 2).

Proposition 2 *Define the operators* $\widetilde{W}_{dd'}(u-v)$ $(d,d'=1,2)$ *by*

$$\widetilde{W}_{21}(u-v) := W(\varpi_1^u\otimes\varpi_1^{u-\hbar},\varpi_1^v),\ \widetilde{W}_{12}(u-v) := W(\varpi_1^u,\varpi_1^v\otimes\varpi_1^{v-\hbar}) \tag{2.8}$$

and

$$\widetilde{W}_{22}(u-v) := W(\varpi_1^u\otimes\varpi_1^{u-\hbar},\varpi_1^v\otimes\varpi_1^{v-\hbar}).$$

We have

$$\widetilde{W}_{dd'}(u-v)(\widehat{\mathcal{P}}(\varpi_d^u\otimes\varpi_{d'}^v)_\lambda^\mu)\subset\widehat{\mathcal{P}}(\varpi_{d'}^v\otimes\varpi_d^u)_\lambda^\mu,$$

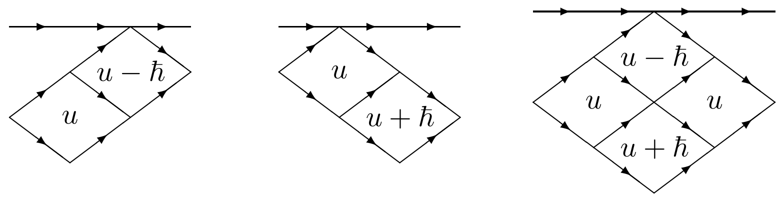


Figure 2.4: $W_{21}(u)$, $W_{12}(u)$ and $W_{22}(u)$.

Proof. From the definition of $\pi_{\varpi_2^u}$ (2.5) and the YBE (0.19),

$$\begin{aligned} &W^{1,2}(u-v)W^{2,3}(u-v-\hbar)(\pi_{\varpi_2^u}\otimes \mathrm{id}) \\ &\quad = (\mathrm{id}\otimes \pi_{\varpi_2^u})W^{1,2}(u-v-\hbar)W^{2,3}(u-v). \end{aligned} \tag{2.9}$$

Applying this to the definition of $\widetilde{W}_{21}(u-v)$, we get

$$\widetilde{W}_{21}(u-v)(\widehat{\mathcal{P}}(\varpi_2^u\otimes\varpi_1^v)_\lambda^\mu)\subset\widehat{\mathcal{P}}(\varpi_1^v\otimes\varpi_2^u)_\lambda^\mu.$$

By the same argument, we have

$$\begin{aligned} &W^{2,3}(u-v+\hbar)W^{1,2}(u-v)(\mathrm{id}\otimes \pi_{\varpi_2^u}) \\ &\quad = (\pi_{\varpi_2^u}\otimes \mathrm{id})W^{2,3}(u-v)W^{1,2}(u-v+\hbar), \end{aligned} \tag{2.10}$$

and

$$\widetilde{W}_{12}(u-v)(\widehat{\mathcal{P}}(\varpi_1^u\otimes\varpi_2^v)_\lambda^\mu)\subset\widehat{\mathcal{P}}(\varpi_2^v\otimes\varpi_1^u)_\lambda^\mu.$$

Together with the equations (2.9),(2.10) and the definition of $\widetilde{W}_{22}(u-v)$, we obtain

$$\widetilde{W}_{22}(u-v)(\widehat{\mathcal{P}}(\varpi_2^u\otimes\varpi_2^v)_\lambda^\mu)\subset\widehat{\mathcal{P}}(\varpi_2^v\otimes\varpi_2^u)_\lambda^\mu.$$

□

We denote by $W_{dd'}(u-v)$ the restricted operators $\widetilde{W}_{dd'}(u-v)|_{\widehat{\mathcal{P}}(\varpi_d^u\otimes\varpi_{d'}^v)}$. By the construction, the operators $W_{dd'}(u-v)$ clearly satisfy the YBE in the operator form (2.4) on $\widehat{\mathcal{P}}(\varpi_d\otimes\varpi_{d'}\otimes\varpi_{d''})$. We introduce their matrix coefficients by the following equation

$$W_{dd'}(u-v)\, g_\lambda^\mu(u)\otimes g_\mu^\nu(v)=\sum_{\kappa\in\mathfrak{h}^*} W_{dd'}\left(\begin{array}{cc|c} \lambda & \mu & \\ \kappa & \nu & \end{array} u-v\right) g_\lambda^\kappa(v)\otimes g_\kappa^\nu(u). \tag{2.11}$$

then, their coefficients $W_{dd'}\left(\begin{array}{cc}\lambda & \mu \\ \kappa & \nu\end{array}\middle| u-v\right)$ satisfies the following type of YBE:

$$\begin{aligned}&\sum_\eta W_{dd'}\left(\begin{array}{cc}\rho & \eta \\ \sigma & \kappa\end{array}\middle| u-v\right) W_{dd''}\left(\begin{array}{cc}\lambda & \mu \\ \rho & \eta\end{array}\middle| u-w\right) W_{d'd''}\left(\begin{array}{cc}\mu & \nu \\ \eta & \kappa\end{array}\middle| v-w\right)\\ &=\sum_\eta W_{d'd''}\left(\begin{array}{cc}\lambda & \eta \\ \rho & \sigma\end{array}\middle| v-w\right) W_{dd''}\left(\begin{array}{cc}\eta & \nu \\ \sigma & \kappa\end{array}\middle| u-w\right) W_{dd'}\left(\begin{array}{cc}\lambda & \mu \\ \eta & \nu\end{array}\middle| u-v\right). \qquad (2.12)\end{aligned}$$

Remark. In general, one can think of more general fused weights $W_{VV'}$, where V, V' stands for some finite dimensional representations of $\mathfrak{g}$. In the above we treated the case $V = L(\varpi_d)$, $V' = L(\varpi_{d'})$ for $\mathfrak{g}$ $(d, d' = 1$ or $2)$. In this setting, for $\lambda, \mu \in \mathfrak{h}^*$, we have

$$\dim \hat{\mathcal{P}}(\varpi_d)^\mu_\lambda \le 1 \quad (d = 1, 2).$$

However, for the general representation of $\mathfrak{g}$, the fused Boltzmann weights depend not only on the difference of elements in $\mathfrak{h}^*$ attached at the both ends of the path, but also on the path which connects the ends. In general, for a representation V of $\mathfrak{g}$, the dimension of the corresponding path space $\widehat{\mathcal{P}}(V)^\mu_\lambda$ is equal to the multiplicity of weight $\mu-\lambda$ in V. If the multiplicity for V or V' is greater than one for some weight(s), then for the fused weights $W_{VV'}$, the "matrix coefficients" $W_{VV'}\left(\begin{array}{cc}\lambda & \mu \\ \eta & \nu\end{array}\middle| u-v\right)$, which can be defined similarly as in (2.11), becomes an matrix rather than a scalar:

$$W_{VV'}\left(\begin{array}{cc}\lambda & \mu \\ \eta & \nu\end{array}\middle| u-v\right) : \widehat{\mathcal{P}}(V)^\mu_\lambda \otimes \widehat{\mathcal{P}}(V')^\nu_\mu \to \widehat{\mathcal{P}}(V')^\kappa_\lambda \otimes \widehat{\mathcal{P}}(V)^\nu_\kappa.$$

In this situation, the YBE written in terms of the "matrix coefficients" also becomes an equation of matrices, whereas the equation (2.12) is an equation of scalar functions.

2.2 Formula for the fused Boltzmann weight

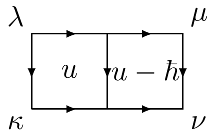
For $p, r \in \mathcal{P}_d$ and $s, q \in \mathcal{P}_{d'}$ $(d, d' = 1, 2)$ such that $p + q = r + s$ we write for brevity (as far as confusion will not arise)

$$s \overset{p}{\underset{r}{\boxed{u}}} q = W_{dd'}\left(\begin{array}{cc}\lambda & \lambda+\widehat{p} \\ \lambda+\widehat{s} & \lambda+\widehat{p}+\widehat{q}\end{array}\middle| u\right). \qquad (2.13)$$

2.2.1 $(2,1)$ case

We calculate the coefficients of the operator $W_{21}(u)$.

$$W_{21}\left(\begin{array}{cc}\lambda & \mu \\ \kappa & \nu\end{array} \middle| u\right) =$$



In what follows, we will often omit the dependence of $g_\lambda^\mu(u) \in \widehat{\mathcal{P}}(\varpi_d^u)$ on u (the spectral parameter) for brevity. Let $p \in \mathcal{P}_1$. From the definitions of f_λ^λ and $\widetilde{W}_{21}$ (2.7,2.8) we have

$$\begin{aligned}
&W_{21}(u)\, f_\lambda^\lambda \otimes e_\lambda^{\lambda+\widehat{p}} = \widetilde{W}_{21}(u)\, f_\lambda^\lambda \otimes e_\lambda^{\lambda+\widehat{p}} \\
=&\widetilde{W}_{21}(u)\left(\sum_{r\in\mathcal{P}_1}[2\lambda_r+2\hbar]\, e_\lambda^{\lambda+\widehat{r}} \otimes e_{\lambda+\widehat{r}}^{\lambda} \otimes e_\lambda^{\lambda+\widehat{p}}\right) \\
=&\sum_{q\in\mathcal{P}_1} e_\lambda^{\lambda+\widehat{q}} \otimes \left(\sum_{\substack{s,t\in\mathcal{P}_1 \\ s+t=p-q}} V_q(\lambda;s,t;u) e_{\lambda+\widehat{q}}^{\lambda+\widehat{q}+\widehat{s}} \otimes e_{\lambda+\widehat{q}+\widehat{s}}^{\lambda+\widehat{p}}\right),
\end{aligned}$$

where we denote by $V_q(\lambda;s,t;u)$ the following function

$$\sum_{r\in\mathcal{P}_1}[2\lambda_r+2\hbar]\, W_{11}\left(\begin{array}{cc}\lambda & \lambda+\widehat{r} \\ \lambda+\widehat{q} & \lambda+\widehat{q}+\widehat{s}\end{array} \middle| u\right) W_{11}\left(\begin{array}{cc}\lambda+\widehat{r} & \lambda \\ \lambda+\widehat{q}+\widehat{s} & \lambda+\widehat{p}\end{array} \middle| u-\hbar\right).$$

If $q \in \mathcal{P}_1$ satisfies $q \neq \pm p$, then the functions $V_q(\lambda;s,t;u)$ vanish except for $(s,t) = (p,-q)$ or $(-q,p)$, and one can easily show that

$$\frac{V_q(\lambda;p,-q;u)}{[(\lambda+\widehat{q})_{p+q}+\hbar]} = \frac{V_q(\lambda;-q,p;u)}{[(\lambda+\widehat{q})_{-q-p}+\hbar]}. \tag{2.14}$$

This equation implies that the vector

$$V_q(\lambda;p,-q;u)\, e_{\lambda+\widehat{q}}^{\lambda+\widehat{q}+\widehat{p}} \otimes e_{\lambda+\widehat{q}+\widehat{p}}^{\lambda+\widehat{p}} + V_q(\lambda;-q,p;u)\, e_{\lambda+\widehat{q}}^{\lambda} \otimes e_\lambda^{\lambda+\widehat{p}}$$

is proportional to $f_{\lambda+\widehat{q}}^{\lambda+\widehat{p}}$ and its coefficient (the both hands sides of (2.14)) is calculated as

$$\frac{[u-\hbar]\,[u+\hbar]\,[u+3\hbar]\,[2\hbar]}{[-3\hbar]^2\,[\hbar]^2} \frac{[\lambda_{q-p}-\hbar-u]\,[2\lambda_q+2\hbar]}{[\lambda_{q-p}-\hbar]\,[\lambda_{q+p}+\hbar]}$$

by using the three term identity (1.18). This function will be denoted as (see (2.13))

$$\begin{array}{ccc} & 0 & \\ q & \boxed{u} & p \\ & p-q & \end{array} \quad (q \neq \pm p).$$

Let us consider the term for $q = p$. For all $s \in \mathcal{P}_1$ we have from the three term identity

$$\frac{V_p(\lambda; s, -s; u)}{[2(\lambda+\widehat{p})_s + 2\hbar]} = \frac{[u-\hbar]\,[u+\hbar]\,[u+3\hbar]}{[-3\hbar]^2\,[\hbar]} \frac{[u+\hbar]}{[\hbar]} \prod_{\substack{r\in\mathcal{P}_1 \\ r\neq \pm p}} \frac{[\lambda_{p+r}+2\hbar]}{[\lambda_{p+r}+\hbar]}. \tag{2.15}$$

The right hand side of this equation is independent of $s \in \mathcal{P}_1$. Thus we see that the vector

$$\sum_{s\in\mathcal{P}_1} V_p(\lambda; s, -s; u) e^{\lambda+\widehat{p}+\widehat{s}}_{\lambda+\widehat{p}} \otimes e^{\lambda+\widehat{p}}_{\lambda+\widehat{p}+\widehat{s}}$$

is proportional to $f^{\lambda+\widehat{p}}_{\lambda+\widehat{p}}$ and its coefficient is equal to the right hand side of (2.15), which is labeled by

$$\begin{array}{c} 0 \\ p\,\boxed{u}\,p\,. \\ 0 \end{array}$$

Here we write all fused Boltzmann weights (the coefficients of the operator $W_{21}(u)$). They are obtained by the three term identity (1.18). We assume $p, q \in \mathcal{P}_1$ satisfy $p \neq \pm q$. The common factor

$$\frac{[u-\hbar]\,[u+\hbar]\,[u+3\hbar]}{[-3\hbar]^2\,[\hbar]}$$

is denoted by $C(u)$.

$$
\begin{array}{ccc} & p+q & \\ q & \boxed{u} & q \\ & p+q & \end{array} = C(u)\frac{[u+2\hbar]}{[\hbar]},
$$

$$
\begin{array}{ccc} & p-q & \\ q & \boxed{u} & q \\ & p-q & \end{array} = C(u)\frac{[u]}{[\hbar]}\frac{[2\lambda_q+2\hbar]}{[2\lambda_q]}\frac{[\lambda_{p-q}-\hbar]}{[\lambda_{p-q}+\hbar]},
$$

$$
\begin{array}{ccc} & 0 & \\ q & \boxed{u} & q \\ & 0 & \end{array} = C(u)\frac{[u+\hbar]}{[\hbar]}\prod_{\substack{r\in\mathcal{P}_1 \\ r\neq\pm q}}\frac{[\lambda_{q+r}+2\hbar]}{[\lambda_{q+r}+\hbar]}, \tag{2.16}
$$

$$
\begin{array}{ccc} & q-p & \\ q & \boxed{u} & p \\ & 0 & \end{array} = C(u)\frac{[\lambda_{q-p}-u]\,[\lambda_{q+p}+2\hbar]}{[2\lambda_p]\,[\lambda_{q-p}+\hbar]}, \tag{2.17}
$$

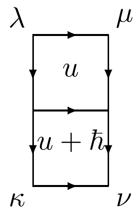
$$
\begin{array}{ccc} & 0 & \\ q & \boxed{u} & p \\ & p-q & \end{array} = C(u)\frac{[2\hbar]}{[\hbar]}\frac{[\lambda_{q-p}-\hbar-u]\,[2\lambda_q+2\hbar]}{[\lambda_{q-p}-\hbar]\,[\lambda_{q+p}+\hbar]}, \tag{2.18}
$$

$$
\begin{array}{ccc} & p+q & \\ q & \boxed{u} & -q \\ & p-q & \end{array} = C(u)\frac{[2\hbar]}{[\hbar]}\frac{[2\lambda_q-u]\,[\lambda_{p-q}-\hbar]}{[2\lambda_q]\,[\lambda_{p+q}+\hbar]}.
$$

2.2.2 $(1,2)$ case

Next we give the example of W_{12}.

$$
W_{12}\left(\begin{array}{cc|c} \lambda & \mu & \\ \kappa & \nu & u \end{array}\right) =
$$



In this case, let us denote the common factor

$$
\frac{[u]\,[u+2\hbar]\,[u+4\hbar]}{[-3\hbar]^2\,[\hbar]}
$$

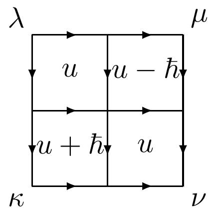
by $D(u)$. To obtain these results, we use only the three-term identity (1.18).

$$
\begin{aligned}
\begin{array}{ccc} & p & \\ p+q & \boxed{u} & p+q \\ & p & \end{array} &= D(u)\frac{[u+3\hbar]}{[\hbar]}, \\
\begin{array}{ccc} & p & \\ q-p & \boxed{u} & q-p \\ & p & \end{array} &= D(u)\frac{[u+\hbar]}{[\hbar]}\frac{[2\lambda_p-2\hbar]}{[2\lambda_p]}\frac{[\lambda_{q-p}+2\hbar]}{[\lambda_{q-p}]}, \\
\begin{array}{ccc} & p & \\ 0 & \boxed{u} & 0 \\ & p & \end{array} &= D(u)\frac{[u+2\hbar]}{[\hbar]}\prod_{\substack{r\in\mathcal{P}_1 \\ r\neq\pm p}}\frac{[\lambda_{p+r}-\hbar]}{[\lambda_{p+r}]}, \\
\begin{array}{ccc} & p & \\ 0 & \boxed{u} & q-p \\ & q & \end{array} &= D(u)\frac{[\lambda_{p-q}-2\hbar-u]\,[\lambda_{p+q}-\hbar]}{[2\lambda_p]\,[\lambda_{q-p}]}, \\
\begin{array}{ccc} & p & \\ p-q & \boxed{u} & 0 \\ & q & \end{array} &= D(u)\frac{[2\hbar]}{[\hbar]}\frac{[\lambda_{p-q}-\hbar-u]\,[2\lambda_q-2\hbar]}{[\lambda_{q-p}]\,[\lambda_{p+q}]}, \\
\begin{array}{ccc} & p & \\ p+q & \boxed{u} & q-p \\ & -p & \end{array} &= D(u)\frac{[2\hbar]}{[\hbar]}\frac{[2\lambda_p-\hbar-u]\,[\lambda_{p+q}+2\hbar]}{[2\lambda_p]\,[\lambda_{q-p}]}.
\end{aligned}
\tag{2.19}
$$

2.2.3 $(2,2)$ case

Finally we give the example of W_{22}.

$$W_{22}\left(\begin{array}{cc|c} \lambda & \mu & \\ \kappa & \nu & u \end{array}\right) =$$



They are equivalent to the Boltzmann weights associated to the vector representation of the type B_2 Lie algebra. We write only two cases as example, which will be used to define the difference operator $M_2(u)$. We set

$$G(u) := \frac{[u-\hbar]\,[u]^2\,[u+\hbar]\,[u+2\hbar]\,[u+3\hbar]^2\,[u+4\hbar]}{[-3\hbar]^4\,[\hbar]^4}. \tag{2.20}$$

$$\begin{array}{ccc} & p+q & \\ 0 & \boxed{u} & 0 \\ & p+q & \end{array} = G(u)\frac{[\lambda_{p+q}-\hbar]}{[\lambda_{p+q}+\hbar]}, \tag{2.21}$$

$$\begin{array}{c} 0 \\ 0\,\boxed{u}\,0 \\ 0 \end{array} = G(u)\frac{[2\hbar]}{[6\hbar]}$$

$$\times \left(\sum_{\substack{r=\pm\varepsilon_1 \\ s=\pm\varepsilon_2}} \frac{[2\lambda_r + 2\hbar]\,[2\lambda_s + 2\hbar]}{[2\lambda_r]\,[2\lambda_s]} \frac{[\lambda_{r+s} - 5\hbar]\,[\lambda_{r+s} + 2\hbar]}{[\lambda_{r+s}]\,[\lambda_{r+s} + \hbar]} - \frac{[u + 6\hbar]\,[u - 3\hbar]}{[u]\,[u + 3\hbar]} \right). \tag{2.22}$$

The formulas (2.19), (2.21) and (2.22) together give the explicit form of $\widetilde{M}_d$ (Theorem 2 (ii)).

We explain how to calculate the fused Boltzmann weight $\begin{array}{c} 0 \\ 0\,\boxed{u}\,0 \\ 0 \end{array}$. According to the definition of the operator $W_{22}(u)$ and the vector f_λ^λ (2.7), the coefficient of $W_{22}(u) f_\lambda^\lambda \otimes f_\lambda^\lambda$ with respect to $f_\lambda^\lambda \otimes f_\lambda^\lambda$ is equal to

$$\frac{1}{[2\lambda_p + 2\hbar]} \sum_{r \in \mathcal{P}_1} [2\lambda_r + 2\hbar]\, W_{21}\left(\begin{array}{cc|c} \lambda & \lambda & \\ \lambda + \widehat{p} & \lambda + \widehat{r} & u \end{array} \right) W_{21}\left(\begin{array}{cc|c} \lambda + \widehat{p} & \lambda + \widehat{r} & \\ \lambda & \lambda & u + \hbar \end{array} \right). \tag{2.23}$$

In this summation, if r is equal to $-p$, then $W_{21}\left(\begin{array}{cc|c} \lambda & \lambda & \\ \lambda + \widehat{p} & \lambda - \widehat{p} & u \end{array} \right) = 0$ and therefore (2.23) can be rewritten as

$$W_{21}\left(\begin{array}{cc|c} \lambda & \lambda & \\ \lambda + \widehat{p} & \lambda + \widehat{p} & u \end{array} \right) W_{21}\left(\begin{array}{cc|c} \lambda + \widehat{p} & \lambda + \widehat{p} & \\ \lambda & \lambda & u + \hbar \end{array} \right)$$
$$+ \sum_{\substack{q \in \mathcal{P}_1 \\ q \neq \pm p}} \frac{[2\lambda_q + 2\hbar]}{[2\lambda_p + 2\hbar]} W_{21}\left(\begin{array}{cc|c} \lambda & \lambda & \\ \lambda + \widehat{p} & \lambda + \widehat{q} & u \end{array} \right) W_{21}\left(\begin{array}{cc|c} \lambda + \widehat{p} & \lambda + \widehat{q} & \\ \lambda & \lambda & u + \hbar \end{array} \right).$$

By means of (2.16), (2.17) and (2.18), this function is equal to

$$\frac{[u - \hbar]\,[u]\,[u + \hbar]\,[u + 2\hbar]\,[u + 3\hbar]\,[u + 4\hbar]\,[2\hbar]}{[-3\hbar]^3\,[\hbar]^4}$$
$$\times \left(\frac{[u + \hbar]\,[u + 2\hbar]}{[2\hbar]\,[-3\hbar]} \prod_{\substack{q \in \mathcal{P}_1 \\ q \neq \pm p}} \frac{[\lambda_{p+q} - \hbar][\lambda_{p+q} + 2\hbar]}{[\lambda_{p+q}][\lambda_{p+q} + \hbar]} \right.$$
$$\left. + \frac{[\hbar]}{[-3\hbar]} \sum_{\substack{q \in \mathcal{P}_1 \\ q \neq \pm p}} \frac{[2\lambda_q - 2\hbar]}{[2\lambda_q]} \frac{[\lambda_{p+q} + 2\hbar + u]\,[\lambda_{p+q} - \hbar - u]\,[\lambda_{p-q} - \hbar]}{[\lambda_{p+q}]\,[\lambda_{p+q} - \hbar]\,[\lambda_{p-q} + \hbar]} \right).$$

To obtain the formula (2.22), we use the following lemma.

Lemma 5 *For any $p \in \mathcal{P}_1$, we have*

$$\begin{aligned}
&\frac{[u+\hbar]\,[u+2\hbar]}{[2\hbar]\,[-3\hbar]} \prod_{\substack{q\in\mathcal{P}_1\\ q\neq\pm p}} \frac{[\lambda_{p+q}-\hbar]}{[\lambda_{p+q}]} \frac{[\lambda_{p+q}+2\hbar]}{[\lambda_{p+q}+\hbar]} \\
&+\frac{[\hbar]}{[-3\hbar]} \sum_{\substack{q\in\mathcal{P}_1\\ q\neq\pm p}} \frac{[2\lambda_q-2\hbar]}{[2\lambda_q]} \frac{[\lambda_{p+q}+2\hbar+u]\,[\lambda_{p+q}-\hbar-u]\,[\lambda_{p-q}-\hbar]}{[\lambda_{p+q}]\,[\lambda_{p+q}-\hbar]\,[\lambda_{p-q}+\hbar]} \\
=&\frac{[u][u+3\hbar]}{[6\hbar][-3\hbar]} \sum_{\substack{r=\pm\varepsilon_1\\ s=\pm\varepsilon_2}} \frac{[2\lambda_r+2\hbar][2\lambda_s+2\hbar]}{[2\lambda_r][2\lambda_s]} \frac{[\lambda_{r+s}-5\hbar][\lambda_{r+s}+2\hbar]}{[\lambda_{r+s}][\lambda_{r+s}+\hbar]} + \frac{[u+6\hbar][u-3\hbar]}{[6\hbar][u+3\hbar]}.
\end{aligned} \tag{2.24}$$

Proof. Let $f(\lambda_p)$ be (the left hand side) $-$ (the right hand side) of (2.24), regarded as a function of λ_p. It is doubly periodic function with the periods $1, \tau$. Let us show that it is entire. The apparent poles of $f(\lambda_p)$ are located at

$$\lambda_p = \lambda_q,\ \lambda_p = \lambda_q \pm \hbar\,(p, q \in \mathcal{P}_1,\ p+q \neq 0),\ \lambda_p = 0\,(p \in \mathcal{P}_1).$$

Note that the left hand side of (2.24) is clearly invariant under $\lambda_q \mapsto -\lambda_q$, and the right-hand side is W-invariant. In view of the symmetry, it suffices to check the regularity at $\lambda_p = \lambda_q,\ \lambda_p = \lambda_q - \hbar$ and $\lambda_p = 0$. By the three-term identity (1.18), it is easy to see that the residue of $f(\lambda_p)$ at $\lambda_p = \lambda_q - \hbar$ vanishes. Manifestly, the point $\lambda_p = \lambda_q$ and $\lambda_p = 0$ is regular.

Now we have proved that $f(\lambda_p)$ is independent of λ_p. We will show $f(-\lambda_q - 2\hbar) = 0$. This can be directly checked by using the identity (1.18) twice, and the proof completes. $\square$

To conclude this section, we will give another expression of the weight $\begin{matrix} & 0 & \\ 0 & \boxed{u} & 0 \\ & 0 & \end{matrix}$, which is found after the publication of [HIK].

Lemma 6 *We have*

$$\sum_{\substack{p=\pm\varepsilon_1\\ q=\pm\varepsilon_2}} U(\lambda_p, \lambda_q) - \sum_{\substack{p=\pm\varepsilon_1\\ q=\pm\varepsilon_2}} \frac{[\lambda_{p+q}-\hbar]\,[\lambda_{p+q}+2\hbar]}{[\lambda_{p+q}]\,[\lambda_{p+q}+\hbar]} = K, \tag{2.25}$$

where K is a constant given by

$$K = \frac{[8\hbar]\,[\hbar]}{[6\hbar]\,[5\hbar]} + \frac{[5\hbar]\,[2\hbar]}{[4\hbar]\,[3\hbar]} + \frac{[6\hbar]\,[3\hbar]}{[5\hbar]\,[4\hbar]} + \frac{[4\hbar]\,[\hbar]}{[3\hbar]\,[2\hbar]}.$$

Proof. Let $f(\lambda_p)$ be the left hand side of (2.25), regarded as a function of λ_p $(p \in I)$. It is doubly periodic function of the periods $1, \tau$. Let us show that it is entire. The apparent poles of $f(\lambda_p)$ are located at

$$\lambda_p = \lambda_q,\ \lambda_p = \lambda_q - \hbar\ (p, q \in I,\ p + q \neq 0),\ \lambda_p = 0\ (p \in I).$$

Note that $f(\lambda_p)$ is W-invariant, then the points $\lambda_p = \lambda_q$ and $\lambda_p = 0$ are regular. Also, the residue of $f(\lambda_p)$ at $\lambda_p = -\lambda_q - \hbar$ is

$$\frac{[2\hbar]}{[6\hbar]} \frac{[-2\lambda_q]\,[2\lambda_q + 2\hbar]}{[-2\lambda_q - 2\hbar]\,[2\lambda_q]} \frac{[-6\hbar]\,[\hbar]}{[-\hbar]} - \frac{[-2h]\,[\hbar]}{[-\hbar]} = 0.$$

Now we have proved that $f(\lambda_p)$ is independent of λ_p, then we consider $g(\lambda_q) = f(-\lambda_q - 2\hbar)$ as a function of λ_q $(q \neq p \in I)$:

$$\begin{aligned} g(\lambda_q) \;=\; & \frac{[2\hbar]}{[6\hbar]} \left(\frac{[2\lambda_q + 2\hbar]\,[2\lambda_q - 2\hbar]\,[2\lambda_q + 7\hbar]}{[2\lambda_q + 4\hbar]\,[2\lambda_q + 2\hbar]\,[2\lambda_q + \hbar]} \right. \\ & + \frac{[2\lambda_q + 6\hbar]\,[2\lambda_q + 2\hbar]\,[2\lambda_q - 3\hbar]}{[2\lambda_q]\,[2\lambda_q + 2\hbar]\,[2\lambda_q + 3\hbar]} \\ & \left. + \frac{[2\lambda_q + 6\hbar]\,[2\lambda_q - 2\hbar]\,[-3\hbar]\,[4\hbar]}{[2\lambda_q + 4\hbar]\,[2\lambda_q]\,[2\hbar]\,[3\hbar]} \right) \\ & - \frac{[-2\lambda_q - 3\hbar]\,[-2\lambda_q]}{[-2\lambda_q - 2\hbar]\,[-2\lambda_q - \hbar]} - \frac{[2\lambda_q + \hbar]\,[2\lambda_q + 4\hbar]}{[2\lambda_q + 2\hbar]\,[2\lambda_q + 3\hbar]} - \frac{[\hbar]\,[4\hbar]}{[2\hbar]\,[3\hbar]}. \end{aligned}$$

By the same argument we can show that $g(\lambda_q)$ is independent of λ_q. Therefore we get K by putting $\lambda_q = \hbar$ in $g(\lambda_q)$ and the proof completes. $\square$

Chapter 3

The difference operators of type C_2

3.1 Construction of the commuting difference operators of type C_2

From now on, the Jacobi's first theta function is denoted by $\theta_1(u)$ instead of $[u]$ (See Appendix C). We define the difference operators $M_d(u)$ $(u \in \mathbb{C}, d = 1, 2)$ acting on the functions on $\mathfrak{h}^*$ by means of the Boltzmann weights of type $(1, 2)$ and $(2, 2)$.

$$(M_d(u)f)(\lambda) := \sum_{p \in \mathcal{P}_d} W_{d2}\left(\begin{array}{cc} \lambda & \lambda + 2\hbar p \\ \lambda & \lambda + 2\hbar p \end{array} \middle| u \right) T^{\hbar}_{2p} f(\lambda). \tag{3.1}$$

Here the shift operator $T^{\hbar}_{2p}$ is defined as

$$T^{\hbar}_{2p} f(\lambda) := f(\lambda + 2\hbar p\,).$$

We recall that for $\lambda \in \mathfrak{h}^*$ and $p \in \mathcal{P}_d\,(d = 1, 2)$, we put

$$\lambda_p := (\lambda, p).$$

Note that if we denote $\lambda_i = (\lambda, \varepsilon_i)$ $(i = 1, 2)$ and $f(\lambda) = f(\lambda_1, \lambda_2)$, then

$$T^{\hbar}_{\pm 2\varepsilon_1} f(\lambda_1, \lambda_2) = f(\lambda_1 \pm \hbar, \lambda_2), \quad T^{\hbar}_{\pm 2\varepsilon_2} f(\lambda_1, \lambda_2) = f(\lambda_1, \lambda_2 \pm \hbar).$$

Theorem 2 (i) *For each* $u, v \in \mathbb{C}$, *we have* $M_d(u)M_{d'}(v) = M_{d'}(v)M_d(u)$ $(d, d' = 1, 2)$.
(ii) *The explicit form of* $M_d(u)$ *are as follows* :

$$M_1(u) = F(u) \sum_{p \in \mathcal{P}_1} \prod_{\substack{q \in \mathcal{P}_1 \\ q \neq \pm p}} \frac{\theta_1(\lambda_{p+q} - \hbar)}{\theta_1(\lambda_{p+q})} T^{\hbar}_{2p}, \tag{3.2}$$

$$M_2(u) = G(u)\left(\sum_{\substack{p=\pm\varepsilon_1\\ q=\pm\varepsilon_2}}\left(\frac{\theta_1(\lambda_{p+q}-\hbar)}{\theta_1(\lambda_{p+q}+\hbar)}T^{\hbar}_{2p}T^{\hbar}_{2q} + U(\lambda_p,\lambda_q)\right) - H(u)\right). \tag{3.3}$$

Here $U(\lambda_p, \lambda_q)$ *is given by* :

$$U(\lambda_p,\lambda_q) = \frac{\theta_1(2\hbar)}{\theta_1(6\hbar)}\frac{\theta_1(2\lambda_p+2\hbar)\,\theta_1(2\lambda_q+2\hbar)}{\theta_1(2\lambda_p)\,\theta_1(2\lambda_q)}\frac{\theta_1(\lambda_{p+q}-5\hbar)\,\theta_1(\lambda_{p+q}+2\hbar)}{\theta_1(\lambda_{p+q})\,\theta_1(\lambda_{p+q}+\hbar)},$$

$G(u)$ *is given in* (2.20) and $F(u), H(u)$ are the following functions depend only on u and $\hbar$:

$$F(u) := \frac{\theta_1(u)\,\theta_1(u+2\hbar)^2\,\theta_1(u+4\hbar)}{\theta_1(-3\hbar)^2\,\theta_1(\hbar)^2},$$

$$\text{and}\qquad H(u) := \frac{\theta_1(u+6\hbar)\,\theta_1(u-3\hbar)\,\theta_1(2\hbar)}{\theta_1(u)\,\theta_1(u+3\hbar)\,\theta_1(6\hbar)}.$$

Proof. The explicit formulas (ii) are given directly by (2.19), (2.21) and (2.22). Now we show the commutativity condition (i). For $t \in \mathcal{P}_d + \mathcal{P}_{d'}$ we will introduce the matrices $A_t(\lambda|u,v)$ and $B_t(\lambda|v,u)$ whose index set is $I_t := \{(p,q) \in \mathcal{P}_d \times \mathcal{P}_{d'} \mid p+q=t\}$:

$$A_t(\lambda|u,v)^{(p,q)}_{(r,s)} := W_{d2}\left(\begin{array}{cc} \lambda & \lambda+\widehat{p} \\ \lambda & \lambda+\widehat{r} \end{array}\middle| u\right) W_{d'2}\left(\begin{array}{cc} \lambda+\widehat{p} & \lambda+\widehat{t} \\ \lambda+\widehat{r} & \lambda+\widehat{t} \end{array}\middle| v\right),$$

$$B_t(\lambda|v,u)^{(p,q)}_{(r,s)} := W_{d'2}\left(\begin{array}{cc} \lambda & \lambda+\widehat{q} \\ \lambda & \lambda+\widehat{s} \end{array}\middle| v\right) W_{d2}\left(\begin{array}{cc} \lambda+\widehat{q} & \lambda+\widehat{t} \\ \lambda+\widehat{s} & \lambda+\widehat{t} \end{array}\middle| u\right).$$

With these matrices, we can write down both the left and right hand sides as

$$M_d(u)M_{d'}(v) = \sum_{t\in\mathcal{P}_d+\mathcal{P}_{d'}} \operatorname{trace} A_t(\lambda|u,v)\; T^{\hbar}_{2t},$$
$$M_{d'}(v)M_d(u) = \sum_{t\in\mathcal{P}_d+\mathcal{P}_{d'}} \operatorname{trace} B_t(\lambda|v,u)\; T^{\hbar}_{2t}.$$

Let us also define the matrix $W_t(\lambda|u-v)$ with the same index set:

$$W_t(\lambda|u-v)^{(p,q)}_{(r,s)} := W_{dd'}\left(\begin{array}{cc} \lambda & \lambda+\widehat{p} \\ \lambda+\widehat{s} & \lambda+\widehat{t} \end{array}\middle| u-v\right).$$

The YBE (2.12) implies

$$W_t(\lambda|u-v)A_t(\lambda|u,v) = B_t(\lambda|v,u)W_t(\lambda|u-v)$$

(Fig. 3.1). By the inversion relation (1.13), it can be seen that $W_t(\lambda|u-v)$ is invertible for generic $u, v \in \mathbb{C}$. It follows that trace $A_t(\lambda|u,v) =$ trace $B_t(\lambda|v,u)$ for all $u, v \in \mathbb{C}$. Hence we have $M_d(u)M_{d'}(v) = M_{d'}(v)M_d(u)$ for all $u, v \in \mathbb{C}$. □

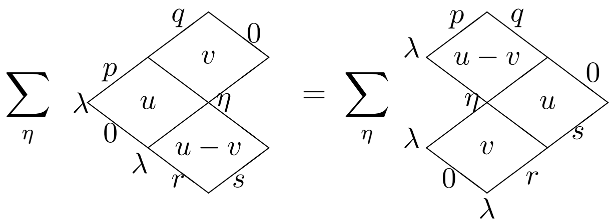


Figure 3.1: The left hand side is the matrix element $W_t(\lambda|u-v)A_t(\lambda|u,v)^{(p,q)}_{(r,s)}$ and the right hand side is $B_t(\lambda|v,u)W_t(\lambda|u-v)^{(p,q)}_{(r,s)}$.

3.2 Identification with van Diejen's system

Thanks to the formula (2.25) for $U(\lambda_p, \lambda_q)$, our operators $M_d(u)$ $(d = 1, 2)$ looks quite similar to the operator obtained in van Diejen's work. Here we will establish the exact formula among these operators.

We define the difference operators $\widetilde{M_d}$ to be the components of $M_d(u)$ independent of u:

$$\widetilde{M_1} = \sum_{p\in\mathcal{P}_1} \prod_{\substack{q\in\mathcal{P}_1\\ q\neq\pm p}} \frac{\theta_1(\lambda_{p+q}-\hbar)}{\theta_1(\lambda_{p+q})} T^{\hbar}_{2p}, \tag{3.4}$$

$$\widetilde{M_2} = \sum_{\substack{p=\pm\varepsilon_1\\ q=\pm\varepsilon_2}} \left(\frac{\theta_1(\lambda_{p+q}-\hbar)}{\theta_1(\lambda_{p+q}+\hbar)} T^{\hbar}_{2p} T^{\hbar}_{2q} + \frac{\theta_1(\lambda_{p+q}-\hbar)\,\theta_1(\lambda_{p+q}+2\hbar)}{\theta_1(\lambda_{p+q})\,\theta_1(\lambda_{p+q}+\hbar)} \right) \tag{3.5}$$

More general commuting difference operators $\mathcal{H}_1, \mathcal{H}_2$ are obtained by van Diejen [vD2] and later by Komori-Hikami [KH2] in a different way. The operators $\mathcal{H}_1, \mathcal{H}_2$ in [vD2] depend on nine complex parameters μ, μ_r, μ'_r $(r = 0, 1, 2, 3)$ satisfying the condition

$$\sum_r (\mu_r + \mu'_r) = 0 \tag{3.6}$$

and are defined by

$$\begin{aligned} \mathcal{H}_1 \;&=\; \sum_{\varepsilon=\pm1} w(\varepsilon x_1) v(\varepsilon x_1 + x_2) v(\varepsilon x_1 - x_2) T^{\gamma}_{\varepsilon 1} \\ &\quad + \sum_{\varepsilon=\pm1} w(\varepsilon x_2) v(\varepsilon x_2 + x_1) v(\varepsilon x_2 - x_1) T^{\gamma}_{\varepsilon 2} \;+\; U_{\{1,2\},1}, \end{aligned}$$

$$
\begin{aligned}
\mathcal{H}_2 \;=\; & \sum_{\varepsilon,\varepsilon'=\pm 1} w(\varepsilon x_1)w(\varepsilon' x_2)v(\varepsilon x_1+\varepsilon' x_2)v(\varepsilon x_1+\varepsilon' x_2+\gamma)T^{\gamma}_{\varepsilon 1}T^{\gamma}_{\varepsilon' 2}\\
& + U_{\{2\},1}\sum_{\varepsilon=\pm 1} w(\varepsilon x_1)v(\varepsilon x_1+x_2)v(\varepsilon x_1-x_2)T^{\gamma}_{\varepsilon 1}\\
& + U_{\{1\},1}\sum_{\varepsilon=\pm 1} w(\varepsilon x_2)v(\varepsilon x_2+x_1)v(\varepsilon x_2-x_1)T^{\gamma}_{\varepsilon 2} \;+\; U_{\{1,2\},2}.
\end{aligned}
$$

Here $T^{\gamma}_{\pm i}$ $(i=1,2)$ stand for the shift operators

$$
T^{\gamma}_{\pm 1}f(x_1,x_2)=f(x_1\pm\gamma,x_2),\quad T^{\gamma}_{\pm 2}f(x_1,x_2)=f(x_1,x_2\pm\gamma)
$$

and

$$
v(z):=\frac{\sigma(z+\mu)}{\sigma(z)},\quad w(z):=\prod_{0\le r\le 3}\frac{\sigma_r(z+\mu_r)\,\sigma_r(z+\mu'_r+\gamma/2)}{\sigma_r(z)\,\sigma_r(z+\gamma/2)}, \tag{3.7}
$$

where $\sigma(z)=\sigma_0(z)$ denotes the Weierstrass sigma function with two quasi periods $\omega_1,\ \omega_2$ and $\sigma_r(z)$ $(r=1,2,3)$ the associated function obtained by the shift of argument by the half periods (See Appendix C for more detail). The functions $U_{\{j\},1}, U_{\{1,2\},j}$ $(j=1,2)$ are defined as follows :

$$
U_{\{j\},1}=-w(x_j)-w(-x_j)\quad (j=1,2),
$$

$$
U_{\{1,2\},1}=\sum_{0\le r\le 3} c_r \prod_{j=1,2}\frac{\sigma_r(\mu-\gamma/2+x_j)\,\sigma_r(\mu-\gamma/2-x_j)}{\sigma_r(-\gamma/2+x_j)\,\sigma_r(-\gamma/2-x_j)},
$$

where

$$
c_r=\frac{2}{\sigma(\mu)\,\sigma(\mu-\gamma)}\prod_{0\le s\le 3}\sigma_s(\mu_{\pi_r(s)}-\gamma/2)\,\sigma_s(\mu'_{\pi_r(s)}),
$$

with π_r denoting the permutation $\pi_0=id$, $\pi_1=(01)(23)$, $\pi_2=(02)(13)$, $\pi_3=(03)(12)$.

$$
U_{\{1,2\},2}=\sum_{\varepsilon,\varepsilon'\in\{1,-1\}} w(\varepsilon x_1)w(\varepsilon' x_2)v(\varepsilon x_1+\varepsilon' x_2)v(-\varepsilon x_1-\varepsilon' x_2-\gamma) \tag{3.8}
$$

We mention that the Komori-Hikami system in [KH2] is of more complicated form and has nine arbitrary parameters, that is, they removed the condition (3.6).

To identify our operators with $\mathcal{H}_1$ and $\mathcal{H}_2$, we specialize parameters μ,μ_r,μ'_r $(r=0,1,2,3)$ in $\mathcal{H}_1,\ \mathcal{H}_2$ as $\mu=-\gamma$, $\mu_r=\mu'_r=0$. Then $w(z)=1$ and $U_{\{1,2\},1}=0$. Let us denote these specialized operators by $\bar{\mathcal{H}}_1,\bar{\mathcal{H}}_2$ respectively. Because of these simplifications, we immediately obtain the following from Lemma 6, giving the identification of our system $\{\widetilde{M}_1,\ \widetilde{M}_2\}$ and van Diejen's $\{\bar{\mathcal{H}}_1,\ \bar{\mathcal{H}}_2\}$.

Theorem 3 *For a function $f(\lambda) = f(\lambda_1, \lambda_2)$ on $\mathfrak{h}$, define $\sigma(f)(x) = \sigma(f)(x_1, x_2)$ by*

$$\sigma(f)(x) := \exp \frac{\eta_1(x_1^2 + x_2^2)}{\omega_1} f(\frac{x_1}{2\omega_1}, \frac{x_2}{2\omega_1}),$$

and set $\gamma = 2\omega_1\hbar$. Then we have

$$\begin{aligned}\sigma(\widetilde{M}_1 f)(x) &= e^{2\eta_1\gamma^2/\omega_1}\bar{\mathcal{H}}_1\sigma(f)(x),\\ \sigma(\widetilde{M}_2 f)(x) &= e^{2\eta_1\gamma^2/\omega_1}\left(\bar{\mathcal{H}}_2 + 2\bar{\mathcal{H}}_1\right)\sigma(f)(x).\end{aligned}$$

Proof. Use the connection between the theta function and sigma function in Appendix C (4.8) and (2.25) to compare (3.3) and (3.8). □

3.3 Differential limit

In this section, let us clarify the connection between our system of difference operators and a quantization of the Inozemtsev Hamiltonian [IM], [I]. By expanding in $\hbar$ one infers that

$$\begin{aligned}\widetilde{M}_1 &= 4 + M_{1,2}\hbar^2 + M_{1,4}\hbar^4 + O(\hbar^5),\\ \widetilde{M}_2 &= 8 + M_{2,2}\hbar^2 + M_{2,4}\hbar^4 + O(\hbar^5).\end{aligned}$$

We will abbreviate a function $f(\lambda_{\varepsilon_1\pm\varepsilon_2})$ as $f(\pm)$, $\partial_i = \partial/\partial\lambda_i$ $(i = 1, 2)$, and $\theta_1'(z) = d\theta_1/dz(z)$ etc. We have

$$\begin{aligned}M_{1,2} =&\partial_1^2 + \partial_2^2\\ &- 2\left(\frac{\theta_1'}{\theta_1}(+) + \frac{\theta_1'}{\theta_1}(-)\right)\partial_1 - 2\left(\frac{\theta_1'}{\theta_1}(+) - \frac{\theta_1'}{\theta_1}(-)\right)\partial_2\\ &+ 2\left(\frac{\theta_1''}{\theta_1}(+) + \frac{\theta_1''}{\theta_1}(-)\right),\\ M_{2,2} =&2M_{1,2},\end{aligned}$$

and

$$
\begin{aligned}
&M_{2,4} - 2M_{1,4} \\
&\quad = \partial_1^2\partial_2^2 \\
&\quad - 2\left(\frac{\theta_1'}{\theta_1}(+) - \frac{\theta_1'}{\theta_1}(-)\right)\partial_1^2\partial_2 - 2\left(\frac{\theta_1'}{\theta_1}(+) + \frac{\theta_1'}{\theta_1}(-)\right)\partial_1\partial_2^2 \\
&\quad + \left\{2\left(\left(\frac{\theta_1'}{\theta_1}\right)^2(+) + \left(\frac{\theta_1'}{\theta_1}\right)^2(-)\right) - \left(\frac{\theta_1''}{\theta_1}(+) + 2\frac{\theta_1'}{\theta_1}(+)\frac{\theta_1'}{\theta_1}(-) + \frac{\theta_1''}{\theta_1}(-)\right)\right\}\partial_1^2 \\
&\quad + \left\{2\left(\left(\frac{\theta_1'}{\theta_1}\right)^2(+) + \left(\frac{\theta_1'}{\theta_1}\right)^2(-)\right) - \left(\frac{\theta_1''}{\theta_1}(+) - 2\frac{\theta_1'}{\theta_1}(+)\frac{\theta_1'}{\theta_1}(-) + \frac{\theta_1''}{\theta_1}(-)\right)\right\}\partial_2^2 \\
&\quad + 4\left(\left(\frac{\theta_1'}{\theta_1}\right)^2(+) - \left(\frac{\theta_1'}{\theta_1}\right)^2(-)\right)\partial_1\partial_2 \\
&\quad + \left\{2\left(\frac{\theta_1'\theta_1''}{\theta_1^2}(+) + \frac{\theta_1'\theta_1''}{\theta_1^2}(-)\right) + 2\left(\frac{\theta_1''}{\theta_1}(+)\frac{\theta_1'}{\theta_1}(-) + \frac{\theta_1'}{\theta_1}(+)\frac{\theta_1''}{\theta_1}(-)\right)\right. \\
&\qquad \left. - 4\left(\left(\frac{\theta_1'}{\theta_1}\right)^3(+) + \left(\frac{\theta_1'}{\theta_1}\right)^3(-)\right)\right\}\partial_1 \\
&\quad + \left\{2\left(\frac{\theta_1'\theta_1''}{\theta_1^2}(+) - \frac{\theta_1'\theta_1''}{\theta_1^2}(-)\right) - 2\left(\frac{\theta_1''}{\theta_1}(+)\frac{\theta_1'}{\theta_1}(-) - \frac{\theta_1'}{\theta_1}(+)\frac{\theta_1''}{\theta_1}(-)\right)\right. \\
&\qquad \left. - 4\left(\left(\frac{\theta_1'}{\theta_1}\right)^3(+) - \left(\frac{\theta_1'}{\theta_1}\right)^3(-)\right)\right\}\partial_2 \\
&\quad + \frac{1}{2}\left(\frac{\theta_1^{(4)}}{\theta_1}(+) + \frac{\theta_1^{(4)}}{\theta_1}(-)\right) - 4\left(\frac{\theta_1'''\theta_1'}{\theta_1^2}(+) + \frac{\theta_1'''\theta_1'}{\theta_1^2}(-)\right) \\
&\quad + 2\left(\frac{\theta_1''\theta_1'^2}{\theta_1^3}(+) + \frac{\theta_1''\theta_1'^2}{\theta_1^3}(-)\right) - 2\frac{\theta_1''}{\theta_1}(+)\frac{\theta_1''}{\theta_1}(-),
\end{aligned}
$$

We set $\Delta = \theta_1(+)\theta_1(-)$, then

$$
\begin{aligned}
\Delta^{-1} \cdot M_{2,2} \cdot \Delta &= \partial_1^2 + \partial_2^2 + 4\left(\left(\frac{\theta_1''}{\theta_1} - \frac{\theta_1'^2}{\theta_1^2}\right)(+) + \left(\frac{\theta_1''}{\theta_1} - \frac{\theta_1'^2}{\theta_1^2}\right)(-)\right) \\
&= \partial_1^2 + \partial_2^2 + 4\left((\log\theta_1)''(+) + (\log\theta_1)''(-)\right),
\end{aligned} \tag{3.9}
$$

$$
\begin{aligned}
& \Delta^{-1}\cdot(M_{2,4}-2M_{1,4})\cdot\Delta \\
= \ & \partial_1^2\partial_2^2 \\
& +4\left(\left(\frac{\theta_1''}{\theta_1}+\frac{\theta_1'^{\,2}}{\theta_1^2}\right)(+)-\left(\frac{\theta_1''}{\theta_1}+\frac{\theta_1'^{\,2}}{\theta_1^2}\right)(-)\right)\partial_1\partial_2 \\
& +2\left(\left(\frac{\theta_1'''}{\theta_1}-3\frac{\theta_1''\theta_1'}{\theta_1^2}+2\frac{\theta_1'^{\,3}}{\theta_1^3}\right)(+)+\left(\frac{\theta_1'''}{\theta_1}-3\frac{\theta_1''\theta_1'}{\theta_1^2}+2\frac{\theta_1'^{\,3}}{\theta_1^3}\right)(-)\right)\partial_1 \\
& +2\left(\left(\frac{\theta_1'''}{\theta_1}-3\frac{\theta_1''\theta_1'}{\theta_1^2}+2\frac{\theta_1'^{\,3}}{\theta_1^3}\right)(+)-\left(\frac{\theta_1'''}{\theta_1}-3\frac{\theta_1''\theta_1'}{\theta_1^2}+2\frac{\theta_1'^{\,3}}{\theta_1^3}\right)(-)\right)\partial_2 \\
& +2\left(\frac{\theta_1^{(4)}}{\theta_1}(+)+\frac{\theta_1^{(4)}}{\theta_1}(-)\right)-8\left(\frac{\theta_1'''\theta_1'}{\theta_1^2}(+)+\frac{\theta_1'''\theta_1'}{\theta_1^2}(-)\right) \\
& -2\left(\left(\frac{\theta 1_1''}{\theta 1_1}\right)^2(+)+\left(\frac{\theta_1''}{\theta_1}\right)^2(-)\right)-8\frac{\theta_1''}{\theta_1}(+)\frac{\theta_1''}{\theta_1}(-) \\
& +16\left(\frac{\theta_1''\theta_1'^{\,2}}{\theta_1^3}(+)+\frac{\theta_1''\theta_1'^{\,2}}{\theta_1^3}(-)\right)+8\left(\frac{\theta_1'^{\,2}}{\theta_1^2}(+)\frac{\theta_1''}{\theta_1}(-)+\frac{\theta_1''}{\theta_1}(+)\frac{\theta_1'^{\,2}}{\theta_1^2}(-)\right) \\
& -8\left(\left(\frac{\theta_1'}{\theta_1}\right)^4(+)+\left(\frac{\theta_1'}{\theta_1}\right)^4(-)\right)+8\left(\frac{\theta_1'}{\theta_1}\right)^2(+)\left(\frac{\theta_1'}{\theta_1}\right)^2(-) \\
= \ & \partial_1^2\partial_2^2 \\
& +4\{(\log\theta_1)''(+)-(\log\theta_1)''(-)\}\partial_1\partial_2 \\
& +2\{(\log\theta_1)'''(+)+(\log\theta_1)'''(-)\}\partial_1+2\{(\log\theta_1)'''(+)-(\log\theta_1)'''(-)\}\partial_2 \\
& +2\{(\log\theta_1)^{(4)}(+)+(\log\theta_1)^{(4)}(-)\} \\
& +4\{(\log\theta_1)''(+)-(\log\theta_1)''(-)\}^2 \\
= \ & \{\partial_1\partial_2+2\left((\log\theta_1)''(+)-(\log\theta_1)''(-)\right)\}^2 .
\end{aligned}
$$

The completely integrable Hamiltonian of type BC_n is introduced by Olshanetsky-Perelomov [OP1], and later generalized Inozemtsev-Meshcheryakov [IM] [I]. In the rank two case, the Hamiltonian is given by

$$
\begin{aligned}
H = & -\frac{1}{2}(\partial_1^2+\partial_2^2)+g(g-1)\left(\wp(x_1+x_2)+\wp(x_1-x_2)\right) \\
& +\sum_{0\le r\le 3} g_r(g_r-1)(\wp(\omega_r+x_1)+\wp(\omega_r+x_2)),
\end{aligned}
$$

where $\wp(x)$ denotes the Weierstrass $\wp$-function with two periods $2\omega_1$ and $2\omega_2$, and $\omega_0 = 0$, $\omega_3=-\omega_1-\omega_2$. By the connection between the theta function and the $\wp$ function (4.9) in Appendix C, our differential limit (3.9) is identified with this Hamiltonian for the special coupling constants $g(g-1)=2$, and $g_r(g_r-1)=0$ $(0\le r\le 3)$.

Appendix B: Commuting difference systems of type A_{n-1}, B_n

As we have seen in Section 3.1, if we construct the system of difference operators by using the fused Boltzmann weights (3.1), the commutativity of these operators relies on the Yang-Baxter equation (2.12). One more key is the fact that the multiplicity of the weight zero subspace in the representation $L(\varpi_2)$ is one. If we take general representation V instead of $L(\varpi_2)$, the resulting commuting operators will become operators for V_0-valued functions on $\mathfrak{h}^*$, where V_0 denotes the weight 0 subspace of V. Thus, if we choose the representation satisfying $\dim V_0 = 1$, the commuting system for scalar functions can be constructed by the same manner. There are two other well-known cases satisfying this property. One is the symmetric tensor representation for type A_{n-1} and the other is the vector representation for type B_n. For type A_{n-1} case, Felder-Varchenko constructed the Ruijsenaars operators (difference operators (0.12) in Introduction) by the same method [FV2]. Actually it turns out that Hasegawa's operator $\widetilde{L}$ (formula (38) in [H3]) coincides with the fused Boltzmann weight in Felder-Varchenko method when the parameters involved are set appropriately. For reader's convenience, the explicit formula for $\widetilde{L}$ is as follows:

$$\begin{aligned}\widetilde{L}(c|u)^i_j &= \sum_{k=1}^{n} \bar{\phi}(u)_\lambda^{\lambda+\hbar\bar{\varepsilon}_j,k} \phi(u+c\hbar)_{\lambda,k}^{\lambda+\hbar\bar{\varepsilon}_i} T_i^\hbar \\ &= \left(\frac{\theta(c\hbar/n + u + \lambda_{ji})}{\theta(u)} \prod_{k\neq j} \frac{\theta(c\hbar/n + \lambda_{ki})}{\theta(\lambda_{kj})} \right) T_i^\hbar. \end{aligned} \tag{3.10}$$

Here $\phi(u)_{\mu,j}^{\mu+\hbar\bar{\varepsilon}_k}$ is the intertwining vector (0.26) and $\bar{\phi}(u)_\mu^{\mu+\hbar\bar{\varepsilon}_k,j}$ is the elements of the inverse matrix to $[\phi(u)_{\mu,j}^{\mu+\hbar\bar{\varepsilon}_k}]_{j,k=1,\ldots,n}$:

$$\sum_{j=1}^{n} \bar{\phi}(u)_\mu^{\mu+\hbar\bar{\varepsilon}_k,j} \phi(u)_{\mu,j}^{\mu+\hbar\bar{\varepsilon}'_k} = \delta_{k,k'}, \quad \sum_{j=1}^{n} \phi(u)_{\mu,j}^{\mu+\hbar\bar{\varepsilon}_k} \bar{\phi}(u)_\mu^{\mu+\hbar\bar{\varepsilon}_k,j'} = \delta_{j,j'},$$

Let us consider the B_n case. We use the Boltzmann weights of type (1.3) and (1.8):

$$0\,\underset{p}{\overset{p}{\boxed{u}}}\,0 = \frac{[c-u]\,[u]}{[c]\,[\hbar]} \frac{[\lambda_p - \hbar/2]}{[\lambda_p + \hbar/2]}, \quad (p \neq 0)$$

$$0\,\underset{0}{\overset{0}{\boxed{u}}}\,0 = \frac{[c+u]\,[2c-u]}{[c]\,[2c]} - \frac{[u]\,[c-u]}{[c]\,[2c]} \left(\sum_{q\neq 0} \frac{[\lambda_q + \hbar/2 + 2c]}{[\lambda_q + \hbar/2]} G_{\lambda_q} \right).$$

With these functions, we set

$$M_1(u) = \sum_{p \in \mathcal{P}} \begin{array}{c} \lambda\, p \\ 0\boxed{u}0 \\ p \end{array} T_p^{\hbar}$$

$$= \frac{[c-u]\,[u]}{[c]\,[\hbar]} \sum_{p \in \mathcal{P}-\{0\}} \left(\frac{[\lambda_p - \hbar/2]}{[\lambda_p + \hbar/2]} T_p - \frac{[\hbar]\,[\lambda_p + \hbar/2 + 2c]}{[2c]\,[\lambda_p + \hbar/2]} G_{\lambda_p} \right) + F(u).$$

$$F(u) = \frac{[c+u]\,[2c-u]\,[\hbar]}{[c-u]\,[u]\,[2c]}$$

$$\mathcal{P} := \{\pm\varepsilon_1, \cdots, \pm\varepsilon_n, 0\}, \quad T_p f(\lambda) := f(\lambda + \hbar p).$$

This operator also corresponds to van Diejen's operator with special coupling constants. By the following lemma, we see the operator $M_1(u)$ separate to the difference operators which depend on one variable.

Lemma 7

$$\sum_{p \neq 0} \frac{[\hbar]\,[\lambda_p + \hbar/2 + 2c]}{[2c]\,[\lambda_p + \hbar/2]} G_{\lambda_p} + \sum_{p \neq 0} \frac{[\lambda_p - \hbar/2]\,[\lambda_p + \hbar]}{[\lambda_p + \hbar/2]\,[\lambda_p]} = K. \tag{3.11}$$

Here K is a constant independent of $\lambda_p\,(p \in \mathcal{P})$:

Proof. The left-hand side, regarded as a function of λ_p $(p \in \mathcal{P})$, is doubly periodic function of the periods $1, \tau$. It is W-invariant, then the point $\lambda_p = \lambda_q$ and $\lambda_p = 0$ is regular. The residue at $\lambda_p = -\hbar/2$ is 0. $\square$

We define

$$\widetilde{M_1} := \sum_{p \in \mathcal{P}-\{0\}} \left(\frac{[\lambda_p - \hbar/2]}{[\lambda_p + \hbar/2]} T_p - \sum_{p \neq 0} \frac{[\lambda_p - \hbar/2]\,[\lambda_p + \hbar]}{[\lambda_p + \hbar/2]\,[\lambda_p]} \right).$$

By sending the step size $\hbar$ to zero, we obtain the differential operator in the same way as in C_2 case. First we have

$$\frac{[\lambda_p - \hbar/2]}{[\lambda_p + \hbar/2]} f(\ldots, \lambda_p + \hbar, \ldots)$$
$$= f + \left(\partial_p - \frac{\theta'}{\theta}(\lambda_p) \right) f \cdot \hbar + \left(\frac{1}{2}\partial_p^2 - \frac{\theta'}{\theta}(\lambda_p)\partial_p + \frac{1}{2}\frac{\theta'^2}{\theta^2}(\lambda_p) \right) f \cdot \hbar^2 + O(\hbar^3).$$

Then,

$$\begin{aligned}
&\frac{[\lambda_p - \hbar/2]}{[\lambda_p + \hbar/2]} f(\ldots, \lambda_p + \hbar, \ldots) + \frac{[-\lambda_p - \hbar/2]}{[-\lambda_p + \hbar/2]} f(\ldots, \lambda_p - \hbar, \ldots) \\
&= 2f + \left(\partial_p^2 - 2\frac{\theta'}{\theta}(\lambda_p)\partial_p + \frac{\theta'^2}{\theta^2}(\lambda_p) \right) f \cdot \hbar^2 + O(\hbar^3).
\end{aligned}$$

We put

$$\Delta := \prod_p \theta(\lambda_p).$$

Then we have

$$\begin{aligned}
\Delta^{-1} \cdot \left(\partial_p^2 - 2\frac{\theta'}{\theta}(\lambda_p)\partial_p + \frac{\theta'^2}{\theta^2}(\lambda_p) \right) \cdot \Delta &= \partial_p^2 + \left(\frac{\theta''}{\theta} - \frac{\theta'^2}{\theta^2} \right)(\lambda_p) \\
&= \partial_p^2 + (\log\theta)''(\lambda_p).
\end{aligned}$$

Moreover, the function multiplication term of $\widetilde{M}_1$ goes over to

$$\frac{[\lambda_p - \hbar/2]\,[\lambda_p + \hbar]}{[\lambda_p + \hbar/2]\,[\lambda_p]} = 1 + \frac{1}{2}\left(\frac{\theta''}{\theta} - \frac{\theta'^2}{\theta^2} \right)(\lambda_p) \cdot \hbar^2 + O(\hbar^3).$$

Then we have

$$\begin{aligned}
\widetilde{M}_1 + K &= \sum_{p \neq 0} \left(\frac{[\lambda_p - \hbar/2]}{[\lambda_p + \hbar/2]} T_p^{\hbar} + \frac{[\lambda_p - \hbar/2]\,[\lambda_p + \hbar]}{[\lambda_p + \hbar/2]\,[\lambda_p]} \right) \\
&= 4n + M_1^{(2)}\hbar^2 + O(\hbar^3),
\end{aligned}$$

where

$$\Delta^{-1} \cdot M_1^{(2)} \cdot \Delta = \sum_p \left(\partial_p^2 + 2(\log\theta)''(\lambda_p) \right).$$

Chapter 4

Diagonalization of the system

4.1 The space of theta functions

Let Q and $Q^\vee$ be the root and coroot lattice, P and $P^\vee$ the weight and coweight lattice respectively. Under the identification $\mathfrak{h} = \mathfrak{h}^*$ via the form $(\ ,\ )$, they are given by

$$P = \sum_{j=1,2} \mathbb{Z}\varepsilon_j, \quad Q^\vee = \sum_{j=1,2} \mathbb{Z}2\varepsilon_j, \tag{4.1}$$

and

$$P^\vee = \ Q = \mathbb{Z}2\varepsilon_1 + \mathbb{Z}2\varepsilon_2 + \mathbb{Z}(\varepsilon_1 + \varepsilon_2).$$

For $\beta \in \mathfrak{h}^*$, we introduce the following operators $S_{\tau\beta}, S_\beta$ acting on the functions on $\mathfrak{h}^*$:

$$\begin{aligned}(S_\beta f)(\lambda) &:= f(\lambda + \beta),\\ (S_{\tau\beta} f)(\lambda) &:= \exp\left[2\pi i\left((\lambda, \beta) + \frac{(\beta,\beta)}{2}\tau\right)\right] f(\lambda + \tau\beta)\end{aligned}$$

They satisfy Heisenberg's relations

$$S_\beta S_\gamma = S_\gamma S_\beta, \quad S_{\tau\beta} S_{\tau\gamma} = S_{\tau\gamma} S_{\tau\beta}, \quad S_\gamma S_{\tau\beta} = e^{2\pi i(\gamma,\beta)} S_{\tau\beta} S_\gamma \tag{4.2}$$

$(\gamma, \beta, \in \mathfrak{h}^*)$. We define the space of theta functions (of level 1) by

$$Th_1 := \left\{f \text{ is holomorphic on } \mathfrak{h}^* \mid S_{\tau\alpha} f = S_\alpha f = f \quad (\forall \alpha \in Q^\vee)\right\}.$$

For each $\mu \in P$ and fixed $\tau \in \mathfrak{H}_+$, we define the classical theta function $\Theta_\mu(\lambda)$ of $\lambda \in \mathfrak{h}^*$ by

$$\Theta_\mu(\lambda) := \sum_{\gamma \in \mu + Q^\vee} \exp\left[2\pi i\left((\gamma, \lambda) + \frac{(\gamma,\gamma)}{2}\tau\right)\right].$$

It is easy to see that the set

$$\{\Theta_\mu(\lambda) \mid \mu \equiv 0,\, \varepsilon_1,\, \varepsilon_2,\, \varepsilon_1+\varepsilon_2 \bmod Q^\vee\}$$

gives a basis for Th_1 over $\mathbb{C}$ [KP].

Let $W \subset GL(\mathfrak{h}^*)$ denote the Weyl group for $(\mathfrak{g},\mathfrak{h})$, and consider the W-invariants in Th_1:

$$Th_1^W := \{f \in Th_1 \mid f(w\lambda) = f(\lambda)\ (\forall w \in W)\}\,.$$

Theorem 4 *The operators $\widetilde{M_1}$, $\widetilde{M_2}$ preserves Th_1^W.*

Lemma 8 *For all $\beta \in P^\vee$ and $d = 1, 2$, we have*

$$[S_{\tau\beta}, M_d(u)] = [S_\beta, M_d(u)] = 0. \tag{4.3}$$

Proof. Note that if $p, q \in \mathcal{P}_1$ ($q \neq \pm p$) and $\beta \in P^\vee$ then $\beta_{p+q} \in \mathbb{Z}$. By the quasi-periodicity (4.1),(4.2), we have

$$\frac{\theta_1((\lambda+\beta)_{p+q}-\hbar)}{\theta_1((\lambda+\beta)_{p+q})} = \frac{\theta_1(\lambda_{p+q}-\hbar)}{\theta_1(\lambda_{p+q})}, \quad \frac{\theta_1((\lambda+\tau\beta)_{p+q}-\hbar)}{\theta_1((\lambda+\tau\beta)_{p+q})} = e^{2\pi i\beta_{p+q}\hbar}\frac{\theta_1(\lambda_{p+q}-\hbar)}{\theta_1(\lambda_{p+q})}.$$

Using these equations, we have for all $p \in \mathcal{P}_1$

$$\begin{aligned}
&S_{\tau\beta}\prod_{q\neq\pm p}\frac{\theta_1(\lambda_{p+q}-\hbar)}{\theta_1(\lambda_{p+q})}T_{2p}^{\hbar}f(\lambda)\\
&\quad= e^{2\pi i((\lambda,\beta)+\tau(\beta,\beta)/2)}\prod_{q\neq\pm p}\frac{\theta_1((\lambda+\tau\beta)_{p+q}-\hbar)}{\theta_1((\lambda+\tau\beta)_{p+q})}f(\lambda+\tau\beta+\widehat{p})\\
&\quad= e^{2\pi i((\lambda,\beta)+\tau(\beta,\beta)/2+2\beta_p\hbar)}\prod_{q\neq\pm p}\frac{\theta_1(\lambda_{p+q}-\hbar)}{\theta_1(\lambda_{p+q})}f(\lambda+\tau\beta+\widehat{p})\\
&\quad= \prod_{q\neq\pm p}\frac{\theta_1(\lambda_{p+q}-\hbar)}{\theta_1(\lambda_{p+q})}T_{2p}^{\hbar}S_{\tau\beta}f(\lambda),
\end{aligned}$$

and

$$\begin{aligned}
&S_\beta\prod_{q\neq\pm p}\frac{\theta_1(\lambda_{p+q}-\hbar)}{\theta_1(\lambda_{p+q})}T_{2p}^{\hbar}f(\lambda) = \prod_{q\neq\pm p}\frac{\theta_1((\lambda+\beta)_{p+q}-\hbar)}{\theta_1((\lambda+\beta)_{p+q})}f(\lambda+\beta+\widehat{p})\\
&\quad= \prod_{q\neq\pm p}\frac{\theta_1(\lambda_{p+q}-\hbar)}{\theta_1(\lambda_{p+q})}f(\lambda+\beta+\widehat{p}) = \prod_{q\neq\pm p}\frac{\theta_1(\lambda_{p+q}-\hbar)}{\theta_1(\lambda_{p+q})}T_{2p}^{\hbar}S_\beta f(\lambda).
\end{aligned}$$

Note that $2\beta_p\hbar = (\widehat{p}, \beta)$ etc. Hence we have $[S_{\tau\beta}, M_1(u)] = [S_\beta, M_1(u)] = 0$. In the same way, we can see that the principal part of $\widetilde{M_2}$ commutes with $S_{\tau\beta}$ and S_β, using the equations

$$\frac{\theta_1((\lambda+\beta)_{p+q}-\hbar)}{\theta_1((\lambda+\beta)_{p+q}+\hbar)} = \frac{\theta_1(\lambda_{p+q}-\hbar)}{\theta_1(\lambda_{p+q}+\hbar)},$$
$$\frac{\theta_1((\lambda+\tau\beta)_{p+q}-\hbar)}{\theta_1((\lambda+\tau\beta)_{p+q}+\hbar)} = e^{2\pi i(2\beta_{p+q}\hbar)}\frac{\theta_1(\lambda_{p+q}-\hbar)}{\theta_1(\lambda_{p+q}+\hbar)}.$$

Using (4.1) and (4.2), it is easy to see that the function

$$C_{p,q}(\lambda) := \frac{\theta_1(2\hbar)}{\theta_1(6\hbar)}\frac{\theta_1(2\lambda_p+2\hbar)}{\theta_1(2\lambda_p)}\frac{\theta_1(2\lambda_q+2\hbar)}{\theta_1(2\lambda_q)}\frac{\theta_1(\lambda_{p+q}-5\hbar)}{\theta_1(\lambda_{p+q}+\hbar)}\frac{\theta_1(\lambda_{p+q}+2\hbar)}{\theta_1(\lambda_{p+q})}$$

$(p, q \in \mathcal{P}_1, p+q \neq 0)$ satisfies $C_{p,q}(\lambda+\beta) = C_{p,q}(\lambda+\tau\beta) = C_{p,q}(\lambda)$ $(\forall\beta \in P^\vee)$. This means that $S_{\tau\beta},\ S_\beta(\beta \in P^\vee)$ commute with the multiplication by $C_{p,q}(\lambda)$. $\square$

Lemma 9 *For all $\gamma \in P^\vee$, we have*

$$S_{\tau\gamma}Th^W \subset Th^W, S_\gamma Th^W \subset Th^W. \tag{4.4}$$

Proof. Let $f \in Th^W$ and $\gamma \in P^\vee$. Since the bilinear form (,) is W-invariant, we have $(S_{\tau\gamma}f)(w\lambda) = (S_{\tau w^{-1}(\gamma)}f)(\lambda)$. Using (4.2), we can write this as $(S_{\tau\gamma}S_{\tau(w^{-1}(\gamma)-\gamma)}f)(\lambda)$, which is equal to $S_{\tau\gamma}f(\lambda)$ because $w^{-1}(\gamma)-\gamma \in Q^\vee$. In the same way, we can show that $(S_\gamma f)(w\lambda) = (S_\gamma f)(\lambda)$.

Evidently $S_{\tau\gamma}f$ and $S_\gamma f$ are holomorphic. For all $\alpha \in Q^\vee$, using (4.2) and $(\gamma, \alpha) \in \mathbb{Z}$, it can be seen that the operators $S_\alpha, S_{\tau\alpha}$ commute with $S_\gamma, S_{\tau\gamma}$. Hence $S_{\tau\gamma}f$ or $S_\gamma f$ are fixed by $S_{\tau\alpha}$ and S_α. $\square$

Now we prove Theorem 4.

Let f be any function in Th^W. In view of (4.3), we have $S_\alpha\widetilde{M}_d f = S_{\tau\alpha}\widetilde{M}_d f = \widetilde{M}_d f$ for all $\alpha \in Q^\vee \subset P^\vee$. It is clear from the explicit form of $\widetilde{M}_d$ that $(\widetilde{M}_d f)(w\lambda) = (\widetilde{M}_d)f(\lambda)$ for all $w \in W$.

Let us show that the function $\widetilde{M}_d f$ is holomorphic on $\mathfrak{h}^*$. For $\mu \in \mathfrak{h}^*$ and $z \in \mathbb{C}$, we denote by D_μ^z the line in $\mathfrak{h}^*$ defined by

$$D_\mu^z := \{\lambda \in \mathfrak{h}^* \,|\, (\lambda, \mu) + z = 0\}.$$

The coefficients of the difference operators $\widetilde{M}_d$ have their possible simple poles along $D + P^\vee + \tau P^\vee$, where we put

$$D := \bigcup_{p\in R_+} D_p^0 \cup \bigcup_{q\in\mathcal{P}_2-\{0\}} D_q^\hbar$$

and R_+ is a fixed set of positive roots.

Next we will show that for any function f in Th^W, $\widetilde{M}_d f$ is regular along D. Let us consider the meromorphic function $g := \left(\prod_{p\in R_+} \theta_1(\lambda_p)\right) \widetilde{M}_d f$, which is regular along $D^0 := \bigcup_{p\in R_+} D_p^0$. Since $\widetilde{M}_d f$ is W-invariant, it is clear that g is W-anti-invariant. This implies that g has zero along D^0 and hence $\widetilde{M}_d f$ is regular along D^0.

The holomorphy along $\bigcup_{q\in\mathcal{P}_2-\{0\}} D_q^\hbar$ is somewhat nontrivial. Let $p = \pm\varepsilon_1$, $q = \pm\varepsilon_2$. Clearly, $\widetilde{M}_1 f$ is regular along $D_{p+q}^\hbar$. Let us consider the function $\widetilde{M}_2 f$. It suffices to show that the following function is regular along $D_{p+q}^\hbar$:

$$\begin{aligned}&\frac{\theta_1(\lambda_{p+q}-\hbar)}{\theta_1(\lambda_{p+q}+\hbar)} T_{2p}^\hbar T_{2q}^\hbar f(\lambda)\\ &\quad+\frac{\theta_1(2\hbar)}{\theta_1(6\hbar)}\frac{\theta_1(2\lambda_p+2\hbar)}{\theta_1(2\lambda_p)}\frac{\theta_1(2\lambda_q+2\hbar)}{\theta_1(2\lambda_q)}\frac{\theta_1(\lambda_{p+q}-5\hbar)}{\theta_1(\lambda_{p+q}+\hbar)}\frac{\theta_1(\lambda_{p+q}+2\hbar)}{\theta_1(\lambda_{p+q})}f(\lambda).\end{aligned}$$

We note that, for any W-invariant function f, we have $\left(T_{2p}^\hbar T_{2q}^\hbar f - f\right)|_{D_{p+q}^\hbar} = 0$. In view of this fact, the residue of the above function along $D_{p+q}^\hbar$ is easily seen to vanish. Thus we have proved that for any function f in Th^W, the functions $\widetilde{M}_d f (d = 1, 2)$ are regular along D.

For $\beta, \gamma \in P^\vee$, we have, by the definitions of $S_{\tau\beta}, S_\gamma$ and (4.3),

$$\begin{aligned}\widetilde{M}_d f(\lambda+\beta\tau+\gamma) &= e^{-2\pi i((\lambda,\beta)+\tau(\beta,\beta)/2)} S_{\tau\beta}S_\gamma \widetilde{M}_d f(\lambda)\\ &= e^{-2\pi i((\lambda,\beta)+\tau(\beta,\beta)/2)} \widetilde{M}_d S_{\tau\beta}S_\gamma f(\lambda). \qquad (4.5)\end{aligned}$$

Since $S_{\tau\beta}S_\gamma f$ belongs to Th^W by (4.4), $\widetilde{M}_d S_{\tau\beta}S_\gamma f$ is regular along D. Then (4.5) implies that $\widetilde{M}_d f$ is regular along $D + \beta\tau + \gamma$. The proof is completed. $\square$

For $\mu \in P$, we define $W_\mu := \{w \in W \mid w\mu = \mu\}$ and introduce the following symmetric sum of theta functions [KP],

$$\mathcal{S}_\mu(\lambda) := \frac{1}{|W_\mu|}\sum_{w\in W} \Theta_{w(\mu)}(\lambda).$$

Then

$$\{\mathcal{S}_\mu(\lambda) \mid \mu \equiv 0,\ \varpi_1\,(=\varepsilon_1),\ \varpi_2\,(=\varepsilon_1+\varepsilon_2) \bmod Q^\vee\}$$

forms a basis for Th_1^W over $\mathbb{C}$.

It is known that Th_1^W is also spanned by the level 1 characters of the affine Lie algebra $\widehat{\mathfrak{sp}}_4(\mathbb{C})$. Note that $\Theta_{-\mu} = \Theta_\mu$ and $\Theta_{\varepsilon_1+\varepsilon_2} = \Theta_{\varepsilon_1-\varepsilon_2}$. We have

$$\mathcal{S}_0 = \Theta_0,\ \mathcal{S}_{\varpi_1} = 2(\Theta_{\varepsilon_1}+\Theta_{\varepsilon_2}),\ \mathcal{S}_{\varpi_2} = 4\Theta_{\varepsilon_1+\varepsilon_2}.$$

4.2 Diagonalization of $\widetilde{M}_d$

Now we are in the position to diagonalize the operators $\widetilde{M}_d$ (3.4, 3.5) on the space Th_1^W. We set

$$f_1 := \Theta_0 + \Theta_{\varepsilon_1+\varepsilon_2}, \quad f_2 := \Theta_{\varepsilon_1} + \Theta_{\varepsilon_2}, \text{ and } \quad f_3 := \Theta_0 - \Theta_{\varepsilon_1+\varepsilon_2}.$$

They are linearly independent in the space Th_1^W.

Theorem 5 *The functions $f_i(\lambda)$ $(i = 1, 2, 3)$ are common eigenfunctions of $\widetilde{M}_d$:*

$$\widetilde{M}_d f_i(\lambda) = E_{d,i} f_i(\lambda) \quad (d = 1, 2, \ i = 1, 2, 3).$$

The eigenvalues are given by

$$E_{1,i} = \frac{\theta_1(2\hbar)\theta_{i+1}(0)}{\theta_1(\hbar)\theta_{i+1}(\hbar)}$$

and $E_{2,i} = 2E_{1,i}$, where the Jacobi theta functions $\theta(z) = \theta_i(z|\tau)$ $(i = 2, 3, 4)$ are defined as in Appendix C.

We will prove this theorem by using the following three lemmas. First, we show that the operators $\widetilde{M}_d$ (surprisingly) split into two A_1-type components.

Lemma 10 *Let us denote $\lambda_\pm := (\lambda, \varepsilon_1 \pm \varepsilon_2)$ and define*

$$H_\pm := \frac{\theta_1(\lambda_\pm - \hbar)}{\theta_1(\lambda_\pm)} T^\hbar_{\varepsilon_1 \pm \varepsilon_2} + \frac{\theta_1(-\lambda_\pm - \hbar)}{\theta_1(-\lambda_\pm)} T^\hbar_{-(\varepsilon_1 \pm \varepsilon_2)}.$$

Then we have

$$\widetilde{M}_1 = H_+ H_-, \quad \widetilde{M}_2 = H_+^2 + H_-^2. \tag{4.6}$$

Proof. To prove the first identity, we note that

$$\begin{aligned}
&\frac{\theta_1(\lambda_+ - \hbar)}{\theta_1(\lambda_+)} T^\hbar_{\varepsilon_1+\varepsilon_2} \frac{\theta_1(\lambda_- - \hbar)}{\theta_1(\lambda_-)} T^\hbar_{\varepsilon_1-\varepsilon_2} \\
&= \frac{\theta_1(\lambda_+ - \hbar)}{\theta_1(\lambda_+)} \frac{\theta_1((\lambda + \hbar(\varepsilon_1+\varepsilon_2))_- - \hbar)}{\theta_1((\lambda + \hbar(\varepsilon_1+\varepsilon_2))_-)} T^\hbar_{\varepsilon_1+\varepsilon_2} T^\hbar_{\varepsilon_1-\varepsilon_2} \\
&= \frac{\theta_1(\lambda_+ - \hbar)}{\theta_1(\lambda_+)} \frac{\theta_1(\lambda_- - \hbar)}{\theta_1(\lambda_-)} T^\hbar_{2\varepsilon_1}.
\end{aligned}$$

Here we used the identity $(\varepsilon_1+\varepsilon_2, \varepsilon_1-\varepsilon_2)=0$. The second identity follows from, for instance,

$$
\begin{aligned}
&\frac{\theta_1(\lambda_+-\hbar)}{\theta_1(\lambda_+)}T^{\hbar}_{\varepsilon_1+\varepsilon_2}\frac{\theta_1(\lambda_+-\hbar)}{\theta_1(\lambda_+)}T^{\hbar}_{\varepsilon_1+\varepsilon_2} \\
&=\frac{\theta_1(\lambda_+-\hbar)}{\theta_1(\lambda_+)}\frac{\theta_1((\lambda+\hbar(\varepsilon_1+\varepsilon_2))_+-\hbar)}{\theta_1((\lambda+\hbar(\varepsilon_1+\varepsilon_2))_+)}T^{\hbar}_{\varepsilon_1+\varepsilon_2}T^{\hbar}_{\varepsilon_1+\varepsilon_2} \\
&=\frac{\theta_1(\lambda_+-\hbar)}{\theta_1(\lambda_+)}\frac{\theta_1(\lambda_++\hbar-\hbar)}{\theta_1(\lambda_++\hbar)}T^{\hbar}_{2(\varepsilon_1+\varepsilon_2)} \\
&=\frac{\theta_1(\lambda_+-\hbar)}{\theta_1(\lambda_++\hbar)}T^{\hbar}_{2\varepsilon_1}T^{\hbar}_{2\varepsilon_2}.
\end{aligned}
$$

Here we used the identity $(\varepsilon_1+\varepsilon_2, \varepsilon_1+\varepsilon_2)=1$. □

Second, we consider the eigenvalue problem for the A_1-type difference operator (difference Lamé or two-body Ruijsenaars operator)

$$\frac{\theta_1(z-\ell\hbar)}{\theta_1(z)}f(z+\hbar)+\frac{\theta_1(z+\ell\hbar)}{\theta_1(z)}f(z-\hbar)=Ef(z). \tag{4.7}$$

Lemma 11 *For the special coupling constant* $\ell=1$,

$$\theta_i(z) \quad (i=2,3,4)$$

are solutions of the equation (4.7) with eigenvalues

$$E=E_i=\frac{\theta_1(2\hbar)\,\theta_i(0)}{\theta_1(\hbar)\,\theta_i(\hbar)} \quad (i=2,3,4).$$

Proof. This is the special case of Felder-Varchenko's study [FV1]. They expressed the solutions of (4.7) in terms of the algebraic Bethe Ansatz method, which is originally developed and applied to the spin chain model. In fact, the operator in the left hand side of (4.7) can be regarded as the transfer matrix of the simplest spin chain, that is, it consists of only one site of freedom with spin $\ell=1$. In this case, the Bethe Ansatz equation is

$$\frac{\theta_1(t-3\hbar/2)}{\theta_1(t+\hbar/2)}=e^{2\hbar c}. \tag{4.8}$$

The solution (t,c) of this equation gives an eigenfunction

$$f(z)=e^{cz}\theta_1(z+t-\hbar/2)$$

with eigenvalue

$$\epsilon=e^{-\hbar c}\frac{\theta_1(2\hbar)}{\theta_1(\hbar)}\frac{\theta_1(t-\hbar/2)}{\theta_1(t+\hbar/2)}.$$

Now one checks directly that the equation (4.8) has solutions

$$(t,c) = \left(\frac{\hbar}{2}+\frac{1}{2},0\right),\ \left(\frac{\hbar}{2}+\frac{1+\tau}{2},\pi i\right),\ \left(\frac{\hbar}{2}+\frac{\tau}{2},\pi i\right). \tag{4.9}$$

In fact, for example, if $t=\hbar/2+\tau/2$ we have

$$\frac{\theta_1(t-3\hbar/2)}{\theta_1(t+\hbar/2)} = \frac{\theta_1(-\hbar+\tau/2)}{\theta_1(\hbar+\tau/2)} = \frac{ie^{-\pi i(-\hbar+\tau/4)}\theta_4(-\hbar)}{ie^{-\pi i(\hbar+\tau/4)}\theta_4(\hbar)} = e^{2\pi i\hbar}.$$

Here we used the relations of the theta functions (4.3) in Appendix C, and the fact that $\theta_4(z)$ is even. Thus we have the eigenfunctions

$$\theta_2(z),\ \theta_3(z),\ \theta_4(z)$$

corresponding to each solution (4.9), respectively. □

Therefore, the product of the theta functions

$$\theta_i(\lambda_-)\theta_j(\lambda_+) \quad (i,j=2,3,4)$$

are simultaneous eigenfunctions of the operators H_+^2, H_-^2, H_+H_- . Finally, we shall establish the relationship of these Bethe Ansatz solutions and the bases of Th_1^W.

Lemma 12 *The functions $f_i(\lambda)\in Th_1^W$ are expressed in terms of the Jacobi theta functions as follows*:

$$f_1(\lambda)=\theta_3(\lambda_-)\theta_3(\lambda_+),\quad f_2(\lambda)=\theta_2(\lambda_-)\theta_2(\lambda_+),\quad f_3(\lambda)=\theta_4(\lambda_-)\theta_4(\lambda_+).$$

Proof. Because of the definitions of coroot lattice $Q^\vee$ (4.1) and Killing form (2.1), each basis of Th_1 is expressed as

$$\begin{aligned}
\Theta_0(\lambda) &= \theta_3(2\lambda_1|2\tau)\theta_3(2\lambda_2|2\tau),\\
\Theta_{\varepsilon_1}(\lambda) &= \theta_3(2\lambda_1|2\tau)\theta_2(2\lambda_2|2\tau),\\
\Theta_{\varepsilon_2}(\lambda) &= \theta_2(2\lambda_1|2\tau)\theta_3(2\lambda_2|2\tau),\\
\Theta_{\varepsilon_1+\varepsilon_2}(\lambda) &= \theta_2(2\lambda_1|2\tau)\theta_2(2\lambda_2|2\tau).
\end{aligned}$$

Here $\lambda_i=\lambda_{\varepsilon_i}$ $(i=1,2)$. Therefore we can prove this lemma by using the identities of theta functions (addition theorems) (4.4), (4.5), (4.6), (4.7) in Appendix C. □

Appendix C : Theta function

We establish notations and identities on the theta functions [WW]. The Jacobi theta functions are defined for $\tau \in \mathfrak{H}_+$ as follows:

$$\theta_1(z|\tau) = \theta_{1/2,1}\left(z+\frac{1}{2},\tau\right) = \sum_{k\in\mathbb{Z}} \exp\left[2\pi i\left(\left(z+\frac{1}{2}\right)\left(k+\frac{1}{2}\right)+\frac{1}{2}\left(k+\frac{1}{2}\right)^2\tau\right)\right]$$

$$\theta_2(z|\tau) = \theta_{1/2,1}(z,\tau) = \sum_{k\in\mathbb{Z}} \exp\left[2\pi i\left(z\left(k+\frac{1}{2}\right)+\frac{1}{2}\left(k+\frac{1}{2}\right)^2\tau\right)\right]$$

$$\theta_3(z|\tau) = \theta_{0,1}(z,\tau) = \sum_{k\in\mathbb{Z}} \exp\left[2\pi i\left(zk+\frac{k^2}{2}\tau\right)\right]$$

$$\theta_4(z|\tau) = \theta_{0,1}\left(z+\frac{1}{2},\tau\right) = \sum_{k\in\mathbb{Z}} \exp\left[2\pi i\left(\left(z+\frac{1}{2}\right)k+\frac{k^2}{2}\tau\right)\right]$$

Note that $\theta_1(z)$ is odd and the other three are even. These functions has quasi-periodicity:

$$\theta_1(z+m|\tau) = (-1)^m\theta_1(z|\tau), \tag{4.1}$$
$$\theta_1(z+m\tau|\tau) = (-1)^m e^{-\pi i m^2\tau-2\pi i m z}\theta_1(z|\tau), \tag{4.2}$$

$(m \in \mathbb{Z})$, while other three can be expressed by $\theta_1(z)$

$$\begin{aligned}\theta_1\left(z+\frac{1}{2}\middle|\,\tau\right) &= \theta_2(z|\tau),\\ \theta_1\left(z+\frac{\tau}{2}\middle|\,\tau\right) &= ie^{-\pi i(z+\tau/4)}\theta_4(z|\tau),\\ \theta_1\left(z+\frac{1}{2}+\frac{\tau}{2}\middle|\,\tau\right) &= e^{-\pi i(z+\tau/4)}\theta_3(z|\tau).\end{aligned} \tag{4.3}$$

We use these identities in the computations in Lemma 12.

$$\theta_4(x|\tau)\theta_4(y|\tau) = \theta_3(x+y|2\tau)\theta_3(x-y|2\tau) - \theta_2(x+y|2\tau)\theta_2(x-y|2\tau), \tag{4.4}$$
$$\theta_3(x|\tau)\theta_3(y|\tau) = \theta_3(x+y|2\tau)\theta_3(x-y|2\tau) + \theta_2(x+y|2\tau)\theta_2(x-y|2\tau), \tag{4.5}$$
$$\theta_2(x|\tau)\theta_2(y|\tau) = \theta_3(x+y|2\tau)\theta_2(x-y|2\tau) + \theta_2(x+y|2\tau)\theta_3(x-y|2\tau), \tag{4.6}$$
$$\theta_1(x|\tau)\theta_1(y|\tau) = \theta_3(x+y|2\tau)\theta_2(x-y|2\tau) - \theta_2(x+y|2\tau)\theta_3(x-y|2\tau). \tag{4.7}$$

The sigma function $\sigma(z)$ is an entire, odd, and quasi-periodic function with two primitive quasi-periods $2\omega_1, 2\omega_2$.

$$\sigma(z+2n\omega_1+2m\omega_2) = (-1)^{n+m+nm}e^{(2n\eta_1+2m\eta_2)(z+n\omega_1+m\omega_2)}\sigma(z)$$

with $\eta_i = \zeta(\omega_i)$ $(i=1,2)$, where $\zeta(z) = \sigma'(z)/\sigma(z)$ denotes the Weierstrass ζ-function. The connection between the Jacobi theta functions and the sigma functions are

$$\sigma(z) = \left(\exp\frac{\eta_1 z^2}{2\omega_1}\right)\frac{\theta_1(z/2\omega_1)}{\theta_1'(0)},$$

$$\sigma_r(z) = \left(\exp\frac{\eta_1 z^2}{2\omega_1}\right)\frac{\theta_{r+1}(z/2\omega_1)}{\theta_{r+1}(0)}\ (r=1,2,3).$$

Then, for the function $v(z)$ in van Diejen's system (3.7), we have

$$v(z) := \frac{\sigma(z+\mu)}{\sigma(z)} = \left(\exp\frac{\eta_1(2z\mu+\mu^2)}{2\omega_1}\right)\frac{\theta_1((z+\mu)/2\omega_1)}{\theta_1(z/2\omega_1)}. \tag{4.8}$$

The connection with $\wp$ function is

$$\wp(z) = -\frac{d^2}{dz^2}\log\sigma(z) = -\frac{1}{4\omega_1^2}\left(\frac{d^2}{dz^2}\log\theta_1(z/2\omega_1)\right) - \frac{\eta_1}{\omega_1}. \tag{4.9}$$

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