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Asymptotic behavior of eigenfunctions and eigenvalues for ergodic and periodic systems

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Abstract

The purpose of this article is to study the influence of dynamical behavior of classical Hamilton systems with ergodic and periodic properties on asymptotic behavior of eigenfunctions and eigenvalues of the corresponding positive elliptic operator on a compact Riemannian manifold, and conversely, to investigate the asymptotic properties of eigenfunctions or eigenvalues which make the corresponding classical mechanics ergodic or periodic.

We will give an estimate of the off-diagonal asymptotics of quantum observables for quantum ergodic systems and a regularity result on limit measures associated with quantum observables for systems with homogeneous Lebesgue spectrum. We will also give necessary and sufficient conditions for ergodicity and weak-mixing property of the classical Hamilton systems, which are obtained by a reduction procedure with symmetry, in terms of semi-classical asymptotic properties of eigenfunctions. Finally, a result on the structure of the set of cluster points for the differences of eigenvalues in a certain semi-classical sense is given, which is considered as a semi-classical analogy of Helton's theorem.

Contents

Introduction

Asymptotic behavior of the eigenvalues of an elliptic operator on a compact Riemannian manifold has been vigorously investigated. For instance, Duistermaat–Guillemin [8] investigated the influence of the periodic orbits of the Hamilton flow on the asymptotic behavior of the counting function of eigenvalues. Helton [17] and Guillemin [11] studied, as we will see in Section 4, the relationship between the periodicity of the classical mechanics and the structure of the set of the cluster points for the differences of eigenvalues.

On asymptotic behavior of eigenfunctions, a remarkable result was established by Shnirelman [26], Zelditch [37] and Colin de Verdierè [6]. The result of S–Z–C states, roughly speaking, that the eigenfunctions of the Laplacian on a compact Riemannian manifold with ergodic geodesic flow are asymptotically uniformly distributed in the high energy level. In 1994, Sunada [28] introduced the concept of *quantum ergodicity at infinite energy level* and, as we will state precisely in Section 2, obtained a necessary and sufficient condition in terms of asymptotic properties of eigenfunctions of a positive elliptic operator in order that the corresponding classical Hamilton system is ergodic. Furthermore his method was used by Zelditch [42] to formulate the notion of quantum weak-mixing.

In the classical ergodic theory, there are many other concepts which are sufficient conditions of classical ergodicity, and these have been intensively investigated. Accordingly, it seems natural to ask how the dynamical behavior of the classical mechanics with more chaotic property than ergodicity affects asymptotic behavior of eigenfunctions or eigenvalues.

Moreover, the classical systems which are dealt in the above works are homogeneous Hamilton systems on a cotangent bundle, and hence the dynamical system at an energy level is isomorphic to it at every other energy level. However, in case where the flow is not homogeneous, the dynamical behavior is different at different energy levels. Indeed, there is the following important example ([14], [29]).

Let M be a compact Riemann surface with constant negative curvature -1 and Ω the symplectic form on the cotangent bundle T^*M over M defined by $\Omega = \Omega_M - \pi_M^* \boldsymbol{B}$, where Ω_M is the canonical symplectic form, $\boldsymbol{B}$ is the volume 2-form on M, and π_M is the projection from T^*M onto M. The term of the 2-form $\boldsymbol{B}$ in the definition of the symplectic form Ω introduces a *magnetic field* on M. Consider the Hamiltonian $H(x,\xi) = \|\xi\|$. Let φ_t be the Hamilton flow determined by (H,Ω), which is called *magnetic flow* under the magnetic field $\boldsymbol{B}$. We denote by ω_e the Liouville measure on the hypersurface $\Sigma_e = H^{-1}(e)$ with $e > 0$. Then the dynamical system $(\Sigma_e, \varphi_t, \omega_e)$ is ergodic if $e \geq 1$ and periodic if $e < 1$. Ergodicity of such dynamical system as a magnetic flow affects *semi-classical* asymptotic behavior of the eigenfunctions for *reduced* quantum Hamiltonian ([18], [27], [39]).

Taking the above background into account, we will consider the following problems.

Problem A Investigate the influence of the dynamical behavior of the classical mechanics satisfying a sufficient condition of ergodicity, such a mixing property, on asymptotic behavior of the eigenfunctions of the corresponding elliptic operator.

Problem B Formulate quantum ergodicity and weak-mixing for the quantum mechanics corresponding to the classical mechanics obtained by a reduction procedure of a homogeneous Hamilton flow with symmetry, and investigate the relationships between classical ergodicity and quantum ergodicity.

Problem C For the same dynamical systems as in Problem B, examine the influence of the periodicity of classical mechanics on the structure of the set of cluster points in a certain semi-classical sense for the differences of eigenvalues.

Of course, one can consider many other problems on the relation between asymptotic behavior of eigenvalues or eigenfunctions and dynamical behavior of the corresponding Hamilton flow. We refer the reader to the recent article of Zelditch [44].

We will mention the contents of this article.

In Section 1, we will review some aspects of the classical ergodic theory.

Problem A is one of the central subject in the area of quantum chaos, and it remains unsettled. In Section 2, we will present some results on this problem. Here we will state the main theorems in Section 2.

Let M be a compact connected Riemannian manifold without boundary, $\hat{H}$ a first order self-adjoint non-negative elliptic pseudodifferential operator (ψDO for short) on M with positive principal symbol $H > 0$. Let φ_t be the Hamilton flow generated by the Hamiltonian H and the canonical symplectic form Ω_M. Let ω be the Liouville measure on $\Sigma = H^{-1}(1)$. We thus obtain the classical dynamical system $(\Sigma, \varphi_t, \omega)$. We denote by $0 \leq e_1 \leq e_2 \leq \cdots \uparrow \infty$ and $\{\varphi_j\}_{j=1}^{\infty}$ the eigenvalues and an orthonormal basis of eigenfunctions of $\hat{H}$, respectively: $\hat{H}\varphi_j = e_j\varphi_j$. We set $N(\lambda) = \sharp\{j \in \mathbf{N}\,;\, e_j \leq \lambda\}$.

Theorem 2.5 *Assume that the Hamilton flow φ_t on Σ is transitive Anosov. Then, for* *every ψDO A of order zero, we have*

$$\limsup_{\lambda\to\infty} N(\lambda)^{-1} \sum_{\substack{j \\ e_j\le\lambda}} \sum_{\substack{k \\ 0<|e_j-e_k|<\delta}} |\langle A\varphi_j,\, \varphi_k \rangle|^2 = O(\delta).$$

It is well-known that the left hand side tends to zero as $\delta \to 0$ if the dynamical system $(\Sigma, \varphi_t, \omega)$ is ergodic. We note that the transitive Anosov flow is ergodic with respect to the Liouville measure.

Theorem 2.6 *Assume that the dynamical system $(\Sigma, \varphi_t, \omega)$ has homogeneous Lebesgue spectrum. Then, for every ψDO A of order zero with $\langle \sigma_0(A) \rangle = 0$, there exists an integrable function p_A on $\mathbf{R}$ such that, for any $a < b$, we have*

$$\lim_{\lambda\to\infty} N(\lambda)^{-1} \sum_{e_j\le\lambda} \sum_{\substack{k \\ a<e_k-e_j<b}} |\langle A\varphi_j,\, \varphi_k \rangle|^2 = \int_a^b p_A(\lambda)\, d\lambda,$$

where $\langle \sigma_0(A) \rangle$ denotes the space average of the principal symbol $\sigma_0(A)$ of A:

$$\langle \sigma_0(A) \rangle = \omega(\Sigma)^{-1} \int_\Sigma \sigma_0(A)\, d\omega.$$

See Section 1 for the definition of *transitive Anosov flow* and *homogeneous Lebesgue spectrum.*

In Section 3, the dynamical system obtained by the *reduction* of a homogeneous Hamilton flow with symmetry, which is the same dynamical system as in [39], will be formulated and some results on Problem B are given. We will give a brief account of them.

Let $\pi : P \to M$ be a compact connected principal bundle over a compact Riemannian manifold M with structure group G, a compact connected Lie group. Choosing a bi-invariant metric on G and a connection 1-form on P, we have a unique G-invariant metric on P which makes the bundle $\pi : P \to M$ into a Riemannian submersion, with fibers isometric to G. We fix such a metric. Let $\hat{H}$ be a self-adjoint non-negative elliptic ψDO of order one on P commuting with G-action and let $H = \sigma_1(\hat{H})$ be its principal symbol. The action of G on the cotangent bundle T^*P is Hamiltonian and we will denote its moment map by $\Phi : T^*P \to \mathcal{G}^*$ with $\mathcal{G}^*$, the dual space of the Lie algebra of G. Let λ be the highest weight of an irreducible representation of G and let $\mathcal{O}_\lambda$ $(\subset \mathcal{G}^*)$ be the coadjoint orbit through λ. It is well-known ([12]) that there is a natural symplectic form on $X_\lambda = \Phi^{-1}(\mathcal{O}_\lambda)/G$ induced by the canonical symplectic forms on T^*P and $\mathcal{O}_\lambda$. Since the function H on T^*P is G-invariant, it induces the Hamiltonian H_λ on X_λ. Let φ_t^λ be the Hamilton flow on X_λ, and let ω_e^λ be the Liouville measure on the hypersurface $H_\lambda^{-1}(e) = \Sigma_e^\lambda$. Then we obtain the Hamilton system $\mathrm{CD}_e^\lambda = (\Sigma_e^\lambda, \varphi_t^\lambda, \omega_e^\lambda)$.

The action of G on the Hilbert space $L^2(P)$ breaks it into a direct sum of the form

$$L^2(P) = \bigoplus_{\mu} \mathcal{L}_{\mu},$$

where μ runs over dominant integral weights and $\mathcal{L}_{\mu}$ is the isotypical subspace associated with the irreducible representation (π_{μ}, V_{μ}) corresponding to the dominant integral weight μ. We denote by $\hat{H}_{\lambda}$ the restriction of $\hat{H}$ on the *ladder subspace* ([13], [15])

$$\mathcal{H}_{\lambda} = \bigoplus_{m=1}^{\infty} \mathcal{L}_{m\lambda} \ (\subset L^2(P)),$$

and call $\hat{H}_{\lambda}$ the *reduced* quantum Hamiltonian. Let $e_1(m) \leq e_2(m) \leq \cdots$ be the eigenvalues of $\hat{H}_{\lambda}$ and let $\{\nu_j^m\}_{j,m\in\mathbf{N}}$ be an orthonormal basis for $\mathcal{H}_{\lambda}$ consisting of the eigenfunctions of $\hat{H}$: $\hat{H}\nu_j^m = e_j(m)\nu_j^m$. For a fixed constant $c > 0$, we set

$$\begin{aligned} \mathcal{N}_m(e,c) &= \{j \in \mathbf{N}; |e_j(m) - me| \leq c\}, \\ N_m(e,c) &= \sharp\mathcal{N}_m(e,c). \end{aligned}$$

The quantity $N_m(e,c)$ plays the same role as the counting function $N(\lambda)$ in the high energy case. Before going to state the main theorem in Section 3, we need to prepare the following two conditions on the dynamical system CD_e^{λ}.

(H1) *The Hamilton vector field, X_H, of H is not tangent to the G-orbit through any point in $\tilde{\Sigma}_e^{\lambda} = Z_e \cap \Phi^{-1}(\mathcal{O}_{\lambda})$, where $Z_e = H^{-1}(e) \subset T^*P$.*

(H2) *The set of periodic points of the reduced flow φ_t^{λ} on Σ_e^{λ} has Liouville measure zero.*

The condition (H1) is used to extend the functions on Σ_e^{λ} to the homogeneous G-invariant functions on T^*P. Under the assumption (H2), the existence of the quantum space average $\langle A \rangle_e^{\lambda}$ of zeroth order ψDO A commuting with G-action, which is defined by

$$\langle A \rangle_e^{\lambda} = \lim_{m\to\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \langle A\nu_j^m, \nu_j^m\rangle,$$

is guaranteed by the semi-classical asymptotic formula due to Guillemin–Uribe [14], [15] and Zelditch [39]. Note that the condition (H1) is fulfilled for the dynamical system generated by the principal symbol of the Laplacian with respect to the fixed metric. We also note that, if the dynamical system CD_e^{λ} is ergodic, then the condition (H2) is satisfied. For a bounded operator A on $L^2(P)$, we define the quantum (long) time average $\bar{A}$ by

$$\bar{A} = \operatorname*{w-lim}_{t\to\infty} \frac{1}{t} \int_0^t e^{is\hat{H}} A e^{-is\hat{H}} \, ds.$$

Now the main theorems in Section 3 can be stated as follows.

Theorem 3.1 *Assume that the dynamical system* CD_e^λ *satisfies the conditions* (H1) *and* (H2). *Then the dynamical system* CD_e^λ *is ergodic if and only if the following two conditions hold.*

(1) *For every* $A \in \mathcal{A}_0$ *and for every orthonormal basis* $\{\nu_j^m\}_{j,m=1}^\infty$ *for* $\mathcal{H}_\lambda$ *consisting of eigenfunctions of* $\hat{H}_\lambda$, *we have*

$$\lim_{m\to\infty} N_m(e,c)^{-1} \sum_{\substack{j,k\in\mathcal{N}_m(e,c)\\ e_j(m)=e_k(m)}} \left|\langle A\nu_j^m, \nu_k^m \rangle\right|^2 = \left|\mathrm{vol}(\Sigma_e^\lambda)^{-1}\int_{\Sigma_e^\lambda} \sigma_0(A)\, d\omega_e^\lambda\right|^2 .$$

(2) *For every* A, $\{\nu_j^m\}$ *as above, we have*

$$\lim_{\delta\downarrow 0}\limsup_{m\uparrow\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \sum_{\substack{k\\ 0<|e_j(m)-e_k(m)|<\delta}} \left|\langle A\nu_j^m, \nu_k^m \rangle\right|^2 = 0.$$

Theorem 3.3 *Suppose that the condition* (H2) *is satisfied. Then the following three conditions are equivalent.*

(S) *For every* $A \in \mathcal{A}_0$, *we have*

$$\lim_{m\to\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \|(\bar{A} - \langle A \rangle_e^\lambda)\nu_j^m\|^2 = 0,$$

where $\|\cdot\|$ *is the* L^2*-norm and* $\{\nu_j^m\}_{j,m}$ *is an orthonormal basis for* $\mathcal{H}_\lambda$ *consisting of eigenfunctions of* $\hat{H}_\lambda$.

(Z) *For every* $A \in \mathcal{A}_0$ *and for every orthonormal basis* $\{\nu_j^m\}_{j,m}$ *for* $\mathcal{H}_\lambda$ *consisting of eigenfunctions of* $\hat{H}_\lambda$, *we have*

$$\lim_{m\to\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \left|\langle A\nu_j^m, \nu_j^m \rangle - \mathrm{vol}(\Sigma_e^\lambda)^{-1}\int_{\Sigma_e^\lambda} \sigma_0(A)\, d\omega_e^\lambda\right| = 0.$$

(C) *For every* A, $\{\nu_j^m\}$ *as in* (Z), *there exists a family* $\{J_m\}_{m\in\mathbf{N}}$ *of subsets in* $\mathcal{N}_m(e,c)$ *satisfying*

$$\lim_{m\to\infty} \frac{\sharp J_m}{N_m(e,c)} = 1$$

such that

$$\lim_{m\to\infty} \max_{j\in J_m} \left|\langle A\nu_j^m, \nu_j^m \rangle - \mathrm{vol}(\Sigma_e^\lambda)^{-1}\int_{\Sigma_e^\lambda} \sigma_0(A)\, d\omega_e^\lambda\right| = 0.$$

Theorem 3.5 *Assume that the conditions* (H1) *and* (H2) *are fulfilled. Then the classical dynamical system* CD_e^λ *is weak-mixing if and only if the following two conditions hold.*

(1) *For every* $A \in \mathcal{A}_0$, $\tau \in \mathbf{R}$ *and every orthonormal basis* $\{\nu_j^m\}_{j,m=1}^\infty$ *for* $\mathcal{H}_\lambda$ *consisting of eigenfunctions of* $\hat{H}_\lambda$, *we have*

$$\lim_{m\uparrow\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \sum_{\substack{k \\ e_k(m)=e_j(m)+\tau}} \left|\langle A\nu_j^m,\ \nu_k^m\rangle\right|^2$$
$$= \left|\mathrm{vol}(\Sigma_e^\lambda)^{-1}\int_{\Sigma_e^\lambda} \sigma_0(A)\, d\omega_e^\lambda\right|^2 \delta_{\tau,0}.$$

(2) *For every* A, τ *and* $\{\nu_j^m\}_{j,m=1}^\infty$ *as above, we have*

$$\lim_{\delta\downarrow 0}\limsup_{m\uparrow\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \sum_{\substack{k \\ 0<|e_k(m)-e_j(m)-\tau|<\delta}} \left|\langle A\nu_j^m,\ \nu_k^m\rangle\right|^2 = 0.$$

Problem C is also still open. However, in Sections 4, 5, we will point out that the periodicity of the classical mechanics, defined as above, relates to the structure of the set of the cluster points in a certain *semi-classical* sense. We will also mention a result which can be considered as an analogy of Helton's theorem [17]. Here we will give the definition of the notion of the cluster point and state the main theorem in Section 4.

For an open interval I and a positive constant $c > 0$, we set

$$N_m(e,c\,;I) = \sharp\{\,(j,k)\in\mathcal{N}_m(e,c)\times\mathbf{N}\,;\, e_k(m)-e_j(m)\in I\,\}.$$

Definition 4.2 *A real number* τ *is said to be the cluster point of the set* $\{\, e_k(m) - e_j(m)\,;\,(j,k)\in\mathcal{N}_m(e,c)\times\mathbf{N},\ m\in\mathbf{Z}\,\}$ *in the semi-classical sense at energy level* e *if, for some constant* $c > 0$,

$$\lim_{m\to\infty} N_m(e,c\,;I) = \infty$$

holds for any open interval I *containing* τ. *We denote by* $s\text{-}D\sigma_e$ *the set of all cluster points at the energy level* e *in the above sense.*

Theorem 4.3 *Assume that the conditions* (H1) *and* (H2) *are satisfied. Then the set* $s\text{-}D\sigma_e$ *of all cluster points in the sense of Definition* 4.2 *is whole real line:*

$$s\text{-}D\sigma_e = \mathbf{R}.$$

In Section 5, we will give some examples for the case where the reduced flow φ_t^λ on Σ_e^λ is periodic.

1 Review of ergodic theory

In this section, we will review briefly some aspects of the classical ergodic theory, and collect several facts which will be used in the following sections.

1.1 Ergodicity and weak-mixing

Let $(\Sigma, \varphi_t, \omega)$ be a dynamical system on a compact manifold Σ, where φ_t is a flow such that $(x,t) \mapsto \varphi_t(x)$ is measurable and ω is an invariant probability measure. For every square integrable function $a \in L^2(\Sigma) = L^2(\Sigma, \omega)$, we denote its space average by $\langle a \rangle$:

$$\langle a \rangle = \int_\Sigma a \, d\omega.$$

For every real number $\tau \in \mathbf{R}$ and positive number $t > 0$, we set

$$a_t(\tau) = \frac{1}{t} \int_0^t e^{-i\tau s} a \circ \varphi_s \, ds \ (\in L^2(\Sigma)).$$

For $\tau = 0$, we will write a_t instead of $a_t(0)$, and we will call it the time average of a up to time $t > 0$.

Theorem 1.1 (von Neumann mean ergodic theorem) *Let $\mathcal{H}$ be a separable Hilbert space, $\{V_t\}_{t \in \mathbf{R}}$ a strongly continuous one-parameter group of unitary operators on $\mathcal{H}$. Let $\mathcal{H}_0$ be the closed subspace in $\mathcal{H}$ consisting of all the vectors invariant under V_t, P the orthogonal projection onto $\mathcal{H}_0$. Then for every $a \in \mathcal{H}$, we have*

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t V_s a \, ds = Pa, \quad in \ \ \mathcal{H}. \tag{1.1}$$

For a proof of Theorem 1.1, we refer the reader to [7]. Applying Theorem 1.1 for $\mathcal{H} = L^2(\Sigma)$ and $V_t a = e^{-i\tau t} a \circ \varphi_t$, the limit $\bar{a}(\tau) := \lim_{t \to \infty} a_t(\tau)$ exists in $L^2(\Sigma)$ for every $a \in L^2(\Sigma)$ and $\tau \in \mathbf{R}$, and it satisfies $\bar{a}(\tau) \circ \varphi_t = e^{i\tau t} \bar{a}(\tau)$. For $\tau = 0$, we will write $\bar{a}$ instead of $\bar{a}(0)$, and call it the long time average of a. Clearly we have $\langle \bar{a} \rangle = \langle a \rangle$ and $\langle \bar{a}(\tau) \rangle = 0$ for $\tau \neq 0$.

Definition 1.1 (1) *The dynamical system $(\Sigma, \varphi_t, \omega)$ is said to be ergodic if, for every $a \in L^2(\Sigma)$, we have $\bar{a} = \langle a \rangle$, a.e.*

(2) *The dynamical system $(\Sigma, \varphi_t, \omega)$ is said to have the weak-mixing property if, for every $a \in L^2(\Sigma)$ and $\tau \in \mathbf{R}$, we have $\bar{a}(\tau) = \langle \bar{a}(\tau) \rangle$, a.e.*

Remark 1.1 The dynamical system $(\Sigma, \varphi_t, \omega)$ is said to have the *mixing property* if

$$\lim_{t\to\infty} \int (a\circ\varphi_t) b\, d\omega = \int a\, d\omega \int b\, d\omega \tag{1.2}$$

for all $a, b \in L^2(\Sigma)$. The mixing property implies weak-mixing property, and the weak-mixing property implies ergodicity. Note that, if the invariant measure of an ergodic system has positive measure for every non-empty open set, then the set of periodic points has measure zero ([36]).

Lemma 1.1 (1) *For every $a \in L^2(\Sigma)$ and $\tau \in \mathbf{R}$, we have the following.*

(i) $\lim_{t\to\infty} \langle\, |a_t(\tau)|^2 \,\rangle = \langle\, |\bar{a}(\tau)|^2 \,\rangle$,

(ii) $\langle\, |\bar{a}(\tau)|^2 \,\rangle \geq |\langle\, a \,\rangle|^2 \delta_{\tau,0}$, *where* $\delta_{\tau,0} = 1$ *if* $\tau = 0$, $\delta_{\tau,0} = 0$ *if* $\tau \neq 0$.

(2) *The dynamical system $(\Sigma, \varphi_t, \omega)$ is ergodic if and only if we have*

$$\langle\, |\bar{a}|^2 \,\rangle = |\langle\, a \,\rangle|^2, \tag{1.3}$$

for every smooth function $a \in C^\infty(\Sigma)$.

(3) *The dynamical system $(\Sigma, \varphi_t, \omega)$ has weak-mixing property if and only if we have*

$$\langle\, |\bar{a}(\tau)|^2 \,\rangle = |\langle\, \bar{a}(\tau) \,\rangle|^2, \tag{1.4}$$

for every $a \in C^\infty(\Sigma)$.

Proof. For $a \in L^2(\Sigma)$, we denote the L^2-norm by $\|a\|$. Clearly we have $\|a\|^2 = \langle\, |a|^2 \,\rangle$. Therefore the inequality $|\|\bar{a}(\tau)\| - \|a_t(\tau)\|| \leq \|\bar{a}(\tau) - a_t(\tau)\|$ implies (1), (i). For any $b \in L^2(\Sigma)$, we obtain the following:

$$\langle\, |b|^2 \,\rangle - |\langle\, b \,\rangle|^2 = \langle\, |b - \langle\, b \,\rangle|^2 \,\rangle. \tag{1.5}$$

Combining this for $b = \bar{a}(\tau)$ with the identity $\langle\, \bar{a}(\tau) \,\rangle = \langle\, a \,\rangle \delta_{\tau,0}$, we obtain (1), (ii). To prove (2), we note that, by the equalities (1.5) and $\langle\, \bar{a} \,\rangle = \langle\, a \,\rangle$, if $(\Sigma, \varphi_t, \omega)$ is ergodic then (1.3) holds for every $a \in L^2(\Sigma)$.

Conversely, assume that (1.3) holds for every $a \in C^\infty(\Sigma)$. Let $b \in L^2(\Sigma)$. One can take a sequence $b_n \in C^\infty(\Sigma)$ such that $\|b_n - b\| \to 0$ as $n \to \infty$. By (1.3) and (1.5), one has $\|\bar{b_n} - \langle\, b_n \,\rangle\| = 0$. Hence one obtains

$$\|\bar{b} - \langle\, b \,\rangle\| \leq \|b - b_n\| + |\langle\, b_n \,\rangle - \langle\, b \,\rangle| \leq 2\|b - b_n\| \to 0 \ (n \to \infty).$$

Therefore $\bar{b} = \langle\, b \,\rangle$, a.e., and hence the dynamical system $(\Sigma, \varphi_t, \omega)$ is ergodic. One can prove (3) in a similar way. ∎

1.2 Homogeneous Lebesgue spectrum

Let $\mathcal{H}$ be the orthogonal complement in $L^2(\Sigma)$ of the one-dimensional subspace of the constant functions, and let $U_t a = a\circ\varphi_t$ for $a \in \mathcal{H}$. The operators $\{U_t\}$ form a strongly continuous one-parameter group of unitary operators. Let

$$U_t = \int e^{itx}\, dE(x) \tag{1.6}$$

be its spectral resolution.

Theorem 1.2 (Hellinger-Hahn) *Let $\mathcal{H}$ be a separable Hilbert space and let E be an spectral measure on $\mathcal{H}$. Then there is a constant κ, $1 \le \kappa \le \infty$, and an orthonormal system $\{h_n\}_{n=1}^{\kappa}$ such that if we set*

$$d\mu_n(x) = d\|E(x)h_n\|^2, \tag{1.7}$$

$$H_n = \left\{a \in \mathcal{H}\,;\, a = \int_{\mathbf{R}} f(x)\, dE(x)h_n,\ f \in L^2(\mathbf{R}, \mu_n)\right\}, \tag{1.8}$$

then the closed subspaces H_n are invariant under the unitary operators $\{U_t\}$, the Hilbert space $\mathcal{H}$ decomposes into the direct sum of the subspaces H_n:

$$\mathcal{H} = \bigoplus_{n=1}^{\kappa} H_n, \tag{1.9}$$

and the measures μ_n satisfy:

$$\mu_1 \gg \mu_2 \gg \mu_3 \gg \cdots, \tag{1.10}$$

where, for two measures μ and ν, $\mu \gg \nu$ means that ν is absolutely continuous with respect to μ. Furthermore, this decomposition is unique in the sense that if another sequence $\{h'_n\}$ satisfies (1.7)–(1.10)*, then the measure μ'_n associated with h'_n is equivalent to the measure μ_n for all n.*

Definition 1.2 *The dynamical system $(\Sigma, \varphi_t, \omega)$ is said to have the homogeneous Lebesgue spectrum with multiplicity κ if the measures μ_n described in Theorem* 1.2 *are equivalent to the Lebesgue measure on* $\mathbf{R}$ *for all n.*

Note that, the above definition of homogeneous Lebesgue spectrum is equivalent to the following, which is adopted as the definition in [23]: there are the subspaces H_n in $\mathcal{H}$ $(1 \le n \le \kappa)$ such that

$$\mathcal{H} = \bigoplus_{n=1}^{\kappa} H_n, \qquad U_t H_n = H_n$$

for all $t \in \mathbf{R}$, and, for each n, there exists an isomorphism Φ_n of the subspace H_n onto $L^2(\mathbf{R}, dx)$ such that, for $a \in H_n$, we have

$$\Phi_n U_t a(x) = (\Phi_n a)(x - t).$$

Remark 1.2 Here we will give the definition of K-system (Kolmogorov system). Let $\mathcal{F}$ be the complete σ-algebra obtained by the completion of the Borel σ-algebra with respect to the measure ω. Then the dynamical system $(\Sigma, \varphi_t, \omega)$ is said to be a *K-system* if there exists a complete σ-subalgebra $\mathcal{F}_0$ satisfying

(1) $\mathcal{F}_0 \subset \varphi_t \mathcal{F}_0$ for all $t > 0$,

(2) $\bigvee_{t \in \mathbf{R}} \varphi_t \mathcal{F}_0 = \mathcal{F}$, where $\bigvee_{t \in \mathbf{R}} \varphi_t \mathcal{F}_0$ is the smallest σ-algebra containing $\varphi_t \mathcal{F}_0$ for all $t \in \mathbf{R}$,

(3) $\bigcap_{t \in \mathbf{R}} \varphi_t \mathcal{F}_0 = \mathcal{F}(\nu)$, where $\mathcal{F}(\nu)$ is the σ-subalgebra of the set of measure 0 or 1.

It is well-known ([23]) that a K-system has homogeneous Lebesgue spectrum, and a system with homogeneous Lebesgue spectrum has mixing property.

1.3 CLT for transitive Anosov flows

In this subsection, we will assume that the map $(p, t) \mapsto \varphi_t(p)$ is smooth. We will also assume that the compact manifold Σ is endowed with a Riemannian metric and the invariant measure ω is absolutely continuous with respect to the Riemannian volume measure. Finally we will assume that $\dim \Sigma \geq 3$.

Definition 1.3 *The flow φ_t is said to be an Anosov flow if the following are satisfied.*

(1) *The vector field X generating the flow φ_t does not vanish.*

(2) *For every point $p \in \Sigma$, the tangent space $T_p\Sigma$ splits into the direct sum*

$$T_p\Sigma = E^0(p) \oplus E^c(p) \oplus E^e(p),$$

where $E^0(p)$ is the one-dimensional subspace spanned by X_p, $\dim E^c(p) = k \neq 0$, $\dim E^c(p) = k \neq 0$, and the subspaces E^c and E^e satisfy that there are constants α, β, $\gamma > 0$ independent of $p \in \Sigma$ such that, for every $p \in \Sigma$ and $t > 0$, we have the following.

(i) *Each $v \in E^c(p)$ satisfies*

$$\|d\varphi_t v\| \leq \alpha e^{-\gamma t} \|v\|, \quad \|d\varphi_{-t} v\| \geq \beta e^{\gamma t} \|v\|,$$

(ii) *Each* $v \in E^e(p)$ *satisfies*

$$\|d\varphi_t v\| \geq \beta e^{\gamma t}\|v\|, \quad \|d\varphi_{-t} v\| \leq \alpha e^{-\gamma t}\|v\|.$$

It is well-known ([1], [2]) that the constants k, l are independent of $p \in \Sigma$. It is also well-known that the tangent distributions $p \mapsto E^c(p)$, $E^e(p)$ are continuous and completely integrable, and hence these generate the foliations $\mathcal{F}^c$, $\mathcal{F}^e$ whose leaves are C^1-manifolds.

Definition 1.4 *The Anosov flow* φ_t *is said to be transitive if the leaves of the foliations* $\mathcal{F}^c$, $\mathcal{F}^e$ *are dense in* Σ.

Remark 1.3 Though an Anosov flow is automatically ergodic ([1]), it does not necessarily have weak-mixing property. However, it is known ([25]) that a transitive Anosov flow with an invariant measure absolutely continuous with respect to a Riemannian volume measure is a K-system, and hence has homogeneous Lebesgue spectrum.

For a real-valued function $a \in L^\infty(\Sigma)$, let

$$\omega_t = \left(\frac{t}{\sqrt{D_t(a)}}(a_t - \langle\, a \,\rangle) \right) * \omega \tag{1.11}$$

be the push-forward measure on $\mathbf{R}$ of the measure ω, where $D_t(a)$ denotes the variance of a:

$$D_t(a) = t^2 \langle\, |a_t - \langle\, a \,\rangle|^2 \,\rangle. \tag{1.12}$$

Definition 1.5 *A real-valued function* $a \in L^\infty(\Sigma)$ *is said to obey the central limit theorem* (CLT) *relative to the flow* φ_t *if the measure* ω_t *on* $\mathbf{R}$ *converges* (*in the dual space of the space of bounded continuous functions on* $\mathbf{R}$) *weakly to the Gaussian distribution* $(2\pi)^{-1/2}e^{-x^2/2}dx$.

Equivalently, a real-valued function $a \in L^\infty(\Sigma)$ obeys the CLT if and only if

$$\lim_{t\to\infty} \omega \left(z \in \Sigma\, ; \; \frac{t(a_t(z) - \langle\, a \,\rangle)}{\sqrt{D_t(a)}} < \alpha \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-x^2/2}\, dx. \tag{1.13}$$

In Section 2, we will use the following theorems.

Theorem 1.3 (Ratner [21]) *Let* φ_t *be a transitive Anosov flow and let* X *be the vector field generating the flow* φ_t*. Then, for every real-valued function* $a \in C^\infty(\Sigma)$*, we have the following.*

(i) *If the equation* $a - \langle a \rangle = Xb$ *has no solution in* $L^2(\Sigma)$, *then there is a positive constant* $\sigma = \sigma_a$ *such that* $D_t(a) \sim \sigma_a t$ *and the function* a *obeys the* CLT.

(ii) *If the equation* $a - \langle a \rangle = Xb$ *has a solution in* $L^2(\Sigma)$, *then* $D_t(a) = O(1)$.

Theorem 1.4 (Zelditch [41]) *Let* φ_t *be a transitive Anosov flow. Then for every* $a \in C^\infty(\Sigma)$ *and every positive integer* k, *we have*

$$\langle\, |a_t - \langle a \rangle|^{2k} \,\rangle = O(t^{-k}). \tag{1.14}$$

Remark 1.4 The geodesic flow on a compact Riemannian manifold of (possibly variable) negative curvature is a transitive Anosov flow ([1]), and hence we can apply Theorems 1.3, 1.4 with the Liouville measure.

2 Off-diagonal asymptotics in quantum ergodicity

In this section, we will consider Problem A in Introduction.

Before going to discuss this problem, we will describe the notion of *quantum ergodicity* and *quantum weak-mixing at infinite energy level* introduced by Sunada [28] and Zelditch [42], which are quantum analogies of Lemma 1.1.

2.1 Quantum ergodicity and weak-mixing

Let M be a compact connected Riemannian manifold without boundary and let $\hat{H}$ be a first order self-adjoint non-negative elliptic pseudodifferential operator (ψDO for short) on M with positive principal symbol $H > 0$. H is a smooth homogeneous function of degree one on the punctured cotangent bundle $T^*M \setminus 0$. Let φ_t be the Hamilton flow generated by the Hamiltonian H and the canonical symplectic form Ω_M. Let ω be the Liouville measure on $\Sigma := H^{-1}(1)$. Since the flow φ_t can be restricted on Σ, we have the classical dynamical system $(\Sigma, \varphi_t, \omega)$.

Let $0 \le e_1 \le e_2 \le \cdots \uparrow \infty$ be the eigenvalues of $\hat{H}$ counting with the repetition according to the multiplicity, and let $N(\lambda) = \sharp\{j \in \mathbf{N}\,;\, e_j \le \lambda\}$. Let $\mathcal{A}_0$ be the set of all ψDO's of order zero on M, which is considered as the $*$-algebra of quantum observables. We denote by $\sigma_0(A)$ the principal symbol of $A \in \mathcal{A}_0$.

The following theorem is due to Sunada [28].

Theorem 2.1 (Sunada) *The dynamical system $(\Sigma, \varphi_t, \omega)$ is ergodic if and only if the following conditions hold. .*

(1) *For every $A \in \mathcal{A}_0$ and every orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ for $L^2(M)$ consisting of eigenfunctions of $\hat{H}$, we have*

$$\lim_{\lambda\to\infty} N(\lambda)^{-1} \sum_{\substack{j,k \\ e_j = e_k \le \lambda}} |\langle A\varphi_j,\, \varphi_k\rangle|^2 = \left|\int_\Sigma \sigma_0(A)\, d\omega\right|^2. \tag{2.1}$$

(2) *For every A, $\{\varphi_j\}$ as above, we have*

$$\lim_{\delta\to 0} \limsup_{\lambda\to\infty} N(\lambda)^{-1} \sum_{\substack{j \\ e_j \le \lambda}} \sum_{\substack{k \\ 0 < |e_j - e_k| < \delta}} |\langle A\varphi_j,\, \varphi_k\rangle|^2 = 0. \tag{2.2}$$

In [28], (2.1) and (2.2) are called *near-diagonal* and *off-diagonal asymptotics*, respectively. Zelditch [42] obtained the following theorem by using the method of [28].

Theorem 2.2 (Zelditch) *The dynamical system $(\Sigma, \varphi_t, \omega)$ has weak-mixing property if and only if the following conditions hold.*

(1) *For every $A \in \mathcal{A}_0$, $\tau \in \mathbf{R}$ and every orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ for $L^2(M)$ consisting of eigenfunctions of $\hat{H}$, we have*

$$\lim_{\lambda \to \infty} N(\lambda)^{-1} \sum_{\substack{j \\ e_j \leq \lambda}} \sum_{\substack{k \\ e_k = e_j + \tau}} |\langle A\varphi_j, \varphi_k \rangle|^2 = \left| \int_\Sigma \sigma_0(A)\, d\omega \right|^2 \delta_{\tau,0}. \tag{2.3}$$

where $\delta_{\tau,0} = 0$ if $\tau \neq 0$, $\delta_{\tau,0} = 1$ if $\tau = 0$.

(2) *For every A, τ and $\{\varphi_j\}$ as above, we have*

$$\lim_{\delta \to 0} \limsup_{\lambda \to \infty} N(\lambda)^{-1} \sum_{\substack{j \\ e_j \leq \lambda}} \sum_{\substack{k \\ 0 < |e_k - e_j - \tau| < \delta}} |\langle A\varphi_j, \varphi_k \rangle|^2 = 0. \tag{2.4}$$

Sunada introduced the notion of *quantum ergodicity at infinite energy level* and obtained Theorem 2.1 by examining the relationship between this notion and classical ergodicity. We will describe this notion briefly.

For a bounded operator A on $L^2(M)$, we define the bounded operator $A_t(\tau)$ ($\tau \in \mathbf{R}$, $t > 0$) by

$$A_t(\tau) = \frac{1}{t} \int_0^t e^{-is\tau} e^{is\hat{H}} A e^{-is\hat{H}}\, ds. \tag{2.5}$$

We will write A_t instead of $A_t(0)$ and call it the time average of A up to time $t > 0$. The bounded operator $A_t(\tau)$ converges weakly as $t \to \infty$ to the bounded operator $\bar{A}(\tau)$ given by

$$\bar{A}(\tau) = \sum_{\substack{e, \\ e, e+\tau \in \mathrm{Spec}(\hat{H})}} P_{e+\tau} A P_e, \tag{2.6}$$

where P_e is the orthogonal projection onto the eigenspace with the eigenvalue $e \in \mathrm{Spec}(\hat{H})$. We will also write $\bar{A}$ instead of $\bar{A}(0)$ and call it the long time average of A. The quantum space average $\langle A \rangle$ of the bounded operator A is defined by

$$\langle A \rangle = \lim_{\lambda \to \infty} N(\lambda)^{-1} \sum_{\substack{j \\ e_j \leq \lambda}} \langle A\varphi_j, \varphi_j \rangle, \tag{2.7}$$

if the limit in (2.7) exists. The statements (1) and (2) in the following lemma is well-known and these are called Szegö limit formula and Egorov theorem, respectively. See [10], [34] for the proof of the following lemma.

Lemma 2.1 *For every quantum observable $A \in \mathcal{A}_0$, we have the following.*

(1) *The space average* $\langle A \rangle$ *of* A *exists and equals the space average* $\langle \sigma_0(A) \rangle$ *of its principal symbol* $\sigma_0(A) \in C^\infty(\Sigma)$.

(2) *The bounded operator* $e^{it\hat{H}}Ae^{-it\hat{H}}$, *and hence* $A_t(\tau)$, *is a* ψDO *of order zero and their principal symbol are given by* $U_t\sigma_0(A)$ *and* $\sigma_0(A)_t(\tau)$, *respectively.*

Now we will consider the triple $(\mathcal{V}, \hat{H}, \mathcal{A}_0)$ as the quantum dynamical system associated with the classical dynamical system $(\Sigma, \varphi_t, \omega)$, where $\mathcal{V}$ is the unit sphere in $L^2(M)$.

Definition 2.1 (Sunada, Zelditch) (1) *The quantum dynamical system* $(\mathcal{V}, \hat{H}, \mathcal{A}_0)$ *is said to be quantum ergodic at infinite energy level if, for every zeroth order* ψDO $A \in \mathcal{A}_0$, *the quantity* $\langle \bar{A}^*\bar{A} \rangle$ *exists and satisfies*

$$\langle \bar{A}^*\bar{A} \rangle = |\langle A \rangle|^2. \tag{2.8}$$

(2) *The quantum dynamical system* $(\mathcal{V}, \hat{H}, \mathcal{A}_0)$ *is said to be quantum weak-mixing at infinite energy level if, for every* ψDO $A \in \mathcal{A}_0$ *of order zero and every* $\tau \in \mathbf{R}$, *the quantity* $\langle \bar{A}(\tau)^*\bar{A}(\tau) \rangle$ *exists and satisfies*

$$\langle \bar{A}(\tau)^*\bar{A}(\tau) \rangle = |\langle A \rangle|^2\delta_{\tau,0}. \tag{2.9}$$

The following lemma is due to Sunada [28], which is used in the next subsection.

Lemma 2.2 *the dynamical system* $(\Sigma, \varphi_t, \omega)$ *is ergodic if and only if the following two conditions hold.*

(1) *The quantum dynamical system* $(\mathcal{V}, \hat{H}, \mathcal{A}_0)$ *is quantum ergodic at infinite energy level.*

(2) *For every quantum observable* $A \in \mathcal{A}_0$, *we have* $\lim_{t\to\infty} \langle A_t^*A_t \rangle = \langle \bar{A}^*\bar{A} \rangle$.

Sunada [28] also obtained the following theorem.

Theorem 2.3 *The quantum dynamical system* $(\mathcal{V}, \hat{H}, \mathcal{A}_0)$ *is quantum ergodic at infinite energy level if and only if for every orthonormal basis* $\{\varphi_j\}$ *of eigenfunction of* $\hat{H}$ *there exists a subsequence* $J \subset \mathbf{N}$ *such that*

$$\lim_{\lambda\to\infty} N(\lambda)^{-1}\sharp\{j \in J\,;\, e_j \leq \lambda\} = 1, \tag{2.10}$$

and for every $A \in \mathcal{A}_0$ *we have*

$$\lim_{j\in J,\, j\to\infty} \langle A\varphi_j,\, \varphi_j \rangle = \int_\Sigma \sigma_0(A)\, d\omega. \tag{2.11}$$

The results of Shnirelman [26], Zelditch [37] and Colin de Verdierè [6] follow from Theorem 2.3 and Lemma 2.2.

2.2 More on the off-diagonal asymptotics

Theorems 2.1, 2.2 shows that the ergodicity or weak-mixing property of classical mechanics have an impact on the asymptotic behavior of the matrix elements $\langle A\varphi_j, \varphi_k \rangle$ of an observable $A \in \mathcal{A}_0$. Zelditch [41] showed that, in case where the system $(\Sigma, \varphi_t, \omega)$ has more chaotic property than ergodicity, the near-diagonal asymptotics (2.1) has a logarithmic order. That is, he obtained the following.

Theorem 2.4 (Zelditch) *Let M be a compact Riemannian manifold of negative curvature, and let $\hat{H}$ be the square root of the Laplacian on M. Then, for every $A \in \mathcal{A}_0$ and positive integer k, we have*

$$N(\lambda)^{-1} \sum_{e_j \leq \lambda} |\langle A\varphi_j, \varphi_j \rangle - \langle \sigma_0(A) \rangle|^k = O((\log \lambda)^{-k/2}). \tag{2.12}$$

By Theorems 2.1, 2.3, we know that the left hand side of (2.12) tends to 0 as $\lambda \to \infty$ if the dynamical system $(\Sigma, \varphi_t, \omega)$ is ergodic. Zelditch used the moment estimates for transitive Anosov flows (Theorems 1.3, 1.4) to prove this theorem. Our main theorems state that the dynamical assumption on $(\Sigma, \varphi_t, \omega)$ also affects the off-diagonal asymptotics (2.2). That is we have the following.

Theorem 2.5 *Assume that the Hamilton flow φ_t on Σ is transitive Anosov. Then for every ψDO $A \in \mathcal{A}_0$ of order zero, we have*

$$\limsup_{\lambda \to \infty} N(\lambda)^{-1} \sum_{\substack{j \\ e_j \leq \lambda}} \sum_{\substack{k \\ 0 < |e_j - e_k| < \delta}} |\langle A\varphi_j, \varphi_k \rangle|^2 = O(\delta). \tag{2.13}$$

Theorem 2.6 *Assume that the dynamical system $(\Sigma, \varphi_t, \omega)$ has homogeneous Lebesgue spectrum. Then, for every ψDO $A \in \mathcal{A}_0$ of order zero with $\langle A \rangle = 0$ there exists an integrable function p_A on $\mathbf{R}$ such that, for any $a < b$, we have*

$$\lim_{\lambda \to \infty} N(\lambda)^{-1} \sum_{e_j \leq \lambda} \sum_{\substack{k \\ a < e_k - e_j < b}} |\langle A\varphi_j, \varphi_k \rangle|^2 = \int_a^b p_A(\lambda)\, d\lambda. \tag{2.14}$$

We note that, Theorem 2.6 has proved in [42] for $\hat{H}$ the square root of the Laplacian on a compact hyperbolic manifolds. Zelditch has proved that, on a compact hyperbolic manifold, one can take the function p_A to be smooth. He has used the fact that the correlation functions have an exponential decay. However the assumption of Theorem 2.6 is somewhat weak. Indeed, there is a metric on the two-dimensional sphere whose geodesic flow is a K-system ([3]), and hence it has homogeneous Lebesgue spectrum.

2.3 Proof of Theorem 2.5

We will give a proof of Theorem 2.5. We note that the Liouville measure ω is equivalent to the Riemannian volume measure, and hence we can apply Theorem 1.4. We also note that the dynamical system $(\Sigma, \varphi_t, \omega)$ is ergodic.

Let $A \in \mathcal{A}_0$ be a ψDO of order zero. We may assume that A is self-adjoint, and hence its principal symbol $\sigma_0(A)$ is real. For each $k \in \mathbf{N}$, we set

$$R_k(A,t) = \langle A_t^k \rangle - \langle A \rangle^k. \tag{2.15}$$

Since $\langle A_t \rangle = \langle A \rangle$, we have

$$R_k(A,t) = \sum_{r=2}^{k} \begin{pmatrix} k \\ r \end{pmatrix} \langle (A_t - \langle A \rangle)^r \rangle \langle A \rangle^{k-r}. \tag{2.16}$$

By Lemma 2.1 and Theorem 1.4, if the exponent r is even then we obtain

$$\langle (A_t - \langle A \rangle)^r \rangle = \langle (\sigma_0(A)_t - \langle \sigma_0(A) \rangle)^r \rangle = O\left(\left(\frac{1}{t}\right)^{r/2}\right). \tag{2.17}$$

If r is odd then, by Cauchy-Schwarz inequality, we have

$$\langle (A_t - \langle A \rangle)^r \rangle \leq \langle (A_t - \langle A \rangle)^{2r} \rangle^{1/2} = O\left(\left(\frac{1}{t}\right)^{r/2}\right). \tag{2.18}$$

Applying these estimates to the terms in the sum of the expression (2.16), we have

$$R_k(A,t) = O\left(\frac{1}{t}\right) \tag{2.19}$$

for all $k \in \mathbf{N}$. We use this estimate for $k = 2$. A direct calculation leads us to

$$A_t \varphi_j = \frac{1}{t} \sum_{\substack{k \\ e_k \neq e_j}} \frac{e^{it(e_k - e_j)} - 1}{i(e_k - e_j)} \langle A\varphi_j,\, \varphi_k \rangle \varphi_k + \sum_{\substack{k \\ e_j = e_k}} \langle A\varphi_j,\, \varphi_k \rangle \varphi_k, \tag{2.20}$$

and hence

$$\begin{aligned} \langle A_t^2 \rangle \;\; &= \;\; \lim_{\lambda \to \infty} N(\lambda)^{-1} \sum_{e_j \leq \lambda} \sum_{\substack{k \\ e_j \neq e_k}} S(t(e_k - e_j)) |\langle A\varphi_j,\, \varphi_k \rangle|^2 \\ &\quad + \lim_{\lambda \to \infty} N(\lambda)^{-1} \sum_{e_j \leq \lambda} \sum_{\substack{k \\ e_j = e_k}} |\langle A\varphi_j,\, \varphi_k \rangle|^2, \end{aligned} \tag{2.21}$$

where, $S(x) = (|e^{ix} - 1|/x)^2 = (\sin^2 x/2)/(x/2)^2$. Note that the second term of (2.21) equals $\langle \bar{A}^* \bar{A} \rangle$. Therefore, applying Lemma 2.2, we obtain

$$R_2(A,t) = \lim_{\lambda \to \infty} N(\lambda)^{-1} \sum_{e_j \leq \lambda} \sum_{\substack{k \\ e_j \neq e_k}} S(t(e_k - e_j)) |\langle A\varphi_j,\, \varphi_k \rangle|^2 \tag{2.22}$$

Now, we can take $\delta_0 > 0$ so that $|x| < \delta_0$ implies $1/2 < R(x)$. Then

$$\begin{aligned}\limsup_{\lambda\to\infty} N(\lambda)^{-1} \sum_{e_j \le \lambda} \sum_{\substack{k \\ 0<|e_k-e_j|<\delta_0/t}} |\langle A\varphi_j,\, \varphi_k \rangle|^2 &\le 2R_2(A,t) \\ &= O\left(\left(\frac{1}{t}\right)\right),\end{aligned}$$

which completes the proof of Theorem 2.5. ▮

2.4 Proof of Theorem 2.6

Before proceeding to the proof of Theorem 2.6, we have to mention the *spectral measure lemma* established by Zelditch [42], [43].

Let A be a bounded operator on $L^2(M)$. For every compactly supported smooth function $f \in C_0^\infty(\mathbf{R})$ on $\mathbf{R}$, we set

$$m_A(f) := \lim_{\lambda\to\infty} N(\lambda)^{-1} \sum_{e_j\le\lambda} \sum_k f(e_k - e_j)\, |\langle A\varphi_j,\, \varphi_k \rangle|^2 \,, \tag{2.23}$$

if the limit exists.

Lemma 2.3 *Let $A \in \mathcal{A}_0$. Then the limit in* (2.23) *exists for all $f \in C_0^\infty(\mathbf{R})$.*

Proof. If a bounded operator B is approximated by ψDO's in the operator norm, then the space average $\langle B \rangle$ exists. Indeed, for a bounded operator B and a positive number λ, we set

$$S(B;\lambda) = N(\lambda)^{-1} \sum_{\substack{j \\ e_j\le\lambda}} \langle B\varphi_j,\, \varphi_j \rangle. \tag{2.24}$$

Then, by definition, we have $\langle B \rangle = \lim_{\lambda\to\infty} S(B;\lambda)$ if the limit exists. By (2.24), we have $S(B;\lambda) \le \|B\|$. Assume that $B_n \in \mathcal{A}_0$ and $\|B_n - B\| \to 0$. Then we have $|\langle B_n \rangle - \langle B_m \rangle| \le \|B_n - B_m\| \to 0$ as $n, m \to \infty$, and hence the limit $c = \lim_{n\to\infty} \langle B_n \rangle$ exists. Therefore

$$\begin{aligned}|c - S(B;\lambda)| &\le |c - \langle B_n \rangle| + |\langle B_n \rangle - S(B_n;\lambda)| + |S(B_n;\lambda) - S(B;\lambda)| \\ &\le |c - \langle B_n \rangle| + |\langle B_n \rangle - S(B_n;\lambda)| + \|B_n - B\| \end{aligned} \tag{2.25}$$

for all $n \in \mathbf{N}$. Letting $\lambda \to \infty$ in (2.25), we obtain

$$\limsup_{\lambda\to\infty} |c - S(B;\lambda)| \le |c - \langle B_n \rangle| + \|B_n - B\|. \tag{2.26}$$

Since (2.26) holds for arbitrary n, we conclude that $\langle B \rangle$ exists and equals $c = \lim_{n\to\infty} \langle B_n \rangle$.
For $A \in \mathcal{A}_0$, $f \in C_0^\infty(\mathbf{R})$ and $R > 0$, we set

$$A_f = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(t) e^{it\hat{H}} A e^{-it\hat{H}}\, dt, \tag{2.27}$$

$$A_f^R = \frac{1}{2\pi}\int_{-R}^{R} \hat{f}(t) e^{it\hat{H}} A e^{-it\hat{H}}\, dt, \tag{2.28}$$

where $\hat{f}$ is the Fourier transform of f. By Lemma 2.1 (2), the operator A_f^R is a ψDO of order zero for $R > 0$, and converges to the bounded operator A_f as $R \to +\infty$ in the operator norm. Therefore the quantity $\langle A^* A_f \rangle$ exists and equals $\lim\limits_{R\to\infty} \langle A^* A_f^R \rangle$. By a direct computation using the spectral decomposition for the operator $e^{it\hat{H}}$, we have

$$S(A^* A_f; \lambda) = N(\lambda)^{-1} \sum_{e_j \leq \lambda} \sum_k f(e_k - e_j) \left| \langle A\varphi_j, \varphi_k \rangle \right|^2, \tag{2.29}$$

and hence $\langle A^* A_f \rangle = m_A(f)$. ▮

By Lemma 2.3, m_A is well-defined as a positive linear functional on $C_0^\infty(\mathbf{R})$, and hence it defines a Borel measure on $\mathbf{R}$. We also denote this measure by m_A. We note that, by the inequality $m_A(f) \leq \|A\|^2 \|f\|_\infty$ ($f \in C_0^\infty(\mathbf{R})$), the measure m_A is a finite measure.

There is another finite measure on $\mathbf{R}$ associated with a ψDO $A \in \mathcal{A}_0$. Let $\tilde{U}_t$ be the one-parameter group of unitary operators on $L^2(\Sigma)$ defined by $\tilde{U}_t a = a \circ \varphi_t$. Let

$$\tilde{U}_t = \int e^{itx} d\tilde{E}(x) \tag{2.30}$$

be the spectral resolution for $\tilde{U}_t$. Note that the one-parameter group of unitary operators U_t defined in Section 1.2 is the restriction of $\tilde{U}_t$ to the closed subspace $\mathcal{H}$ which is the orthogonal complement of the one-dimensional subspace of the constant functions. Thus we have $P\tilde{U}_t = U_t P$, where P is the orthogonal projection onto $\mathcal{H}$: $Pa = a - \langle a \rangle$, $a \in L^2(\Sigma)$. For every $a \in L^2(\Sigma)$, let $\tilde{\mu}_a$ be the spectral measure associated with a and $\tilde{E}$, that is $\tilde{\mu}_a(\Lambda) = \|\tilde{E}(\Lambda) a\|^2$ for every Borel set $\Lambda \subset \mathbf{R}$. Since a constant function is an eigenfunction of $\tilde{U}_t$ with eigenvalue 1, we have

$$\mu_{Pa}(\Lambda) = \tilde{\mu}_a(\Lambda) - |\langle a \rangle|^2 \delta_0, \tag{2.31}$$

where $\mu_{Pa}(\Lambda) = \|E(\Lambda) Pa\|^2$ is the spectral measure associated with $Pa \in \mathcal{H}$ and E, and δ_0 is the Dirac measure. The following lemma is called *spectral measure lemma*, due to Zelditch [42], [43].

Lemma 2.4 (Zelditch) *Let $A \in \mathcal{A}_0$ be a ψDO of order zero. Then we have*

$$m_A = \tilde{\mu}_{\sigma_0(A)} = \mu_{P\sigma_0(A)} + |\langle A \rangle|^2 \delta_0 \tag{2.32}$$

Proof. By lemma 2.1, we have $(\tilde{U}_t \sigma_0(A),\, \sigma_0(A))_{L^2(\Sigma)} = \langle\, A^* e^{it\hat{H}} A e^{-it\hat{H}} \,\rangle$. Therefore we obtain

$$\begin{aligned}\int f\, d\tilde{\mu}_{\sigma_0(A)} &= \frac{1}{2\pi}\int \hat{f}(t)(\tilde{U}_t\sigma_0(A),\, \sigma_0(A))_{L^2(\Sigma)}\, dt \\ &= \frac{1}{2\pi}\int \hat{f}(t)\langle\, A^* e^{it\hat{H}} A e^{-it\hat{H}} \,\rangle\, dt \\ &= \langle\, A^* A_f \,\rangle \\ &= \int f\, dm_A,\end{aligned}$$

for every $f \in C_0^\infty(\mathbf{R})$, which completes the proof. ▮

Proof of Theorem 2.6 Let $A \in \mathcal{A}_0$ with $\langle\, A \,\rangle = \langle\, \sigma_0(A) \,\rangle = 0$, and let h_n, μ_n be the orthonormal system and the measures described in the Hellinger-Hahn theorem (Theorem 1.2), respectively. By the decomposition (1.9), there are functions $g_{n,A} \in L^2(\mu_n)$ such that

$$\sigma_0(A) = \sum_n \int g_{n,A}(x)\, dE(x) h_n. \tag{2.33}$$

Then we have

$$\mu_{\sigma_0(A)}(\Lambda) = \sum_n \int \chi_\Lambda(x)|g_{n,A}(x)|^2\, d\mu_n(x). \tag{2.34}$$

Since $\{U_t\}$ has homogeneous Lebesgue spectrum, the measure μ_n is equivalent to Lebesgue measure, and hence, for each n, there is a function $e_n \in L^1(\mathbf{R}, dx)$ such that $e_n > 0$ (dx-a.e.) and $d\mu_n(x) = e_n(x)dx$. Therefore we obtain

$$\mu_{\sigma_0(A)}(\Lambda) = \sum_n \int \chi_\Lambda(x)|g_{n,A}(x)|^2 e_n(x)\, dx. \tag{2.35}$$

The above equation (2.35) implies that the measure $\mu_{\sigma_0(A)}$ is absolutely continuous with respect to Lebesgue measure. Combining this with Lemma 2.4, we conclude that the measure dm_A is absolutely continuous with respect to Lebesgue measure.

To deduce (2.14) from the assertion proved as above, we need to take care because it is not trivial that one can replace f in (2.23) by the characteristic function χ of the interval (a, b). However it can be shown as follows. Let $0 \le f_n \le 1$ be a smooth function on $\mathbf{R}$ such that $f_n = 1$ on the interval $[a + 1/n, b - 1/n]$ and $f_n = 0$ on the outside of the interval (a, b). By Lebesgue convergence theorem one has

$$\int \chi\, dm_A = \lim_{n\to\infty} \int f_n\, dm_A. \tag{2.36}$$

For a function f on $\mathbf{R}$, set

$$S_A(f, \lambda) = N(\lambda)^{-1} \sum_{e_j \le \lambda} \sum_k f(e_k - e_j)|\langle\, A\varphi_j,\, \varphi_k \,\rangle|^2. \tag{2.37}$$

Then one obtain

$$\begin{aligned}
&\left|\int \chi\, dm_A - S_A(\chi,\lambda)\right| \\
&\quad \leq \left|\int \chi\, dm_A - \int f_n\, dm_A\right| + \left|\int f_n\, dm_A - S_A(f_n,\lambda)\right| \\
&\quad\quad + \left|S_A(f_n,\lambda) - S_A(\chi,\lambda)\right|. \qquad (2.38)
\end{aligned}$$

By the typical choice of the function f_n as above, one can easily deduce that

$$\begin{aligned}
|S_A(f_n,\lambda) - S_A(\chi,\lambda)| \;\leq\; & N(\lambda)^{-1} \sum_{e_j\leq\lambda} \sum_{\substack{k \\ 0<e_k-e_j-a<1/n}} |\langle\, A\varphi_j,\, \varphi_k\,\rangle|^2 \\
& + N(\lambda)^{-1} \sum_{e_j\leq\lambda} \sum_{\substack{k \\ -1/n<e_k-e_j-b<0}} |\langle\, A\varphi_j,\, \varphi_k\,\rangle|^2 . \qquad (2.39)
\end{aligned}$$

Note that the dynamical system with homogeneous Lebesgue spectrum has mixing, and hence weak-mixing, property. So, after taking the $\limsup$ in λ in the inequality (2.39), one can apply quantum weak-mixing theorem (Theorem 2.2) to the right hand side of (2.39). Thus one obtains that the $\limsup$ in λ of the third term of (2.38) tends to zero as n goes to infinity. Combining this and (2.36), one concludes that the left hand side of (2.38) tends to zero as λ goes to infinity. From this, (2.14) follows. ▌

3 Quantum ergodicity at a finite energy level

This section is devoted to the discussion on Problem B in Introduction.

In Section 2.1, we have described the notion of quantum ergodicity at infinite energy level. The classical system investigated in Section 2 is homogeneous Hamilton flow, that is the flow which commutes with $\mathbf{R}_+$-action on the cotangent bundle. However, there are natural classical systems which are not homogeneous. For example, the magnetic flow under the uniform magnetic field on a compact Riemann surface with constant negative curvature -1 has different behavior on different energy surfaces. (See [14], [29]. See also Section 5.) This phenomenon arises from the effect of the magnetic field. Ergodicity of such dynamical systems affects the *semi-classical* asymptotic behavior of the eigenfunctions of corresponding quantum Hamiltonian ([18], [27], [39]).

Our purposes of this section are to formulate a notion of quantum ergodicity for the quantum mechanics corresponding to the classical system such a magnetic flow by using Sunada's method described in Section 2, and investigate the relationship between classical and quantum ergodicity. We will call the notion introduced in this section quantum ergodicity *at a finite energy level* because we take the dependence of dynamical behavior on the energy level into consideration.

We will give a brief account of the dynamical system discussed in this section. The precise formulation of the dynamical system, which is the same as in [39], is described in the following subsection.

We note that the magnetic flow is obtained by the *reduction* of the geodesic flow on a compact S^1-bundle with a connection 1-form and with a Riemannian metric which is invariant under S^1-action. (See Section 5 for the definition of the magnetic flow.) In this case, the magnetic field is represented by the curvature 2-form of a connection form. Accordingly, we will consider the reduced dynamical system of the Hamilton flow generated by the Hamiltonian which is invariant under the group action on the cotangent bundle over a compact principal bundle. The corresponding quantum mechanics is generated by a first order positive elliptic pseudodifferential operator (ψDO for short) which commutes with the group action. However, as the case of classical mechanics, we need to consider a *reduced* quantum mechanics. More precisely, we consider the operator restricted to a *ladder subspace* associated with a fixed irreducible representation of the structure group as a reduced quantum Hamiltonian. We will define the notion of quantum ergodicity at a finite energy level for the quantum mechanics generated by the reduced Hamiltonian. To study the relationship between classical and quantum ergodicity, we will use the semi-classical trace formula due to Guillemin–Uribe [14], [15] and Zelditch [39]. We will also describe the notion of *quantum weak-mixing at a finite energy level*, which is a semi-classical analogy of Definition 2.1, (2).

3.1 Formulation of dynamical systems

Let $\pi : P \to M$ be a compact connected principal bundle over a compact Riemannian manifold $(M, \langle\, ,\, \rangle_M)$ with structure group G, a compact connected Lie group. We fix a connection 1-form Θ on P and an adjoint-invariant inner product $\langle\, ,\, \rangle_{\mathcal{G}}$ on the Lie algebra $\mathcal{G}$ of the Lie group G. We define the Riemannian metric $\langle\, ,\, \rangle_P$ by

$$\langle\, u, v\,\rangle_P = \langle\, d\pi(u), d\pi(v)\,\rangle_M + \langle\, \Theta(u), \Theta(v)\,\rangle_{\mathcal{G}}, \quad u, v \in TP. \tag{3.1}$$

Let $\hat{H}$ be a first order self-adjoint non-negative elliptic ψDO on P commuting with G-action and let $H = \sigma_1(\hat{H})$ be its principal symbol. Since $\hat{H}$ commutes with G-action, H is a G-invariant smooth function on the punctured cotangent bundle $T^*P \setminus 0$. We will assume that the principal symbol H is positive. The (left) action of G on T^*P is Hamiltonian and its equivariant moment map $\Phi : T^*P \to \mathcal{G}^*$ is given by

$$\langle \Phi(p, \zeta), A\rangle = \zeta(A^*_p), \quad (p, \zeta) \in T^*P, A \in \mathcal{G}, \tag{3.2}$$

where $\mathcal{G}^*$ is the dual space of the Lie algebra $\mathcal{G}$ of G. Let (π_λ, V_λ) be an irreducible representation of G with the highest weight λ in the positive Weyl chamber of a dual Cartan subalgebra, and let $\mathcal{O}_\lambda$ be the coadjoint orbit through λ. Since the differential map $d\Phi$ of the moment map Φ is surjective at each point, $\Phi^{-1}(\mathcal{O}_\lambda)$ is a submanifold in T^*P, and G acts freely on it. The leaves of the null-foliation of the G-invariant closed 2-form $\iota^*_\lambda\Omega_P - \Phi^*\omega_\lambda$ on $\Phi^{-1}(\mathcal{O}_\lambda)$ are just the G-orbits, where Ω_P is the canonical symplectic form on T^*P, $\iota_\lambda : \Phi^{-1}(\mathcal{O}_\lambda) \hookrightarrow T^*P \setminus 0$ is the inclusion, and ω_λ is the Kostant-Kirillov symplectic form on $\mathcal{O}_\lambda$. Thus it induces the symplectic form Ω_λ on $X_\lambda = \Phi^{-1}(\mathcal{O}_\lambda)/G$. The symplectic manifold $(X_\lambda, \Omega_\lambda)$ is called the *reduced* phase space. The G-invariant Hamiltonian H defines the Hamiltonian H_λ on X_λ such that H_λ satisfies the relation $q^*_\lambda H_\lambda = H$ on $\Phi^{-1}(\mathcal{O}_\lambda)$, where $q_\lambda : \Phi^{-1}(\mathcal{O}_\lambda) \to X_\lambda$ is the projection. Let φ^λ_t denotes the Hamilton flow generated by $(H_\lambda, \Omega_\lambda)$. The flow φ^λ_t can be restricted on the energy surface $\Sigma^\lambda_e = H^{-1}_\lambda(e)$, which preserves the Liouville measure ω^λ_e. Therefore, we have the classical dynamical system $\mathrm{CD}^\lambda_e = (\Sigma^\lambda_e, \varphi^\lambda_t, \omega^\lambda_e)$.

A quantum counterpart of the dynamical system CD^λ_e will be described as follows. The Lie group G acts on $L^2(P)$, the Hilbert space of square integrable functions on P, by the identity

$$R_g\varphi(p) = \varphi(p.g), \quad g \in G, \ p \in P, \ \varphi \in L^2(P). \tag{3.3}$$

This action breaks $L^2(P)$ into a direct sum of Hilbert spaces,

$$L^2(P) = \bigoplus_{\mu} \mathcal{L}_\mu, \tag{3.4}$$

where μ runs over the dominant integral weights and $\mathcal{L}_\mu$ is the isotypical subspaces associated with the irreducible representation (π_μ, V_μ) corresponding to the dominant integral weight μ. More precisely, the Hilbert space $\mathcal{L}_\mu$ is the closure of the image of the evaluation map, $\mathrm{Hom}_G(V_\mu, C^\infty(P)) \otimes V_\mu \to L^2(P)$. The Hilbert space $\mathcal{L}_\mu$ is also obtained by the following way. Since the operator $\hat{H}$ is elliptic and the manifold P is compact, the Hilbert space $L^2(P)$ is the direct sum of finite dimensional eigenspaces of $\hat{H}$. Since the operator $\hat{H}$ commutes with G-action, G acts on each eigenspace, and hence each eigenspace is decomposed into irreducible representations. Then the Hilbert space $\mathcal{L}_\mu$ is the direct sum of the representations which is equivalent to the irreducible representation corresponding to μ. We set

$$\mathcal{H}_\lambda = \bigoplus_{m=1}^{\infty} \mathcal{L}_{m\lambda} \ (\subset L^2(P)), \tag{3.5}$$

and denote the unit sphere in $\mathcal{H}_\lambda$ by $\mathcal{V}_\lambda$. The subspace $\mathcal{H}_\lambda$ is called the *ladder space* associated with the irreducible representation λ ([13], [15]).

Now we set up the triple $\mathrm{QD}^\lambda = (\mathcal{V}_\lambda, \hat{H}_\lambda, \mathcal{A}_0^\lambda)$ as a quantum dynamical system where $\hat{H}_\lambda$ is the restriction of $\hat{H}$ to $\mathcal{H}_\lambda$, $\mathcal{A}_0^\lambda$ is the $*$-algebra of operators on $\mathcal{H}_\lambda$ which are the restriction of the elements in $\mathcal{A}_0$, the $*$-algebra of all ψDO of order zero commuting with G-action, to $\mathcal{H}_\lambda$. We will regard the $*$-algebra $\mathcal{A}_0$ as the algebra of quantum observables.

We will call the dynamical system QD^λ the *reduced* quantum dynamical system.

3.2 Some properties of the dynamical system CD_e^λ

Let ψ_t be the Hamilton flow generated by H and the canonical symplectic form Ω_P on T^*P. The flow ψ_t commutes with G-action. We note that the reduced flow φ_t^λ can be obtained from the Hamilton flow ψ_t.

Lemma 3.1 *The flow ψ_t can be restricted on $\Phi^{-1}(\mathcal{O}_\lambda)$, and the following diagram is commutative:*

$$\begin{tikzcd}
\Phi^{-1}(\mathcal{O}_\lambda) \arrow[r, "\psi_t"] \arrow[d, "q_\lambda"'] & \Phi^{-1}(\mathcal{O}_\lambda) \arrow[d, "q_\lambda"] \\
X_\lambda \arrow[r, "\varphi_t^\lambda"] & X_\lambda
\end{tikzcd}$$

where q_λ is the projection.

Proof. First, we will show that the flow ψ_t can be restricted to $\Phi^{-1}(\mathcal{O}_\lambda)$. Let X_H be the Hamilton vector field on $T^*P \setminus 0$ generated by (H, Ω_P). The differential map $d\Phi_z$ at each point $z \in T^*P$ is surjective, and hence we have $T_z\Phi^{-1}(\mathcal{O}_\lambda) = d\Phi_z^{-1}(T_{\Phi(z)}\mathcal{O}_\lambda)$ for $z \in \Phi^{-1}(\mathcal{O}_\lambda)$. By the definition of the moment map, we have

$$\langle\, d\Phi_z,\, A\,\rangle = i(A_z^\sharp)\Omega_P$$

for every $A \in \mathcal{G}$, where

$$A_z^\sharp = \left.\frac{d}{dt}\right|_{t=0} (\exp tA)z,$$

and $i(\cdot)$ denotes the interior product. Since the Hamiltonian H is invariant under G-action, we obtain

$$\langle\, d\Phi_z(X_H),\, A\,\rangle = \Omega_P(A_z^\sharp, X_H) = -(dH)(A_z^\sharp) = 0,$$

for every $A \in \mathcal{G}$. This implies $d\Phi_z(X_H) = 0$, and hence $X_H \in T_z\Phi^{-1}(\mathcal{O}_\lambda)$. Therefore the flow ψ_t can be restricted to $\Phi^{-1}(\mathcal{O}_\lambda)$.

Next, we will show that the diagram is commutative. Let $u \in T_z\Phi^{-1}(\mathcal{O}_\lambda)$ $(z \in \Phi^{-1}(\mathcal{O}_\lambda))$. By the identity $q_\lambda^* H_\lambda = H$ on $\Phi^{-1}(\mathcal{O}_\lambda)$, we have

$$\begin{aligned} q_\lambda^*(i((q_\lambda)_* X_H)\Omega_\lambda)(u) &= (q_\lambda^*\Omega_\lambda)(X_H, u) \\ &= \Omega_P(X_H, u) \quad (\text{since } d\Phi_z(X_H) = 0) \\ &= (q_\lambda^* dH_\lambda)(u). \end{aligned}$$

Since q_λ is a submersion, we have $i((q_\lambda)_* X_H)\Omega_\lambda = dH_\lambda$ at $z \in \Phi^{-1}(\mathcal{O}_\lambda)$. Therefore we obtain that $(q_\lambda)_* X_H = X_{H_\lambda}$, where X_{H_λ} is the Hamilton vector field on X_λ determined by $(H_\lambda, \Omega_\lambda)$. Hence we conclude that $q_\lambda \circ \psi_t = \varphi_t^\lambda \circ q_\lambda$. ∎

Next, we will consider the following condition.

(H1) *The Hamilton vector field, X_H, of H is not tangent to the G-orbit through any point in $\tilde{\Sigma}_e^\lambda := Z_e \cap \Phi^{-1}(\mathcal{O}_\lambda)$, where $Z_e = H^{-1}(e) \subset T^*P$.*

Note that, for example, the dynamical system generated by the Riemannian norm function with respect to the fixed metric (see Section 3.1) satisfies the condition (H1) if $e > |\lambda|$. The condition (H1) makes us to obtain the following lemmas.

Lemma 3.2 *Suppose that the condition* (H1) *is satisfied. Then the subset $\tilde{\Sigma}_e^\lambda$ is a submanifold in T^*P, and hence $\tilde{\Sigma}_e^\lambda$ is a principal G-bundle over Σ_e^λ.*

Proof. Since the differential map $d\Phi_z$ at $z \in \tilde{\Sigma}_e^\lambda$ is surjective, we have $T_z\Phi^{-1}(\mathcal{O}_\lambda) = d\Phi_z^{-1}(T_{\Phi(z)}\mathcal{O}_\lambda)$. By the equivariance of Φ, we obtain

$$d\Phi_z^{-1}(T_{\Phi(z)}\mathcal{O}_\lambda) = \mathcal{G}(z) + \mathcal{G}(z)^\perp,$$

where $\mathcal{G}(z) = \{A_z^* \in T_zT^*P\,;\, A \in \mathcal{G}\}$ and "$\perp$" denotes the annihilator of $\mathcal{G}(z)$ with respect to Ω_P. Since H is G-invariant, we have $\mathcal{G}(z) \subset (X_H)^\perp = T_zZ_e$. Therefore we obtain

$$T_zZ_e + T_z\Phi^{-1}(\mathcal{O}_\lambda) = (X_H)^\perp + \mathcal{G}(z)^\perp,$$

and hence

$$(T_zZ_e + T_z\Phi^{-1}(\mathcal{O}_\lambda))^\perp = (X_H) \cap \mathcal{G}(z). \tag{3.6}$$

By the assumption (H1), the right hand side of (3.6) is zero. So the submanifolds Z_e and $\Phi^{-1}(\mathcal{O}_\lambda)$ intersect transversally. Thus we conclude the assertion. ∎

Lemma 3.3 *For each smooth function a on Σ_e^λ, there exists a smooth function $\tilde{a}$ on $T^*P \setminus 0$ which is G-invariant, homogeneous of degree zero such that*

$$q_\lambda^* a = \tilde{a} \quad \text{on} \quad \tilde{\Sigma}_e^\lambda, \tag{3.7}$$

where q_λ is the projection from $\tilde{\Sigma}_e^\lambda$ onto Σ_e^λ.

Proof. The function $q_\lambda^* a$ is a G-invariant smooth function on $\tilde{\Sigma}_e^\lambda$, and it can be extended to a smooth function a_0 on Z_e. Averaging a_0 on the G-orbits and extending to a smooth function on $T^*P \setminus 0$ of degree zero, we obtain a desired function $\tilde{a}$. ∎

Remark 3.1 We note that a G-invariant smooth function a on $T^*P\setminus 0$ defines a smooth function on X_λ. We will continue to denote it by a. If a is G-invariant on T^*P, then $a\circ\psi_t$ is also G-invariant, and hence, by Lemma 3.1, the function on X_λ induced by the G-invariant function $a\circ\psi_t$ coincides with $a\circ\varphi_t^\lambda$.

3.3 Statements of main theorems

Let $e_1(m) \leq e_2(m) \leq \cdots$ be the eigenvalues of the restriction of the operator $\hat{H}$ to $\mathcal{L}_{m\lambda}$, and let $\{\nu_j^m\}_{j\in\mathbf{N}}$ be the orthonormal basis for $\mathcal{L}_{m\lambda}$ of the eigenfunctions of $\hat{H}$: $\hat{H}\nu_j^m = e_j(m)\nu_j^m$. For a fixed constant $c > 0$, let

$$\begin{aligned} \mathcal{N}_m(e,c) &= \{j \in \mathbf{N}; |e_j(m) - me| \leq c\}, \\ N_m(e,c) &= \sharp\mathcal{N}_m(e,c). \end{aligned}$$

Then our first theorem can be stated as follows. (See Section 3.4 for the assumption (H2).)

Theorem 3.1 *Assume that the dynamical system* CD_e^λ *satisfies the conditions* (H1), (H2). *Then the dynamical system* CD_e^λ *is ergodic if and only if the following two conditions hold.*

(1) *For every* $A \in \mathcal{A}_0$ *and for every orthonormal basis* $\{\nu_j^m\}_{j,m=1}^\infty$ *for* $\mathcal{H}_\lambda$ *consisting of eigenfunctions of* $\hat{H}_\lambda$, *we have*

$$\lim_{m\to\infty} N_m(e,c)^{-1} \sum_{\substack{j,k\in\mathcal{N}_m(e,c)\\ e_j(m)=e_k(m)}} \left|\langle A\nu_j^m, \nu_k^m \rangle\right|^2 = \left|\mathrm{vol}(\Sigma_e^\lambda)^{-1}\int_{\Sigma_e^\lambda}\sigma_0(A)\, d\omega_e^\lambda\right|^2. \tag{3.8}$$

(2) *For every* A, $\{\nu_j^m\}$ *as above, we have*

$$\lim_{\delta\downarrow 0}\limsup_{m\uparrow\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \sum_{\substack{k\\ 0<|e_j(m)-e_k(m)|<\delta}} \left|\langle A\nu_j^m, \nu_k^m \rangle\right|^2 = 0. \tag{3.9}$$

This theorem is a semi-classical analogy of Sunada's theorem (Theorem 2.1).

Before going to state our second theorem, we refer to Zelditch's result ([39]).

Theorem 3.2 (Zelditch) *Assume that the dynamical system* CD_e^λ *is ergodic. Then for every orthonormal basis* $\{\nu_j^m\}$ *and for every* ψDO A *of order zero, we have*

$$\lim_{m\to\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \left|\langle A\nu_j^m, \nu_j^m \rangle - \int_{\tilde{S}_e^\lambda}\tilde{\sigma_0}(A)\, d\mu_e^\lambda\right| = 0. \tag{3.10}$$

Remark 3.2 We will give a brief explanation for the integral in (3.10). For details, see [12], [13], [15], [39]. Let $(T^*P)_{C(\mathcal{O}_\lambda)}$ be the space of the leaves of the null-foliation on $\Phi^{-1}(C(\mathcal{O}_\lambda))$ determined by the canonical symplectic form Ω_P, where $C(\mathcal{O}_\lambda)$ is the cone through the orbit $\mathcal{O}_\lambda$, $C(\mathcal{O}_\lambda) = \{rf\,;\, f\in\mathcal{O}_\lambda,\ r>0\}$. Note that the orbit $\mathcal{O}_\lambda$ is integral, that is, for $f\in\mathcal{O}_\lambda$, there is a character $\chi_f : G_f \to S^1$ (G_f is the stabilizer of f) such that $d\chi_f(A) = 2\pi i\langle f, A\rangle$ for every $A\in\mathcal{G}_f$ ($\mathcal{G}_f$ is the Lie algebra of G_f). Then the leaf of the null-foliation through $z\in\Phi^{-1}(C(\mathcal{O}_\lambda))$ is the orbit through z under the action of the identity component of the kernel, $\ker\chi_f$, of χ_f. The function $\tilde{\sigma_0}(A)$ is the one on $(T^*P)_{C(\mathcal{O}_\lambda)}$ obtained by integrating $\sigma_0(A)$ over the fibers.

The natural action of G on the symplectic manifold $(T^*P)_{C(\mathcal{O}_\lambda)}$ is Hamiltonian. Let $\Psi : (T^*P)_{C(\mathcal{O}_\lambda)} \to \mathcal{G}^*$ be the moment map of the above action, and let $p = |\Psi|$. Then the Hamilton flow of p on $(T^*P)_{C(\mathcal{O}_\lambda)}$ is periodic with constant period, and hence it induces an S^1-action on $(T^*P)_{C(\mathcal{O}_\lambda)}$. This S^1-action is obtained by regarding S^1 as $G_f/\ker\chi_f$. The level surface $p^{-1}(|\lambda|)$ is an S^1-bundle over the Kazhdan–Kostant–Sternberg reduction

$X_\lambda^\sharp$ with respect to the orbit $\mathcal{O}_\lambda$, which is the leaf-space of the null-foliation on $\Phi^{-1}(\mathcal{O}_\lambda)$ of Ω_P. The surface $\tilde{S}_e^\lambda$ in (3.10) is the intersection $\tilde{S}_e^\lambda = p^{-1}(|\lambda|) \cap \tilde{H}_\lambda^{-1}(e)$ in $(T^*P)_{C(\mathcal{O}_\lambda)}$, where $\tilde{H}_\lambda$ is the function on $(T^*P)_{C(\mathcal{O}_\lambda)}$ induced by G-invariant Hamiltonian H on T^*P. The measure μ_e^λ in (3.10) is the normalized Liouville measure on $\tilde{S}_e^\lambda$.

Remark 3.3 In case where the function $\sigma_0(A)$ is invariant under the action of G, the integral in the right hand side of (3.10) is reduced to the integral over $\Sigma_e^\lambda \subset X_\lambda$ of the function induced by $\sigma_0(A)$. We explain this as follows.

The level surface $\tilde{S}_e^\lambda$ is an S^1-bundle over the level surface $S_e^\lambda = (H_\lambda^\sharp)^{-1}(e)$ in $X_\lambda^\sharp$, where the function $H_\lambda^\sharp$ is the one on $X_\lambda^\sharp$ induced by H. Note that $X_\lambda^\sharp$ is symplectically diffeomorphic to the product $X_\lambda^\sharp = X_\lambda \times \mathcal{O}_\lambda$ ([12]), and the action of G on $X_\lambda^\sharp$ is interpreted as the action only on the second component of the product. Since $H_\lambda^\sharp$ is G-invariant, we have $S_e^\lambda = \Sigma_e^\lambda \times \mathcal{O}_\lambda$. Therefore the integral in (3.10) is reduced to the integral over Σ_e^λ.

To state our second theorem, which relates Theorem 3.1 to Zelditch's theorem (Theorem 3.2), we need to prepare some notation. For every quantum observable $A \in \mathcal{A}_0$, we define the quantum space average $\langle A \rangle_e^\lambda$ of A by

$$\langle A \rangle_e^\lambda = \lim_{m\to\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \langle A\nu_j^m, \nu_j^m \rangle. \tag{3.11}$$

The existence of the above limit and the independence of the choice of the constant c are guaranteed by the semi-classical trace formula due to V. Guillemin-A. Uribe ([14], [15]) and S. Zelditch ([39]) under the assumption (H2). (See Theorem 3.4 in Section 3.4.) We also define the quantum time average $\bar{A}$ of $A \in \mathcal{A}_0$ by

$$\bar{A} = \operatorname*{w-lim}_{t\to\infty} \frac{1}{t} \int_0^t e^{is\hat{H}} A e^{-is\hat{H}}\, ds. \tag{3.12}$$

Now we can state our second theorem as follows.

Theorem 3.3 *Suppose that the condition* (H2) *is satisfied. Then the following three conditions are equivalent.*

(S) *For every $A \in \mathcal{A}_0$, we have*

$$\lim_{m\to\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \|(\bar{A} - \langle A \rangle_e^\lambda)\nu_j^m\|^2 = 0, \tag{3.13}$$

where $\|\cdot\|$ is the L^2-norm and $\{\nu_j^m\}_{j,m}$ is an orthonormal basis for $\mathcal{H}_\lambda$ consisting of eigenfunctions of $\hat{H}_\lambda$.

(Z) *For every $A \in \mathcal{A}_0$ and for every orthonormal basis $\{\nu_j^m\}_{j,m}$ for $\mathcal{H}_\lambda$ consisting of eigenfunctions of $\hat{H}_\lambda$, we have*

$$\lim_{m\to\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \left| \langle A\nu_j^m, \nu_j^m \rangle - \mathrm{vol}(\Sigma_e^\lambda)^{-1} \int_{\Sigma_e^\lambda} \sigma_0(A)\, d\omega_e^\lambda \right| = 0. \tag{3.14}$$

(C) *For every A, $\{\nu_j^m\}$ as in* (Z)*, there exists a family $\{J_m\}_{m\in\mathbf{N}}$ of subsets in $\mathcal{N}_m(e,c)$ satisfying*

$$\lim_{m\to\infty} \frac{\sharp J_m}{N_m(e,c)} = 1 \tag{3.15}$$

such that

$$\lim_{m\to\infty} \max_{j\in J_m} \left| \langle A\nu_j^m, \nu_j^m \rangle - \mathrm{vol}(\Sigma_e^\lambda)^{-1} \int_{\Sigma_e^\lambda} \sigma_0(A)\, d\omega_e^\lambda \right| = 0. \tag{3.16}$$

We note that the conditions (1) in Theorem 1 and (S) in Theorem 3.3 are equivalent to quantum ergodicity of QD^λ at energy level e defined in Section 3.4. (See Lemma 3.5.) Note also that the conditions (Z) and (C) in Theorem 3.3 are equivalent without assuming the condition (H2). (See Proposition 3.4 in Section 3.6.) The condition (C) in Theorem 3.3 is a semi-classical analogy of the convergence theorem (Theorem 2.3).

3.4 Quantum ergodicity at a finite energy level

This subsection is devoted to defining the notion of quantum ergodicity for QD^λ at energy level $e > 0$, following the method in [28]. Let A be a bounded operator on $L^2(P)$ which commutes with G-action. Then the quantum time average of A is defined by

$$\bar{A} = \operatorname*{w-lim}_{t\to\infty} A_t, \quad A_t = \frac{1}{t}\int_0^t e^{is\hat{H}} A e^{-is\hat{H}} ds. \tag{3.17}$$

The above weak limit exists, and the bounded operators A_t, $\bar{A}$ commute with G-action. Furthermore the operators $\bar{A}$ and $\hat{H}$ commute. By the spectral theorem, we have

$$\hat{H} = \sum_\mu \sum_{e(\mu)} e(\mu) P_{e(\mu)}, \quad e^{it\hat{H}} = \sum_\mu \sum_{e(\mu)} e^{ite(\mu)} P_{e(\mu)}, \tag{3.18}$$

where μ runs over irreducible representations of G, $e(\mu)$ runs over eigenvalues of the restriction of $\hat{H}$ to $\mathcal{L}_\mu$, and $P_{e(\mu)}$ is the projection onto the eigenspace with the eigenvalue $e(\mu)$. Using the expression (3.18), we obtain that the time average $\bar{A}$ of A has the form

$$\bar{A} = \sum_\mu \sum_{e(\mu)} P_{e(\mu)} A P_{e(\mu)}. \tag{3.19}$$

The quantum space average of A is defined by

$$\langle A \rangle_e^\lambda = \lim_{m\to\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \langle A\nu_j^m,\ \nu_j^m\rangle, \tag{3.20}$$

if the above limit exists, where $N_m(e,c)$, $\mathcal{N}_m(e,c)$ and ν_j^m are described in Section 3.3. Note that $\langle A \rangle_e^\lambda = \langle \bar{A} \rangle_e^\lambda$ if the left hand side exists. To guarantee the existence of the space average of $A \in \mathcal{A}_0$, we need the following condition.

(H2) *The set of periodic points of φ_t^λ on Σ_e^λ has Liouville measure zero.*

Under the condition (H2), we have the following semi-classical asymptotic formula due to Guillemin-Uribe ([14], [15]) and Zelditch ([39]).

Theorem 3.4 (Guillemin-Uribe, Zelditch) *Suppose that the condition* (H2) *is satisfied. Then for every $A \in \mathcal{A}_0$ we have the following formula.*

$$\sum_{j\in\mathcal{N}_m(e,c)} \langle A\nu_j^m, \nu_j^m\rangle = 2c\left(\frac{m}{2\pi}\right)^{n+d-1} \int_{\Sigma_e^\lambda} \sigma_0(A)d\omega_e^\lambda \ +\ o(m^{n+d-1}), \tag{3.21}$$

where $n = \dim M$ and $2d = \dim \mathcal{O}_\lambda$.

We refer to [5], [14], [15] and [39] for the proof of this formula.

Next, we will prepare some notation on the classical mechanics CD_e^λ. For each L^2-function $a \in L^2(\Sigma_e^\lambda)$, let a_t be the time average of a up to time $t > 0$:

$$a_t = \frac{1}{t}\int_0^t a\circ\varphi_s^\lambda\, ds,$$

and let $\langle a \rangle_e^\lambda$ be the space average of a:

$$\langle a \rangle_e^\lambda = \mathrm{vol}(\Sigma_e^\lambda)^{-1} \int_{\Sigma_e^\lambda} a\, d\omega_e^\lambda.$$

By Theorem 1.1, the long time average $\bar{a} = \lim_{t\to\infty} a_t$ exists in $L^2(\Sigma_e^\lambda)$. The following lemma is the direct consequence of Egorov's theorem (Lemma 2.1, (2)) and Theorem 3.4.

Lemma 3.4 (1) *For every $A \in \mathcal{A}_0$, we have $e^{it\hat{H}}Ae^{-it\hat{H}} \in \mathcal{A}_0$, and hence $A_t \in \mathcal{A}_0$. The principal symbols of the operators $e^{it\hat{H}}Ae^{-it\hat{H}}$ and A_t are given by $\sigma_0(A)\circ\psi_t$ and*

$$\sigma_0(A_t) = \frac{1}{t}\int_0^t \sigma_0(A)\circ\psi_s\, ds, \tag{3.22}$$

respectively, where ψ_t is the Hamilton flow on T^*P.

(2) *If the condition* (H2) *is fulfilled, then for every $A \in \mathcal{A}_0$ we have $\langle A \rangle_e^\lambda = \langle \sigma_0(A) \rangle_e^\lambda$.*

Now we will define quantum ergodicity at a finite energy level, which is an analogy of Definition 2.1, (1).

Definition 3.1 *The reduced quantum dynamical system* QD^λ *is said to be quantum ergodic at energy level* e *if, for every observable* $A \in \mathcal{A}_0$*, the space average* $\langle \bar{A}^* \bar{A} \rangle_e^\lambda$ *of the operator* $\bar{A}^*\bar{A}$ *exists and satisfy*

$$\langle \bar{A}^* \bar{A} \rangle_e^\lambda = |\langle A \rangle_e^\lambda|^2. \tag{3.23}$$

Lemma 3.5 *Assume that the condition* (H2) *is satisfied. Then the reduced quantum mechanics* QD^λ *is quantum ergodic at energy level* e *if and only if the condition* (S) *in the statement of Theorem* 3.3 *holds.*

Proof. Note that $\langle \bar{A}^* \bar{A} \rangle_e^\lambda$ exists and satisfies (3.23) if and only if the following is satisfied:

$$\langle (\bar{A} - \langle A \rangle_e^\lambda)^* (\bar{A} - \langle A \rangle_e^\lambda) \rangle_e^\lambda = 0, \tag{3.24}$$

and, by the definition of the quantum space average, (3.24) holds in and only if (3.13) in the condition (S) holds. ∎

Proposition 3.1 *Assume that the condition* (H2) *is satisfied. Then* QD^λ *is quantum ergodic at energy level* e *if and only if the condition* (1) *in Theorem* 3.1 *holds.*

Proof. Let $A \in \mathcal{A}_0$ be a zeroth order ψDO commuting with G-action and let $\{\nu_j^m\}$ be an orthonormal basis of eigenfunctions for $\mathcal{H}_\lambda$. Since the bounded operator $\bar{A}$ commutes with $\hat{H}$ and G-action, $\bar{A}\nu_j^m$ is an eigenfunction of $\hat{H}$ with eigenvalue $e_j(m)$. Therefore, by (3.19), we have

$$\bar{A}\nu_j^m = \sum_{\substack{k \\ e_k(m)=e_j(m)}} \langle A\nu_j^m,\ \nu_k^m \rangle \nu_k^m, \tag{3.25}$$

and hence we obtain

$$\langle \bar{A}^* \bar{A} \rangle_e^\lambda = \lim_{m\to\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \sum_{\substack{k \\ e_k(m)=e_j(m)}} |\langle A\nu_j^m,\ \nu_k^m \rangle|^2. \tag{3.26}$$

Thus the assertion follows from Lemma 3.4, (2) and Definition 3.1. ∎

In order to prove Theorem 3.1, we shall prepare the following proposition.

Proposition 3.2 *Assume that the condition* (H2) *is satisfied. Then every quantum observable $A \in \mathcal{A}_0$ satisfies*

$$\lim_{t\to\infty} \langle\, A_t^* A_t \,\rangle_e^\lambda = \langle\, \bar{A}^* \bar{A} \,\rangle_e^\lambda \tag{3.27}$$

if and only if (3.9) *in the condition* (2) *of Theorem* 3.1 *holds.*

Proof. This proposition is obtained by the similar way to the proof of Lemma 2-2 in [28]. However, we will recall it just to make sure. Note that, by the assumption (H2) and Lemmas 3.4, 1.1, $\lim_{t\to\infty} \langle\, A_t^* A_t \,\rangle_e^\lambda$ exists. A direct computation leads us to

$$A_t \nu_j^m = \frac{1}{t} \sum_{\substack{k \\ e_k(m)\neq e_j(m)}} \frac{(e^{it(e_k(m)-e_j(m))}-1)}{i(e_k(m)-e_j(m))} \langle\, A\nu_j^m,\, \nu_k^m \,\rangle \nu_k^m + \bar{A}\nu_j^m, \tag{3.28}$$

and hence

$$\begin{aligned} \langle\, A_t^* A_t \nu_j^m,\, \nu_j^m \,\rangle &= \frac{1}{t^2} \sum_{\substack{k \\ e_k(m)\neq e_j(m)}} \frac{|e^{it(e_k(m)-e_j(m))}-1|^2}{|e_k(m)-e_j(m)|^2} |\langle\, A\nu_j^m,\, \nu_k^m \,\rangle|^2 \\ &\quad + \langle\, \bar{A}^* \bar{A} \nu_j^m,\, \nu_j^m \,\rangle. \end{aligned} \tag{3.29}$$

We set $S(x) = x^{-2}|e^{ix}-1|^2 = 2x^{-2}(1-\cos x)$ and

$$S_t = \limsup_{m\uparrow\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \sum_{\substack{k \\ e_j(m)\neq e_k(m)}} S(t(e_j(m)-e_k(m)))\, |\langle\, A_m \nu_j^m,\, \nu_k^m \,\rangle|^2. \tag{3.30}$$

We observe that (3.27) holds if and only if

$$\lim_{t\to\infty} S_t = 0. \tag{3.31}$$

Indeed, under the assumption (3.27), (3.31) follows directly from (3.29). Conversely, assume that (3.31) holds. Since $\langle\, A_t^* A_t^* \,\rangle_e^\lambda$ exists, the existence of $\langle\, \bar{A}^* \bar{A} \,\rangle_e^\lambda$ follows from (3.29), and hence we have

$$\langle\, A_t^* A_t^* \,\rangle_e^\lambda = S_t + \langle\, \bar{A}^* \bar{A} \,\rangle_e^\lambda \tag{3.32}$$

By taking $t \to \infty$ in (3.32), we obtain (3.27).

Note that there exists $\alpha > 0$ such that $S(x) \geq 1/2$ if $|x| < \alpha$. Then we have

$$\begin{aligned} S_t &\geq \limsup_{m\uparrow\infty} N_m(e,c)^{-1} \times \\ &\quad \sum_{j\in\mathcal{N}_m(e,c)} \sum_{\substack{k \\ 0<|e_j(m)-e_k(m)|\leq\alpha/t}} S(t(e_j(m)-e_k(m)))\, |\langle\, A\nu_j^m,\, \nu_k^m \,\rangle|^2 \\ &\geq \frac{1}{2} \limsup_{m\uparrow\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \sum_{\substack{k \\ 0<|e_j(m)-e_k(m)|\leq\alpha/t}} |\langle\, A\nu_j^m,\, \nu_k^m \,\rangle|^2. \end{aligned} \tag{3.33}$$

Therefore (3.31) implies (3.9).

Conversely, we will assume (3.9). For any $\varepsilon > 0$, there exists $T > 0$ such that $S(x) < \varepsilon$ if $|x| > T$. Then we have

$$\begin{aligned}
& N_m(e,c)^{-1} \sum_{j \in \mathcal{N}_m(e,c)} \sum_{\substack{k \\ e_j(m) \neq e_k(m)}} S(t(e_j(m) - e_k(m))) \, |\langle A\nu_j^m, \, \nu_k^m \rangle|^2 \\
\leq \ & \varepsilon N_m(e,c)^{-1} \sum_{j \in \mathcal{N}_m(e,c)} \sum_{\substack{k \\ |e_j(m) - e_k(m)| > T/t}} |\langle A\nu_j^m, \, \nu_k^m \rangle|^2 \\
& \qquad + N_m(e,c)^{-1} \sum_{j \in \mathcal{N}_m(e,c)} \sum_{\substack{k \\ 0 < |e_j(m) - e_k(m)| \leq T/t}} |\langle A\nu_j^m, \, \nu_k^m \rangle|^2 \\
\leq \ & \varepsilon \|A\|^2 + N_m(e,c)^{-1} \sum_{j \in \mathcal{N}_m(e,c)} \sum_{\substack{k \\ 0 < |e_j(m) - e_k(m)| \leq T/t}} |\langle A\nu_j^m, \, \nu_k^m \rangle|^2. \qquad (3.34)
\end{aligned}$$

Therefore we obtain

$$S_t \leq \varepsilon \|A\|^2 + \limsup_{m \uparrow \infty} N_m(e,c)^{-1} \sum_{j \in \mathcal{N}_m(e,c)} \sum_{\substack{k \\ 0 < |e_j(m) - e_k(m)| \leq T/t}} |\langle A\nu_j^m, \, \nu_k^m \rangle|^2. \qquad (3.35)$$

Letting $t \to \infty$ in (3.35), we have $\limsup_{t \to \infty} S_t \leq \varepsilon \|A\|^2$. Since $\varepsilon > 0$ is arbitrary, we conclude that $\lim_{t \to \infty} S_t = 0$, and hence (3.27). ∎

3.5 Proof of Theorem 3.1

In the preceding subsection, we have defined quantum ergodicity of the reduced quantum dynamical system QD^λ at a finite energy level. This notion plays an important role in the proof of Theorem 3.1 (stated in Section 3.3). Indeed, in view of Propositions 3.1, 3.2, we only need to prove the following proposition for the proof of Theorem 3.1.

Proposition 3.3 *Assume that the conditions* (H1) *and* (H2) *are fulfilled. Then the dynamical system* CD_e^λ *is ergodic if and only if the following two conditions hold:*

(1) *The reduced quantum dynamical system* QD^λ *is quantum ergodic at energy level* e.

(2) *For every observable* $A \in \mathcal{A}_0$*, we have* (3.27) *in Proposition* 3.2.

Proof. We take an arbitrary $A \in \mathcal{A}_0$. Then we have

$$\begin{aligned}
|\langle A \rangle|^2 &= |\langle \sigma_0(A) \rangle| \quad \text{(Lemma 3.4, (2))} \\
&= \langle |\overline{\sigma_0(A)}|^2 \rangle \quad \text{(ergodicity)}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{t\to\infty} \langle\, |\sigma_0(A)_t|^2 \,\rangle \quad \text{(Lemma 1.1, (1), (i))} \\
&= \lim_{t\to\infty} \langle\, \sigma_0(A_t^* A_t) \,\rangle \quad \text{(Lemma 3.4, (1))} \\
&= \lim_{t\to\infty} \langle\, A_t^* A_t \,\rangle \quad \text{(Lemma 3.4, (2))}.
\end{aligned}
$$

On the one hand, by (3.29), we have $\langle\, A_t^* A_t \nu_j^m,\, \nu_j^m \,\rangle \geq \langle\, \bar{A}^* \bar{A} \nu_j^m,\, \nu_j^m \,\rangle$, and hence

$$
\langle\, A_t^* A_t \,\rangle \geq \limsup_{m\uparrow\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \langle\, \bar{A}^* \bar{A} \nu_j^m,\, \nu_j^m \,\rangle.
$$

On the other hand,

$$
\begin{aligned}
0 &\leq \liminf_{m\uparrow\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \|(\bar{A} - \langle\, A \,\rangle)\nu_j^m\|^2 \\
&= \liminf_{m\uparrow\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \langle\, \bar{A}^* \bar{A} \nu_j^m,\, \nu_j^m \,\rangle - |\langle\, A \,\rangle|^2.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
|\langle\, A \,\rangle|^2 &= \lim_{t\to\infty} \langle\, A_t^* A_t \,\rangle \\
&\geq \limsup_{m\uparrow\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \langle\, \bar{A}^* \bar{A} \nu_j^m,\, \nu_j^m \,\rangle \\
&\geq \liminf_{m\uparrow\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \langle\, \bar{A}^* \bar{A} \nu_j^m,\, \nu_j^m \,\rangle \\
&\geq |\langle\, A \,\rangle|^2.
\end{aligned}
$$

This implies that $\langle\, \bar{A}^* \bar{A} \,\rangle$ exists and

$$
|\langle\, A \,\rangle|^2 = \lim_{t\to\infty} \langle\, A_t^* A_t \,\rangle = \langle\, \bar{A}^* \bar{A} \,\rangle.
$$

We will prove the converse. Let $A \in \mathcal{A}_0$. Then we have

$$
\begin{aligned}
|\langle\, A \,\rangle|^2 &= |\langle\, \sigma_0(A) \,\rangle|^2 \quad \text{(Lemma 3.4, (2))} \\
&\leq \langle\, |\overline{\sigma_0(A)}|^2 \,\rangle \quad \text{(Lemma 1.1, (1), (ii))} \\
&= \lim_{t\to\infty} \langle\, |\sigma_0(A)_t|^2 \,\rangle \quad \text{(Lemma 1.1, (1), (i))} \\
&= \lim_{t\to\infty} \langle\, \sigma_0(A_t^* A_t) \,\rangle \quad \text{(Lemma 3.4, (1))} \\
&= \lim_{t\to\infty} \langle\, A_t^* A_t \,\rangle \quad \text{(Lemma 3.4, (2))} \\
&= \langle\, \bar{A}^* \bar{A} \,\rangle \quad \text{(Assumption (1))} \\
&= |\langle\, A \,\rangle|^2 \quad \text{(Assumption (2))}.
\end{aligned}
$$

Thus for every smooth function $\sigma_0(A)$ which is the principal symbol of $A \in \mathcal{A}_0$, the equation (1.3) in Lemma 1.1 (2) holds. Now, by Lemma 3.3, for every $a \in C^\infty(\Sigma_e^\lambda)$, there exists a smooth function $\tilde{a}$ on $T^*P \setminus 0$ which is G-invariant, homogeneous of degree zero and $q_\lambda^* a = \tilde{a}$ on $\tilde{\Sigma}_e^\lambda$. Let A_0 be the ψDO of order zero whose principal symbol is $\tilde{a}$. Then the operator $A = \int_G gA_0g^{-1}\,dg$ is in $\mathcal{A}_0$ whose principal symbol is $\tilde{a}$, and hence $\langle\, |\overline{\sigma_0(A)}|^2 \,\rangle = \langle\, |\bar{a}|^2 \,\rangle$. Therefore the dynamical system CD_e^λ is ergodic. ▮

3.6 Proof of Theorem 3.3

Now we will proceed to the proof of Theorem 3.3. For this sake, we will define auxiliary notions.

Definition 3.2 (1) *A family $\{\, \mathcal{S}_m\,;\, \mathcal{S}_m \subset \sigma_m(e,c) \,\}$ of subsets in $\sigma_m(e,c) = \{\, \lambda \in \sigma(\hat{H}|_{\mathcal{L}_{m\lambda}})\,;\, |\lambda - me| \le c \,\}$ is said to satisfy the condition* (D1) *if it satisfies*

$$\lim_{m\to\infty} N_m(e,c)^{-1} \sum_{\lambda\in\mathcal{S}_m} (\dim V_\lambda) = 1, \tag{3.36}$$

where V_λ is the eigenspace of an eigenvalue λ of $\hat{H}|_{\mathcal{L}_{m\lambda}}$.

(2) *A family $\{\, J_m\,;\, J_m \subset \mathcal{N}_m(e,c) \,\}$ of subsets in $\mathcal{N}_m(e,c) = \{\, j \in \mathbf{N}\,;\, e_j(m) \in \sigma_m(e,c) \,\}$ is said to satisfy the condition* (D2) *if we have*

$$\lim_{m\to\infty} N_m(e,c)^{-1} \sharp J_m = 1. \tag{3.37}$$

Let $\boldsymbol{x} = \{x^m\}_{m\in\mathbf{Z}}$ be a family of sequences $x^m = \{x_j^m\}_{j\in\mathcal{N}_m(e,c)}$ of non-negative numbers such that $0 \le x_j^m \le K$ for all m, j, for some constant $K > 0$. For each $\lambda \in \sigma_m(e,c)$, we set

$$x_\lambda^m = (\dim V_\lambda)^{-1} \sum_{\substack{j \\ e_j(m)=\lambda}} x_j^m, \tag{3.38}$$

so that

$$N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} x_j^m = N_m(e,c)^{-1} \sum_{\lambda\in\sigma_m(e,c)} (\dim V_\lambda)\, x_\lambda^m. \tag{3.39}$$

Lemma 3.6 *Let $\boldsymbol{x} = \{x^m\}_{m\in\mathbf{Z}}$ be a family of sequences $x^m = \{x_j^m\}_{j\in\mathcal{N}_m(e,c)}$ as above. Then $\boldsymbol{x} = \{x^m\}_{m\in\mathbf{Z}}$ satisfies*

$$\lim_{m\to\infty} N_m(e,c)^{-1} \sum_{\lambda\in\sigma_m(e,c)} (\dim V_\lambda)\, x_\lambda^m = 0, \tag{3.40}$$

if and only if there exists a family $\{\, \mathcal{S}_m\,;\, \mathcal{S}_m \subset \sigma_m(e,c) \,\}$ satisfying the condition (D1) *such that*

$$\lim_{m\to\infty} \max_{\lambda\in\mathcal{S}_m} x_\lambda^m = 0. \tag{3.41}$$

Proof. Since " if " part is obvious, we will only give a proof of " only if " part. Assume that $\boldsymbol{x} = \{x^m\}_{m\in\mathbf{Z}}$ satisfies (3.40). Then one can find a sequence $\{l_m\}$ of natural numbers which is monotone increasing and goes to infinity as $m \to \infty$ such that

$$N_m(e,c)^{-1} \sum_{\lambda\in\sigma_m(e,c)} (\dim V_\lambda)\ x_\lambda^m < \frac{K}{2^{l_m}}$$

for every $m \in \mathbf{N}$. We define

$$\mathcal{S}_m = \{\ \lambda \in \sigma_m(e,c)\ ;\ x_\lambda^m < \frac{1}{l_m}\ \}.$$

It is clear that $\mathcal{S}_m$ satisfies (3.41). Furthermore $\{\mathcal{S}_m\}$ satisfies (D1). Indeed we have

$$\begin{aligned} \frac{K}{2^{l_m}} &> N_m(e,c)^{-1} \sum_{\lambda\in\sigma_m(e,c)} (\dim V_\lambda)\, x_\lambda^m \\ &\geq \{l_m N_m(e,c)\}^{-1} \sum_{\lambda\in\sigma_m(e,c)\setminus\mathcal{S}_m} (\dim V_\lambda), \end{aligned}$$

and hence

$$1 - N_m(e,c)^{-1} \sum_{\lambda\in\mathcal{S}_m} (\dim V_\lambda) < K\frac{l_m}{2^{l_m}}.$$

This implies (3.36). ∎

Lemma 3.7 *Let $\boldsymbol{x} = \{x^m\}_{m\in\mathbf{Z}}$ be a family of sequences $x^m = \{x_j^m\}_{j\in\mathcal{N}_m(e,c)}$ of non-negative numbers as above. Then the following conditions are equivalent.*

(1) *There exists a family $\{\mathcal{S}_m\}_m$ satisfying* (D1) *such that it satisfies* (3.41).

(2) *There exists a family $\{J_m\}_m$ satisfying* (D2) *such that*

$$\lim_{m\uparrow\infty} \max_{j\in J_m} x_j^m = 0. \tag{3.42}$$

Proof. First, we will assume the condition (1). Then one can find a sequence $\{l_m\}$ of natural numbers which is monotone increasing and goes to infinity as $m \to \infty$ such that for all $\lambda \in \mathcal{S}_m$

$$x_\lambda^m = (\dim V_\lambda)^{-1} \sum_{\substack{j \\ e_j(m)=\lambda}} x_j^m < \frac{K}{2^{l_m}}.$$

We define $J_m \subset \mathcal{N}_m(e,c)$ by

$$J_m = \{\ j \in \mathcal{N}_m(e,c)\ ;\ e_j(m) \in \mathcal{S}_m \ \text{ and } x_j^m < \frac{1}{l_m}\ \}.$$

This family clearly satisfies (3.42). Note that we have

$$
\begin{aligned}
&1 - N_m(e,c)^{-1}\sharp J_m \\
&\quad = N_m(e,c)^{-1}\sum_{\lambda\in\mathcal{S}_m}\sum_{\substack{j\in\mathcal{N}_m(e,c)\setminus J_m\\ e_j(m)=\lambda}} 1 \ + \ N_m(e,c)^{-1}\sum_{\lambda\in\sigma_m(e,c)\setminus\mathcal{S}_m}\sum_{\substack{j\\ e_j(m)=\lambda}} 1 \\
&\quad = \mathrm{I} + \mathrm{I\!I} \quad \text{(say)}.
\end{aligned}
$$

Since $\{\mathcal{S}_m\}$ satisfies the condition (D1), $\mathrm{I\!I}$ goes to zero as $m\to\infty$. On the other hand,

$$\mathrm{I} = N_m(e,c)^{-1}\sum_{\lambda\in\mathcal{S}_m}(\dim V_\lambda)\, S_\lambda^m,$$

where we set

$$S_\lambda^m = (\dim V_\lambda)^{-1}\sum_{\substack{j\in\mathcal{N}_m(e,c)\setminus J_m\\ e_j(m)=\lambda}} 1.$$

If $e_j(m)=\lambda\in\mathcal{S}_m$ and $j\notin J_m$, then $x_j^m \ge l_m^{-1}$. Thus, for $\lambda\in\mathcal{S}_m$, we have

$$\frac{K}{2^{l_m}} \ge (\dim V_\lambda)^{-1}\sum_{\substack{j\in\mathcal{N}_m(e,c)\setminus J_m\\ e_j(m)=\lambda}} x_j^m \ge l_m^{-1} S_\lambda^m.$$

This implies that $\mathrm{I} < K l_m/2^{l_m} \to 0 \quad (m\to\infty)$. Hence the family $\{J_m\}$ satisfies (D2).

Next, we shall prove the converse. In view of Lemma 3.6, it suffices to prove that the condition (2) implies (3.40). Combining (3.39) and the assumption (2), we have

$$
\begin{aligned}
N_m(e,c)^{-1}\sum_{\lambda\in\sigma_m(e,c)}(\dim V_\lambda)\, x_\lambda^m &\le N_m(e,c)^{-1}\sum_{j\in J_m} x_j^m + K\left[1 - N_m(e,c)^{-1}\sharp J_m\right] \\
&\le \max_{j\in J_m} x_j^m + K\left[1 - N_m(e,c)^{-1}\sharp J_m\right].
\end{aligned}
$$

Letting $m\to\infty$ in the above inequality, the condition (3.40) follows. ▮

Proposition 3.4 *The conditions* (Z) *and* (C) *in the statement of Theorem* 3.3 *are equivalent.*

Proof. We take an $A\in\mathcal{A}_0$ and set

$$x_j^m = |\langle A\nu_j^m, \nu_j^m\rangle - \langle\sigma_0(A)\rangle_e^\lambda|$$

for $j\in\mathcal{N}_m(e,c)$. Note that $0\le x_j^m \le \|A\| + |\langle\sigma_0(A)\rangle_e^\lambda|^2$. Hence, by Lemmas 3.6, 3.7 we conclude the assertion. ▮

The conditions (Z) and (C) are equivalent without assuming (H2). Next we will prove the equivalence of (S) and (C). For this sake, we prepare the following lemma.

Lemma 3.8 *Consider a family* $\{ J_m\,;\, J_m \subset \mathcal{N}_m(e,c) \}$, *and set* $J_m(\lambda) = \{ j \in J_m\,;\, e_j(m) = \lambda \}$. *Then* $\{J_m\}_m$ *satisfies* (D2) *if and only if there exists a family* $\{\mathcal{S}_m\,;\, \mathcal{S}_m \subset \sigma_m(e,c) \}$ *satisfying* (D1) *such that*

$$\lim_{m\uparrow\infty} \max_{\lambda\in\mathcal{S}_m} (\dim V_\lambda)^{-1} \sharp J_m(\lambda) = 1. \tag{3.43}$$

Proof. Set

$$x_j^m = \begin{cases} 0 & \text{if} \quad j \in J_m \\ 1 & \text{if} \quad j \in \mathcal{N}_m(e,c) \setminus J_m. \end{cases}$$

Then for all $\lambda \in \sigma_m(e,c)$, we have $x_\lambda^m = 1 - (\dim V_\lambda)^{-1}\sharp J_m(\lambda)$. Therefore

$$1 - N_m(e,c)^{-1}\sharp J_m = N_m(e,c)^{-1} \sum_{\lambda\in\sigma_m(e,c)} (\dim V_\lambda) x_\lambda^m.$$

Hence by Lemma 3.7, $\{J_m\}_m$ satisfies (D2) if and only if there exists a family $\{\mathcal{S}_m\}_m$ satisfying (D1) such that

$$\lim_{m\to\infty} \min_{\lambda\in\mathcal{S}_m} x_\lambda^m \le \lim_{m\to\infty} \max_{\lambda\in\mathcal{S}_m} x_\lambda^m = 0.$$

Since $\min_{\lambda\in\mathcal{S}_m} x_\lambda^m = 1 - \max_{\lambda\in\mathcal{S}_m} (\dim V_\lambda)^{-1}\sharp J_m(\lambda)$, we conclude the assertion. ∎

Finally, we will prove the following proposition, which completes the proof of Theorem 3.3.

Proposition 3.5 *Assume that* (H2) *is satisfied. Then the conditions* (S) *and* (C) *in Theorem* 3.3 *are equivalent.*

Proof. We will assume that the condition (S) holds. Note that $\langle A \rangle_e^\lambda = \langle \sigma_0(A) \rangle_e^\lambda$ by Lemma 3.4, (2). Therefore by setting $x_j^m = \|(\bar{A} - \langle A \rangle_e^\lambda)\nu_j^m\|$, the condition (C) follows from Lemmas 3.6, 3.7 and the inequality

$$|\langle A\nu_j^m, \nu_j^m \rangle - \langle \sigma_0(A) \rangle_e^\lambda| \le \|(\bar{A} - \langle A \rangle_e^\lambda)\nu_j^m\|.$$

We will prove the converse. We may assume, without loss of generality, that $A \in \mathcal{A}_0$ is self-adjoint. Since the time average $\bar{A}$ commutes with $\hat{H}$ and G-action, we can take an orthonormal basis $\{\nu_j^m\}$ for $\mathcal{L}_{m\lambda}$ consisting of eigenfunctions of $\hat{H}$ such that $\bar{A}\nu_j^m = \mu_j^m \nu_j^m$ for some $\mu_j^m \in \mathbf{R}$. Note that $\langle \bar{A}\nu_j^m, \nu_j^m \rangle = \langle A\nu_j^m, \nu_j^m \rangle$. Then we have

$$\|(\bar{A} - \langle A \rangle_e^\lambda)\nu_j^m\|^2 = |\mu_j^m - \langle A \rangle_e^\lambda|^2 = |\langle A\nu_j^m, \nu_j^m \rangle - \langle \sigma_0(A) \rangle_e^\lambda|^2. \tag{3.44}$$

Let $\{J_m\}_m$ be a family described in the condition (C). By Lemma 3.8, we can find a family $\mathcal{S}_m$ satisfying (D1) such that (3.43) holds. In view of Lemma 3.6, we only need to prove that this family $\{\mathcal{S}_m\}$ satisfies

$$\lim_{m\uparrow\infty}\max_{\lambda\in\mathcal{S}_m}(\dim V_\lambda)^{-1}\sum_{\substack{j\\ e_j(m)=\lambda}}\|(\bar{A}-\langle A\rangle_e^\lambda)\nu_j^m\|^2=0. \tag{3.45}$$

By (3.43), for arbitrary $\varepsilon>0$ we can find a positive number N_1 such that $m\geq N_1$ implies $(\dim V_\lambda)^{-1}[(\dim V_\lambda)-\sharp J_m(\lambda)]<\varepsilon$ for all $\lambda\in\mathcal{S}_m$. By the condition (C) and (3.44), there is a positive number N_2 such that $m\geq N_2$ implies $\|(\bar{A}-\langle A\rangle_e^\lambda)\nu_j^m\|^2<\varepsilon$ for all $j\in J_m$. Therefore if $m\geq\max\{N_1,N_2\}$ then for all $\lambda\in\mathcal{S}_m$ we have

$$\begin{aligned}
&(\dim V_\lambda)^{-1}\sum_{\substack{j\\ e_j(m)=\lambda}}\|(\bar{A}-\langle A\rangle_e^\lambda)\nu_j^m\|^2\\
&=\;(\dim V_\lambda)^{-1}\sum_{\substack{j\in J_m\\ e_j(m)=\lambda}}\|(\bar{A}-\langle A\rangle_e^\lambda)\nu_j^m\|^2\\
&\quad+(\dim V_\lambda)^{-1}\sum_{\substack{j\in\mathcal{N}_m(e,c)\setminus J_m\\ e_j(m)=\lambda}}\|(\bar{A}-\langle A\rangle_e^\lambda)\nu_j^m\|^2\\
&\leq\;(1+K)\varepsilon,
\end{aligned}$$

where $K=\|A\|+|\langle A\rangle_e^\lambda|^2$. Since $\varepsilon>0$ is arbitrary, we obtain (3.45). ∎

3.7 Quantum weak-mixing at a finite energy level

Theorem 3.1 says that ergodicity of classical dynamical system CD_e^λ is related to the semi-classical asymptotic behavior of near-diagonal components of quantum observables. Thus it is natural to ask which property of classical mechanics affects the asymptotic behavior of the components far from the diagonal. For this problem, Zelditch [42] showed that classical weak-mixing is equivalent to the notion of *quantum weak-mixing* (see Section 2.2) plus an additional condition, and he obtained Theorem 2.2.

In this section, we will discuss quantum weak-mixing of QD^λ at a finite energy level. We will begin with recalling some notation in Section 1.1.

For every $\tau\in\mathbf{R}$ and every $a\in L^2(\Sigma_e^\lambda)$, we define $a_t(\tau)\in L^2(\Sigma_e^\lambda)$ by

$$a_t(\tau)=\frac{1}{t}\int_0^t e^{-i\tau s}a\circ\varphi_s^\lambda\,ds.$$

By von Neumann's ergodic theorem (Theorem 1.1), the function $a_t(\tau)$ converges in L^2-sense to the function $\bar{a}(\tau)\in L^2(\Sigma_e^\lambda)$ satisfying $\bar{a}(\tau)\circ\varphi_t^\lambda=e^{i\tau t}\bar{a}(\tau)$ as $t\to\infty$. The dynamical system CD_e^λ is said to be *weak-mixing* if

$$\bar{a}(\tau) = \langle\, a \,\rangle_e^\lambda \delta_{\tau,0}\,, \qquad \text{a.e.},$$

or equivalently

$$\langle\, |\bar{a}(\tau)|^2 \,\rangle_e^\lambda = |\langle\, \bar{a}(\tau) \,\rangle_e^\lambda|^2$$

for all $a \in C^\infty(\Sigma_e^\lambda)$. (See Lemma 1.1, (3).)

We will describe a quantum analogue of this notion. (See Section 2.1 for the high energy case.) For every quantum observable $A \in \mathcal{A}_0$ and for every $\tau \in \mathbf{R}$, we define the bounded operator $\bar{A}(\tau)$ by

$$\bar{A}(\tau) = \operatorname*{w-lim}_{t\to\infty} A_t(\tau), \quad A_t(\tau) = \frac{1}{t}\int_0^t e^{-i\tau s} e^{is\hat{H}} A e^{-is\hat{H}}\, ds.$$

The bounded operator $\bar{A}(\tau)$ commutes with G-action and has the following form

$$\bar{A}(\tau) = \sum_\mu \sum_{e(\mu)\in\sigma(\hat{H}|_{\mathcal{L}_\mu})} P_{e(\mu)+\tau} A P_{e(\mu)}.$$

By Egorov's theorem (Lemma 3.4, (1)), the operator $A_t(\tau)$ is in $\mathcal{A}_0$ and its principal symbol is given by

$$\sigma_0(A_t(\tau)) = \frac{1}{t}\int_0^t e^{-i\tau s}\sigma_0(A)\circ\psi_s\, ds.$$

Definition 3.3 *The reduced quantum dynamical system* QD^λ *is said to be quantum weak-mixing at energy level* $e > 0$ *if for every observable* $A \in \mathcal{A}_0$ *and every* $\tau \in \mathbf{R}$*,* $\langle\, \bar{A}(\tau)^*\bar{A}(\tau) \,\rangle_e^\lambda$ *and* $\langle\, A \,\rangle_e^\lambda$ *exist and satisfy*

$$\langle\, \bar{A}(\tau)^*\bar{A}(\tau) \,\rangle_e^\lambda = |\langle\, A \,\rangle_e^\lambda|^2 \delta_{\tau,0}\ ,$$

or equivalently,

$$\langle\, \bar{A}(\tau)^*\bar{A}(\tau) \,\rangle_e^\lambda = |\langle\, \bar{A}(\tau) \,\rangle_e^\lambda|^2.$$

The following proposition and theorem can be obtained by a method similar to the proofs of Proposition 3.3 and Theorem 3.1, respectively.

Proposition 3.6 *Assume that the conditions* (H1) *and* (H2) *are fulfilled. Then the classical dynamical system* CD_e^λ *is weak-mixing if and only if the following two conditions hold.*

(1) *The reduced quantum dynamical system* QD^λ *is quantum weak-mixing at energy level* e*.*

(2) *For every observable* $A \in \mathcal{A}_0$ *and for every* $\tau \in \mathbf{R}$, *we have*

$$\lim_{t\to\infty} \langle\, A_t(\tau)^* A_t(\tau)\,\rangle_e^\lambda = \langle\, \bar{A}(\tau)^* \bar{A}(\tau)\,\rangle_e^\lambda.$$

Theorem 3.5 *Assume that the conditions* (H1) *and* (H2) *are fulfilled. Then the classical dynamical system* CD_e^λ *is weak-mixing if and only if the following two conditions hold.*

(1) *For every* $A \in \mathcal{A}_0$, $\tau \in \mathbf{R}$ *and orthonormal basis* $\{\nu_j^m\}_{j,m=1}^\infty$ *for* $\mathcal{H}_\lambda$ *consisting of eigenfunctions of* $\hat{H}_\lambda$, *we have*

$$\lim_{m\uparrow\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \sum_{\substack{k \\ e_k(m)=e_j(m)+\tau}} \left|\langle A\nu_j^m,\ \nu_k^m\rangle\right|^2$$
$$= \left|\mathrm{vol}(\Sigma_e^\lambda)^{-1}\int_{\Sigma_e^\lambda} \sigma_0(A)\, d\omega_e^\lambda\right|^2 \delta_{\tau,0}.$$

(2) *For every* A, τ *and* $\{\nu_j^m\}_{j,m=1}^\infty$ *as above, we have*

$$\lim_{\delta\downarrow 0}\limsup_{m\uparrow\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \sum_{\substack{k \\ 0<|e_k(m)-e_j(m)-\tau|<\delta}} \left|\langle A\nu_j^m,\ \nu_k^m\rangle\right|^2 = 0.$$

4 A semi-classical analogy of Helton's theorem

In this section, we will discuss Problem C in Introduction.

The theorem of Helton–Guillemin [17], [11] gives a necessary and sufficient condition in terms of the cluster points for the differences of eigenvalues of the Laplacian on a compact Riemannian manifold in order that the geodesic flow is periodic. The purpose of this section is to investigate the relationship between the structure of the set of cluster points in a certain *semi-classical sense* for the differences of eigenvalues of the reduced quantum Hamiltonian $\hat{H}_\lambda$ and periodicity of the reduced Hamilton flow φ_t^λ described in Section 3.1.

4.1 Helton's theorem

Let M be a compact Riemannian manifold and let $\hat{H}$ be a first order self-adjoint non-negative elliptic pseudodifferential operator on M. We denote the eigenvalues of the operator $\hat{H}$ by $e_1 \leq e_2 \leq e_3 \leq \cdots$. For a positive real number λ and an open interval I, we set

$$\begin{aligned} \mathcal{N}(\lambda) &= \{j \in \mathbf{N}\,;\, e_j \leq \lambda\}, \\ N(\lambda; I) &= \sharp\{(j,k) \in \mathcal{N}(\lambda) \times \mathbf{N}\,;\, e_k - e_j \in I\}. \end{aligned}$$

Definition 4.1 *A real number τ is said to be a cluster point of the set $\{e_k - e_j\}$ if*

$$\lim_{\lambda \to \infty} N(\lambda; I) = \infty \tag{4.1}$$

holds for any open interval I containing τ. We denote the set of all cluster points by $D\sigma(\hat{H})$.

Remark 4.1 Note that the notion of the *cluster point* in the sense of Definition 4.1 is differ from that of the *accumulate point* in the usual sense. Indeed, zero is always a cluster point in the sense of Definition 4.1. However, let M be the standard sphere of dimension n and $\hat{H}$ be the square root of the Laplacian with respect to the standard metric. The eigenvalues of $\hat{H}$ are given by $\sqrt{p(p+n-1)}$ with p non-negative integers. Then the inequality

$$|\sqrt{p(p+n-1)} - \sqrt{q(q+n-1)}| \geq |p-q|$$

shows that zero is not an accumulate point of the set $\{e_k - e_j\}$.

With Definition 4.1, Helton's theorem [17] (see also [11]), which is well-known in spectral geometry, can be stated as follows.

Theorem 4.1 (Helton) *If the Hamilton flow generated by the principal symbol of $\hat{H}$ is not periodic, then we have $D\sigma(\hat{H}) = \mathbf{R}$.*

Theorem 4.2 (Helton-Guillemin) *Let $\hat{H}$ be the positive square root of the Laplacian on M. Then the geodesic flow is periodic if and only if there exists a positive constant T such that $D\sigma(\hat{H}) = \{\frac{2\pi n}{T}\,;\, n \in \mathbf{Z}\}$. In this case, the positive constant T is the least common period of the periodic geodesic flow.*

Remark 4.2 The cause of the restriction of the operator $\hat{H}$ to the Laplacian in Theorem 4.2 is the fact that the periodic geodesic flow has the common period ([35]) though the periodic Hamilton flow does not necessarily have the common period.

The notion of the cluster point defined in the following subsection (Definition 4.2) is a semi-classical analogy of Definition 4.1. The set of cluster points in the sense of Definition 4.2 depends on the energy level. However, it is natural since the dynamical behavior of the reduced flow φ_t^λ depend on the energy level.

Unfortunately, our main theorem stated in Section 4.2 does not completely clarify the relation between the structure of the set of the cluster point in the sense of Definition 4.2 and periodicity of the reduced flow φ_t^λ. Particularly, in case where the flow φ_t^λ has quite different behavior on different energy surfaces, such as a *magnetic flow* on a compact Riemann surface with constant negative curvature -1 (see Section 5), it is not made clear to what extent the structure of the set of the cluster points in the sense of Definition 4.2 depends on the energy level.

However, as we will see some examples in the next section, the structure of the set of the cluster points in the sense of Definition 4.2 will be closely related to the periodicity of the dynamical system CD_e^λ.

4.2 Cluster points in the semi-classical sense

As described in Section 3, for each integer m, let $e_1(m) \leq e_2(m) \leq e_3(m) \leq \cdots$ be the eigenvalues of the operator $\hat{H}_\lambda$ on $\mathcal{L}_{m\lambda}$. We fix $e > 0$ as an energy level. For an open interval I, we set

$$\begin{aligned} \mathcal{N}_m(e,c) &= \{\, j \in \mathbf{N}\,;\, |e_j(m) - me| \leq c\,\}, \\ N_m(e,c\,;I) &= \sharp\{\,(j,k) \in \mathcal{N}_m(e,c) \times \mathbf{N}\,;\, e_k(m) - e_j(m) \in I\,\}, \end{aligned}$$

where $c > 0$ is a positive constant.

Definition 4.2 *A real number τ is said to be a cluster point of the set* $\{ e_k(m) - e_j(m) \,;\, (j,k) \in \mathcal{N}_m(e,c) \times \mathbf{N},\ m \in \mathbf{Z} \}$ *in the semi-classical sense at energy level e if, for some constant $c > 0$,*

$$\lim_{m\to\infty} N_m(e, c\,; I) = \infty \tag{4.2}$$

holds for any open interval I containing τ. We denote by s-$D\sigma_e$ the set of all cluster points at the energy level e in the above sense.

We will investigate the relation between the structure of the set s-$D\sigma_e$ and the periodicity of the reduced flow φ_t^λ on Σ_e^λ. Note that, if the operator $\hat{H}$ is the Laplacian on P with respect to a Riemannian metric, then Theorem 4.2 can be applied for the set of cluster points $D\sigma(\hat{H})$ in the sense of Definition 4.1.

Lemma 4.1 *For every e, the set s-$D\sigma_e$ of cluster points in the sense of Definition* 4.2 *is a subset of the set $D\sigma(\hat{H})$ of cluster points in the sense of Definition* 4.1.

Proof. Set $\lambda_m = me + 1$. Then for any open interval I we have $N_m(e; I) \le N(\lambda_m; I)$. Note that the number $N(\lambda; I)$ is monotone increasing in λ. Hence the lemma follows. ∎

Corollary *Let $\hat{H}$ be the square root of the Laplacian on P with respect to the Riemannian metric described in Section* 3.1. *Suppose that the geodesic flow ψ_t on the cotangent bundle $T^*P \setminus 0$ is periodic with period $T > 0$. Then s-$D\sigma_e \subset \dfrac{2\pi}{T}\mathbf{Z}$.*

Our main theorem of this section is the following. (See Sections 3.2, 3.4 for the conditions (H1), (H2), respectively.)

Theorem 4.3 *Assume that the conditions* (H1) *and* (H2) *are satisfied. Then the set s-$D\sigma_e$ of all cluster points in the sense of Definition* 4.2 *is whole real line:*

$$s\text{-}D\sigma_e = \mathbf{R}.$$

By Theorem 4.3, we obtain the following.

Corollary *Assume that the conditions* (H1) *and* (H2) *are satisfied. Then for every real number τ, there are sequences m_l, j_l, $k_l \in \mathbf{N}$ with $m_l \uparrow \infty$ such that we have*

$$\frac{e_{j_l}(m_l)}{m_l} \to e \quad \textit{and} \quad e_{k_l}(m_l) - e_{j_l}(m_l) \to \tau \quad (l \to \infty).$$

4.3 Proof of Theorem 4.3

To begin with, we will recall the notation in Section 3. For $e > 0$, we set $\Sigma_e^\lambda = H_\lambda^{-1}(e) \subset X_\lambda$ where H_λ is the reduced Hamiltonian. Let $L^2(\Sigma_e^\lambda)$ be the Hilbert space of L^2-functions on Σ_e^λ with respect to the normalized Liouville measure, $d\omega_e^\lambda$, and let U_t^e be the one-parameter family of unitary operators on $L^2(\Sigma_e^\lambda)$ defined by $U_t^e a = a \circ \varphi_t^\lambda$, $a \in Ł^2(\Sigma_e^\lambda)$. Let

$$U_t^e = \int e^{itx}\, dE_e(x) \tag{4.3}$$

be the spectral resolution of U_t^e. (We have written the above spectral measure by $\tilde{E}$ in Section 1. However, in this subsection, we will write it E_e for simplicity.) We set

$$S_e^\lambda = \int x\, dE_e(x). \tag{4.4}$$

The self-adjoint operator S_e^λ is an L^2-extension of the restriction on Σ_e^λ of $-\sqrt{-1}$ times the Hamilton vector field.

Lemma 4.2 *If the reduced flow φ_t^λ is not periodic on Σ_e^λ, then the spectrum* $\mathrm{Spec}(S_e^\lambda)$ *of the self-adjoint operator S_e^λ is whole real line.*

Proof. The following proof is due to Guillemin [11]. However we will repeat it to make sure. Suppose that the orbit of the flow φ_t^λ through $z_0 \in \Sigma_e^\lambda$ is not periodic. Consider the bounded operator

$$f(S_e^\lambda)a = \int f(x)\, dE_e(x)a = \frac{1}{2\pi}\int \hat{f}(t) U_t^\lambda a\, dt, \quad a \in L^2(\Sigma_e^\lambda), \tag{4.5}$$

for $f \in C_0^\infty(\mathbf{R})$, where $\hat{f}$ is the Fourier transform of f. We will show that, $f(S_e^\lambda) = 0$ implies $f = 0$. This claim concludes $\mathrm{Spec}(S_e^\lambda) = \mathbf{R}$.

First we note that, by (4.5), for every continuous function a, the function $f(S_e^\lambda)a$ is also continuous and its value at $z \in \Sigma_e^\lambda$ is given by

$$f(S_e^\lambda)a(z) = \frac{1}{2\pi}\int \hat{f}(t) U_t^\lambda a(z)\, dt. \tag{4.6}$$

Second we claim that, for large $K > 0$ and $g \in C_0^\infty(\mathbf{R})$ with $\mathrm{supp}\, g \subset (-K, K)$, there are smooth functions $a_K \in C^\infty(\Sigma_e^\lambda)$ and $g_K \in C^\infty(\mathbf{R})$ such that these are satisfy $a_K(\varphi_t^\lambda z_0) = g(t) + g_K(t)$, $\|g_K\|_\infty \le 2\|g\|_\infty$ and $\mathrm{supp}\, g_K \cap [-K, K] = \emptyset$.

Indeed, let

$$\gamma_K = \{\varphi_t^\lambda z_0\,;\, t \in [-K, K]\}.$$

Since the Hamilton vector field X_{H_λ} generating the flow φ_t^λ is non-vanishing, the segment γ_K is diffeomorphic to the interval $[-K, K]$. Let Γ_K be a tubular neighborhood of γ_K

such that Γ_K is diffeomorphic to $[-K, K] \times D$ with $\gamma_K \cong [-K, K] \times \{0\}$, where D is sufficiently small disk of dimension $\dim \Sigma_e^\lambda - 1$. We will take a function $\rho \in C_0^\infty(D)$ with $0 \le \rho \le 1$, $\rho(0) = 1$. Set

$$a_0(s, x) = g(s)\rho(x), \quad (s, x) \in [-K, K] \times D,$$

which is a compactly supported smooth function on Γ_K, and extend a_0 to the smooth function a_K on Σ_e^λ which is zero on the outside of Γ_K. If we set $g_K(t) = a_K(\varphi_t^\lambda z_0) - g(t)$, then we have $g_K(t) = 0$ for $|t| \le K$, and hence $\operatorname{supp} g_K \cap [-K, K] = \emptyset$. Furthermore we obtain

$$|g_K(t)| = |a_K(\varphi_t^\lambda z_0) - g(t)| \le 2|g(t)|$$

for all $t \in \mathbf{R}$, since $|a(z)| \le |g(s)|$ for $z = (s, x) \in \Gamma_K \cong [-K, K] \times D$. Then the functions a_K, g_K have the required properties.

Now, suppose that $f \in C_0^\infty(\mathbf{R})$ satisfies $f(S_e^\lambda) = 0$. By (4.6), for every $a \in C^\infty(\Sigma_e^\lambda)$, $g \in C_0^\infty(\mathbf{R})$ and $K > 0$, we have

$$0 = f(S_e^\lambda) a_K(z_0) = \frac{1}{2\pi} \int \hat{f}(t) g(t)\, dt + \frac{1}{2\pi} \int_{\mathbf{R} \setminus [-K,K]} \hat{f}(t) g_K(t)\, dt. \tag{4.7}$$

The second term in the right hand side of (4.7) tends to zero as $K \to \infty$ by the inequality

$$\left| \int_{\mathbf{R} \setminus [-K,K]} \hat{f}(t) g_K(t)\, dt \right| \le 2\|g\|_\infty \int_{\mathbf{R} \setminus [-K,K]} |\hat{f}(t)|\, dt.$$

Therefore we obtain

$$\int \hat{f}(t) g(t)\, dt = 0$$

for arbitrary $g \in C_0^\infty(\mathbf{R})$, which implies $f = 0$. ▮

Before proceeding the proof of the main theorem, we need to give an account of the spectral measure lemma. For a function $a \in L^2(\Sigma_e^\lambda)$, we denote by $d\mu_a$ the spectral measure corresponding to a, that is $\mu_a(\Lambda) = \|E_e(\Lambda)a\|^2$ for Borel subset $\Lambda \subset \mathbf{R}$. Let $\{\nu_j^m\}_{j,m \in \mathbf{N}}$ be an orthonormal basis for $\mathcal{H}_\lambda$ consisting of eigenfunctions of the operator $\hat{H}_\lambda$. We set

$$\langle A \rangle_e^\lambda = \lim_{m \to \infty} N_m(e, c)^{-1} \sum_{j \in \mathcal{N}_m(e,c)} \langle A\nu_j^m, \nu_j^m \rangle, \tag{4.8}$$

where $N_m(e, c) = \sharp \mathcal{N}_m(e, c)$.

By Theorem 3.4, we have $\langle A \rangle_e^\lambda = \int_{\Sigma_e^\lambda} \sigma_0(A)\, d\omega_e^\lambda$ if the condition (H2) is satisfied. We set

$$m_A^\lambda(f) := \lim_{m \to \infty} N_m(e, c)^{-1} \sum_{j \in \mathcal{N}_m(e)} \sum_k f(e_k(m) - e_j(m)) |\langle A\nu_j^m, \nu_k^m \rangle|^2 \tag{4.9}$$

By Theorem 3.4, the existence of the limit in (4.9) can be proved by the same fashion as the proof of Lemma 2.2.

The functional m_A^λ on $C_0^\infty(\mathbf{R})$ defines a measure on $\mathbf{R}$. We will also denote this measure by dm_A^λ.

Lemma 4.3 (Zelditch) *Assume that the condition* (H2) *holds. Then for every pseudodifferential operator A of order zero on P commuting with the action of G, we have $dm_A^\lambda = d\mu_{\sigma_0(A)}$.*

This lemma can be proved by a method similar to the proof of Lemma 2.3.

We will prove the following proposition, which completes the proof of Theorem 4.3.

Proposition 4.1 *Assume that the conditions* (H1) *and* (H2) *are satisfied. Then the spectrum* $\mathrm{Spec}(S_e^\lambda)$ *of the self-adjoint operator S_e^λ is contained in s-$D\sigma_e$.*

Proof. Let τ be not in s-$D\sigma_e$. Then, for every positive constant $c > 0$, one can take a sequence m_l of positive integers and an open interval I containing τ such that $m_l \uparrow \infty$ as $l \to \infty$ and $N_{m_l}(e, c\,; I) \leq C$ for some constant $C > 0$. For any smooth function f on $\mathbf{R}$ with compact support containing the open interval I, we consider the bounded operator $f(S_e^\lambda)$. Let A be a pseudodifferential operator on P of order zero commuting with G-action, and let a be the smooth function on Σ_e^λ induced by the principal symbol of A. On the one hand, by Lemma 4.3, we have

$$\begin{aligned}(f(S_e^\lambda)a,\, a)_{L^2(\Sigma_e^\lambda)} &= \int f\, d\mu_a \\ &= \int f\, dm_A^\lambda \\ &= \lim_{m\to\infty} S_m(A,f) \\ &= \lim_{l\to\infty} S_{m_l}(A,f),\end{aligned}$$

where

$$S_m(A,f) = N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \sum_{k\in\mathbf{N}} f(e_k(m) - e_j(m)) |\langle\, A\nu_j^m,\, \nu_k^m\,\rangle|^2.$$

On the other hand,

$$\begin{aligned}|S_{m_l}(A,f)| &\leq \|A\|^2 \sup_{x\in\mathbf{R}} |f(x)|^2 \frac{N_{m_l}(e,c\,;I)}{N_{m_l}(e,c)} \\ &\leq C\|A\|^2 \sup_{x\in\mathbf{R}} |f(x)|^2 N_{m_l}(e,c)^{-1}. \qquad (4.10)\end{aligned}$$

Since $N_{m_l}(e,c) \to \infty$ as $l \to \infty$, the last term in (4.10) tends to zero as $l \to \infty$. Thus we have $(f(S_e^\lambda)a,\, a)_{L^2(\Sigma_e^\lambda)} = 0$. For every smooth function, a, on Σ_e^λ, there exists a pseudodifferential operator A on P of order zero commuting with the action of G such that the function on Σ_e^λ induced by the principal symbol of A coincides with a (see Lemma 3.3). Hence we obtain $(f(S_e^\lambda)a,\, a)_{L^2(\Sigma_e^\lambda)} = 0$ for every smooth function a on Σ_e^λ. By the polarization identity

$$(f(S_e^\lambda)a,\, b)_{L^2(\Sigma_e^\lambda)} = \frac{1}{4}\sum_{n=0}^{3} i^n (f(S_e^\lambda)(a + i^n b),\, a + i^n b)_{L^2(\Sigma_e^\lambda)} = 0,$$

we have $f(S_e^\lambda) = 0$. Since f is arbitrary as far as its support is contained in the open interval I, we conclude that τ is not in the spectrum of S_e^λ, which completes the proof of this proposition. ▮

5 Examples for the circle bundle case

Here we will discuss the circle bundle case, that is the case where $G = S^1$. This case is particularly important since there are interesting examples. In Section 5.1, we will recall the definition of the reduced quantum and classical dynamical system for this case. We will give some examples in Section 5.2.

5.1 Magnetic Schrödinger operator

Let $\pi : P \to M$ be a principal S^1-bundle over a compact Riemannian manifold $(M, \langle\ ,\ \rangle_M)$ and let Θ be a connection 1-form on P. Let $\boldsymbol{B}$ be the curvature 2-form of Θ. Note that $\boldsymbol{B}$ is a closed 2-form on M. We fix a strictly positive function V on M. Then the metric $\langle\ ,\ \rangle_P$ on P, defined by the identity

$$\langle u, v \rangle_P = \langle d\pi(u), d\pi(v) \rangle_M + V^{-2}\Theta(u)\Theta(v) \quad u, v \in TP, \tag{5.1}$$

is invariant under the action of S^1. Now we consider the first order self-adjoint non-negative elliptic ψDO $\hat{H}$ on P defined by

$$\hat{H} = \sqrt{D^*D - V^2\partial_\theta^2}, \tag{5.2}$$

where D is the covariant exterior differentiation with respect to the connection Θ, and ∂_θ is the infinitesimal generator of S^1-action. The operator $\hat{H}$ commutes with S^1-action. Note that the principal symbol $\tilde{H}$ of $\hat{H}$ is the Riemannian norm function on $T^*P \setminus 0$, and hence corresponding Hamilton flow ψ_t is the geodesic flow. The S^1-action on P lifts to the Hamiltonian (left) action on T^*P:

$$z(p, \zeta) = (pz^{-1}, z^*\zeta), \quad z \in S^1,\ (p, \zeta) \in T^*P, \tag{5.3}$$

and its moment map is given by

$$\Phi : T^*P \to \mathbf{R},\ \Phi(p, \zeta) = \zeta(\partial_\theta). \tag{5.4}$$

We take the irreducible representation $\lambda = 1 \in \mathbf{R}$ of S^1, that is the multiplication by elements of S^1. Then the corresponding reduced phase space (X_1, Ω_1), $X_1 = \Phi^{-1}(1)/S^1$ is symplectically diffeomorphic to $(T^*M, \Omega_M - \pi_M^*\boldsymbol{B})$, where Ω_M is the canonical symplectic form on T^*M. The Hamiltonian H on $X_1 = T^*M$, which is induced by $\tilde{H}$, is of the form

$$H(x, \xi) = \sqrt{\|\xi\|^2 + V(x)}. \tag{5.5}$$

The reduced flow φ_t is generated by $(H, \Omega_M - \pi_M^*\boldsymbol{B})$ and it is called *electro-magnetic flow* associated with the magnetic field $\boldsymbol{B}$ and the electric potential V. Let ω_e be the Liouville measure on $\Sigma_e = H^{-1}(e)$ $(e > \max V)$. Then the dynamical system CD_e formulated in Section 3 is the triple $(\Sigma_e, \varphi_t, \omega_e)$.

If we set $(x(t), \xi(t)) = \varphi_t(x_0, \xi_0)$ for $(x_0, \xi_0) \in \Sigma_e$, then the curve $x(t)$ on M satisfies the equation

$$\frac{D}{dt}\frac{dx}{dt} = \frac{1}{e}J(\frac{dx}{dt}) - \frac{1}{2e^2}\mathrm{grad}\,(V^2), \tag{5.6}$$

with the initial conditions

$$x(0) = x_0, \quad \frac{dx}{dt}(0) = \frac{1}{e}\xi_0, \tag{5.7}$$

where D/dt is the covariant differentiation associated with the given metric on M, $J : TM \to TM$ is the skew symmetric operator characterized by the identity

$$\boldsymbol{B}_x(u, v) = \langle\, u,\ J(v)\,\rangle. \tag{5.8}$$

Note that, in (5.7), we have identified T^*M with TM by the Riemannian metric $\langle\, ,\, \rangle_M$.

Next We will recall the formulation of the reduced quantum dynamical system QD described in Section 3.1.

The action of S^1 on $L^2(P)$ breaks it into the following direct sum of Hilbert spaces:

$$L^2(P) = \bigoplus_{m\in\mathbf{Z}} \mathcal{L}_m, \tag{5.9}$$

where the closed subspace $\mathcal{L}_m$ is defined by

$$\mathcal{L}_m = \{\, f \in L^2(P)\,;\, f(pz) = z^{-m}f(p) \ \ z \in S^1,\ p \in P\}. \tag{5.10}$$

Now, for every integer $m \in \mathbf{Z}$, we define the *magnetic Schrödinger operator* $\hat{H}_m$ by the restriction of the operator $\hat{H}$ to the subspace $\mathcal{L}_m$.

We will give another description of the magnetic Schrödinger operator $\hat{H}_m$. For every integer $m \in \mathbf{Z}$, let $L^m \to M$ be the Hermitian line bundle associated with P via the character $z \mapsto z^m$ of the group S^1. Then there is a natural unitary isomorphism $L^2(M, L^m) \cong \mathcal{L}_m$ from the Hilbert space $L^2(M, L^m)$ of L^2-sections of L^m onto the Hilbert space $\mathcal{L}_m$. By this unitary isomorphism, the operator $\hat{H}_m^2$ is unitarily equivalent to the second order positive elliptic operator

$$(\nabla_m^*\nabla_m) + m^2V^2, \tag{5.11}$$

where $V = \|\partial/\partial\theta\|^{-1}$ and ∇_m is the connection on the line bundle L^m induced by Θ. The elliptic operator $\nabla_m^*\nabla_m$ is locally expressed by the following form:

$$\nabla_m^*\nabla_m = -\frac{1}{\sqrt{G}}\sum_{i,j=1}^{n}\left(\frac{\partial}{\partial x^i} + m\sqrt{-1}A_i\right) g^{ij}\sqrt{G}\left(\frac{\partial}{\partial x^j} + m\sqrt{-1}A_j\right), \tag{5.12}$$

where G is the determinant of the matrix (g_{ij}) of the component of the Riemannian metric on M, $(g^{ij}) = (g_{ij})^{-1}$ and $\boldsymbol{A} = \sum_i A_i dx^i$ is the pull-back of the connection 1-form Θ by a local section of $\pi : P \to M$, and hence it satisfies $d\boldsymbol{A} = \boldsymbol{B}$. The local 1-form $\boldsymbol{A}$ represents a vector potential of the magnetic field $\boldsymbol{B}$.

5.2 Examples

Example 1 Let M be a Riemann surface with constant negative curvature -1 and let $\boldsymbol{B}$ be the volume form on M. Then, by Gauss-Bonnet theorem, the integral of $(2\pi)^{-1}$ times the volume form $\boldsymbol{B}$ is an integer, and hence there exists a principal S^1-bundle P and a connection 1-form Θ whose curvature form is $\boldsymbol{B}$. We take $V \equiv 1$. Then, for $e > 1$, the Liouville measure ω_e on the energy surface $\Sigma_e = H^{-1}(e)$ is given by the direct product of the canonical measure on the unit sphere and the volume measure dV_M on M up to constant multiple. For an integer m, let $\{\nu_j^m\}$ be an orthonormal basis of eigenfunctions of the Magnetic Schrödinger operator $\hat{H}_m$ for the Hilbert space $\mathcal{L}_m$. Let f be a smooth function on M and $A_f \in \mathcal{A}_0$ be the multiplication operator by the lift of f on P. The principal symbol of A_f is given by the lift of f. Therefore we have

$$\langle\, A_f \nu_j^m,\, \nu_j^m \,\rangle = \int_M f |\nu_j^m|^2 \, dV_M, \tag{5.13}$$

$$\langle\, \sigma_0(A_f) \,\rangle = \mathrm{vol}(M)^{-1} \int_M f \, dV_M. \tag{5.14}$$

It is well-known ([29]) that the dynamical system $(\Sigma_e, \varphi_t, \omega_e)$ is ergodic if $e \geq \sqrt{2}$, periodic if $1 < e < \sqrt{2}$. Therefore we obtain the following corollary of Theorems 3.1, 3.3.

Corollary *Let M be a compact Riemann surface with constant negative curvature -1, $\mathbf{B}$ the volume 2-form and $e \geq \sqrt{2}$. Then for every orthonormal basis $\{\nu_j^m\}$ of eigenfunctions of $\hat{H}$, there exists a family $\{J_m\}$ of subsets in $\mathcal{N}_m(e,c)$ satisfying*

$$\lim_{m\to\infty} \frac{\sharp J_m}{N_m(e,c)} = 1 \tag{5.15}$$

such that for all $f \in C^\infty(M)$ we have

$$\lim_{m\to\infty} \max_{j\in J_m} \left| \int_M f |\nu_j^m|^2 dV_M - \mathrm{vol}(M)^{-1} \int_M f \, dV_M \right| = 0. \tag{5.16}$$

Proof. In view of Theorems 3.1 and 3.3, we only need to prove that a family $\{J_m\}$ can be taken independently of the choice of a smooth function f. For this sake, let $\{\varphi_p\}$ be an orthonormal basis for $L^2(M)$ consisting of eigenfunctions of the Laplacian. For every $l \in \mathbf{N}$, let $\{J_m(l)\}$ be a family satisfying (5.15), (5.16) for all $f = \varphi_p$ with $p \leq l$. We may assume $J_m(l+1) \subset J_m(l)$ for all l. We can find a sequence $\{l_m\}_{m\in\mathbf{N}}$ of natural numbers which is monotone increasing and tends to infinity as m goes to infinity such that

$$1 - \frac{1}{2^{l_m}} \leq \frac{\sharp J_m(l_m)}{N_m(e,c)}. \tag{5.17}$$

We set $J_m = J_m(l_m)$. We will show that the family $\{J_m\}$, $J_m \subset \mathcal{N}_m(e,c)$ is a required one. In view of (5.17), (5.15) is obvious. We will write

$$A_j^m(f) = \int_M f|\nu_j^m|^2 dV_M - \mathrm{vol}(M)^{-1}\int_M f\, dV_M,$$

for each smooth function $f \in C^\infty(M)$. Since $\lim_{m\to\infty}\max_{j\in J_m(l)}|A_j^m(\varphi_p)| = 0$, for $p \leq l$ and $l_m \uparrow \infty$ as $m \to \infty$, we have

$$\lim_{m\to\infty}\max_{j\in J_m}|A_j^m(\varphi_p)| = 0 \tag{5.18}$$

for all $p \in \mathbf{N}$. For every smooth function f, and every positive integer n, we set $f_n = \sum_{p=1}^{n}\langle f, \varphi_p\rangle\varphi_p$, which converges to f in $L^2(M)$ as $n \to \infty$. Note that, since $\{\varphi_p\}$ is an orthonormal basis of eigenfunctions, the function f_n is also smooth and converges to f uniformly on M. Therefore, for any $\varepsilon > 0$, we can take $n \in \mathbf{N}$ so that $\|f - f_n\|_\infty < \varepsilon$. Then, for every $j \in J_m$, we have

$$\begin{aligned}
|A_j^m(f)| &\leq \int_M |f-f_n||\nu_j^m|^2 dV_M + |A_j^m(f_n)| + \mathrm{vol}(M)^{-1}\int_M |f-f_n| dV_M \\
&\leq 2\|f-f_n\|_\infty + \sum_{p=1}^{n}|\langle f, \varphi_p\rangle||A_j^m(\varphi_p)| \\
&\leq 2\varepsilon + \|f\|_{L^2}\left(\sum_{p=1}^{n}|A_j^m(\varphi_p)|^2\right)^{1/2},
\end{aligned}$$

and hence

$$\max_{j\in J_m}|A_j^m(f)| \leq 2\varepsilon + \|f\|_{L^2}\left(\sum_{p=1}^{n}\max_{j\in J_m}|A_j^m(\varphi_p)|^2\right)^{1/2}. \tag{5.19}$$

By (5.18), we obtain

$$\limsup_{m\to\infty}\max_{j\in J_m}|A_j^m(f)| \leq 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the family $\{J_m\}$ satisfies (5.16) for every $f \in C^\infty(M)$. ▮

Example 2 The Hopf fibration $S^3 \to S^2$. Let $P = S^3 = \mathrm{SU}(2)$ and $M = S^2$. We fix the inner product $\langle A, B\rangle_{su(2)} = 2\mathrm{Tr}(B^*A)$ on the Lie algebra $su(2)$ of SU(2). Then

$$e_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

form an orthonormal basis which satisfy the relation

$$[e_1, e_2] = e_3,\ \ [e_2, e_3] = e_1,\ \ [e_3, e_1] = e_2.$$

We consider M as the adjoint orbit through e_1, and hence it equipped with the metric $\langle\, ,\, \rangle_M$ which is isometric to the standard 2-sphere with radius 1. The natural projection $\pi : P \to M$ is an S^1-bundle. Let Θ be the connection 1-form on P defined by

$$\Theta(A) = \frac{1}{2}\langle\, A, e_1\,\rangle E, \quad A \in su(2),$$

where $E = 2e_1$. We regard $\mathbf{R} \cdot E$ as the Lie algebra of S^1. Note that the curvature form of the connection Θ equals (1/2) times the volume form on M. For positive constant $V > 0$, we define the metric $\langle\, ,\, \rangle_{P,V}$ by

$$\langle\, ,\, \rangle_{P,V} = \pi^*\langle\, ,\, \rangle_M + \frac{1}{4V^2}\langle\, \Theta, \Theta\,\rangle_{su(2)}.$$

Especially, the Riemannian manifold $(P, \langle\, ,\, \rangle_{P,1/2})$ is isometric to the standard 3-sphere with radius 2.

Lemma 5.1 *For every $e > V$, the magnetic flow φ_t on Σ_e is periodic with period $T(e) = 2\pi\sqrt{\dfrac{e^2}{e^2 - d}}$ where $d = V^2 - 1/4$.*

Proof. We will show that the solution curve of the equation (5.6), (5.7) is periodic with with period $T(e)$. The magnetic field $\boldsymbol{B}$ is (1/2) times the volume form on S^2. Thus $\boldsymbol{B}$ is of the form

$$\boldsymbol{B}_x(u, v) = \frac{1}{2}\langle\, u \times v,\ x\,\rangle,$$

where $\langle\, ,\, \rangle$ is the canonical metric on $\mathbf{R}^3 \cong su(2)$, and we regard TS^2 as

$$TS^2 = \{(x, u) \in \mathbf{R}^3 \times \mathbf{R}^3\,;\ |x| = 1,\ \langle\, x,\ u\,\rangle = 0\}.$$

Therefore the map $J : TS^2 \to TS^2$ defined in (5.8) is given by $J(v) = (1/2)v \times x$. Since V is constant, the equation (5.6) comes into the following form:

$$\frac{D}{dt}\dot{x} = \frac{1}{2e}\dot{x} \times x. \tag{5.20}$$

Note that the covariant differentiation of $\dot{x}$ is obtained by the projection of $\ddot{x}$ onto the subspace perpendicular to x. Thus we have

$$\frac{D}{dt}\dot{x} = \ddot{x} - \langle\, \ddot{x},\ x\,\rangle x.$$

Since $|\dot{x}(t)|$ is constant, we have $\langle \ddot{x}, x \rangle = -\alpha^2$, where we set $\alpha = |\dot{x}(0)| = (1/e)\|\xi\|$. Clearly we have

$$\alpha = \sqrt{\frac{e^2 - V^2}{e^2}}. \tag{5.21}$$

Therefore the equation (5.20) is written as

$$\ddot{x} = \frac{1}{2e}\dot{x} \times x - \alpha^2 x. \tag{5.22}$$

We will seek for a solution of the form

$$x(t) = a\cos(ct)e_1 + a\sin(ct)e_2 + be_3. \tag{5.23}$$

Substituting (5.23) into (5.22), we obtain that, if

$$a = \sqrt{\frac{e^2 - V^2}{e^2 - d}}, \; b = -\frac{1}{2\sqrt{e^2 - d}}, \; c = \sqrt{\frac{e^2 - d}{e^2}},$$

then (5.23) is a solution of (5.22). Since the isometry group of S^2 acts transitively on Σ_e, every solution of (5.22) is an isometric image of (5.23). Clearly the solution (5.23) is periodic with period $2\pi/c = T(e)$. ▮

Remark 5.1 Lemma 5.1 is essentially obtained by Sunada [30].

It is easy to see that, the eigenvalues of the magnetic Schrödinger operator $\hat{H}_m^V$ are given by

$$\lambda_p^V(m) = \frac{1}{2}\sqrt{(2p + |m| + 1)^2 + 4dm^2 - 1}, \quad p = 0, 1, 2, \ldots,$$

and the multiplicity of $\lambda_p^V(m)$ equals $2p + |m| + 1$. (See, for instance, [19].) Then we will claim the following.

Proposition 5.1 *We have $s\text{-}D\sigma_e = \frac{2\pi}{T(e)}\mathbf{Z}$ for all $e > V$.*

Proof. We note that, for $e > V$, $|\lambda_p^V(m) - me| \leq c$ if and only if

$$C_-(m) \leq p \leq C_+(m), \tag{5.24}$$

where we set

$$C_\pm(m) = \frac{\sqrt{4(em \pm c)^2 - 4dm^2 + 1} - m - 1}{2}. \tag{5.25}$$

It is easy to see that

$$C_+(m) - C_-(m) \geq \frac{2cem}{\sqrt{4(e^2 - d)m^2 + 4c^2 + 1}}.$$

We take constants $m_0 > 0$ and $c > 0$ such that $e^2 m_0^2 - 1 > 0$ and

$$c > \sqrt{\frac{4(e^2 - d)m_0^2 + 1}{4(e^2 m_0^2 - 1)}}.$$

Then we have $C_+(m) - C_-(m) > 1$ for every $m > m_0$, and hence we can take a positive integer q_m satisfying (5.24) for every $m > m_0$.

For an integer d, we set $p_m = q_m + d$. Then we have

$$\lambda^V_{p_m}(m) - \lambda^V_{q_m}(m) \to \frac{2\pi}{T(e)} d$$

as $m \to \infty$. Therefore $(2\pi/T(e))\mathbf{Z} \subset s\text{-}D\sigma_e$.

Conversely, let $\tau \in s\text{-}D\sigma_e$ and let $I = (a, b)$ be an open interval containing τ such that $b - a < 2\pi/T(e)$. Then, for every positive integer m, there are positive integers q_m, p_m such that q_m satisfies (5.1) and $a < \lambda^V_{p_m}(m) - \lambda^V_{q_m}(m) < b$. Let $d_m = p_m - q_m$. We will assume that $\tau > 0$, and hence $a > 0$, $d_m > 0$. By the inequality

$$\lambda^V_{p+1}(m) - \lambda^V_p(m) > \frac{1}{1 + 2\sqrt{d}}$$

(if $d \leq 0$ then the right hand side of this inequality can be replaced by 1), we have

$$b > \lambda^V_{p_m}(m) - \lambda^V_{q_m}(m) > \frac{d_m}{1 + 2\sqrt{d}} > 0.$$

Note that d_m is a positive integer. Therefore, by taking a subsequence of $\{d_m\}$ if necessary, we can assume that there is an integer d such that $d_m = d$ for sufficiently large m. Then, by the same argument as above, we obtain that $\lambda^V_{p_m}(m) - \lambda^V_{q_m}(m) \to (2\pi/T(e))d$ as $m \to \infty$. Thus by $b - a < 2\pi/T(e)$, we conclude that $\{(2\pi/T(e))d\} = I \cap (2\pi/T(e))\mathbf{Z}$, and hence $\tau = (2\pi/T(e))d$. We can prove it in a similar fashion for $\tau < 0$. Since the open interval I is arbitrary as far as it contains τ, we conclude that $\tau = (2\pi/T(e))d$, which completes the proof. ▮

Example 3 Let $\mathbf{H}_1^{\mathrm{red}}$ be the reduced Heisenberg group of dimension 3 (see [9]). The group multiplication of $\mathbf{H}_1^{\mathrm{red}} = \mathbf{R}^2 \times S^1$ is defined by the identity

$$(x, y, e^{2\pi i t}) \cdot (x', y', e^{2\pi i t'}) = (x + x', y + y', e^{2\pi i (t + t' + (1/2)(x'y - xy'))}).$$

Let Γ be the lattice $\Gamma = \{(k, n, e^{\pi i k n}) \in \mathbf{H}_1^{\mathrm{red}};\, k, n \in \mathbf{Z}\}$. Then the nilmanifold $P = \mathbf{H}_1^{\mathrm{red}}/\Gamma$ is an S^1-bundle over the flat torus $\mathrm{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$. We fix the connection 1-form

$$\Theta = 2\pi(dt + \frac{1}{2}(xdy - ydx))$$

on P, whose curvature 2-form is 2π times the volume form on T^2. We fix the invariant metric on P obtained by setting $V = 2\pi$ in (5.1).

Lemma 5.2 *For every $e > 2\pi$, the magnetic flow φ_t on Σ_e is periodic with period e.*

This lemma can be proved by solving directly the equation (5.6).

It is not hard to see that the eigenvalues of the magnetic Schrödinger operator $\hat{H}_m$ are given by

$$\lambda_p(m) = \sqrt{2\pi|m|(2p+1) + 4\pi^2 m^2}, \quad p = 0, 1, 2, \dots,$$

and the multiplicity of $\lambda_p(m)$ is equal to $|m|$. (See [16].)

Proposition 5.2 *For every $e > 2\pi$, we have $s\text{-}D\sigma_e = \frac{2\pi}{e}\mathbf{Z}$.*

Proof. Note that $|\lambda_p(m) - me| \le c$ if and only if

$$C_-(m) \le p \le C_+(m), \tag{5.26}$$

where we set

$$C_\pm(m) = \frac{(me \pm c)^2}{4\pi m} - \frac{2\pi m + 1}{2}.$$

We also note that for non-negative integers p, q, we have

$$\lambda_p(m) - \lambda_q(m) = \frac{2\sqrt{2\pi}(p-q)}{\sqrt{(2p+1)/m + 2\pi} + \sqrt{(2q+1)/m + 2\pi}} \tag{5.27}$$

Since $C_+(m) - C_-(m) = ce/\pi$, if we take a constant $c > 0$ such that $c > \pi/e$, then for every positive integer m we can take a positive integer q_m satisfying (5.26). For arbitrary integer d, let $p_m = q_m + d$. Then, by (5.27), we have

$$\lambda_{p_m}(m) - \lambda_{q_m}(m) \to \frac{2\pi}{e}d,$$

and hence $\frac{2\pi}{e}d \in s\text{-}D\sigma_e$.

Conversely, let $\tau \in s\text{-}D\sigma$ and $I = (a, b)$ be an open interval containing τ. Let p_m, q_m be two non-negative integers such that q_m satisfies (5.26) and $\lambda_{p_m}(m) - \lambda_{q_m}(m) \in I$. We set $r_m = (2q_m + 1)/m + 2\pi$ and $d_m = p_m - q_m$. Then $\lambda_{p_m}(m) - \lambda_{q_m}(m) \in I$ if and only if

$$\frac{a}{2}\sqrt{\frac{r_m}{2\pi}} < \frac{d_m}{\sqrt{1 + 2d_m/mr_m} + 1} < \frac{b}{2}\sqrt{\frac{r_m}{2\pi}}. \tag{5.28}$$

Assume that $\tau > 0$, and hence $a > 0$. Note that $\sqrt{r_m/2\pi} \to e^2/2\pi$ as $m \to \infty$ and $d_m \in \mathbf{Z}$. Then, by the inequality (5.28), we have $a < (2\pi/e)d_m < b$ for sufficiently large m. Therefore if we take $b - a$ sufficiently small, then there is an integer d such that $d_m = d$ for sufficiently large m. By the same argument as in Example 2, we have $\tau = (2\pi/e)d$. We can prove it by the same way in case where $\tau < 0$. ∎

Examples 2, 3 suggest that, if the magnetic flow φ_t^λ is periodic on Σ_e^λ with period $T(e)$, then $s\text{-}D\sigma_e = (2\pi/T(e))\mathbf{Z}$. However, it has not been proved yet.

Appendix: Weak limits of eigenfunctions

Throughout this section, we will follow the notation of Section 2. In Section 2, we have mentioned quantum ergodicity. Ergodicity of the classical dynamical system $(\Sigma, \varphi_t, \omega)$ affects the asymptotic behavior of eigenfunctions φ_j in the high energy level. Especially, Theorem 2.3 says that, if the dynamical system $(\Sigma, \varphi_t, \omega)$ is ergodic then, for every orthonormal basis $\{\varphi_j\}$ of eigenfunctions of $\hat{H}$, there exists a subsequence of full density such that it converges weakly to the Liouville measure. Then it is natural to ask whether the sequence $\{\varphi_j\}$ converges without taking a subsequence. Though there are several results on this problem ([22], [44]), it seems very hard to solve it completely. Here we will give some generalities on the weak limit points of the eigenfunctions, which are called *quantum limits*, and some results obtained by using the methods in this article.

A.1 C^*-algebra $\mathcal{A}$ and its properties

Let $\mathcal{A}$ be the closure of the algebra $\mathcal{A}_0$ with respect to the operator norm. $\mathcal{A}$ is a C^*-subalgebra of the algebra of all bounded operators on $L^2(M)$.

Lemma A.1 *There exists a unique $*$-homomorphism $\sigma : \mathcal{A} \to C(\Sigma)$ from $\mathcal{A}$ onto the commutative C^*-algebra $C(\Sigma)$ of all continuous functions on Σ such that the restriction $\sigma_{|\mathcal{A}_0}$ of σ to $\mathcal{A}_0$ coincides with the principal symbol map σ_0.*

Proof. For every $A \in \mathcal{A}$, we can choose a sequence $A_n \in \mathcal{A}_0$ such that $\|A - A_n\| \to 0$ as $n \to \infty$. Note that the sequence $\{\sigma_0(A_n)\}_{n\in\mathbf{N}}$ in $C(\Sigma)$ converges to a continuous function. Indeed, by the formula (see [34])

$$\|\sigma_0(A_n)\|_\infty = \inf_{K:\,\mathrm{compact}} \|A_n + K\|, \tag{A.1}$$

we have

$$\|\sigma_0(A_n) - \sigma_0(A_m)\|_\infty \leq \|A_n - A_m\| \to \infty \ (n, m \to \infty).$$

We define

$$\sigma(A) = \lim_{n\to\infty} \sigma_0(A_n). \tag{A.2}$$

Let $B_n \in \mathcal{A}_0$ be another sequence such that $\|A - B_n\| \to 0$. Let $b = \lim_{n\to\infty} \sigma_0(B_n)$. Then we have

$$\|\sigma(A) - b\|_\infty \leq \|\sigma(A) - \sigma_0(A_n)\|_\infty + \|A_n - B_n\| + \|b - \sigma_0(B_n)\|_\infty \to 0,$$

and hence the limit in (A.2) is independent of the choice of the sequence $A_n \in \mathcal{A}_0$ as far as it converges to A. Since the principal symbol map σ_0 is a $*$-homomorphism, the map σ is also a $*$-homomorphism. Clearly we have $\sigma_{|\mathcal{A}_0} = \sigma_0$. To show the surjectivity of σ, we note that σ_0 is surjective onto the $*$-subalgebra $C^\infty(\Sigma)$, and hence the image of σ is dense in $C(\Sigma)$. It is well-known (see Proposition 2.3.1 in [4]) that the image of a $*$-homomorphism from a C^*-algebra to another is closed. Thus the image of σ is closed, and hence the assertion follows. ∎

Before going to discuss the weak limits of eigenfunctions, we will mention some properties of the C^*-algebra $\mathcal{A}$. Let $\mathcal{K}$ be the C^*-algebra of all compact operators on $L^2(M)$. Let $A \in \mathcal{K}$. Then, for any $\varepsilon > 0$, we can take a finite-rank operator B such that $\|A - B\| < \varepsilon$. Let $\{e_n\}_{n=1}^N$ be an orthonormal basis for the image of B ($N = \dim \mathrm{Im} B$). The operator B is an integral operator with the kernel

$$K(x,y) = \sum_{n=1}^{N} e_n(x)\overline{B^* e_n(y)} \in L^2(M \times M).$$

We take $L \in C^\infty(M \times M)$ such that $\|K - L\|_{L^2(M\times M)} < \varepsilon$. We denote by T_L the integral operator with the kernel L. The operator T_L is a smoothing operator, and hence $T_L \in \mathcal{A}_0 \subset \mathcal{A}$ with $\sigma(T_L) = 0$. Furthermore we have

$$\|A - T_L\| \leq \|A - B\| + \|B - T_L\| \leq \|A - B\| + \|K - L\|_{L^2(M\times M)} \leq 2\varepsilon.$$

Since T_L is a compact operator, so is A. Thus we obtain

$$\mathcal{K} \subset \sigma^{-1}(0) \subset \mathcal{A}. \tag{A.3}$$

(A.3) makes us to obtain the following proposition.

Proposition A.1 *The following sequence is exact:*

$$0 \longrightarrow \mathcal{K} \xrightarrow{\ i\ } \mathcal{A} \xrightarrow{\ \sigma\ } C(\Sigma) \longrightarrow 0, \tag{A.4}$$

where $i : \mathcal{K} \to \mathcal{A}$ is the inclusion.

Proof. Let $A \in \mathcal{A}$ with $\sigma(A) = 0$. Then one can take a sequence $A_n \in \mathcal{A}_0$ such that $\|A - A_n\| \to 0$. Since σ is continuous, we have

$$\|\sigma_0(A_n)\|_\infty \to 0. \tag{A.5}$$

By (A.1), for each n, there is a compact operator $K_n \in \mathcal{K}$ such that

$$\|\sigma(A_n)\|_\infty \leq \|A_n - K_n\| \leq \|\sigma(A_n)\|_\infty + \frac{1}{n} \to 0. \tag{A.6}$$

From (A.6), it follows that

$$\|A - K_n\| \le \|A - A_n\| + \|A_n - K_n\| \to 0.$$

Since $K_n \in \mathcal{K}$, we have $A \in \mathcal{K}$, and hence $\sigma^{-1}(0) = \mathcal{K}$. Therefore the sequence (A.4) is exact. ▮

Lemma A.2 *The C^*-algebra $\mathcal{A}$ is separable.*

Proof. Let $\{a_n\}_{n\in\mathbf{N}}$ be a countable dense set in $C(\Sigma)$, and let $A_n \in \sigma^{-1}(a_n)$. Note that the C^*-algebra $\mathcal{K}$ of all compact operators is separable. Thus we can take a countable dense set $\{K_m\}_{n\in\mathbf{N}}$ in $\mathcal{K}$. Set $B_{m,n} = A_n + K_m$. We will show that the countable set $\{B_{m,n}\}_{m,n\in\mathbf{N}}$ is dense in $\mathcal{A}$. Let $A \in \mathcal{A}$. For arbitrary $\varepsilon > 0$, there is a positive number n such that

$$\|\sigma(A - A_n)\|_\infty = \inf_{K:\,\mathrm{compact}} \|A - A_n + K\| < \varepsilon.$$

Let K be a compact operator such that $\|A - A_n + K\| < \varepsilon$. Then we can take a positive number m such that $\|K + K_m\| < \varepsilon$. Therefore we obtain

$$\|A - B_{m,n}\| < \|A - A_n + K\| + \|K + K_m\| < 2\varepsilon.$$

This shows that the countable set $\{B_{m,n}\}_{m,n\in\mathbf{N}}$ is dense in $\mathcal{A}$. ▮

A.2 Quantum limits

Let $\{\varphi_j\}$ be an orthonormal basis of eigenfunction of the operator $\hat{H}$. We will consider φ_j as a vector state on the C^*-algebra $\mathcal{A}$ defined by $\varphi_j(A) = \langle A\varphi_j, \varphi_j \rangle$, $A \in \mathcal{A}$. Since $\{\varphi_j\}$ is bounded in $\mathcal{A}^*$, it is relatively compact with respect to the weak*-topology. Combining this with Lemma A.2, we can find a convergent subsequence of $\{\varphi_j\}$ in the weak*-topology.

Lemma A.3 *Let $\{\varphi_j\}_{j\in J}$ $(J \subset \mathbf{N})$ be a convergent subsequence of the fixed orthonormal basis of eigenfunctions in the weak*-topology. Then there exists a measure μ_J on Σ which is invariant under the Hamilton flow φ_t such that*

$$\lim_{J\ni j\to\infty} \varphi_j(A) = \int_\Sigma \sigma(A)\, d\mu_J, \tag{A.7}$$

for every $A \in \mathcal{A}$.

Proof. First, we note that the orthonormal basis $\{\varphi_j\}$ converges weakly to zero, and a compact operators map a weakly convergent sequence to a strongly convergent one. Thus the limit $\underset{J \ni j \to \infty}{\text{w}^*\text{-lim}}\, \varphi_j$ is constant on each fiber of the exact sequence (A.4). In view of the surjectivity of σ, it defines a linear functional μ_J on $C(\Sigma)$ by the identity

$$\mu_J(\sigma(A)) = \lim_{J \ni j \to \infty} \varphi_j(A).$$

Next, we will show the positivity of the linear functional μ_J. Let $a \in C^\infty(\Sigma)$ is non-negative. Then we can take an operator $Op(a) \in \sigma^{-1}(a) \subset \mathcal{A}_0$ such that $Op(a) \geq 0$. (See [6] or Chapter VII in [34].) Thus we obtain

$$\mu_J(a) = \lim_{J \ni j \to \infty} \varphi_j(Op(a)) \geq 0.$$

This shows that the linear functional μ_J is positive, and hence it defines a Borel measure on Σ. Finally, we will prove the invariance of the measure μ_J. We define the $*$-automorphism $\alpha_t : \mathcal{A} \to \mathcal{A}$ by $\alpha_t A = e^{it\hat{H}} A e^{-it\hat{H}}, \quad A \in \mathcal{A}$. By Egorov's theorem (Lemma 2.1, (2)), we have

$$\sigma(\alpha_t A) = \sigma(A) \circ \varphi_t, \tag{A.8}$$

for every $A \in \mathcal{A}$. Let $a \in C(\Sigma)$ and $A \in \sigma^{-1}(a)$. Then, by (A.8), we obtain

$$\mu_J(a \circ \varphi_t) = \lim_{J \ni j \to \infty} \varphi_j(\alpha_t A) = \lim_{J \ni j \to \infty} \varphi_j(A) = \mu_J(a).$$

This shows that the measure μ_J is invariant. ▮

Definition A.1 *The invariant measure μ_J which is a weak*-limit of a subsequence $\{\varphi_j\}_{j \in J}$ is called a quantum limit with respect to the subsequence $J \subset \mathbf{N}$. We will denote by $\mathcal{Q}$ the set of all quantum limits.*

Note that $\mathcal{Q}$ is a subset of the set $\mathcal{M}_I(\Sigma)$ of all invariant probability measures.

Remark A.1 Let $\mu_J \in \mathcal{Q}$ be a quantum limit with respect to a subsequence $J \in \mathbf{N}$. Let $\pi_M : \Sigma \to M$ be the projection. For a smooth function $f \in C^\infty(M)$, we will denote by $A_f \in \mathcal{A}$ the multiplication operator by f. Then we have

$$\int_M f |\varphi_j|^2 \, dV_M = \varphi_j(A_f) \to \int_\Sigma \pi_M^* f \, d\mu_J = \int_M f \, d(\pi_M)_* \mu_J, \tag{A.9}$$

as $J \ni j \to \infty$. Therefore the measure $(\pi_M)_*\mu_J$ on M is the weak limit of the measures $\{|\varphi_j|^2 dV_M\}_{j \in J}$. We also note that, if the dynamical system $(\Sigma, \varphi_t, \omega)$ with ω the normalized Liouville measure is ergodic, then by Theorem 2.3 we have $\omega \in \mathcal{Q}$ with respect to a subsequence $J \subset \mathbf{N}$ of full-density.

A.3 Ergodicity of quantum limits

Let $\mu_J \in \mathcal{Q}$ be a quantum limit with respect to a subsequence $J \subset \mathbf{N}$. Then we have the dynamical system $(\Sigma, \varphi_t, \mu_J)$. The following theorem gives a condition for the ergodicity of $(\Sigma, \varphi_t, \mu_J)$ in terms of asymptotic properties of eigenfunctions. In the following, we will denote by $\mu_J(a)$ the integral of $a \in L^2(\mu_J)$ by the measure μ_J.

Proposition A.2 *Let $\mu_J \in \mathcal{Q}$ be a quantum limit with respect to $J = \{j_k\} \subset \mathbf{N}$. Then the dynamical system $(\Sigma, \varphi_t, \mu_J)$ is ergodic if and only if the following two conditions hold.*

(1) *For every $A \in \mathcal{A}$, we have*

$$\lim_{k\to\infty} \sum_{\substack{j \\ e_j = e_{j_k}}} |\langle A\varphi_{j_k}, \varphi_j \rangle|^2 = |\mu_J(\sigma(A))|^2. \tag{A.10}$$

(2) *For every $A \in \mathcal{A}$, we have*

$$\lim_{\delta\to 0} \limsup_{k\to\infty} \sum_{\substack{j \\ 0<|e_j - e_{j_k}|<\delta}} |\langle A\varphi_{j_k}, \varphi_j \rangle|^2 = 0. \tag{A.11}$$

Proposition A.3 *The dynamical system $(\Sigma, \varphi_t, \mu_J)$ has weak-mixing property if and only if the following two conditions hold.*

(1) *For every $A \in \mathcal{A}$ and $\tau \in \mathbf{R}$, we have*

$$\lim_{k\to\infty} \sum_{\substack{j \\ e_j = e_{j_k}+\tau}} |\langle A\varphi_{j_k}, \varphi_j \rangle|^2 = |\mu_J(\sigma(A))|^2 \delta_{\tau,0}. \tag{A.12}$$

(2) *For every $A \in \mathcal{A}$ and $\tau \in \mathbf{R}$, we have*

$$\lim_{\delta\to 0} \limsup_{k\to\infty} \sum_{\substack{j \\ 0<|e_j - e_{j_k}-\tau|<\delta}} |\langle A\varphi_{j_k}, \varphi_j \rangle|^2 = 0. \tag{A.13}$$

Proof. We will give a proof only for Proposition A.3. Suppose that the dynamical system $(\Sigma, \varphi_t, \mu_J)$ has weak mixing property. Let $A \in \mathcal{A}$ and $\tau \in \mathbf{R}$. Then we have

$$\begin{aligned} |\mu_J(\sigma(A))|^2 \delta_{\tau,0} &= |\mu_J(\overline{\sigma(A)}(\tau))|^2 \\ &= \mu_J(|\overline{\sigma(A)}(\tau)|^2) \quad \text{(weak-mixing property)} \\ &= \lim_{t\to\infty} \mu_J(\sigma(A_t(\tau)^* A_t(\tau))). \end{aligned} \tag{A.14}$$

By the definition of $A_t(\tau)$ (see Section 2.1), we have

$$A_t(\tau)\varphi_l = \frac{1}{t}\sum_{\substack{j \\ e_j \neq e_l+\tau}} \frac{e^{it(e_j-e_l-\tau)}-1}{i(e_j-e_l-\tau)}\langle A\varphi_l,\, \varphi_j\rangle\varphi_j + \sum_{\substack{j \\ e_j = e_l+\tau}} \langle A\varphi_l,\, \varphi_j\rangle\varphi_j. \tag{A.15}$$

This implies

$$\begin{aligned} \|A_t(\tau)\varphi_{j_k}\|^2 &= \sum_{\substack{j \\ e_j \neq e_{j_k}+\tau}} S(t(e_j-e_{j_k}-\tau))|\langle A\varphi_{j_k},\, \varphi_j\rangle|^2 + \|\bar{A}(\tau)\varphi_{j_k}\|^2 & \text{(A.16)} \\ &\geq \|\bar{A}(\tau)\varphi_{j_k}\|^2, & \text{(A.17)} \end{aligned}$$

where $\bar{A}(\tau)$ is the operator defined by

$$\bar{A}(\tau) = \sum_{\substack{e, \\ e, e+\tau \in \mathrm{Spec}(\hat{H})}} P_{e+\tau}AP_e,$$

and $S(x) = (|e^{ix}-1|/x)^2$. Note that the bounded operator $\bar{A}(\tau)$ is not necessarily an element of $\mathcal{A}$. From (A.17), it follows that

$$\mu_J(\sigma(A_t(\tau)^*A_t(\tau))) = \lim_{k\to\infty}\|A_t(\tau)\varphi_{j_k}\|^2 \geq \limsup_{k\to\infty}\|\bar{A}(\tau)\varphi_{j_k}\|^2. \tag{A.18}$$

On the other hand, we have

$$\begin{aligned} 0 &\leq \liminf_{k\to\infty}\|(\bar{A}(\tau)-\mu_J(\overline{\sigma(A)}(\tau)))\varphi_{j_k}\|^2 & \\ &= \liminf_{k\to\infty}\|\bar{A}(\tau)\varphi_{j_k}\|^2 - |\mu_J(\overline{\sigma(A)}(\tau))|^2. & \text{(A.19)} \end{aligned}$$

Therefore we obtain

$$|\mu_J(\overline{\sigma(A)}(\tau))|^2 = \lim_{k\to\infty}\|\bar{A}(\tau)\varphi_{j_k}\|^2 = \lim_{t\to\infty}\mu_J(\sigma(A_t(\tau)^*A_t(\tau))). \tag{A.20}$$

Since

$$\lim_{k\to\infty}\|\bar{A}(\tau)\varphi_{j_k}\|^2 = \lim_{k\to\infty}\sum_{\substack{j \\ e_j = e_{j_k}+\tau}} |\langle A\varphi_{j_k},\, \varphi_j\rangle|^2,$$

we conclude (A.12). Furthermore, by (A.20) and (A.16), we have

$$\lim_{t\to\infty}\limsup_{k\to\infty}\sum_{\substack{j \\ e_j \neq e_{j_k}+\tau}} S(t(e_j-e_{j_k}-\tau))|\langle A\varphi_{j_k},\, \varphi_j\rangle|^2 = 0. \tag{A.21}$$

By the same argument as in the proof of Proposition 3.2, we conclude (A.13).

Conversely, we will assume (A.12) and (A.13). We set

$$S_{j_k}(t, A) = \sum_{\substack{j \\ e_j \neq e_{j_k}+\tau}} S(t(e_j-e_{j_k}-\tau))|\langle A\varphi_{j_k},\, \varphi_j\rangle|^2$$

From the assumption (A.12), it follows that

$$\lim_{k\to\infty} \|\bar{A}(\tau)\varphi_{j_k}\|^2 = |\mu_J(\overline{\sigma(A)}(\tau))|^2.$$

By (A.13) and the same argument as in the proof of Proposition 3.2, we obtain (A.21). Combining (A.16) with $\lim_{k\to\infty} \|A_t(\tau)\varphi_{j_k}\|^2 = \mu_J(|\sigma(A)_t(\tau)|^2)$, we have

$$\lim_{k\to\infty} S_{j_k}(t, A) = \mu_J(|\sigma_t(\tau)|^2) - |\mu_J(\overline{\sigma(A)}(\tau))|^2$$

Letting $t \to \infty$, we conclude

$$\mu_J(|\overline{\sigma(A)}(\tau)|^2) = |\mu_J(\overline{\sigma(A)}(\tau))|^2.$$

Since σ is surjective, the dynamical system $(\Sigma, \varphi_t, \mu_J)$ has weak-mixing property. ∎

A.4 Cluster points and quantum limits

Let $\mu_J \in \mathcal{Q}$ be a quantum limit with respect to a subsequence $J = \{j_k\} \subset \mathbf{N}$. For a bounded open interval $I \subset \mathbf{R}$, we set

$$\begin{aligned} \mathcal{N}_J(I) &= \{(j, j_k) \in \mathbf{N} \times J\,;\, e_j - e_{j_k} \in I\}, \\ N_J(I) &= \sharp\mathcal{N}_J(I). \end{aligned}$$

Definition A.2 *A real number $\tau \in \mathbf{R}$ is said to be a cluster point with respect to the subsequence $J \subset \mathbf{N}$ if $N_J(I) = +\infty$ for every bounded open interval I containing τ. We will denote by $D\sigma_J(\hat{H})$ the set of all cluster points with respect to J.*

Let $L^2(\mu_J) = L^2(\Sigma, \mu_J)$ be the Hilbert space of L^2-functions with respect to the measure μ_J. Note that, since μ_J is a finite measure, $C(\Sigma)$, and hence $C^\infty(\Sigma)$, is dense in $L^2(\mu_J)$. Let U_t^J be the strongly continuous one-parameter group of unitary operators on $L^2(\mu_J)$ defined by $U_t^J a = a \circ \varphi_t$, $a \in L^2(\mu_J)$, and let

$$U_t^J = \int e^{itx}\, dE_J(x)$$

be its spectral resolution. We define the self-adjoint operator S_J on $L^2(\mu_J)$ by

$$S_J = \int x\, dE_J(x).$$

We will denote by $\langle\,\cdot, \cdot\,\rangle_J$ the L^2-inner product on $L^2(\mu_J)$. The following proposition is an analogy of Proposition 4.1.

Proposition A.4 *We have* $\mathrm{Spec}(S_J) \subset D\sigma_J(\hat{H})$.

Proof. For $a \in C(\Sigma)$ we take an operator $A \in \sigma^{-1}(a) \subset \mathcal{A}$. For every $f \in C_0^\infty(\mathbf{R})$, we have

$$f(S_J)a = \frac{1}{2\pi} \int \hat{f}(t) U_t^J a \, dt = \sigma(A_f),$$

where $A_f \in \mathcal{A}$ is the operator defined by (see Section 2.5)

$$A_f = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(t) e^{it\hat{H}} A e^{-it\hat{H}} \, dt$$

with $\hat{f}$ the Fourier transform of f.

Now, let τ is not in $D\sigma_J(\hat{H})$. Then we can choose an open interval I such that $N_J(I)$ is finite. Let $f \in C_0^\infty(I)$. A direct calculation by using the spectral decomposition for the operator $e^{it\hat{H}}$ leads us to obtain that

$$\langle\, f(S_J)a,\, a \,\rangle_J = \lim_{k\to\infty} \varphi_{j_k}(A^* A_f) = \lim_{k\to\infty} \sum_j f(e_j - e_{j_k}) |\langle\, A\varphi_{j_k},\, \varphi_j \,\rangle|^2. \tag{A.22}$$

Note that $e_{j_k} \to \infty$ as $k \to \infty$. By the assumption $N_J(I) < \infty$, if we take k large enough, then $e_j - e_{j_k}$ is not contained in I for every $j \in \mathbf{N}$. Since the support of f is contained in I, the right hand side of (A.22) equals zero. This implies that $f(S_J) = 0$ as far as the support of f is contained in the open interval I, and hence it concludes that τ is not contained in $\mathrm{Spec}(S_J)$. ∎

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