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Dirichlet problem at infinity for harmonic maps

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Dirichlet problem at infinity for harmonic maps

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Daishi WATABE

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Contents

Introduction

A harmonic map $u\colon M \to M'$ between Riemannian manifolds M and M', equipped with metrics locally given by $g = \sum_{i,j=1}^{m} g_{ij} ds^i ds^j$ and $g = \sum_{\beta,\gamma=1}^{m'} g'_{\beta\gamma} du^\beta du^\gamma$, is a solution to the harmonic map equation:

$$\Delta_g u^\alpha = -\sum_{i,j=1}^{m} \sum_{\beta,\gamma=1}^{m'} g^{ij} {\Gamma'}^{\alpha}_{\beta\gamma}(u) \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\gamma}{\partial x^j} \qquad (\alpha = 1, \ldots, m),$$

where $\{\Gamma^{\alpha}_{\beta\gamma}{}'\}$ are the Christoffel symbols of g' and Δ_g is the Laplacian of g. The existence of a harmonic map between compact Riemannian manifolds was established in the sixties by Eells and Sampson in their seminal paper [**10**], when a target manifold M' is of nonpositive curvature. In the seventies, under the same curvature assumption, Hamilton [**13**] investigated the Dirichlet problem for harmonic maps between compact Riemannian manifolds with boundary. However, people have only recently begun to investigate the noncompact analogue, for instance, the Dirichlet problem for harmonic maps between Cartan-Hadamard manifolds which we state explicitly below. Recall that for this problem under consideration, the manifolds M are simply connected, complete, Riemannian manifolds M of nonpositive sectional curvature. In particular, these are not compact, but can be compactified by adding the spheres at infinity ∂M of M defined by the asymptotic classes of geodesic rays in M, thus giving us the compactifications of M, which will be denoted by $\overline{M} = M \cup \partial M$. This leads us to the following Dirichlet problem at infinity for harmonic maps between Cartan-Hadamard manifolds.

Dirichlet problem at infinity for harmonic maps: *For given Cartan-Hadamard manifolds M, M' and a continuous map $f\colon \partial M \to \partial M'$, find a map $u\colon \overline{M} \to \overline{M'}$ satisfying the conditions*:

(1) $u|_{\partial M} = f$ *and,*
(2) $u|_M\colon M \to M'$ *is a solution to the harmonic map equation.*

INTRODUCTION

In the early nineties, Li and Tam [**18**], [**19**] and, simultaneously, Akutagawa [**2**] in the two-dimensional case, carried out ground breaking work by introducing new techniques for existence arguments, when M and M' are both real hyperbolic spaces. Recently, in order to simplify these arguments, Bando [**3**] combined Green's function with Hamilton's method [**13**] as a basis for his own argument. On the other hand, Donnelly [**6**], [**7**] extended Li and Tam's results to prove the existence of harmonic maps between all rank one symmetric spaces of noncompact type [namely, real, complex, quaternion hyperbolic spaces and the Cayley hyperbolic plane]. In order to refine his own results, Donnelly [**7**] later used Graham's Hölder space [**12**], where derivatives are assigned weights depending on the direction. Recently, a few attempts were made to generalize their results as can be seen in Nishikawa and Ueno [**21**] and Ueno [**24**]. The uniqueness of a solution belonging to $C^1(\overline{M}, \overline{M}')$, in the work of Li and Tam, and $C^2(\overline{M}, \overline{M}')$ or C^3_β, $\beta > 2$, in the work of Donnelly, has been established for a given non-degenerate boundary value. In order to confirm that these regularity assumptions up to the boundaries are necessary for obtaining the uniqueness, we need to construct more than one harmonic map [for instance, those provided by a family] which induce a given boundary value and are only Hölder continuous when being viewed as maps from $\overline{M}$ to $\overline{M'}$.

With regard to this problem, Li and Tam [**19**] provided an explicit example of a family of harmonic maps between real hyperbolic planes, which induce the identity map on the boundary; these maps are only Hölder continuous with exponent of $1/2$ when being viewed as self-maps of $\overline{M}$. Hence, they have verified that the assumption of regulrity cannot be removed from their uniqueness theorem [**19**] when the dimension is two. They have constructed this example by, firstly, reducing the harmonic map equation to a nonlinear ordinary differential equation; and, secondly, giving explicitly expressed solutions. In this rare example, we can express solutions for a nonlinear differential equation *explicitly*; but we cannot generally expect this to be the case. Subsequently, Economakis [**9**] generalized Li and Tam's example to higher dimensional cases by using a contraction mapping theorem, but yielding no explicitly expressed solution; his examples constructed abstractly are only Hölder continuous with exponents of less than $1/2$ when being viewed as self-maps of $\overline{M}$. Thereby, he also verified that the assumption of regularity cannot be removed from the uniqueness theorem when dimensions are greater than two.

In accordance with these studies, the following problem was suggested by Nishikawa:

INTRODUCTION

Problem 0.1. *Can we find a family of proper harmonic maps between complex hyperbolic spaces which do not satisfy the assumption of regularity in Donnelly's uniqueness theorems ?*

In the present thesis, we shall resolve this problem by extending Li and Tam's and Economakis's results to other rank one symmetric spaces. In fact, our approach is somewhat different from the one they used on the following points. Firstly, instead of using a contraction mapping theorem as in Economakis's existence argument, we shall simplify the nonlinear ordinary differential equation into a translation invariant equation and then utilize the comparison argument of solutions in conjunction with a diagonal method. We can also use the latter method, with an additional slight modification, in the construction of diverse families of harmonic maps between various tube domains as in Chapters 2 and 3. Secondly, in order to estimate the Hölder regularity of solutions at the boundary, Li and Tam, and Economakis utilized the fact that the Jacobian matrix of a geodesic symmetry of the real hyperbolic space is expressed as a product of an orthogonal matrix and a scalar function. However, this fact is no longer true for other rank one symmetric spaces of noncompact type. Therefore it is necessary to provide an argument which works even in the cases where the Jacobian matrix can not be expressed as the product described above.

Throughout this thesis, we shall identify the boundary of a rank one symmetric space of non-compact type [other than the real hyperbolic space] with the one-point compactification of a two-step nilpotent Lie group, whose Lie algebra admits a natural filtration. Regarding this identification, Donnelly [**6**, Proposition 3.4] showed that harmonic maps extending to $C^1(\overline{M}, \overline{M'})$ maps induce boundary values which preserve this natural filtration. [As was pointed out by Donnelly [**6**], these boundary values are *contact* transformations when both M and M' are complex hyperbolic spaces of the same dimension, and, more commonly, when they are non-degenerate maps (cf. Definition 3.2). Hence, later in [**8**], Donnelly utilized the term "*contact*" instead of "*filtration preserving*".] This observation attracted a great deal of attention because this statement implies that not all maps can be boundary values of harmonic maps. Following this, Nishikawa and Ueno [**21**] defined generalizations of rank one symmetric spaces, which they call k-term Carnot spaces (cf. Definition 1.1); and studied the filtration preserving properties of the boundary values of harmonic maps between k-term Carnot spaces.

In connection with the study mentioned above, Donnelly [**8**] provided a family of Hölder continuous self-maps of the compactification $\overline{M}$ of a rank one symmetric space of noncompact type, which are harmonic on the interior and assume a C^1 boundary value, being given by a homomorphism which does not preserve the filtration.

Considering that the above harmonic maps are not uniquely determined for a given boundary value, Donnelly suggested the following question through a personal discussion:

PROBLEM 0.2. *Can we show the uniqueness of harmonic maps inducing a given boundary value which is filtration preserving?*

As for supporting evidences leading to an affirmative answer, we firstly refer to Donnelly's uniqueness theorem [**6**] of harmonic maps, which extend to C^2 maps up to the boundary and induce an assigned non-degenerate boundary value; and, secondly, we refer to the filtration preserving property of the boundary values of harmonic maps which can be viewed as maps in $C^2(\overline{M}, \overline{M}) \subset C^l(\overline{M}, \overline{M})$ by Donnelly [**6**, Proposition 3.4]. Regarding the problem stated above, we shall prove the following:

THEOREM 0.1. *Let M aud M' be rank one symmetric spaces of noncompact type. Suppose that $h\colon \partial M \to \partial M'$ is a map obtained by extening a non-trivial filtration preserving homomorphism which is possibly degenerate. Then, there exists a family of harmonic maps which assume h on the boundary ∂M.*

Given this theorem, the answer to the problem above is not affirmative as shall be illustrated in Examples 1 through 6 in Chapter 3, leading us to search for a stricter condition than that of contactness.

At this point, it should be further noted that our theorem provides harmonic maps not only for *non-degenerate* boundary values but also for *degenerate* ones. Hence, our theorem is also useful for constructing harmonic maps between complex hyperbolic spaces even in the case where the dimension of the target is less than that of the source; boundary values of which are *always automatically degenerate*, as the author was personally informed by Donnelly. We shall provide families of these maps in Examples 3 through 5 in Chapter 3. In this respect, we also refer to Donnelly's existence theorem [**6**, Theorem 6.2] for *non-degenerate* boundary values.

Up to this point, we have discussed the Dirichlet problem between *rank one* symmetric spaces of noncompact type, the compactifications of which are C^∞*manifolds* [not necessarily Riemannian ones]. In contrast, the compactifications of *rank* q (> 1) symmetric spaces of noncompact type have the so-called corners on the boundary and, consequently, they are *not even* C^∞ *manifolds.* Hence, it is important, at this point, to consider harmonic maps between rank q (> 1) symmetric spaces of noncompact type, a problem which has neither seemed tractable nor been explored in the past. In this respect, the following problem naturally arises:

PROBLEM 0.3. *Can we construct harmonic maps between rank two symmetric spaces?*

In this thesis, we shall establish the existence of harmonic self-maps on rank two symmetric spaces of noncompact type; namely, $D_{2,2}(\mathbb{C})$, the space of 2×2 complex matrices Z satisfying $I_2 - {}^t\bar{Z}Z$ equipped with the metric $i\partial\overline{\partial} \log \det(I_2 - {}^t\bar{Z}Z)$.

THEOREM 0.2. *There exists a family of harmonic self-maps of* $D_{2,2}(\mathbb{C})$ *that induce the identity map on the corner of* $\overline{D_{2,2}(\mathbb{C})}$.

For more detailed definition of the terminology in the statement, we refer to Definition 8.1.

This thesis is organized as follows:

- In Chapter 1, seeking particular forms of solutions, we shall simplify the harmonic map equation between warped products such as k-term Carnot spaces, thereby obtaining an ordinary differential equation.
- In Chapter 2, this ordinary differential equation shall be examined and a family of solutions for this equation shall be constructed.
- In Chapter 3, we shall apply our constructions from the previous two chapters to the investigation of the Dirichlet problem at infinity, thereby providing examples of harmonic maps inducing filtration preserving boundary values which may possibly be degenerate.
- In Chapter 4, in order to study the regularity of these constructed maps in the following chapters, we shall review relevant facts about hyperbolic geometry.
- In Chapter 5, by utilizing a slightly different analysis from those in Chapter 2, we shall reconstruct one of the harmonic maps we constructed which induce the identity map on the boundary. This approach is more convenient for regularity estimations and for the analysis of asymptotic behavior.
- In Chapter 6, making full use of the results from Chapters 4 and 5, we shall make regularity estimations of our harmonic maps constructed in Chapter 4, thereby

verifying that the assumption of regularity is essential for Donnelly's uniqueness theorem [**6**, Theorem 3.13].

- In Chapter 7, we shall re-examine the regularity of these harmonic maps by utilizing Graham's non-isotropic Hölder spaces. [Here derivatives are assigned weights depending on the directions and the elements of these spaces have boundary values belonging to Folland and Stein's Hölder space defined by means of the Heisenberg distance function.] Thereby, we shall verify that the assumption of regularity cannot be removed from Donnelly's uniqueness theorem [**7**, Theorem 2.8].
- In Chapter 8, we shall extend our investigation of the Dirichlet problem at infinity for harmonic maps between rank one symmetric spaces, thus taking into account rank q (> 1) symmetric spaces. By utilizing thorough computations of $\mathfrak{su}(2,2)$, we shall reduce the harmonic map equation to an ordinary differential equation which can be studied by using the result from either Chapter 2 or 5. Thereby, we prove the existence of the family of harmonic self-maps on $D_{2,2}(\mathbb{C})$.

Throughout this thesis, $C = C(*, \dots, *)$ will always denote a constant depending only on the quantities appearing in parenthesis. In a given context, the same letter C will, in general, be used to denote different constants depending on the same set of arguments.

CHAPTER 1

The harmonic map equation between k-term Carnot spaces

To begin with, we fix our notation and terminology. First, let S denote a simply connected solvable Lie group satisfying the following conditions:

1. S is a semidirect product of a nilpotent Lie group N and a one-dimensional abelian Lie group A.
2. If $\mathfrak{n}$ and $\mathfrak{s} = \mathfrak{n} + \mathbb{R}\{H\}$ denote the Lie algebras of N and S respectively, then $\mathfrak{n}$ has a decomposition $\mathfrak{n} = \sum_{i=1}^{k} \mathfrak{n}_i$ into k-subspaces given by

$$\mathfrak{n}_i = \{X \in \mathfrak{n}_i \mid \mathrm{ad}(H)(X) = iX\}, \quad i = 1, \dots, k,$$

where $\mathrm{ad}(H)$ denotes the adjoint representation. Since $\mathrm{ad}(H)$ is a Lie algebra homomorphism, the above decomposition of $\mathfrak{n}$ defines a graded Lie algebra structure of $\mathfrak{n}$, that is, $[\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_{i+j}$ with the convention $\mathfrak{n}_i = \{0\}$ for $i > k$; thereby, yielding a corresponding filtration: $\mathfrak{n}^{(1)} \subset \mathfrak{n}^{(2)} \ldots \subset \mathfrak{n}^{(k)}$ given by $\mathfrak{n}^{(i)} = \sum_{l=1}^{i} \mathfrak{n}_l$.

Utilizing these properties of S and denoting by $\Psi \colon N \times A \ni (\mathbf{n}, \exp(tH)) \mapsto \mathbf{n} \cdot \exp(tH) \in N \cdot A$ the multiplication of S, Nishikawa and Ueno observed that the pull-back metric $g = \Psi^*(\tilde{g})$ of any left invariant metric $\tilde{g}$ on S is expressed as follows:

$$g = e^{-2t} g_{\mathfrak{n}_1} + \cdots + e^{-2kt} g_{\mathfrak{n}_k} + dt^2,$$

where $g_{\mathfrak{n}_1} + \cdots + g_{\mathfrak{n}_k}$ denotes a left invariant metric of N satisfying

$$g_{\mathfrak{n}_j}(X, X) \neq 0 \ (X (\neq 0) \in \mathfrak{n}_j), \quad g_{\mathfrak{n}_i}(X, X) = 0 \ (X \in \mathfrak{n}_j, i \neq j),$$
$$g_{\mathfrak{n}_l}(X, Y) = 0 \ (X \in \mathfrak{n}_i, Y \in \mathfrak{n}_j, i \neq j).$$

When $\log(\rho)/2$ is substituted for t, the metric g on $N \times \mathbb{R}_+$ is realized as

$$g = \frac{g_{\mathfrak{n}_1}}{\rho} + \cdots + \frac{g_{\mathfrak{n}_k}}{\rho^k} + \frac{d\rho^2}{4\rho^2}.$$

This leads us to utilize an orthogonal left invariant frame $\{e_i\}_{i=1}^m$ satisfying:

$$e_m = \partial/\partial\rho,\ [e_m, e_\alpha] = 0,\ [e_\alpha, e_\beta] = \textstyle\sum_{\gamma=1}^{m-1} a_{\alpha\beta}^\gamma e_\gamma,\ a_{\beta\gamma}^\alpha = 0 \text{ unless } \alpha \in I_\lambda, \beta \in I_\mu, \gamma \in I_{\lambda+\mu},$$
$$g_{ii} = \rho^{-l} \quad (i \in I_l), \quad g_{mm} = \rho^{-2}4^{-1}, \quad g_{ij} = 0, \quad (i \neq j) \text{ for } g_{ij} = g(e_i, e_j).$$

Here, the index set I_l corresponding to the gradation is given by

$$I_l = \{j \mid 1 + \textstyle\sum_{1\le i\le l-1} n_i \le j \le \sum_{1\le i\le l} n_i\}, \qquad n_i = \dim(\mathfrak{n}_i).$$

Although we do not need to specify the sign of the curvature up to this point, it is known that there exists a left invariant metric g of S whose sectional curvatures are negative [**15**].

Definition 1.1 (Nishikawa and Ueno [**21**]). The warped product manifold $M = (S, g)$ described above is called a *k-term Carnot space* if its sectional curvature is negative.

Nishikawa and Ueno arrived at the above notion of k-term Carnot spaces as a generalization of rank one symmetric spaces of noncompact type. For example, real hyperbolic spaces are 1-term Carnot spaces, and complex or quaternion hyperbolic spaces and the Cayley hyperbolic plane are 2-term Carnot spaces. With these symmetric spaces included, all k-term Carnot spaces M are homogeneous Cartan-Hadamard manifolds. The $\mathbb{R}_+$ directions ($\mathbf{n} = \text{Constant}, \rho$) define asymptotic geodesics of this space, thereby yielding a point at infinity $\infty \in \partial M$ and allowing us to identify $\partial M \setminus \{\infty\}$ with $N \times \{0\}$.

As is well-known, a map $u\colon M \to M'$ between Riemannian manifolds M and M' equipped with frames $\{e_i\}_{i=1}^m$ and $\{e_\gamma\}_{\gamma=1}^{m'}$, respectively, is harmonic if the tension field $\tau(u)$ of u given by the following formula vanishes identically on M:

$$\begin{aligned}\tau(u) &= \sum_{i,j=1}^m g^{ij}\Big(\tilde\nabla_{e_i} u_*(e_j) - u_*\big(\nabla^M_{e_i} e_j\big)\Big)\\
&= \sum_{\alpha=1}^{m'}\sum_{i,j=1}^m g^{ij}\Big(u^\alpha_{ij} + \sum_{\beta,\gamma=1}^{m'} {e'}^*_\alpha(\nabla^{M'}_{{e'}_\beta} e'_\gamma) u^\beta_i u^\gamma_j - \sum_{l=1}^m e^*_l(\nabla^M_{e_i} e_j) u^\alpha_l\Big) e'_\alpha \qquad (1)\\
&= \sum_{\alpha=1}^{m'} \tau^\alpha(u) {e'}_\alpha.\end{aligned}$$

Here $g_{ij} = g(e_i, e_j)$ is the components of the metric g of M; $(g_{ij})^{-1} = (g^{ij})$; $u_*(e_i) = \sum_{\gamma=1}^{m'} u_i^\gamma e'_\gamma$ and $u^\gamma_{ij} = e_i u^\gamma_j$; ∇^M and $\nabla^{M'}$ are Levi-Civita connections on TM and TM', respectively; and $\tilde\nabla$ is an induced connection on $u^{-1}(TM')$ (see [**6**, (2.1)]).

Hence, in order to compute the tension field of a map u between warped product manifolds $M = (S, g)$ and $M' = (S', g')$ described above, (preserving the prime $'$ and Greek-indices for the target, and letting $\rho'\circ u$ denote the ρ' component of u), noting first that g_{ij} is diagonal, we start by simplifying the components $\tau^\alpha(u)$ of the tension field (1) as follows:

$$\begin{aligned}\tau^\alpha(u) &= \sum_{j=1}^{m} g^{jj}\Big(u^\alpha_{jj} - \sum_{l=1}^{m} e^*_l(\nabla^M_{e_j} e_j)u^\alpha_l + \sum_{\beta=1}^{m'} e'^*_\alpha(\nabla^{M'}_{e'_\beta} e'_\beta)u^\beta_j u^\beta_j \\ &\quad + \sum_{1\le\beta<\gamma\le m'} \big(e'^*_\alpha(\nabla^{M'}_{e'_\beta} e'_\gamma) + e'^*_\alpha(\nabla^{M'}_{e'_\gamma} e'_\beta)\big)u^\beta_j u^\gamma_j\Big).\end{aligned}$$

This leads us to examine the following components of the Levi-Civita connections of M and M':

$$\begin{aligned}&e^*_m(\nabla^M_{e_j} e_j) = 2l\rho^{-\lambda+1}, \quad e^*_i(\nabla^M_{e_j} e_j) = 0, \quad e^*_m(\nabla^M_{e_m} e_m) = -\rho^{-1},\\ &e^*_\alpha(\nabla^{M'}_{e_\beta} e_\gamma) + e^*_\alpha(\nabla^{M'}_{e_\gamma} e_\beta) = -(\rho\circ u)^\mu\big((\rho'\circ u)^{-\lambda}a'^\beta_{\gamma\alpha} + (\rho'\circ u)^{-\mu-\lambda}a'^\gamma_{\beta\alpha}\big) = -(\rho'\circ u)^{-\lambda}a'^\gamma_{\beta\alpha},\\ &e^*_\alpha(\nabla^{M'}_{e_\beta} e_{m'}) + e^*_\alpha(\nabla^{M'}_{e_{m'}} e_\beta) = -\lambda(\rho'\circ u)^{-1},\\ &e^*_{m'}(\nabla^{M'}_{e_\beta} e_\beta) = 2\lambda(\rho'\circ u)^{-\lambda+1}, \quad e^*_\delta(\nabla^{M'}_{e_\beta} e_\beta) = 0, \quad e^*_{m'}(\nabla^{M'}_{e_{m'}} e_{m'}) = -(\rho'\circ u)^{-1}\end{aligned}$$

for $\alpha \in I'_\mu, \beta \in I'_\lambda, \gamma \in I'_{\mu+\lambda}$ $(1 \le \lambda, \mu \le k')$, $j \in I_l$ $(1 \le l \le k)$, $(i \ne m)$, $(\delta \ne m')$. Thereby we can calculate the components of the tension field $\tau(u)$ of a map u as follows:

$$\begin{aligned}\tau^{m'}(u) &= \sum_{j=1}^{m} g^{jj}u^{m'}_{jj} - (2\sum_{l=1}^{k} ln_l - 4)\rho u^{m'}_m - \sum_{l=1}^{k}\sum_{j\in I_l}\rho^l(\rho'\circ u)^{-1}(u^{m'}_j)^2\\ &\quad -4\rho^2(\rho'\circ u)^{-1}(u^{m'}_m)^2 + 2\sum_{j=1}^{m} g^{jj}\Big(\sum_{\lambda=1}^{k'}\lambda(\rho'\circ u)^{-\lambda+1}\sum_{\gamma\in I'_\lambda}(u^\gamma_j)^2\Big),\\ \tau^\alpha(u) &= \sum_{j=1}^{m} g^{jj}u^\alpha_{jj} - (2\sum_{l=1}^{k} ln_l - 4)\rho u^\alpha_m - 2\sum_{l=1}^{k}\rho^l\sum_{j\in I_l}\mu(\rho'\circ u)^{-1}u^{m'}_j u^\alpha_j\\ &\quad -8\rho^2(\rho'\circ u)^{-1}u^{m'}_m u^\alpha_m - \sum_{j=1}^{m} g^{jj}\sum_{\lambda=1}^{k'}\sum_{\beta\in I'_\lambda,\gamma\in'_{\lambda+\mu}}(\rho'\circ u)^{-\lambda}a'^\gamma_{\beta\alpha}u^\beta_j u^\gamma_j\end{aligned}$$

for $\alpha \in I'_\mu$ $(1 \le \mu \le k')$. Hence, denoting $\sum_{l=1}^{k} ln_l$ as $\mathcal{N}$, we have the following

LEMMA 1.1. *Let M and M' be as above. A map $u\colon M \ni (\mathbf{n}, \rho) \mapsto (h(\mathbf{n}), \psi(\rho)) \in M'$ is harmonic if the following conditions hold*:

$$\begin{aligned}
0 &= \rho^2 \frac{d\psi(\rho)}{d\rho^2} - (\frac{1}{2}\mathcal{N} - 1)\rho \frac{d\psi(\rho)}{d\rho} \\
&+ \frac{1}{2}\sum_{l=1}^{k} \rho^l \Big(\sum_{\lambda=1}^{k'} \lambda \psi^{-\lambda+1}(\rho) \sum_{j\in I_l, \gamma \in I'_\lambda} (h_j^\gamma)^2\Big) - \rho^2 \psi(\rho)^{-1}\Big(\frac{d\psi(\rho)}{d\rho}\Big)^2 \quad (\rho > 0),
\end{aligned} \tag{2}$$

$$\sum_{j\in I_l} \sum_{\beta \in I'_\lambda, \gamma \in I'_{\lambda+\mu}} a'^{\gamma}_{\beta\alpha} h_j^\beta h_j^\gamma = 0 \quad (\alpha \in I_\mu) \quad \textit{on } N \tag{3}$$

for $1 \leq l \leq k,\ 1 \leq \mu, \lambda \leq k'$,

$$h_j^\gamma = \text{Constant} \qquad (1 \leq j \leq m-1,\ 1 \leq \gamma \leq m'-1) \quad \textit{on } N. \tag{4}$$

REMARK 1.1. When M and M' are both one-dimensional complex hyperbolic spaces $\mathbb{C}H^1$, it holds that $n_1 = 0, n_2 = 1$ and that the equations (2) through (4) reduce to the following:

$$0 = \rho^2 \frac{d\psi(\rho)}{d\rho^2} + \rho^2 \psi(\rho)^{-1}(h_1^1)^2 - \rho^2\psi(\rho)^{-1}\Big(\frac{d\psi(\rho)}{d\rho}\Big)^2,$$

where h_1^1 is a constant. $\psi(\rho) = |h_1^1|^{-1}\sinh(C\rho)/C$ is a solution for each $C > 0$ and $u\colon (t, \rho) \to (h_1^1 t, |h_1^1|^{-1}\sinh(C\rho)/C)$ is a family of harmonic maps parametrized by C. This is Li and Tam's [**19**] example when $h_1^1 = 1$.

REMARK 1.2. When h is the identity map of N, it holds that $h_j^\gamma = \delta_j^\gamma$ for $j, \gamma \geq 1$, and thereby (3) follows immediately and the first equation (2) reduces to the following:

$$0 = \rho^2 \frac{d\psi(\rho)}{d\rho^2} - (\frac{1}{2}\mathcal{N} - 1)\rho\frac{d\psi(\rho)}{d\rho} + \frac{1}{2}\sum_{\lambda=1}^{k} \lambda n_\lambda \rho^\lambda \psi^{-\lambda+1}(\rho) - \rho^2\psi(\rho)^{-1}\Big(\frac{d\psi(\rho)}{d\rho}\Big)^2. \tag{5}$$

By providing the solutions of this equation, we can obtain a family of harmonic maps of the form $u\colon M \ni (\mathbf{n}, \rho) \mapsto (\mathbf{n}, \psi(\rho)) \in M$ as was studied in [**25**], which utilized an argument similar to that described in Theorem 5.2. By substituting ρ for $\psi(\rho)$ in (5), one verifies that the identity map is a particular member of the family, recalling that the identity map is a harmonic map. In Chapter 2, by means of a slightly different approach, we shall prove the existence of solutions for (2) based on the construction in Chapter 2, thereby obtaining these harmonic maps as a particular case, as in Example 1 in Chapter 3. It should be noted that, for this particular example, we have not assumed that sectional curvatures of g are non-positive, and that M is not necessarily a Cartan-Hadamard manifold.

CHAPTER 2

An existence theorem

In this chapter, in order to provide harmonic maps, for any given h satisfying (3) and (4), we shall construct a one-parameter family of solutions to the equation (2) of the particular form $\psi(\rho) = C^{-1}\rho^a \exp(f(\log(|\rho|)))$ for positive functions f. To begin with, we observe that

$$\frac{d\psi(\rho)}{d\rho} = C^{-1}(a+f')\rho^{a-1}e^f, \quad \frac{d^2\psi(\rho)}{d\rho^2} = C^{-1}\rho^{a-2}e^f(f''+(f'+a)^2-a-f'),$$

where $f' = df(t)/dt,\ f'' = d^2f(t)/dt^2$. By substituting the above into each respective term of (2), we can obtain

$$\rho^2\frac{d^2\psi(\rho)}{d\rho^2} - (\frac{1}{2}\mathcal{N}-1)\rho\frac{d\psi(\rho)}{d\rho} - \rho^2\psi(\rho)^{-1}\Big(\frac{d\psi(\rho)}{d\rho}\Big)^2 = C^{-1}\rho^a e^f(f'' - \frac{1}{2}\mathcal{N}a - \frac{1}{2}\mathcal{N}f'),$$

$$\begin{aligned}\frac{1}{2}\sum_{l=1}^{k}\rho^l\sum_{\lambda=1}^{k'}\lambda\psi(\rho)^{-\lambda+1}c_{l\lambda} &= \frac{1}{2}\psi(\rho)\sum_{l=1}^{k}e^{lt}\sum_{\lambda=1}^{k'}\lambda\psi(\rho)^{-\lambda}c_{l\lambda}\\ &= \frac{1}{2}C^{-1}\rho^a e^f\Big(\sum_{\lambda=1}^{k'}\sum_{l=1}^{k}\lambda c_{l\lambda}C^\lambda e^{\lambda(l/\lambda-a)t-\lambda f}\Big).\end{aligned}$$

Here $c_{l\lambda}$ is a constant given by

$$c_{l\lambda} = \sum_{j\in I_l,\gamma\in I'_\lambda}(h_j^\gamma)^2. \tag{6}$$

Hence, we can conclude the following:

Given a solution $f \geq 0$ for

$$f''(t) = \frac{1}{2}\mathcal{N}f'(t) + \frac{1}{2}\mathcal{N}a - \frac{1}{2}\sum_{\lambda=1}^{k'}\sum_{l=1}^{k}\lambda c_{l\lambda}C^\lambda e^{\lambda(l/\lambda-a)t-\lambda f(t)}, \qquad t\in\mathbb{R}, \tag{7}$$

we are able to construct a solution for (2) as $\psi(\rho) = C^{-1}\rho^a\exp(f(\log(|\rho|)))$. When $c_{l\lambda}$ are all zero, we find that there is an explicit solution given by $f(t) = f(0)e^{\mu t}$ for

$\mu = (\mathcal{N} + (\mathcal{N}^2 + 4\mathcal{N}a)^{1/2})/4$. Accordingly, we shall assume that at least one of $c_{l\lambda}$ is not zero.

Continuing our construction, we shall now set

$$a = \min\{l/\lambda \mid c_{l\lambda} \neq 0,\ 1 \leq l \leq k,\ 1 \leq \lambda \leq k'\} \tag{8}$$

so that $l/\lambda - a \geq 0$ for all l, λ satisfying $c_{l\lambda} \neq 0$, and let C be the mnimum positive number satisfying

$$\mathcal{N}a - \sum_{l/\lambda = a} \lambda c_{l\lambda} C^{\lambda} = 0.$$

Then, for positive numbers $a_i, b_i, \alpha_j, \beta_j, \gamma_j$ ($i \in \Lambda_A$, $j \in \Lambda_B$, where Λ_A and Λ_B are index sets of finite number of elements), we can express the right-hand side of (7) as

$$\begin{aligned}
&\frac{1}{2}\mathcal{N}f'(t) + \frac{1}{2}\mathcal{N}a - \frac{1}{2}\sum_{l/\lambda=a} \lambda c_{l\lambda} C^{\lambda} e^{-\lambda f(t)} - \frac{1}{2}\sum_{l/\lambda \neq a} \lambda c_{l\lambda} C^{\lambda} e^{\lambda(l/\lambda - a)t - \lambda f(t)} \\
&= \frac{1}{2}\mathcal{N}f'(t) + \frac{1}{2}\mathcal{N}a - \frac{1}{2}\sum_{l/\lambda=a} \lambda c_{l\lambda} C^{\lambda} + \frac{1}{2}\sum_{l/\lambda=a} \lambda c_{l\lambda} C^{\lambda}(1 - e^{-\lambda f(t)}) \\
&\qquad - \frac{1}{2}\sum_{l/\lambda \neq a} \lambda c_{l\lambda} C^{\lambda} e^{\lambda(l/\lambda - a)t - \lambda f(t)} \\
&= \frac{1}{2}\mathcal{N}f'(t) + \sum_{i \in \Lambda_A} a_i(1 - e^{-b_i f(t)}) - \sum_{j \in \Lambda_B} \alpha_j e^{\beta_j t - \gamma_j f(t)}.
\end{aligned}$$

This leads us to examine the following:

Theorem 2.1. *For positive numbers* $\mathcal{N}, a_i, b_i, \alpha_j, \beta_j, \gamma_j, (i \in \Lambda_A, j \in \Lambda_B)$, *satisfying* $\mathcal{N}/2 > \max_{j \in \Lambda_B}(\beta_j)$ *or* $\Lambda_B = \emptyset$, *there exists a solution to the following equation*:

$$f''(t) = \frac{1}{2}\mathcal{N}f'(t) + \sum_{i \in \Lambda_A} a_i(1 - e^{-b_i f(t)}) - \sum_{j \in \Lambda_B} \alpha_j e^{\beta_j t - \gamma_j f(t)} \qquad (t \in \mathbb{R}) \tag{9}$$

satisfying: $f(t) \to 0$ $(t \to -\infty), f(t) \to \infty$ $(t \to \infty)$ *and* $f(0) > \sum_{j \in \Lambda_B} \alpha_j/\beta_j(\mathcal{N}/2 - \beta_j)$.

Upon completing the proof of this theorem, we can then establish the existence of a solution $\psi(\rho)$ for (2) on $\mathbb{R}_+$ as $\psi(\rho) = C^{-1}\rho^a \exp(f(\log(|\rho|)))$ under the assumption that $\max_{j \in \Lambda_B}(\beta_j) < \mathcal{N}/2$ or that $\Lambda_B = \emptyset$. These are reasonable assumptions to be satisfied in many applications concerning the construction of harmonic maps. [For example, when $h = \mathrm{id}$, it holds that $\Lambda_B = 0$. Furthermore, when M and M' are rank one symmetric spaces of noncompact type, $\max_{j \in \Lambda_B}(\beta_j) < \mathcal{N}/2$ holds true for most of the homomorphisms h

as shall be described in Theorem 2.2 and Lemma 3.1.] As we shall see, these assumptions allow us to rule out the resonance condition of a barrier function (15) as a solution for (13).

In the following, $f(t) \to 0$ $(t \to -\infty)$ and $f(t) \to \infty$ $(t \to \infty)$ are abbreviated to $f(-\infty) = 0$ and $f(\infty) = \infty$, respectively.

PROOF. In the first step, we shall construct a solution $f(t)$ of (9) for $t \leq 0$ satisfying $f(-\infty) = 0$ and $f(0) = f_0$ for a given sufficiently large positive number f_0. In the second step, we shall make a continuation of this solution to $t \geq 0$.

Rewriting the equation (9) above as

$$(f'(t)e^{-t\mathcal{N}/2})' = e^{-\mathcal{N}/2}\Big(\sum_{i\in\Lambda_A} a_i(1 - e^{-b_i f(t)}) - \sum_{j\in\Lambda_B} \alpha_j e^{\beta_j t - \gamma_j f(t)}\Big), \tag{10}$$

we use the following well-known result from a two-point boundary value problem of an ordinary differential equation.

LEMMA 2.1. ([**27**, p.262-266], [**11**, 175C]). *Let $p(t) > 0$ be in $C^1([t_1, t_2])$ and $q(t) \in C^0([t_1, t_2])$. For a function $f \in C^2([t_1, t_2])$, set $Lf = (p(t)f')' + q(t)f$. Consider the following second-order equation*

$$Lf = F(t, f) \tag{11}$$

with boundary conditions: $f(t_1) = \eta_1$ and $f(t_2) = \eta_2$. Suppose that we have C^2 functions $\underline{f}(t)$ and $\overline{f}(t)$ satisfying $\underline{f}(t) \leq \overline{f}(t), \underline{f}(t_1) \leq \eta_1 \leq \overline{f}(t_1)$, $\underline{f}(t_2) \leq \eta_2 \leq \overline{f}(t_2)$ and

$$\begin{aligned} L\underline{f}(t) &\geq F(t, \underline{f}), \\ L\overline{f}(t) &\leq F(t, \overline{f}), \end{aligned} \qquad (t_1 \leq t \leq t_2). \tag{12}$$

(Inequalities in this instance are reversed when compared with the comparison theorems in the initial value problem.) Assume also that $F(t, f)$ is continuous for $t_1 \leq t \leq t_2$ and $\underline{f} \leq f \leq \overline{f}$. Then, there exists a solution $f(t)$ for (11) with boundary values $f(t_1) = \eta_1, f(t_2) = \eta_2$ and satisfying

$$\underline{f}(t) \leq f(t) \leq \overline{f}(t)$$

for $t_0 \leq t \leq t_1$. Moreover, the solution is unique if $F(t, f)$ is a monotone increasing function for f.

In order to utilize the lemma stated above for our purpose, let us firstly abbreviate the right-hand side of (10) to $F(t,f)$. Note that $F(t,f)$ is continuous for $-\infty < t \le 0$, $-\infty < f < \infty$, and is monotone increasing for f. Next, we define $\overline{f}$ and $\underline{f}$ as solutions for

$$\overline{f}''(t) = \frac{1}{2}\mathcal{N}\overline{f}'(t) - \sum_{j\in\Lambda_B} \alpha_j e^{\beta_j t}, \tag{13}$$

$$\underline{f}''(t) = \frac{1}{2}\mathcal{N}\underline{f}'(t) + \sum_{i\in\Lambda_A} a_i b_i \underline{f}(t) \tag{14}$$

with boundary conditions: $\overline{f}(0) = \underline{f}(0) = f_0$ and $\overline{f}(-\infty) = \underline{f}(-\infty) = 0$. These equations have unique solutions, which can be explicitly expressed as follows:

$$\begin{aligned} \overline{f}(t) &= \Big(f_0 - \sum_{j\in\Lambda_B} \frac{\alpha_j}{\beta_j(\mathcal{N}/2-\beta_j)}\Big)e^{\mathcal{N}t/2} + \sum_{j\in\Lambda_B} \frac{\alpha_j}{\beta_j(\mathcal{N}/2-\beta_j)} e^{\beta_j t}, \\ \underline{f}(t) &= f_0 e^{\lambda t}, \qquad \lambda = \Big(\mathcal{N} + \big(\mathcal{N}^2 + 16\sum_{i\in\Lambda_A} a_i b_i\big)^{1/2}\Big). \end{aligned} \tag{15}$$

From these explicitly expressed solutions, we can observe that

$$\begin{aligned} \underline{f}(t) \le \overline{f}(t) \text{ for } t \le 0, \\ \overline{f}(t) \le \underline{f}(t) \text{ for } t \ge 0, \end{aligned} \tag{16}$$

because it holds that $\lambda \ge \mathcal{N}/2 > \beta_j$. Furthermore, noting that $(1-e^{-b_i x})/x$ is a monotone decreasing function of $x \ge 0$, we find that

$$1 - e^{b_i \underline{f}(t)} = \big((1 - e^{-b_i \underline{f}(t)})/\underline{f}(t)\big)\underline{f}(t) \le b_i \underline{f}(t), \tag{17}$$

and thereby it follows that

$$\begin{aligned} (e^{-t\mathcal{N}/2}\underline{f}'(t))' &= F(t,\underline{f}) - e^{-t\mathcal{N}/2}\Big(\sum_{i\in\Lambda_A} a_i\big((1-e^{-b_i\underline{f}(t)}) - b_i\underline{f}(t)\big) - \sum_{j\in\Lambda_B} \alpha_j e^{\beta_j t - \gamma_j \underline{f}(t)}\Big) \\ &\ge F(t,\underline{f}(t)), \\ (e^{-t\mathcal{N}/2}\overline{f}'(t))' &= F(t,\overline{f}) - e^{-t\mathcal{N}/2}\Big(\sum_{i\in\Lambda_A} a_i(1-e^{-b_i\overline{f}(t)}) + \sum_{j\in\Lambda_B} \alpha_j e^{\beta_j t}(1-e^{-\gamma_j\overline{f}(t)})\Big) \\ &\le F(t,\overline{f}(t)). \end{aligned}$$

Given the verifications made above, we can use Lemma 2.1 in order to obtain a unique solution $f_\kappa(t)$ for (9) with boundary values $f_\kappa(0) = f_0$ and $f_\kappa(-\kappa) = \overline{f}(-\kappa)$, for each positive integer κ. Thus we can obtain a sequence $\{f_\kappa\}$ which satisfies

$$\underline{f}(t) \le f_\kappa(t) \le \overline{f}(t) \text{ on } [-\kappa, 0]. \tag{18}$$

It should be noted that the behavior of $f_\kappa(t)$ is controlled by $\underline{f}(t)$ and $\overline{f}(t)$ independently of κ. By using a well-known lemma [**14**, p. 428 Lemma 5.1] for (9), we have a constant C such that $\sup_{-\kappa\le t\le 0}|f'_\lambda| < C$ for all $\lambda > \kappa$. Combining this with (9), we have constants C' and C'' such that $\sup_{-\kappa\le t\le 0}|f'_\lambda| < C'$ and $\sup_{-\kappa\le t\le 0}|f''_\lambda| < C''$ for all $\lambda \ge \kappa$. Therefore, when we examine $\{f_\kappa\}$ firstly on $[-1,0]$, by using the Ascoli-Arzelà theorem, we have a sub-sequence $\{f_{\kappa_i}\}$ of $\{f_\kappa\}$ converging in $C^2([-1,0])$ topology. Focusing on $[-2,0]$ secondly, utilizing the Ascoli-Arzelà theorem once again, we see that there is a sub-sequence $\{f_{\kappa_{i_l}}\}$ of $\{f_{\kappa_i}\}$ converging in $C^2([-2,0])$ topology. Continuing on, we obtain a diagonal sub-sequence converging to f_∞ locally in $C^2((-\infty,0])$. Thus we have obtained a solution $f(t) = \overline{f}(t)$ of (9) satisfying

$$\underline{f}(t) \le f(t) \le \overline{f}(t) \text{ on } (-\infty, 0].$$

Now we shall consider the continuation of $f(t)$ for $t \ge 0$ to obtain a global solution $f(t)$ on $(-\infty,\infty)$. In order to prove the global existence of f on $t \ge 0$, it suffices to find functions which restrict the behavior of both f and f' from above and below; thereby, we show that the local solutions f and f' do not diverge within a finite time. To this end, we firstly note that $\overline{f}(t) \le \underline{f}(t)$ for $t \ge 0$ from (16). This reversal of inequality for $t \ge 0$ when compared to that for $t \le 0$ leads us to utilize solutions for (13) and (14) as a sub-solution and a super-solution, respectively. If this seems to be contradictory, let us recall that the directions of the inequalities (12) in Lemma 2.1 are reversed when compared with the comparison theorem in the initial value problem. (See [**4**, p. 73, Theorem 5.1] and [**11**, 175C, 176E]. See also [**27**, p. 246, p. 140].)

Moreover, since we assume $f_0 > \sum_{j\in\Lambda_B}\alpha_j/\beta_j(\mathcal{N}/2-\beta_j)$, we find that $\overline{f}, \underline{f} > 0$ for all $t \ge 0$.

Hence, by utilizing (7), (13), (14) and (17), we can obtain the following differential inequalities:

$$\begin{aligned}(f'-\overline{f}')' &= \frac{1}{2}\mathcal{N}(f'-\overline{f}') + \sum_{i\in\Lambda_A} a_i e^{-b_i\overline{f}}(1-e^{-b_i(f-\overline{f})}) + \sum_{j\Lambda_B}\alpha_j e^{\beta_j t-\gamma_j\overline{f}}(1-e^{-\gamma_j(f-\overline{f})})\\ &\quad + \sum_{i\in\Lambda_A} a_i(1-e^{-b_i\overline{f}}) + \sum_{j\in\Lambda_B}\alpha_j e^{\beta_j t}(1-e^{-\gamma_j\overline{f}})\\ &\ge \frac{1}{2}\mathcal{N}(f'-\overline{f}') + \sum_{i\in\Lambda_A} a_i e^{-b_i\overline{f}}(1-e^{-b_i(f-\overline{f})}) + \sum_{j\in\Lambda_B}\alpha_j e^{\beta_j t-\gamma_j\overline{f}}(1-e^{-\gamma_j(f-\overline{f})}),\end{aligned}$$

$$
\begin{aligned}
(\underline{f}' - f')' &= \frac{1}{2}\mathcal{N}(\underline{f}' - f') + \sum_{i\in\Lambda_A} a_i e^{-b_i\underline{f}+b_i(\underline{f}-f)}(1 - e^{-b_i(\underline{f}-f)}) \\
&\quad + \sum_{i\in\Lambda_A} a_i b_i \underline{f} - \sum_{i\in\Lambda_A} a_i(1 - e^{-b_i\underline{f}}) + \sum_{j\in\Lambda_B} \alpha_j e^{\beta_j t-\gamma_j\underline{f}+\gamma_j(\underline{f}-f)} \\
&\geq \frac{1}{2}\mathcal{N}(\underline{f}' - f') + \sum_{i\in\Lambda_A} a_i e^{-b_i f}(1 - e^{-b_i(\underline{f}-f)}).
\end{aligned}
$$

Thereby, we find that $f' - \overline{f}'$, $\underline{f}' - f'$ and $f - \overline{f}$, $\underline{f} - f$ are all monotone non-decreasing, and in particular, they are all non-negative for $t \geq 0$; furthermore, they are bounded by $\underline{f}' - \overline{f}'$ and $\underline{f} - \overline{f}$, respectively, for all $t \geq 0$, and in particular, none of these differences diverge at a finite time. Hence we obtain

$$
\begin{aligned}
&\underline{f}(t) \geq f(t) \geq \overline{f}(t), \\
&\underline{f}'(t) \geq f(t) \geq \overline{f}'(t)
\end{aligned}
$$

for all $t \geq 0$, thus concluding that $f(t)$ and $f'(t)$, which exist locally, do not diverge at a finite time for $t \geq 0$. Since $C_1 > 0$, we find that $\overline{f}(\infty) = \infty$ and $f(\infty) = \infty$, thereby completing the proof. □

Consequently, we have the following:

THEOREM 2.2. *There exists a one-parameter family of solutions to* (2) *satisfying* $\psi(0) = 0$, $\psi(\infty) = \infty$ *and* $\psi(\rho) > 0$ *for* $\rho > 0$ *when* $k - a < \mathcal{N}/2$ *or* $\Lambda_B = \emptyset$.

PROOF. By utilizing the theorem proved above, we can obtain a family of solutions for (2) as $\psi(\rho) = C^{-1}\rho^a \exp(f(\log(|\rho|)))$ parameterized by $f_0 > \sum_{j\in\Lambda_B} \alpha_j/\beta_j(\mathcal{N}/2 - \beta_j)$, if $\max_{j\in\Lambda_B}(\beta_j) < \mathcal{N}/2$ or $\Lambda_B = \emptyset$.

When $\Lambda_B \neq \emptyset$, since $\beta_j \in \{l - \lambda a \mid 1 \leq \lambda \leq k,\ 1 \leq \lambda \leq k'\}$, it follows that $\beta_j \leq k - a$; thereby, $k - a < \mathcal{N}/2$ implies that $\max_{j\in\Lambda_B}(\beta_j) < \mathcal{N}/2$. □

CHAPTER 3

Applications to the Dirichlet problem

When M and M' are k-term Carnot spaces, we can use the result from previous chapters to the Dirichlet problem at infinity for harmonic maps between them. Typical examples of k-term Carnot spaces are rank one symmetric spaces of noncompact type, which can be described as follows:

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{C}a$ denote real, complex, quaternion or Cayley number field. Let us set $d = \dim_{\mathbb{R}}(\mathbb{K})$ and $\mathrm{Im}(\mathbb{K}) = \{a - \bar{a} \mid a \in \mathbb{K}\}$. To begin with, we define N to be a Lie group whose underlying manifold is $\mathbb{K}^n \times \mathrm{Im}(\mathbb{K})$ with coordinate $(\mathbf{x}, \mathbf{t}) = (\mathbf{x}^1, \dots, \mathbf{x}^n, \mathbf{t})$, where the group law is given by

$$(\mathbf{x}, \mathbf{t}) \cdot (\mathbf{x}', \mathbf{t}') = (\mathbf{x} + \mathbf{x}', \mathbf{t} + \mathbf{t}' + 2\mathrm{Im}(\mathbf{x} \cdot \bar{\mathbf{x}}')).$$

When $\mathbb{K} = \mathbb{C}$, N is the Lie group called the Heisenberg group. The left translation of N by $(\mathbf{x}, \mathbf{t})$ shall be denoted as $\tau_{(\mathbf{x},\mathbf{t})}$. Next, we define $S = N \cdot \mathbb{R}_+$ to be a semidirect product of N and $\mathbb{R}_+$ given by the dilation $\rho \cdot (\mathbf{x}, \mathbf{t}) = (\rho^{1/2}\mathbf{x}, \rho\mathbf{t})$. Let $\tau_{\mathbf{s}}$ denote the left translation of $N \cdot \mathbb{R}_+$ by $\mathbf{s} = (\mathbf{x}, \mathbf{t}, \rho)$. Now we endow S with a left invariant metric g on S so that $M = (S, g)$ becomes a symmetric space, which is called a real, complex or quaternion hyperbolic space, denoted by $\mathbb{K}H^{n+1}$ where $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, or the Cayley hyperbolic plane denoted by $\mathbb{C}aH^2$.

In the following, we shall examine the explicit formula of the metric g on S in conjunction with the canonical generator of the Lie algebra of S when $(S, g) = \mathbb{C}H^{n+1}$ or $\mathbb{H}H^{n+1}$.

To begin with, let $\{\mathbf{e}^j\}_{j=1}^d$ denote the canonical generator of $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$ given respectively by $\mathbf{e}^1 = 1$ and $\mathbf{e}^2 = \sqrt{-1}$ when $\mathbb{K} = \mathbb{C}$, and $\mathbf{e}^1 = 1$, $\mathbf{e}^2 = \mathbf{i}$, $\mathbf{e}^3 = \mathbf{j}$, $\mathbf{e}^4 = \mathbf{k}$, $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ when $\mathbb{K} = \mathbb{H}$. Utilizing these, for the coordinate $(\mathbf{x}, \mathbf{t}) = (\mathbf{x}^1, \dots, \mathbf{x}^n, \mathbf{t})$ of N, we can express $\mathbf{x}^j$ as $\sum_{l=1}^d x^{jl}\mathbf{e}^l$ and $\mathbf{t}$ as $\sum_{l=2}^d t^l\mathbf{e}^l$. For $i > 0$ we let $\mathrm{Im}_i(\mathbf{x}^j) = x^{ji}$ denote the $\mathbf{e}^i$ component of $\mathbf{x}^j$. Then, the left invariant extensions in N of tangent vectors $\partial/\partial x^{jl}$ $(1 \le j \le n, 1 \le l \le d)$, $2\partial/\partial t^l$ $(2 \le l \le d)$ at $\mathbf{o} = (0, 0, 1) \in \mathbb{K}^n \times \mathrm{Im}(\mathbb{K}) \times \mathbb{R}_+$ can be computed as follows:

For $\mathbf{x}' \in \mathbb{K}^n$ which is of the form

$$\mathbf{x}' = (0, \ldots, 0, \underset{j\text{-th}}{\hat{\varepsilon \mathbf{e}^l}}, 0, \ldots, 0), \qquad \varepsilon \in \mathbb{R},$$

we can easily observe that

$$\begin{aligned} \tau_{(\mathbf{x},\mathbf{t})*}\big(\partial/\partial x^{jl}\big) f &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} f((\mathbf{x},\mathbf{t})\cdot(\mathbf{x}',0)) \\ &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} f(\mathbf{x}^1, \ldots, \mathbf{x}^j + \varepsilon\mathbf{e}^l, \ldots, \mathbf{x}^n, \mathbf{t} + 2\mathrm{Im}(\mathbf{x}^j \varepsilon \bar{\mathbf{e}}^l)) \\ &= \Big(\frac{\partial}{\partial x^{jl}} + 2\sum_{i=2}^{d} \mathrm{Im}_i(\mathbf{x}^j \bar{\mathbf{e}}^l)\frac{\partial}{\partial t^i}\Big) f \\ &=: e_{d(j-1)+l} f, \end{aligned}$$

$$\tau_{(\mathbf{x},\mathbf{t})*}\big(2\partial/\partial t^l\big) f = 2\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} f((\mathbf{x},\mathbf{t})\cdot(0,\varepsilon\mathbf{e}^l)) = 2\frac{\partial}{\partial t^l} f =: e_{dn+l-1} f.$$

Similarly, we observe that the left invariant extensions of $\partial/\partial x^{jl}$ $(1 \le j \le n, 1 \le l \le d)$, $2\partial/\partial t^l$ $(2 \le l \le d)$ and $2e_m = 2\partial/\partial\rho$ in $N \cdot \mathbb{R}_+$ are given, respectively, by

$$\begin{aligned} L_{d(j-1)+l} &= \rho^{1/2} e_{d(j-1)+l} \qquad (1 \le j \le n,\ 1 \le l \le d), \\ L_{dn+l-1} &= \rho e_{dn+l-1} \qquad (2 \le l \le d), \\ L_m &= 2\rho e_m. \end{aligned}$$

By utilizing these, we define $\mathfrak{n}_1$ and $\mathfrak{n}_2$ by

$$\mathfrak{n}_1 = \mathrm{Span}_{\mathbb{R}}\{L_{d(j-1)+l}\}_{1\le j\le n, 1\le l\le d}, \qquad \mathfrak{n}_2 = \mathrm{Span}_{\mathbb{R}}\{L_{nd+l-1}\}_{2\le l\le d}.$$

Then, for $H = L_m$, we have the following decomposition of the Lie algebra of S:

$$\mathfrak{s} = \mathbb{R}_+\{H\} + \mathfrak{n}_1 + \mathfrak{n}_2,$$

where $\mathfrak{n}_l = \{X \in \mathfrak{s} \mid [H,X] = lX\}$ $(l = 1,2)$. Furthermore, for $m = n_1 + n_2 + 1$, we have

$$[e_\alpha, e_\beta] = \sum_{\gamma=1}^{m-1} a_{\alpha\beta}^{\gamma} e_\gamma,$$

where $a_{\alpha\beta}^{\gamma} = 0$ unless $\alpha, \beta \in I_1$ $\gamma \in I_2$, and thereby it holds that $\mathfrak{n}_2 \subset [\mathfrak{n}, \mathfrak{n}]$.

Having obtained the explicit formula of the canonical generator of the Lie algebra $\mathfrak{s}$, we shall consider the metric g of S. Firstly, since S acts on $N \cdot \mathbb{R}_+$ transitively, an inner product $\langle \cdot, \cdot \rangle$ of the tangent space $T_{\mathbf{o}}(S)$ at $\mathbf{o} = (0,0,1) \in S$, define the left invariant metric g by assigning $g_{\mathbf{s}\cdot\mathbf{o}}(V, V') = \langle \tau_{\mathbf{s}*}^{-1} V, \tau_{\mathbf{s}*}^{-1} V' \rangle$ for $V, V' \in T_{\mathbf{s}\cdot\mathbf{o}}(S)$ at each $\mathbf{s} \in S$. At this point, by taking the inner product above as

$$\langle \cdot, \cdot \rangle = |d\mathbf{x}|^2 + |d\mathbf{t}|^2/4 + d\rho^2/4,$$

we see that $M = (S, g)$ gives rise to a symmetric space according to Lemma 4.1 below. Secondly, since L_j are left invariant (as defined by $\tau_{\mathbf{s}*}(L_j) = L_j$), we can see that $\tau_{\mathbf{s}*}^{-1}(L_j|_{\mathbf{s}\cdot\mathbf{o}}) = L_j|_{\mathbf{o}}$. Thirdly, noting that $\{L_j|_{\mathbf{o}}\}_{j=1}^m = \{\partial/\partial x^{11}, \dots, \partial/\partial x^{nd}, 2\partial/\partial t^2, \dots, 2\partial/\partial t^d, 2\partial/\partial\rho\}$ is an orthonormal basis for the inner product $\langle \cdot, \cdot \rangle$ (as defined above), we find that

$$g_{\mathbf{s}\cdot\mathbf{o}}(L_i|_{\mathbf{s}\cdot\mathbf{e}}, L_j|_{\mathbf{s}\cdot\mathbf{o}}) = g_{\mathbf{o}}(\tau_{\mathbf{s}*}^{-1}(L_i|_{\mathbf{s}\cdot\mathbf{o}}), \tau_{\mathbf{s}*}^{-1}(L_j|_{\mathbf{s}\cdot\mathbf{o}})) = \langle L_i|_{\mathbf{o}}, L_j|_{\mathbf{o}} \rangle = \delta_{ij},$$

which implies that $\{L_j\}$ is an orthonormal frame of g. Hence, the explicit formula of left invariant metric g as determined by $\langle \cdot, \cdot \rangle$ is finally expressed as follows:

$$\begin{aligned} g = \sum_{l=1}^{m} (L_l^*)^2 &= \frac{\sum_{l \in I_1} (e_l^*)^2}{\rho} + \frac{\sum_{l \in I_2} (e_l^*)^2}{\rho^2} + \frac{d\rho^2}{4\rho^2} \\ &= \frac{|d\mathbf{x}|^2}{\rho} + \frac{|d\mathbf{t} - 2\mathrm{Im}(\mathbf{x} \cdot d\bar{\mathbf{x}})|^2}{4\rho^2} + \frac{d\rho^2}{4\rho^2}. \end{aligned} \tag{19}$$

We have deduced this last equality by noting that

$$\begin{aligned} &(d\mathbf{t} - 2\mathrm{Im}(\mathbf{x} \cdot d\bar{\mathbf{x}}))\Big(\frac{\partial}{\partial x^{jl}} + 2\sum_{i=2}^{d} \mathrm{Im}_i(\mathbf{x}^j \bar{\mathbf{e}}^l) \frac{\partial}{\partial t^i}\Big) \\ &= \ 2\sum_{i=2}^{d} \mathrm{Im}_i(\mathbf{x}^j \bar{\mathbf{e}}^l)\mathbf{e}^i - 2\mathrm{Im}(\mathbf{x}^j \bar{\mathbf{e}}^l) = 0. \end{aligned}$$

Consequently, for $g_{ij} = g(e_i, e_j)$, we have

$$\begin{aligned} g_{ll} &= \rho^{-1} \quad (l \in I_1), \quad & g_{ll} &= \rho^{-2} \quad (l \in I_2), \\ g_{mm} &= \rho^{-2} 4^{-1}, & g_{ij} &= 0 \quad (i \neq j). \end{aligned}$$

At this point it should be noted that $\mathfrak{n}_1 = \mathrm{Ker}(d\mathbf{t} - 2\mathrm{Im}(\mathbf{x} \cdot d\bar{\mathbf{x}}))$ defines a codimension $d-1$ distribution in $\mathfrak{n}$. When $\mathbb{K} = \mathbb{C}$, this distribution and $d\mathbf{t} - 2\mathrm{Im}(\mathbf{x} \cdot d\bar{\mathbf{x}})$ are also called a contact structure and a contact form of N, respectively.

Definition 3.1. When $n_1, n_1', n_2, n_2' > 0$, a map $h \colon N \to N'$ is said to be a *filtration preserving* map if the components of the differential map of h given by $h_*(e_j) = \sum_{\gamma=1}^{m'-1} h_j^\gamma e_\gamma'$ satisfy

$$h_j^\gamma = 0 \text{ for } j \in I_1, \gamma \in I_2'. \tag{20}$$

DEFINITION 3.2. Suppose $n_2, n'_2 > 0$. A map $h\colon N \to N'$ is said to be *non-degenerate* if

$$\sum_{j\in I_2,\gamma\in I'_2} (h_j^\gamma)^2 > 0 \text{ on } N.$$

h is said to be degenerate if the left hand side vanishes at a point on N. A proper harmonic map from $N\cdot\mathbb{R}_+$ to $N'\cdot\mathbb{R}_+$ is said to be non-degenerate if its boundary value is non-degenerate.

A non-degenerate filtration preserving map $h\colon N \to N'$ is referred to as a *contact* map when both $N\cdot\mathbb{R}_+$ and $N'\cdot\mathbb{R}_+$ are complex hyperbolic spaces of the same dimension.

LEMMA 3.1. *Let $N\cdot\mathbb{R}_+$ and $N'\cdot\mathbb{R}_+$ be rank one symmetric spaces of noncompact type. Suppose that $h\colon N \to N'$ is any Lie group homomorphism satisfying*

$$0 = \sum_{j\in I_1,\beta\in I'_1,\gamma\in I'_2} {a'}^\gamma_{\beta\alpha} h_j^\beta h_j^\gamma \qquad \text{for any } \alpha \in I'_1. \tag{21}$$

Then, there exists a family of harmonic maps which assume h on the boundary N, except in the cases when $n_1 = 1, k = 1, a = 1/2$ and $n_1 = 0, n_2 = 1, k = 2, a = 1$, where $a = \min\{l/\lambda \mid \sum_{j\in I_l}\sum_{\gamma\in I'_\lambda}(h_j^\gamma)^2 > 0,\ \ 1 \le l \le k,\ 1 \le \lambda \le k'\}$ as in (8).

PROOF. To begin with, from the definition of a, we find that $a = 1/2, 1, 2$ when both the domain and range are rank one symmetric spaces of noncompact type. Consequently, we obtain $k - a < \mathcal{N}/2 = n_1/2 + n_2$ except in the cases when $n_1 = 1, k = 1, a = 1/2$ and $n_1 = 0, n_2 = 1, k = 2, a = 1$. [The case $n_1 = 0, n_2 = 1, k = 2, a = 1/2$ does not occur according to the definition of a.]

Next, since h is a Lie group homomorphism, the differential map h_* preserves the Lie bracket product, thus it maps the center $\mathfrak{n}_2$ into $\mathfrak{n}'_2$. This implies that $h_j^\gamma = 0$ for $j \in I_2$ and $\gamma \in I'_1$ except in the case when $n_1 = 0, n_2 = 1, k = 2$. Consequently, we have

$$0 = \sum_{j\in I_2,\beta\in I'_1,\gamma\in I'_2} {a'}^\gamma_{\beta\alpha} h_j^\beta h_j^\gamma \qquad \text{for any } \alpha \in I'_1,$$

thereby, we can reduce the condition (3) into

$$0 = \sum_{j\in I_1,\beta\in I'_1,\gamma\in I'_2} {a'}^\gamma_{\beta\alpha} h_j^\beta h_j^\gamma \qquad \text{for any } \alpha \in I'_1.$$

Moreover, since h is a Lie group homomorphism, the components of the differential map of h are constant, see, e.g., Donnelly [**8**, Proposition 3.1], and thereby (4) follows automatically.

Now utilizing Lemma 1.1 in conjunction with Theorem 2.2 we have completed the proof. □

When h is a filtration preserving homomorphism, the condition (21) follows automatically and $a = 1$. Therefore Lemma 3.1 implies the following:

THEOREM 3.1. *Let M and M' be rank one symmetric spaces of noncompact type. Suppose that $h\colon N \to N'$ is any nontrivial filtration preserving homomorphism. Then, there exists a family of harmonic maps which assume h on the boundary $\partial M \cap N$.*

We shall now illustrate how one can apply our Lemma 3.1 and Theorem 3.1 in the following examples:

Example 1. The identity map of N preserves the filtration; thereby we can construct harmonic maps that induce the identity map on the boundary:

$$u\colon (\mathbf{x}, \mathbf{t}, \rho) \to (\mathbf{x}, \mathbf{t}, \psi(\rho)).$$

The case where $M = M' = \mathbb{R}H^n$ was studied by Economakis [**9**] but his approach was different from our's, which will be provided in Chapter 2 and again in Chapter 5 with a slight modification to that of Chapter 2. Their Hölder regularity will be discussed in Chapters 6 and 7.

Example 2. More generally, we have the following examples of harmonic maps that assume non-degenerate filtration preserving homomorphisms on the boundary:

$$u : (\mathbf{x}, \mathbf{t}, \rho) \to (C\mathbf{x}, C^2\mathbf{t}, \psi(\rho)).$$

Example 3. We can construct a harmonic map $u\colon \mathbb{K}H^n \to \mathbb{R}H^{n'}$ $(n' \leq n)$ of the following form:

$$u\colon (\mathbf{x}^1, \dots, \mathbf{x}^n, \mathbf{t}, \rho) \to (x^{11}, x^{21}, \dots, x^{n'1}, \psi(\rho)).$$

Example 4. Let us consider a degenerate filtration preserving map between the boundaries of $M = \mathbb{K}H^n$ and $M = \mathbb{K}H^{n'}$ $(n' \leq n)$ given by

$$h_*(e_{d(j-1)+1}) = e'_{d(j-1)+1'},\ h_*(e_{d(j-1)+1}) = 0 \quad (2 \leq l \leq d, 1 \leq j \leq n'),$$
$$h * (e_{dn+l-1}) = 0 \ (2 \leq l \leq d).$$

This clearly preserves the Lie bracket product and hence is a degenerate filtration preserving homomorphism of the Lie algebra. By using this h, we can construct a harmonic map u of the form:

$$u\colon (\mathbf{x}, \mathbf{t}, \rho) \to (h(\mathbf{x}, \mathbf{t}), \psi(\rho)).$$

We can also consider a filtration preserving map given by

$$h_*(e_{d(j-1)+1}) = e'_{d(j-1)+1} \;\; (1 \leq j \leq n'), \qquad h_*(e_{dn+l-1}) = 0 \;\; (2 \leq l \leq d).$$

This is another example of a degenerate Lie homomorphism that preserves the filtration. Utilizing this homomorphism h, we obtain the harmonic maps of the form:

$$u\colon (\mathbf{x}, \mathbf{t}, \rho) \to (h(\mathbf{x}, \mathbf{t}), \psi(\rho)).$$

Example 5. A map between the boundaries of $M = \mathbb{K}H^n$ and $M = \mathbb{K}H^{n'}$ $(n' \leq n)$

$$h\colon (\mathbf{x}^1, \dots, \mathbf{x}^n, \mathbf{t}) \to (\zeta_1 \sum_{l=1}^{d} x^{1l}, \dots, \zeta_{n'} \sum_{l=1}^{d} x^{n'l}, 0) \qquad (\zeta_1, \dots, \zeta_{n'} \in \mathbb{K})$$

is also an example of a degenerate filtration preserving Lie homomorphism. Therefore we can utilize Theorem 3.1 to construct its harmonic extensions.

Example 6. We have an example of harmonic maps u assuming a filtration preserving map on the boundary in the case where the dimension of the target is greater than that of the source:

$$u\colon (\mathbf{x}, \mathbf{t}, \rho) \to (\mathbf{x}, \mathbf{x}, \mathbf{t}, \psi(\rho)).$$

Example 7. Let us consider a homomorphism that does not preserve the filtration of the boundary of complex hyperbolic spaces given by

$$h : (\mathbf{x}^1, \dots, \mathbf{x}^n, t) \to (0, \dots, 0, x^{11}).$$

This map satisfies $h_1^{n'_1+1} = 1$ and $a = 1/2$. Hence we can utilize Lemma 3.1 to construct harmonic maps inducing h on the boundary. The images of these harmonic maps are contained in $\mathbb{C}H^1$'s which are totally geodesic submanifolds in $\mathbb{C}H^{n'}$. This example was inspired by Donnelly [**8**] obtained through a suitable composition of homomorphisms: $\mathfrak{n} \to \mathfrak{n}_1 \to \mathfrak{n}$. Donnelly's approach [**8**] to prove the existence of harmonic maps is analogous to that of Economakis. In his construction of harmonic maps, Donnelly [**8**] assumed that h does not preserve the filtration, namely that $c_{12} \neq 0$ (cf. (6)); but, our construction in Lemma 3.1 does not assume this and we can therefore obtain harmonic maps that induce filtration preserving homomorphisms as well. Furthermore, Donnelly provided other homomorphisms that do not preserve the filtration in [**8**] and our construction also works when we utilize these boundary values.

CHAPTER 4

The Cayley transform

In this chapter, we review some facts from hyperbolic geometry. In the following, $\mathbb{K}$ denotes $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

1. The homogeneous model of $\mathbb{K}H^n$

Let $V^{1,n}(\mathbb{K})$ be the vector space $\mathbb{K}^{n+1}$ together with the unitary structure given by

$$\Phi(z, w) = -\bar{z}_0 w_0 + \bar{z}_1 w_1 + \cdots + \bar{z}_n w_n.$$

The group $G = SO(1, n)$, $SU(1, n)$ or $Sp(1, n)$ is a subgroup of $SL(n+1, \mathbb{K})$ which satisfies

$$\Phi(g(z), g(w)) = \Phi(z, w) \qquad \forall g \in G.$$

Define V_{-1} by

$$V_{-1} = \{\zeta = (\zeta^0, \ldots, \zeta^n) \in \mathbb{K}^{n+1} | \Phi(\zeta, \zeta) = -1\}.$$

A projection map $P\colon V_{-1} \to V_{-1}/\sim$ is induced by the following equivalence relation:

$$\zeta \sim \zeta' \text{ if and only if there exists a } \lambda \in \mathbb{K} \setminus \{0\} \text{ such that } \zeta = \zeta'\lambda.$$

Since $\Phi(\zeta, \zeta) = -1 < 0$ implies that

$$|\zeta^1|^2 + \cdots + |\zeta^n|^2 < |\zeta^0|^2,$$

we have $|\zeta^0|^2 \neq 0$. Hence, $P(V_-)$ is identified with

$$B^n_{\mathbb{K}} = \left\{ w = (w^1, \ldots, w^n) \in \mathbb{K}^n \;\Big|\; \sum_{j=1}^n |w^j|^2 < 1 \right\}$$

by assigning $[\zeta] \in P(V_-)$ to $w \in B^n_{\mathbb{K}}$, where $w^j = \zeta^j(\zeta^0)^{-1}$, $j = 1, \cdots, n$. Then, in the coordinate representation, the map $P\colon V_{-1} \to B^n_{\mathbb{K}}$ is given by

$$P(\zeta) = w, \quad w^j = \zeta^j(\zeta^0)^{-1}.$$

2. The Cayley transform as determined by $\pm k_n \in \partial\mathbb{K}H^n$

Let $\{k_1, \dots, k_n\}$ and $\{e_0, \dots, e_n\}$ be the standard basis of $P(\mathbb{K}^{n+1})$ and $\mathbb{K}^{n+1}$, respectively. An element g in G is said to be parabolic if g leaves exactly one point on the boundary fixed.

Firstly, we shall focus on the particular boundary point $k_n = (0, \dots, 0, 1) \in \partial\mathbb{K}H^n$. Since $k_n = P(e_0+e_n)$, an element $g \in G$ leaves k_n fixed if and only if $g(e_0+e_n) = (e_0+e_n)\lambda$ for $\lambda \in \mathbb{K} \setminus \{0\}$. Because of this, in order to examine the parabolic subgroup, a different basis $\hat{e}_j = \sum_{i=1}^n e_i d_{ij}$ which contains a multiple of $e_0 + e_n$ is often of use. Following this, we shall change the standard basis as follows:

$$\begin{aligned} \hat{e}_0 &= (e_0 - e_n)/\sqrt{2}, \\ \hat{e}_n &= (e_0 + e_n)/\sqrt{2}, \\ \hat{e}_j &= e_j, \quad 1 \leq j \leq n-1, \end{aligned}$$

which is provided, in the form of a matrix, by

$$D = \begin{pmatrix} 1/\sqrt{2} & & -1/\sqrt{2} \\ & E_{n-1} & \\ 1/\sqrt{2} & & 1/\sqrt{2} \end{pmatrix},$$

where E_{n-1} is the identity matrix of the degree $n-1$. The linear transformation $C = D^{-1}$ or the projective transformation which it induces is called a Cayley transform. In the coordinate representation, C is given by

$$C\colon (\zeta^0, \zeta^1, \dots, \zeta^{n-1}, \zeta^n) \to \big((\zeta^0 - \zeta^n)/\sqrt{2}, \zeta^1, \dots, \zeta^{n-1}, (\zeta^0 + \zeta^n)/\sqrt{2})\big).$$

Viewed as a projective transformation,

$$\begin{aligned} \eta^j &= -\sqrt{2}w^j(1-w^n)^{-1}, \\ \eta^n &= (1+w^n)(1-w^n)^{-1}, \end{aligned}$$

which maps an open ball

$$B^n_{\mathbb{K}} = \Big\{ w \in \mathbb{K}^n \;\Big|\; \sum_{k=1}^n |w^k|^2 < 1 \Big\}$$

to the Siegel domain of type II

$$\Sigma = \Big\{ \eta \in \mathbb{K}^n \;\Big|\; \mathrm{Re}(\eta^n) > \sum_{j=1}^{n-1} |\eta^j|^2/2 \Big\}.$$

Our convention in the case of $\mathbb{K} = \mathbb{C}$ differs from that of Graham's [**12**, p. 444]. Note that his z^n is $\sqrt{-1}\eta^n$, z^j is $\sqrt{2}\eta^j$ and his w is $-w$, respectively. Following Graham's change of variable [**12**, p. 444], setting a new coordinate $(\mathbf{x}, \mathbf{t}, \rho)$ by

$$\mathbf{x}^j = \eta^j/\sqrt{2}, \quad \rho = \mathrm{Re}(\eta^n) - \textstyle\sum_{j=1}^{n-1}|\eta^j|^2/2, \quad -\mathbf{t} = \mathrm{Im}(\eta^n), \tag{22}$$

we obtain a diffeomorphism from $B^n_{\mathbb{K}}$ to $\mathbb{K}^{n-1} \times \mathrm{Im}(\mathbb{K}) \times \mathbb{R}_+$ given as follows:

$$\begin{aligned} \mathbf{x}^j &= -w^j(1-w^n)^{-1}, \\ -\mathbf{t} + \rho &= (1+w^n)(1-w^n)^{-1} - \sum_{j=1}^{n-1}|w^j|^2|1-w^n|^{-2}. \end{aligned} \tag{23}$$

At this point it should be noted that the group $\hat{G} = D^{-1}GD$ preserves $D^{-1}(V_{-1})$, thereby the action of G on $B^n_{\mathbb{K}}$ is converted by the Cayley transform C into the action of $\hat{G}$ in Σ. Furthermore, C maps k_n to the point $\infty = C(k_n) \in \partial\Sigma$. Thus, the isotropy group of k_n in G corresponds to that of ∞ in $\hat{G}$.

Secondly, we shall focus on another boundary point $-k_n = (0, \dots, 0, -1) \in \partial\mathbb{K}H^n$. Noting that $P(-e_0 + e_n) = -k_n$, we can perform the same computation by replacing k_n with $-k_n$. In the coordinate representation, we have

$$C\colon (\zeta^0, \zeta^1, \dots, \zeta^{n-1}, \zeta^n) \to \Big(\big((-\zeta^0) - \zeta^n\big)/\sqrt{2}, \zeta^1, \dots, \zeta^{n-1}, \big((-\zeta^0) + \zeta^n\big)/\sqrt{2}\Big).$$

Viewed as a projective transformation,

$$\begin{aligned} \eta^j &= -\sqrt{2}(-w^j)\big(1-(-w^n)\big)^{-1}, \\ \eta^n &= \big(1+(-w^n)\big)\big(1-(-w^n)\big)^{-1}. \end{aligned}$$

If we set a new coordinate $(\mathbf{x}, \mathbf{t}, \rho)$ according to (22) again, then the formula above is given in the following form:

$$\begin{aligned} \mathbf{x}^j &= -(-w^j)\big(1-(-w^n)\big)^{-1}, \\ -\mathbf{t} + \rho &= \big(1+(-w^n)\big)\big(1-(-w^n)\big)^{-1} - \sum_{j=1}^{n-1}|w^j|^2|1-(-w^n)|^{-2}. \end{aligned} \tag{24}$$

Thus, we constructed Cayley transforms as determined by the two boundary points k_n and $-k_n$. We shall denote these as Ψ_{k_n} and Ψ_{-k_n}, respectively. They satisfy $\Psi_{k_n} = (-\mathrm{Id})\circ\Psi_{-k_n}$. Hence, we have the following boundary charts of the compactification $\overline{B^n_{\mathbb{K}}} =$

$B^n_{\mathbb{K}} \cup S^{n_1+n_2}$ ($S^{n_1+n_2}$ being a n_1+n_2 dimensional sphere) given by the Cayley transforms Ψ_{k_n} and Ψ_{-k_n}:

$$\begin{aligned} \Psi_{k_n} \ &: \mathbb{R}^{n_1+n_2} \times \mathbb{R}_{\geq 0} \to \overline{B}^n_{\mathbb{K}} \setminus \{k_n\}, \\ \Psi_{-k_n} &: \mathbb{R}^{n_1+n_2} \times \mathbb{R}_{\geq 0} \to \overline{B}^n_{\mathbb{K}} \setminus \{-k_n\}. \end{aligned} \tag{25}$$

3. A linear fractional transformation

$G = SO(1,n)_o,\ SU(1,n)$, or $Sp(1,n)$ acts on $P(V_-) = B^n_{\mathbb{K}}$, as linear fractional transformations $B^n_{\mathbb{K}} \ni w \to s(w) \in B^n_{\mathbb{K}}$ given by

$$w^i{\circ}s(w) = (s_{i0} + \sum_{j=1}^{n} s_{ij}w^j)(s_{00} + \sum_{j=1}^{n} s_{0j}w^j)^{-1} \text{ for } s = (s_{ij}) \in G.$$

There is an Iwasawa decomposition $G = KAN$, where K coincides with a stabilizer subgroup of G that leaves the origin of $P(V_-)$ fixed. Noting that $S = NA$ is diffeomorphic to $G/K \cong P(V_-)$, we shall examine the Lie algebra $\mathfrak{s}$ of S. The $\mathfrak{a}$-gradation of $\mathfrak{s}$ is given by $\mathfrak{s} = \mathbb{R}\{\mathbf{H}\} + \mathfrak{n}_1 + \mathfrak{n}_2$,

$$\mathbf{H} = \begin{pmatrix} & & 1 \\ & & \\ 1 & & \end{pmatrix}, \ \mathbf{X}_l = \begin{pmatrix} & -\bar{\mathbf{x}}^l & \\ -\mathbf{x}^l & & \mathbf{x}^l \\ & -\bar{\mathbf{x}}^l & \end{pmatrix}, \ \mathbf{T} = \frac{1}{2}\begin{pmatrix} -\mathbf{t} & & \mathbf{t} \\ & & \\ -\mathbf{t} & & \mathbf{t} \end{pmatrix},$$

$$\begin{aligned} \mathfrak{n}_1 &= \Big\{ \sum_{l=1}^{n-1} \mathbf{X}_l \ \Big| \ (\mathbf{x}^1, \dots, \mathbf{x}^{n-1}) \in \mathbb{K}^{n-1} \Big\}, \quad \mathfrak{n}_2 = \big\{ \mathbf{T} \ \big| \ \mathbf{t} \in \mathrm{Im}(\mathbb{K}) \big\}, \\ \mathfrak{n}_i &= \Big\{ X \in \mathfrak{s} \ \Big| \ \mathrm{ad}(\mathbf{H})X = iX \Big\} \quad (i = 1, 2), \end{aligned}$$

where, each $\mathbf{X}_l$ has four entries depending on $\mathbf{x}^j$, $-\mathbf{x}^j$ and $-\bar{\mathbf{x}}^j$ placed in the $(l+1)$-st column and $(l+1)$-st row and the other entries being zero. Since any element of $\mathfrak{n}$ is provided as linear combination $\sum_{l=1}^{n-1} \mathbf{X}^l + \mathbf{T}$, we can express each element of S [being defined as $(s_{ij}) = \exp(\sum_{l=1}^{n-1} \mathbf{X}^l + \mathbf{T}) \exp(s\mathbf{H})$] as follows:

$$(s_{ij}) = \begin{pmatrix} \mathrm{Ch}(s) + e^{-s}(|\mathbf{x}|^2 - \mathbf{t})/2 & -\bar{\mathbf{x}}^1 & \cdots & -\bar{\mathbf{x}}^{n-1} & \mathrm{Sh}(s) + s^{-s}(-|\mathbf{x}|^2 + \mathbf{t})/2 \\ -e^{-s}\mathbf{x}^1 & & & & e^{-s}\mathbf{x}^1 \\ \cdots & 1 & & & \cdots \\ \cdots & & & 1 & \cdots \\ -e^{-s}\mathbf{x}^{n-1} & & & & e^{-s}\mathbf{x}^{n-1} \\ \mathrm{Sh}(s) + e^{-s}(|\mathbf{x}|^2 - \mathbf{t})/2 & -\bar{\mathbf{x}}^1 & \cdots & -\bar{\mathbf{x}}^{n-1} & \mathrm{Ch}(s) + s^{-s}(-|\mathbf{x}|^2 + \mathbf{t})/2 \end{pmatrix}.$$

Recognizing that an action of an element of S to $B^n_{\mathbb{K}}$ is a linear fractional transformation, we can observe that (s_{ij}) maps the origin of $B^n_{\mathbb{K}}$ to

$$\begin{aligned} w^l &= -e^{-s}\mathbf{x}^l\big(\mathrm{Ch}(s)+e^{-s}(|\mathbf{x}|^2-\mathbf{t})/2\big)^{-1} \\ &= -2\mathbf{x}^l\big(e^{2s}+1+|\mathbf{x}|^2-\mathbf{t}\big)^{-1}, \\ w^n &= \big(\mathrm{Sh}(s)+e^{-s}(|\mathbf{x}|^2-\mathbf{t})/2\big)\big(\mathrm{Ch}(s)+e^{-s}(|\mathbf{x}|^2-\mathbf{t})/2\big)^{-1} \\ &= \big(e^{2s}-1+|\mathbf{x}|^2-\mathbf{t}\big)\big(e^{2s}+1+|\mathbf{x}|^2-\mathbf{t}\big)^{-1}. \end{aligned}$$

Hence, by substituting ρ for e^{2s}, we obtain a diffeomorphism $M \ni (\mathbf{x},\mathbf{t},\rho) \mapsto (w^1,\dots,w^n) \in P(V_-)$ given by

$$\begin{aligned} w^l &= -2\mathbf{x}^l\big(|\mathbf{x}|^2-\mathbf{t}+\rho+1\big)^{-1} \quad (1 \le l \le n-1), \\ w^n &= \big(|\mathbf{x}|^2-\mathbf{t}+\rho-1\big)\big(|\mathbf{x}|^2-\mathbf{t}+\rho+1\big)^{-1}. \end{aligned}$$

Since the point ∞ determined by $\mathbb{R}_+$ directions is mapped to $k_n \in \partial B^n_{\mathbb{K}}$, this diffeomorphism shall be denoted by Φ_{k_n}. Recognizing that

$$\begin{aligned} 1-w^n &= 2\big(|\mathbf{x}|^2-\mathbf{t}+\rho+1\big)^{-1}, \\ 1+w^n &= 2\big(|\mathbf{x}|^2-\mathbf{t}+\rho\big)\big(|\mathbf{x}|^2-\mathbf{t}+\rho+1\big)^{-1}, \end{aligned}$$

the inverse of this diffeomorphism is given by

$$\begin{aligned} \mathbf{x}^l &= -w^l\big(1-w^n\big)^{-1}, \\ -\mathbf{t}+\rho &= \big(1+w^n\big)\big(1-w^n\big)^{-1} - \sum_{l=1}^{n-1}|w^l|^2\big|1-w^n\big|^{-2}. \end{aligned}$$

At this point, it should be noted that this diffeomorphism is identical to the Cayley transform (23). By setting $\Phi_{-k_n} = (-\mathrm{Id})\circ\Phi_{k_n}$, we have boundary charts of the compactification $\overline{B^n_{\mathbb{K}}} = B^n_{\mathbb{K}} \cup S^{n_1+n_2}$ given by

$$\begin{aligned} \Phi_{k_n} &: \mathbb{R}^{n_1+n_2}\times\mathbb{R}_{\ge 0} \to \overline{B}^n_{\mathbb{K}} \setminus \{k_n\}, \\ \Phi_{-k_n} &: \mathbb{R}^{n_1+n_2}\times\mathbb{R}_{\ge 0} \to \overline{B}^n_{\mathbb{K}} \setminus \{-k_n\}. \end{aligned} \tag{26}$$

As we see from Chen and Greenberg [**5**, Proposition 2.3.1], the left invariant metric on $P(V_-) = B^n_{\mathbb{K}}$ is $dw{\cdot}d\bar{w}$ at the origin $\mathbf{0} = (0,\dots,0) \in B^n_{\mathbb{K}}$. Furthermore, we can verify the following

LEMMA 4.1.

$$d(w\circ\Psi_{k_n}){\cdot}\overline{d(w\circ\Psi_{k_n})}|_{\mathbf{o}} = d(w\circ\Psi_{-k_n}){\cdot}\overline{d(w\circ\Psi_{-k_n})}|_{\mathbf{o}} = d\mathbf{x}{\cdot}d\bar{\mathbf{x}} + d\mathbf{t}d\bar{\mathbf{t}}/4 + d\rho^2/4$$

for $\mathbf{o} = (0,0,1) = \Psi_{k_n}^{-1}(\mathbf{0})$. *This determines the canonical left invariant metric* g *so that* $(N \cdot \mathbb{R}_+, g)$ *is a symmetric space.*

Since the geodesic symmetry of the ball $B_{\mathbb{K}}^n$ at the origin $\mathbf{0} \in B_{\mathbb{K}}^n$ is $B_{\mathbb{K}}^n \ni w \mapsto -w \in B_{\mathbb{K}}^n$, the geodesic symmetry of $N \cdot \mathbb{R}_+$ at $\mathbf{o} = (0,0,1) = \Phi_{k_n}^{-1}(\mathbf{0}) = \Phi_{-k_n}^{-1}(\mathbf{0}) \in N \cdot \mathbb{R}_+$ is provided by

$$\sigma = \Phi_{k_n}^{-1} \circ \Phi_{-k_n} = \Phi_{k_n}^{-1} \circ (-\mathrm{Id}) \circ \Phi_{k_n} : \mathbb{R}^{n_1+n_2} \times \mathbb{R}_{\geq 0} \setminus \{(0,0)\} \to \mathbb{R}^{n_1+n_2} \times \mathbb{R}_{\geq 0} \setminus \{(0,0)\}.$$

This will be the coordinate transformation of the coordinate system as given in (25). Regarding the estimation in Proposition 6.3, it should be noted that $\sigma^*(g) = g$, since $w \mapsto -w$ is an isometry of $(B_{\mathbb{K}}^n, g_B)$ and it holds that $g = \Psi_{k_n}^*(g_B) = \Psi_{-k_n}^*(g_B)$.

Lemma 4.2. *The explicit formula of* σ *is given as follows*:

$$\begin{aligned} \mathbf{x}^l \circ \sigma &= -\mathbf{x}^l \big(|\mathbf{x}|^2 - \mathbf{t} + \rho\big)^{-1}, \\ -\mathbf{t} \circ \sigma + (\rho \circ \sigma) &= (\mathbf{t} + \rho)\big||\mathbf{x}|^2 + \mathbf{t} + \rho\big|^{-2}. \end{aligned}$$

Proof. Recognizing that

$$\begin{aligned} (1+w^n)(1-w^n)^{-1} &= \big(1+w^n\big)\big(1-\overline{w}^n\big)\big(1-\overline{w}^n\big)^{-1}(1-w^n)^{-1} \\ &= \big(1-\overline{w}^n + w^n + |w^n|^2\big)|1-w^n|^{-2}, \\ (1-w^n)^{-1}(1+w^n) &= (1-w^n)^{-1}\big(1-\overline{w}^n\big)^{-1}\big(1-\overline{w}^n\big)(1+w^n) \\ &= |1-w^n|^{-2}\big(1-\overline{w}^n + w^n + |w^n|^2\big), \end{aligned}$$

we obtain

$$|\mathbf{x}|^2 - \mathbf{t} + \rho = (1+w^n)(1-w^n)^{-1} = (1-w^n)^{-1}(1+w^n),$$

and thereby

$$\big(|\mathbf{x}|^2 - \mathbf{t} + \rho\big)^{-1} = (1-w^n)(1+w^n)^{-1} = (1+w^n)^{-1}(1-w^n).$$

Utilizing the above, we find that

$$\begin{aligned}
\mathbf{x}^l\circ\sigma &= -(-w^l)\big(1-(-w^n)\big)^{-1} \\
&= -(-w^l)\big(1-w^n\big)^{-1}\big(1-w^n\big)\big(1+w^n\big)^{-1} \\
&= -\mathbf{x}^l\big(|\mathbf{x}|^2-\mathbf{t}+\rho\big)^{-1}, \\
-\mathbf{t}\circ\sigma+\rho\circ\sigma &= \big(1-(-w^n)\big)^{-1}\big(1+(-w^n)\big)-|\mathbf{x}\circ\sigma|^2 \\
&= \big(1+w^n\big)^{-1}\big(1-w^n\big)-|\mathbf{x}\circ\sigma|^2 \\
&= \big(|\mathbf{x}|^2-\mathbf{t}+\rho\big)^{-1}-|\mathbf{x}\circ\sigma|^2 \\
&= (\mathbf{t}+\rho)\big||\mathbf{x}|^2+\mathbf{t}+\rho\big|^{-2}.
\end{aligned}$$

□

Lemma 4.3. *Defining the Heisenberg inversion as* $\tilde{\sigma} := \sigma|_{\rho=0}$*, when* $\mathbb{K}=\mathbb{C}$*, we have*

$$\tilde{\sigma}^*(d\mathbf{t}-2\mathrm{Im}(\mathbf{x}\cdot d\bar{\mathbf{x}})) = (d\mathbf{t}-2\mathrm{Im}(\mathbf{x}\cdot d\bar{\mathbf{x}}))||\mathbf{x}|^2-\mathbf{t}|^{-2}.$$

In particular, on the overlapping region $S^{n_1+n_2}\setminus\{k_n,-k_n\}$*,* $\mathrm{Ker}(d\mathbf{t}-2\mathrm{Im}(\mathbf{x}\cdot d\bar{\mathbf{x}}))\subset T(S^{n_1+n_2}\setminus\{k_n\})$ *on one chart given in* (25) *coexists with* $\mathrm{Ker}(d\mathbf{t}-2\mathrm{Im}(\mathbf{x}d\cdot\bar{\mathbf{x}}))\subset T(S^{n_1+n_2}\setminus\{-k_n\})$ *on the other.*

Proof. Note that

$$\mathbf{x}\circ\tilde{\sigma} = -\mathbf{x}\big(|\mathbf{x}|^2-\mathbf{t}\big)^{-1}, \qquad \mathbf{t}\circ\tilde{\sigma} = \mathbf{t}\big||\mathbf{x}|^2-\mathbf{t}\big|^{-2}.$$

Since $d\big||\mathbf{x}|^2-\mathbf{t}\big|^2 = d|\mathbf{x}|^4+d|\mathbf{t}|^2$, we have

$$d\mathbf{t}\circ\tilde{\sigma} = -d\mathbf{t}\big||\mathbf{x}|^2-\mathbf{t}\big|^{-2}+\mathbf{t}\big(d|\mathbf{x}|^4+d|\mathbf{t}|^2\big)\big||\mathbf{x}|^2-\mathbf{t}\big|^{-4}. \tag{27}$$

Next, we obtain

$$d\mathbf{x}\circ\tilde{\sigma} = -d\mathbf{x}\big(|\mathbf{x}|^2-\mathbf{t}\big)^{-1}+\mathbf{x}\big(d|\mathbf{x}|^2+d\mathbf{t}\big)\big(|\mathbf{x}|^2-\mathbf{t}\big)^{-2},$$

and furthermore

$$d\overline{\mathbf{x}\circ\tilde{\sigma}} = -\big(|\mathbf{x}|^2-\mathbf{t}\big)^{-1}d\overline{\mathbf{x}}+\big(|\mathbf{x}|^2+\mathbf{t}\big)^{-2}\big(d|\mathbf{x}|^2+d\mathbf{t}\big)\overline{\mathbf{x}}.$$

By using $(|\mathbf{x}|^2+\mathbf{t})^{-1} = \big||\mathbf{x}|^2-\mathbf{t}\big|^{-2}\big(|\mathbf{x}|^2-\mathbf{t}\big)$, we have

$$\mathbf{x}\circ\tilde{\sigma}\cdot d\overline{\mathbf{x}\circ\sigma} = \mathbf{x}\cdot d\overline{\mathbf{x}}\big||\mathbf{x}|^2-\mathbf{t}\big|^{-2}-\mathbf{x}\cdot\big||\mathbf{x}|^2-\mathbf{t}\big|^{-4}\big(|\mathbf{x}|^2-\mathbf{t}\big)\big(d|\mathbf{x}|^2+d\mathbf{t}\big)\overline{\mathbf{x}}.$$

In order to compute $\mathrm{Im}(\mathbf{x}\circ\tilde{\sigma}\cdot d\overline{\sigma\circ\tilde{\sigma}})$, we must observe that

$$\begin{aligned}
&2\mathrm{Im}\big(-\mathbf{x}\cdot(|\mathbf{x}|^2-\mathbf{t})(d|\mathbf{x}|^2+d\mathbf{t})\overline{\mathbf{x}}\big)\\
&= \mathbf{x}\cdot\big(|\mathbf{x}|^2-\mathbf{t}\big)\big(d|\mathbf{x}|^2+d\mathbf{t}\big)\overline{\mathbf{x}}-\mathbf{x}\cdot\big(d|\mathbf{x}|^2-d\mathbf{t}\big)\big(|\mathbf{x}|^2+\mathbf{t}\big)\overline{\mathbf{x}}\\
&= \mathbf{x}\cdot\big(2|\mathbf{x}|^2d\mathbf{t}-2\mathbf{t}d|\mathbf{x}|^2-\mathbf{t}d\mathbf{t}+d\mathbf{t}\mathbf{t}\big)\overline{\mathbf{x}}.
\end{aligned}$$

Combining this with the second term of (27) $\times\big||\mathbf{x}|^2-\mathbf{t}\big|^4$, we have

$$\begin{aligned}
&\mathbf{t}\big(d|\mathbf{x}|^4+d|\mathbf{t}|^2\big)+\mathbf{x}\cdot\big(2|\mathbf{x}|^2d\mathbf{t}-2\mathbf{t}d|\mathbf{x}|^2-\mathbf{t}d\mathbf{t}+d\mathbf{t}\mathbf{t}\big)\overline{\mathbf{x}}\\
&= \mathbf{t}d|\mathbf{t}|^2+2\mathbf{x}\cdot|\mathbf{x}|^2d\mathbf{t}\overline{\mathbf{x}}\\
&\quad +\mathbf{t}d|\mathbf{x}|^4-2\mathbf{x}\cdot\mathbf{t}d|\mathbf{x}|^2\overline{\mathbf{x}}+\mathbf{x}\cdot\big(-\mathbf{t}d\mathbf{t}+d\mathbf{t}\mathbf{t}\big)\overline{\mathbf{x}}.
\end{aligned}$$

In the case where $\mathbb{K}=\mathbb{C}$,

$$\mathbf{t}d|\mathbf{t}|^2+2\mathbf{x}\cdot|\mathbf{x}|^2d\mathbf{t}\overline{\mathbf{x}}=2d\mathbf{t}\big(|\mathbf{x}|^4+|\mathbf{t}|^2\big)=2d\mathbf{t}\big||\mathbf{x}|^2-\mathbf{t}\big|^2,$$

and other terms cancel out each other. □

CHAPTER 5

Harmonic maps inducing the identity map on the boundary

1. Preliminary computations

Once again, we shall prove the existence of harmonic maps in Example 1, by utilizing a slightly different method from those in Chapter 2. This method is more convenient for regularity estimation as well as for the analysis of asymptotic behavior.

Recall from Chapter 1 that the components of the tension field of the map $u\colon M \ni (\mathbf{x}, \mathbf{t}, \rho) \mapsto (\mathbf{x}, \mathbf{t}, \psi(\rho)) \in M$ are given by

$$\begin{aligned}\tau^m(u) &= 4\rho^2\frac{d^2\psi(\rho)}{d\rho^2} - (2\sum_{j=1}^{k} jn_j - 4)\rho\frac{d\psi(\rho)}{d\rho} \\ &\quad +2\sum_{l=1}^{k} l\rho^l\psi(\rho)^{1-l}n_l - 4\rho^2\psi(\rho)^{-1}\Big(\frac{d\psi}{d\rho}\Big)^2\end{aligned}$$

and $\tau^1(u) = \cdots = \tau^{n_1+n_2}(u) \equiv 0$. Remark that we observe these by noting that $u^m_m = d\psi/d\rho$ and $u^\gamma_i = \delta_{i\gamma}$ $(i, \gamma \neq m)$, which are valid because $u_*(e_m) = d\psi/d\rho\, e_m$ and $u_*(e_j) = e_j$ for $j = 1, \ldots, n_1 + n_2$. Once these are observed, setting $\mathcal{N} = \sum_{j=1}^k jn_j$, $\dot\psi = d\psi/d\rho$ and $\ddot\psi = d^2\psi/d\rho^2$, we have the following

Lemma 5.1. *Suppose $\psi(\rho)$ is a solution to*

$$\begin{cases} \rho\ddot\psi(\rho) - (\frac{1}{2}\mathcal{N} - 1)\dot\psi(\rho) + \frac{1}{2}\sum_{l=1}^{k} ln_l\left(\frac{\rho}{\psi(\rho)}\right)^{l-1} - (\dot\psi(\rho))^2\frac{\rho}{\psi(\rho)} = 0, \\ \psi(0) = 0, \quad \dot\psi(0) = 1, \quad \psi(\rho) = -\psi(-\rho) > 0 \text{ for } \rho > 0. \end{cases} \tag{28}$$

Then $u\colon (\mathbf{x}, \mathbf{t}, \rho) \mapsto (\mathbf{x}, \mathbf{t}, \psi(\rho))$ is a harmonic self-map of M inducing the identity map on the boundary ∂M.

In the next section, we shall establish the existence of a one-parameter family of global solutions to the equation (28) and study their asymptotic behavior. The growth estimates in Proposition 5.2 will be used in Section 3 of Chapter 6 in order to prove Proposition 6.1.

2. An asymptotic analysis of the translation invariant equation

THEOREM 5.1. *There exists a one-parameter family of global solutions $\psi(\rho) = \psi_\lambda(\rho)$ parameterized by $\lambda \geq 0$ to the equation* (28).

The translation invariance of the equation (29) in the proposition below is a key to the non-uniqueness of solutions for (28). By means of the following proposition, in order to prove Theorem 5.1, it suffices to show that there exists a nontrivial global solution $f(t)$ to the equation (29).

PROPOSITION 5.1. *$\psi(\rho) = \rho\exp\big(f(\log(|\rho|))\big)$ is a solution to* (28) *if and only if f satisfies*

$$\begin{cases} f''(t) - \frac{1}{2}\mathcal{N}f'(t) - \frac{1}{2}\sum_{l=1}^{k} ln_l\big(1 - e^{-lf(t)}\big) = 0, \\ f(t), f'(t) > 0, \lim_{x\to-\infty} f(t) = 0, \lim_{x\to-\infty} f'(t) = 0, \end{cases} \tag{29}$$

where $f' = df/dt$ and $f'' = d^2f/dt^2$. Given a solution $f(t)$ to (29)*, we have the solution $f(t + \log(\lambda))$ satisfying* (29) *for each $\lambda > 0$. In consequence, we see that $\psi_\lambda(\rho) = \rho\exp\big(f(\log(|\rho|\lambda))\big)$ $(\lambda \geq 0)$ form a one-parameter family of solutions of* (28) *parametrized by $\lambda \geq 0$.*

PROOF. Note that $\log(|\rho|)' = \mathrm{sgn}(\rho)/|\rho| = 1/\rho$. By substituting $\rho\exp\big(f(\log(|\rho|))\big)$ for $\psi(\rho)$ in equation (28), we have

$$\begin{aligned} &\rho\ddot{\psi}(\rho) - (\frac{1}{2}\mathcal{N} - 1)\dot{\psi}(\rho) + \frac{1}{2}\sum_{l=1}^{k} ln_l\left(\frac{\rho}{\psi(\rho)}\right)^{l-1} - (\dot{\psi}(\rho))^2\frac{\rho}{\psi(\rho)} \\ = \;& \rho(f''(t) + (f'(t)+1)f'(t))\rho^{-1}e^{f(t)} - (\frac{1}{2}\mathcal{N} - 1)(1 + f'(t))e^{f(t)} \\ &+\frac{1}{2}\sum_{l=1}^{k} ln_l\left(\frac{\rho}{\rho e^{f(t)}}\right)^{l-1} - \big((1 + f'(t))e^{f(t)}\big)^2\frac{\rho}{\rho e^{f(t)}} \\ = \;& e^{f(t)}\Big(f''(t) - \frac{1}{2}\mathcal{N}(1 + f'(t)) + \frac{1}{2}\sum_{l=1}^{k} ln_l e^{-lf(t)}\Big) = 0, \end{aligned}$$

where $t = \log(|\rho|)$. Since $f(t) > 0$ and $f(t) \to 0$ as $t \to -\infty$, we have $\psi(0) = 0$ and $\psi(\rho) \to \infty$ as $\rho \to \infty$. Moreover, it follows from $f(t) \to 0$ as $t \to -\infty$ and $f'(t) \to 0$ that $\dot{\psi}(0) = 1$.

Conversely, if $\psi(\rho)$ satisfies (28), then we can verify that $f(t) = \log(\psi(\exp(t))) - t$ satisfies (29). Indeed, for $\rho = e^t$, $\dot{\psi}(0) = 1$ and $\psi(0) = 0$ being the case, it holds that $f(t) = \log(\psi(e^t)/e^t) \to 0$ and $f'(t) = \dot{\psi}(e^t)e^t/\psi(e^t) - 1 \to 0$ as $t \to -\infty$. Since $e^{-f(t)} = \rho/\psi(\rho)$, we have

$$\begin{aligned} & f''(t) - \frac{1}{2}\mathcal{N} f'(t) - \frac{1}{2}\sum_{l=1}^{k}(1 - e^{-lf(t)})ln_l \\ = & \frac{\rho}{\psi(\rho)}\Big(\rho\ddot{\psi}(\rho) - (\frac{1}{2}\mathcal{N} - 1)\dot{\psi}(\rho) + \frac{1}{2}\sum_{l=1}^{k}\Big(\frac{\rho}{\psi(\rho)}\Big)^{l-1} ln_l - \frac{\rho}{\psi(\rho)}\dot{\psi}(\rho)^2\Big) = 0. \end{aligned}$$

□

Proof of Theorem 5.1. By setting $X(t) = f(t)$ and $Y(t) = f'(t)$, we can express the equation above as a system of first-order ordinary differential equations:

$$\begin{aligned} \frac{dY}{dt}(t) &= \frac{1}{2}\mathcal{N} Y(t) + \frac{1}{2}\sum_{l=1}^{k}(1 - e^{-lX(t)})ln_l, \\ \frac{dX}{dt}(t) &= Y(t). \end{aligned}$$

Consequently, we have

$$\begin{aligned} \frac{dY}{dX} &= \frac{\mathcal{N}}{2} + \frac{\sum_{l=1}^{k}(1 - e^{-lX})ln_l}{2Y} \\ &= \frac{\mathcal{N}}{2} + E(X)\frac{X}{Y}, \end{aligned}$$

where

$$E(X) = \frac{\sum_{l=1}^{k}(1 - e^{-lX})ln_l}{2X}.$$

At this point, it should be noted that $E(X)$ is a monotone decreasing function of $X > 0$.

In the following, in order to show the global existence of the solution $Y(X)$ satisfying $Y(0) = 0$, we shall solve the following equation:

$$\begin{cases} \dfrac{dY}{dX} = \dfrac{\mathcal{N}}{2} + E(X)\dfrac{X}{Y}, \\ Y(X) \to 0,\ dY/dX \to a\ \ (X \to 0), \quad a = \mathcal{N}/2 + E(0)/a > 0. \end{cases} \tag{30}$$

Remark that the condition $dY/dX \to a \ (X \to 0)$ corresponds to the requirement that $\lim_{X\to 0} dY/dX = \lim_{X\to 0}(\mathcal{N}/2 + E(X)X/Y)$. We shall define a constant c to be

$$c = E(0) = \frac{1}{2}\sum_{l=1}^{k} l^2 n_l.$$

Then a constant a in (30) is given by

$$a = (\mathcal{N} + \sqrt{\mathcal{N}^2 + 16c})/4.$$

Our strategy to complete the proof of Theorem 5.1 is as follows: in Step 1, it will be shown that $Y(X)$ exists globally; in Step 2, by using $Y(X)$, we shall solve $f'(t) = Y(f(t))$ with boundary values $f, f' \to 0 \ \ t \to -\infty$, and thereby we will establish the global existence of a solution $f(t)$ to the equation (29). Once the global existence of $f(t)$ is established, the proof of Theorem 5.1 will be completed by using Proposition 5.1.

Step 1: Our method to show the global existence of $Y(X)$ to (30) is as follows: Note that the right-hand side is C^∞ for the variables $Y > 0$ and $X > 0$. This means that for any $\varepsilon_0 > 0$, the solution $Y(X)$ with an initial value $Y(\varepsilon_0) > 0$ exists locally for $X > \varepsilon_0$ and that dY/dX does not diverge at finite X as long as $X(> 0)$ and $Y(> 0)$ are finite. Accordingly, in order to prove that $Y(X)$ exists globally on $[\varepsilon_0, \infty)$, it suffices to construct positive functions which restrict the behavior of $Y(X)$ from above and below for all $X > \varepsilon_0$. We can then verify that neither $Y(X)$ nor dY/dX diverges at any finite time.

Supposing that $\overline{c} > c > \underline{c} \geq 0$ and

$$\underline{a} = \mathcal{N}/2 + \underline{c}/\underline{a} > 0, \quad \overline{a} = \mathcal{N}/2 + \overline{c}/\overline{a} > 0,$$

we have $\underline{a} < a < \overline{a}$. Next, given any $\varepsilon_0 > 0$, we shall solve

$$\frac{d\overline{Y}(X)}{dX} = \frac{\mathcal{N}}{2} + \overline{c}\frac{X}{\overline{\overline{Y}}},$$
(31)
$$\frac{dY(X)}{dX} = \frac{\mathcal{N}}{2} + E(X)\frac{X}{Y},$$
$$\frac{d\underline{Y}(X)}{dX} = \frac{\mathcal{N}}{2} + \underline{c}\frac{X}{\underline{Y}}$$

with initial values:

$$\underline{a}\varepsilon_0 = \underline{Y}(\varepsilon_0) < Y(\varepsilon_0) < \overline{Y}(\varepsilon_0) = \overline{a}\varepsilon_0.$$

Clearly, $\underline{Y}(X) = \underline{a}X$ and $\overline{Y}(X) = \overline{a}X$ are solutions for the first and third equations.

This being understood, let us observe the following lemma:

LEMMA 5.2. *Given any $T_0 > \varepsilon_0$, it holds that*

$$\underline{Y}(X) \quad < Y(X) < \quad \overline{Y}(X) \tag{32}$$

on $[\varepsilon_0, T_0]$ for all $\underline{c}$ and $\overline{c}$ satisfying

$$0 \leq \underline{c} < E(T_0), \quad c < \overline{c}.$$

PROOF. Since $E(X)X/Y > 0$, we have $dY/dX > \mathcal{N}/2$, and thereby $Y(X) > 0$ is monotone increasing. Thus we can conclude that Y is bounded by X axis and that dY/dX does not diverge at a finite time from (30). Consequently, let us next assume that $\overline{Y}$ cannot bound Y from above, and accordingly, there exists an initial intersection of $\overline{Y}$ and Y at $X_0 < \infty$.

Then, since $\overline{Y}(\varepsilon_0) > Y(\varepsilon_0)$ and $\overline{Y}(X)$ meets $Y(X)$ for the first time at X_0, we have

$$\left[\frac{d(\overline{Y} - Y)}{dX}\right]\Big|_{X=X_0} \leq 0.$$

On the other hand, since $E(X)$ is a monotone decreasing function of X and $E(0) = c < \overline{c}$, it follows that

$$\left[\frac{d\overline{Y}}{dX} - \frac{dY}{dX}\right]\Big|_{X=X_0} = \big(\overline{c} - E(X_0)\big)\frac{X_0}{\overline{Y}(X_0)} > 0.$$

Hence we obtain a contradiction, which implies the global existence of Y on $[\varepsilon_0, \infty)$.

Next, in order to verify (32), let us further assume that $\underline{Y}$ cannot bound Y from below, and accordingly, there exists an initial intersection of $\underline{Y}$ and Y at $X_0 < T_0$. Then we obtain $[d(Y - \underline{Y})/dX]|_{X=X_0} \leq 0$. On the contrary, our assumption $E(X_0) \geq E(T_0) > \underline{c}$ provides

$$\left[\frac{dY}{dX} - \frac{d\underline{Y}}{dX}\right]\Big|_{X=X_0} = \big(E(X_0) - \underline{c}\big)\frac{X_0}{\underline{Y}(X_0)} > 0.$$

Therefore, we obtain another contradiction. □

By making use of the Lemma above, for each integer $j > 0$, we can obtain the solution $Y = Y_j$ which has an initial value $Y(\varepsilon_0) = a\varepsilon_0$ on $[\varepsilon_0, 1]$, defined as $\varepsilon_0 = 1/j$. It follows from the conclusions made above that each element of the sequence $\{Y_j\}$ satisfies:

$$\underline{Y} < Y_j < \overline{Y} \qquad \text{on} \quad [1/j, 1].$$

Furthermore, it should be noted that the equation (30) combined with (32) provides the upper and lower bounds for dY/dX and d^2Y/dX^2. Given these, when we examine

$\{Y_j\}$ firstly on $[1/2, 1]$, by using the Ascoli-Arzelà theorem, we have a sub-sequence $\{Y_{j_i}\}$ converging in $C^1([1/2, 1])$. Then, secondly, focusing on $[1/3, 1]$, by utilizing the Ascoli-Arzelà theorem once again, we can select a sub-sequence $\{Y_{j_{i_l}}\}$ converging in $C^1([1/3, 1])$. Upon continuing, we obtain a diagonal sub-sequence converging to Y_∞ locally in $C^1((0, 1])$. Thus we obtain a solution of (30) satisfying

$$\underline{Y}(X) < Y_\infty(X) < \overline{Y}(X) \qquad \text{on } (0, 1].$$

Having already established the existence of solution Y for (30) with an initial value $Y(1) = Y_\infty(1)$ satisfying $\underline{Y}(1) < Y_\infty(1) < \overline{Y}(1)$ as in Lemma 5.2, the continuation of Y_∞ provides a solution Y on $(0, \infty)$.

At this point, it should be noted that, when dividing (32) by X, it holds that

$$\underline{a} < \frac{Y(X)}{X} < \overline{a}.$$

By setting $0 < X < T_0 \to 0$ in order to let $\overline{c}, \underline{c} \to c$ and $\overline{a}, \underline{a} \to a$, we observe that

$$\frac{Y(X)}{X} \to a \qquad X \to 0.$$

Hence, we obtained our desired global solution $Y(X)$ to (30) defined on $(0, \infty)$.

Step 2: It is important to note that $Y(X) \in C^\infty$ by induction: Firstly, (30) implies $Y(X) \in C^1$, and secondly, $Y(X) \in C^k$ $(k \geq 1)$ implies that the left-hand side of (30) [that is, dY/dX] is also C^k thereby giving $Y(X) \in C^{k+1}$. Consequently, we see that a solution for $f'(t) = Y(f(t))$ exists locally and $f'(t)$ does not diverge unless $f(t)$ diverges. We shall prove that $f(t)$ exists globally on $\mathbb{R}$ by showing that $f(t)$ does not diverge at a finite time t.

Given $t_0 \in \mathbb{R}$, let us firstly solve

$$\overline{f}'(t) = \overline{a}\overline{f}(t), \quad f'(t) = Y(f(t)), \quad \underline{f}'(t) = \underline{a}\,\underline{f}(t)$$

for $t > t_0$ with initial values: $\overline{f}(t_0) = \overline{f}_0 > f(t_0) = f_0 > \underline{f}(t_0) = \underline{f}_0 > 0$. According to (32), as long as $0 < f < T_0$, it holds that

$$\overline{f}' - f' = \overline{a}\overline{f} - Y(f) \geq \overline{a}(\overline{f} - f), \qquad f' - \underline{f}' = Y(f) - \underline{a}\underline{f} \geq \underline{a}(f - \underline{f}),$$

and hence

$$\overline{f} - f \geq e^{(t-t_0)\overline{a}}(\overline{f}_0 - f_0), \qquad f - \underline{f} \geq e^{(t-t_0)\underline{a}}(f_0 - \underline{f}_0). \tag{33}$$

Substituting both $\overline{f} = \overline{f}_0 e^{(t-t_0)\overline{a}}$ and $\underline{f} = \underline{f}_0 e^{(t-t_0)\underline{a}}$ for each respective term of (33), and noting that $f_0 e^{(t-t_0)\overline{a}} \leq T_0$ implies

$$t \leq \frac{1}{\overline{a}} \log(T_0/f_0) + t_0,$$

we find that

$$f_0 e^{(t-t_0)\overline{a}} \geq f(t) \geq f_0 e^{(t-t_0)\underline{a}} \tag{34}$$

for the time interval $[t_0, \overline{a}^{-1} \log(T_0/f_0) + t_0]$. Since T_0 can be infinity, we can observe that $f(t)$ and $f'(t)$, which exist locally, do not diverge at a finite time. Therefore we obtain the global solution $f(t)$ for the time interval $[t_0, \infty)$.

Secondly, in order to see the behavior of $f(t)$ for $t \leq t_0$, by setting $\tilde{t}_0 = -t_0$, we shall solve the following equations:

$$-\overline{g}'(t) = \overline{a}\,\overline{g}(t), \quad -g'(t) = Y(g(t)), \quad -\underline{g}'(t) = \underline{a}\,\underline{g}(t)$$

for $t \geq \tilde{t}_0$ with initial values: $\overline{g}(\tilde{t}_0) = \overline{g}_0 > g(\tilde{t}_0) = g_0 > \underline{g}(\tilde{t}_0) = \underline{g}_0 > 0$. By adapting the argument given in the above for $t \geq t_0$ to $t \geq \tilde{t}_0$, it holds that

$$-(\overline{g}' - g') = \overline{a}\,\overline{g} - Y(g) \geq \overline{a}(\overline{g} - g), \qquad -(g' - \underline{g}') = Y(g) - \underline{a}\,\underline{g} \geq \underline{a}(g - \underline{g}),$$

from which we obtain

$$\overline{g} - g \leq e^{-(t-\tilde{t}_0)\overline{a}}(\overline{g}_0 - g_0), \qquad g - \underline{g} \leq e^{-(t-\tilde{t}_0)\underline{a}}(g_0 - \underline{g}_0).$$

By substituting both $\overline{g} = \overline{g}_0 e^{-(t-\tilde{t}_0)\overline{a}}$ and $\underline{g} = \underline{g}_0 e^{-(t-\tilde{t}_0)\underline{a}}$ for each respective term in the above, we have

$$g_0 e^{-(t-\tilde{t}_0)\underline{a}} \leq g(t) \leq g_0 e^{-(t-\tilde{t}_0)\overline{a}} \tag{35}$$

for $t > \tilde{t}_0$. Since $f(-t) = g(t)$ when $t > \tilde{t}_0$, we obtain the solution $f(t)$ for the time interval $(-\infty, t_0]$. Hence, we have a positive solution $f(t)$ for the equation (29) for all $t \in \mathbb{R}$, which is in C^∞ by an argument similar to that in the case of $Y(X)$. By means of Proposition 5.1, we are led to the family of solutions $\psi_\lambda(\rho)$ for the equation (28), and thereby we have completed the proof of Theorem 5.1. □

THEOREM 5.2. *There exists a family of harmonic diffeomorphisms $u_\lambda|_M \colon M \to M$ which induce the identity map on the boundary.*

PROOF. Since $\dot{\psi}_\lambda(\rho) > 0$ for all $\rho > 0$, we have $Ju = (\partial s^i \circ u/\partial s^j) > 0$ for all $(\mathbf{x}, \mathbf{t}, \rho) \in N \cdot \mathbb{R}_+$, where $(s^1, \dots, s^m) = (x^{11}, \dots, t^d, \rho)$. Hence the result follows from Theorem 5.1 combined with Lemma 5.1. $\square$

PROPOSITION 5.2. *For* $C_0 > 0$, $\mathcal{N} = \sum_{l=1}^k l n_l$ *and* $\delta > 1$, *there exist positive constants* $C_1(C_0, \mathcal{N}), C_2(C_0, \mathcal{N})$ *and* $C_3(C_0, \mathcal{N}, \delta)$ *so that, for* $\rho > C_0$, *the following inequalities hold*:

$$\psi(\rho) \geq \rho \exp(C_1 \rho^{\mathcal{N}/2}), \tag{36}$$

$$\dot{\psi}(\rho) \leq C_2 \rho^{3\mathcal{N}/2} \psi(\rho), \tag{37}$$

$$\dot{\psi}(\rho) \leq C_3 \psi(\rho)^\delta \qquad (\delta > 1). \tag{38}$$

PROOF. Firstly, let us set $f_0 := f(\log(t_0))$ for $t_0 = \log(\lambda C_0)$.

By setting $\underline{c} = 0$ and $\underline{a} = \mathcal{N}/2$, accordingly, we may set $T_0 = \infty$, so that the inequality (34) holds for the time interval $[t_0, \infty)$. Owing to this inequality, we have

$$\psi(\rho) \geq \rho \exp\bigl(f_0 e^{\mathcal{N}(\log(\lambda\rho) - \log(\lambda C_0))/2}\bigr) = \rho \exp(f_0 (C_0^{-1}\rho)^{\mathcal{N}/2}) = \rho \exp(C_1 \rho^{\mathcal{N}/2}),$$

where $C_1 = f_0 C_0^{-\mathcal{N}/2}$. Thus, we obtain the inequality (36).

Secondly, since each n_l is a non-negative integer, it holds that

$$\sum_{l=1}^k l^2 n_l \leq \sum_{l=1}^k l^2 n_l^2 \leq (\sum_{l=1}^k l n_l)^2 = \mathcal{N}^2,$$

from which it follows that

$$a = \Bigl(\mathcal{N} + \sqrt{\mathcal{N}^2 + 8\textstyle\sum_{l=1}^k l^2 n_l}\Bigr)/4 \leq \mathcal{N}.$$

Recognizing that $\overline{a}$ in (32) and (34) can be arbitrarily close to a, we may set $\overline{a}$ so that

$$\mathcal{N} < \overline{a} \leq 3\mathcal{N}/2$$

holds. Since $C_0^{-1}\rho \geq 1$ from our assumption $\rho \geq C_0$, we have

$$C_0^{-(3\mathcal{N}/2) + \overline{a} - 1} \rho^{(3\mathcal{N}/2) - \overline{a} + 1} \geq 1. \tag{39}$$

Moreover, by using (32) and (34), we have $f'(t) = Y(f(t)) \le \overline{a} f(t) \le \overline{a} f_0 e^{\overline{a}(t-t_0)}$. Hence,

$$\begin{aligned}\dot{\psi}(\rho) &= (1 + f'(\log(\lambda\rho)))\rho^{-1}\psi(\rho)\\ &\le \big(1 + \overline{a} f_0 e^{(\log(\lambda\rho)-\log(\lambda C_0))\overline{a}}\big)\rho^{-1}\psi(\rho)\\ &\le \big(1 + \max(1, \overline{a} f_0)\big(\rho C_0^{-1}\big)^{\overline{a}}\big)\rho^{-1}\psi(\rho)\\ &\le 2\max(1, \overline{a} f_0) C_0^{-\overline{a}} \rho^{\overline{a}-1}\psi(\rho)\\ &\le C_4 C_0^{-(3\mathcal{N}/2)-1}\rho^{3\mathcal{N}/2}\psi(\rho) = C_2\rho^{3\mathcal{N}/2}\psi(\rho),\end{aligned}$$

where $C_4 = 2\max(1, \overline{a} f_0)$ and $C_2 = C_4 C_0^{-(3\mathcal{N}/2)-1}$. Thus, we obtain (37).

Thirdly, combining the inequality (36) with our assumption $\rho > C_0$, we have $\psi(\rho) \ge \rho \exp(C_1\rho^{\mathcal{N}/2}) > C_0 \exp(C_1\rho^{\mathcal{N}/2})$. Hence

$$\rho^{\mathcal{N}/2} \le C_1^{-1} \log(\psi(\rho) C_0^{-1}). \tag{40}$$

Combining (40) with (37) and setting $\eta(\rho) = \psi(\rho)C_0^{-1}$, we have a constant C_3 so that

$$\begin{aligned}\dot{\psi}(\rho) &\le C_2\rho^{3\mathcal{N}/2}\psi(\rho) \le C_2 C_0^{-1}\big(C_1^{-1}\log(\eta(\rho))\big)^3(\eta(\rho))^{1-\delta}\eta(\rho)^{\delta}\\ &= C_2 C_0^{-1} C_1^{-3}\left(\frac{3}{\delta-1}\frac{\log(\eta(\rho)^{\frac{\delta-1}{3}})}{\eta(\rho)^{\frac{\delta-1}{3}}}\right)^3 \eta(\rho)^{\delta} \le C_3\psi(\rho)^{\delta} \qquad (\delta > 1),\end{aligned}$$

where the last inequality is deduced by observing that $\eta(\rho) = C_0^{-1}\psi(\rho) \ge \exp(C_1 C_0^{\alpha\mathcal{N}})$ for $\rho > C_0$ according to (36), and that $\log(x)/x$ is a bounded function on $x > C_0$. Hence, we have (38). $\square$

LEMMA 5.3. *There exists $C_5 = C_5(\mathcal{N})$ so that*

$$\lim_{\rho\to\infty} \frac{f(\log(\lambda\rho))}{(\mathcal{N}/2)^{-1}\lambda^{\mathcal{N}/2}\rho^{\mathcal{N}/2}} = C_5.$$

PROOF. Let h be the solution for $h''(t) = \mathcal{N}h'(t)/2 + \mathcal{N}/2$ with initial values: $h'(t_0) = f'(t_0) > 0$ and $h(t_0) = f(t_0) > 0$. Noting that

$$\frac{d(h'-f')}{dt} = \frac{1}{2}\mathcal{N}(h'-f') + \frac{1}{2}\sum_{l=1}^{k} e^{-lf(t)} l n_l \ge \frac{1}{2}\mathcal{N}(h'-f'),$$

we find that $h'(t) \ge f'(t)$ for $t \ge t_0$, and hence

$$h'(t) = e^{\mathcal{N}(t-t_0)/2}(f'(t_0)+1) - 1 \ge f'(t).$$

Dividing the above by $e^{\mathcal{N}t/2}$, we see that $f'(t)e^{-\mathcal{N}t/2}$ is bounded from above. Moreover, since $f(t) > 0$, we have

$$\frac{d(f'(t)e^{-\mathcal{N}t/2})}{dt} = e^{-\mathcal{N}t/2}\big(f''(t) - \frac{1}{2}\mathcal{N}f'(t)\big) = \frac{1}{2}e^{-\mathcal{N}t/2}\sum_{l=1}^{k}\big(1 - e^{-lf(t)}\big)ln_l > 0,$$

which implies that $f'(t)e^{-\mathcal{N}t/2}$ is monotone increasing. It follows from these observations that there is a constant C_5 such that

$$C_5 = \lim_{t\to\infty}\frac{f'(t)}{e^{\mathcal{N}t/2}} = \lim_{t\to\infty}\frac{f(t)}{(\mathcal{N}/2)^{-1}e^{\mathcal{N}t/2}}. \tag{41}$$

□

THEOREM 5.3. *Let* $\mathbf{n} = (\mathbf{x}, \mathbf{t})$ *denote a point on* $\mathbb{K}^n \times \mathrm{Im}(\mathbb{K}) = N$*. Then, for* $\lambda \neq \lambda'$ *and fixed* $\mathbf{s}_0 = (\mathbf{n}_0, \rho_0) \in N \cdot \mathbb{R}_+$*, we have* $\mathrm{dist}(u_\lambda(\mathbf{s}), u_{\lambda'}(\mathbf{s})) \sim \exp(\mathcal{N}\mathrm{dist}(\mathbf{s}_0, \mathbf{s}))$ *as* $\mathbf{s} := (\mathbf{n}, \rho) \to \infty$*, while bounding* $|\mathbf{n}|$*.*

PROOF. Let $(\mathbf{n}, \rho(t))$ be the geodesic of $N \cdot \mathbb{R}_+$ joining $(\mathbf{n}, \rho_0)$ and $\mathbf{s} := (\mathbf{n}, \rho)$, satisfying $\rho(0) = \rho_0$ and $\rho(1) = \rho$. Noting that

$$\mathrm{dist}((\mathbf{n}, \rho_0), (\mathbf{n}, \rho)) = \left|\int_0^1 \sqrt{\frac{\rho'(t)^2}{4\rho(t)^2}}dt\right| = \left|\frac{1}{2}\int_{\rho_0}^{\rho}\frac{d\rho}{\rho}\right| = \left|\log(\rho^{1/2}/\rho_0^{1/2})\right| \tag{42}$$

by using Lemma 5.3, we have

$$\begin{aligned}\frac{\mathrm{dist}(u_\lambda(\mathbf{s}), u_{\lambda'}(\mathbf{s}))}{\rho^{\mathcal{N}/2}} &= \left|\frac{\log(\psi(\lambda\rho))}{\rho^{\mathcal{N}/2}} - \frac{\log(\psi(\lambda'\rho))}{\rho^{\mathcal{N}/2}}\right| = \left|\frac{f(\log(\lambda\rho))}{\rho^{\mathcal{N}/2}} - \frac{f(\log(\lambda'\rho))}{\rho^{\mathcal{N}/2}}\right| \\ &\to C_5(\mathcal{N}/2)^{-1}|(\lambda^{\mathcal{N}/2} - \lambda'^{\mathcal{N}/2})| \quad \text{as } \mathbf{s} = (\mathbf{n}, \rho) \to \infty.\end{aligned} \tag{43}$$

Since the line segment joining $(\mathbf{n}, \rho_0)$ and $(\mathbf{n}, \rho)$ is a geodesic of $N \cdot \mathbb{R}_+$, we have

$$\begin{aligned}\mathrm{dist}((\mathbf{n}, \rho_0), (\mathbf{n}, \rho)) &\leq \mathrm{dist}((\mathbf{n}_0, \rho_0), (\mathbf{n}, \rho)) \\ &\leq \mathrm{dist}((\mathbf{n}_0, \rho_0), (\mathbf{n}, \rho_0)) + \mathrm{dist}((\mathbf{n}, \rho_0), (\mathbf{n}, \rho)).\end{aligned}$$

Applying the exponential function $\exp(*)$ to the above, we have a constant C_6 such that

$$\begin{aligned}\exp\big(\mathcal{N}\mathrm{dist}((\mathbf{n}, \rho_0), (\mathbf{n}, \rho))\big) &\leq \exp\big(\mathcal{N}\mathrm{dist}((\mathbf{n}_0, \rho_0), (\mathbf{n}, \rho))\big) \\ &\leq C_6\exp\big(\mathcal{N}\mathrm{dist}((\mathbf{n}, \rho), (\mathbf{n}, \rho_0))\big),\end{aligned}$$

since $\mathrm{dist}((\mathbf{n}_0, \rho_0), (\mathbf{n}, \rho_0)) < \infty$ from our assumption $|\mathbf{n}| < \infty$ when $(\mathbf{n}, \rho) \to \infty$. Furthermore, combining this with (42), we obtain

$$\exp\big(\mathcal{N}\mathrm{dist}((\mathbf{n}_0, \rho_0), (\mathbf{n}, \rho))\big) \sim \exp\big(\mathcal{N}\mathrm{dist}((\mathbf{n}, \rho), (\mathbf{n}, \rho_0))\big) = \rho^{\mathcal{N}/2}/\rho_0^{\mathcal{N}/2},$$

when we let $(\mathbf{n},\rho) \to \infty$ while keeping $|\mathbf{n}|$ finite. Combining the above with (43), we obtain the result. $\square$

PROPOSITION 5.3. *Near $\rho = 0$, we have*

$$\psi(\rho) = \rho + o(\rho|\rho|^{\underline{a}}) \qquad \text{for any } \underline{a} < (\mathcal{N} + \sqrt{\mathcal{N}^2 + 8\textstyle\sum_{l=1}^{k} l^2 n_l}\,)\Big/4. \tag{44}$$

PROOF. Fix a solution $f(-t) = g(t)$ and let $t = -\log(|\rho|)$. Dividing (34) by $e^{-t\underline{a}}$, we have

$$g(t)e^{t\underline{a}} \geq g(t_0)e^{\tilde{t}_0\underline{a}}\, e^{t(\underline{a}-\overline{a})}.$$

Consequently, we see that $g(t)e^{t\underline{a}}$ is bounded from below. Furthermore, we have

$$\frac{d(e^{\underline{a}t}g(t))}{dt} = \underline{a}e^{t\underline{a}}g(t) + e^{t\underline{a}}g'(t) = e^{t\underline{a}}\Big(\underline{a}g(t) + g'(t)\Big) = e^{t\underline{a}}\Big(\underline{Y}(g) - Y(g)\Big) < 0.$$

Hence $g(t)e^{t\underline{a}}$ is monotone decreasing.

It then follows from these observations that there is a constant $C_7(\underline{c})$ such that

$$C_7 = \lim_{t\to\infty} \frac{g(t)}{e^{-t\underline{a}}}.$$

Now, suppose $C_7 \neq 0$. By using de l'Hôpetal's theorem, we have

$$a = \lim_{X\to 0} \frac{dY}{dX} = -\lim_{t\to\infty} \frac{g''(t)}{g'(t)} = -\lim_{t\to\infty} \frac{g'(t)}{g(t)} = -\lim_{t\to\infty} \frac{g'(t)}{e^{-t\underline{a}}} \lim_{t\to\infty} \frac{e^{-t\underline{a}}}{g(t)} = \underline{a},$$

which contradicts $\underline{a} < a$. Hence, we must have $C_7(\underline{c}) = 0$ and $\lim_{\rho\to 0} f(\log(|\rho|))/|\rho|^{\underline{a}} = 0$. Therefore we have

$$\begin{aligned}
\exp(f(\log(|\rho|))) &= 1 + f(\log(|\rho|))\sum_{n=1}^{\infty} \frac{f(\log(|\rho|))^{n-1}}{n!} \\
&= 1 + |\rho|^{\underline{a}}\Big(f(\log(|\rho|))/|\rho|^{\underline{a}}\Big)\sum_{n=1}^{\infty} \frac{f(\log(|\rho|))^{n-1}}{n!} \\
&= 1 + o(|\rho|^{\underline{a}}).
\end{aligned}$$

Furthermore, for a given t_0 and T_0 satisfying $T_0 > f(t_0)$, we have

$$\begin{aligned}
f_0e^{(t-t_0)\underline{a}} \leq\ & f(t) \leq f_0e^{(t-t_0)\overline{a}} \quad \text{on} \quad [t_0, \overline{a}^{-1}\log(T_0/f_0) + t_0], \\
f_0e^{(t-t_0)\overline{a}} \leq\ & f(t) \leq f_0e^{(t-t_0)\underline{a}} \quad \text{on} \quad (-\infty_0, t_0]
\end{aligned}$$

for all $\underline{c}$ satisfying $0 \leq \underline{c} < E(T_0)$. Since $f(t_0) \to 0$ as $t_0 \to -\infty$, we can let $T_0 \to 0$ so that $E(T_0) \to c$ as $t_0, t \to -\infty$. This leads us to conclude the following:

$$\psi(\rho) = \rho + o(\rho|\rho|^{\underline{a}}), \qquad \underline{a} < \left(\mathcal{N} + \sqrt{\mathcal{N}^2 + 8\sum_{l=1}^{k} l^2 n_l}\,\right)\Big/ 4.$$

□

CHAPTER 6

The boundary regularity

1. Regularity on $\overline{B^n_{\mathbb{K}}} \setminus \{k_n\}$

Keeping the notation in previous Chapters, we denote by $u_\lambda\colon \mathbb{K}^n \times \mathrm{Im}(\mathbb{K}) \times \mathbb{R}_+ \to \mathbb{K}^n \times \mathrm{Im}(\mathbb{K}) \times \mathbb{R}_+$ the maps constructed in Chapter 4; and by $\Phi_{k_n} : \mathbb{K}^n \times \mathrm{Im}(\mathbb{K}) \times \mathbb{R}_+ \to B^n_{\mathbb{K}}$ the Cayley transform defined in Chapter 5.

Let us first remark that

$$\Phi_{k_n} \circ u_\lambda \circ \Phi_{k_n}^{-1} =: u_{B^n_{\mathbb{K}},\lambda}$$

is a harmonic map from $B^n_{\mathbb{K}}$ to itself. Recall that $\overline{B^n_{\mathbb{K}}}$ is equipped with boundary charts as in (26). Since $u_\lambda|_{\partial M} = \mathrm{id}$ and $\sigma^2 = \mathrm{id}$, the regularity of the self-maps $u_{B^n_{\mathbb{K}},\lambda}$ of $\overline{B^n_{\mathbb{K}}}$ is equivalent to that of

$$\begin{aligned} \Phi_{k_n}^{-1} \circ (\Phi_{k_n} \circ u_\lambda \circ \Phi_{k_n}^{-1}) \circ \Phi_{k_n} &= u_\lambda, \\ \Phi_{-k_n}^{-1} \circ (\Phi_{k_n} \circ u_\lambda \circ \Phi_{k_n}^{-1}) \circ \Phi_{-k_n} &= \sigma \circ u_\lambda \circ \sigma \end{aligned}$$

viewed as maps from $\mathbb{R}^{n_1+n_2} \times \mathbb{R}_{\geq 0}$ to itself. By using the asymptotic expansion (44), we obtain the regularity of u_λ which then implies the following

Corollary 6.1. *$u_{B^n_{\mathbb{K}}\lambda} \in C^{1+\underline{a}}(\overline{B^n_{\mathbb{K}}} \setminus \{k_n\}, \overline{B^n_{\mathbb{K}}} \setminus \{k_n\})$ for*

$$\underline{a} < \left(\mathcal{N} + \sqrt{\mathcal{N}^2 + 8\sum_{l=1}^k l^2 n_l}\right)/4.$$

The regularity is invariant under the coordinate transformation on an overlapping area within an atlas. Thereby, the regularity on $\overline{B^n_{\mathbb{K}}} \setminus \{k_n, -k_n\}$ is also given by Corollary 6.1. Hence it suffices to estimate the regularity of $\tilde{u}_\lambda$ only near a small neighborhood of $(0, 0, 0)$.

In the following, we suppose $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$ and shall abbreviate u_λ, $\tilde{u}_\lambda$ and $u_{B^n_{\mathbb{K}},\lambda}$ to $\tilde{u}$ and $u_{B^n_{\mathbb{K}}}$, respectively.

2. Regularity of $\tilde{u}$

DEFINITION 6.1. For $\mathbf{s} = (\mathbf{x}, \mathbf{t}, \rho) \in \mathbb{K}^n \times \mathrm{Im}(\mathbb{K}) \times \mathbb{R}_+$, let $\|\cdot\|_H \colon M \to \mathbb{R}_{\geq 0}$ be

$$\|\mathbf{s}\|_H := \left||\mathbf{x}|^2 + \mathbf{t} + \rho\right|^{1/2} = \left((|\mathbf{x}|^2 + \rho)^2 + |\mathbf{t}|^2\right)^{1/4}.$$

Here $|\cdot|$ denotes the Euclidean norm.

REMARK 6.1. It holds that $\rho{\circ}\sigma = \rho/\|\mathbf{s}\|_H^4$.

The following observation will be useful in order to prove Lemma 6.8.

LEMMA 6.1. $\|\sigma(\mathbf{s})\|_H = 1/\|\mathbf{s}\|_H$.

PROOF.

$$\begin{aligned}\|\sigma(\mathbf{s})\|_H^2 &= \left||\mathbf{x}{\circ}\sigma|^2 + \mathbf{t}{\circ}\sigma + \rho{\circ}\sigma\right| \\ &= \left||\mathbf{x}|^2 - \mathbf{t} + \rho\right|\left||\mathbf{x}|^2 + \mathbf{t} + \rho\right|^{-2} = \|\mathbf{s}\|_H^{-2}.\end{aligned}$$

□

LEMMA 6.2. *For* $|\mathbf{s}| \leq 5^{-1/2}$, *we have*

$$C_8|\mathbf{s}| \leq \|\mathbf{s}\|_H \leq C_9|\mathbf{s}|^{1/2}. \tag{45}$$

PROOF. It is immediate that

$$\begin{aligned}\|\mathbf{s}\|_H^4 &= (|\mathbf{x}|^2 + \rho)^2 + |\mathbf{t}|^2 \\ &\leq (|\mathbf{s}|^2 + |\mathbf{s}|)^2 + |\mathbf{s}|^2 \leq 5|\mathbf{s}|^2 \ (\leq 1),\end{aligned} \tag{46}$$

$$\begin{aligned}|\mathbf{s}|^2 &= |\mathbf{x}|^2 + |\mathbf{t}|^2 + \rho^2 \\ &\leq \|\mathbf{s}\|_H^2 + \|\mathbf{s}\|_H^4 + \|\mathbf{s}\|_H^4 \leq 3\|\mathbf{s}\|_H^2.\end{aligned} \tag{47}$$

This completes the estimate. □

LEMMA 6.3. $\|(u{\circ}\sigma)(\mathbf{s})\|_H^{-1}\|\mathbf{s}\|_H^{-1} \leq 1$.

PROOF. By using Definition 6.1, we have $\|(\mathbf{x}, \mathbf{t}, \rho')\|_H \geq \|(\mathbf{x}, \mathbf{t}, \rho)\|_H$ when $\rho' \geq \rho$. Since $\psi(\rho) \geq \rho$, by using Lemma 6.1, we have

$$\begin{aligned}\|u{\circ}\sigma(\mathbf{s})\|_H &= \|(\mathbf{x}{\circ}\sigma, \mathbf{t}{\circ}\sigma, \psi(\rho{\circ}\sigma))\|_H \geq \|(\mathbf{x}{\circ}\sigma, \mathbf{t}{\circ}\sigma, \rho{\circ}\sigma)\|_H \\ &= \|\sigma(\mathbf{s})\|_H = \|\mathbf{s}\|_H^{-1}.\end{aligned}$$

□

LEMMA 6.4. *For* $|\mathbf{s}| \leq 5^{-1/2}$, *we have*

$$|\tilde{u}(\mathbf{s})| \leq C_{10}|\mathbf{s}|^{1/2}.$$

PROOF. By using Lemmas 6.1 and 6.3, we have

$$\|\tilde{u}\|_H = \|\sigma\circ u\circ\sigma\|_H = \|u\circ\sigma\|_H^{-1} = \|u\circ\sigma\|_H^{-1}\|\mathbf{s}\|_H^{-1}\|\mathbf{s}\|_H \leq \|\mathbf{s}\|_H. \tag{48}$$

Note that we have $\|\tilde{u}\|_H \leq 5^{1/4}|\mathbf{s}|^{1/2}$ by using the inequalities (46) and (48). In particular, we also have $\|\tilde{u}\|_H \leq 1$. Hence, by using the inequality (47), we have $3^{-1/2}|\tilde{u}| \leq \|\tilde{u}\|_H$. Summing up, we have obtained $C_8|\tilde{u}| \leq C_9|\mathbf{s}|^{1/2}$, thus completing the proof of the lemma. □

PROPOSITION 6.1. *For $|\mathbf{s}| \leq 5^{-1/2}$, we have*

$$\|J\tilde{u}\| \leq C_{11}|\mathbf{s}|^{-(\delta+2)} \text{ for any } \delta > 1.$$

Here $\|J\tilde{u}\| = \left(\mathrm{Tr}({}^tJ\tilde{u}\cdot J\tilde{u})\right)^{1/2} = \left(\sum_{ll'}|\partial(s^l\circ\tilde{u})/\partial s^{l'}|^2\right)^{1/2}$ *for* $(s^1,\dots,s^m) = (x^{11},\dots,t^d,\rho)$.

The proof of this proposition shall be made in the next section.

LEMMA 6.5. *Let $|\mathbf{s}_1| \leq |\mathbf{s}_2| \leq 1/2$. Let $\mathbf{s}$ be any point on the line segment joining $\mathbf{s}_1$ and $\mathbf{s}_2$. Suppose that we have*

$$\|J\tilde{u}\| \leq C_{12}|\mathbf{s}|^{-\beta}, \quad |\tilde{u}(\mathbf{s})| \leq C_{13}|\mathbf{s}|^{\gamma}$$

with $\beta+\gamma \geq 1+\varepsilon$ for $\varepsilon > 0$. Then

$$|s^i\circ\tilde{u}(\mathbf{s}_1) - s^i\circ\tilde{u}(\mathbf{s}_2)| < C_{14}|\mathbf{s}_1-\mathbf{s}_2|^{\gamma/(\beta+\gamma)}, \quad i = 1,\dots,m.$$

PROOF. In the case of $|\mathbf{s}_2|^{\beta+\gamma} \leq |\mathbf{s}_1-\mathbf{s}_2|$, it holds that

$$|\tilde{u}(\mathbf{s}_1)-\tilde{u}(\mathbf{s}_2)| \leq |\tilde{u}(\mathbf{s}_1)| + |\tilde{u}(\mathbf{s}_2)| \leq C_{13}|\mathbf{s}_1|^{\gamma} + C_{13}|\mathbf{s}_2|^{\gamma} \leq 2C_{13}|\mathbf{s}_1-\mathbf{s}_2|^{\gamma/(\beta+\gamma)}.$$

When $|\mathbf{s}_2|^{\beta+\gamma} \geq |\mathbf{s}_1-\mathbf{s}_2|$, for any point $\mathbf{s}$ on the line segment from $\mathbf{s}_1$ to $\mathbf{s}_2$, we have $|\mathbf{s}| \geq |\mathbf{s}_2| - |\mathbf{s}-\mathbf{s}_2| \geq |\mathbf{s}_2| - |\mathbf{s}_1-\mathbf{s}_2| \geq |\mathbf{s}_2|(1-|\mathbf{s}_2|^{\beta+\gamma-1}) \geq |\mathbf{s}_2|(1-2^{-\varepsilon})$, and thereby $|\mathbf{s}|^{-\beta} \leq C_{15}|\mathbf{s}_1-\mathbf{s}_2|^{-\beta/(\beta+\gamma)}$. By using the mean value inequality, we have

$$|\tilde{u}(\mathbf{s}_1)-\tilde{u}(\mathbf{s}_2)| \leq \|J\tilde{u}(\mathbf{s})\||\mathbf{s}_1-\mathbf{s}_2| \leq C_{12}|\mathbf{s}|^{-\beta}|\mathbf{s}_1-\mathbf{s}_2| \leq C_{15}C_{12}|\mathbf{s}_1-\mathbf{s}_2|^{\gamma/(\beta+\gamma)}.$$

□

Combining Lemmas 6.4 and 6.5 with Proposition 6.1, and noting that $2^{-1}/(2^{-1}+2+\delta) = 1/(5+2\delta)$, we can prove the following:

PROPOSITION 6.2. $u_{B^n_{\mathbb{K}}} \in C^{\varepsilon}(\overline{B^n_{\mathbb{K}}}, \overline{B^n_{\mathbb{K}}})$ *for $\varepsilon < 1/7$ and $\mathbb{K} = \mathbb{C}, \mathbb{H}$.*

3. The estimate of $\|J\tilde{u}\|$

The purpose of this section is to prove Proposition 6.1.

LEMMA 6.6. *If $\mathbf{s}_t$ is a path satisfying $|\mathbf{s}_t| \le 5^{-1/2}$ and $\rho{\circ}\sigma(\mathbf{s}_t) \to 0$ as $t \to \infty$, then we have $J\tilde{u} \to \mathrm{Id}$ (the identity matrix) as $t \to \infty$.*

PROOF. Let $\mathbf{s}_\infty = (\mathbf{x}_\infty, \mathbf{t}_\infty, \rho_\infty)$ be any point in $\overline{M}$ so that $\rho{\circ}\sigma(\mathbf{s}_t) \to 0$ as $\mathbf{s}_t \to \mathbf{s}_\infty$. Since $\rho{\circ}\sigma(\mathbf{s}_\infty) = 0$, it holds that $\mathbf{x}_\infty \ne 0$ or $\mathbf{t}_\infty \ne 0$. Moreover, since $|\mathbf{s}_t| \le 5^{-1/2}$, we also have $\mathbf{x}_\infty{\circ}\sigma \ne 0$ or $\mathbf{t}_\infty{\circ}\sigma \ne 0$. In addition, $\sigma \in C^\infty$ near $\mathbf{s}_\infty = (\mathbf{x}_\infty, \mathbf{t}_\infty, \rho_\infty)$ when $\mathbf{x}_\infty \ne 0$ or $\mathbf{t}_\infty \ne 0$. We also have $u = \big(\mathbf{x}, \mathbf{t}, \rho + o(\rho^{\underline{a}+1})\big)$ according to Proposition 5.3.

If a function f is in C^∞ near $\mathbf{x}, \mathbf{t} \ne 0$, it holds that

$$f\big(\mathbf{x}, \mathbf{t}, \rho + o(\rho^{\underline{a}+1})\big) = f(\mathbf{x}, \mathbf{t}, \rho) + o(\rho^{\underline{a}+1})$$

and we therefore have

$$f\big(\mathbf{x}{\circ}\sigma, \mathbf{t}{\circ}\sigma, \rho{\circ}\sigma + o\big((\rho{\circ}\sigma)^{\underline{a}+1}\big)\big) = f(\mathbf{x}{\circ}\sigma, \mathbf{t}{\circ}\sigma, \rho{\circ}\sigma) + o\big((\rho{\circ}\sigma)^{\underline{a}+1}\big).$$

By applying this observation to $\partial(s^i{\circ}\sigma)/\partial s^l$, we have

$$\frac{\partial(s^i{\circ}\sigma)}{\partial s^l}\big(\mathbf{x}{\circ}\sigma, \mathbf{t}{\circ}\sigma, \rho{\circ}\sigma + o\big((\rho{\circ}\sigma)^{\underline{a}+1}\big)\big) = \frac{\partial(s^i{\circ}\sigma)}{\partial s^l}(\mathbf{x}{\circ}\sigma, \mathbf{t}{\circ}\sigma, \rho{\circ}\sigma) + o\big((\rho{\circ}\sigma)^{\underline{a}+1}\big).$$

Utilizing the above and the chain rule, we have

$$\begin{aligned}
\frac{\partial(s^i{\circ}\tilde{u})}{\partial s^j} &= \sum_{l,k=1}^{m} \frac{\partial(s^i{\circ}\sigma)}{\partial s^l}(u{\circ}\sigma)\cdot\frac{\partial(s^l{\circ}u)}{\partial s^k}(\sigma)\cdot\frac{\partial(s^k{\circ}\sigma)}{\partial s^j} \\
&= \sum_{l=1}^{m-1} \frac{\partial(s^i{\circ}\sigma)}{\partial s^l}(u{\circ}\sigma)\cdot\frac{\partial(s^l{\circ}\sigma)}{\partial s^j} + \frac{\partial(s^i{\circ}\sigma)}{\partial s^m}(u{\circ}\sigma)\cdot\frac{\partial(s^m{\circ}\sigma)}{\partial s^j}\cdot\dot{\psi}(\rho{\circ}\sigma) \\
&= \sum_{l=1}^{m-1} \Big(\frac{\partial(s^i{\circ}\sigma)}{\partial s^l}(\sigma) + o\big((\rho{\circ}\sigma)^{\underline{a}+1}\big)\Big)\cdot\frac{\partial(s^l{\circ}\sigma)}{\partial s^j} \\
&\qquad + \Big(\frac{\partial(s^i{\circ}\sigma)}{\partial s^m}(\sigma) + o\big((\rho{\circ}\sigma)^{\underline{a}+1}\big)\Big)\cdot\frac{\partial(s^m{\circ}\sigma)}{\partial s^j}\cdot\big(1 + o\big((\rho{\circ}\sigma)^{\underline{a}}\big)\big) \\
&= \sum_{l=1}^{m} \frac{\partial(s^i{\circ}\sigma)}{\partial s^l}(\sigma)\cdot\frac{\partial(s^l{\circ}\sigma)}{\partial s^j} + o\big((\rho{\circ}\sigma)^{\underline{a}+1}\big) \\
&= \delta_{ij} + o\big((\rho{\circ}\sigma)^{\underline{a}+1}\big).
\end{aligned}$$

This completes the proof of the lemma. ∎

REMARK 6.2. Given Lemma 6.6, in order to prove Proposition 6.1, it suffices to estimate $\|J\tilde{u}\|$ for sufficiently large $\rho{\circ}\sigma$. In what follows, we shall assume $\rho{\circ}\sigma = \rho/\|\mathbf{s}\|_H^4 \geq C_0$ for a constant $C_0 > 0$, accordingly.

In the following, we prove Proposition 6.3, which is a key to the estimation of $\|J\tilde{u}\|$. To begin with, let us note that the left invariant orthonormal frames obtained in the previous section can be expressed as follows:

When $\mathbb{K}H^{n+1} = \mathbb{C}H^{n+1}$,

$$\begin{cases} L_{2j-1} = \rho^{1/2}\left(\partial/\partial x^{j1} + 2x^{j2}\partial/\partial t^2\right), \\ L_{2j} \quad = \rho^{1/2}\left(\partial/\partial x^{j2} - 2x^{j1}\partial/\partial t^2\right), \\ L_{2n+1} = 2\rho\partial/\partial t^2, \\ L_{2n+2} = 2\rho\partial/\partial\rho. \end{cases} \qquad (1 \leq j \leq n).$$

When $\mathbb{K}H^{n+1} = \mathbb{H}H^{n+1}$,

$$\begin{cases} L_{4j-3} = \rho^{1/2}\left(\partial/\partial x^{j1} + 2x^{j2}\partial/\partial t^2 + 2x^{j3}\partial/\partial t^3 + 2x^{j4}\partial/\partial t^4\right), \\ L_{4j-2} = \rho^{1/2}\left(\partial/\partial x^{j2} - 2x^{j1}\partial/\partial t^2 - 2x^{j4}\partial/\partial t^3 + 2x^{j3}\partial/\partial t^4\right), \\ L_{4j-1} = \rho^{1/2}\left(\partial/\partial x^{j3} + 2x^{j4}\partial/\partial t^2 - 2x^{j1}\partial/\partial t^3 - 2x^{j2}\partial/\partial t^4\right), \\ L_{4j} \quad = \rho^{1/2}\left(\partial/\partial x^{j4} - 2x^{j3}\partial/\partial t^2 + 2x^{j2}\partial/\partial t^3 - 2x^{j1}\partial/\partial t^4\right), \\ L_{4n+l-1} = 2\rho\partial/\partial t^l \quad (2 \leq l \leq 4), \\ L_{4n+4} = 2\rho\partial/\partial\rho. \end{cases} \qquad (1 \leq j \leq n).$$

Regarding these, for a point $\mathbf{s} \in N \cdot \mathbb{R}_+$, we define the matrices $\mathcal{T} = (\tau_{ij}(\mathbf{s}))$ in the case of $\mathbb{C}H^{n+1}$ by

$$(\tau_{ij}(\mathbf{s})) = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \cdots & & & \\ & & & 1 & & \\ -2x^{12} & 2x^{11} & \cdots & 2x^{n1} & 2\rho^{1/2} & \\ & & & & & 2\rho^{1/2} \end{pmatrix},$$

and in the case of $\mathbb{H}H^{n+1}$ by

$$(\tau_{ij}(\mathbf{s})) = \begin{pmatrix} 1 &&&&&&&&& \\ & 1 &&&&&&&& \\ && 1 &&&&&&& \\ &&& 1 &&&&&& \\ &&&& \cdots & \cdots &&&& \\ &&&&& 1 &&&& \\ -2x^{12} & 2x^{11} & -2x^{14} & 2x^{13} & \cdots & 2x^{n3} & 2\rho^{1/2} &&& \\ -2x^{13} & 2x^{14} & 2x^{11} & -2x^{12} & \cdots & -2x^{n2} && 2\rho^{1/2} && \\ -2x^{14} & -2x^{13} & 2x^{12} & 2x^{11} & \cdots & 2x^{n1} &&& 2\rho^{1/2} & \\ &&&&&&&&& 2\rho^{1/2} \end{pmatrix}.$$

Let $\mathcal{T}^{-1} = (\tau^{ij}(\mathbf{s}))$ denote their inverse matrices. Utilizing these, we can express the frame $\{L_i\}$ as

$$L_i = \rho^{1/2} \sum_{l=1}^{m} \tau_{li}(\mathbf{s}) \frac{\partial}{\partial s^l},$$

and the dual frame $\{L_j^*\}$ as

$$L_j^* = \rho^{-1/2} \sum_{l=1}^{m} \tau^{jl}(\mathbf{s}) ds^l,$$

where $(s^1, \ldots, s^m) = (x^{11}, \ldots, t^d, \rho)$. Indeed, we can see the following:

$$L_i(L_j^*) = \sum_{l=1}^{m} \tau_{li}\tau^{jl} = \delta_i^j.$$

Next, for a column vector $d\mathbf{s} = {}^t(ds^1, \ldots, ds^m)$, we define the column vector $\mathbb{L}^*$ by

$$\mathbb{L}^* = \begin{pmatrix} L_1^* \\ \vdots \\ L_m^* \end{pmatrix} = \rho^{-1/2}\mathcal{T}^{-1}(\mathbf{s}) \begin{pmatrix} ds^1 \\ \vdots \\ ds^m \end{pmatrix} = \rho^{-1/2}\mathcal{T}^{-1}(\mathbf{s}) \cdot d\mathbf{s}.$$

Utilizing this notation, we can express $g = {}^t\mathbb{L}^* \cdot \mathbb{L}^*$ because $\{L_j\}$ is an orthonormal frame. Moreover, since σ is an isometry, we have

$$g = {}^t\,\mathbb{L}^* \cdot \mathbb{L}^* = {}^t\left(\big(\rho\circ\sigma(\mathbf{s})\big)^{-1/2}\mathcal{T}(\sigma(\mathbf{s}))^{-1} d\sigma\right) \cdot \left(\big(\rho\circ\sigma(\mathbf{s})\big)^{-1/2}\mathcal{T}(\sigma(\mathbf{s}))^{-1} d\sigma\right). \tag{49}$$

Furthermore, for the Jacobi matrix $J\sigma = (\partial(s^i{\circ}\sigma)/\partial s^j)$, we can express the column vector $d\sigma = {}^t(d(s^1{\circ}\sigma), \dots, d(s^m{\circ}\sigma))$ as follows:

$$d\sigma = \begin{pmatrix} d(s^1{\circ}\sigma) \\ \vdots \\ d(s^m{\circ}\sigma) \end{pmatrix} = J\sigma \cdot d\mathbf{s} = J\sigma \cdot \mathcal{T}(\mathbf{s})\rho^{1/2}\rho^{-1/2}\mathcal{T}(\mathbf{s})^{-1}d\mathbf{s} = J\sigma \cdot \mathcal{T}(\mathbf{s})\rho^{1/2}\mathbb{L}^*.$$

By substituting the above $d\sigma$ to each respective term in (49), we can obtain the following

PROPOSITION 6.3.

$$\rho^{1/2}\big(\rho{\circ}\sigma(\mathbf{s})\big)^{-1/2}\mathcal{T}(\sigma(\mathbf{s}))^{-1}{\cdot}J\sigma(\mathbf{s}){\cdot}\mathcal{T}(\mathbf{s})$$

is an orthogonal matrix.

REMARK 6.3. Observing that

$$\begin{aligned} \tau_{\mathbf{s}*}(\sum_{i=1}^m a_i\partial/\partial s^i) &= \sum_{i=1}^m a_i\tau_{\mathbf{s}*}(\partial/\partial s^i) \\ &= \sum_{j\in I_1} a^j\rho^{1/2}\sum_{l=1}^m \tau_{lj}\partial/\partial s^l + 2^{-1}\sum_{j\in I_2}\rho^{1/2}a^j\sum_{l=1}^m \tau_{lj}\partial/\partial s^l \\ &\quad +2^{-1}a^m\rho^{1/2}\sum_{l=1}^m \tau_{lm}\partial/\partial s^l, \end{aligned}$$

it is easy to confirm the following well-known fact, since $(\tau_{lj}(\mathbf{s}))$ is non-singular: The linear map

$$\tau_{\mathbf{s}*}\colon T_o(M) \ni (a^1, \dots, a^m) \to (a^1, \dots, a^m)\begin{pmatrix} \rho^{1/2} & & & & \\ & \cdots & & & \\ & & 2^{-1}\rho^{1/2} & & \\ & & & \cdots & \\ & & & & 2^{-1}\rho^{1/2} \end{pmatrix}(\tau_{lj}(\mathbf{s})) \in T_{\mathbf{s}\cdot o}(M)$$

is non-singular; this fact is in consistent with the fact that multiplication $\tau_{\mathbf{s}}$ (the left translation) is a diffeomorphism of the Lie group.

By means of Proposition 6.3, we have the following lemma:

LEMMA 6.7. *For $|\mathbf{s}| < 5^{-1/2}$ and $\rho/\|\mathbf{s}\|_H^4 > C_0$, we have*

$$\|J\tilde{u}\| \;\leq\; C_{17}\|u{\circ}\sigma\|_H^{-2}\|\mathbf{s}\|_H^{-2}\psi(\rho{\circ}\sigma)^{-1/2}\dot{\psi}(\rho{\circ}\sigma)|\mathbf{s}|^{-2}.$$

PROOF. To begin with, by using the chain rule, we have

$$J\tilde{u} = J\sigma(u\circ\sigma)\cdot Ju(\sigma)\cdot J\sigma.$$

Then, Proposition 6.3 leads us to rewrite the above as follows:

$$\begin{aligned} J\tilde{u} &= \big(\rho\circ\sigma(u\circ\sigma)\big)^{1/2}\big(\rho\circ u(\sigma)\big)^{-1/2}\mathcal{T}(\sigma(u\circ\sigma)) \\ &\quad \cdot\big(\rho\circ u(\sigma)\big)^{1/2}\big(\rho\circ\sigma(u\circ\sigma)\big)^{-1/2}\cdot\mathcal{T}(\sigma(u\circ\sigma))^{-1}\cdot J\sigma(u\circ\sigma)\cdot\mathcal{T}(u\circ\sigma) \quad (50)\\ &\quad \cdot\mathcal{T}(u\circ\sigma)^{-1}\cdot Ju(\sigma)\cdot\big(\rho\circ\sigma(\mathbf{s})\big)^{1/2}\rho^{-1/2}\cdot\mathcal{T}(\sigma(\mathbf{s})) \\ &\quad \cdot\rho^{1/2}\big(\rho\circ\sigma(\mathbf{s})\big)^{-1/2}\mathcal{T}^{-1}(\sigma(\mathbf{s}))\cdot J\sigma(\mathbf{s})\cdot\mathcal{T}(\mathbf{s}) \quad (51)\\ &\quad \cdot\mathcal{T}(\mathbf{s})^{-1}. \end{aligned}$$

Here, factors (50) and (51) in the above product are orthogonal matrices by Proposition 6.3. Hence, by using the triangle inequality, we find that

$$\begin{aligned} \|J\tilde{u}\| &\le m^2\big(\rho\circ\sigma(u\circ\sigma)\big)^{1/2}\big(\rho\circ u(\sigma)\big)^{-1/2}(\rho\circ\sigma)^{1/2}\rho^{-1/2} \\ &\quad \cdot\|\mathcal{T}(\sigma(u\circ\sigma))\|\|\mathcal{T}(u\circ\sigma)^{-1}\cdot Ju(\sigma)\cdot\mathcal{T}(\sigma)\|\|\mathcal{T}(\mathbf{s})^{-1}\|, \end{aligned}$$

where $m = n_1 + n_2 + 1$.

Next, we shall evaluate each term of the above inequality individually.
Firstly, by using Remark 6.1, we have

$$\big(\rho\circ\sigma(u\circ\sigma)\big)^{1/2}\big(\rho\circ u(\sigma)\big)^{-1/2}(\rho\circ\sigma)^{1/2}\rho^{-1/2} = \|u\circ\sigma\|_H^{-2}\|\mathbf{s}\|_H^{-2}.$$

Secondly, we can observe that

$$\|\mathcal{T}(\mathbf{s})\|^2 = \mathrm{Tr}({}^t\mathcal{T}(\mathbf{s})\cdot\mathcal{T}(\mathbf{s})) = n_1 + 4n_2(|\mathbf{x}|^2 + \rho) + 4\rho.$$

Furthermore, according to the proof of Lemma 6.4, it holds that $|\rho\circ\tilde{u}| \le |\mathbf{x}\circ\tilde{u}|^2 + |\rho\circ\tilde{u}| \le \|\tilde{u}\|_H^2 \le 1$. Hence we have

$$\|\mathcal{T}(\sigma(u\circ\sigma))\|^2 \le n_1 + 4n_2 + 4.$$

Thirdly, we can easily see that $\mathcal{T}(u\circ\sigma)^{-1}\cdot Ju(\sigma)\cdot\mathcal{T}(\sigma)$ is a diagonal matrix with entries:

$$1, \quad \big(\rho\circ\sigma/\psi(\rho\circ\sigma)\big)^{1/2}, \quad \big(\rho\circ\sigma/\psi(\rho\circ\sigma)\big)^{1/2}\dot{\psi}(\rho\circ\sigma).$$

In the first case, by utilizing our construction of ψ, we find that $\rho\circ\sigma/\psi(\rho\circ\sigma) = \exp\big(-f(\log(\lambda\rho\circ\sigma))\big) \le 1$, thereby we have

$$\big(\rho\circ\sigma/\psi(\rho\circ\sigma)\big)^{1/2} \le 1.$$

In the second case, since $\psi(\rho) = \rho \exp(f(\log(\lambda\rho)))$ and $f' > 0$, it holds that

$$\big(\rho\circ\sigma/\psi(\rho\circ\sigma)\big)^{1/2}\dot{\psi}(\rho\circ\sigma) = \big(1 + f'(\log(\lambda\,\rho\circ\sigma))\big)\exp(2^{-1}f(\log(\lambda\,\rho\circ\sigma))) > 1.$$

Comparing them, we have

$$\|\mathcal{T}(u\circ\sigma)^{-1}\cdot Ju(\sigma)\cdot\mathcal{T}(\sigma(\mathbf{s}))\| \leq m\dot{\psi}(\rho\circ\sigma)\,\big(\rho\circ\sigma/\psi(\rho\circ\sigma)\big)^{1/2}.$$

Fourthly, because $|\mathbf{s}| < 5^{-1/2}$ implies $\rho < 1$ and $|\mathbf{x}| < 1$, we may observe

$$\begin{aligned}\|\mathcal{T}(\mathbf{s})^{-1}\|^2 &= n_1 + 4^{-1}n_2(|\mathbf{x}|^2+1)\rho^{-1} + \rho^{-1}4^{-1}\\ &\leq n_1 + (3n_2+1)4^{-1}\rho^{-1}\\ &\leq \max(n_1, (3n_2+1)4^{-1})\rho^{-1}.\end{aligned}$$

Finally, by using the inequality (45), we obtain

$$(\rho\circ\sigma)^{1/2}\rho^{-1/2} = 1/\|\mathbf{s}\|_H^2 \leq C_8^2|\mathbf{s}|^{-2}.$$

Summing up these, we have completed the proof of the lemma. □

LEMMA 6.8. *Given $\delta > 1$, for $|\mathbf{s}| < 5^{-1/2}$ and $\rho/\|\mathbf{s}\|_H^4 > C_0$, we have*

$$\|(u\circ\sigma)(\mathbf{s})\|_H^{-2}\|\mathbf{s}\|_H^{-2}\dot{\psi}(\rho\circ\sigma)\psi(\rho\circ\sigma)^{-1/2} \leq C_{19}/|\mathbf{s}|^{2\delta-1}. \tag{52}$$

PROOF. It should be noted that it suffices to get the estimate (52) for $3/2 > \delta > 1$, since we assume $1 \leq 1/|\mathbf{s}|$ (as in $|\mathbf{s}| < 5^{-1/2}$).

By using Definition 6.1, we have the following:

$$\begin{aligned}&\|u\circ\sigma\|_H^{-2}\|\mathbf{s}\|_H^{-2}\psi(\rho\circ\sigma)^{-1/2}\dot{\psi}(\rho\circ\sigma)\\ &= \frac{\dot{\psi}(\rho\circ\sigma)\psi(\rho\circ\sigma)^{-1/2}}{\Big(\big(\|\mathbf{s}\|_H^2|\mathbf{x}\circ\sigma|^2 + \|\mathbf{s}\|_H^2\psi(\rho\circ\sigma)\big)^2 + |\mathbf{t}\circ\sigma|^2\|\mathbf{s}\|_H^4\Big)^{1/2}} =: R_3.\end{aligned}$$

In what follows, for the sake of simplicity, we shall set $\eta = \|\mathbf{s}\|_H^2\psi(\rho\circ\sigma)$.

In order to complete our estimation, we shall discuss the following two cases separately, with careful consideration of the following identity as in Lemma 6.1:

$$1 = \|\sigma(\mathbf{s})\|_H^4\|\mathbf{s}\|_H^4 = \|\mathbf{s}\|_H^4|\mathbf{t}\circ\sigma|^2 + (\|\mathbf{s}\|_H^2|\mathbf{x}\circ\sigma|^2 + \rho/\|\mathbf{s}\|_H^2)^2. \tag{53}$$

In the first case when $\|\mathbf{s}\|_H^4|\mathbf{t}\circ\sigma|^2 \geq 1/2$, by using (38) and (45) we obtain,

$$\begin{aligned} R_3 &\leq \frac{C_3\psi(\rho\circ\sigma)^{\delta-1/2}}{(1/2+\eta^2)^{1/2}} = \frac{C_3\eta^{\delta-1/2}}{(1/2+\eta^2)^{1/2}}\frac{1}{\|\mathbf{s}\|_H^{2\delta-1}} \\ &\leq \frac{C_3C_{21}}{\|\mathbf{s}\|_H^{2\delta-1}} \leq \frac{C_4C_{21}C_{22}}{|\mathbf{s}|^{2\delta-1}}, \end{aligned}$$

where $C_{22} = C_8^{1-2\delta}$.

In the second case when $\|\mathbf{s}\|_H^4|\mathbf{t}\circ\sigma|^2 \leq 1/2$, it follows that

$$R_3 \leq \frac{\dot\psi(\rho/\|\mathbf{s}\|_H^4)\psi(\rho/\|\mathbf{s}\|_H^4)^{-1/2}}{\|\mathbf{s}\|_H^2|\mathbf{x}\circ\sigma|^2 + \|\mathbf{s}\|_H^2\psi(\rho\circ\sigma)} =: R_4.$$

Firstly, when $\|\mathbf{s}\|_H^2|\mathbf{x}\circ\sigma|^2 \geq 1/2$, by using (38) and (45) once again, we have

$$R_4 \leq \frac{C_3\psi(\rho\circ\sigma)^{\delta-1/2}}{1/2+\eta} = \frac{C_3\eta^{\delta-1/2}}{1/2+\eta}\frac{1}{\|\mathbf{s}\|_H^{2\delta-1}} \leq \frac{C_3C_{23}}{\|\mathbf{s}\|_H^{2\delta-1}} \leq \frac{C_3C_{23}C_{22}}{|\mathbf{s}|^{2\delta-1}}.$$

Secondly, when $\|\mathbf{s}\|_H^2|\mathbf{x}\circ\sigma|^2 \leq 1/2$, by utilizing the identity (53) together with $\|\mathbf{s}\|_H^4|\mathbf{t}\circ\sigma|^2 \leq 1/2$, we have $(\|\mathbf{s}\|_H^2|\mathbf{x}\circ\sigma|^2 + \rho/\|\mathbf{s}\|_H^2)^2 \geq 1/2$, which implies that $\|\mathbf{s}\|_H^2|\mathbf{x}\circ\sigma|^2 + \rho/\|\mathbf{s}\|_H^2 \geq 1/\sqrt{2}$. Hence it follows that

$$\rho/\|\mathbf{s}\|_H^2 \geq 1/\sqrt{2} - \|\mathbf{s}\|_H^2|\mathbf{x}\circ\sigma|^2.$$

Combining this with $\|\mathbf{s}\|_H^2|\mathbf{x}\circ\sigma|^2 \leq 1/2$, we have $\rho/\|\mathbf{s}\|_H^2 \geq 1/\sqrt{2} - 1/2$, and hence

$$\|\mathbf{s}\|_H^2/\rho \leq (1/\sqrt{2} - 1/2)^{-1}.$$

Making use of this inequality together with (36) and (37), we have

$$\begin{aligned} R_4 &\leq \frac{C_2\big(\rho/\|\mathbf{s}\|_H^4\big)^{3\mathcal{N}/2}}{\psi(\rho/\|\mathbf{s}\|_H^4)^{1/2}\|\mathbf{s}\|_H^2} \leq \frac{C_2\big(\rho/\|\mathbf{s}\|_H^4\big)^{3\mathcal{N}/2+1}\big(\rho/\|\mathbf{s}\|_H^4\big)^{-1}}{\big(\rho/\|\mathbf{s}\|_H^4\big)^{1/2}\exp\big(2^{-1}C_1\big(\rho/\|\mathbf{s}\|_H^4\big)^{\mathcal{N}/2}\big)\|\mathbf{s}\|_H^2} \\ &= C_2\big(\|\mathbf{s}\|_H^2/\rho\big)\big(\rho/\|\mathbf{s}\|_H^4\big)^{3\mathcal{N}/2+1/2}\exp\big(-2^{-1}C_1\big(\rho/\|\mathbf{s}\|_H^4\big)^{\mathcal{N}/2}\big) < \infty. \end{aligned}$$

Summing up these, we have verified (52). ☐

Proof of Proposition 6.1. By combining the estimates in Lemmas 6.7 and 6.8, we can prove Proposition 6.1. ☐

4. $u_{B^n_{\mathbb{K}}} \notin C^\varepsilon(\overline{B^n_{\mathbb{K}}}, \overline{B^n_{\mathbb{K}}})$ for $\varepsilon > 1/2$

Proposition 6.4. *$u_{B^n_{\mathbb{K}}} \notin C^\varepsilon(\overline{B^n_{\mathbb{K}}}, \overline{B^n_{\mathbb{K}}})$ for $\varepsilon > 1/2$ and $\mathbb{K} = \mathbb{C}, \mathbb{H}$.*

Define two paths $\mathbf{s}_i(\tau)$ $(i = 1, 2)$ by

$$\begin{aligned} &\rho\circ\mathbf{s}_1(\tau) = \tau^4\psi^{-1}(1/\tau^2), \rho\circ\mathbf{s}_2(\tau) = 0, \\ &\|\mathbf{s}_1(\tau)\|_H = \tau, \|\mathbf{s}_2(\tau)\|_H = (t^2\circ\mathbf{s}_1(\tau))^{1/2} = (t^2\circ\mathbf{s}_2(\tau))^{1/2}, \\ &\mathbf{x}\circ\mathbf{s}_i(\tau) = t^3\circ\mathbf{s}_i(\tau) = t^4\circ\mathbf{s}_i(\tau) = 0, \quad i = 1, 2. \end{aligned}$$

First, we note that $\|\mathbf{s}_1(\tau)\|_H^4 = \tau^8\psi^{-1}(1/\tau^2)^2 + (t^2\circ\mathbf{s}_1(\tau))^2$ implies

$$(t^2\circ\mathbf{s}_1(\tau)/\|\mathbf{s}_1(\tau)\|_H^2)^2 = 1 - \tau^4\psi^{-1}(1/\tau^2)^2.$$

Since $\rho\circ\tilde{u}(\mathbf{s}_1) = \psi(\rho\circ\sigma(\mathbf{s}_1))/(\psi(\rho\circ\sigma(\mathbf{s}_1))^2 + (t^2\circ\mathbf{s}_1(\tau)/\|\mathbf{s}_1(\tau)\|_H^4)^2)$, we have

$$\begin{aligned} |\tilde{u}(\mathbf{s}_1(\tau)) - \tilde{u}(\mathbf{s}_2(\tau))| &\geq |\rho\circ\tilde{u}(\mathbf{s}_1) - \rho\circ\tilde{u}(\mathbf{s}_2)| \\ &= \rho\circ\tilde{u}(\mathbf{s}_1) = \tau^2/\big(2 - \tau^4\psi^{-1}(1/\tau^2)^2\big). \end{aligned}$$

If $\tilde{u}$ is ε-Hölder continuous, we have

$$|\tilde{u}(\mathbf{s}_1(\tau)) - \tilde{u}(\mathbf{s}_2(\tau))| \leq C_{28}|\mathbf{s}_1(\tau) - \mathbf{s}_2(\tau)|^\varepsilon = C_{28}\tau^{4\varepsilon}\psi^{-1}(1/\tau^2)^\varepsilon.$$

which implies that

$$\tau^2/(2 - \tau^4\psi^{-1}(1/\tau^2)^2) \leq C_{28}\tau^{4\varepsilon}\big(\psi^{-1}(1/\tau^2)\big)^\varepsilon.$$

Hence we obtain

$$\begin{aligned} 1 &\leq (2 - \tau^4\psi^{-1}(1/\tau^2)^2)C_{28}\tau^{2(2\varepsilon-1)}\big(\psi^{-1}(1/\tau^2)\big)^\varepsilon \\ &\leq 2C_{28}\tau^{2(2\varepsilon-1)}\big(\psi^{-1}(1/\tau^2)\big)^\varepsilon. \end{aligned} \tag{54}$$

For $\rho' = \rho\circ\sigma(\mathbf{s}_1) = \rho\circ\mathbf{s}_1/\tau^4$, we have $1/\tau^2 = \psi(\rho') = \rho'\exp(f(\log(\lambda\rho')))$. Therefore we obtain $\log(1/\tau^2) = \log(\rho') + f(\log(\lambda\rho'))$. Thus, by using (41) we have

$$\begin{aligned} C_5 &= \lim_{\rho'\to\infty} \frac{f(\log(\rho'))}{(\mathcal{N}/2)^{-1}e^{\mathcal{N}\log(\rho')/2}} \\ &= \lim_{\rho'\to\infty} \frac{\log(1/\tau^2) - \log(\rho')}{(\mathcal{N}/2)^{-1}\rho'^{\mathcal{N}/2}} \\ &= \lim_{\rho'\to\infty} \frac{\log(1/\tau^2)}{(\mathcal{N}/2)^{-1}\rho'^{\mathcal{N}/2}}. \end{aligned}$$

Hence it follows that

$$\psi^{-1}(1/\tau^2) = \rho' \sim \big(\log(1/\tau^2)\big)^{2/\mathcal{N}}.$$

Combining the above with (54) and further supposing that $\varepsilon > 1/2$, we observe that the right-hand side of (54) tends to be 0 when $\tau \to 0$. However, the left-hand side is one. This leads to a contradiction. $\square$

CHAPTER 7

Graham's non-isotropic Hölder spaces

In this chapter, for coordinate functions of those harmonic maps we constructed in Chapter 5, we shall estimate their regularity in terms of Graham's non-isotropic Hölder spaces.

To begin with, let $\mathbf{n} = (\mathbf{x}, \mathbf{t})$ denote a point of $N = \mathbb{K}^n \times \mathrm{Im}(\mathbb{K})$ and define the Heisenberg distance function d_N of N by

$$d_N(\mathbf{n}_0, \mathbf{n}_1) \;=\; \left(|\mathbf{x}_0 - \mathbf{x}_1|^4 + |\mathbf{t}_0 - \mathbf{t}_1 - 2\mathrm{Im}(\mathbf{x}_0 \cdot \bar{\mathbf{x}}_1)|^2\right)^{1/4},$$

which has a good property for scaling, that is,

$$\rho d_N(\mathbf{n}, \mathbf{n}') = d_N(\rho \cdot \mathbf{n}, \rho \cdot \mathbf{n}'), \tag{55}$$

where $\rho \cdot \mathbf{n} = \rho \cdot (\mathbf{x}, \mathbf{t}) = (\rho^{1/2}\mathbf{x}, \rho\mathbf{t})$ is the dilation. Then, by utilizing d_N, Folland and Stein's Hölder space Γ_β is defined to be the set of functions f on N satisfying:

$$|f(\mathbf{n}_1) - f(\mathbf{n}_2)| \le C_{29} d_N(\mathbf{n}_1, \mathbf{n}_2)^\beta \qquad \text{for all } \mathbf{n}_1, \mathbf{n}_2 \in N.$$

Extensive research has been made regarding the properties of this Γ_β space. Well-known inclusion relationships are given below:

Lemma 7.1.

$$C^\beta \subset \Gamma_\beta \subset C^{\beta/2}.$$

This can be proved by noting that

$$C_{30} d_N(\mathbf{n}_1, \mathbf{n}_2) \le |\mathbf{n}_1 - \mathbf{n}_2| \le C_{31} d_N(\mathbf{n}_1, \mathbf{n}_2)^{1/2} \tag{56}$$

for small $\mathbf{n}_1, \mathbf{n}_2$.

Given these spaces, let us consider the boundary value h of coordinate functions of our harmonic maps. We can easily observe the following:

$$\begin{aligned} |t^l{\circ}h(\mathbf{n}_1) - t^l{\circ}h(\mathbf{n}_2)| &\le C_{32} d_N(\mathbf{n}_1, \mathbf{n}_2)^\beta && \text{for all } \beta \le 2 && (2 \le l \le d), \\ |x^{jl}{\circ}h(\mathbf{n}_1) - x^{jl}{\circ}h(\mathbf{n}_2)| &\le C_{33} d_N(\mathbf{n}_1, \mathbf{n}_2)^\beta && \text{for all } \beta \le 1 && (1 \le j \le n, 1 \le l \le d), \end{aligned}$$

which concludes the following:

COROLLARY 7.1. *$u|_{\partial M} \in \Gamma_\beta$ for $\beta \leq 1$.*

Taking spaces Γ_β into account, Graham [**12**] defined Hölder spaces on $M = N \cdot \mathbb{R}_+$ whose members have boundary values belonging to Γ_β, by utilizing a discretization of the invariant metric g, that is, the distance function:

$$d(\mathbf{s}_1, \mathbf{s}_2)^2 = \frac{|\Delta \mathbf{x}|^2}{\rho} + \frac{|\Delta \rho|^2}{\rho^2} + \frac{|\Delta t - 2\mathrm{Im}(\mathbf{x}_1 \cdot \Delta \bar{\mathbf{x}})|^2}{\rho^2}$$

for $\mathbf{s}_i = (\mathbf{x}_i, \mathbf{t}_i, \rho_i) \in M, (i = 1, 2)$, where

$$\rho = \min(\rho_1, \rho_2), \quad \Delta \mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2, \quad \Delta \mathbf{t} = \mathbf{t}_1 - \mathbf{t}_2 \quad \text{and} \quad \Delta \rho = \rho_1 - \rho_2.$$

In order to identify the above with Graham's expression [**12**, (6.2)], we may compute the following:

$$\begin{aligned} d(\mathbf{s}_1, \mathbf{s}_2)^2 &= |(\rho^{-1/2} \Delta \mathbf{x}, \rho^{-1} \Delta \rho, \rho^{-1}(\Delta \mathbf{t} - 2\mathrm{Im}(\mathbf{x}_1 \cdot \Delta \bar{\mathbf{x}})))|^2 \\ &= |\rho^{-1} \cdot (\mathbf{n}_1 \cdot \mathbf{n}_2^{-1}, \rho_1 - \rho_2)|^2. \end{aligned}$$

By using this distance function, Graham defined a two-parameter family of Hölder spaces Γ_α^β as follows: for $-\infty < \beta \leq \alpha$, $0 < \alpha < 1$, a function f on M is in Γ_α^β if

$$|f(\mathbf{s}_1) - f(\mathbf{s}_2)| \leq C_{34} \rho^{\beta/2} d(\mathbf{s}_1, \mathbf{s}_2)^\alpha \qquad \text{for all } \mathbf{s}_1, \mathbf{s}_2 \in M.$$

In this section, we shall assume that all functions on $M \cup \{\rho = 0\}$ are compactly supported, by multiplying smooth cut-off functions if required.

Now let us recall the following theorem by Graham [**12**, Theorem 6.17, Proposition 6.7]:

THEOREM 7.1. *Suppose that $\alpha \geq \beta > 0$. Then we have $C^\alpha \subset \Gamma_\alpha^\beta$. Moreover, f being in Γ_α^β implies that $f(\cdot, \rho)$ belongs to Γ_β uniformly in ρ. Consequently, f has a boundary value $f(\cdot, 0)$ belonging to Γ_β.*

As a consequence of this theorem, we may make a characterization of Γ_α^β as a space of functions which belong to C^α in the interior and whose boundary values belong to Γ_β. The space C_k^β analogous to Γ_α^β was examined by Graham, where the interior regularity was measured in terms of C^k norms, rather than C^α norms. This space is defined in the following way:

For multi indices $\gamma = (\gamma^1, \ldots, \gamma^m)$, let us set

$$D^\gamma = e_1^{\gamma^1} \cdots e_m^{\gamma^m}, \quad |\gamma| = \sum_{i=1}^m \gamma^i, \quad \mathrm{wt}(\gamma) = \sum_{i \in I_1} \gamma^i + \sum_{i \in I_2} 2\gamma^i + 2\gamma^m.$$

Then C^β_k is defined to be the space of functions f satisfying

$$|D^\gamma f| \le C_{35}\rho^{(\beta-\mathrm{wt}(\gamma))/2}$$

for all multi indices γ satisfying $|\gamma| \le k$, taking into account the appropriate weight for each derivative.

PROPOSITION 7.1 (Proposition 6.15 [**12**]). $C^\beta_1 \subset \Gamma^\beta_\alpha$.

It should be remarked that these spaces Γ^β_α and C^β_k are invariant under group actions.

PROPOSITION 7.2 (Proposition 6.7 [**12**]). *$f \in \Gamma^\beta_\alpha$ if and only if $f\circ\tau_{\mathbf{s}} \in \Gamma^\beta_\alpha$, and $f \in C^\beta_k$ if and only if $f\circ\tau_{\mathbf{s}} \in C^\beta_k$.*

In order to study the Dirichlet problem at infinity for harmonic maps, Donnelly [**7**] adopted Graham's space C^β_k and proved the uniqueness of the solution within C^β_3 for $\beta > 2$. Following them, let us estimate the regularity of the coordinate functions of our harmonic maps $\tilde{u}$ near the origin on the chart given in (25).

First, we remark the following:

$$\|J\tilde{u}\cdot\mathcal{T}(\mathbf{s})\rho^{1/2}\|^2 = \sum_{i=1}^{m}\sum_{j=1}^{m}\Big(L_j s^i\circ\tilde{u}\Big)^2 \ge \sum_{i=1}^{m}\sum_{|\gamma|=1}\Big(|D^\gamma s^i\circ\tilde{u}|\rho^{\mathrm{wt}(\gamma)/2}\Big)^2.$$

Similar to Lemma 6.7, supposing that $\rho/\|\mathbf{s}\|^4_H > C_0$ [as given in Remark 6.2], we have

LEMMA 7.2.

$$\|J\tilde{u}\cdot\mathcal{T}(\mathbf{s})\rho^{1/2}\| \le C_7\|u\circ\sigma\|^{-2}_H\|\mathbf{s}\|^{-2}_H\dot{\psi}(\rho\circ\sigma)\big(\rho\circ\sigma/\psi(\rho\circ\sigma)\big)^{1/2}\rho^{1/2}.$$

Here, it should be remarked that, since $\mathcal{T}(\mathbf{s})$ is multiplied to $J\tilde{u}$ from the right, the troublesome term $\|\mathcal{T}^{-1}(\mathbf{s})\|$ is not present at this time as it is in Lemma 6.7.

Let us now evaluate the right-hand side of the above. Firstly, $\|\mathbf{s}\|^4_H > \rho^2$ implies that

$$\big(\rho\circ\sigma\big)^{1/2}\rho^{1/2} = (\rho/\|\mathbf{s}\|^4_H)^{1/2}\rho^{1/2} \le 1.$$

Secondly, by utilizing the proof of Lemma 6.8, we have

$$\|u\circ\sigma\|^{-2}_H\|\sigma\|^{-2}_H\dot{\psi}(\rho\circ\sigma)\psi(\rho\circ\sigma)^{-1/2} \le C_{38}\|\mathbf{s}\|^{1-2\delta}_H.$$

Finally, combining $\|\mathbf{s}\|^{1-2\delta}_H \le \rho^{(1-2\delta)/2}$ with the above, we obtain the following:

PROPOSITION 7.3. *$\tilde{u} \in C^\beta_1$ for $\beta < -1$.*

Proposition 7.3 shows that the assumption on regularity in Donnelly's theorem [**7**] cannot be removed.

Next, let us consider the space Γ^{α}_{β}. First, we remark the following:

LEMMA 7.3. ([**12**, Lemma 6.4]) *Let* $\mathbf{n}_1, \mathbf{n}_2 \in N$*, and suppose that* $\mathbf{n}_1$ *varies over a bounded set. Then* $C_{36}|\mathbf{n}_1 - \mathbf{n}_2| \leq |\mathbf{n}_1 \cdot \mathbf{n}_2^{-1}| \leq C_{37}|\mathbf{n}_1 - \mathbf{n}_2|$.

Utilizing the above as in Graham's proof [**12**, Proposition 6.8], we have the following rough estimate:

$$
\begin{aligned}
&|s^i \circ \tilde{u}(\mathbf{s}_1) - s^i \circ \tilde{u}(\mathbf{s}_2)| \\
&\leq \; C_{38}(|\mathbf{x}_1 - \mathbf{x}_2|^2 + |\mathbf{t}_1 - \mathbf{t}_2|^2 + |\rho_1 - \rho_2|^2)^{\alpha/2} \\
&\leq \; C_{39}(|\mathbf{x}_1 - \mathbf{x}_2|^2 + |\mathbf{t}_1 - \mathbf{t}_2 - 2\mathrm{Im}(\mathbf{x}_1 \cdot (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2))|^2 + |\rho_1 - \rho_2|^2)^{\alpha/2} \\
&= \; C_{39}\rho^{\alpha/2}\Big(\frac{|\mathbf{x}_1 - \mathbf{x}_2|^2}{\rho} + \frac{\rho|\mathbf{t}_1 - \mathbf{t}_2 - 2\mathrm{Im}(\mathbf{x}_1 \cdot (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2))|^2}{\rho^2} + \frac{\rho|\rho_1 - \rho_2|^2}{\rho^2}\Big)^{\alpha/2} \\
&\leq \; C_{40}\rho^{\alpha/2}\Big(\frac{|\mathbf{x}_1 - \mathbf{x}_2|^2}{\rho} + \frac{|\mathbf{t}_1 - \mathbf{t}_2 - 2\mathrm{Im}(\mathbf{x}_1 \cdot (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2))|^2}{\rho^2} + \frac{|\rho_1 - \rho_2|^2}{\rho^2}\Big)^{\alpha/2}
\end{aligned}
$$

for $\alpha < 1/7$ near the origin. In the last inequality, we remark that the right-hand side may diverge. Hence, we find that near $\rho = 0$, each coordinate function of $\tilde{u}$ is in Γ^{β}_{α} for $-\infty < \beta \leq \alpha$ for $\alpha < 1/7$. Thereby, once again, we can verify that the assumption on regularity cannot be removed from Donnelly's theorem [**7**].

CHAPTER 8

Harmonic maps between rank two symmetric spaces of noncompact type

The pourpose of this chapter is to establish the existence of harmonic self-maps on rank two symmetric spaces of noncompact type; namely, $D_{2,2}(\mathbb{C})$, the space of 2×2 complex matrices Z satisfying $I_2 - {}^t\bar{Z}Z$ equipped with metric $i\partial\overline{\partial}\log\det(I_2 - {}^t\bar{Z}Z)$.

THEOREM 8.1. *There exists a family of harmonic self-maps of $D_{2,2}(\mathbb{C})$ that induce the identity map on the nilpotent Lie group N.*

Before we start our discussion, we shall explain N in the statement above in conjunction with the outline of this Chapter. To begin with, we recall that an element in $SU(2,2)$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

acts on $D_{2,2}(\mathbb{C}) = \{Z \in M_2(\mathbb{C}) \mid I_2 - {}^t\bar{Z}Z > 0\}$ isometrically as a linear fractional transformation:

$$Z \to (AZ+B)(CZ+D)^{-1}.$$

In Section 8.1, we describe an Iwasawa decomposition of $\mathfrak{su}(2,2)$, corresponding to the Iwasawa decomposition of $SU(2,2) = KAN$, whose K coincides with the isotropy subgroup which fixes the zero matrix. Thus, $D_{2,2}(\mathbb{C})$ is diffeomorphic to AN through linear fractional transformation. N in the statement above is the locus $N \subset AN$ in this realization of $D_{2,2}(\mathbb{C})$. In Section 8.2, we compute the invariant metric of $D_{2,2}(\mathbb{C})$ and, in Section 8.3, we compute the tension field. Finally in Section 8.4, we prove Theorem 8.1. This N appears on the ideal boundary of $D_{2,2}(\mathbb{C})$ as codimension one subspace. In particular, this proves Theorem 0.2 in the introduction.

1. An Iwasawa decomposition of $\mathfrak{su}(2,2)$

Firstly, we recall that an element in $\mathfrak{su}(2,2)$ is of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ -\bar{a}_{12} & a_{22} & a_{23} & a_{24} \\ \bar{a}_{13} & \bar{a}_{23} & a_{33} & a_{34} \\ \bar{a}_{14} & \bar{a}_{24} & -\bar{a}_{34} & a_{44} \end{pmatrix},$$

where a_{11}, a_{22}, a_{33} and a_{44} are in $\sqrt{-1}\mathbb{R}$ and satisfy $a_{11}+a_{22}+a_{33}+a_{44}=0$. Next, we set

$$I_{2,2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

making $\Theta g = I_{2,2} g I_{2,2}$ a Cartan involution. The decomposition $\mathfrak{k}+\mathfrak{p}$ of $\mathfrak{su}(2,2)$, through the utilization of this Cartan involution, is given by

$$\begin{pmatrix} a_{11} & a_{12} & & \\ -\bar{a}_{12} & a_{22} & & \\ & & a_{33} & a_{34} \\ & & -\bar{a}_{34} & a_{44} \end{pmatrix} + \begin{pmatrix} & & a_{13} & a_{14} \\ & & a_{23} & a_{24} \\ \bar{a}_{13} & \bar{a}_{23} & & \\ \bar{a}_{14} & \bar{a}_{24} & & \end{pmatrix}.$$

Given this $\mathfrak{p} \subset \mathfrak{su}(2,2)$, we can select two generators of the maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$ given by

$$H_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Furthermore, considering the eigenspace decomposition of $\mathfrak{su}(2,2)$ by $\mathrm{ad}(H_1)$ and $\mathrm{ad}(H_2)$, we set

$$X_{11} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \qquad X_{21} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix},$$

$$X_{22} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & i & 0 \\ i & 0 & 0 & -i \\ -i & 0 & 0 & i \\ 0 & i & i & 0 \end{pmatrix}, \qquad X_{12} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & -i & 0 \\ i & 0 & 0 & -i \\ i & 0 & 0 & -i \\ 0 & i & -i & 0 \end{pmatrix},$$

$$T_1 = \frac{1}{2} \begin{pmatrix} -i & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & i \end{pmatrix}, \qquad T_2 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -i & i & 0 \\ 0 & -i & i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Consequently, we have the following Lie bracket relation:

$$\begin{aligned}
&[H_1, X_{11}] = X_{11},\ [H_1, X_{21}] = X_{21},\ [H_1, X_{22}] = X_{22}, \\
&[H_1, X_{12}] = X_{12},\ [H_1, T_1] = 2T_1, \\
&[H_2, X_{11}] = -X_{11},\ [H_2, X_{21}] = X_{21},\ [H_2, X_{22}] = -X_{22}, \\
&[H_2, X_{12}] = X_{12},\ [H_2, T_2] = 2T_2, \\
&[X_{11}, X_{12}] = -4T_1,\ [X_{21}, X_{22}] = -4T_1, \\
&[X_{11}, T_2] = -X_{12},\ [X_{22}, T_2] = X_{21},
\end{aligned} \tag{57}$$

and all the other Lie bracket products vanish.

Furthermore, let us observe that $\mathfrak{n} = \mathrm{Span}_{\mathbb{R}}\{X_{11}, X_{12}, X_{21}, X_{22}, T_1, T_2\}$ is closed under the Lie bracket product $[*, *]$ and that

$$\begin{aligned}
\mathfrak{n}^{(1)} &= [\mathfrak{n}, \mathfrak{n}] = \mathrm{Span}_{\mathbb{R}}\{T_1, X_{21}, X_{12}\}, \\
\mathfrak{n}^{(2)} &= [\mathfrak{n}, \mathfrak{n}^{(1)}] = \mathbb{R}\{T_1\}, \\
\mathfrak{n}^{(3)} &= [\mathfrak{n}, \mathfrak{n}^{(2)}] = \{0\},
\end{aligned}$$

which verifies that $\mathfrak{n}$ is a nilpotent Lie algebra. Similarly, we can check that $\mathfrak{a} + \mathfrak{n}$ is a solvable Lie algebra. Furthermore, we can also observe that there is a vector space direct sum decomposition of $\mathfrak{g} = \mathfrak{su}(2,2)$ in the following form:

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}.$$

This is an Iwasawa decomposition. If we denote K, A and N the subgroup of $G = SU(2,2)$ corresponding to the Lie algebra $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$, respectively, then, $K \times A \times N \ni (k,a,n) \to kan \in G$ is a diffeomorphism. Hence $S = A \cdot N$ is diffeomorphic to G/K. We should note here that K is the isotropy subgroup of $G = SU(2,2)$ at the origin $Z = O$ (zero matrix). Hence, we can identify $D_{2,2}(\mathbb{C})$ with $N \cdot A$ through the fractional transformation to the origin. That is, we have obtained a diffeomorphism

$$\mathbb{R}^6 \times \mathbb{R}_+^2 \ni (x_{11}, x_{12}, x_{21}, x_{22}, t_1, t_2, \rho_1, \rho_2) \to Z = BD^{-1} \in D_{2,2}(\mathbb{C}),$$

where B and D are submatrix of

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \exp\Big(\sum_{1 \le i,j \le 2} x_{ij} X_{ij} + \sum_{1 \le i \le 2} t_i T_i\Big) \cdot \exp\Big(\sum_{1 \le i \le 2} 2^{-1} \log(\rho_i) H_i\Big).$$

2. The invariant metric

At this point, we should note that the left invariant metric $i\partial\overline{\partial} \log\det(I - {}^t\,\overline{Z}Z)$ of $D_{2,2}(\mathbb{C})$ coincides with $\mathrm{Trace}({}^t d\bar{Z} \cdot dZ)$ at the origin $Z = O$ (zero matrix).

Given the computation made above, we shall substitute $Z = (Z_{ij}) = BD^{-1}$ to the origin. If we set $\mathbf{o} = (0,0,0,0,0,0,1,1) \in \mathbb{R}^6 \times \mathbb{R}_+^2$, then $(Z_{ij})|_{\mathbf{o}} = O$. Then we find that

$$\mathrm{Trace}({}^t d\bar{Z} \cdot dZ)|_{\mathbf{o}} = \sum_{1 \le i,j \le 2} (dx_{ij})^2 + \sum_{1 \le i \le 2} (dt_i)^2/4 + \sum_{1 \le i \le 2} (d\rho_i)^2/4.$$

Accordingly, an orthonormal basis of this inner product $\langle *, * \rangle = \mathrm{Trace}({}^t d\bar{Z} \cdot dZ)|_{\mathbf{o}}$ is given by

$$\{\partial/\partial x_{11}, \partial/\partial x_{22}, \partial/\partial x_{12}, \partial/\partial x_{21}, 2\partial/\partial t_1, 2\partial/\partial t_2, 2\partial/\partial \rho_1, 2\partial/\partial \rho_2\}.$$

Then we obtain the left invariant extensions of the orthonormal basis given above, which we can denote as follows:

$$X_{11} = \rho_1^{1/2}\rho_2^{-1/2} e_3,\ X_{22} = \rho_1^{1/2}\rho_2^{-1/2} e_4,\ X_{12} = \rho_1^{1/2}\rho_2^{1/2} e_5,$$
$$X_{21} = \rho_1^{1/2}\rho_2^{1/2} e_6,\ T_1 = 2\rho_1 e_7,\ T_2 = 2\rho_2 e_8,\ H_1 = 2\rho_1 e_1,\ H_2 = 2\rho_2 e_2.$$

Here $e_1 = \partial/\partial\rho_1, e_2 = \partial/\partial\rho_2$; and $e_3, \dots, e_8$ span the nilpotent Lie algebra of left invariant vector fields corresponding to $\mathfrak{n}$. Since $G = SU(2,2)$ acts on $D_{2,2}(\mathbb{C})$ transitively, an inner product $\langle \cdot, \cdot \rangle$ of the tangent space $T_{\mathbf{o}}(D_{2,2}(\mathbb{C}))$ at $\mathbf{o} \in D_{2,2}(\mathbb{C})$ defines a left invariant metric g by assigning $g_{\mathbf{s}\cdot\mathbf{o}}(V, V') = \langle \tau_{\mathbf{s}}^{-1}(V), \tau_{\mathbf{s}}^{-1}(V') \rangle$ for $V, V' \in T_{\mathbf{s}\cdot\mathbf{o}}(D_{2,2}(\mathbb{C}))$ at each

$\mathbf{s}\cdot\mathbf{o} \in D_{2,2}(\mathbb{C})$ for $\mathbf{s} \in SU(2,2)$, where $\tau_{\mathbf{s}}$ is the left translation. Hence, the above invariant vector fields are orthonormal frame. Thus, the components of the metric $g_{ij} = g(e_i, e_j)$, $(g^{ij}) = (g_{ij})^{-1}$ are given by

$$\begin{aligned} g_{11} &= 4^{-1}\rho_1^{-2}, \\ g_{22} &= 4^{-1}\rho_2^{-2}, \\ g_{33} = g_{44} &= \rho_2\rho_1^{-1}, \\ g_{55} = g_{66} &= \rho_2^{-1}\rho_1^{-1}, \\ g_{77} &= 4^{-1}\rho_1^{-2}, \\ g_{88} &= 4^{-1}\rho_2^{-2}, \\ g_{ij} &= g^{ij} = 0, \quad (i \neq j). \end{aligned}$$

3. The tension field

Noting that g_{ij} is diagonal, we start by simplifying the components $\tau^\alpha(u)$ of the tension field as follows:

$$\begin{aligned} \tau^\alpha(u) \;\; = \;\; & \sum_{j=1}^{8} g^{jj}\Big(u^\alpha_{jj} - \sum_{l=1}^{8} e^*_l(\nabla^M_{e_j} e_j)u^\alpha_l + \sum_{\beta=1}^{8} {e'}^*_\alpha(\nabla^{M'}_{{e'}_\beta} e'_\beta)u^\beta_j u^\beta_j \\ & +\Big(\sum_{1\le\beta\le2,\beta<\gamma\le8} + \sum_{3\le\beta<\gamma\le8}\Big)\big({e'}^*_\alpha(\nabla^{M'}_{{e'}_\beta} e'_\gamma) + {e'}^*_\alpha(\nabla^{M'}_{{e'}_\gamma} e'_\beta)\big)u^\beta_j u^\gamma_j\Big). \end{aligned}$$

This leads us to examine the following components of the Levi-Civita connection with respect to this frame:

$$\begin{aligned} &\nabla_{e_1} e_1 = -\rho_1^{-1} e_1, \\ &\nabla_{e_2} e_2 = -\rho_2^{-1} e_2, \\ &\nabla_{e_3} e_3 = \nabla_{e_4} e_4 = 2\rho_2 e_1 - 2\rho_2^2\rho_1^{-1} e_2, \\ &\nabla_{e_5} e_5 = \nabla_{e_6} e_6 = 2\rho_2^{-1} e_1 + 2\rho_1^{-1} e_2, \\ &\nabla_{e_7} e_7 = \rho_1^{-1} e_1, \\ &\nabla_{e_8} e_8 = \rho_2^{-1} e_2, \\ &\nabla_{e_1} e_j = -\rho_1^{-1} e_j/2, \quad j = 3, \dots, 6, \\ &\nabla_{e_2} e_j = \rho_2^{-1} e_j/2, \quad j = 3, 4, \\ &\nabla_{e_2} e_j = -\rho_2^{-1} e_j/2, \quad j = 5, 6, \end{aligned}$$

and for $\alpha, \beta, \gamma \ge 3$,

$$e^*_\alpha(\nabla_{e_\beta} e_\gamma) + e^*_\alpha(\nabla_{e_\gamma} e_\beta) = {g'}^{\alpha\alpha} g'_{\gamma\gamma} {a'}^\gamma_{\alpha\beta} + {g'}^{\alpha\alpha} g'_{\beta\beta} {a'}^\beta_{\alpha\gamma} =: Q^\alpha_{\beta\gamma},$$

where

$$[e_\beta, e_\gamma] = \sum_{\alpha=3}^{8} {a'}^{\alpha}_{\beta\gamma} e'_\alpha, \qquad 3 \le \beta, \gamma \le 8.$$

Since $e_3, \ldots, e_8$ span the nilpotent Lie algebra of left invariant vector fields corresponding to the nilpotent Lie algebra $\mathfrak{n}$, we find that $a^i_{ij} = 0$. Thereby, we find that

$$\begin{aligned}
\tau^1(u) &= \sum_{j=1}^{8} g^{jj} u^1_{jj} - 8\rho_1 u^1_1 \\
&\quad + \sum_{j=1}^{8} g^{jj} \Big(-(u^1_j)^2 (\rho_1 \circ u)^{-1} + 2\big((u^3_j)^2 + (u^4_j)^2\big)(\rho_2 \circ u) \\
&\quad + 2\big((u^5_j)^2 + (u^6_j)^2\big)(\rho_2 \circ u)^{-1} + (u^7_j)^2 (\rho_1 \circ u)^{-1} \Big), \\
\tau^2(u) &= \sum_{j=1}^{8} g^{jj} u^2_{jj} - 8\rho_1 u^2_1 \\
&\quad + \sum_{j=1}^{8} g^{jj} \Big(-(u^2_j)^2 (\rho_2 \circ u)^{-1} - 2\big((u^3_j)^2 + (u^4_j)^2\big)(\rho_1 \circ u)^{-1}(\rho_2 \circ u)^2 \\
&\quad + 2\big((u^5_j)^2 + (u^6_j)^2\big)(\rho_1 \circ u)^{-1} + (u^8_j)^2 (\rho_2 \circ u)^{-1} \Big), \\
\tau^\alpha(u) &= \sum_{j=1}^{8} g^{jj} u^\alpha_{jj} - 8\rho_1 u^\alpha_1 \\
&\quad + \sum_{j=1}^{8} g^{jj} \Big(R_{\alpha 1} u^\alpha_j u^1_j + R_{\alpha 2} u^\alpha_j u^2_j + \sum_{3 \le \beta < \gamma \le 8} Q^\alpha_{\beta\gamma} u^\beta_j u^\gamma_j \Big) \\
\text{for} &\quad 3 \le \alpha \le 8,
\end{aligned}$$

where $R_{\alpha 1}$ and $R_{\alpha 2}$ are functions of $\rho_1 \circ u$ and $\rho_2 \circ u$.

4. Proof of the Theorems 8.1 and 8.2

Proof of the Theorem 8.1:

Given these computations, we shall consider the tension field of the map u in the following form:

$$u \colon (\mathbf{n}, \rho_1, \rho_2) \to (h(\mathbf{n}), \psi_1(\rho_1), C_2^2 \rho_2),$$

where h is a homomorphism satisfying $h^3_3 = h^4_4 = C_1C_2^{-1}$, $h^5_5 = h^6_6 = C_1C_2$, $h^7_7 = C_1^2$, $h^8_8 = C_2^2$, and $h^\gamma_j = 0$ $(j \neq \gamma)$. Here C_1 and C_2 are constants. When $C_1 = C_2 = 1$, $h = \mathrm{id}$. From $\tau^1(u) = 0$, we must have

$$0 = 4\rho_1^2\frac{d^2\psi_1(\rho_1)}{d\rho_1^2} - 8\rho_1\frac{d\psi_1(\rho_1)}{d\rho_1} - \left(\frac{d\psi_1(\rho_1)}{d\rho_1}\right)^2\frac{4\rho_1^2}{\psi_1(\rho_1)} + \frac{4C_1^4\rho_1^2}{\psi_1(\rho_1)} + 8C_1^2\rho_1.$$

Next, we find that $\tau^2(u) = -4\rho_2^2C_2^4(C_2^2\rho_2)^{-1} - 4\rho_1\rho_2^{-1}(C_1C_2^{-1})^2(C_2^2\rho_2)^2\psi_1(\rho_1)^{-1} + 4\rho_1\rho_2(C_1C_2)^2\psi_1(\rho_1)^{-1} + 4\rho_2^2C_2^4(C_2^2\rho_2)^{-1} = 0$. Moreover, the other components of the tension fields vanish.

By substituting $\rho_1C_1^2\exp(f(\log(\rho_1)))$ for $\psi_1(\rho_1)$, we have

$$0 = f'' - 3f' + (e^{-2f} - 1) + 2(e^{-f} - 1).$$

The existence of the solution for this equation is confirmed by Theorem 2.1; alternatively, this being a translation invariant equation, the proof method of Theorem 5.1 or [**25**, Proposition 2.1] can be used in order to establish the global existence of the solution.

□

REMARK 8.1. $D_{n,n}(\mathbb{C})$ is rank n, but its cohomogeneity is one.

REMARK 8.2. $\nabla_{\rho_1e_1}\rho_1e_1 = \nabla_{\rho_2e_2}\rho_2e_2 = 0$ together with $g_{11} = 4^{-1}\rho_1^{-2}$ and $g_{22} = 4^{-1}\rho_2^{-2}$ implies that $(\{\rho_1 = 0\} \cup \{\rho_2 = 0\}) \cup (\{\rho_1 = \infty\} \cup \{\rho_2 = \infty\})$ correspond to the ideal boundary.

DEFINITION 8.1. We call the locus $(\{\rho_1 = 0\} \cap \{\rho_2 = 0\}) \cup (\{\rho_1 = \infty\} \cap \{\rho_2 = \infty\})$ the corner of $\overline{D_{2,2}(\mathbb{C})}$.

Proof of the Theorem 0.2:

The statement follows immediately from Theorem 8.1 and Definition 8.1. □

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