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by

Kazuya MATSUMI

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Kazuya MATSUMI

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ARITHMETIC OF THREE-DIMENSIONAL COMPLETE REGULAR LOCAL RINGS OF POSITIVE CHARACTERISTICS

KAZUYA MATSUMI

Contents

1. Introduction

In this paper, we prove a variety of interesting arithmetics enjoyed by three-dimensional complete regular local rings of positive characteristics. We may take $A := \mathbb{F}_p[[X, Y, Z]]$ as such a ring.

First, we explain Hasse principle. The classical Hasse principle is a basic theorem which relates the Brauer group of algebraic number fields, or one-variable function fields over finite fields, with the Brauer groups of their localizations. It was generalized to two-dimensional global fields by K. Kato in [Ka4], as the exactness of a certain complex (he calls Hasse principle complex) defined for each two-dimensional global field. Moreover, he was able to formulate this complex even for arbitrary global fields and conjectured its exactness (called cohomological Hasse principle for those fields). Until now, works of J. Colliot-Thélène, U. Jannsen or S. Saito have given affirmative answers to this conjecture (cf. [CT], [J2], [Sa3]). In particular, Colliot-Thélène proved it for arbitrary three-dimensional global fields of positive characteristic with divisible coefficients (cf. loc. cit.).

On the other hand, for an arbitrary two-dimensional complete regular local ring R having finite residue field, S. Saito proved the exactness of a certain complex associated to

R in [Sa2] which is quite similar to that of one-dimensional global fields (a cohomological Hasse principle for R). This theorem of Saito tells us that even for local rings, there exists a formalism connecting local and global arithmetics like one-dimensional global fields.

Then occurs a natural question whether it is possible to generalize this result of Saito to three-dimensional local rings which also can be seen as an analogy of a cohomological Hasse principle for two-dimensional global fields proved by Kato (cf. loc. cit.).

We show this is indeed possible at least in the positive characteristic cases. Let us state the main result.

Theorem 1.1. (**Theorem 3.1**) *Let A be an arbitrary three-dimensional complete regular local ring of positive characteristic with finite residue field. Then for an arbitrary natural number m prime to the characteristic of A, the following Hasse principle complex for A becomes exact*:

$$0 \to H^4_{ét}(K, \mu_m^{\otimes 3}) \to \bigoplus_{\mathfrak{p}\in P^1_A} H^3_{ét}(\kappa(\mathfrak{p}), \mu_m^{\otimes 2}) \to \bigoplus_{\mathfrak{m}\in P^2_A} H^2_{ét}(\kappa(\mathfrak{m}), \mu_m) \to \mathbb{Z}/m \to 0,$$

where P^i_A denotes the set of height i prime ideals of A and $\kappa(\mathfrak{p})$, $\kappa(\mathfrak{m})$ denote the residue fields of $\mathfrak{p}$, $\mathfrak{m}$*, respectively.*

We remark that the degree 'four' of the first cohomology is obtained by adding one to the Krull-dimension of A, and the twist number three of $\mu_m^{\otimes 3}$ comes from the Krull-dimension of A. This is one of the most peculiar formalism in a cohomological Hasse principle, which origins in the classical Hasse principle of the Brauer group of one-dimensional global fields (one can verify this by the isomorphism $\mathrm{Br}(K)_m \cong H^2_{\mathrm{Gal}}(K, \mu_m^{\otimes 1})$ for an arbitrary one-dimensional global field K, where $\mathrm{Br}(K)_m$ denotes the subgroup of $\mathrm{Br}(K)$ consisting of m-torsion elements).

We now explain briefly how to prove Theorem 1.1. First, in Section 2, we review the absolute purity theorem by Fujiwara-Gabber (Theorem 2.1) which enables us to interpret each term in the above complex as the local cohomology of certain henselian local rings. We also review a result of Gabber on the cohomological dimension for fractional fields of complete regular local rings over fields (Theorem 2.2). In Section 3, we prove the main result. We consider the coniveau-spectral sequence for A. Then, thanks to Theorem 2.1, we can reduce the exactness of the above Hasse principle complex to the vanishing of each $E_2^{p,q}$-term of the coniveau-spectral sequence ($p+q=4$). There, Theorem 2.2 is fully used and we finally reduce the vanishing of $E_2^{0,4}$, which is the most hard task, to the exactness of the Gersten-Quillen complex of A. But it is exact by Quillen (cf. [Q]), thus the proof is completed.

Finally, we remark the relation of our result with the famous Bloch-Ogus complex(which we do not review here). This complex is believed to be exact for an arbitrary henselian regular local ring over a field, and it is proved only for those rings obtained from the henselization of smooth local rings (roughly, the localization of affine rings of smooth varieties over a field). The above complex is the special case of the Bloch-Ogus complex for A, so our result could be seen as a partial answer to this conjecture for complete regular local rings.

Next, we explain class field theory. The motivation for this study origins in higher-dimensional class field theory established by Bloch, Kato-Saito and Parshin (cf. [Bl], [Ka-Sa1], [Ka-Sa2], [P4]). Although Kato and Saito accomplished many magnificent results, there still remain many problems unsolved. Indeed, the important progress in this direction was recently accomplished by Jannsen-Saito (cf. [J-S]). Our result in this paper can be seen also as new progress in this area, and at the same time it becomes a generalization of local class field theory. Such generalization was first accomplished successfully by S. Saito and A. N. Parshin independently by proving class field theory for two-dimensional complete regular local rings (cf. [Sa1], [P4]), but for complete local rings whose dimensions are truly bigger than two, there was no result on their class field theory at all.

Let us take a positive characteristic three-dimensional complete regular local ring having finite residue field, and denote it by A. We also denote its fractional field by K. We may, for example, take $A = \mathbb{F}_p[[X, Y, Z]]$. Our main interest is to describe the structure of the Galois group $\mathrm{Gal}(K^{ab}/K)$ by using only the geometric information of K, where K^{ab} denotes the maximal abelian extension of K in the fixed algebraic closure $\overline{K}$ of K. More precisely, we should construct the idele class group C_K and the reciprocity map

$$\rho_K \colon C_K \to \mathrm{Gal}(K^{ab}/K)$$

which approximates $\mathrm{Gal}(K^{ab}/K)$ reasonably by C_K. This formalism is well-known by class field theory of algebraic number fields. We define the idele class group C_K in Section 1, where at the same time we put a certain nice topology on it. An important fact is that the topology of our idele class group C_K is extremely plain and understandable. Next, the reciprocity map is defined in rather traditional way in Sections 4 and 5. Then, this reciprocity map turns out to describe the structure of $\mathrm{Gal}(K^{ab}/K)$ quite successfully. Now, we state our main theorems.

Theorem 1.2 (**Theorem 7.1**). *Let A be an arbitrary three-dimensional complete regular local ring of odd characteristic with finite residue field. We denote by K its fractional field. Then, the topological idele class group C_K is canonically attached to K, and it holds the following dual reciprocity isomorphism*:

$$\rho_K^*\colon H^1_{\mathrm{Gal}}(K,\,\mathbb{Q}_p/\mathbb{Z}_p) \xrightarrow{\sim} \mathrm{Hom}_c(C_K,\,\mathbb{Q}_p/\mathbb{Z}_p),$$

where $\mathrm{Hom}_c(C_K,\,\mathbb{Q}_p/\mathbb{Z}_p)$ *denotes the set of all continuous homomorphisms of finite order from* C_K *to* $\mathbb{Q}_p/\mathbb{Z}_p$.

Remark 1. We can eliminate the assumption of $\mathrm{ch}(K) \neq 2$ in the theorem if we assume the exactness of a certain complex of Gersten-Quillen type for Milnor K-groups. For details, see Remark 7.

This theorem is the most hard part in this paper and is established in Section 5 by using all results in Sections 3 and 4. The most important key for the proof of Theorem 1.2 is the exact sequence (7.71) which is obtained by the deep study of the various étale cohomologies of Spec$A$.

Next, we state prime to p parts.

Theorem 1.3 (**Theorem 8.1**). *Let A be an arbitrary three-dimensional complete regular local ring of positive characteristic with finite residue field and K be its fractional field* (*we may allow the case of* $\mathrm{ch}(K) = 2$). *Then, under the Bloch-Milnor-Kato conjecture* (*see below*), *we have the dual reciprocity isomorphism*

$$\rho_K^*\colon H^1_{\mathrm{Gal}}(K,\,\mathbb{Q}_l/\mathbb{Z}_l) \xrightarrow{\sim} \mathrm{Hom}_c(C_K,\,\mathbb{Q}_l/\mathbb{Z}_l)$$

for an arbitrary prime $l \neq p$, where Hom_c *denotes the set of all continuous homomorphisms of finite order.*

We explain the Bloch-Milnor-Kato conjecture. For an arbitrary field F and a natural number m, it asserts the bijectivity of the Galois symbol $K_m^M(F)/n \xrightarrow{\sim} H^m_{\mathrm{Gal}}(F,\,\mu_n^{\otimes m})$ for an arbitrary natural number n prime to $\mathrm{ch}(K)$. For the proof of Theorem 1.3, we use this conjecture by putting $m = 3$. The proof of Theorem 1.3 is much easier than that of Theorem 1.2, but we mention that a cohomological Hasse principle by S. Saito (cf. [Sa1]) plays a key role in the proof.

By combining above Theorem 1.2 with Theorem 1.3, we get the following class field theory of K.

Theorem 1.4 (**Theorem 8.2**). *Let A be an arbitrary three-dimensional complete regular local ring of positive odd characteristic with finite residue field. Let K be its fractional* *field. Then, under the Bloch-Milnor-Kato conjecture, we have the following dual reciprocity isomorphism*:

$$\rho_K^* \colon H^1_{\mathrm{Gal}}(K,\, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}_c(C_K,\, \mathbb{Q}/\mathbb{Z}),$$

where Hom_c *denotes the set of all continuous homomorphisms of finite order.*

This theorem proves class field theory for K in its most desirable form. Indeed, through the reciprocity isomorphism ρ_K^*, we can perfectly understand the structure of the galois group of an arbitrary finite abelian extension of K (see corollaries below). Further, Theorem 1.4 involves the so-called existence theorem. We give some corollaries.

Corollary 1.5 (Corollary 8.3). *Let A, K and the assumptions be as in* Theorem 1.4. *Then the reciprocity homomorphism*

$$\rho_K \colon C_K \to \mathrm{Gal}(K^{ab}/K)$$

has a dense image in $\mathrm{Gal}(K^{ab}/K)$ *by the Krull topology.*

This follows immediately from Theorem 1.4 by considering dual.

Next, we state the explicit reciprocity isomorphism for certain finite abelian extensions which is familiar in the one-dimensional local or global class field theory.

Corollary 1.6 (Corollary 8.4). *Let A, K and the assumptions be as in* Theorem 1.4. *Then, for an arbitrary finite abelian extension L/K such that the integral closure of A in L is regular, there exists the following canonical reciprocity isomorphism*:

$$\rho_K \colon C_K/N_{L/K}(C_L) \xrightarrow{\sim} \mathrm{Gal}(L/K).$$

This result can be seen as the first successful explicit description for the galois group of finite abelian extension of semi-global fields in the sense of Kato-Saito (cf. [Ka-Sa2]).

Finally, in a forthcoming paper [Ma1], we prove class field theory for the fractional field of an arbitrary power series ring $\mathbb{F}_q[[X_1, ..., X_n]]$ with $n \geq 4$.

Convention. Through the paper, for an arbitrary commutative ring, we always denote by P_R^i the set of height i prime ideals of R.

2. The absolute purity theorem by Fujiwara-Gabber

In this section, we recall the absolute purity of Fujiwara and Gabber (cf. [Fu]).

Theorem 2.1. (*Fujiwara-Gabber*) *Let X be an arbitrary noetherian excellent regular scheme over a field k, and Z be its regular closed subscheme over k of codimension c.*

Then, for an arbitrary natural number m prime to the characteristic k, we have the following canonical isomorphism:

$$H^i_Z(X_{\acute{e}t}\,,\mu_m^{\otimes j}) \cong H^{i-2c}_{\acute{e}t}(Z\,,\mu_m^{\otimes(j-c)}). \tag{2.1}$$

Remark 2. This theorem is proved by O. Gabber and K. Fujiwara. It was conjectured by M. Artin in SGA 4 (tome 3, XIX) about 30 years ago. Fujiwara proved this theorem using his theory of tubular neighborhoods in rigid analytic geometry. For more details, see Fujiwara's paper [Fu], especially Theorem 7.1.1 and sentences just below Corollary 7.1.7 in it.

Next, we explain another useful result of Gabber on the cohomological dimension for the fractional field of complete regular local ring $k\,[[X_1, X_2, \cdots X_n]]$ $(n \geq 1)$. His proof was lectured at I.H.P.(cf. [Ga2]), and is given below. The author heartily thank him for informing me of it.

Theorem 2.2. (*Gabber*) *Let K_n be the fractional field of a n-dimensional complete regular local ring $k\,[[X_1, X_2, \cdots X_n]]$. Then for an arbitrary prime number $l\,(l \neq p)$, the l-cohomological dimension $\mathrm{cd}_l\, K_n$ of K_n satisfies*

$$cd_l\, K_n \leq cd_l\, k + n.$$

Proof. We use induction on n. First, we define

$A_{n-1} := k\,[[X_1, X_2, \ldots X_{n-1}]]$ $(n \geq 1)$,
$K_{n-1} :=$ The fractional field of A_{n-1},
$A'_n :=$ the henselization of $A_{n-1}[X_n]$ at the maximal ideal $(X_1, X_2, \cdots, X_{n-1}, X_n)$,
$K'_n :=$ The fractional field of A'_n,

The theorem is clear when $n = 0$.

As a first step, we show

$$\mathrm{cd}_l\, K'_n \leq \mathrm{cd}_l\, k + n. \tag{2.2}$$

From the definition of the henselization, there exists a filtering index category Λ and an inductive system of domains B_λ $(\lambda \in \Lambda)$ which is étale over the localization of $A_{n-1}[X_n]$ at $(X_1, ..., X_n)$ such that A'_n can be written as the inductive limit of the form

$$A'_n = \varinjlim_{\lambda \in \Lambda} B_\lambda. \tag{2.3}$$

So, we have

$$K'_n = \varinjlim_{\lambda \in \Lambda} Q(B_\lambda), \tag{2.4}$$

where $Q(B_\lambda)$ denotes the fractional field of B_λ. Since $Q(B_\lambda) = Q(K_{n-1}\otimes_{A_{n-1}} B_\lambda)$ and $Q(K_{n-1}\otimes_{A_{n-1}} B_\lambda) = \varinjlim_{f\neq 0,\, f\in B_\lambda} K_{n-1}\otimes_{A_{n-1}} B_\lambda[1/f]$, there exist some index category Λ' and index system B'_λ such that each B'_λ is a normal domain and we have

$$K'_n = \varinjlim_{\lambda\in\Lambda'} B'_\lambda. \tag{2.5}$$

Then, we see that $\operatorname{Spec} B'_\lambda$ is an affine curve over K_{n-1}.

Next, assume M to be a finite $\mathrm{Gal}(\overline{K'_n}/K'_n)$-module of l-power order. Since we have $\mathrm{Gal}(\overline{K'_n}/K'_n) = \varprojlim \pi_1^{alg}(\operatorname{Spec} B'_\lambda)$, it follows that

$$H^i_{\mathrm{Gal}}(K'_n, M) = \varinjlim_{\lambda\in\Lambda} H^i_{\mathrm{Gal}}(\pi_1^{alg}(\operatorname{Spec} B'_\lambda), M) = \varinjlim_{\lambda\in\Lambda} H^i_{\acute{e}t}(\operatorname{Spec} B'_\lambda, M). \tag{2.6}$$

By putting $\overline{B'_\lambda} = B'_\lambda \otimes_{K_{n-1}} \overline{K_{n-1}}$, we obtain the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p_{\mathrm{Gal}}(K_{n-1}, H^q_{\acute{e}t}(\operatorname{Spec} \overline{B'_\lambda}, M)) \Longrightarrow H^{p+q}_{\acute{e}t}(\operatorname{Spec} B'_\lambda, M). \tag{2.7}$$

Further, the induction assumption together with the weak Lefschetz theorem shows that the $E_2^{p,q}$-term vanishes when $p > \mathrm{cd}_l\, k + (n-1)$, or $q > 1$. Hence for $i > \mathrm{cd}_l\, k + n$, we obtain $H^i_{\acute{e}t}(\operatorname{Spec} B'_\lambda, M) = 0$, hence $H^i_{\acute{e}t}(K'_n, M) = 0$ by (2.6). This proves $\mathrm{cd}_l\, K'_n \leq cd_l\, k + n$.

Next, we show $\mathrm{cd}_l\, K_n \leq \mathrm{cd}_l\, k + n$. We see $K_n = \varinjlim_f A_n\,[1/f]$, where f runs over all non-zero elements of A_n. Put $\widehat{U_f} = \operatorname{Spec} A_n\,[1/f]$. Let M be a finite $\mathrm{Gal}(\overline{K_n}/K_n)$-module of l-power order. Then, M is a $\pi_1(\widehat{U_f})$-module, and we have

$$H^i_{\mathrm{Gal}}(K_n, M) = \varinjlim_f H^i_{\acute{e}t}(\pi_1(\widehat{U_f}), M). \tag{2.8}$$

But for an l-primary torsion module M ($l \neq p = ch\, \widehat{U_f}$), it is easily seen that

$$H^i_{\acute{e}t}(\pi_1(\widehat{U_f}), M) \cong H^i_{\acute{e}t}(\pi_1(\widehat{U_f})^{\mathrm{tame}}, M), \tag{2.9}$$

where $\pi_1(\widehat{U_f})^{\mathrm{tame}} := \varprojlim \mathrm{Aut}\,(V/\widehat{U_f})$, where V runs over all tamely ramified Galois coverings of $\widehat{U_f}$. This is easily seen, for example, by the fact that for a pro-p subgroup P of the pro-finite group G, $H^l_{\mathrm{Gal}}(P, M) = 0$ for $l > 0$ by the assumption of M, so the Hochschild-Serre spectral sequence $E_2^{k,l} = H^k_{\mathrm{Gal}}(G/P, H^l_{\mathrm{Gal}}(P, M)) \Longrightarrow H^i_{\mathrm{Gal}}(G, M)$ degenerates giving an isomorphism $H^i_{\mathrm{Gal}}(G, M) \cong H^i_{\mathrm{Gal}}(G/P, M^P)$. Now, we get (2.9) by putting $G = \pi_1(\widehat{U_f})$, $G/P = \pi_1(\widehat{U_f})^{\mathrm{tame}}$ in the just above isomorphism. Thus, we have

$$H^i_{\mathrm{Gal}}(K_n, M) = \varinjlim_f H^i_{\acute{e}t}(\pi_1(\widehat{U_f}), M) \cong \varinjlim_f H^i_{\acute{e}t}(\pi_1(\widehat{U_f})^{\mathrm{tame}}, M). \tag{2.10}$$

Assume $i > \mathrm{cd}_l\, k + n$, and let $e \in H^i_{\mathrm{Gal}}(K_n, M)$ be an arbitrary element of $H^i_{\mathrm{Gal}}(K_n, M)$. Then from (2.10), e can be represented by an element $\widehat{e_f}$ of $H^i_{\acute{e}t}(\pi_1(\widehat{U_f})^{\mathrm{tame}}, M)$ for some

f, and by the Weierstrass Preparation theorem, we may assume $f \in A'_n$. Put $U'_f = \operatorname{Spec} A'_n[1/f]$. Now, we have the following result of M. Artin.

Theorem 2.3 (Artin, [A] *Th*.5.1)**.** *Let R be an arbitrary excellent equi-characteristic henselian local ring, $X/\operatorname{Spec} R$ proper, $Y \subset X$ a closed subscheme, and $U = X - Y$. Let $\widehat{V}$ be an étale covering of $\widehat{U}$, where $\widehat{U}$ is the fibre product $U \times_{\operatorname{Spec} R} \operatorname{Spec} \widehat{R}$, where $\widehat{R}$ denotes the completion of R by the maximal ideal of R. Further, assume that X is normal. Then, if the covering $\widehat{V}$ is tamely ramified at points of $\widehat{Y}$ which are of codimension 1 in $\widehat{X}$, then $\widehat{V}$ is induced by an étale covering V of U.*

From this theorem, we immediately obtain an isomorphism $\pi_1(\widehat{U_f})^{\text{tame}} \xrightarrow{\sim} \pi_1(U'_f)^{\text{tame}}$. Consequently, we get an isomorphism

$$H^i_{\acute{e}t}(\pi_1(U'_f)^{\text{tame}}, M) \xrightarrow{\sim} H^i_{\acute{e}t}(\pi_1(\widehat{U_f})^{\text{tame}}, M). \tag{2.11}$$

By this isomorphism (2.11), $\widehat{e_f}$ is sent to $e'_f \in H^i_{\acute{e}t}(\pi_1(U'_f)^{\text{tame}}, M)$.

On the other hand, we have

$$H^i_{\text{Gal}}(K'_n, M) = \varinjlim_{f \in A'_n, f \neq 0} H^i_{\acute{e}t}(\pi_1(U'_f), M) = \varinjlim_{f \in A'_n, f \neq 0} H^i_{\acute{e}t}(\pi_1(U'_f)^{\text{tame}}, M), \tag{2.12}$$

because an arbitrary element of M is annihilated by natural number prime to the characteristic of U'_f. We have proved in the first step that the left hand side is zero. So, there exists some element $g \in A'_n$ such that the image of e'_f by the inflation map $H^i_{\acute{e}t}(\pi(U'_f)^{\text{tame}}, M) \to H^i_{\acute{e}t}(\pi(U'_{fg})^{\text{tame}}, M)$ is zero. It follows that also the image of $\widehat{e_f}$ by the inflation map $H^i_{\acute{e}t}(\pi_1(\widehat{U_f})^{\text{tame}}, M) \to H^i_{\acute{e}t}(\pi_1(\widehat{U_{fg}})^{\text{tame}}, M)$ is zero, because the isomorphism (2.11) commutes with the inflation maps of the étale cohomology in the inductive limits,viz.

$$\begin{array}{ccc} H^i_{\acute{e}t}(\pi_1(U'_f)^{\text{tame}}, M) & \xrightarrow{\sim} & H^i_{\acute{e}t}(\pi_1(\widehat{U_f})^{\text{tame}}, M) \\ \downarrow{\scriptstyle \text{Inflation}} & & \downarrow{\scriptstyle \text{Inflation}} \\ H^i_{\acute{e}t}(\pi_1(U'_{fg})^{\text{tame}}, M) & \xrightarrow{\sim} & H^i_{\acute{e}t}(\pi_1(\widehat{U_{fg}})^{\text{tame}}, M). \end{array} \tag{2.13}$$

Especially by (2.10), e is zero as an element of $H^i_{\text{Gal}}(K_n, M)$, which shows that $H^i_{\text{Gal}}(K_n, M) = 0$. Therefore the desirable result $\operatorname{cd}_l K_n \leq \operatorname{cd}_l k + n$ is proved. □

Remark 3. Though it seems much more difficult to prove, Theorem 2.2 would hold even in the mixed characteristic cases. That is, for the quotient field L of a complete regular local ring $\mathbb{Z}_p[[X_1, \ldots X_{n-1}]]$, it would hold that $\operatorname{cd}_l L \leq n + 1$. Indeed, for the case of $n = 2$, i.e. $L = Q(\mathbb{Z}_p[[X]])$, the theorem $\operatorname{cd}_l L \leq 3$ was proved by K. Kato (cf. [Sa1]) using his deep calculations of Milnor K-groups for higher local fields and certain results in [MS]. See also [Ka1].

3. Proof of Hasse principle

In this section we prove a cohomological Hasse principle for A.

Theorem 3.1. *Let A be the three-dimensional formal power series ring over finite field. Then for an arbitrary natural number m prime to the characteristic of A, the following Hasse principle complex for A becomes exact:*

$$0 \to H^4_{ét}(K, \mu_m^{\otimes 3}) \to \bigoplus_{\mathfrak{p}\in P^1_A} H^3_{ét}(\kappa(\mathfrak{p}), \mu_m^{\otimes 2}) \to \bigoplus_{\mathfrak{m}\in P^2_A} H^2_{ét}(\kappa(\mathfrak{m}), \mu_m) \to \mathbb{Z}/m \to 0, \tag{3.1}$$

where P^i_A denotes the set of height i prime ideals of A and $\kappa(\mathfrak{p})$, $\kappa(\mathfrak{m})$ denotes the residue fields of $\mathfrak{p}$, $\mathfrak{m}$, respectively.

Proof. For the proof, we use the coniveau-spectral sequence. Let us denote $X :\ = \operatorname{Spec} A$. Consider the following coniveau spectral sequence for X:

$$E_1^{p,q} = \bigoplus_{x\in X^{(p)}} H_x^{p+q}(X_{\text{ét}},\, \mu_m^{\otimes 3}) \Rightarrow H_{\text{ét}}^{p+q}(X,\, \mu_m^{\otimes 3}), \tag{3.2}$$

where $X^{(p)}$ denotes the set of codimension p primes of X. We have an isomorphism $H_x^{p+q}(X_{\text{ét}},\, \mu_m^{\otimes 3}) \cong H_x^{p+q}(X_x^h,\, \mu_m^{\otimes 3})$ by excision, where X_x^h denotes the henselization of X at its prime point x. As X_x^h and its maximal point x satisfy the assumption of Theorem 2.1 with $Z = x$ and $X = X_x^h$, we can apply this theorem and find that each component $H_x^{p+q}(X_{\text{ét}},\, \mu_m^{\otimes 3})$ in $E_1^{p,q}$-term of (3.2) can be replaced by $H_{\text{ét}}^{q-p}(\kappa(x), \mu_m^{\otimes(3-p)})$. So we can rewrite the above coniveau spectral sequence (3.2) as,

$$E_1^{p,q} = \bigoplus_{x\in X^{(p)}} H_{\text{ét}}^{q-p}(\kappa(x), \mu_m^{\otimes(3-p)}) \Rightarrow H_{\text{ét}}^{p+q}(X, \mu_m^{\otimes 3}). \tag{3.3}$$

We see that each $E_1^{p,q}$-term vanishes if $p > 3$, or $p > q$. Especially, the following complex in $E_1^{p,q}$-terms

$$0 \to E_1^{0,4} \xrightarrow{d_1^{0,4}} E_1^{1,4} \xrightarrow{d_1^{1,4}} E_1^{2,4} \xrightarrow{d_1^{2,4}} E_1^{3,4} \to 0 \tag{3.4}$$

becomes

$$\begin{aligned} 0 \to H^4_{\text{ét}}(K, \mu_m^{\otimes 3}) &\xrightarrow{\iota_1} \bigoplus_{\mathfrak{p}\in X^{(1)}} H^3_{\text{ét}}(\kappa(\mathfrak{p}), \mu_m^{\otimes 2}) \\ &\xrightarrow{\iota_2} \bigoplus_{\mathfrak{m}\in X^{(2)}} H^2_{\text{ét}}(\kappa(\mathfrak{m}), \mu_m) \xrightarrow{addition} (\mathbb{Z}/m) \to 0, \end{aligned}$$

which is nothing but a Hasse principle complex (3.1). So, Theorem 3.1 follows from the following proposition:

Proposition 3.2. $E_2^{0,4} = E_2^{1,4} = E_2^{2,4} = E_2^{3,4} = 0.$

As stated above, by the theory of spectral sequence, we have only to show this proposition.

Proof of Proposition 3.2. First, we state a lemma.

Lemma 3.3. *For $p+q>1$, the convergent term satisfies*
$E^{p+q} = H_{ét}^{p+q}(X\,, \mu_m^{\otimes 3}) = H_{ét}^{p+q}(\mathrm{Spec}\, A,\, \mu_m^{\otimes 3}) = 0$*, hence* $E_\infty^{p,q} = 0$ *for* $p+q>1$.

Proof. This follows from an isomorphism $H_{\text{ét}}^{p+q}(\mathrm{Spec}\, A,\, \mu_m^{\otimes 3}) \cong H_{\text{ét}}^{p+q}(A\,/\mathfrak{m}_A,\, \mu_m^{\otimes 3})$, which is deduced from the comparison theorem of the Henselian local rings (cf. [Ga1]), and the latter group is 0 for $p+q>1$, because the cohomological dimension of the finite field $A\,/\mathfrak{m}_A$ for torsion sheaves is 1. □

We return to the proof of Proposition 3.2.

1). First, we prove $E_2^{3,4} = 0$. Just below, we see that both $E_2^{5,3}$ and $E_2^{1,5}$ are zero. For $E_2^{5,3} = 0$, because there is no codimension 5 prime in X. And for the proof of $E_2^{1,5} = 0$, we need the following results:

Theorem 3.4. (*Nagata,* [Na] *Chap.*V*,Cor.*31.6) *If R is a complete local integral domain, then R contains a complete regular ring S such that R is a finite S-module and such that $S = I\,[[X_1, \ldots, X_r]]$ with a coefficient ring I of R and analytically independent elements $X_1, \ldots, X_r$.*

Theorem 3.5. (*Serre,* [Se] *Chap.*I*, Prop.*14) *Let H be a closed subgroup of the pro-finite group G. Then, one has $cd_p\,(H) \le cd_p\,(G)$, where p is an arbitrary prime integer.*

It is enough to prove $E_1^{1,5} = 0$. But $E_1^{1,5} = \bigoplus_{\mathfrak{p}\,\in X^{(1)}} H_{\text{ét}}^4(\kappa(\mathfrak{p}),\, \mu_m^{\otimes 2})$ and we will prove each component $H_{\text{ét}}^4(\kappa(x),\, \mu_m^{\otimes 2})$ in $E_1^{1,5}$ exactly vanishes.

We apply Theorem 3.4 by putting $I = \mathbb{F}_p$, $r = 2$ which shows that $\kappa(\mathfrak{p})$ ($\mathfrak{p} \in X^{(1)}$) is a finite extension of $\mathrm{Frac}(\mathbb{F}_p[[X,Y]])$. But we already know $\mathrm{cd}_l\,\mathrm{Frac}\,(\mathbb{F}_p[[X,Y]]) = 3$ by Theorem 2.2, and Theorem 3.5 tells us that $\mathrm{cd}_l\,\kappa(\mathfrak{p}) \leqq \mathrm{cd}_l\,\mathrm{Frac}\,(\mathbb{F}_p[[X,Y]]) = 3$, because $\mathrm{Gal}\,(\overline{\kappa(\mathfrak{p})}/\kappa(\mathfrak{p}))$ is the closed subgroup of $\mathrm{Gal}\,(\overline{\mathrm{Frac}(\mathbb{F}_p[[X,Y]])}/\mathrm{Frac}\,(\mathbb{F}_p[[X,Y]]))$. Thus, we obtain $H_{\text{ét}}^4(\kappa(\mathfrak{p}),\, \mu_m^{\otimes 2}) = 0$.

It follows that $E_3^{3,4} = \mathrm{Ker}(E_2^{3,4} \xrightarrow{d_2^{3,4}} E_2^{5,3})/\mathrm{Im}(E_2^{1,5} \xrightarrow{d_2^{1,5}} E_2^{3,4}) = E_2^{3,4}$. But, it is easily seen that $E_3^{3,4} = \ldots = E_\infty^{3,4} = 0$. Hence we get the desired vanishing of $E_2^{3,4}$.

2). Next, we show $E_2^{2,4} = 0$. This is proved as follows. First, it holds that $E_2^{4,3} = E_2^{0,5} = 0$ which follows from the vanishing of $E_1^{4,3} = E_1^{0,5} = 0$. Of course, $E_1^{4,3} = 0$ follows from the inequality $p = 4 > 3 = q$ and $E_1^{0,5} = 0$ follows again from Theorem 2.2. So, we have

$$E_3^{2,4} = \mathrm{Ker}(\mathrm{E}_2^{2,4} \xrightarrow{\mathrm{d}_2^{2,4}} \mathrm{E}_2^{4,3})/\mathrm{Im}(\mathrm{E}_2^{0,5} \xrightarrow{\mathrm{d}_2^{0,5}} \mathrm{E}_2^{2,4}) = \mathrm{E}_2^{2,4}. \tag{3.5}$$

But it follows that $E_3^{2,4} = \cdots = E_\infty^{2,4} = 0$. Hence we are done.

3). Thirdly, we prove $E_2^{1,4} = 0$. We need a lemma.

Lemma 3.6. *We have the vanishing* $E_2^{3,3} = 0$.

Proof. By definition, $E_2^{3,3} = \mathrm{Ker}(E_1^{3,3} \xrightarrow{d_1^{3,3}} E_1^{4,3})/\mathrm{Im}(E_1^{2,3} \xrightarrow{d_1^{2,3}} E_1^{3,3}) = \mathrm{Coker}(E_1^{2,3} \xrightarrow{d_1^{2,3}} E_1^{3,3})$, where the second equality follows from the vanishing $E_1^{4,3} = 0$. But this group is rewritten as

$$E_2^{3,3} = \mathrm{Coker}(E_1^{2,3} \xrightarrow{d_1^{2,3}} E_1^{3,3}) = \mathrm{Coker}\left(\bigoplus_{\mathfrak{m} \in X^{(2)}} H^1_{et}(\kappa(\mathfrak{m}), \mu_m) \xrightarrow{d_1^{2,3}} \mathbb{Z}/m\right). \tag{3.6}$$

But from the Kummer theory, we can rewrite this group as,

$$E_2^{3,3} = \mathrm{Coker}\left(\bigoplus_{\mathfrak{m} \in X^{(2)}} \kappa(\mathfrak{m})^*/m \xrightarrow{d_1^{2,3}} \mathbb{Z}/m\right). \tag{3.7}$$

But each map $\kappa(\mathfrak{m})^*/m \to \mathbb{Z}/m$ is nothing but the valuation map of one-dimensional complete discrete valuation field $\kappa(\mathfrak{m})$. So if we choose a height two prime $\mathfrak{m}$ such that $A/\mathfrak{m}$ is regular, it becomes surjective, hence $d_1^{2,3}$ is also surjective which proves $E_2^{3,3} = 0$. □

Now we proceed as follows. It is easily seen that $E_3^{1,4} = \ldots = E_\infty^{1,4} = 0$. Further, $E_3^{1,4} = \mathrm{Ker}(E_2^{1,4} \xrightarrow{d_2^{1,4}} E_2^{3,3})$. But Lemma 3.6 shows $E_2^{3,3} = 0$ and it follows immediately that $E_2^{1,4} = 0$.

4). Finally, the proof of Proposition 3.2 is finished by showing $E_2^{0,4} = 0$. For this, we first show that $E_3^{0,4} = 0$. It is easily seen that $E_4^{0,4} = \ldots = E_\infty^{0,4} = 0$. But $E_4^{0,4} = \mathrm{Ker}(E_3^{0,4} \xrightarrow{d_3^{0,4}} E_3^{3,2})$ and $E_3^{3,2} = 0$, which results from $E_1^{3,2} = 0\,(p = 3 > 2 = q)$. Putting all together, we have $E_3^{0,4} = E_4^{0,4} = 0$. On the other hand, $E_3^{0,4} = \mathrm{Ker}(E_2^{0,4} \xrightarrow{d_2^{0,4}} E_2^{2,3})$, so our desirable vanishing $E_2^{0,4} = 0$ follows directly from $E_2^{2,3} = 0$.

Claim 3.7. *We have the vanishing* $E_2^{2,3} = 0$.

This is the most hard task for a Hasse principle. We prove this by the exactness of Gersten-Quillen complex for A.

Proof of Claim 3.7. By definition,

$$E_2^{2,3} = \mathrm{Ker}(E_1^{2,3} \overset{d_1^{2,3}}{\to} E_1^{3,3})/\mathrm{Im}(E_1^{1,3} \overset{d_1^{1,3}}{\to} E_1^{2,3}).$$

So, the exactness of the following sequence furnishes this Claim:

$$E_1^{1,3} \overset{d_1^{1,3}}{\to} E_1^{2,3} \overset{d_1^{2,3}}{\to} E_1^{3,3}. \tag{3.8}$$

This sequence (3.8) can be rewritten as

$$\bigoplus_{\mathfrak{p}\in X^{(1)}} H^2(\kappa(\mathfrak{p}), \mu_m^{\otimes 2}) \to \bigoplus_{\mathfrak{m}\in X^{(2)}} H^1(\kappa(\mathfrak{m}),\, \mu_m) \to \mathbb{Z}/m. \tag{3.9}$$

Further, by the theorem of Mercurjev-Suslin and the Kummer theory, (3.9) is rewritten as

$$\bigoplus_{\mathfrak{p}\in X^{(1)}} K_2^Q(\kappa(\mathfrak{p}))\,/m \to \bigoplus_{\mathfrak{m}\in X^{(2)}} K_1^Q(\kappa(\mathfrak{m}))/m \to \mathbb{Z}/m. \tag{3.10}$$

(we used the fact that $K_i^M(k) \cong K_i^Q(k)$ $(i = 1, 2)$ for a field k). So everything is reduced to prove the exactness of the sequence (3.10), but this is nothing but the Gersten-Quillen complex for A which is proved exact by Quillen in [Q]. So we finished the proof of Claim 3.7, hence Proposition 3.2. Thus, the proof of Theorem 3.1 is now completely finished. □

4. Construction of the idele class group C_K

In this section, we introduce the K-theoretic topological idele class group C_K which plays the central role in this paper. First, we review the basic results. For an arbitrary field k, the n-th Milnor K-group $K_n^M(k)$ is defined as follows:

Definition 1. Milnor's K-group $K_q^M(k)$ for a field k is defined by

$$K_q^M(k) := ((k^\times)^{\otimes q})/J,$$

where J is the subgroup of the q-fold tensor product $(k^*)^{\otimes k}$ of $k^\times$ (as a $\mathbb{Z}$-module) generated by elements of the form $a_1 \otimes \ldots \otimes a_q$ satisfying $a_i + a_j = 1$ for some $i \neq j$.

Especially, for an arbitrary discrete valuation field F, we define the subgroups $U^i K_n^M(F)$ for $i \geq 0$ as

$$U^i K_n^M(F){:}\ = \{\mathrm{Image}{:}\ x_1 \otimes x_2 \otimes \cdots \otimes x_n \mapsto K_n^M(F) \mid \text{s.t. } x \in U^{(i)}(F), x_2, \ldots, x_n \in K^*\}, \tag{4.1}$$

where $U^{(i)}(F)$ denotes the multiplicative group $(1 + u_F^i\, \mathcal{O}_F)^* \subset \mathcal{O}_F^*$. Here, u_F and $\mathcal{O}_F$ denote the uniformizing parameter and the valuation ring of F, respectively.

Next, we recall the definition of higher dimensional local fields (cf. [Ka1], [P1]).

Definition 2. A complete discrete valuation field k_n is said to be n-dimensional local if there exists the following sequence of fields k_i $(1 \leq i \leq n)$:
each k_i is a complete discrete valuation field having k_{i-1} as the residue field of the valuation ring $\mathcal{O}_{k_i}$ of k_i, and k_0 is defined to be a finite field.

Now, we review class field theory for higher dimensional local fields established by K. Kato and A. N. Parshin (cf. [Ka1], [P1]).

Theorem 4.1 (Kato, Parshin). *For an arbitrary n-dimensional local field k_n, there exists the canonical reciprocity map*

$$\rho_{k_n} : K_n^M(k_n) \to \mathrm{Gal}(k_n^{ab}/k_n)$$

which satisfies the following two conditions:
i) for an arbitrary finite abelian extension k'/k_n, ρ_{k_n} induces an isomorphism

$$\rho_{k_n} : K_n^M(k_n)/N_{k'/k_n}(K_n^M(k')) \xrightarrow{\sim} \mathrm{Gal}(k'/k_n),$$

ii) the correspondence $k' \mapsto N_{k'/k_n}(K_n^M(k'))$ is a bijection between the set of all finite abelian extensions of k_n and the set of all open subgroups of $K_n^M(k_n)$ of finite index.

Next, we define the idele class group C_K. For this, we fix the following notation.

Notations.
$P_A^2 :=$ the set of all height 2 primes of A,
$P_A^1 :=$ the set of all height 1 primes of A,
$P_{\mathfrak{m}}^1 :=$ the set of all height 1 primes of $A_{\mathfrak{m}}$,
$A_{\mathfrak{m}} := \varprojlim_n A_{(\mathfrak{m})}/\mathfrak{m}^n$ ($A_{(\mathfrak{m})}$ denotes the localization of A at $\mathfrak{m}$),
$A_{\mathfrak{p}} := \varprojlim_n A_{(\mathfrak{p})}/\mathfrak{p}^n$,
$A_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}} := \varprojlim_n A_{\mathfrak{m}(\mathfrak{p}_{\mathfrak{m}})}/\mathfrak{p}_{\mathfrak{m}}^n$,
$K_{\mathfrak{m}} := \mathrm{Frac}\, A_{\mathfrak{m}}$, $K_{\mathfrak{p}} := \mathrm{Frac}\, A_{\mathfrak{p}}$, $K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}} = \mathrm{Frac}\, A_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}$.

Remark 4. In the above notation, $A_{\mathfrak{m}}$ becomes a two-dimensional complete regular local ring whose residue field $\kappa(\mathfrak{m})$ is one-dimensional local. We also remark that $K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}$ is a three-dimensional local field defined above.

Under these notations, we introduce a modulus M as follows:

Definition 3. A modulus M is a formal sum

$$M\colon = \Sigma_{\mathfrak{p}\in P_A^1} n_{\mathfrak{p}} \overline{(\mathfrak{p})}$$

of prime divisors $\overline{(\mathfrak{p})}$ defined by $(\mathfrak{p} = 0)$ in $\operatorname{Spec} A$ and $n_{\mathfrak{p}}$ is a positive integer which is zero for almost all $\mathfrak{p}$.

Next, for an arbitrary modulus M and each $\mathfrak{m} \in P_A^2$, we define the group $C_{\mathfrak{m}}(M)$ by

$$C_{\mathfrak{m}}(M)\colon = \operatorname{Coker}\left(K_3^M(K_{\mathfrak{m}}) \stackrel{diagonal}{\longrightarrow} \bigoplus_{\mathfrak{p}_{\mathfrak{m}}\in P_{\mathfrak{m}}^1} \left(K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}) / U^{M(\mathfrak{p}_{\mathfrak{m}})} K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}) \right) \right), \tag{4.2}$$

where $M(\mathfrak{p}_{\mathfrak{m}})$ is defined to be $n_{\mathfrak{p}}$ if $\mathfrak{p}_{\mathfrak{m}} \mapsto \mathfrak{p}$ under the canonical map $\operatorname{Spec} A_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}} \to \operatorname{Spec} A$.

Now, by using these $C_{\mathfrak{m}}(M)$ for an arbitrary height two prime $\mathfrak{m} \in P_A^2$, we define the topological group $C_K(M)$ as

$$C_K(M)\colon = \operatorname{Coker}\left(\bigoplus_{\mathfrak{p}\in P_A^1} K_3^M(K_{\mathfrak{p}}) \to \bigoplus_{\mathfrak{m}\in P_A^2} C_{\mathfrak{m}}(M) \right), \tag{4.3}$$

and we put the *discrete* topology on $C_K(M)$.

We must check the well-definedness of the above definition of $C_K(M)$ in (4.3). That is, the image of each group $K_3^M(K_{\mathfrak{p}})$ in $\prod_{\mathfrak{m}\in P_A^2} C_{\mathfrak{m}}(M)$ actually lies in the direct sum. We prove this as the following lemma.

Lemma 4.2. *For each $\mathfrak{p} \in P_A^1$, the image of $K_3^M(K_{\mathfrak{p}})$ in $\prod_{\mathfrak{m}\in P_A^2, \mathfrak{m}\supset\mathfrak{p}} C_{\mathfrak{m}}(M)$ lies in $\bigoplus_{\mathfrak{m}\in P_A^2, \mathfrak{m}\supset\mathfrak{p}} C_{\mathfrak{m}}(M)$.*

Proof. Take an arbitrary height one prime $\lambda \in P_A^1$. Then, each element of $K_3^M(K_\lambda)$ lands in only such component $C_{\mathfrak{m}}(M)$ of $\prod_{\mathfrak{m}\in P_A^2} C_{\mathfrak{m}}(M)$ as $\mathfrak{m} \supset \lambda$. So, we have only to prove that an arbitrary element $\alpha \in K_3^M(K_\lambda)$ vanishes in $C_M(\mathfrak{m})$ for almost all $\mathfrak{m}$ which contains λ. In the below, we denote by $u_{\lambda_{\mathfrak{m}}}$ the regular parameter of $\lambda_{\mathfrak{m}} \in P_{\mathfrak{m}}^1$. Take an arbitrary element α from $K_3^M(K_\lambda)$. First, it is easily found that there exists a surjection $K_3^M(K) \twoheadrightarrow K_3^M(K_\lambda)/F^{n_\lambda}K_3^M(K_\lambda)$ for an arbitrary $n_\lambda \geq 0$. So, we may assume $\alpha \in K_3^M(K)$. If we write $\alpha = (a_1, b_1, c_1) \ldots (a_n, b_n, c_n)$, paying attention to the fact that A is uniquely factorized domain, we may assume that all a_i, b_i, c_i lies in $A[\frac{1}{u_{\mathfrak{p}_1}}, \ldots, \frac{1}{\mathfrak{p}_m}]$ with finitely many height one primes $\mathfrak{p}_j$ $(j = 1, \ldots, m)$. But if $\lambda \neq \mathfrak{p}_j$, only finite height two primes of A can lie over both λ and $\mathfrak{p}_j$. Thus, we may assume that $\alpha \in K_3^M(A_{\mathfrak{m}}[\frac{1}{\lambda}])$ except for finite height two primes. Further, the definition of a modulus M shows that for almost all $\mathfrak{p}$, its coefficient $n_{\mathfrak{p}}$ is zero.

So, if we complete A at a height-two prime $\mathfrak{m}$ on λ except finitely many height two primes which lie over both λ and some $\mathfrak{p}$ having non-zero modulus $n_{\mathfrak{p}}$, we have $M(\mathfrak{p}_{\mathfrak{m}}) = 0$ for every height one prime $\mathfrak{p}_{\mathfrak{m}} \neq \lambda_{\mathfrak{m}} \in P^1_{A_{\mathfrak{m}}}$. Moreover, a prime u_λ in A remains as a prime in $A_{\mathfrak{m}}$ except finitely many $A_{\mathfrak{m}}$. So, putting all together, except for finitely many height two primes in P^2_A which are explained above, α lies in the subgroup $K_3^M(A_{\mathfrak{m}}[\frac{1}{u_{\lambda_{\mathfrak{m}}}}])$ in $K_3^M(K_{\mathfrak{m},\lambda_{\mathfrak{m}}})$. Moreover, the above discussion assures that we may assume each modulus $M(\mathfrak{p}_{\mathfrak{m}})$ of $\mathfrak{p}_{\mathfrak{m}}(\neq \lambda_{\mathfrak{m}})$ to be zero for such $\mathfrak{m}$. Thus, in this situation, our assertion immediately follows from the next sub-lemma.

Sub-lemma 4.3. *For each $\mathfrak{m} \in P^2_A$, the group $K_3^M(A_{\mathfrak{m}}[\frac{1}{u_{\lambda_{\mathfrak{m}}}}])$ vanishes in $C_{\mathfrak{m}}(M)$ if each modulus $M(\mathfrak{p}_{\mathfrak{m}})$ is zero for all $\mathfrak{p}_{\mathfrak{m}} \in P^1_{\mathfrak{m}}$ such that $\mathfrak{p}_{\mathfrak{m}} \neq \lambda_{\mathfrak{m}}$.*

For the proof, we need the Gersten-Quillen complex in Milnor K-theory.

Proof of Sub-lemma 4.3. Consider the Gersten-Quillen complex

$$K_3^M(A_{\mathfrak{m}}[\frac{1}{u_{\lambda_{\mathfrak{m}}}}]) \to K_3^M(K_{\mathfrak{m}}) \overset{\oplus\partial_{\mathfrak{p}_{\mathfrak{m}}}}{\to} \bigoplus_{\mathfrak{p}_{\mathfrak{m}} \neq \lambda_{\mathfrak{m}}} K_2^M(\kappa(\mathfrak{p}_{\mathfrak{m}})) \to 0, \tag{4.4}$$

where $\partial_{\mathfrak{p}_{\mathfrak{m}}}$ is the boundary map $\partial_{\mathfrak{p}_{\mathfrak{m}}} : K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}) \to K_2^M(\kappa(\mathfrak{p}_{\mathfrak{m}}))$ in algebraic K-theory. By Theorem 4.6 below, the kernel of the boundary map $\partial_{\mathfrak{p}_{\mathfrak{m}}}$ coincides with $U^0K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})$. So, the above sequence (4.4) shows that $K_3^M(A_{\mathfrak{m}}[\frac{1}{u_{\lambda_{\mathfrak{m}}}}])$ lies in $U^0K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})$ for all $\mathfrak{p}_{\mathfrak{m}}$ such that $\mathfrak{p}_{\mathfrak{m}} \neq \lambda_{\mathfrak{m}}$. Thus, in the definition of $C_{\mathfrak{m}}(M)$, if we move an arbitrary element $\alpha \in K_3^M(A_{\mathfrak{m}}[\frac{1}{u_{\lambda_{\mathfrak{m}}}}]) \subset K_3^M(K_{\mathfrak{m},\lambda_{\mathfrak{m}}})$ into $\bigoplus_{\mathfrak{p}_{\mathfrak{m}} \neq \lambda_{\mathfrak{m}}} \left(K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}) / U^0 K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}) \right)$ by the diagonal image of $\alpha \in K_3^M(K_{\mathfrak{m}})$ in $C_{\mathfrak{m}}(M)$, it becomes 0 in $U^0K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})$ for all $\mathfrak{p}_{\mathfrak{m}}$ such that $\mathfrak{p}_{\mathfrak{m}} \neq \lambda_{\mathfrak{m}}$. Hence, we get the desired assertion in the sub-lemma. □

Now, we define the idele class group C_K for K.

Definition 4. We define the topological idele class group C_K as

$$C_K\colon = \varprojlim{}_M C_K(M), \tag{4.5}$$

where $\varprojlim{}_M$ is taken by the surjection $C_K(M') \twoheadrightarrow C_K(M)$ if $M' - M$ is effective ($\Leftrightarrow n'_{\mathfrak{p}} - n_{\mathfrak{p}} \geq 0$), and we endow the inverse limit topology on C_K induced from the discrete topology on each $C_K(M)$ (by definition, the fundamental open subsets of C_K are the inverse images of all open subgroups of $C_K(M)$ in C_K with M running over all moduli).

Remark 5. The inverse limit topology of C_K implies that for each M, the subgroup $\mathrm{Ker}\,(C_K \twoheadrightarrow C_M(K))$ is open in C_K.

Lemma 4.4 (Explicit Representation of C_K)**.** *The above idele class group C_K can be also represented explicitly as follows:*

$$C_K = \varprojlim{}_M D_K / F^M D_K, \tag{4.6}$$

where D_K is defined by

$$D_K :\, = \left(\prod\nolimits'_{\mathfrak{m}\in P_A^2, \mathfrak{p}_\mathfrak{m}\in P_\mathfrak{m}^1} K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}) \right) / \prod\nolimits_{\mathfrak{m}\in P_A^2} K_3^M(K_\mathfrak{m}) \prod\nolimits_{\mathfrak{p}\in P_A^1} K_3^M(K_\mathfrak{p}). \tag{4.7}$$

Here, $\left(\prod'_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}} K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})\right)$ denotes the subgroup of the direct product $\left(\prod_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}} K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})\right)$ such that an arbitrary element $a \in \left(\prod'_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}} K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})\right)$ satisfies the following conditions:

1) each $(\mathfrak{m}, \mathfrak{p}_\mathfrak{m})$-component $a_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}$ of a lies in $U^0K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})$ if $\mathfrak{p}_\mathfrak{m} \mapsto \mathfrak{p}$ for almost all $\mathfrak{p} \in P_A^1$,

2) for an arbitrary element $\mathfrak{p} \in P_A^1$, $(\mathfrak{m}, \mathfrak{p}_\mathfrak{m})$-component $a_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}$ of a lies in $K_3^M(A_\mathfrak{m}[\frac{1}{u_{\mathfrak{p}_\mathfrak{m}}}])$ for almost all $\mathfrak{p}_\mathfrak{m}$ such that $\mathfrak{p}_\mathfrak{m} \mapsto \mathfrak{p}$.

Each group $F^M D_K$ in (4.6) is defined by

$$F^M D_K :\, = \mathrm{Image}\left(\prod_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}{}' U^{M(\mathfrak{p}_\mathfrak{m})} K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}) \;\to\; D_K \right), \tag{4.8}$$

where $M(\mathfrak{p}_\mathfrak{m})$ is defined in (4.2). Moreover, we have an isomorphism:

$$D_K / F^M D_K \xrightarrow{\sim} C_K(M). \tag{4.9}$$

Proof. This is easily proved in the similar way as in Sub-lemma 2.3. □

Next, we define the subgroup F^0C_K of C_K which plays an important role in Section 5. First, for each $\mathfrak{m} \in P_A^2$ we define the group $F^0C_\mathfrak{m}(M)$ ($\subset C_\mathfrak{m}(M)$) as follows:

$$F^0C_\mathfrak{m}(M) :\, = \mathrm{Image}\left(\bigoplus_{\mathfrak{p}_\mathfrak{m}\in P_\mathfrak{m}^1} U^0K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}) \to C_\mathfrak{m}(M) \right). \tag{4.10}$$

Definition 5. We define F^0C_K by the inverse limit

$$F^0C_K :\, = \varprojlim{}_M F^0C_K(M), \tag{4.11}$$

where

$$F^0C_K(M) :\, = \mathrm{Image}\left(\bigoplus_{\mathfrak{m}\in P_A^2} F^0C_\mathfrak{m}(M) \to C_K(M) \right). \tag{4.12}$$

By an easy check, we see the isomorphism

$$F^0D_K/F^M D_K \cong F^0C_K(M). \tag{4.13}$$

In Section 5 and Section 6, it is proved that this group F^0C_K corresponds to the maximal unramified extension of K by the reciprocity map ρ_K. So, if the residue field of A is $\mathbb{F}_q$, it follows that $\mathrm{Coker}(F^0C_K \to C_K) \cong \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \cong \widehat{\mathbb{Z}}$.

Next, we state the Bloch-Milnor-Kato conjecture which is strongly believed to hold and plays an important role in the proof for prime to p parts in Section 5.

Conjecture 1 (Bloch-Milnor-Kato)**.** For an arbitrary field K and a natural number n prime to the characteristic of K, there holds the following Galois symbol isomorphism:

$$K_n^M(K)\,/m \xrightarrow{\sim} H^n_{\mathrm{Gal}}(K,\, \mu_m^{\otimes n}).$$

Here, we state the p primary version of this conjecture which is the celebrated theorem by Bloch-Gabber-Kato.

Theorem 4.5 (Bloch-Gabber-Kato)**.** *For an arbitrary field K of positive characteristic p, the following differential symbol becomes an isomorphism:*

$$K_n^M(K)\,/p^m \xrightarrow{\sim} H^0_{\mathrm{Gal}}(K,\, W_m\Omega^n_{K,\log}), \tag{4.14}$$

where $W_m\Omega^n_{K,\log}$ denotes the logarithmic Hodge-Witt sheaves of length m.

This theorem is also indispensable for the proof of Theorem 1.2 accomplished in Section 5.

Finally, we review the following extremely useful theorem by Kato.

Theorem 4.6 (Kato, [Ka1] I, II)**.** *Let k be a discrete valuation field with residue field F such that $(F : F^p) = p^d$, and $K_n^M(k), U^iK_n^M(k)$ be defined as above. Further, we denote by $k_n^M(k)$ the group $K_n^M(k)/K_n^M(k)^p$, and denote by $U^ik_n^M(k)$ the image of $U^iK_n^M(k)$ in $k_n^M(k)$. Then, each sub-quotient $Gr^ik_n^M(k) := U^ik_n^M(k)\,/U^{i+1}k_n^M(k)$ is completely calculated as follows:*

(1) *There exist the following exact sequence:*

$$0 \to K_n^M(F) \to K_n^M(k)\,/\,U^1K_n^M(k) \xrightarrow{\partial} K_{n-1}^M(F) \to 0, \tag{4.15}$$

and $\mathrm{Ker}\,(K_n^M(k) \xrightarrow{\partial} K_{n-1}^M(F))$ *coincides with $U^0K_n^M(k)$.*

Consequently, we have the following isomorphism:

$$U^0K_n^M(k)\,/\,U^1K_n^M(k) \cong K_n^M(F). \tag{4.16}$$

Hereafter, we assume $i > 0$, and u_k denotes a uniformizing parameter of the valuation ring $\mathcal{O}_k$.

(2) *if* $p \nmid i$, *there exists the following isomorphism*:

$$\Omega_F^{n-1} \xrightarrow{\sim} Gr^i k_n^M(k) \tag{4.17}$$

by $\Omega_F^{n-1} \ni s dt_1/t_1 \wedge \dots dt_{n-1}/t_{n-1} \mapsto \overline{(1+su_k^i, t_1, \dots, t_{n-1})} \in Gr^i k_n^M(k)$.

(3) *if* $p \mid i$, *there exists the following isomorphism*:

$$\Omega_F^{n-1}/\Omega_{F,d=0}^{n-1} \oplus \Omega_F^{n-2}/\Omega_{F,d=0}^{n-2} \xrightarrow{\sim} Gr^i k_n^M(k) \tag{4.18}$$

by $\Omega_F^{n-2}/\Omega_{F,d=0}^{n-2} \ni s dt_1/t_1 \wedge \dots dt_{n-2}/t_{n-2} \mapsto \overline{(1+s\,u_k^i, t_1, \dots, t_{n-2}, u_k)} \in Gr^i k_n^M(k)$, *and the map from* $\Omega_F^{n-1}/\Omega_{F,d=0}^{n-1}$ *to* $Gr^i k_n^M(k)$ *is the same one defined in* (2). ($\Omega_{F,d=0}^i$ *denotes the set of* d*-closed* i*-forms of* F).

(4) *if* $ch.k = 0$ *and* $i = ep/(p-1)$, *there exists the following isomorphism*:

$$\Omega_F^{n-1}/(1-C)\,\Omega_{F,d=0}^{n-1} \oplus \Omega_F^{n-2}/(1-C)\,\Omega_{F,d=0}^{n-2} \xrightarrow{\sim} Gr^i k_n^M(k), \tag{4.19}$$

where the maps from the left hand side to $Gr^i k_n^M(k)$ *are the same ones defined in* (3).

5. Duality for two dimensional complete Gorenstein local rings

In this section, we establish the duality theorems for two-dimensional complete normal Gorenstein local rings with finite residue field. For an arbitrary complete normal local ring R over a field k, the Grothendieck duality theorem states that its k dual with respect to the $\mathfrak{m}_R$-adic topology is represented by the local hypercohomology of the dualizing complex D_R of R. What we will do in this section is to rewrite this local hypercohomology explicitly by using differential forms, or more precisely, differential ideles. The basic reference in this section is [H]. First, we review the following Grothendieck duality theorem.

Theorem 5.1 (Grothendieck)**.** *Let* R *be an arbitrary complete normal local ring of Krull dimension* n *with residue field* k. *Then there exists a unique complex* $D_R^\bullet$ *called a normalized dualizing complex of* R *in the derived category of* R, *and satisfies the following isomorphism for an arbitrary finite* R*-module* M:

$$\mathrm{Ext}_R^{n-i}(M, D_R^\bullet) \cong \mathrm{Hom}_{\mathfrak{m}\text{–adic}}(\mathbb{H}_{\mathfrak{m}_R}^i(R,\, M),\, k), \tag{5.1}$$

where $\mathbb{H}_{\mathfrak{m}_R}^i(R,\, \mathcal{F}^\bullet)$ *denotes the hypercohomology of the bounded complex* $\mathcal{F}^\bullet$ *with support at the maximal pont* $\mathfrak{m}_R$ *of* R.

Remark 6. Let R be a complete Cohen-Macaulay local ring, then the above dualizing complex $D_R^\bullet$ becomes very simple. That is, $D_R^\bullet$ becomes a single sheaf. Moreover if R is Gorenstein, this sheaf becomes the sheaf of n-forms of R, hence we have the isomorphism $\mathcal{F}_R \cong \Omega_R^n[n]$.

Now, we return to our complete local ring R which is normal Gorenstein with finite residue field. Our task is to rewrite the local cohomology $H^2_{\mathfrak{m}}(R,\, \Omega^2_R)$ explicitly by using the localization sequence in étale cohomology. Let us state the first main result.

Theorem 5.2. *For an arbitrary two-dimensional normal complete Gorenstein local ring R with finite residue field, we denote by F its fractional field. Then there exists the following canonical isomorphism*:

$$R \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{cont}}\left(\left(\bigoplus_{\mathfrak{q} \in P^1_R} \left(\Omega^2_{F_{\mathfrak{q}}} / \Omega^2_{R_{\mathfrak{q}}} \right) \right) / (\Omega^2_F / \Omega^2_R),\, \mathbb{Z}/p \right), \tag{5.2}$$

where $R_{\mathfrak{q}}$ denotes the completion of the localized ring $R_{(\mathfrak{q})}$ at its maximal prime $\mathfrak{q}$, and $F_{\mathfrak{q}}$ denotes the fractional field of $R_{\mathfrak{q}}$. In the above, $\left(\bigoplus_{\mathfrak{q} \in P^1_R} \left(\Omega^2_{F_{\mathfrak{q}}} / \Omega^2_{R_{\mathfrak{q}}} \right) \right) / (\Omega^2_F / \Omega^2_R)$ is considered as a discrete module.

Proof. Let us consider the following localization sequence which is exact:

$$\begin{aligned} \cdots \to H^1_{\mathfrak{m}_R}(\mathrm{Spec}\, R,\, \Omega^2_{\mathrm{Spec}\, R}) \;\to\; H^1(\mathrm{Spec}\, R,\, \Omega^2_{\mathrm{Spec}\, R}) \;\to\; H^1(X,\, \Omega^2_X) \;\to \\ \to\; H^2_{\mathfrak{m}_R}(\mathrm{Spec}\, R,\, \Omega^2_{\mathrm{Spec}\, R}) \;\to\; H^2(\mathrm{Spec}\, R,\, \Omega^2_{\mathrm{Spec}\, R}) \;\to\; H^2(X,\, \Omega^2_X) \cdots, \end{aligned} \tag{5.3}$$

where $X = \mathrm{Spec}\, R \setminus \mathfrak{m}_R$ with the maximal ideal $\mathfrak{m}_R$ of R. We see that both groups $H^1(\mathrm{Spec}\, R,\, \Omega^2_{\mathrm{Spec}\, R})$ and $H^2(\mathrm{Spec}\, R,\, \Omega^2_{\mathrm{Spec}\, R})$ in (5.3) vanish, so we obtain the isomorphism

$$H^1\,(X,\, \Omega^2_X) \;\cong\; H^2_{\mathfrak{m}_R}(\mathrm{Spec}\, R,\, \Omega^2_{\mathrm{Spec}\, R}). \tag{5.4}$$

We analyze the group $H^1(X,\, \Omega^2_X)$. First we notice that the Krull dimension of X is 1. By considering the localization sequence in étale cohomology of X paying attentions to the fact that each height one prime of X corresponds bijectively to the unique height one prime of R, we obtain the following exact sequence:

$$\cdots \to H^0(X,\, \Omega^2_X) \;\to\; H^0(F,\, \Omega^2_F) \;\to\; \bigoplus_{\mathfrak{q} \in X^{(1)}} H^1_{\mathfrak{q}}(X,\, \Omega^2_X) \;\to\; H^1(X,\, \Omega^2_X) \;\to\; 0. \tag{5.5}$$

The final 0 is obtained by replacing the group $H^1(F, \Omega^2_F)$ by 0. From (5.5), we can find that the group $H^1(X,\, \Omega^2_X)$ is explicitly expressed as

$$H^1(X,\, \Omega^2_X) \;\cong\; \left(\bigoplus_{\mathfrak{q} \in X^{(1)}} H^1_{\mathfrak{q}}(X,\, \Omega^2_X) \right) / \left(H^0(F,\, \Omega^2_F) / H^0(X,\, \Omega^2_X) \right). \tag{5.6}$$

By replacing $H^0(F,\, \Omega^2_F)$ and $H^0(X,\, \Omega^2_X)$ with Ω^2_F and Ω^2_R respectively, we obtain

$$H^1(X,\, \Omega^2_X) \;\cong\; \left(\bigoplus_{\mathfrak{q} \in X^{(1)}} H^1_{\mathfrak{q}}(X,\, \Omega^2_X) \right) /(\, \Omega^2_F / \Omega^2_R\,). \tag{5.7}$$

We have $H^1_{\mathfrak{q}}(X,\,\Omega^2_X)\;\cong\;H^1_{\mathfrak{q}}\,(\operatorname{Spec} R_{(\mathfrak{q})},\,\Omega^2_{\operatorname{Spec} R_{(\mathfrak{q})}})$ it holds that

$$H^1_{\mathfrak{q}}(\operatorname{Spec} R_{(\mathfrak{q})},\Omega^2_{\operatorname{Spec} R_{(\mathfrak{q})}})\cong H^0(\operatorname{Spec} F,\,\Omega^2_{\operatorname{Spec} F})/H^0(\operatorname{Spec} R_{(\mathfrak{q})},\,\Omega^2_{\operatorname{Spec} R_{(\mathfrak{q})}})\cong(\Omega^2_F\,/\Omega^2_{R_{(\mathfrak{q})}})$$
$$\cong(\Omega^2_{F_{\mathfrak{q}}}\,/\Omega^2_{R_{\mathfrak{q}}}). \tag{5.8}$$

Thus by considering (5.6), (5.7) and (5.8), Theorem 5.1, applied with $M=\Omega^2_R, i=n=2$, provides the desired perfectness of (5.2). □

Next, we state the duality theorem for $R\,/R^p$.

Theorem 5.3. *Let R be the same ring as in* Theorem 5.2. *Then, there exists the following isomorphism*:

$$(R/R^p)\cong\operatorname{Hom}_{\mathrm{cont}}\left(\left(\bigoplus_{\mathfrak{q}\in P^1_R}(\Omega^1_{F_{\mathfrak{q}}}/(\Omega^1_{F_{\mathfrak{q}},d=0},\,\Omega^1_{R_{\mathfrak{q}}}))\right)/(\Omega^1_F/(\Omega^1_{F,d=0},\Omega^1_R)),\,\mathbb{Z}/p\right), \tag{5.9}$$

where the group $\left(\bigoplus_{\mathfrak{q}\in P^1_R}(\Omega^1_{F_{\mathfrak{q}}}\,/(\,\Omega^1_{F_{\mathfrak{q}},\,d=0}\,,\,\Omega^1_{R_{\mathfrak{q}}}))\right)/(\;\Omega^1_F\,/(\,\Omega^1_{F,\,d=0},\;\Omega^1_R)\;)$ *is assumed to have the discrete topology.*

Proof. For the proof, we need the Cartier operator C. Consider the exact sequences

$$0\to R\overset{x\mapsto x^p}{\to}R\;\to\;R\,/R^p\to 0 \tag{5.10}$$

$$0\to(R/R^p)^*\to\left(\bigoplus_{\mathfrak{q}\in P^1_R}\left(\Omega^2_{F_{\mathfrak{q}}}/\Omega^2_{R_{\mathfrak{q}}}\right)\right)/(\Omega^2_F/\Omega^2_R)\;\overset{C}{\to}\;\left(\bigoplus_{\mathfrak{q}\in P^1_R}\left(\Omega^2_{F_{\mathfrak{q}}}/\Omega^2_{R_{\mathfrak{q}}}\right)\right)/(\Omega^2_F/\Omega^2_R)\to 0, \tag{5.11}$$

where (5.11) is obtained by taking the Pontryagin dual of (5.10) and Theorem 3.2. But the property of the Cartier operator C shows that

$$\operatorname{Ker}(\,C\colon\Omega^2_{F_{\mathfrak{q}}}\;\to\;\Omega^2_{F_{\mathfrak{q}}}\,)\;=\;d\,\Omega^1_{F_{\mathfrak{q}}}\;\cong\;(\Omega^1_{F_{\mathfrak{q}}}\,/\Omega^1_{F_{\mathfrak{q}},d=0}) \tag{5.12}$$
$$\operatorname{Ker}(\,C\colon\Omega^2_F\;\to\;\Omega^2_F\,)\;=\;d\,\Omega^1_F\;\cong\;(\Omega^1_F\,/\Omega^1_{F,d=0}), \tag{5.13}$$

where d denotes the differential operator. So, it follows that

$$\operatorname{Ker}(\,C\colon\Omega^2_{F_{\mathfrak{q}}}\,/\Omega^2_{R_{\mathfrak{q}}}\;\to\Omega^2_{F_{\mathfrak{q}}}\,/\Omega^2_{R_{\mathfrak{q}}}\,)\;=\;\Omega^1_{F_{\mathfrak{q}}}\,/(\,\Omega^1_{F_{\mathfrak{q}},d=0},\;\Omega^1_{R_{\mathfrak{q}}}\,), \tag{5.14}$$
$$\operatorname{Ker}(\,C\colon\Omega^2_F\,/\Omega^2_R\;\to\;\Omega^2_F\,/\Omega^2_R\,)\;=\;\Omega^1_F\,/(\,\Omega^1_{F,d=0},\;\Omega^1_R\,). \tag{5.15}$$

Now from (5.14) and (5.15), easy arguments show

$$\mathrm{Ker}\Big(C\colon (\bigoplus_{\mathfrak{q}\in P^1_R} (\Omega^2_{F_\mathfrak{q}}/\Omega^2_{R_\mathfrak{q}}))/(\Omega^2_F/\Omega^2_R) \to (\bigoplus_{\mathfrak{q}\in P^1_R} (\Omega^2_{F_\mathfrak{q}}/\Omega^2_{R_\mathfrak{q}}))/(\Omega^2_F/\Omega^2_R)\Big)$$

$$= \left(\bigoplus_{\mathfrak{q}\in P^1_R} (\,\Omega^1_{F_\mathfrak{q}}\,/\,(\,\Omega^1_{F_\mathfrak{q},\,d=0},\,\Omega^1_{R_\mathfrak{q}}\,)\,)\right)/(\,\Omega^1_F\,/(\Omega^1_{F,\,d=0},\,\Omega^1_R)\,). \tag{5.16}$$

Thus, we get the following commutative diagram:

$$\begin{array}{ccccc} 0\longrightarrow & R & \xrightarrow{x\mapsto x^p} & R & \\ & \downarrow\cong & & \downarrow\cong & \\ 0\to & \Big(\Big(\bigoplus_{\mathfrak{q}\in P^1_R}\Big(\Omega^2_{F_\mathfrak{q}}/\Omega^2_{R_\mathfrak{q}}\Big)\Big)/(\Omega^2_F/\Omega^2_R)\Big)^* & \to & \Big(\Big(\bigoplus_{\mathfrak{q}\in P^1_R}\Big(\Omega^2_{F_\mathfrak{q}}/\Omega^2_{R_\mathfrak{q}}\Big)\Big)/(\Omega^2_F/\Omega^2_R)\Big)^* & \end{array}$$

$$\begin{array}{ccc} \longrightarrow & R\,/R^p & \longrightarrow 0 \\ & \downarrow f & \\ \to & \Big(\Big(\bigoplus_{\mathfrak{q}\in P^1_R}(\Omega^1_{F_\mathfrak{q}}/(\Omega^1_{F_\mathfrak{q},\,d=0},\,\Omega^1_{R_\mathfrak{q}}))\Big)/(\,\Omega^1_F\,/(\Omega^1_{F,\,d=0},\,\Omega^1_R)\,)\Big)^* & \to 0, \end{array} \tag{5.17}$$

where the lower row in the above diagram (5.17) is exact from (5.16). A diagram chase shows that the vertical arrow f in diagram (5.17) becomes bijective, which is the desired result. □

Here, we state the duality theorem for F, F/F^p where F is the fractional field of R. For this, we explain the differential idele class group. It is defined by

$$(\underline{\prod}_{\mathfrak{q}\in P^1_R} \Omega^2_{F_\mathfrak{q}})/\Omega^2_F \tag{5.18}$$

$$\text{or}\quad (\underline{\prod}_{\mathfrak{q}\in P^1_R} (\Omega^1_{F_\mathfrak{q}}/\Omega^1_{F_\mathfrak{q},d=0}))/(\,\Omega^1_F/\Omega^1_{F,d=0}\,), \tag{5.19}$$

where the restricted product in (5.18) is defined by the condition that any element in it lies in the group $\left(\prod_{\mathfrak{q}\in U}\Omega^2_{R_\mathfrak{q}}\right)\oplus\left(\bigoplus_{\mathfrak{q}\notin U}\Omega^2_{F_\mathfrak{q}}\right)$ for some open $U\subset \mathrm{Spec}\,R$. The definition of (5.19) is given similarly to (5.18). Now, the duality results for F, F/F^p are stated as follows:

Theorem 5.4. *Let R be the same ring as in* Theorem 5.2 *and F be its fractional field. Then, there exists the following isomorphism*:

$$F \cong \mathrm{Hom}_{\mathrm{cont}}\left((\underline{\prod}_{\mathfrak{q}\in P^1_R}\Omega^2_{F_\mathfrak{q}})/\Omega^2_F,\ \mathbb{Z}/p\right), \tag{5.20}$$

where $\mathrm{Hom}_{\mathrm{cont}}$ *denotes the set of homomorphism* $\chi\colon (\underline{\prod}_{\mathfrak{q}\in P^1_R} \Omega^2_{F_\mathfrak{q}})/\Omega^2_F \to \mathbb{Z}/p$ *such that* χ *annihilates* $\mathfrak{q}^{n_\mathfrak{q}}\Omega^2_{R_\mathfrak{q}}$ *for each* $\mathfrak{q}$ *with some* $n_\mathfrak{q} \geq 0$ *and almost all* $n_\mathfrak{q} = 0$.

Theorem 5.5. *For an arbitrary complete local ring* R *with fractional field* F *which satisfies the condition in* Theorem 5.2, *there exists the following canonical isomorphism*:

$$(F/F^p) \cong \mathrm{Hom}_{\mathrm{cont}}\left(\left(\underline{\prod_{\mathfrak{q}\in P^1_R}} (\Omega^1_{F_\mathfrak{q}}/\Omega^1_{F_\mathfrak{q},d=0})\right)/(\Omega^1_F/\Omega^1_{F,d=0}),\ \mathbb{Z}/p\right), \tag{5.21}$$

where $\mathrm{Hom}_{\mathrm{cont}}$ *denotes the same meaning as in* Theorem 5.4.

These theorems are immediately obtained from Theorem 5.2, Theorem 5.3 respectively by considering the fact that $F = \varinjlim_{f\in R} R[\frac{1}{f}]$.

6. The complete discrete valuation field $K_\mathfrak{p}$

The aim in this section is to construct the idele class group $C_{K_\mathfrak{p}}$ for each complete discrete valuation field $K_\mathfrak{p}$ with $\mathfrak{p} \in P^1_A$. The main theorem is proved at the end of this section. For the definition of the idele class group $C_{K_\mathfrak{p}}$, we need to introduce the group $C^i_{K_\mathfrak{p}}$ for each natural number i (≥ 0). In this section, we always denote by $u_\mathfrak{p}$ the regular parameter of $\mathfrak{p}$ in A.

Definition 6. We define the group $C^i_{K_\mathfrak{p}}$ for each natural number i (≥ 0) as follows:

$$C^i_{K_\mathfrak{p}} := \mathrm{Coker}\left(K^M_3(K_\mathfrak{p}) \overset{diagonal}{\longrightarrow} \prod\nolimits'_{\mathfrak{q}\in P^1_{\widetilde{A/u_\mathfrak{p}}}} \left(K^M_3(K_{\mathfrak{p},\mathfrak{q}})/U^iK^M_3(K_{\mathfrak{p},\mathfrak{q}})\right)\right), \tag{6.1}$$

where $\widetilde{A/u_\mathfrak{p}}$ denotes the normalization of two-dimensional complete local ring $A/u_\mathfrak{p}$, and each $K_{\mathfrak{p},\mathfrak{q}}$ is the unique complete discrete valuation field satisfying

1) $K_\mathfrak{p} \subset K_{\mathfrak{p},\mathfrak{q}}$ and $\mathfrak{m}_{K_\mathfrak{p}}\mathcal{O}_{K_{\mathfrak{p},\mathfrak{q}}} = \mathfrak{m}_{K_{\mathfrak{p},\mathfrak{q}}}$, where $\mathfrak{m}_{K_\mathfrak{p}}$ and $\mathfrak{m}_{K_{\mathfrak{p},\mathfrak{q}}}$ denote the maximal ideals of the valuation rings $\mathcal{O}_{K_\mathfrak{p}}$ and $\mathcal{O}_{K_{\mathfrak{p},\mathfrak{q}}}$, respectively.

2) The residue field $\mathcal{O}_{K_{\mathfrak{p},\mathfrak{q}}}/\mathfrak{m}_{K_{\mathfrak{p},\mathfrak{q}}}$ of the valuation ring $\mathcal{O}_{K_{\mathfrak{p},\mathfrak{q}}}$ of $K_{\mathfrak{p},\mathfrak{q}}$ coincides with the fractional field of the complete discrete valuation ring $(\widetilde{A/u_\mathfrak{p}})_\mathfrak{q}$ defined by $(\widetilde{A/u_\mathfrak{p}})_\mathfrak{q} := \varprojlim_n (\widetilde{A/u_\mathfrak{p}})_{(\mathfrak{q})}/\mathfrak{q}^n$.

We remark that each residue field of $\mathcal{O}_{K_{\mathfrak{p},\mathfrak{q}}}$ in 2) is a two-dimensional local field.

Next, we give the definition of the group $\prod'_{\mathfrak{q}\in P^1_{\widetilde{A/u_\mathfrak{p}}}} \left(K^M_3(K_{\mathfrak{p},\mathfrak{q}})/U^iK^M_3(K_{\mathfrak{p},\mathfrak{q}})\right)$. If we write $\iota\colon \mathcal{O}_{K_{\mathfrak{p},\mathfrak{q}}} \twoheadrightarrow \mathcal{O}_{K_{\mathfrak{p},\mathfrak{q}}}/\mathfrak{m}_{K_{\mathfrak{p},\mathfrak{q}}}$, it is easily seen that $\iota^{-1}((\widetilde{A/u_\mathfrak{p}})_\mathfrak{q})$ is the subring of $\mathcal{O}_{K_{\mathfrak{p},\mathfrak{q}}}$ by the condition 2). Now, under these preparation, the definition of $\prod'$ is given as follows.

For an arbitrary element a in $\prod'_{\mathfrak{q}\in P^1_{\widetilde{A/u_\mathfrak{p}}}} \left(K_3^M(K_{\mathfrak{p},\mathfrak{q}})/U^i K_3^M(K_{\mathfrak{p},\mathfrak{q}})\right)$, its $\mathfrak{q}$-component $a_\mathfrak{q}$ belongs to $\mathrm{Image}\Big(K_3^M(\iota^{-1}((\widetilde{A/u_\mathfrak{p}})_\mathfrak{q})[\frac{1}{u_\mathfrak{p}}]) \overset{canonical}{\longrightarrow} K_3^M(K_{\mathfrak{p},\mathfrak{q}})/U^i K_3^M(K_{\mathfrak{p},\mathfrak{q}})\Big)$ for almost all $\mathfrak{q}$, and if a height one prime $\mathfrak{q}$ in $P^1_{\widetilde{A/u_\mathfrak{p}}}$ lies over a *non*-regular height one prime $\mathfrak{q}'$ of $A/u_\mathfrak{p}$ (this means that the localization $(A/u_\mathfrak{p})_{(\mathfrak{q}')}$ of $(A/u_\mathfrak{p})$ at $\mathfrak{q}'$ does not become regular), we put no condition on $\mathfrak{q}$-component $a_\mathfrak{q}$ of a.

Now, under these preparations, we can define the idele class group $C_{K_\mathfrak{p}}$ as follows:

Definition 7. We define the idele class group $C_{K_\mathfrak{p}}$ by

$$C_{K_\mathfrak{p}} = \varprojlim{}_{i\geq 0}\, C^i_{K_\mathfrak{p}}. \tag{6.2}$$

Now, by using $C_{K_\mathfrak{p}}$ defined above, we construct the reciprocity pairing

$$H^1_{\mathrm{Gal}}(K_\mathfrak{p},\, \mathbb{Q}/\mathbb{Z}) \times C_{K_\mathfrak{p}} \to \mathbb{Q}/\mathbb{Z}. \tag{6.3}$$

We begin by defining the pairing

$$H^1_{\mathrm{Gal}}(K_\mathfrak{p},\, \mathbb{Z}/p) \times C_{K_\mathfrak{p}}/p \to \mathbb{Z}/p$$

which would be established at (6.15) below. First, we choose an (non-canonical) isomorphism

$$K_\mathfrak{p} \;\cong\; \kappa(\mathfrak{p})\,((u_\mathfrak{p})). \tag{6.4}$$

Further, if we also choose an (non-canonical) isomorphism

$$\mathrm{Frac}((\widetilde{A/u_\mathfrak{p}})_\mathfrak{q}) \;\cong\; \mathbb{F}_q\,((s_\mathfrak{q}))((t_\mathfrak{q})), \tag{6.5}$$

$K_{\mathfrak{p},\mathfrak{q}}$ is explicitly rewritten as

$$K_{\mathfrak{p},\mathfrak{q}} \cong \mathbb{F}_q\,((s_\mathfrak{q}))((t_\mathfrak{q}))((u_\mathfrak{p})). \tag{6.6}$$

On the other hand, for each three-dimensional local field $K_{\mathfrak{p},\mathfrak{q}}$, there exists the following residue pairing by Kato-Parshin:

$$H^1_{\mathrm{Gal}}(K_{\mathfrak{p},\mathfrak{q}},\, \mathbb{Z}/p) \,\times\, K_3^M(K_{\mathfrak{p},\mathfrak{q}})/p \,\to\, \mathbb{Z}/p \tag{6.7}$$

defined by

$$\Big(\chi_{\mathfrak{p},\mathfrak{q}}, \overline{(a_{\mathfrak{p},\mathfrak{q}}, b_{\mathfrak{p},\mathfrak{q}}, c_{\mathfrak{p},\mathfrak{q}})}\Big) \mapsto \mathrm{Res}_{\frac{ds_\mathfrak{q}}{s_\mathfrak{q}}\wedge\frac{dt_\mathfrak{q}}{t_\mathfrak{q}}\wedge\frac{du_\mathfrak{p}}{u_\mathfrak{p}}}\,(\chi_{\mathfrak{p},\mathfrak{q}}\frac{da_{\mathfrak{p},\mathfrak{q}}}{a_{\mathfrak{p},\mathfrak{q}}} \wedge \frac{db_{\mathfrak{p},\mathfrak{q}}}{b_{\mathfrak{p},\mathfrak{q}}} \,\wedge\, \frac{dc_{\mathfrak{p},\mathfrak{q}}}{c_{\mathfrak{p},\mathfrak{q}}}) \,\in \mathbb{Z}/p, \tag{6.8}$$

where $\mathrm{Res}_{\frac{ds_\mathfrak{p}}{s_\mathfrak{p}}\wedge\frac{dt_\mathfrak{q}}{t_\mathfrak{q}}\wedge\frac{du_\mathfrak{q}}{u_\mathfrak{q}}}$ denotes the value of the trace $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$ of the coefficient $c_{-1,-1,-1}$ of the logarithmic form $\frac{ds_\mathfrak{p}}{s_\mathfrak{p}} \wedge \frac{dt_\mathfrak{q}}{t_\mathfrak{q}} \wedge \frac{du_\mathfrak{q}}{u_\mathfrak{q}}$ in the differential three-form $(\chi_{\mathfrak{p},\mathfrak{q}}\,\frac{da_{\mathfrak{p},\mathfrak{q}}}{a_{\mathfrak{p},\mathfrak{q}}} \wedge \frac{db_{\mathfrak{p},\mathfrak{q}}}{b_{\mathfrak{p},\mathfrak{q}}} \wedge \frac{dc_{\mathfrak{p},\mathfrak{q}}}{c_{\mathfrak{p},\mathfrak{q}}})$.

So, by summing these residue pairings over all $K_{\mathfrak{p},\mathfrak{q}}$, we get the following pairing:

$$H^1_{\mathrm{Gal}}(K_{\mathfrak{p}},\,\mathbb{Z}/p)\,\times\,\varprojlim_i \prod\nolimits'_{\mathfrak{q}\in P^1_{\widetilde{A/u_{\mathfrak{p}}}}}\ (\ K_3^M(K_{\mathfrak{p},\mathfrak{q}})\,/U^iK_3^M(K_{\mathfrak{p},\mathfrak{q}})\ )\,/p\ \rightarrow\ \mathbb{Z}/p. \tag{6.9}$$

Here, we used the restriction homomorphism

$$r_{\mathfrak{p},\mathfrak{q}}\colon H^1_{\mathrm{Gal}}(K_{\mathfrak{p}},\,\mathbb{Z}/p)\ \rightarrow\ H^1_{\mathrm{Gal}}(K_{\mathfrak{p},\mathfrak{q}},\,\mathbb{Z}/p) \tag{6.10}$$

in Galois cohomology. The well-definedness of the above pairing (6.9) is the consequence of the definition of the product $\prod'$. An important fact is that each element $\chi_{\mathfrak{p},\mathfrak{q}}\in H^1_{\mathrm{Gal}}(K_{\mathfrak{p},\mathfrak{q}},\,\mathbb{Z}/p)$ annihilates $U^iK_3^M(K_{\mathfrak{p},\mathfrak{q}})\,/p$ for some i. On the other hand, Theorem 4.6 states that any element $\overline{(a_{\mathfrak{p}},b_{\mathfrak{p}},c_{\mathfrak{p}})}\in K_3^M(K_{\mathfrak{p}})\,/p$ can be written as

$$\overline{(a_{\mathfrak{p}},b_{\mathfrak{p}},c_{\mathfrak{p}})}=\overline{(1+\alpha\,u^j_{\mathfrak{p}},\,\beta,\,\gamma)} \tag{6.11}$$

with $\alpha,\,\beta,\,\gamma\ \in\ \kappa(\mathfrak{p})$. We use this fact in the proof of the following proposition. That is, we prove the reciprocity law for $K_{\mathfrak{p}}$ as

Proposition 6.1. *For an arbitrary element $a\in K_3^M(K_{\mathfrak{p}})$, its diagonal image into the right hand side of* (6.9) *is annihilated by the pairing* (6.9).

Proof. First, we see that if we use the representation of $K_{\mathfrak{p}}$ in (6.4), each element $\chi_{\mathfrak{p}}\in H^1_{\mathrm{Gal}}(K_{\mathfrak{p}},\,\mathbb{Z}/p)\ \cong K_{\mathfrak{p}}/(\mathfrak{P}-1)K_{\mathfrak{p}}\ \big((\mathfrak{P}-1)x:=x^p-x\big)$ is explicitly represented as

$$\chi_{\mathfrak{p}}\ =\sum_{n\gg-\infty}\overline{\delta_n\,u^n_{\mathfrak{p}}}, \tag{6.12}$$

where each $\delta_n\in\kappa(\mathfrak{p})$ and $\overline{\delta_n\,u^n_{\mathfrak{p}}}$ denotes the image of $\delta_n\,u^n_{\mathfrak{p}}$ in $K_{\mathfrak{p}}/(\mathfrak{P}-1)K_{\mathfrak{p}}$. But it is easily seen that the pair $\Big(\overline{\delta_n\,u^n_{\mathfrak{p}}},\,\overline{(a_{\mathfrak{p}},b_{\mathfrak{p}},c_{\mathfrak{p}})}\Big)$ goes to 0 for $n\geq 1$ under the pairing (6.9). Thus, we have only to check the above proposition in the case

$$\chi_{\mathfrak{p}}=\overline{(\frac{\delta_n}{u^n_{\mathfrak{p}}})} \tag{6.13}$$

for an arbitrary $n\geq 0$. Then, we find that the residue pairing $\Big(\overline{(\frac{\delta_n}{u^n_{\mathfrak{p}}})},\,\overline{(1+\alpha\,u^j_{\mathfrak{p}},\,\beta,\,\gamma)}\Big)$ becomes 0 in this situation if $j\nmid n$. Otherwise, $n=kj$ and in this case, the residue pairing is explicitly calculated as

$$\Bigg(\overline{(\frac{\delta_{kj}}{u^{kj}_{\mathfrak{p}}})},\,\overline{(1+\alpha\,u^j_{\mathfrak{p}},\,\beta,\,\gamma)}\Bigg)\ \mapsto\ \sum_{\mathfrak{q}\in P^1_{\widetilde{A/u_{\mathfrak{p}}}}}\mathrm{Res}_{\frac{ds_{\mathfrak{q}}}{s_{\mathfrak{q}}}\wedge\frac{dt_{\mathfrak{q}}}{t_{\mathfrak{q}}}}\Bigg(\big(\,(-1)^{k-1}j\overline{\delta_{kj}}\alpha^k\,\big)\,\frac{d\,\beta}{\beta}\,\wedge\,\frac{d\,\gamma}{\gamma}\Bigg). \tag{6.14}$$

But it is found that the right hand side of (6.14) is nothing but the residue pairing $\Big(\,(\,(-1)^{k-1}j\overline{\delta_{kj}}\alpha^k\,),\,\overline{(\beta,\,\gamma)}\Big)$ with $(\,(-1)^{k-1}j\overline{\delta_{kj}}\alpha^k\,)\ \in\ H^1_{\mathrm{Gal}}(\kappa(\mathfrak{p}),\,\mathbb{Z}/p)$ and $\overline{(\beta,\,\gamma)}\ \in$

$K_2^M(\kappa(\mathfrak{p}))\,/p$ in the class field theory for the field $\kappa(\mathfrak{p})$, which becomes zero by Kato's reciprocity law proved as Proposition 7 in [Ka3]. Hence, we are done. $\square$

Now, Proposition 6.1 shows that the right hand side of the pairing (6.9) factors through $C_{K_\mathfrak{p}}\,/p$. That is, the pairing (6.9) is rewritten as

$$H^1_{\mathrm{Gal}}(K_\mathfrak{p},\, \mathbb{Z}\,/p) \,\times\, C_{K_\mathfrak{p}}\,/p \,\rightarrow\, \mathbb{Z}\,/p. \tag{6.15}$$

By induction, we get the pairing

$$H^1_{\mathrm{Gal}}(K_\mathfrak{p},\, \mathbb{Z}\,/p^m) \,\times\, C_{K_\mathfrak{p}}\,/p^m \,\rightarrow\, \mathbb{Z}\,/p^m \tag{6.16}$$

for an arbitrary natural number m. Similarly by using the Kummer theory, we get the pairing

$$H^1_{\mathrm{Gal}}(K_\mathfrak{p},\, \mathbb{Z}\,/l^m) \,\times\, C_{K_\mathfrak{p}}\,/l^m \,\rightarrow\, \mathbb{Z}\,/l^m \tag{6.17}$$

for an arbitrary prime l and m (key ingredients for the construction of this pairing (6.17) are the canonical isomorphism $H^4_{\mathrm{Gal}}(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},\, \mu_{l^m}^{\otimes 3}) \,\cong\, \mathbb{Z}\,/l^m$ (cf. [Ka1] I, II) and the Galois symbol isomorphism $K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})\,/l^m \,\cong\, H^3_{\mathrm{Gal}}(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},\, \mu_{l^m}^{\otimes 3})$ which follows from the Bloch-Milnor-Kato conjecture. We omit details). By taking the inductive limit of (6.16) and (6.17), we get the following reciprocity pairing:

$$H^1_{\mathrm{Gal}}(K_\mathfrak{p},\, \mathbb{Q}\,/\mathbb{Z}) \,\times\, C_{K_\mathfrak{p}} \,\rightarrow\, \mathbb{Q}\,/\mathbb{Z}. \tag{6.18}$$

From this, we get the dual reciprocity homomorphism

$$\rho^*_{K_\mathfrak{p}} : H^1_{\mathrm{Gal}}(K_\mathfrak{p},\, \mathbb{Q}\,/\mathbb{Z}) \,\rightarrow\, \mathrm{Hom}\,(C_{K_\mathfrak{p}},\, \mathbb{Q}\,/\mathbb{Z}). \tag{6.19}$$

Next, we give the definition of certain filtrations on C_K and $H^1_{\mathrm{Gal}}(K_\mathfrak{p},\, \mathbb{Z}/p)$.

Definition 8. For an arbitrary positive integer $n \,\geq\, 0$, we define the filtration $F^n\, C_{K_\mathfrak{p}}$ on $C_{K_\mathfrak{p}}$ and $F_n H^1_{\mathrm{Gal}}(K_\mathfrak{p},\, \mathbb{Z}/p)$ on $H^1_{\mathrm{Gal}}(K_\mathfrak{p},\, \mathbb{Z}/p)$ as follows:

1) $F^n\, C_{K_\mathfrak{p}} :\, = \varprojlim_i F^n\, C^i_{K_\mathfrak{p}},$ (6.20)

where each $F^n\, C^i_{K_\mathfrak{p}}$ is defined by $F^n\, C^i_{K_\mathfrak{p}} := \mathrm{Image}\left({\prod}'_{\mathfrak{q}\in P^1_{\widetilde{A\,/u_\mathfrak{p}}}} U^n K_3^M(K_{\mathfrak{p},\mathfrak{q}}) \,\rightarrow\, C^i_{K_\mathfrak{p}} \right)$

($F^n\, C^i_{K_\mathfrak{p}} = 0$ if $i \leq n$).

2) $F_n H^1_{\mathrm{Gal}}(K_\mathfrak{p},\, \mathbb{Z}/p) :\, = \{\mathrm{Im}\colon x \mapsto (K_\mathfrak{p}\,/(\mathfrak{P}-1)K_\mathfrak{p}) \mid\, x \in K_\mathfrak{p}$ satisfies $v_{u_\mathfrak{p}}(x) \geq -n\}$,

where $(\mathfrak{P}-1)$ is defined by $(\mathfrak{P}-1)\,(x) \,=\, x^p - x$. ($F_n$ is the increasing filtration). (6.21)

We define the filtration $F^n(C_{K_\mathfrak{p}}/m)$ on $C_{K_\mathfrak{p}}/m$ for a natural number m (>1) by

$$F^n(C_{K_\mathfrak{p}}/m) := \text{Image}\left(F^n C_{K_\mathfrak{p}} \to C_{K_\mathfrak{p}}/m\right). \tag{6.22}$$

We put

$$Gr^n(C_{K_\mathfrak{p}}/p) := F^n(C_{K_\mathfrak{p}}/p)/F^{n+1}(C_{K_\mathfrak{p}}/p) \tag{6.23}$$

$$Gr_n H^1_{\text{Gal}}(K_\mathfrak{p}, \mathbb{Z}/p) := F_n H^1_{\text{Gal}}(K_\mathfrak{p}, \mathbb{Z}/p)/F_{n-1} H^1_{\text{Gal}}(K_\mathfrak{p}, \mathbb{Z}/p). \tag{6.24}$$

We see that there exists an isomorphism

$$F_\infty H^1_{\text{Gal}}(K_\mathfrak{p}, \mathbb{Z}/p)/F_0 H^1_{\text{Gal}}(K_\mathfrak{p}, \mathbb{Z}/p) \cong H^2_\mathfrak{p}(A_\mathfrak{p}, \mathbb{Z}/p) \tag{6.25}$$

which follows from the localization sequence

$$0 \to H^1_{\text{Gal}}(A_\mathfrak{p}, \mathbb{Z}/p) \to H^1_{\text{Gal}}(K_\mathfrak{p}, \mathbb{Z}/p) \to H^2_\mathfrak{p}(A_\mathfrak{p}, \mathbb{Z}/p) \to 0.$$

There is a lemma.

Lemma 6.2. *There hold the following isomorphisms*:

1.*if* $p \nmid n$, $\kappa(\mathfrak{p}) \xrightarrow{\sim} Gr_n H^1_{\text{Gal}}(K_\mathfrak{p}, \mathbb{Z}/p)$ by $\kappa(\mathfrak{p}) \ni a \mapsto \overline{(\frac{a}{u_\mathfrak{p}^n})} \in Gr_n H^1_{\text{Gal}}(K_\mathfrak{p}, \mathbb{Z}/p)$ (6.26)

2.*if* $p \mid n$, $\kappa(\mathfrak{p})/\kappa(\mathfrak{p})^p \xrightarrow{\sim} Gr^n H^1_{\text{Gal}}(K_\mathfrak{p}, \mathbb{Z}/p)$ by $\kappa(\mathfrak{p}) \ni a \mapsto \overline{(\frac{a}{u_\mathfrak{p}^n})} \in Gr_n H^1_{\text{Gal}}(K_\mathfrak{p}, \mathbb{Z}/p)$. (6.27)

This lemma is checked without any difficulty by the explicit calculations.

We state another useful lemma.

Lemma 6.3. *There exist the following surjections*:

1. *if* $p \nmid n$
$$\left(\underline{\prod}_{\mathfrak{q} \in P^1_{\widetilde{A/u_\mathfrak{p}}}} \Omega^2_{\kappa(\mathfrak{p})_\mathfrak{q}}\right)/\Omega^2_{\kappa(\mathfrak{p})} \twoheadrightarrow Gr^n(C_{K_\mathfrak{p}}/p). \tag{6.28}$$

2. *if* $p \mid n$
$$\left(\underline{\prod}_{\mathfrak{q} \in P^1_{\widetilde{A/u_\mathfrak{p}}}} (\Omega^1_{\kappa(\mathfrak{p})_\mathfrak{q}}/\Omega^1_{\kappa(\mathfrak{p})_\mathfrak{q}, d=0})\right)/(\Omega^1_{\kappa(\mathfrak{p})}/\Omega^1_{\kappa(\mathfrak{p}), d=0}) \twoheadrightarrow Gr^n(C_{K_\mathfrak{p}}/p), \tag{6.29}$$

where $\kappa(\mathfrak{p})_\mathfrak{q} := Frac((\widetilde{A/u_\mathfrak{p}})_\mathfrak{q})$ (*recall* $(\widetilde{A/u_\mathfrak{p}})_\mathfrak{q} := \varprojlim_n (\widetilde{A/u_\mathfrak{p}})_{(\mathfrak{q})}/\mathfrak{q}^n$) *and each left hand side of* (6.28), (6.29) *is the differential ideles defined at* (5.18), (5.19).

Proof. This follows directly from Theorem 4.6. □

Under these preliminaries, we state the main theorem in this section.

Theorem 6.4. *The dual reciprocity homomorphism* $\rho^*_{K_\mathfrak{p}}$ *in* (6.19) *induces the injective homomorphism*

$$\rho^*_{K_\mathfrak{p}}\colon H^2_\mathfrak{p}(K_\mathfrak{p},\, \mathbb{Z}\,/p) \hookrightarrow \mathrm{Hom}(F^0C_{K_\mathfrak{p}},\, \mathbb{Z}\,/p), \tag{6.30}$$

where Hom *is the set of homomorphisms between discrete groups.*

Proof. We consider the pairing between each gr-quotients:

$$Gr_nH^1_{\mathrm{Gal}}(K_\mathfrak{p},\, \mathbb{Z}/p) \,\times\, Gr^n(C_{K_\mathfrak{p}}/p) \,\to\, \mathbb{Z}/p. \tag{6.31}$$

The well-definedness of this pairing is easily checked. This pairing induces the homomorphism

$$Gr_nH^1_{\mathrm{Gal}}(K_\mathfrak{p},\, \mathbb{Z}/p) \,\to\, \mathrm{Hom}(Gr^n(C_{K_\mathfrak{p}}/p),\, \mathbb{Z}/p). \tag{6.32}$$

We assert that the homomorphism (6.32) is injective. We treat the case $p \nmid n$ (the case $p \mid n$ is proved without any change). By Lemma 6.3. 1, we have an injective homomorphism

$$\mathrm{Hom}(Gr^n(C_{K_\mathfrak{p}}/p),\, \mathbb{Z}/p) \hookrightarrow \mathrm{Hom}\left(\left(\underline{\prod}_{\mathfrak{q}\in P^1_{\widetilde{A\,/u_\mathfrak{p}}}} \Omega^2_{\kappa(\mathfrak{p})_\mathfrak{q}}\right)/\Omega^2_{\kappa(\mathfrak{p})},\, \mathbb{Z}/p\right).$$

Further, Theorem 5.4 gives us another injective homomorphism

$$\kappa(\mathfrak{p}) \hookrightarrow \mathrm{Hom}\left(\left(\underline{\prod}_{\mathfrak{q}\in P^1_{\widetilde{A\,/u_\mathfrak{p}}}} \Omega^2_{\kappa(\mathfrak{p})_\mathfrak{q}}\right)/\Omega^2_{\kappa(\mathfrak{p})},\, \mathbb{Z}/p\right).$$

Now, Lemma 6.2 together with above two injective homomorphisms shows the desired injectivity of (6.32).

We return to the proof of Theorem 6.4. By considering $n = 0, \ldots, \infty$ of the injective homomorphisms (6.32) together with the equality $F_\infty H^1_{\mathrm{Gal}}(K_\lambda,\, \mathbb{Z}/p)/F_0H^1_{\mathrm{Gal}}(K_\lambda,\, \mathbb{Z}/p) = H^2_\mathfrak{p}(K_\mathfrak{p},\, \mathbb{Z}\,/p)$ in (6.25), we get the desired injectivity (6.30) (we remark that $Gr^0(C_{K_\lambda}/p) = 0$ which follows from Lemma 7 in [Ka1], (II)). □

7. Proof of the existence theorem (p primary parts)

In this section, we prove the following existence theorem for p primary parts.

Theorem 7.1. *Let A be an arbitrary three-dimensional complete regular local ring of positive odd characteristic with finite residue field and K be its fractional field. Then there exists the following canonical dual reciprocity isomorphism*:

$$\rho^*_K\colon H^1_{\mathrm{Gal}}(K,\, \mathbb{Q}_p\,/\mathbb{Z}_p) \xrightarrow{\sim} \mathrm{Hom}_c(C_K,\, \mathbb{Q}_p\,/\mathbb{Z}_p), \tag{7.1}$$

where Hom_c *denotes the set of all continuous homomorphisms from* C_K *to* $\mathbb{Q}_p/\mathbb{Z}_p$ *of finite order.*

The definition of ρ_K^* (mod p) is stated in (7.29) below.

Proof. First, we begin with the following lemma which reduces the proof of the bijectivity of ρ_K^* in (7.1) to that of ρ_K^*/p.

Lemma 7.2. *The proof of the isomorphism* (7.1) *is reduced to that of the following isomorphism*:

$$\rho_K^*/p\colon H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p) \xrightarrow{\sim} \mathrm{Hom}_c(C_K, \mathbb{Z}/p). \tag{7.2}$$

Proof. From the short exact sequence

$$0 \to \mathbb{Z}/p^n \to \mathbb{Z}/p^{n+1} \to \mathbb{Z}/p \to 0,$$

we get the following long exact sequence of the Galois cohomology:

$$0 \to H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p^n) \to H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p^{n+1}) \to H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p) \to 0, \tag{7.3}$$

where we used the vanishing $H^2_{\mathrm{Gal}}(K, \mathbb{Z}/p) = 0$ (cf. [SGA 4], X).

On the other hand, we have the exact sequence

$$C_K/p \to C_K/p^{n+1} \to C_K/p^n \to 0. \tag{7.4}$$

Taking the Pontryagin dual of the exact sequence (7.4), we get the exact sequence

$$0 \to \mathrm{Hom}_c(C_K/p^n, \mathbb{Z}/p^n) \to \mathrm{Hom}_c(C_K/p^{n+1}, \mathbb{Z}/p^{n+1}) \to \mathrm{Hom}_c(C_K/p, \mathbb{Z}/p). \tag{7.5}$$

Consider the commutative diagram

$$\begin{array}{ccccccc}
0\to & H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p^n) & \to & H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p^{n+1}) & \to & H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p) & \to 0 \\
& \downarrow{\scriptstyle \rho_K{}^*/p^n} & & \downarrow{\scriptstyle \rho_K{}^*/p^{n+1}} & & \downarrow{\scriptstyle \rho_K{}^*/p} & \\
0\to & \mathrm{Hom}_c((C_K/p^n), \mathbb{Z}/p^n) & \to & \mathrm{Hom}_c((C_K/p^{n+1}), \mathbb{Z}/p^{n+1}) & \to & \mathrm{Hom}_c((C_K/p), \mathbb{Z}/p), &
\end{array} \tag{7.6}$$

where both rows are exact from (7.3), (7.5). Applying the snake lemma to this diagram, we see that the bijectivity of $\rho_K{}^*/p^{n+1}$ is deduced from those of $\rho_K{}^*/p^n$ and $\rho_K{}^*/p$. □

By this lemma, we have only to prove the bijectivity of (7.2).

Next, we state the definition of the (dual) reciprocity map ρ_K^*/p and the method of the proof. This map has the key to analyze $\mathrm{Gal}(K^{ab}/K)$. First, we see that if we construct the pairing

$$H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p) \times C_K/p \to \mathbb{Z}/p, \tag{7.7}$$

then by using the above diagram (7.6), we can inductively define the pairing

$$H^1_{\mathrm{Gal}}(K,\,\mathbb{Z}/p^m)\,\times\,C_K/p^m\,\rightarrow\,\mathbb{Z}/p^m \tag{7.8}$$

for an arbitrary positive natural number m. We will construct the pairing (7.7). The idea is to gather the reciprocity pairing for each three-dimensional local fields $K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}$.

We begin the construction. Take $\chi \in H^1_{\mathrm{Gal}}(K,\,\mathbb{Z}/p)$. Then, by the restriction map $r_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}\colon H^1_{\mathrm{Gal}}(K,\,\mathbb{Z}/p) \rightarrow H^1_{\mathrm{Gal}}(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},\,\mathbb{Z}/p)$ in Galois cohomology, we can send χ into $H^1_{\mathrm{Gal}}(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},\,\mathbb{Z}/p)$. On the other hand, there exists the following canonical reciprocity pairing by Kato-Parshin in three-dimensional local class field theory:

$$H^1_{\mathrm{Gal}}(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},\,\mathbb{Z}/p)\,\times\,K^M_3(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/p\,\rightarrow\,\mathbb{Z}/p \tag{7.9}$$

by which for $\chi_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}} \in H^1_{\mathrm{Gal}}(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},\,\mathbb{Z}/p)$ and $\overline{(a_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},b_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},c_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})} \in K^M_3(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/p$, the pair $(\,\chi_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},\,\overline{(a_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},b_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},c_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})}\,)$ goes as

$$(\,\chi_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},\,\overline{(a_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},b_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},c_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})}\,)\,\mapsto\,\mathrm{Res}_{\frac{ds_{\mathfrak{p}_\mathfrak{m}}}{s_{\mathfrak{p}_\mathfrak{m}}}\wedge\frac{dt_{\mathfrak{p}_\mathfrak{m}}}{t_{\mathfrak{p}_\mathfrak{m}}}\wedge\frac{du_{\mathfrak{p}_\mathfrak{m}}}{u_{\mathfrak{p}_\mathfrak{m}}}}\,(\chi_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}\,\frac{da_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}}{a_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}}\wedge\frac{db_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}}{b_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}}\wedge\frac{dc_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}}{c_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}}\,)\,\in\mathbb{Z}/p. \tag{7.10}$$

Here, we assume that each three-dimensional local field $K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}$ is represented non-canonically as $K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}} = \mathbb{F}_q((s_{\mathfrak{p}_\mathfrak{m}}))((t_{\mathfrak{p}_\mathfrak{m}}))((u_{\mathfrak{p}_\mathfrak{m}}))$. This pairing is well defined because even if we use another representation $K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}} = \mathbb{F}_q((r_{\mathfrak{p}_\mathfrak{m}}))((q_{\mathfrak{p}_\mathfrak{m}}))((p_{\mathfrak{p}_\mathfrak{m}}))$, the residue of $(\chi_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}\,\frac{da_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}}{a_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}}\wedge\frac{db_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}}{b_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}}\wedge\frac{dc_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}}{c_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}})$ does not change. By gathering the pairing (7.9) for each $K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}$, we get the pairing

$$H^1_{\mathrm{Gal}}(K,\,\mathbb{Z}/p)\,\times\,\left(\prod\nolimits'_{\mathfrak{m}\in P^2_A,\mathfrak{p}_\mathfrak{m}\in P^1_\mathfrak{m}}K^M_3(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/p\right)\,\rightarrow\,\mathbb{Z}/p, \tag{7.11}$$

which can be written explicitly as

$$\left(\chi,\overline{(a_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},b_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},c_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})}_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}\right)\,\mapsto\,\sum\nolimits_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}\mathrm{Res}_{\frac{ds_{\mathfrak{p}_\mathfrak{m}}}{s_{\mathfrak{p}_\mathfrak{m}}}\wedge\frac{dt_{\mathfrak{p}_\mathfrak{m}}}{t_{\mathfrak{p}_\mathfrak{m}}}\wedge\frac{du_{\mathfrak{p}_\mathfrak{m}}}{u_{\mathfrak{p}_\mathfrak{m}}}}\,(\chi_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}\,\frac{da_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}}{a_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}}\wedge\frac{db_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}}{b_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}}\wedge\frac{dc_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}}{c_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}})\,\in\mathbb{Z}/p. \tag{7.12}$$

By conditions 1), 2) of $\left(\prod\nolimits'_{\mathfrak{m}\in P^2_A,\mathfrak{p}_\mathfrak{m}\in P^1_\mathfrak{m}}K^M_3(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/p\right)$ stated in Lemma 4.4, the above pairing (7.11) is well defined.

Next, we prove the reciprocity law for the pairing (7.11). That is,

Proposition 7.3 (Reciprocity proposition)**.** *Both $K^M_3(K_\mathfrak{m})/p$ and $K^M_3(K_\mathfrak{p})/p$ are annihilated by an arbitrary element χ of $H^1_{\mathrm{Gal}}(K,\,\mathbb{Z}/p)$ by the pairing* (7.11)*. Here, the above* *two groups are embedded diagonally into* $\prod'_{\mathfrak{p}_\mathfrak{m} \in P^1_\mathfrak{m}} K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/p$, $\prod'_{\mathfrak{p}_\mathfrak{m} \mapsto \mathfrak{p}} K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/p$ $\left(\subset \prod'_{\mathfrak{m} \in P_A^2, \mathfrak{p}_\mathfrak{m} \in P^1_\mathfrak{m}} K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/p\right)$, *respectively.*

Proof. We begin with $K_3^M(K_\mathfrak{m})/p$. In this case, we have to prove that in the pairing

$$H^1_{\mathrm{Gal}}(K_\mathfrak{m}, \mathbb{Z}/p) \times \underline{\prod_{\mathfrak{p}_\mathfrak{m} \in P^1_\mathfrak{m}}} K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/p \to \mathbb{Z}/p, \tag{7.13}$$

where $\underline{\prod}$ is defined by $\mathfrak{p}_\mathfrak{m}$-component $a_{\mathfrak{p}_\mathfrak{m}}$ lies in $U^0 K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})$ for almost all $\mathfrak{p}_\mathfrak{m} \in P^1_\mathfrak{m}$, the diagonal image of $K_3^M(K_\mathfrak{m})/p$ into $\underline{\prod}_{\mathfrak{p}_\mathfrak{m} \in P^1_\mathfrak{m}} K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/p$ is annihilated by an arbitrary $\chi_\mathfrak{m} \in H^1_{\mathrm{Gal}}(K_\mathfrak{m}, \mathbb{Z}/p)$. But this is nothing but Kato's reciprocity law for two-dimensional complete normal local rings whose residue fields are higher dimensional local fields (Proposition 7 in [Ka3]).

Next, we prove the reciprocity law for $K_\mathfrak{p}$. In this case, we have to prove that any pair $\left(\chi_\mathfrak{p}, \overline{(a_\mathfrak{p}, b_\mathfrak{p}, c_\mathfrak{p})}\right)$ with $\chi_\mathfrak{p} \in H^1_{\mathrm{Gal}}(K_\mathfrak{p}, \mathbb{Z}/p)$ and $\overline{(a_\mathfrak{p}, b_\mathfrak{p}, c_\mathfrak{p})} \in K_3^M(K_\mathfrak{p})/p$ goes to zero under the pairing (7.11). By using Theorem 4.5, we can consider $\overline{(a_\mathfrak{p}, b_\mathfrak{p}, c_\mathfrak{p})} \in \Omega^3_{K_\mathfrak{p},\log}$. So, by the cup product

$$H^1_{\mathrm{Gal}}(K_\mathfrak{p}, \mathbb{Z}/p) \times H^0_{\mathrm{Gal}}(K_\mathfrak{p}, \Omega^3_{K_\mathfrak{p},\log}) \to H^1_{\mathrm{Gal}}(K_\mathfrak{p}, \Omega^3_{K_\mathfrak{p},\log}), \tag{7.14}$$

we can consider $\left(\chi_\mathfrak{p}, \overline{(a_\mathfrak{p}, b_\mathfrak{p}, c_\mathfrak{p})}\right) \in H^1_{\mathrm{Gal}}(K_\mathfrak{p}, \Omega^3_{K_\mathfrak{p},\log})$. Then, we prove

Claim 7.4. *There exists the following complex*:

$$H^1_{\mathrm{Gal}}(K_\mathfrak{p}, \Omega^3_{K_\mathfrak{p},\log}) \to \bigoplus_{\substack{\mathfrak{p}_\mathfrak{m} \in P^1_\mathfrak{m},\, \mathfrak{p}_\mathfrak{m} \mapsto \mathfrak{p} \\ \mathfrak{m} \in P^2_A}} H^1_{\mathrm{Gal}}(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}, \Omega^3_{K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},\log}) \xrightarrow{\text{addition}} \mathbb{Z}/p, \tag{7.15}$$

where the first map is the restriction homomorphism in Galois cohomology.

Before the proof, we remark that the reciprocity law for $K_\mathfrak{p}$ is equivalent to the existence of the above complex (7.15).

Proof of Claim 7.4. Consider the coniveau-spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in (\mathrm{Spec}A)^{(p)}} H_x^{p+q}(\mathrm{Spec}A, \Omega^3_{A,\log}[-3]) \Longrightarrow H^{p+q}_{\text{ét}}(\mathrm{Spec}A, \Omega^3_{A,\log}[-3]), \tag{7.16}$$

where $(\mathrm{Spec}A)^{(p)}$ denotes the set of primes of codimension p. By writing down the E_1-term sequence

$$E_1^{1,4} \xrightarrow{d_1^{1,4}} E_1^{2,4} \xrightarrow{d_1^{2,4}} E_1^{3,4} \to 0, \tag{7.17}$$

we get

$$\bigoplus_{\mathfrak{p}\in P_A^1} H^2_{\mathfrak{p}}(A,\, \Omega^3_{A,\,\log}) \to \bigoplus_{\mathfrak{m}\in P_A^2} H^3_{\mathfrak{m}}(A,\, \Omega^3_{A,\,\log}) \to \mathbb{Z}/p \to 0 \tag{7.18}$$

(we used the isomorphism $H^4_{\mathfrak{m}_A}(\operatorname{Spec} A,\, \Omega^3_{\operatorname{Spec} A,\,\log}) \cong \mathbb{Z}/p$). Further, localization sequence provides the isomorphisms

$$H^2_{\mathfrak{p}}(A,\, \Omega^3_{A,\log}) \cong H^2_{\mathfrak{p}}(A^h_{\mathfrak{p}},\, \Omega^3_{A^h_{\mathfrak{p}},\log}) \cong H^1_{\mathrm{Gal}}(K^h_{\mathfrak{p}},\, \Omega^3_{K^h_{\mathfrak{p}},\log}) \cong H^1_{\mathrm{Gal}}(K_{\mathfrak{p}},\, \Omega^3_{K_{\mathfrak{p}},\log}). \tag{7.19}$$

So, we can rewrite (7.18) as

$$\bigoplus_{\mathfrak{p}\in P_A^1} H^1_{\mathrm{Gal}}(K_{\mathfrak{p}},\, \Omega^3_{K_{\mathfrak{p}},\,\log}) \to \bigoplus_{\mathfrak{m}\in P_A^2} H^3_{\mathfrak{m}}(A,\, \Omega^3_{A,\,\log}) \to \mathbb{Z}/p \to 0. \tag{7.20}$$

But by writing $T_{\mathfrak{m}} := \operatorname{Spec} A_{\mathfrak{m}} \backslash \mathfrak{m}$, the cohomology group $H^3_{\mathfrak{m}}(A,\, \Omega_{A,\log})$ in (7.20) is calculated as

$$H^3_{\mathfrak{m}}(A,\, \Omega^3_{A,\log}) \cong H^3_{\mathfrak{m}}(A^h_{\mathfrak{m}},\, \Omega^3_{A^h_{\mathfrak{m}},\log}) \cong H^3_{\mathfrak{m}}(A_{\mathfrak{m}},\, \Omega^3_{A_{\mathfrak{m}},\log}) \cong H^2_{\text{ét}}(T_{\mathfrak{m}},\, \Omega^3_{T_{\mathfrak{m}},\log}), \tag{7.21}$$

where the isomorphism $H^3_{\mathfrak{m}}(A^h_{\mathfrak{m}},\, \Omega^3_{A^h_{\mathfrak{m}},\log}) \cong H^3_{\mathfrak{m}}(A_{\mathfrak{m}},\, \Omega^3_{A_{\mathfrak{m}},\log})$ follows from the fact that both groups have the same order p (this fact is proved by the standard argument involving duality Theorem 3.1). On the other hand, we have the following localization sequence on $T_{\mathfrak{m}}$:

$$H^1_{\mathrm{Gal}}(K_{\mathfrak{m}},\, \Omega^3_{K_{\mathfrak{m}},\,\log}) \to \bigoplus_{\mathfrak{p}_{\mathfrak{m}}\in P_{\mathfrak{m}}^1} H^2_{\mathfrak{p}_{\mathfrak{m}}}(T_{\mathfrak{m}},\, \Omega^3_{X_{\mathfrak{m}},\log}) \to H^2_{\text{ét}}(T_{\mathfrak{m}},\, \Omega^3_{T_{\mathfrak{m}},\log}) \to 0. \tag{7.22}$$

Moreover, there exists the isomorphism

$$H^2_{\mathfrak{p}_{\mathfrak{m}}}(T_{\mathfrak{m}},\, \Omega^3_{X_{\mathfrak{m}},\log}) \cong H^1_{\text{ét}}(K^h_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\, \Omega^3_{K^h_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\log}) \cong H^1_{\text{ét}}(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\, \Omega^3_{K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\log})\ (\cong \mathbb{Z}/p), \tag{7.23}$$

where $K^h_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}$ denotes the fractional field of the henselian local ring $A^h_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}$ obtained by the henselization of $A_{\mathfrak{m}}$ at its height-one prime $\mathfrak{p}_{\mathfrak{m}}$ (for the final isomorphism in (7.23), see [Ka1]).

From (7.21), (7.22), (7.23), the complex (7.20) is rewritten as

$$\bigoplus_{\mathfrak{p}\in P_A^1} H^1_{\mathrm{Gal}}(K_{\mathfrak{p}},\, \Omega^3_{K_{\mathfrak{p}},\log}) \to \bigoplus_{\mathfrak{m}\in P_A^2} \bigoplus_{\mathfrak{p}_{\mathfrak{m}}\in P_{\mathfrak{m}}^1} H^1_{\text{ét}}(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\, \Omega^3_{K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\log}) \xrightarrow{\textit{addition}} \mathbb{Z}/p. \tag{7.24}$$

From this, for each $H^1_{\mathrm{Gal}}(K_{\mathfrak{p}},\, \Omega^3_{K_{\mathfrak{p}},\log})$, we can deduce the following complex:

$$H^1_{\mathrm{Gal}}(K_{\mathfrak{p}},\, \Omega^3_{K_{\mathfrak{p}},\log}) \to \bigoplus_{\substack{\mathfrak{p}_{\mathfrak{m}} \in P^1_{\mathfrak{m}},\, \mathfrak{p}_{\mathfrak{m}} \mapsto \mathfrak{p} \\ \mathfrak{m} \in P^2_A}} H^1_{\text{ét}}(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\, \Omega^3_{K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\log}) \xrightarrow{addition} \mathbb{Z}/p, \tag{7.25}$$

which is nothing but the complex (7.15). □

Thus from Proposition 5.3, the above pairing (7.11) factors as

$$H^1_{\mathrm{Gal}}(K,\, \mathbb{Z}/p) \times \left({\prod}'_{\mathfrak{m}\in P^2_A, \mathfrak{p}_{\mathfrak{m}} \in P^1_{\mathfrak{m}}} K^M_3(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})/p \right) / \prod_{\mathfrak{m}\in P^2_A} K^M_3(K_{\mathfrak{m}})/p \prod_{\mathfrak{p}\in P^1_A} {K_3}^M(K_{\mathfrak{p}})/p \to \mathbb{Z}/p, \tag{7.26}$$

where the right hand side of the pairing (7.26) is D_K/p defined in (4.7). Further, it is found that each element $\chi \in H^1_{\mathrm{Gal}}(K,\, \mathbb{Z}/p)$ annihilates $F^M D_K$ for some modulus M. So, by taking the limit on M, we get the pairing

$$H^1_{\mathrm{Gal}}(K,\, \mathbb{Z}/p) \times \varprojlim{}_M(D_K/F^M D_K)/p \to \mathbb{Z}/p. \tag{7.27}$$

Now, the equation (4.6) rewrites this pairing (7.27) as

$$H^1_{\mathrm{Gal}}(K,\, \mathbb{Z}/p) \times C_K/p \to \mathbb{Z}/p. \tag{7.28}$$

By taking dual of (7.28), we at last get the dual reciprocity homomorphism

$$\rho^*_K/p\colon H^1_{\mathrm{Gal}}(K,\, \mathbb{Z}/p) \to \mathrm{Hom}(C_K,\, \mathbb{Z}/p). \tag{7.29}$$

Next, we explain the method of the proof of bijectivity of (7.2). We consider the scheme $X = \operatorname{Spec} A \setminus \mathfrak{m}_A$ which is the regular excellent scheme of Krull-dimension two. Then, we consider the closed subscheme $Z = \cup_{i=1\ldots m}\, \mathfrak{m}_i$ of X where each $\mathfrak{m}_i$ is a closed point of X (hence Z is codimension two). For the pair (X, Z), we consider the localization sequence ((7.30) below) in the étale cohomology with $\mathbb{Z}/p$-coefficient. This is the first step.

In the second step, we consider the localization sequence obtained from the pair $(X \setminus Z,\, W \setminus Z)$ where $W = \cup_{j=1\ldots n}\, \overline{\mathfrak{p}_j}$ is the union of finite codimension one closed sub-schemes of X ($\overline{\mathfrak{p}_j}$ denotes the closure of $\mathfrak{p}_j$ in X). This is (7.31) below.

Under these settings, we consider the limit $(\cup\, \mathfrak{m}_i) \to P^2_A$ and $(\cup\, \mathfrak{p}_j) \to P^1_A$ set-theoretically, where $\mathfrak{m}_i$ and $\mathfrak{p}_j$ run over all height two primes of A and all height one primes of A, respectively.

The crucial fact is that under these limit procedures, the localization sequence (7.31) below turns out to involve the very important Galois cohomology group $H^1_{\mathrm{Gal}}(K,\,\mathbb{Z}/p)$ which is nothing but the Pontryagin dual of $\mathrm{Gal}(K^{ab}/K)/p$.

Now, we begin the proof. As stated above, we denote by $\mathfrak{m}_i$, $\mathfrak{p}_j$ a height 2 prime and a height 1 prime of X, respectively. First, we consider the following localization sequence:

$$\begin{aligned}\cdots \longrightarrow \bigoplus_{i=1\ldots n} H^1_{\mathfrak{m}_i}(X_{et},\,\mathbb{Z}/p) \longrightarrow H^1_{et}(X,\,\mathbb{Z}/p) \longrightarrow H^1_{et}(X\setminus \cup_i \mathfrak{m}_i,\,\mathbb{Z}/p) \longrightarrow \\ \bigoplus_{i=1\ldots n} H^2_{\mathfrak{m}_i}(X_{et},\,\mathbb{Z}/p) \longrightarrow H^2_{et}(X,\,\mathbb{Z}/p) \longrightarrow H^2_{et}(X\setminus \cup_i \mathfrak{m}_i,\,\mathbb{Z}/p) \longrightarrow \\ \bigoplus_{i=1\ldots n} H^3_{\mathfrak{m}_i}(X_{et},\,\mathbb{Z}/p) \longrightarrow H^3_{et}(X,\,\mathbb{Z}/p) \longrightarrow H^3_{et}(X\setminus \cup_i \mathfrak{m}_i,\,\mathbb{Z}/p) \longrightarrow \cdots .\end{aligned} \tag{7.30}$$

Next, the second localization sequence is stated as follows:

$$\begin{aligned}\to \bigoplus_{j=1\ldots m} H^1_{\overline{\mathfrak{p}_j}\setminus(\cup_i \mathfrak{m}_i)}(X\setminus(\cup_i \mathfrak{m}_i)) \to H^1_{et}(X\setminus(\cup_i \mathfrak{m}_i)) \to H^1_{et}(X\setminus((\cup_i \mathfrak{m}_i)\cup(\cup_j \overline{\mathfrak{p}_j}))) \to \\ \bigoplus_{j=1\ldots m} H^2_{\overline{\mathfrak{p}_j}\setminus(\cup_i \mathfrak{m}_i)}(X\setminus(\cup_i \mathfrak{m}_i)) \to H^2_{et}(X\setminus(\cup_i \mathfrak{m}_i)) \to H^2_{et}(X\setminus((\cup_i \mathfrak{m}_i)\cup(\cup_j \overline{\mathfrak{p}_j}))) \to,\end{aligned} \tag{7.31}$$

where $H^1_{et}(X\setminus(\cup_i \mathfrak{m}_i)) :=H^1_{et}(X\setminus(\underset{i=1,..,n}{\cup} \mathfrak{m}_i),\,\mathbb{Z}/p)$ and $H^1_{et}(X\setminus((\cup_i \mathfrak{m}_i)\cup(\cup_j \overline{\mathfrak{p}_j}))) := H^1_{et}(X\setminus((\underset{i=1,..,n}{\cup} \mathfrak{m}_i)\cup(\underset{j=1,..,m}{\cup} \overline{\mathfrak{p}_j})),\,\mathbb{Z}/p)$, respectively.

We denote by L^k the following limit of the cohomology group:

$$L^k :=\varinjlim_{U_\theta} H^k_{\text{ét}}(U_\theta,\,\mathbb{Z}/p), \tag{7.32}$$

where U_θ runs over all open subschemes of X such that each complement $X\setminus U_\theta$ is a closed subscheme of X of codimension two. Then, under the increasing limit of $(\cup \mathfrak{m}_i) \to P^2_A$, $(\cup \mathfrak{p}_j) \to P^1_A$, we get the following exact sequence from (7.31):

$$\bigoplus_{\mathfrak{p}\in P^1_A} H^1_{\mathfrak{p}}(X_{(\mathfrak{p})},\,\mathbb{Z}/p) \to L^1 \to H^1_{et}(K,\,\mathbb{Z}/p) \to \bigoplus_{\mathfrak{p}\in P^1_A} H^2_{\mathfrak{p}}(X_{(\mathfrak{p})},\,\mathbb{Z}/p) \to L^2 \to 0, \tag{7.33}$$

where the final 0 is obtained by replacing $H^2_{et}(K,\,\mathbb{Z}/p)$ with 0 (cf. [SGA 4], X), and we define $H^i_{\mathfrak{p}}(X_{(\mathfrak{p})},\,\mathbb{Z}/p) := \varinjlim_{U\supset\mathfrak{p}} H^i_{\mathfrak{p}}(U,\,\mathbb{Z}/p)$, where U runs over all open subschemes of X containing $\mathfrak{p}$.

We have a lemma.

Lemma 7.5. *There hold the following isomorphisms*:

1) $H^1_{\mathfrak{p}}(X_{(\mathfrak{p})}, \mathbb{Z}/p) \cong H^1_{\mathfrak{p}}(A_{(\mathfrak{p})}, \mathbb{Z}/p) = 0$ ($A_{(\mathfrak{p})}$ *is the localization of* A *at* $\mathfrak{p}$).

2) $L^1 \cong \mathbb{Z}/p$.

Proof. 1) The first isomorphism is clear and the second equality is proved as follows. Consider the following exact localization sequence:

$$0 \to H^1_{\mathfrak{p}}(A_{(\mathfrak{p})}, \mathbb{Z}/p) \to H^1_{et}(A_{(\mathfrak{p})}, \mathbb{Z}/p) \to H^1_{et}(K, \mathbb{Z}/p) \to H^2_{\mathfrak{p}}(A_{(\mathfrak{p})}, \mathbb{Z}/p) \to \dots . \tag{7.34}$$

Then the restriction map $H^1_{et}(A_{(\mathfrak{p})}, \mathbb{Z}/p) \to H^1_{et}(K, \mathbb{Z}/p)$ in (7.34) becomes injective because $\mathbb{Z}/p$-torsors of $A_{(\mathfrak{p})}$ can be seen as a $\mathbb{Z}/p$-torsors of K naturally. Our assertion follows from this injectivity considering (7.34).

2) For this, we use the exact sequence (7.30). By the standard argument involving the duality (7.45) below, we see that $H^1_{\mathfrak{m}_i}(X_{et}, \mathbb{Z}/p) = H^2_{\mathfrak{m}_i}(X_{et}, \mathbb{Z}/p) = 0$ for an arbitrary height two prime $\mathfrak{m}_i$ (notice that the isomorphism $H^1_{\mathfrak{m}_i}(X_{et}, \mathbb{Z}/p) \cong H^1_{\mathfrak{m}_i}(A^h_{\mathfrak{m}_i}, \mathbb{Z}/p)$ holds which enables us to use duality (7.45)). Hence by (7.30), we get $L^1 \cong H^1_{et}(X, \mathbb{Z}/p)$.

So, our task is to prove the isomorphism $H^1_{et}(X, \mathbb{Z}/p) \cong \mathbb{Z}/p$. By considering the long exact sequence in the étale cohomology deduced from the following short exact sequence

$$0 \to \mathbb{Z}/p \to \mathcal{O}_A \xrightarrow{x \mapsto x^p - x} \mathcal{O}_A \to 0$$

on $\operatorname{Spec} A_{et}$, we get the isomorphism

$$H^1_{et}(A, \mathbb{Z}/p) \cong \mathbb{Z}/p. \tag{7.35}$$

On the other hand, there exists the following exact sequence:

$$\begin{aligned} 0 \to H^0_{\mathfrak{m}_A}(A, \mathcal{O}_A) \to H^1_{\mathfrak{m}_A}(A, \mathbb{Z}/p) \to H^1_{\mathfrak{m}_A}(A, \mathcal{O}_A) \to H^1_{\mathfrak{m}_A}(A, \mathcal{O}_A) \to \\ \to H^2_{\mathfrak{m}_A}(A, \mathbb{Z}/p) \to H^2_{\mathfrak{m}_A}(A, \mathcal{O}_A) \to \dots . \end{aligned} \tag{7.36}$$

But from the Grothendieck duality Theorem 5.1, we get the following isomorphism:

$$H^i_{\mathfrak{m}_A}(\operatorname{Spec} A, \mathcal{O}_A) \cong \operatorname{Hom}_c(\operatorname{Ext}^{3-i}_A(A, \Omega^3_A), \mathbb{Z}/p). \tag{7.37}$$

From (7.37), we get $H^i_{\mathfrak{m}_A}(A, \mathcal{O}_A) = 0$ $(i = 0, 1, 2)$. So, we have $H^1_{\mathfrak{m}_A}(A, \mathbb{Z}/p) = H^2_{\mathfrak{m}_A}(A, \mathbb{Z}/p) = 0$ by (7.36). Now, the exact sequence

$$0 \to H^1_{\mathfrak{m}_A}(A, \mathbb{Z}/p) \to H^1_{et}(A, \mathbb{Z}/p) \to H^1_{et}(X, \mathbb{Z}/p) \to H^2_{\mathfrak{m}_A}(A, \mathbb{Z}/p) \to \dots \tag{7.38}$$

shows that $H^1_{et}(A, \mathbb{Z}/p) \cong H^1_{et}(X, \mathbb{Z}/p)$. But as we have the isomorphism $H^1_{et}(A, \mathbb{Z}/p) \cong \mathbb{Z}/p$ by (7.35), we get the desired isomorphism $H^1_{et}(X, \mathbb{Z}/p) \cong \mathbb{Z}/p$. □

Next, we analyze L^2. For this, we have the following result.

Proposition 7.6. *There exists the canonical injection*

$$L^2 \hookrightarrow \bigoplus_{\mathfrak{m} \in P_A^2} \mathrm{Hom}(K_3^M(A_{\mathfrak{m}}),\, \mathbb{Z}/p). \tag{7.39}$$

Here, we consider $K_3^M(A_{\mathfrak{m}})$ as a discrete module.

Proof. We use the localization sequence (7.30). First, we state a lemma.

Lemma 7.7. *The group $H^2_{et}(X,\, \mathbb{Z}/p) \cong H^3_{\mathfrak{m}_A}(A,\, \mathbb{Z}/p)$ vanishes.*

Proof. The isomorphism in the statement of lemma follows by considering (7.30) and the theorem by M. Artin on the cohomological dimension for p-torsion sheaves (cf. loc.cit.).

Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 \to & H^3_{\mathfrak{m}_A}(A,\, \mathbb{Z}/p) & \to & H^3_{\mathfrak{m}_A}(\mathrm{Spec}\, A,\, \mathcal{O}_A) & \to & H^3_{\mathfrak{m}_A}(\mathrm{Spec}\, A,\, \mathcal{O}_A) & \to & H^4_{\mathfrak{m}_A}(A,\, \mathbb{Z}/p) & \to 0 \\ & \downarrow & & \downarrow \cong & & \downarrow \cong & & \| & \\ & 0 \longrightarrow & & \mathrm{Hom}_c(\Omega^3_A,\, \mathbb{Z}/p) & \to & \mathrm{Hom}_c(\Omega^3_A,\, \mathbb{Z}/p) & \to & \mathrm{Hom}_c(\Omega^3_{A,\log},\, \mathbb{Z}/p) & \to 0, \end{array} \tag{7.40}$$

where the extreme left and right zeros in the upper row are obtained from the vanishings $H^2_{\mathfrak{m}_A}(\mathrm{Spec}\, A,\, \mathcal{O}_A) = H^4_{\mathfrak{m}_A}(\mathrm{Spec}\, A,\, \mathcal{O}_A) = 0$ which follows directly from (7.37). Further, the bottom exact sequence is obtained as follows:

Consider the short exact sequence

$$0 \to \Omega^3_{A,log} \to \Omega^3_A \stackrel{1-C}{\to} \Omega^3_A \to 0, \tag{7.41}$$

where the surjection $\Omega^3_A \stackrel{1-C}{\to} \Omega^3_A$ is proved by the explicit calculation. By applying the Pontryagin dual functor $\mathrm{Hom}_{\mathrm{cont}}(*\,, \mathbb{Q}/\mathbb{Z})$ to the exact sequence (7.41), we get the bottom exact sequence of (7.40) (we put the $\mathfrak{m}_A$-adic topology on Ω^3_A). Now, the desired vanishing $H^3_{\mathfrak{m}_A}(A,\, \mathbb{Z}/p) = 0$ follows from this diagram (7.40) immediately. □

From this lemma combined with (7.30), we get the injectivity $L^2 \hookrightarrow \bigoplus_{\mathfrak{m}} H^3_{\mathfrak{m}}(X,\, \mathbb{Z}/p)$. Further, the excision theorem in étale cohomology provides the isomorphisms

$$H^3_{\mathfrak{m}}(X,\, \mathbb{Z}/p) \cong H^3_{\mathfrak{m}}(X^h_{\mathfrak{m}},\, \mathbb{Z}/p) \cong H^3_{\mathfrak{m}}(A^h_{\mathfrak{m}},\, \mathbb{Z}/p). \tag{7.42}$$

So, Proposition 7.6 follows from

Lemma 7.8. *There exists the following injective homomorphism:*

$$H^3_{\mathfrak{m}}(A^h_{\mathfrak{m}},\, \mathbb{Z}/p) \hookrightarrow \mathrm{Hom}(K_3^M(A_{\mathfrak{m}})/p,\, \mathbb{Z}/p).$$

Proof. From the short exact sequence

$$0 \to \mathbb{Z}/p \to \mathcal{O}_{A^h_{\mathfrak{m}}} \stackrel{x^p - x}{\to} \mathcal{O}_{A^h_{\mathfrak{m}}} \to 0, \tag{7.43}$$

we deduce the long exact sequence

$$H^2_{\mathfrak{m}}(A^h_{\mathfrak{m}}, \mathbb{Z}/p) \to H^2_{\mathfrak{m}}(A^h_{\mathfrak{m}}, \mathcal{O}_{A^h_{\mathfrak{m}}}) \stackrel{x^p - x}{\to} H^2_{\mathfrak{m}}(A^h_{\mathfrak{m}}, \mathcal{O}_{A^h_{\mathfrak{m}}}) \to H^3_{\mathfrak{m}}(A^h_{\mathfrak{m}}, \mathbb{Z}/p) \to 0 \tag{7.44}$$

of the local cohomology, where the final 0 is obtained by the vanishing $H^3_{\mathfrak{m}}(A^h_{\mathfrak{m}}, \mathcal{O}_{A^h_{\mathfrak{m}}})$ which follows from the Grothendieck duality

$$H^i_{\mathfrak{m}}(A^h_{\mathfrak{m}}, \mathcal{O}_{A^h_{\mathfrak{m}}}) \cong \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Ext}^{2-i}_{A_{\mathfrak{m}}}(\mathcal{O}_{A_{\mathfrak{m}}}, \Omega^3_{A_{\mathfrak{m}}}), \kappa(\mathfrak{m})). \tag{7.45}$$

We need a Sub-lemma.

Sub-lemma 7.9. *We put the inverse limit topology on $A_{\mathfrak{m}}$ induced from each locally compact group $A_{\mathfrak{m}}/\mathfrak{m}^n$ (notice that each $A_{\mathfrak{m}}/\mathfrak{m}^n$ $(n \geq 1)$ is a finite vector space over one dimensional local field $\kappa(\mathfrak{m})$, hence has the natural induced topology). Then, with the natural topology on $\Omega^3_{A_{\mathfrak{m}}}$ induced from the above mentioned topology on $A_{\mathfrak{m}}$, we have the following isomorphism*:

$$H^2_{\mathfrak{m}}(A^h_{\mathfrak{m}}, \mathcal{O}_{A^h_{\mathfrak{m}}}) \cong \mathrm{Hom}_{\mathrm{cont}}(\Omega^3_{A_{\mathfrak{m}}}, \mathbb{Z}/p). \tag{7.46}$$

Proof. By Theorem 5.1, we have the isomorphism

$$H^2_{\mathfrak{m}}(A^h_{\mathfrak{m}}, \mathcal{O}_{A^h_{\mathfrak{m}}}) \cong \mathrm{Hom}_{\mathrm{cont}}(\Omega^3_{A_{\mathfrak{m}}}, \kappa(\mathfrak{m})). \tag{7.47}$$

Consider the map

$$\mathrm{Hom}_{\mathfrak{m}_A\text{–adic}}(\Omega^3_{A_{\mathfrak{m}}}, \kappa(\mathfrak{m})) \stackrel{\iota_1}{\stackrel{\sim}{\to}} \mathrm{Hom}_{\mathfrak{m}_A\text{–adic}}(\Omega^3_{A_{\mathfrak{m}}}, \Omega^1_{\kappa(\mathfrak{m})}) \stackrel{\iota_2}{\to} \mathrm{Hom}(\Omega^3_{A_{\mathfrak{m}}}, \mathbb{Z}/p), \tag{7.48}$$

where ι_1 is induced by $\kappa(\mathfrak{m}) \stackrel{\sim}{\to} \Omega^1_{\kappa(\mathfrak{m})}$ $(a \mapsto a\, du_{\kappa(\mathfrak{m})})$ and ι_2 is induced by the residue homomorphism $\Omega^1_{\kappa(\mathfrak{m})} \stackrel{Res}{\to} \mathbb{Z}/p$.

But it is found that the image of $\mathrm{Hom}_{\mathfrak{m}_A\text{–adic}}(\Omega^3_{A_{\mathfrak{m}}}, \kappa(\mathfrak{m}))$ by the composite homomorphism $\iota_2 \circ \iota_1$ in (7.48) lies in $\mathrm{Hom}_{\mathrm{cont}}(\Omega^3_{A_{\mathfrak{m}}}, \mathbb{Z}/p)$ $(\subset \mathrm{Hom}(\Omega^3_{A_{\mathfrak{m}}}, \mathbb{Z}/p))$ by the definition of the inverse limit topology on $A_{\mathfrak{m}}$.

Thus, we can argue after dividing $\Omega^3_{A_{\mathfrak{m}}}$ by $\mathfrak{m}^n_A$ with some n, which becomes finite dimensional vector space over $\kappa(\mathfrak{m})$. But for an arbitrary finite dimensional vector space V over one-dimensional local field $\kappa(\mathfrak{m})$ which has the natural topology induced from that of $\kappa(\mathfrak{m})$, there exists the isomorphism

$$\mathrm{Hom}_{\kappa(\mathfrak{m})\text{–module}}(V, \mathfrak{m}) \cong \mathrm{Hom}_{\mathrm{cont}}(V, \mathbb{Z}/p) \tag{7.49}$$

(exercise!), which shows our assertion. Hence, we are done. □

We return to the proof of Lemma 7.8. By considering the exact sequence

$$0 \to \Omega^3_{A_\mathfrak{m},\log} \to \Omega^3_{A_\mathfrak{m}} \overset{1-C}{\to} \Omega^3_{A_\mathfrak{m}}, \tag{7.50}$$

we get the following commutative diagram:

$$\begin{array}{ccccccc} H^2_\mathfrak{m}(A^h_\mathfrak{m}, \mathcal{O}_{A^h_\mathfrak{m}}) & \longrightarrow & H^2_\mathfrak{m}(A^h_\mathfrak{m}, \mathcal{O}_{A^h_\mathfrak{m}}) & \longrightarrow & H^3_\mathfrak{m}(A^h_\mathfrak{m}, \mathbb{Z}/p) & \longrightarrow & 0 \\ \downarrow\cong & & \downarrow\cong & & \downarrow & & \\ \mathrm{Hom}_c(\Omega^3_{A_\mathfrak{m}}, \mathbb{Z}/p) & \to & \mathrm{Hom}_c(\Omega^3_{A_\mathfrak{m}}, \mathbb{Z}/p) & \to & \mathrm{Hom}_c(\Omega^3_{A_\mathfrak{m},\log}, \mathbb{Z}/p) & \to & 0, \end{array} \tag{7.51}$$

where the vertical isomorphisms follow from Sub-lemma 7.9. From this diagram, we get the isomorphism

$$H^3_\mathfrak{m}(A^h_\mathfrak{m}, \mathbb{Z}/p) \cong \mathrm{Hom}_c(\Omega^3_{A_\mathfrak{m},\log}, \mathbb{Z}/p). \tag{7.52}$$

As $\mathrm{Hom}_c(\Omega^3_{A_\mathfrak{m},\log}, \mathbb{Z}/p) \subset \mathrm{Hom}(\Omega^3_{A_\mathfrak{m},\log}, \mathbb{Z}/p)$ (Hom denotes the set of all homomorphisms between discrete abelian groups), we have the injection

$$H^3_\mathfrak{m}(A^h_\mathfrak{m}, \mathbb{Z}/p) \hookrightarrow \mathrm{Hom}(\Omega^3_{A_\mathfrak{m},\log}, \mathbb{Z}/p). \tag{7.53}$$

Thus, Lemma 7.8 follows from the following claim.

Claim 7.10. *There exists the canonical injective homomorphism*

$$\mathrm{Hom}(\Omega^3_{A_\mathfrak{m},\log}, \mathbb{Z}/p) \hookrightarrow \mathrm{Hom}(K^M_3(A_\mathfrak{m}), \mathbb{Z}/p). \tag{7.54}$$

Proof. By considering dual, we prove that there exists the surjective homomorphism

$$K^M_3(A_\mathfrak{m})/p \twoheadrightarrow \Omega^3_{A_\mathfrak{m},\log} \tag{7.55}$$

between discrete modules. Consider the localization sequence on $T_\mathfrak{m} = \mathrm{Spec}\, A_\mathfrak{m} \setminus \mathfrak{m}$

$$\begin{aligned} 0 \to \bigoplus_{\mathfrak{p}_\mathfrak{m} \in P^1_\mathfrak{m}} H^0_{\mathfrak{p}_\mathfrak{m}}(T_\mathfrak{m}, \Omega^3_{T_\mathfrak{m},\log}) &\to H^0_{et}(T_\mathfrak{m}, \Omega^3_{T_\mathfrak{m},\log}) \to H^0_{et}(K_\mathfrak{m}, \Omega^3_{K_\mathfrak{m},\log}) \to \\ \bigoplus_{\mathfrak{p}_\mathfrak{m} \in P^1_\mathfrak{m}} H^1_{\mathfrak{p}_\mathfrak{m}}(T_\mathfrak{m}, \Omega^3_{T_\mathfrak{m},\log}) &\to H^1_{et}(T_\mathfrak{m}, \Omega^3_{T_\mathfrak{m},\log}) \to H^1_{et}(K_\mathfrak{m}, \Omega^3_{K_\mathfrak{m},\log}) \to \dots . \end{aligned} \tag{7.56}$$

From the vanishing $H^0_{\mathfrak{p}_\mathfrak{m}}(T_\mathfrak{m}, \Omega^3_{T_\mathfrak{m},\log}) = 0$ and the equality $H^0_{et}(T_\mathfrak{m}, \Omega^3_{T_\mathfrak{m},\log}) = \Omega^3_{A_\mathfrak{m},\log}$, we get the following exact sequence from (7.56):

$$0 \to \Omega^3_{A_\mathfrak{m},\log} \to \Omega^3_{K_\mathfrak{m},\log} \to \bigoplus_{\mathfrak{p}_\mathfrak{m} \in P^1_\mathfrak{m}} H^1_{\mathfrak{p}_\mathfrak{m}}(T_\mathfrak{m}, \Omega^3_{T_\mathfrak{m},\log}). \tag{7.57}$$

On the other hand, we have the following purity isomorphism:

Lemma 7.11. *It holds the following isomorphism*:

$$H^1_{\mathfrak{p}_\mathfrak{m}}(T_\mathfrak{m},\, \Omega^3_{T_\mathfrak{m},\log}) \;\cong\; \Omega^2_{\kappa(\mathfrak{p}_\mathfrak{m}),\log}. \tag{7.58}$$

Proof. By excision, there exists an isomorphism

$$H^1_{\mathfrak{p}_\mathfrak{m}}(T_\mathfrak{m},\, \Omega^3_{T_\mathfrak{m},\log}) \cong H^1_{\mathfrak{p}_\mathfrak{m}}(A^h_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},\, \Omega^3_{A^h_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},\log}), \tag{7.59}$$

where $A^h_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}$ denotes the henselian discrete valuation ring obtained by the henselization of $A_\mathfrak{m}$ at its height one prime $\mathfrak{p}_\mathfrak{m}$. So, by localization sequence on Spec $A^h_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}$, we see

$$H^1_{\mathfrak{p}_\mathfrak{m}}(A^h_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},\, \Omega^3_{A^h_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},\log}) \cong \mathrm{Coker}(\Omega^3_{A^h_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},\log} \to \Omega^3_{K^h_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},\log}). \tag{7.60}$$

From (7.60) and the exact residue sequence

$$\Omega^3_{A^h_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},\log} \to \Omega^3_{K^h_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},\log} \overset{\mathrm{Res}}{\to} \Omega^2_{\kappa(\mathfrak{p}_\mathfrak{m}),\log} \to 0, \tag{7.61}$$

our desired result follows immediately. □

From (7.58) just proven above, we have the exact sequence

$$0 \to \Omega^3_{A_\mathfrak{m},\log} \to \Omega^3_{K_\mathfrak{m},\log} \to \bigoplus_{\mathfrak{p}_\mathfrak{m}\in P^1_\mathfrak{m}} \Omega^2_{\kappa(\mathfrak{p}_\mathfrak{m}),\log}. \tag{7.62}$$

On the other hand, there exists the (mod p) Gersten-Quillen complex

$$K_3^M(A_\mathfrak{m})/p \to K_3^M(K_\mathfrak{m})/p \to \bigoplus_{\mathfrak{p}_\mathfrak{m}\in P^1_\mathfrak{m}} K_2^M(\kappa(\mathfrak{p}_\mathfrak{m}))/p \to K_1^M(\kappa(\mathfrak{m}))/p \to 0 \tag{7.63}$$

which becomes exact (cf. [So]). Combining (7.62) and (7.63), we get the commutative diagram

$$\begin{array}{ccccccc} & & K_3^M(A_\mathfrak{m})/p & \longrightarrow & K_3^M(K_\mathfrak{m})/p & \to & \bigoplus_{\mathfrak{p}_\mathfrak{m}\in P^1_\mathfrak{m}} K_2^M(\kappa(\mathfrak{p}_\mathfrak{m}))/p \\ & & \downarrow & & \downarrow\cong & & \downarrow\cong \\ 0 & \longrightarrow & \Omega^3_{A_\mathfrak{m},\log} & \longrightarrow & \Omega^3_{K_\mathfrak{m},\log} & \longrightarrow & \bigoplus_{\mathfrak{p}_\mathfrak{m}\in P^1_\mathfrak{m}} \Omega^2_{\kappa(\mathfrak{p}_\mathfrak{m}),\log}, \end{array} \tag{7.64}$$

where the vertical isomorphisms follow from Theorem 4.5. From this diagram, it is easily seen that the extreme left vertical arrow is surjective, which is nothing but our desired result. Thus, we finished the proof of Claim 7.10, Lemma 7.8 and consequently, Proposition 7.6. □

Remark 7. The exactness of the complex (7.63) follows from the assumption of odd-positive characteristic of K. If the characteristic of $K_{\mathfrak{m}}$ is 2, we must assume its exactness (cf. [So]).

By Lemma 7.5 and Proposition 7.6, the exact sequence (7.33) becomes as follows:

$$0 \to \mathbb{Z}/p \to H^1_{et}(K, \mathbb{Z}/p) \to \bigoplus_{\mathfrak{p}\in P^1_A} H^2_{\mathfrak{p}}(A_{(\mathfrak{p})}, \mathbb{Z}/p) \to \bigoplus_{\mathfrak{m}\in P^2_A} \mathrm{Hom}(K^M_3(A_{\mathfrak{m}}), \mathbb{Z}/p). \tag{7.65}$$

Finally, we analyze the group $H^2_{\mathfrak{p}}(A_{(\mathfrak{p})}, \mathbb{Z}/p)$.

We have a lemma.

Lemma 7.12. *Let A be as above. We denote by $A_{(\mathfrak{p})}$, $A^h_{\mathfrak{p}}$ the localization of A at $\mathfrak{p}$ and the henselization of $A_{(\mathfrak{p})}$ at $\mathfrak{p}$, respectively. Then, there holds the following isomorphisms:*

$$H^2_{\mathfrak{p}}(A_{(\mathfrak{p})}, \mathbb{Z}/p) \cong H^2_{\mathfrak{p}}(A^h_{\mathfrak{p}}, \mathbb{Z}/p) \cong H^2_{\mathfrak{p}}(A_{\mathfrak{p}}, \mathbb{Z}/p), \tag{7.66}$$

This lemma is important, so we give the full detail of the proof.

Proof of Lemma 7.12. The first isomorphism is nothing but the excision in the étale cohomology. And for the proof of the second isomorphism, we proceed as follows. By using the localization sequence

$$0 \to H^1_{et}(A^h_{\mathfrak{p}}, \mathbb{Z}/p) \to H^1_{et}(K^h_{\mathfrak{p}}, \mathbb{Z}/p) \to H^2_{\mathfrak{p}}(A^h_{\mathfrak{p}}, \mathbb{Z}/p) \to 0, \tag{7.67}$$

we get the isomorphism

$$H^2_{\mathfrak{p}}(A^h_{\mathfrak{p}}, \mathbb{Z}/p) \cong \mathrm{Hom}_c\,(\,\mathrm{Gal}((K^h_{\mathfrak{p}})^{ab}/(K^h_{\mathfrak{p}})^{ur}), \mathbb{Z}/p), \tag{7.68}$$

where $(K^h_{\mathfrak{p}})^{ur}$ denotes the maximal unramified extension of $K^h_{\mathfrak{p}}$. In the same way, we get the following isomorphism:

$$H^2_{\mathfrak{p}}(A_{\mathfrak{p}}, \mathbb{Z}/p) \cong \mathrm{Hom}_c\,(\mathrm{Gal}(K^{ab}_{\mathfrak{p}}/K^{ur}_{\mathfrak{p}}), \mathbb{Z}/p), \tag{7.69}$$

where $K^{ur}_{\mathfrak{p}}$ denotes the maximal unramified extension of $K_{\mathfrak{p}}$. But, Artin's approximation theorem in [A] provides the isomorphism

$$\mathrm{Gal}(\overline{K^h_{\mathfrak{p}}}/K^h_{\mathfrak{p}}) \cong \mathrm{Gal}(\overline{K_{\mathfrak{p}}}/K_{\mathfrak{p}}). \tag{7.70}$$

Now, the second isomorphism in (7.66) is immediately follows from (7.68) and (7.69) by (7.70). □

By this lemma, the above exact sequence (7.65) becomes as follows:

$$0 \to \mathbb{Z}/p \to H^1_{et}(K, \mathbb{Z}/p) \to \bigoplus_{\mathfrak{p}\in P^1_A} H^2_{\mathfrak{p}}(A_{\mathfrak{p}}, \mathbb{Z}/p) \to \bigoplus_{\mathfrak{m}\in P^2_A} \mathrm{Hom}(K^M_3(A_{\mathfrak{m}}), \mathbb{Z}/p). \tag{7.71}$$

We state another important proposition.

Proposition 7.13. *There exists the complex*

$$0 \to \mathbb{Z}/p \to \mathrm{Hom}_c(C_K/p, \mathbb{Z}/p) \to \bigoplus_{\mathfrak{p} \in P_A^1} \mathrm{Hom}(F^0(C_{K_\mathfrak{p}}/p), \mathbb{Z}/p) \to \bigoplus_{\mathfrak{m} \in P_A^2} (K_3^M(A_\mathfrak{m})/p)^* \tag{7.72}$$

which is exact at $\mathbb{Z}/p$ *and* $\mathrm{Hom}_c(C_K/p, \mathbb{Z}/p)$.

Before the proof, we remark that this proposition combined with the exact sequence (7.71) provides the key diagram (7.94) below by which we can accomplish the proof of Theorem 7.1.

Proof of Proposition 7.13. For the proof, we use the group F^0C_K in Definition 5. We first prove the exactness of

$$F^0C_K \to C_K \to \mathbb{Z} \to 0. \tag{7.73}$$

By definition of F^0C_K in (4.11), it is sufficient to prove the exactness

$$F^0C_K(M) \to C_K(M) \to \mathbb{Z} \to 0 \tag{7.74}$$

for each modulus M. But this sequence is rewritten as

$$\mathrm{Coker}\left(\bigoplus_{\mathfrak{p} \in P_A^1} K_3^M(K_\mathfrak{p}) \to \bigoplus_{\mathfrak{m} \in P_A^2} F^0C_\mathfrak{m}(M)\right) \to \mathrm{Coker}\left(\bigoplus_{\mathfrak{p} \in P_A^1} K_3^M(K_\mathfrak{p}) \to \bigoplus_{\mathfrak{m} \in P_A^2} C_\mathfrak{m}(M)\right) \to \mathbb{Z}. \tag{7.75}$$

For each $\mathfrak{m} \in P_A^2$, we have the canonical isomorphism

$$\mathrm{Coker}\left(F^0C_\mathfrak{m}(M) \to C_\mathfrak{m}(M)\right) \cong \kappa(\mathfrak{m})^*, \tag{7.76}$$

which immediately follows from Theorem 4.6 (1) and the exact Gersten-Quillen complex

$$K_3^M(K_\mathfrak{m}) \to \bigoplus_{\mathfrak{p}_\mathfrak{m} \in P_\mathfrak{m}^1} K_2^M(\kappa(\mathfrak{p}_\mathfrak{m})) \to \kappa(\mathfrak{m})^* \to 0. \tag{7.77}$$

(we assume the exactness of the complex (7.77) if the characteristic of $K_\mathfrak{m}$ is 2 (cf. [Q], [So])).

Hence, the proof of the exactness of (7.75) is reduced to that of the exactness of

$$\bigoplus_{\mathfrak{p} \in P_A^1} K_3^M(K_\mathfrak{p}) \to \bigoplus_{\mathfrak{m} \in P_A^2} \kappa(\mathfrak{m})^* \to \mathbb{Z}. \tag{7.78}$$

Further, again by Theorem 4.6 (1), we see that each map $K_3^M(K_\mathfrak{p}) \to \bigoplus_{\mathfrak{m} \in P_A^2} \kappa(\mathfrak{m})^*$ factors as $K_3^M(K_\mathfrak{p}) \xrightarrow{\partial} K_2^M(\kappa(\mathfrak{p})) \to \bigoplus_{\mathfrak{m} \in P_A^2} \kappa(\mathfrak{m})^*$, where ∂ is the boundary map in algebraic K-theory. Thus, for the proof of exactness of the sequence in (7.78), we have only to prove the exactness of

$$\bigoplus_{\mathfrak{p}\in P_A^1} K_2^M(\,\kappa(\mathfrak{p})\,) \;\to\; \bigoplus_{\mathfrak{m}\in P_A^2} \kappa(\mathfrak{m})^* \;\to\; \mathbb{Z}. \tag{7.79}$$

But this is nothing but the Gersten-Quillen theorem for A, hence is exact (cf. [Q]). Thus we proved the exactness of (7.73).

By putting $\bigotimes_{\mathbb{Z}} \mathbb{Z}/p$ to (7.73), we get the exact sequence

$$F^0C_K/p \;\to\; C_K/p \;\to\; \mathbb{Z}/p \;\to\; 0. \tag{7.80}$$

By taking the Pontryagin dual of this sequence, we get the exact sequence

$$0 \to \mathbb{Z}/p \to \mathrm{Hom}_c(C_K/p,\, \mathbb{Z}/p) \to\; \mathrm{Hom}_c(F^0C_K/p,\, \mathbb{Z}/p). \tag{7.81}$$

Thus the proof of the exactness of the complex (7.72) at $\mathrm{Hom}_c(C_K/p,\, \mathbb{Z}/p)$ in Proposition 7.13 is obtained from (7.81) and the following lemma.

Lemma 7.14. *For each height one prime $\mathfrak{p} \in P_A^1$, there exists the unique homomorphism*

$$\Psi_{\mathfrak{p}}\colon\; F^0C_{K_{\mathfrak{p}}} \to F^0C_K, \tag{7.82}$$

which induces the injective homomorphism

$$\Psi^*\colon \mathrm{Hom}_c(F^0C_K/p,\, \mathbb{Z}/p) \hookrightarrow \bigoplus_{\mathfrak{p}\in P_A^1} \mathrm{Hom}(F^0(C_{K_{\mathfrak{p}}}/p),\, \mathbb{Z}/p). \tag{7.83}$$

Proof. From the definition of $F^0C_{K_{\mathfrak{p}}}/p$ in (6.20), for the existence of $\Psi_{\mathfrak{p}}$, we have only to construct a homomorphism

$$\Psi_{\mathfrak{p},\mathfrak{q}}\colon U^0K_3^M(K_{\mathfrak{p},\mathfrak{q}}) \;\to\; F^0C_K \tag{7.84}$$

for each $(\mathfrak{p}, \mathfrak{q})$. Now, we have the following useful theorem by Nagata.

Theorem 7.15 (Nagata, cf. [Na], cor. 37.6, 9, 10)**.** *For an arbitrary complete integral local ring R, there exists a one-to-one correspondence between maximal ideals of the normalization $\widetilde{R}$ of R and prime ideals of zero of the completion $\widehat{R}$ of R.*

From this, we get the following dictionary:

For each height one prime $\mathfrak{q}$ of $\widetilde{A/u_{\mathfrak{p}}}$, there exists the unique prime $\mathfrak{p}_{\mathfrak{m}} \in P_{\mathfrak{m}}^1$ for some $\mathfrak{m} \in P_A^2$ such that $\mathfrak{p}_{\mathfrak{m}} \mapsto \mathfrak{p}$. (7.85)

Moreover, the above theorem by Nagata directly implies that if we take the complete discrete valuation ring $\widetilde{(A/u_{\mathfrak{p}})}_{\mathfrak{q}} := \varprojlim_i \widetilde{(A/u_{\mathfrak{p}})}_{(\mathfrak{q})}/\mathfrak{q}^i$ as before, we have an isomorphism

$$\zeta_{\mathfrak{q}} \colon \mathrm{Frac}(\widetilde{(A/u_{\mathfrak{p}})}_{\mathfrak{q}}) \cong \kappa(\mathfrak{p}_{\mathfrak{m}}), \tag{7.86}$$

where Frac means the fractional field. So, it turns out that both $K_{\mathfrak{p},\mathfrak{q}}$ and $K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}$ are complete discrete valuation fields with isomorphic residue fields. Thus, we can construct an isomorphism

$$\zeta_{\mathfrak{p},\mathfrak{q}} \colon K_{\mathfrak{p},\mathfrak{q}} \cong K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}} \tag{7.87}$$

which induces an isomorphism

$$\Psi_{\mathfrak{p},\mathfrak{q}} \colon U^0K_3^M(K_{\mathfrak{p},\mathfrak{q}}) \cong U^0K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}). \tag{7.88}$$

From this map, combined with the natural map $U^0K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}) \to F^0C_K$, we get the desired map $\Psi_{\mathfrak{p},\mathfrak{q}}$ in (7.84) which at the same time also gives us the map $\Psi_{\mathfrak{p}}$ in (7.82).

Now, we show the injectivity (7.83). Let $\chi \in \mathrm{Hom}_c(F^0C_K/p, \mathbb{Z}/p)$ be an arbitrary element. Then, it is easily seen that χ annihilates

$$I_K(M_\chi) := \mathrm{Ker}(F^0C_K \to F^0C_K(M_\chi)) \tag{7.89}$$

for some modulus M_χ by the map $F^0C_K \to F^0C_K/p \xrightarrow{\chi} \mathbb{Z}/p$. Thus, χ belongs to the group $\mathrm{Hom}_c(F^0C_K/I_K(M_\chi), \mathbb{Z}/p)$. But we have the surjective homomorphism

$$F^0D_K \twoheadrightarrow F^0C_K/I_K(M_\chi)\ (\subset F^0C_K(M_\chi)) \tag{7.90}$$

from (4.13).

Further, by considering two definitions (4.8), (6.22), we get the surjective homomorphism

$$\left(\prod_{\mathfrak{p}\in P_A^1} F^0(C_{K_{\mathfrak{p}}}/p)\right) \twoheadrightarrow F^0D_K/p. \tag{7.91}$$

From (7.90), (7.91), we get the surjective homomorphism

$$\Psi_{M_\chi} \colon \left(\prod_{\mathfrak{p}\in P_A^1} F^0(C_{K_{\mathfrak{p}}}/p)\right) \twoheadrightarrow (F^0C_K/I_\chi(K))/p. \tag{7.92}$$

By taking the Pontryagin dual of the surjection (7.92) considering the important fact that groups in the both hands side of (7.92) have the discrete topology (the discreteness of the right hand side comes from that of $C_K(M_\chi)$), we get the injective homomorphism

$$\Psi^*_{M_\chi} \colon \mathrm{Hom}(F^0C_K/I_\chi(K), \mathbb{Z}/p) \hookrightarrow \bigoplus_{\mathfrak{p}\in P_A^1} \mathrm{Hom}(F^0(C_{K_{\mathfrak{p}}}/p), \mathbb{Z}/p). \tag{7.93}$$

As χ is an arbitrary element of $\mathrm{Hom}_c(F^0C_K/p, \mathbb{Z}/p)$, the desired injectivity (7.82) is established. $\square$

From the exact sequences (7.65), (7.72), we get the following commutative diagram:

$$\begin{array}{ccccccc} 0 \to \mathbb{Z}/p \to & H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p) & \longrightarrow & \bigoplus_{\mathfrak{p}} H^2_{\mathfrak{p}}(A_{\mathfrak{p}}, \mathbb{Z}/p) & \longrightarrow & \bigoplus_{\mathfrak{m}} (K^M_3(A_{\mathfrak{m}})/p)^* \\ \| & \downarrow \rho_K{}^*/p & & \downarrow \oplus\, \rho^*_{K_{\mathfrak{p}}}/p & & \| \\ 0 \to \mathbb{Z}/p \to & \mathrm{Hom}_c(C_K/p, \mathbb{Z}/p) & \xrightarrow{\Psi^*} & \bigoplus_{\mathfrak{p}} \mathrm{Hom}(F^0(C_{K_{\mathfrak{p}}}/p), \mathbb{Z}/p) & \to & \bigoplus_{\mathfrak{m}} (K^M_3(A_{\mathfrak{m}})/p)^*, \end{array} \tag{7.94}$$

where $(K^M_3(A_{\mathfrak{m}})/p)^*$ denotes $\mathrm{Hom}(K^M_3(A_{\mathfrak{m}})/p, \mathbb{Z}/p)$ and Ψ^* was defined at (7.83). The bottom row is exact at $\mathbb{Z}/p$ and at $\mathrm{Hom}_c(C_K/p, \mathbb{Z}/p)$ by Proposition 7.13.

Now, we prove the bijectivity of $\rho_K{}^*/p$ by using the diagram (7.94). First, the injectivity of $\rho_K{}^*/p$ immediately follows from that of each $\rho^*_{K_{\mathfrak{p}}}/p$ proved in Theorem 6.4. Next, we show the surjectivity of $\rho_K{}^*/p$, which is the most hard task.

By the diagram chase in (7.94), the surjectivity of ρ^*_K/p is immediately obtained from

Claim 7.16. *For each height one prime $\mathfrak{p} \in P^1_A$, there holds the following inclusion:*

$$\mathrm{Im}\left(H^2_{\mathfrak{p}}(A_{\mathfrak{p}}, \mathbb{Z}/p) \xrightarrow{\rho^*_{K_\lambda}} \mathrm{Hom}(F^0C_{K_\lambda}, \mathbb{Z}/p)\right) \supset \mathrm{Im}\left(\mathrm{Hom}_c(C_K, \mathbb{Z}/p) \xrightarrow{\Psi^*_\lambda} \mathrm{Hom}(F^0C_{K_\lambda}, \mathbb{Z}/p)\right), \tag{7.95}$$

where Im *means 'image'.*

It is easily seen that this claim immediately implies the desired surjectivity of ρ^*_K/p.

Proof of Claim 7.16. For an arbitrary continuous character $\chi \in \mathrm{Hom}_c(C_K, \mathbb{Z}/p)$, we show that the induced homomorphism

$$\chi \circ \Psi_\lambda \colon F^0C_{K_\lambda} \xrightarrow{\Psi_\lambda} C_K \xrightarrow{\chi} \mathbb{Z}/p \tag{7.96}$$

lies in Image $\left(H^2_{\mathfrak{p}}(K_{\mathfrak{p}}, \mathbb{Z}/p) \xrightarrow{\rho_{K_\lambda}{}^*} \mathbb{Z}/p\right)$ for each height one prime $\lambda \in P^1_A$.

First, take an arbitrary $\lambda \in P^1_A$ and $\chi \in \mathrm{Hom}_c(C_K, \mathbb{Z}/p)$. We know that the continuous homomorphism χ annihilates the subgroup $C_K(M_\chi)$ for some modulus M_χ, and we write this modulus M_χ as

$$M_\chi = \sum_{\mathfrak{p} \in P^1_A} n_{\mathfrak{p}}\, (\overline{\mathfrak{p}}). \tag{7.97}$$

We define the support of M and the finite subset (possibly empty) $P_A^2(\lambda, M) \subset P_A^2$, respectively as follows:

$$\text{supp.}\, M:\ = \{\mathfrak{p} \in P_A^1 \mid n_{\mathfrak{p}} \neq 0\}, \tag{7.98}$$

$$P_A^2(\lambda, M):\ = \{\mathfrak{m} \in P_A^2 \mid \mathfrak{m} \supset \lambda \text{ and } \mathfrak{m} \supset \mathfrak{p}_i,\ \text{for some } \mathfrak{p}_i\,(\neq \lambda) \in \text{supp.}\, M\}. \tag{7.99}$$

From the fact that χ annihilates the subgroup $C_K(M_\chi)\,(\subset C_K)$, it follows directly that the subgroup $F^{n_\lambda}C_{K_\lambda}$ ($\subset F^0C_{K_\lambda}$, n_λ is the coefficient of $\overline{\lambda}$ in (7.97)) is annihilated by the composite map $\chi \circ \Psi_\lambda$. Besides, the next lemma holds.

Lemma 7.17. *The composite homomorphism*

$$\chi \circ \Psi_\lambda\colon F^0C_{K_\lambda} \to \mathbb{Z}/p$$

annihilates the subgroup Λ_χ *of* $F^0C_{K_\lambda}$ *defined by*

$$\Lambda_\chi\colon = \text{Image}\left(\left(\left(\prod_{(\lambda_\mathfrak{m},\mathfrak{m})\in P} K_3^M(A_\mathfrak{m}[\frac{1}{u_{\lambda_\mathfrak{m}}}])\right) \oplus \left(\bigoplus_{(\lambda_\mathfrak{m},\mathfrak{m})\in Q} V_{\lambda_\mathfrak{m}}\right)\right) \xrightarrow{\oplus_\mathfrak{q}\Psi_{\lambda,\mathfrak{q}}^{-1}} F^0C_{K_\lambda}\right),$$

where $\Psi_{\lambda,\mathfrak{q}}$ *is an isomorphism defined in* (7.88) *(put* $\mathfrak{p} = \lambda$ *in it),* P, Q *are defined by*

$$P = \{\lambda_\mathfrak{m} \in P_\mathfrak{m}^1, \mathfrak{m} \in P_A^2 \mid \lambda_\mathfrak{m} \mapsto \lambda,\ \mathfrak{m} \notin P_A^2(\lambda, M)\}$$
$$Q = \{\lambda_\mathfrak{m} \in P_\mathfrak{m}^1, \mathfrak{m} \in P_A^2 \mid \lambda_\mathfrak{m} \mapsto \lambda,\ \mathfrak{m} \in P_A^2(\lambda, M)\}$$

and each group $V_{\lambda_\mathfrak{m}}$ $(\subset K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}))$ *is defined by*

$$V_{\lambda_\mathfrak{m}} = \text{Ker}\left(K_3^M(K_\mathfrak{m}) \xrightarrow{diagonal} \bigoplus_{\mathfrak{p}_\mathfrak{m}\in P_\mathfrak{m}^1,\, \mathfrak{p}_\mathfrak{m}\neq\lambda_\mathfrak{m}} \left(K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/U^{M(\mathfrak{p}_\mathfrak{m})}K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})\right)\right).$$

Proof. This follows immediately from the reciprocity law for each $K_3^M(K_\mathfrak{m})$ in C_K proved in Proposition 7.3 (one can argue quite similarly as in the proof of Sub-lemma 4.3). □

Now, we return to the proof of Claim 7.16. From Lemma 7.17 and the fact that $F^{n_\lambda}C_{K_\lambda}$ is annihilated by $\chi \circ \Psi_\lambda$, if we put

$$\Xi\colon = F^0C_{K_\lambda}/(\Lambda_\chi,\ F^{n_\lambda}C_{K_\lambda}), \tag{7.100}$$

the map $\chi \circ \Psi_\lambda$ is considered as a character $\chi \circ \Psi_\lambda\colon \Xi \to \mathbb{Z}/p$. We define the filtration on the group Ξ by

$$F^n\,\Xi\colon = \text{Image}\,(F^nC_{K_\lambda} \to \Xi)\,, \quad F^n(\Xi/p)\colon = \text{Image}\,(F^n\Xi \to \Xi/p)$$

We also put

$$Gr^n(\Xi/p) := F^n(\Xi/p)/F^{n+1}(\Xi/p). \tag{7.101}$$

We remark that $Gr^i(\Xi/p)$ is the discrete module. The following calculation is the key for the proof of Claim 7.16.

Proposition 7.18. *There exist surjective homomorphisms*

1. $$\frac{\left(\bigoplus_{(\lambda_{\mathfrak{m}},\mathfrak{m})\in P}(\Omega^2_{\kappa(\lambda_{\mathfrak{m}})}/\Omega^2_{(A/u_{\lambda_{\mathfrak{m}}})})\right)\oplus\left(\bigoplus_{(\lambda_{\mathfrak{m}},\mathfrak{m})\in Q}(\Omega^2_{\kappa(\lambda_{\mathfrak{m}})}/c_{\lambda_{\mathfrak{m}}}\Omega^2_{(A/u_{\lambda_{\mathfrak{m}}})})\right)}{\Omega^2_{\kappa(\lambda)}} \twoheadrightarrow Gr^n\,(\Xi\,/p) \quad (p\nmid n)$$

2. $$\frac{\left(\bigoplus_{(\lambda_{\mathfrak{m}},\mathfrak{m})\in P}(\Omega^1_{\kappa(\lambda_{\mathfrak{m}})}/(\Omega^1_{\kappa(\lambda_{\mathfrak{m}}),d=0},\Omega^1_{(A/u_{\lambda_{\mathfrak{m}}})}))\right)\oplus\left(\bigoplus_{(\lambda_{\mathfrak{m}},\mathfrak{m})\in Q}(\Omega^1_{\kappa(\lambda_{\mathfrak{m}})}/(\Omega^1_{\kappa(\lambda_{\mathfrak{m}}),d=0},c'_{\lambda_{\mathfrak{m}}}\Omega^1_{(A/u_{\lambda_{\mathfrak{m}}})}))\right)}{(\Omega^1_{\kappa(\lambda)}/\Omega^1_{\kappa(\lambda),d=0})}$$
$$\twoheadrightarrow Gr^n\,(\Xi\,/p) \quad (p\mid n),$$

where $\Omega^1_{*,\,d=0}$ denotes the set of d-closed 1-forms of $*$ and P, Q are the same ones defined in Lemma 7.17.

Proof. Both groups $K_3^M(A_{\mathfrak{m}}[\frac{1}{u_{\lambda_{\mathfrak{m}}}}])/p$, $V_{\lambda_{\mathfrak{m}}}/p$ have natural filtrations $F^n(K_3^M(A_{\mathfrak{m}}[\frac{1}{u_{\lambda_{\mathfrak{m}}}}])/p)$ $F^n(V_{\lambda_{\mathfrak{m}}}/p)$ induced from $U^nK_3^M(K_{\mathfrak{m},\lambda_{\mathfrak{m}}})/p$ (those are subgroups of $K_3^M(K_{\mathfrak{m},\lambda_{\mathfrak{m}}})/p$). We define

$$Gr^n\,(K_3^M(A_{\mathfrak{m}}[\frac{1}{u_{\lambda_{\mathfrak{m}}}}])/p) := F^n(K_3^M(A_{\mathfrak{m}}[\frac{1}{u_{\lambda_{\mathfrak{m}}}}])/p)/F^{n+1}(K_3^M(A_{\mathfrak{m}}[\frac{1}{u_{\lambda_{\mathfrak{m}}}}])/p),$$
$$Gr^n(V_{\lambda_{\mathfrak{m}}}/p) := F^n(V_{\lambda_{\mathfrak{m}}}/p)/F^{n+1}(V_{\lambda_{\mathfrak{m}}}/p).$$

We calculate these gr-quotients explicitly as

$$\begin{aligned}
\Omega^2_{\kappa(\lambda_{\mathfrak{m}})}/\Omega^2_{(A_{\mathfrak{m}}/u_{\lambda_{\mathfrak{m}}})} &\xrightarrow{\sim} Gr^n(K_3^M(A_{\mathfrak{m}}[\frac{1}{u_{\lambda_{\mathfrak{m}}}}])/p) && (p\nmid n)\\
\Omega^1_{\kappa(\lambda_{\mathfrak{m}})}/(\Omega^1_{\kappa(\lambda_{\mathfrak{m}}),d=0},\ \Omega^1_{(A_{\mathfrak{m}}/u_{\lambda_{\mathfrak{m}}})}) &\xrightarrow{\sim} Gr^n(K_3^M(A_{\mathfrak{m}}[\frac{1}{u_{\lambda_{\mathfrak{m}}}}])/p) && (p\mid n)\\
\Omega^2_{\kappa(\lambda_{\mathfrak{m}})}/c_{\lambda_{\mathfrak{m}}}\Omega^2_{(A_{\mathfrak{m}}/u_{\lambda_{\mathfrak{m}}})} &\xrightarrow{\sim} Gr^n(V_{\lambda_{\mathfrak{m}}}/p) && (p\nmid n)\\
\Omega^1_{\kappa(\lambda_{\mathfrak{m}})}/(\Omega^1_{\kappa(\lambda_{\mathfrak{m}}),d=0},\ c'_{\lambda_{\mathfrak{m}}}\Omega^1_{(A_{\mathfrak{m}}/u_{\lambda_{\mathfrak{m}}})}) &\xrightarrow{\sim} Gr^n(V_{\lambda_{\mathfrak{m}}}/p) && (p\mid n),
\end{aligned} \tag{7.102}$$

where $c_{\lambda_{\mathfrak{m}}}$, $c'_{\lambda_{\mathfrak{m}}}$ are certain constants in $\kappa(\lambda_{\mathfrak{m}})$. These isomorphisms are easily calculated from Theorem 4.6. Now, we go as follows. First for each $n \geq 1$, we have the obvious (from definition) surjection

$$\operatorname{Coker}\left(U^n K_3^M(K_\lambda)/p \overset{diagonal}{\longrightarrow} \prod\nolimits'_{\mathfrak{q}\in \widetilde{P^1_{A/u_{\mathfrak{p}}}}} \left((U^n K_3^M(K_{\mathfrak{p},\mathfrak{q}})/p) \,/ (U^{n_\lambda} K_3^M(K_{\mathfrak{p},\mathfrak{q}})/p) \right) \right) \twoheadrightarrow (F^n C_{K_\lambda}/p)/F^{n_\lambda}(C_{K_\lambda}/p).$$

By using the correspondence (7.88), this surjection is rewritten as

$$\operatorname{Coker}\left(U^n K_3^M(K_\lambda)/p \overset{diagonal}{\longrightarrow} \prod'_{(\lambda_{\mathfrak{m}},\mathfrak{m})\in S} \left((U^n K_3^M(K_{\mathfrak{m},\lambda_{\mathfrak{m}}})/p)/(U^{n_\lambda} K_3^M(K_{\lambda,\lambda_{\mathfrak{m}}})/p) \right) \right) \twoheadrightarrow (F^n C_{K_\lambda}/p)/F^{n_\lambda}(C_{K_\lambda}/p), \quad . \tag{7.103}$$

where $S := \{\lambda_{\mathfrak{m}} \in P^1_{\mathfrak{m}}, \mathfrak{m} \in P^2_A \mid \lambda_{\mathfrak{m}} \mapsto \lambda\}$. Thus, by considering the definition of Ξ with a little argument, we get the following surjection from (7.103):

$$\operatorname{Coker}\left(U^n K_3^M(K_\lambda)/p \to \left(\frac{\prod_{(\lambda_{\mathfrak{m}},\mathfrak{m})\in S} U^n K_3^M(K_{\lambda_{\mathfrak{m}},\mathfrak{m}})/p}{\left(\prod_{(\lambda_{\mathfrak{m}},\mathfrak{m})\in P} F^n(K_3^M(A_{\mathfrak{m}}[\frac{1}{u_{\lambda_{\mathfrak{m}}}}])/p) \right) \oplus \left(\bigoplus_{(\lambda_{\mathfrak{m}},\mathfrak{m})\in Q} F^n(V_{\lambda_{\mathfrak{m}}}/p) \right)} \right) \right) \twoheadrightarrow F^n(\Xi/p). \tag{7.104}$$

From (7.104) by using we get the following obvious surjection

$$\operatorname{Coker}\left(Gr^n(K_3^M(K_\lambda)/p) \to \left(\frac{\prod'_{(\lambda_{\mathfrak{m}},\mathfrak{m})\in S} Gr^n(K_3^M(K_{\lambda_{\mathfrak{m}},\mathfrak{m}})/p)}{\left(\prod_{(\lambda_{\mathfrak{m}},\mathfrak{m})\in P} Gr^n(K_3^M(A_{\mathfrak{m}}[\frac{1}{u_{\lambda_{\mathfrak{m}}}}])\,/p) \right) \oplus \left(\bigoplus_{(\lambda_{\mathfrak{m}},\mathfrak{m})\in Q} Gr^n(V_{\lambda_{\mathfrak{m}}}/p) \right)} \right) \right) \twoheadrightarrow Gr^n(\Xi/p). \tag{7.105}$$

By replacing each term in (7.105) with results (7.102), the surjective homomorphisms in Proposition 7.18 follow immediately. □

We recall that the canonical homomorphism

$$Gr_n H^1_{\mathrm{Gal}}(K_\lambda, \mathbb{Z}/p) \to \operatorname{Hom}(Gr^n(C_{K_\lambda}/p), \mathbb{Z}/p).$$

was defined in (6.32). As there exists a natural surjective homomorphism

$$Gr^n(C_{K_\lambda}/p) \twoheadrightarrow Gr^n(\Xi/p) \tag{7.106}$$

between discrete groups, we get the canonical injection

$$\mathrm{Hom}(Gr^n(\Xi/p), \mathbb{Z}/p) \hookrightarrow \mathrm{Hom}(Gr^n(C_{K_\lambda}/p), \mathbb{Z}/p). \tag{7.107}$$

On this subgroup $\mathrm{Hom}(Gr^n(\Xi/p), \mathbb{Z}/p)$, we have the following theorem.

Theorem 7.19. *There exists the surjective homomorphism*

$$\rho^*_{K_\lambda}\colon Gr_n H^1_{\mathrm{Gal}}(K_\lambda, \mathbb{Z}/p) \twoheadrightarrow \mathrm{Hom}\,(Gr^n(\Xi/p), \mathbb{Z}/p) \tag{7.108}$$

for an arbitrary positive natural number $n \geq 1$.

Proof. For brevity, we treats only the case $p \nmid n$ (the case $p \mid n$ is proved quite similarly). We consider the surjective homomorphism

$$\Big(\prod_{(\lambda_{\mathfrak{m}},\mathfrak{m})\in S} \Omega^2_{\kappa(\lambda_{\mathfrak{m}})}\Big)/\Omega^2_{\kappa(\lambda)} \twoheadrightarrow \frac{\left(\bigoplus_{(\lambda_{\mathfrak{m}},\mathfrak{m})\in P}(\Omega^2_{\kappa(\lambda_{\mathfrak{m}})}/\Omega^2_{(A/u_{\lambda_{\mathfrak{m}}})})\right) \oplus \left(\bigoplus_{(\lambda_{\mathfrak{m}},\mathfrak{m})\in Q}(\Omega^2_{\kappa(\lambda_{\mathfrak{m}})}/c_{\lambda_{\mathfrak{m}}}\Omega^2_{(A/u_{\lambda_{\mathfrak{m}}})})\right)}{\Omega^2_{\kappa(\lambda)}}. \tag{7.109}$$

Now, the surjection 1. in Proposition 7.18 and the surjection (7.109) provide the surjection

$$\pi\colon \Big(\prod_{(\lambda_{\mathfrak{m}},\mathfrak{m})\in S} \Omega^2_{\kappa(\lambda_{\mathfrak{m}})}\Big)/\Omega^2_{\kappa(\lambda)} \twoheadrightarrow Gr^n(\Xi/p). \tag{7.110}$$

Further, (7.110) shows that any character $\chi\colon Gr^n(\Xi/p) \to \mathbb{Z}/p$ induces the continuous character

$$\chi\circ\pi\colon \Big(\prod_{(\lambda_{\mathfrak{m}},\mathfrak{m})\in S} \Omega^2_{\kappa(\lambda_{\mathfrak{m}})}\Big)/\Omega^2_{\kappa(\lambda)} \to \mathbb{Z}/p \tag{7.111}$$

in the sense of duality Theorem 5.4. Thus, by applying Theorem 5.4 with $F = \kappa(\lambda)$, we see that the character $\chi\circ\pi$ in (7.111) comes from $\kappa(\lambda)$. But Lemma 6.2 in Section 4 provides an isomorphism $Gr_n H^1_{\mathrm{Gal}}(K_\lambda, \mathbb{Z}/p) \cong \kappa(\lambda)$. Thus, we see that $\chi\circ\pi$ exactly comes from $Gr_n H^1_{\mathrm{Gal}}(K_\lambda, \mathbb{Z}/p)$ which is what is wanted. □

Now, by using Theorem 7.19, we can prove Claim 7.16. We prove this, by downward induction, which is the original method by Kato in his proof of higher dimensional local class field theory [Ka1]. Our task is to show that the map $\chi\circ\Psi_\lambda$, which lives in $\mathrm{Hom}(F^0C_{K_\lambda}/p, \mathbb{Z}/p)$, actually lies in the image of the homomorphism

$$\rho^*_{K_\lambda}\colon H^2_\lambda(A_\lambda, \mathbb{Z}/p) \to \mathrm{Hom}(F^0C_{K_\lambda}, \mathbb{Z}/p). \tag{7.112}$$

First, we see that $\chi \circ \Psi_\lambda$ annihilates $F^{n_\lambda}(\Xi/p)$ (n_λ is the coefficient of λ in the modulus M_χ, see (7.97)). Thus, when it is restricted on $F^{n_\lambda - 1}(\Xi/p)$, it induces the character

$$\chi \circ \Psi_\lambda \colon Gr^{n_\lambda-1}(\Xi/p) \to \mathbb{Z}/p. \tag{7.113}$$

But Theorem 7.19 tells us that there exists a some element $\chi_{\lambda,n_\lambda-1} \in Gr_{n_\lambda-1}H^1_{\mathrm{Gal}}(K_\lambda, \mathbb{Z}/p)$ which coincides with the above character $\chi \circ \Psi_\lambda$. Thus, if we fix one lift $\widetilde{\chi_{\lambda,n_\lambda-1}}$ of the character $\chi_{\lambda,n_\lambda-1}$ in $F_{n_\lambda-1}H^1_{\mathrm{Gal}}(K_\lambda, \mathbb{Z}/p)$, it follows that $\chi \circ \Psi_\lambda - \widetilde{\chi_{\lambda,n_\lambda-1}}$ annihilates the subgroup $F^{n_\lambda-1}(\Xi/p)$. So by restriction to $F^{n_\lambda-2}(\Xi/p)$, $\chi \circ \Psi_\lambda - \widetilde{\chi_{\lambda,n_\lambda-1}}$ can be considered as an element of $\mathrm{Hom}(Gr^{n_\lambda-2}(\Xi/p), \mathbb{Z}/p)$. By taking the same procedure and arguing similarly as above, we obtain a character $\widetilde{\chi_{\lambda,n_\lambda-2}} \in F_{n_\lambda-2}H^1_{\mathrm{Gal}}(K_\lambda, \mathbb{Z}/p)$ such that $\chi \circ \Psi_\lambda - \widetilde{\chi_{\lambda,n_\lambda-1}} - \widetilde{\chi_{\lambda,n_\lambda-2}}$ annihilates $F^{n_\lambda-2}(\Xi/p)$. Continuing like this, we finally reach an element $\widetilde{\chi_{\lambda,0}} \in F_0H^1_{\mathrm{Gal}}(K_\lambda, \mathbb{Z}/p)$ such that $\chi \circ \Psi_\lambda - \widetilde{\chi_{\lambda,n_\lambda-1}} - \widetilde{\chi_{\lambda,n_\lambda-2}} - \cdots - \widetilde{\chi_{\lambda,0}}$ annihilates the subgroup $F^0(\Xi/p) = \Xi/p$ which implies that

$$F_{n_\lambda-1}H^1_{\mathrm{Gal}}(K_\lambda, \mathbb{Z}/p) \ni \widetilde{\chi_{\lambda,n_\lambda-1}} + \widetilde{\chi_{\lambda,n_\lambda-2}} + \cdots + \widetilde{\chi_{\lambda,0}} = \chi \circ \Psi_\lambda \tag{7.114}$$

as characters of Ξ/p. But as $F_0H^1_{\mathrm{Gal}}(K_\lambda, \mathbb{Z}/p)$ annihilates $F^0C_{K_\lambda}/p$, hence Ξ/p, we find that

$$\widetilde{\chi_{\lambda,n_\lambda-1}} + \widetilde{\chi_{\lambda,n_\lambda-2}} + \cdots + \widetilde{\chi_{\lambda,0}} \in F_{n_\lambda-1}H^1_{\mathrm{Gal}}(K_\lambda, \mathbb{Z}/p)/F_0H^1_{\mathrm{Gal}}(K_\lambda, \mathbb{Z}/p). \tag{7.115}$$

Together with (7.114) and (7.115), the isomorphism

$$F_\infty H^1_{\mathrm{Gal}}(K_\lambda, \mathbb{Z}/p)/F_0H^1_{\mathrm{Gal}}(K_\lambda, \mathbb{Z}/p) \cong H^2_\lambda(A_\lambda, \mathbb{Z}p)$$

in (6.25) shows that $\chi \circ \Psi_\lambda (\in F_{n_\lambda-1}H^1_{\mathrm{Gal}}(K_\lambda, \mathbb{Z}/p)/F_0H^1_{\mathrm{Gal}}(K_\lambda, \mathbb{Z}/p))$ actually lies in $H^2_\lambda(A_\lambda, \mathbb{Z}p)$. Thus, the proof of Claim 7.16 is finished, hence now, Theorem 7.1 is proved rigorously. □

8. Proof of the existence theorem (prime to p parts)

In this section, we prove the bijectivity of the dual reciprocity map ρ_K^* modulo arbitrary natural number m prime to the characteristic of K. Though we use some deep results in algebraic K-Theory, the proof of the existence theorem for prime to p parts is much easier than that for p primary parts accomplished in the previous section. The key tools in this section are Saito's Hasse Principle for two-dimensional normal complete local rings which are not necessarily regular, and the Bloch-Milnor-Kato Conjecture. Our purpose in this section is to prove the following theorem:

Theorem 8.1. *Let A be an arbitrary positive characteristic three-dimensional complete regular local ring with finite residue field. Then, under the Bloch-Milnor-Kato conjecture for K (see Conjecture 1), there exists the canonical dual reciprocity isomorphism*

$$\rho_K^*\colon H^1_{\mathrm{Gal}}(K,\, \mathbb{Q}_l/\mathbb{Z}_l) \xrightarrow{\sim} \mathrm{Hom}_c(C_K,\, \mathbb{Q}_l/\mathbb{Z}_l) \tag{8.1}$$

for an arbitrary prime $l \neq p$, where Hom_c denotes the set of all continuous homomorphisms of finite order.

By combining Theorem 7.1 and Theorem 8.1, we get the class field theory for K.

Theorem 8.2. *Let A be an arbitrary positive odd characteristic three-dimensional complete regular local ring with finite residue field. Then, under the Bloch-Milnor-Kato conjecture for K (see loc.cit.), there exists the following canonical dual reciprocity isomorphism:*

$$\rho_K^*\colon H^1_{\mathrm{Gal}}(K,\, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}_c(C_K,\, \mathbb{Q}/\mathbb{Z}), \tag{8.2}$$

where $\mathrm{Hom}_c(C_K,\, \mathbb{Q}/\mathbb{Z})$ means the set of all continuous homomorphisms of finite order from C_K to $\mathbb{Q}/\mathbb{Z}$.

Remark 8. As stated in Remark 7, even for the case of $ch(K) = 2$, we have the same result in Theorem 8.2 if we may assume the exactness of the Gersten-Quillen complex in Milnor K-theory.

We give some corollaries.

Corollary 8.3. *Let A and K be as above. Then under the same assumption in the above Theorem 8.2, the canonical reciprocity map*

$$\rho_K\colon C_K \to \mathrm{Gal}(K^{ab}/K) \tag{8.3}$$

has a dense image in $\mathrm{Gal}(K^{ab}/K)$ by the Krull topology.

Proof. This follows immediately from the injectivity of (8.2) by considering dual. $\square$

Next, we give explicit isomorphisms for certain finite abelian extensions.

Corollary 8.4. *Let A, K and the assumptions be as in Theorem 8.2. Then, for an arbitrary finite abelian extension L/K such that the integral closure of A in L is regular, there exists the following canonical reciprocity isomorphism:*

$$\rho_K\colon C_K/N_{L/K}(C_L) \xrightarrow{\sim} \mathrm{Gal}(L/K). \tag{8.4}$$

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
0 \to & \mathrm{Hom}_c(\mathrm{Gal}(L/K),\, \mathbb{Q}/\mathbb{Z}) & \to & H^1_{\mathrm{Gal}}(K,\, \mathbb{Q}/\mathbb{Z}) & \to & H^1_{\mathrm{Gal}}(L,\, \mathbb{Q}/\mathbb{Z}) & \to 0 \\
 & \downarrow & & \downarrow \cong & & \downarrow \cong & \\
0 \to & \mathrm{Hom}_c(C_K/N_{L/K}(C_L),\, \mathbb{Q}/\mathbb{Z}) & \to & \mathrm{Hom}_c(C_K,\, \mathbb{Q}/\mathbb{Z}) & \to & \mathrm{Hom}_c(C_L,\, \mathbb{Q}/\mathbb{Z}). &
\end{array} \tag{8.5}$$

From this, we get the bijectivity of the extreme left vertical arrow. Corollary follows by taking dual of the isomorphism $\mathrm{Hom}_c(\mathrm{Gal}(L/K),\, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}_c(C_K/N_{L/K}(C_L),\, \mathbb{Q}/\mathbb{Z})$ proved just now noticing that the group $C_K/N_{L/K}(C_L)$ is discrete because $N_{L/K}(C_L)$ contains $\mathrm{Ker}\Big(C_K \to C_K(M)\Big)$ for some modulus M (this follows from [Ka1], II). □

Remark 9. For general abelian extension L/K that does not satisfy the above condition in Corollary 8.4, there would occur the inequality $|C_K/N_{L/K}(C_L)| > [L : K]$. Indeed in the case of class field theory for two-dimensional complete regular local rings, such examples exist ! (cf. [Sh]).

Proof of Theorem 8.1. For verifying the isomorphism (8.1), we have only to prove the following isomorphism:

$$\rho_K^*/l \colon H^1_{\mathrm{Gal}}(K,\, \mathbb{Z}/l) \xrightarrow{\sim} \mathrm{Hom}_c(C_K,\, \mathbb{Z}/l) \tag{8.6}$$

for an arbitrary prime $l \neq p$ (the isomorphism of the reciprocity map ρ_K^*/l^m for $m > 1$ is also proved in the same way to the case $m = 1$ without any essential change).

We begin the proof of the isomorphism (8.6). As l is prime to the characteristic K, the group C_K/l becomes discrete. By norm arguments, we may assume $\mu_l \in K$, where μ_l denotes the group of l-th power of unity. By Kummer theory, we have the isomorphism

$$H^1_{\mathrm{Gal}}(K,\, \mu_l) \cong K^*/K^{*l}. \tag{8.7}$$

Moreover, as A is a unique factorial domain, we have the factorization

$$K^*/K^{*l} \cong \mu_l \times \bigoplus_{\mathfrak{p} \in P_A^1} u_{\mathfrak{p}}^{\mathbb{Z}/l}, \tag{8.8}$$

where μ_l comes from the isomorphism $A^*/A^{*l} \cong \mu_l$ and $u_{\mathfrak{p}}$ denotes the regular parameter of a prime $\mathfrak{p}$. Then, there exists the following exact sequences:

$$F^0 C_K/l \to C_K/l \to \mathbb{Z}/l \to 0, \tag{8.9}$$

$$0 \to \mu_l \to H^1_{\mathrm{Gal}}(K,\, \mu_l) \to \bigoplus_{\mathfrak{p} \in P_A^1} \mathbb{Z}/l \to 0, \tag{8.10}$$

where the exactness of (8.9) is obtained by putting $\bigotimes_{\mathbb{Z}} \mathbb{Z}/l$ to the exact sequence (7.73). And the exactness of (8.10) is obvious from (8.7) and (8.8). Now, we have the key theorem.

Theorem 8.5. *There exists the canonical isomorphism*

$$F^0 C_K/l \cong \prod_{\mathfrak{p} \in P_A^1} \mu_l. \tag{8.11}$$

The proof of this theorem is given below, and we continue to prove Theorem 8.1 assuming Theorem 8.5. By (8.11), we can rewrite (8.9) as

$$\prod_{\mathfrak{p} \in P_A^1} \mu_l \to C_K/l \to \mathbb{Z}/l \to 0. \tag{8.12}$$

By (8.10) and the Pontryagin dual of (8.6) tensored with μ_l, we get the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mu_l & \longrightarrow & H^1_{\mathrm{Gal}}(K, \mu_l) & \longrightarrow & \bigoplus_{\mathfrak{p} \in P_A^1} \mathbb{Z}/l & \to & 0 \\ & & \downarrow \cong & & \downarrow \rho_K^*/l & & \downarrow \cong & & \\ 0 & \to & \mathrm{Hom}_c(\mathbb{Z}/l, \mu_l) & \to & \mathrm{Hom}_c(C_K, \mu_l) & \to & \mathrm{Hom}_c(\prod_{\mathfrak{p} \in P_A^1} \mathbb{Z}/l, \mu_l). & & \end{array} \tag{8.13}$$

The bijectivity of the middle vertical arrow, which is nothing but the reciprocity map ρ_K^*/l, follows immediately from this diagram. □

Now, we begin to prove Theorem 8.5.

Proof of Theorem 8.5. We analyze the group $F^0 C_K/l$. Recall that by using

$$F^0 C_K(M) := \mathrm{Image}\left(\bigoplus_{\mathfrak{m} \in P_A^2} F^0 C_{\mathfrak{m}}(M) \to C_K(M) \right),$$

the group $F^0 C_K$ was defined in (4.11) as

$$F^0 C_K := \varprojlim_M F^0 C_K(M).$$

But it is easily checked by using Theorem 4.6 that we have an isomorphism

$$F^0 C_K(M) \cong \mathrm{Coker}\left(\bigoplus_{\mathfrak{p} \in P_A^1} U^0 K_3^M(K_{\mathfrak{p}}) \to \bigoplus_{\mathfrak{m} \in P_A^2} F^0 C_{\mathfrak{m}}(M) \right). \tag{8.14}$$

We prove Theorem 8.5 by the following two results.

Proposition 8.6. *There holds the isomorphism*

$$F^0 C_{\mathfrak{m}}(M)/l \cong \bigoplus_{\mathfrak{p}_{\mathfrak{m}} \in P_{\mathfrak{m}}^1} K_3^M(\kappa(\mathfrak{p}_{\mathfrak{m}}))/l. \tag{8.15}$$

for an arbitrary modulus M.

Theorem 8.7. *For each* $\mathfrak{p} \in P_A^1$*, we have the isomorphism*

$$\operatorname{Coker}\left(K_3^M(K_{\mathfrak{p}})/l \to \bigoplus_{\substack{\mathfrak{p}_{\mathfrak{m}} \in P^1_{A_{\mathfrak{m}}}, \mathfrak{p}_{\mathfrak{m}} \mapsto \mathfrak{p} \\ \mathfrak{m} \in P_A^2}} K_3^M(\kappa(\mathfrak{p}_{\mathfrak{m}}))/l \right) \cong \mu_l. \tag{8.16}$$

From Proposition 8.6 and Theorem 8.7, the isomorphism (8.14) shows

$$F^0 C_K(M)/l \cong \prod_{\mathfrak{p} \in P_A^1} \mu_l$$

for an arbitrary modulus M. By taking the inverse limit of all M, we have

$$F^0 C_K/l \cong \prod_{\mathfrak{p} \in P_A^1} \mu_l$$

which is nothing but the desired isomorphism (8.11).

Proof of Proposition 8.6. From (4.10), $F^0 C_{\mathfrak{m}}(M)/l$ is rewritten as

$$F^0 C_{\mathfrak{m}}(M)/l :=\operatorname{Image}\left(\bigoplus\nolimits_{\mathfrak{p}_{\mathfrak{m}} \in P^1_{\mathfrak{m}}} U^0 K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})/l \to C_{\mathfrak{m}}(M)/l\right). \tag{8.17}$$

We have a lemma.

Lemma 8.8. *We have the following isomorphism*:

$$U^0 K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})/l \cong K_3^M(\kappa(\mathfrak{p}_{\mathfrak{m}}))/l. \tag{8.18}$$

Proof. By Theorem 4.6 (1), we have the exact sequence

$$U^1 K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}) \to U^0 K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}) \to K_3^M(\kappa(\mathfrak{p}_{\mathfrak{m}})) \to 0. \tag{8.19}$$

By putting $\bigotimes_{\mathbb{Z}} \mathbb{Z}/l$ to (8.19), we get

$$U^1 K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})/l \to U^0 K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})/l \to K_3^M(\kappa(\mathfrak{p}_{\mathfrak{m}}))/l \to 0. \tag{8.20}$$

But as $K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})$ is l-divisible, we have $U^1 K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})/l = 0$. From this, by considering (8.20), we have the desired isomorphism (8.18). □

By Lemma 8.8 and the definition of $C_{\mathfrak{m}}(M)$ stated at (4.2), (8.17) is rewritten as

$$F^0 C_{\mathfrak{m}}(M)/l \cong \operatorname{Coker}\left(K_3^M(K_{\mathfrak{m}})/l \xrightarrow{\text{diagonal}} \bigoplus\nolimits_{\mathfrak{p}_{\mathfrak{m}} \in P^1_{\mathfrak{m}}} K_3^M(\kappa(\mathfrak{p}_{\mathfrak{m}}))/l\right). \tag{8.21}$$

We have the following claim.

Claim 8.9. *The diagonal map in* (8.21) *is the zero map.*

Before the proof, we see that Proposition 8.6 follows immediately from this claim.

Proof of Claim 8.9. As $\kappa(\mathfrak{p}_\mathfrak{m})$ is two-dimensional local field, it has the valuation ring $\mathcal{O}_{\kappa(\mathfrak{p}_\mathfrak{m})}$. If we denote by $k_{\mathfrak{p}_\mathfrak{m}}$ the residue field of $\mathcal{O}_{\kappa(\mathfrak{p}_\mathfrak{m})}$, Theorem 4.6 (1) provides the following exact sequence:

$$U^0K_3^M(\kappa(\mathfrak{p}_\mathfrak{m}))/l \to K_3^M(\kappa(\mathfrak{p}_\mathfrak{m}))/l \to K_2^M(k_{\mathfrak{p}_\mathfrak{m}})/l \to 0. \tag{8.22}$$

Lemma 8.10. *We have the vanishing*

$$U^0K_3^M(\,\kappa(\mathfrak{p}_\mathfrak{m})\,)/l = 0. \tag{8.23}$$

Proof. By applying Theorem 4.6 (1) to $\kappa(\mathfrak{p}_\mathfrak{m})$, we obtain the exact sequence

$$U^1K_3^M(\,\kappa(\mathfrak{p}_\mathfrak{m})\,) \to U^0K_3^M(\,\kappa(\mathfrak{p}_\mathfrak{m})\,) \to K_3^M(k_{\mathfrak{p}_\mathfrak{m}}) \to 0. \tag{8.24}$$

Putting $\bigotimes_\mathbb{Z} \mathbb{Z}/l$ to (8.24), we get the following exact sequence:

$$U^1K_3^M(\,\kappa(\mathfrak{p}_\mathfrak{m})\,)/l \to U^0K_3^M(\,\kappa(\mathfrak{p}_\mathfrak{m})\,)/l \to K_3^M(k_{\mathfrak{p}_\mathfrak{m}})/l \to 0. \tag{8.25}$$

For one-dimensional local field $k_{\mathfrak{p}_\mathfrak{m}}$, Kato proved $K_3^M(k_{\mathfrak{p}_\mathfrak{m}})/l = 0$ in [Ka2]. Thus from (8.25), together with $U^1K_3^M(\,\kappa(\mathfrak{p}_\mathfrak{m})\,)/l = 0$, we get the desired vanishing. □

So, combining (8.22) and (8.23), we get the isomorphism

$$K_3^M(\,\kappa(\mathfrak{p}_\mathfrak{m})\,)/l \cong K_2^M(\,k_{\mathfrak{p}_\mathfrak{m}}\,)/l\ . \tag{8.26}$$

We use the following result of Moore.

Theorem 8.11 (Moore, [Mo])**.** *For an arbitrary one-dimensional local field F, there exists the isomorphism*

$$K_2^M(F) \cong V \oplus \mu(F), \tag{8.27}$$

where V is the uniquely divisible subgroup of $K_2^M(F)$ and $\mu(F)$ denotes the group of all powers of unity contained in F.

By putting $\bigotimes_\mathbb{Z} \mu_l$ to the isomorphism (8.27) with $F = k_{\mathfrak{p}_\mathfrak{m}}$, we get the isomorphism

$$K_2^M(k_{\mathfrak{p}_\mathfrak{m}})\,/l \cong \mu_l, \tag{8.28}$$

where we used the fact that $\mu_l \in k_{\mathfrak{p}_\mathfrak{m}}$. Further, as $\kappa(\mathfrak{m})^*\,/l \cong \mu_l$, it also holds that

$$K_3^M(\kappa(\mathfrak{p}_\mathfrak{m}))/l \cong \kappa(\mathfrak{m})^*/l. \tag{8.29}$$

Now, from the Gersten-Quillen complex (we do not need its exactness)

$$K_3^M(K_{\mathfrak{m}})/l \to \bigoplus_{\mathfrak{p}_{\mathfrak{m}} \in P^1_{\mathfrak{m}}} K_2^M(\kappa(\mathfrak{p}_{\mathfrak{m}}))/l \to \kappa(\mathfrak{m})^*/l \to 0, \tag{8.30}$$

we see $K_3^M(K_{\mathfrak{m}})/l \to \kappa(\mathfrak{m})^*/l$ is the zero map, hence by considering the isomorphism (8.29), we see that the map $K_3^M(K_{\mathfrak{m}})/l \to K_3^M(\,\kappa(\mathfrak{p}_{\mathfrak{m}})\,)/l$ is also zero map. Thus, Claim 8.9 is proved, hence Proposition 8.6 is proved completely. □

Proof of Theorem 8.7. By the correspondence (7.85) and the isomorphism $\kappa(\mathfrak{p}_{\mathfrak{m}}) \cong \mathrm{Frac}((\widetilde{A/u_{\mathfrak{p}}})_{\mathfrak{q}})$ stated in (7.86), we can rewrite (8.16) as

$$\mathrm{Coker}\left(K_3^M(K_{\mathfrak{p}})/l \to \bigoplus_{\mathfrak{q} \in P^1_{\widetilde{A/u_{\mathfrak{p}}}}} K_3^M(\mathrm{Frac}((\widetilde{A/u_{\mathfrak{p}}})_{\mathfrak{q}}))/l \right). \tag{8.31}$$

By using Theorem 4.6 (1) repeatedly, we see that the group (8.31) is isomorphic to

$$\mathrm{Coker}\left(K_3^M(\,\kappa(\mathfrak{p})\,)/l \to \bigoplus_{\mathfrak{q} \in P^1_{\widetilde{A/u_{\mathfrak{p}}}}} K_2^M(\,\kappa(\mathfrak{q})\,)/l \right). \tag{8.32}$$

Thus our task is to prove that the group (8.32) is isomorphic to μ_l. Now, we can use the following cohomological Hasse principle by S. Saito:

Theorem 8.12 (S. Saito, [Sa1]). *For an arbitrary two-dimensional excellent normal complete local ring R with a finite residue field, and an arbitrary natural number m prime to the characteristic of R, the following sequence is exact:*

$$0 \to (\mathbb{Z}/m)^{r_R} \to H^3_{\mathrm{Gal}}(F,\, \mu_m^{\otimes 2}) \to \bigoplus_{\mathfrak{q} \in P^1_R} H^2_{\mathrm{Gal}}(\kappa(\mathfrak{q}),\, \mu_m) \to \mathbb{Z}/m \to 0, \tag{8.33}$$

where F denotes the fractional field of R, P^1_R denotes the set of all height one primes of R, and r_R is the rank of R (for details, we refer the original paper).

As $\mu_l \in R$, by putting $\bigotimes_{\mathbb{Z}} \mu_l$ to (8.33) with $m = l$, we get the following short exact sequence:

$$H^3_{et}(F,\, \mu_l^{\otimes 3}) \to \bigoplus_{\mathfrak{q} \in P^1_R} H^2_{et}(\kappa(\mathfrak{q}),\, \mu_l^{\otimes 2}) \to \mu_l \to 0 \tag{8.34}$$

Now, we have isomorphisms $K_3^M(\kappa(\mathfrak{p}))/l \cong H^3_{\mathrm{Gal}}(\kappa(\mathfrak{p}),\, \mu_l^{\otimes 3})$ by the Bloch-Milnor-Kato conjecture, $K_2^M(\kappa(\mathfrak{q}))/l \cong H^2_{\mathrm{Gal}}(\kappa(\mathfrak{q}),\, \mu_l^{\otimes 2})$ by Merkur'ev-Suslin, and $\kappa(\mathfrak{m}_R)^*/l \cong \mu_l$ by Kummer theory and assumption.

By using these isomorphisms, if we apply the above theorem of Saito with $R = \widetilde{A/u_{\mathfrak{p}}}$, the exact sequence (8.34) is rewritten as

$$K_3^M(\kappa(\mathfrak{p}))/l \to \bigoplus_{\mathfrak{q} \in P^1_{\widetilde{A/u_{\mathfrak{p}}}}} K_2^M(\kappa(\mathfrak{q}))/l \to \mu_l \to 0$$

which immediately shows that the group (8.32) is isomorphic to μ_l. Thus, Theorem 8.7 is proved. Finally, the proof of Theorem 8.1 is now completely finished. □

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Mathematical Institute, Tohoku University, Sendai 980-8578, Japan

E-mail address: `matsumi@math.tohoku.ac.jp`

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