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The evolution of harmonic maps

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The evolution of harmonic maps

A thesis presented

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Contents

Prefaces

The purpose of this thesis is to study partial regularity of the weakly evolutional harmonic maps whose target manifolds are spheres.

Harmonic maps are considered a generalization of the closed geodesics in differential geometry and have been studied by Almgren, Brezis, Eells, Giaquinta, Hardt, Hildebrandt, Lin, Lieb, Nishikawa, Schoen, Struwe and Uhlenbeck, etc. We recall several important results on harmonic maps.

Let B^d and S^{D-1} be the d-dimensional unit ball and the $D-1$-dimensional unit sphere respectively with positive integers d and D more than or equal to 2. $H^{1,2}(B^d;\mathbb{R}^D)$ denotes

$$\begin{aligned}
&H^{1,2}(B^d;\mathbb{R}^D)=\{u=(u^i)(i=1,\dots,D);u^i\in L^2(B^d),\partial u^i/\partial x_\alpha \\
&\quad\in L^2(B^d)(\alpha=1,\dots,d)\},\quad \big(\partial u^i/\partial x_\alpha;\text{weak derivative of }u^i\big),
\end{aligned}$$

the norm $||u||_{H^{1,2}}$ of $H^{1,2}(B^d;\mathbb{R}^D)$ is given by

$$||u||_{H^{1,2}}=\left\{\sum_{i=1}^{D}\int_{B^d}|u^i|^2dx+\sum_{\alpha=1}^{d}\sum_{i=1}^{D}\int_{B^d}\left|\frac{\partial u^i}{\partial x_\alpha}\right|^2dx\right\}^{1/2}.$$

Also

$$\begin{aligned}
&H^{1,2}(B^d;S^{D-1})=\{u\in H^{1,2}(B^d;\mathbb{R}^D);|u|=1,\text{ a.e. }x\in B^d\},\\
&\mathring{H}^{1,2}(B^d;\mathbb{R}^D)=\overline{C_0^\infty(B^d;\mathbb{R}^D)}^{H^{1,2}(B^d;\mathbb{R}^D)}.
\end{aligned}$$

Give $u_0\in H^{1,2}(B^d;S^{D-1})$. We consider the boundary value problem to the following system:

$$\begin{aligned}
&\triangle u+|\nabla u|^2u=0,\qquad \text{in}\quad B^d,\\
&u=u_0\qquad \text{on}\quad \partial B^d.
\end{aligned}$$

A solution of the boundary value problem above is called a (Dirichlet) harmonic map.

Since harmonic maps are the Euler-Lagrange equations of the Dirichlet energy

$$E[u]=\frac{1}{2}\int_{B^d}|\nabla u|^2dx,\qquad \text{with restriction }|u|=1,$$

the existence of a harmonic map follows from the variational problem:

Problem 1. Find a map $u_{\min}$ in $H^{1,2}(B^d;S^{D-1})$ with

$$E[u_{\min}]=\inf_{u\in H^{1,2}_{u_0}(B^d;S^{D-1})}E[u],$$

where $H^{1,2}_{u_0}(B^d;S^{D-1}) = \{u \in H^{1,2}(B^d;S^{D-1})\ ; u - u_0 \in \overset{\circ}{H}{}^{1,2}(B^d;\mathbb{R}^D)\}$.

The map $u_{\min}$ is often called "minimizer." Related to the problem above, the following two results on minimizer are fundamental:

(I) $u_{\min}$ exists in all d, D if $H^{1,2}_{u_0}(B^d;S^{D-1})$ is not empty and it is smooth in $d=2$. (Special case of the results by Morrey, C. B. Jr; see "Multiple integrals in the calculus of variations")

(II) $u_{\min}$ is smooth on an open set whose compliment is a set of at most finite $(d-3)$-Hausdorff measure. (This result is due to Schoen, R and Uhlenbeck, K).

As is well-known, $H^{1,2}_{u_0}(B^d;S^{d-1})$ is not empty and any map $u: B^d \to S^{d-1}$ with $\deg(u|_{\partial B^d}) \neq 0$ has at least one singular point (non-continuous point) if $d \geq 3$. Thus the result by Schoen, R and Uhlenbeck, K is sharp.

$u_{\min}$ is the critical point of the Dirichlet energy E and hence is a harmonic map. The result above shows the existence of a harmonic map for a given boundary value u_0. However, in general harmonic maps are not unique. Then the following problem naturally arises:

Problem 2. For a given map $u_0 \in H^{1,2}_{u_0}(B^d;S^{D-1})$, find all harmonic map with a boundary value u_0.

To solve the problem 2, Eells-Sampson proposed the strategy that considers the following parabolic system:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \triangle u + |\nabla u|^2 u && \text{in} \quad Q_\infty, \\ u(0,x) &= u_0(x) && \text{in} \quad \{0\} \times B^d, \\ u(t,x) &= u_0(x) && \text{in} \quad [0,\infty) \times \partial B^d, \\ & && (Q_\infty = (0,\infty) \times B^d) \end{aligned}$$

shows that a solution $u(t,x)$ of the system above, converges to a harmonic map as $t \to +\infty$. A solution $u(t,x)$ of the system above is called "evolutional harmonic map". By using a certain penalty method, Chen, Y and Struwe, M have constructed a weakly evolutional harmonic map $u(t,x)$, that is, a map satisfying

[A-1] $u \in L^\infty(0,T;H^{1,2}(B^d;S^{D-1})) \cap H^{1,2}(0,T;L^2(B^d;S^{D-1}))$ for any positive number T,

[A-2] $u(t,x)$ satisfies the system above in the distribution sense,

[A-3] $u(t,x) - u_0(x) \in \overset{\circ}{H}{}^{1,2}(B^d;\mathbb{R}^D)$ for almost every $t \in (0,T)$,

[A-4] $\lim_{t \to +0} u(t,\cdot) = u_0(\cdot)$ in $L^2(B^d;\mathbb{R}^D)$;

a penalty method is to examine the following approximate systems:

$$\begin{aligned}
&\frac{\partial u_\lambda}{\partial t}=\triangle u_\lambda-\lambda(|u_\lambda|^2-1)u_\lambda && \text{for} \quad Q_\infty,\\
&u_\lambda(0,x)=u_0(x) && \text{in} \quad \{0\}\times B^d,\\
&u_\lambda(t,x)=u_0(x) && \text{in} \quad [0,\infty)\times\partial B^d,
\end{aligned}$$

where λ is any positive number, u_λ is a map from Q_∞ to $\mathbb{R}^D$ and u_0 is a map from B^d to $\mathbb{R}^D$ with $|u_0|=1$. However, in general, the weakly evolutional harmonic maps are neither unique nor smooth on the whole domain by the topological obstruction. To ensure the uniqueness, a class of solutions proposed by Chen, Y and Struwe, M seems too broad. The main purpose of the thesis is to find a class of solutions with the following two requirements:

(I) There exists a unique weakly evolutional harmonic map in a class of solutions.

(II) The singular set of weakly evolutional harmonic map in this class is smaller than that of the weakly evolutional harmonic maps constructed by Chen, Y and Struwe, M.

To the requirements of (I) and (II), we show the class of solutions $u(t,x)$ that implies the following properties:

[H-1] $\quad u(t,x)$ satisfies [A-1] - [A-4],

[H-2] $$\lim_{h\searrow 0}\frac{1}{h}\int_Q |\langle x+h\mathbf{e},\nabla\rangle u(t,x+h\mathbf{e})-\langle x,\nabla\rangle u(z)|^2\,dz=0,$$

[H-3] $$\lim_{h\searrow 0}\frac{1}{h}\int_Q |\nabla u(t+h,x)-\nabla u(z)|^2\,dz=0,$$

where $z=(t,x)$, $dz=dtdx$, $\mathbf{e}$ is a unit vector in $\mathbb{R}^d$ and any compact set $Q\subset\subset Q_\infty$. Then we state our first main theorem:

Theorem 1. *Let d be 3. If a map $u{:}Q_\infty\to S^{D-1}$ satisfies the hypothesis* [H-1] - [H-3], *then the map u is smooth except for a relative closed set in Q_∞ having at most zero 3-dimensional Hausdorff measure with respect to the parabolic metric where the parabolic metric d_{P} means for any points (t,x) and (s,y) $\in\mathbb{R}_+\times\mathbb{R}^d$,*

$$d_{\mathrm{P}}((t,x),(s,y))=|t-s|^{1/2}+|x-y|.$$

The proof of this theorem is directly performed by the combining the following two theorems:

Theorem 2. *Let u be a map satisfying* [H-1] - [H-3]; *when we set*

$$\Sigma_u=\left\{z_0=(t_0,x_0)\in Q_\infty;\liminf_{r\searrow 0} r^{-3}\int_{Q_r(z_0)}|\nabla u|^2\,dz>0\right\},$$

then Σ_u *is a relative closed set in* Q_∞ *satisfying* $\mathcal{H}^3(\Sigma_u)=0$. *Here the* 3*-dimensional Hausdorff measure with respect to the parabolic metric* $\mathcal{H}^3(\Sigma_u)$ *is given by*

$$\mathcal{H}^3(\Sigma_u)=\sup_{R>0}\left\{\inf_{\text{covering}}\left\{\sum_i r_i^3;\Sigma_u\subset\bigcup_i P_{r_i}(z_i),z_i=(t_i,x_i)\in\Sigma_u,r_i<R\right\}\right\},$$

with $P_R(z_0)=(t_0-R^2,t_0+R^2)\times B_R(x_0)$.

THEOREM 3. *There exist constants* $0<\epsilon_0$, $\tau_0<1$ *such that for any map* u *with* [H-1] - [H-3] *and any* $Q_r(z_0)\subset\subset Q_\infty$,

$$\frac{r^{-3}}{2}\int_{Q_r(z_0)}|\nabla u|^2\,dz<\epsilon_0 \quad \textit{implies}$$
$$\frac{(\tau_0 r)^{-3}}{2}\int_{Q_{\tau_0 r}(z_0)}|\nabla u|^2\,dz\le\frac{1}{2}\frac{r^{-3}}{2}\int_{Q_r(z_0)}|\nabla u|^2\,dz.$$

The following monotonicity lemma has played a crucial role in the proof of Theorem 3:

LEMMA 1. *For all map* u *with* [H-1] - [H-3], *the inequality holds*

$$\frac{1}{r_2^2}\int_{t_0-r_2^2}^{t_0}\frac{dt}{r_1}\int_{B_{r_1}(x_0)}|\nabla u|^2\,dx\le\frac{C_M}{(2r_2)^2}\int_{t_0-(2r_2)^2}^{t_0}\frac{dt}{2r_2}\int_{B_{2r_2}(x_0)}|\nabla u|^2\,dx,$$

where $(t_0-r_2^2,t_0)\times B_{r_1}(x_0)\subset Q_{r_2}(z_0)\subset Q_{2r_2}(z_0)\subset\subset Q_\infty$ *are any concentric cylinders with* $z_0=(t_0,x_0)$ *and* C_M *is a positive constant independent of* r_1,r_2,z_0,u.

To complete the proof of Theorem 1, we apply DeGiorgi-Nash-Moser's iteration technique to Theorem 3 and successively use the Schauder estimates for the parabolic equations and the boot strap argument; we can verify that u with [H-1] - [H-3] belongs to $C^\infty(Q_\infty\backslash\Sigma_u)$. Struwe, M suggests that the class of solutions satisfying [A-1] - [A-4] and the monotonicity formula may permit us to possess the unique solution u to the prescribed initial-boundary condition. The existence of a map with [H-1] - [H-3] will be discussed in the forthcoming paper. This will be stated in Chapter 2.

Chapter 3 demonstrates a new proof of a partial regularity of the weakly evolutional harmonic maps constructed by Chen, Y and Struwe, M. This new method enables us explicitly to estimate various constants appeared in the proof. We show it by combining Giaquinta's and Ladyženskaya-Ural'ceva's iteration techniques with the nonlinear Fefferman-Phong inequality which is instructive in itself. Here the nonlinear Fefferman-Phong inequality is

LEMMA 2. *Let* u_λ *be the smooth solution of the penalty system and* η *a non-negative smooth function with a compact support on a parabolic cube* $D_R(z_0)$ *and* $\sup_{D_R(z_0)}\eta(z)\le 1$; *Then, for the Ginzburg-Landau energy density* $e_\lambda(u_\lambda)=$

$1/2|\nabla u_\lambda|^2+\lambda/4(|u_\lambda|^2-1)^2$,

$$\int_{D_R(z_0)}\left|\left(\left(\sqrt{e_\lambda^{(\kappa)}(u_\lambda)}\right)^p\eta\right)\right|^{2+2/p}dz\leq C_{FP}\Phi_R^2(z_0)$$
$$\times\int_{D_R(z_0)}\left|\nabla\left(\left(\sqrt{e_\lambda^{(\kappa)}(u_\lambda)}\right)^p\eta\right)\right|^2dz$$

holds where κ is any positive number, $e_\lambda^{(\kappa)}=\max\{e_\lambda-\kappa,0\}$, $p=1,1+2/d$, C_{FP} is a positive constant independent of κ, R, z_0, u_λ and η, and

$$\Phi_R^2(z_0)=(10\sqrt{3}e^{1/4}+1)R^{-d}\int_{t_0-4R^2}^{t_0+R^2}dt\int_{D_{(1+\delta(R))R}(x_0)}e_\lambda(u_\lambda)dx$$
$$+10\sqrt{3}R^{2-d}\exp\left(-\frac{\delta^2(R)}{12}\right)\operatorname*{ess\cdot sup}_{0\leq t\leq\infty}\int_{B^d}e_\lambda(u_\lambda)dx$$
$$+5\sqrt{3}R\left(\sup_{x\in\partial B^d}\left(|x|+\frac{|x|^2}{\epsilon_0}\right)\sup_{z\in\{0\}\times B^d\cup[0,\infty)\times\partial B^d}|\nabla_{\tan}u_0|^2\right.$$
$$\left.+2\sup_{z\in\{0\}\times B^d\cup[0,\infty)\times\partial B^d}\left|\frac{\partial u_0}{\partial t}\right|^2\right)\mathcal{L}^{d-1}(\partial B^d)$$
$$\times\sup_{0\leq s\leq\infty}\max(s^{1-d},s^{5-d})\exp(-\frac{1}{4s^2}\inf_{x\in\partial B^d}|x|^2),$$

with $z_0=(t_0,x_0)\in Q_\infty$, $dz=dtdx$, $\delta(R)=\sqrt{(12(d-2)+1)\cdot|\log R|}$ and $\mathcal{L}^{d-1}=$ the $(d-1)$-dimensional Lebesgue measure.

CHAPTER 1

The evolution of Harmonic maps

1. Harmonic maps

Let M be a d-dimensional Riemannian manifold with metric γ and N a compact D-dimensional manifold with a metric g, respectively. By Nash's embedding theorem (See Nash [**87**]) we assume that N can be isometrically imbedded to a $\hat{D}$-dimensional Euclidean space $\mathbb{R}^{\hat{D}}$ for some positive integer $\hat{D}$. For a C^1-map $u=(u^1,\ldots,u^{\hat{D}})\ :\ M\to N\subset\mathbb{R}^{\hat{D}}$ let

$$e(u)=\frac{1}{2}\gamma^{\alpha\beta}(x)u^i_{x_\alpha}u^i_{x_\beta}=:\frac{1}{2}|\nabla u|^2_M$$

be the energy density, written in local coordinates $x=(x_\alpha)_{\alpha=1,\ldots,d}$ on M with $\gamma=(\gamma_{\alpha\beta})$, $(\gamma^{\alpha\beta})=(\gamma_{\alpha\beta})^{-1}$. Repeated Greek indices tacitly will be summed from 1 to d, repeated Latin indices from 1 to $\hat{D}$. Moreover, $u^i_{x_\alpha}=\partial u^i/\partial x_\alpha$ etc. A C^1-variation of u is a family (u_ϵ) of C^1-maps $u_\epsilon\colon M\to N\subset\mathbb{R}^{\hat{D}}$ smoothly depending on a parameter $|\epsilon|<\epsilon_0$, and such that $u_0=u$. A variation (u_ϵ) of u is said to be compactly supported variations if there exists a compact set $\Omega\subset\subset M$ such that $u_\epsilon=u$ on $M\setminus\Omega$ for all $|\epsilon|\ll 1$.

Definition 1.1. A C^1-map $u\colon M\to N\subset\mathbb{R}^{\hat{D}}$ is harmonic if it is stationary for Dirichlet's energy

$$E(u)=\int_M e(u)\,dvol_M \tag{1.1}$$

with respect to compactly supported variations.

Note that in local coordinates $dvol_M=\sqrt{|\gamma|}dx$, where $|\gamma|=|\det(\gamma_{\alpha\beta})|$.

We derive the Euler-Lagrange equation satisfied by a harmonic map u: let $U\subset\mathbb{R}^{\hat{D}}$ be a tubular neighborhood of N and $\pi_N\colon U\to N$ the (smooth) nearest-neighbor projection. By $T_pN(\subset T_p\mathbb{R}^{\hat{D}})$ denote the tangent space to N at a point $p\in N$. Choose $\phi\in C^1_0(M;\mathbb{R}^{\hat{D}})$ satisfy

$$\phi(x)\in T_{u(x)}N$$

for all $x\in M$. ϕ induces a C^1-variation

$$u_\epsilon=\pi_N\circ(u+\epsilon\phi).$$

Since $d\pi_N(p)\Big|_{T_pN}$=id for $p\in\mathbb{N}$, clearly we have

$$\frac{du_\epsilon}{d\epsilon}\bigg|_{\epsilon=0}=(d\pi_N\circ u)\phi=\phi.$$

Suppose that ϕ has support in a single coordinate chart. Then

$$\begin{aligned}\frac{dE}{d\epsilon}[u_\epsilon]\bigg|_{\epsilon=0}&=\int_M\gamma^{\alpha\beta}\sqrt{|\gamma|}u^i_{x_\beta}\phi^i_{x_\alpha}dx\\ &=-\int_M\frac{1}{\sqrt{|\gamma|}}\frac{\partial}{\partial x_\alpha}(\gamma^{\alpha\beta}\sqrt{|\gamma|}\frac{\partial}{\partial x_\beta}u^i)\phi^i\sqrt{|\gamma|}dx\\ &=-\int_M\triangle_M u^i\phi^i dvol_M,\end{aligned}$$

where $\triangle_M=1/\sqrt{|\gamma|}\ \partial/\partial x_\alpha\ (\gamma^{\alpha\beta}\sqrt{|\gamma|}\partial/\partial x_\beta)$ denotes the Laplace-Beltrami operator on M.

Thus, if $u\in C^2$ is harmonic, u satisfies

$$\triangle_M u\perp T_uN \tag{1.2}$$

and conversely. To obtain a more explicit form of (1.2), let $\nu_{D+1},\ldots,\nu_{\hat{D}}$ denote a local orthonormal frame for $(T_pN)^\perp$, the orthogonal complement of T_pN in $\mathbb{R}^{\hat{D}}$, near $p=u(x)\in N$. Then, by (1.2) there exist scalar functions $\lambda^{D+1},\ldots,\lambda^{\hat{D}}$ such that

$$-\triangle_M u=\sum_{k=D+1}^{\hat{D}}\lambda^k(\nu_k\circ u).$$

For any fixed k, multiplying by ν_k, since $\langle u_{x_\alpha},\nu_k\circ u\rangle=0$ for all α, we obtain

$$\begin{aligned}\lambda^k=&-\triangle_M\langle u,(\nu_k\circ u)\rangle=-\mathrm{div}\langle\nabla u,\nu_k(u)\rangle+\gamma^{\alpha\beta}\langle u_{x_\alpha},\frac{\partial}{\partial x_\beta}(\nu_k\circ u)\rangle\\ =&\gamma^{\alpha\beta}A^k(u)(u_{x_\alpha},u_{x_\beta}),\end{aligned}$$

where $A^k=d\nu_k$ denotes the second fundamental form with respect to ν_k. $\langle\cdot,\cdot\rangle$ means the inner product between a map and a map. Thus we find that (1.2) is equivalent to

$$-\triangle_M u=A(u)(\nabla u,\nabla u)_M, \tag{1.3}$$

where in local coordinates

$$A(u)(\nabla u,\nabla u)_M=\sum_{k=D+1}^{\hat{D}}\gamma^{\alpha\beta}A^k(u)(u_{x_\alpha},u_{x_\beta})(\nu_k\circ u).$$

Example 1.1. If $M=T^d=\mathbb{R}^d/\mathbb{Z}^d$, $N=S^d\subset\mathbb{R}^{d+1}$, equation (1.3) simply becomes

$$-\triangle u=|\nabla u|^2u.$$

Harmonic maps are a generalized concept of harmonic functions. Harmonic maps $S^1 \to N$ correspond to closed geodesics on N. Important applications of harmonic maps are in Teichmüller theory in understanding the Weil-Peterson metric on Teichmüller space, see Earle-Eells [**31**], [**32**], Eells [**33**], Fisher-Tromba [**44**], Tromba [**114**], or in proving rigidity theorems for Kähler manifolds (Mostow [**85**], Mostow-Siu [**86**], Siu [**103**]). Comprehensive surveys of harmonic maps and their applications are given in Eells-Lemaire [**34**], [**35**], Hildebrandt [**65**], [**66**], Jost [**70**], [**71**] and Schoen-Yau [**99**]. Also Hardt [**58**] gives a survey of the structure of the singular set of harmonic maps and Smith [**104**] surveys harmonic maps from spheres to spheres. This thesis mainly treats with the case of $M = B_1^d(0) \subset \mathbb{R}^d$ (the d-dimensional unit ball) and of $N = S^{D-1}$ (the $D-1$-dimensional unit sphere) by the following reason:

(i) The sphere is the simplest target manifold in hard situation; the hard situation means, for example, the positive curvature target manifold and the manifold with $\pi_D(N) \neq 0$. If we compare Eells-Sampson [**36**] with Chen-Struwe [**24**], what we should realize is that if we can obtain a result on harmonic maps whose target is the D-dimensional sphere, we can show the corresponding result on one's of the D-dimensional compact Riemannian manifold without a boundary.

(ii) We have two applications to physics: One is liquid crystal theory in $M = B_1^3(0), N = S^2$, another is the instantons in Yang-Mills connections in $M = \mathbb{R}^4, N = S^3 \cong SU(2)$. For instance we refer to Ericksen-Kinderlehrer [**38**] and therein for liquid crystal theory and to Atiyah [**1**] for Yan-Mills connections.

(iii) A higher symmetric structure of the sphere enables us to find "highly symmetric harmonic maps." Later this will be discussed in detail.

Bochner identity

A Bochner identity is very useful to analyze the harmonic maps: It is a certain differential equation satisfied by the energy density $e(u)$ of a harmonic map $u\colon M \to N$. Let $R^{(M)}$, $\mathrm{Ric}^{(M)}$ and $R^{(N)}$ denote the Riemann curvature tensor on M, Ricci curvature on M and the Riemann curvature tensor on N, respectively. Given any point $x_0 \in M$, $R^{(M)} = (R_{\alpha\mu\beta\nu})$ and $\mathrm{Ric}^{(M)} = (R_{\alpha\beta})$ respectively means the coordinate representation of $R^{(M)}$, $\mathrm{Ric}^{(M)}$ in normal coordinates $x = (x_\alpha)$ ($\alpha = 1, \dots, d$) about $x_0 \in M$; while $R^{(N)} = (\tilde{R}_{ikjl})$ does the coordinate representation of $R^{(N)}$ in normal coordinates $u = (u^i)$ around $u_0 = u(x_0) \in N$. We use the notation of Jost [**72**].

Proposition 1.1. *If $u \in C^3(M;N)$ is harmonic, then in local coordinates as above the following holds*:

$$-\triangle_M e(u) + |\nabla du|^2 + R_{\alpha\beta} u^i_{x_\alpha} u^i_{x_\beta} = \tilde{R}_{ikjl} u^i_{x_\alpha} u^j_{x_\alpha} u^k_{x_\beta} u^l_{x_\beta}, \tag{1.4}$$

where ∇ denotes the covariant derivative on $T^*M\otimes u^{-1}TN$. (See Jost [**72**, p. 96f] *for an equivalent expression in general coordinates and an invariant form of* (1.4)).

Proof. The proof relies on the following identities valid at $x_0\in M$ in normal coordinates around x_0 on M and around $u(x_0)$ on N. Let $\gamma^{\mu\nu}_{,\alpha}=(\gamma^{\mu\nu})_{x_\alpha}$, etc. Then the identities are

$$\begin{aligned}
\gamma^{\mu\nu}_{,\alpha\beta}&=-\gamma_{\mu\nu,\alpha\beta};\\
(\triangle_M u)_{x_\mu}-\triangle_M(u_{x_\mu})&=(\gamma^{\alpha\beta}\sqrt{|\gamma|})_{x_\alpha x_\mu}u_{x_\beta}=\gamma^{\alpha\beta}_{,\alpha\mu}u_{x_\beta}+\frac{1}{2}\gamma_{\rho\rho,\alpha\mu}u_{x_\alpha}\\
&=-\gamma_{\alpha\beta,\alpha\mu}u_{x_\beta}+\frac{1}{2}\gamma_{\rho\rho,\alpha\mu}u_{x_\alpha};\\
R_{\alpha\beta}=R^\mu_{\alpha\mu\beta}=\Gamma^\mu_{\alpha\beta,\mu}-\Gamma^\mu_{\alpha\mu,\beta}&=\frac{1}{2}(\gamma_{\alpha\mu,\beta\mu}+\gamma_{\beta\mu,\alpha\mu})-\frac{1}{2}(\gamma_{\alpha\beta,\mu\mu}+\gamma_{\mu\mu,\alpha\beta}).
\end{aligned}$$

Thus, at x_0 we obtain

$$\begin{aligned}
\triangle_M e(u)&=\triangle_M(\frac{1}{2}\gamma^{\mu\nu}g_{ij}(u)u^i_{x_\mu}u^j_{x_\nu})\\
&=(\triangle_M u^i_{x_\mu})u^i_{x_\mu}+\frac{1}{2}\gamma^{\mu\nu}_{,\alpha\alpha}u^i_{x_\mu}u^i_{x_\nu}+u^i_{x_\alpha x_\mu}u^i_{x_\alpha x_\mu}+\frac{1}{2}g_{ij,kl}(u)u^i_{x_\mu}u^j_{x_\mu}u^k_{x_\alpha}u^l_{x_\alpha}\\
&=(\triangle_M u^i)_{x_\mu}u^i_{x_\mu}+u^i_{x_\alpha x_\mu}u^i_{x_\alpha x_\mu}+\gamma_{\alpha\beta,\alpha\mu}u^i_{x_\beta}u^i_{x_\mu}\\
&\quad-\frac{1}{2}(\gamma_{\mu\nu,\alpha\alpha}+\gamma_{\alpha\alpha,\mu\nu})u^i_{x_\mu}u^i_{x_\nu}+\frac{1}{2}g_{ij,kl}(u)u^i_{x_\mu}u^j_{x_\mu}u^k_{x_\alpha}u^l_{x_\alpha}\\
&=(\triangle_M u^i)_{x_\mu}u^i_{x_\mu}+u^i_{x_\alpha x_\mu}u^i_{x_\alpha x_\mu}+R_{\alpha\beta}u^i_{x_\alpha}u^i_{x_\beta}\\
&\quad+\frac{1}{2}(g_{ij,kl}+g_{il,jk}-g_{jl,ik})u^i_{x_\mu}u^j_{x_\mu}u^k_{x_\alpha}u^l_{x_\alpha}.
\end{aligned}\tag{1.5}$$

Finally, note that by (1.3) we have

$$(\triangle_M u^i)_{x_\mu}u^i_{x_\mu}=-\tilde\Gamma^i_{kl,j}(u)u^k_{x_\alpha}u^l_{x_\alpha}u^i_{x_\mu}u^j_{x_\mu}$$

at x_0, because $\tilde\Gamma(u(x_0))=0$ by our choice of coordinates. Moreover, note

$$\begin{aligned}
\tilde R_{iljk}&=\tilde\Gamma^i_{kl,j}-\tilde\Gamma^i_{jl,k},\\
\tilde\Gamma^i_{jl,k}&=\frac{1}{2}(g_{ij,kl}+g_{il,jk}-g_{jl,ik}).
\end{aligned}\tag{1.6}$$

Then we conclude

$$-\triangle_M e(u)+u^i_{x_\alpha x_\mu}u^i_{x_\alpha x_\mu}=-R_{\alpha\beta}u^i_{x_\alpha}u^i_{x_\beta}+\tilde R_{iljk}u^i_{x_\mu}u^j_{x_\mu}u^k_{x_\alpha}u^l_{x_\alpha}.$$

Since $\tilde R_{iljk}=\tilde R_{ikjl}$ the claim follows. $\square$

As a consequence of (1.4), we have the following corollary; see e.g. Jost [**72**].

Proposition 1.2. *If M is compact manifold with $Ric^M\geq 0$ and $\partial M=\emptyset$, and if the sectional curvature of N is non-positive, then any harmonic map $u\in C^\infty(M;N)$ is totally geodesic in the sense that $\nabla du\equiv 0$; that is, du is parallel with respect to the pull-back covariant derivative on $T^*M\otimes u^{-1}TN$. Moreover, if $Ric^M>0$ at a point of M, then $u\equiv\mathrm{const}$. If the sectional curvature K^N of N is negative, then $u\equiv\mathrm{const}$ or $u(M)$ is covered by a closed geodesic ball.*

Proof. Integrate (1.4) over M to obtain $|\nabla du|^2\equiv 0$, $\mathrm{Ric}^{(M)}(du,du)\equiv 0$, $\big\langle \tilde{R}^{(N)}(du,du)du,du\big\rangle\equiv 0$ on M under the above assumptions. □

For most of our purposes it suffices to note a weaker Bochner-type estimate. On account of (1.6), we obtain

$$-\triangle_M e(u)+|\nabla^2 u|^2\leq |Ric^M|e(u)+C\big(e(u)\big)^2. \tag{1.7}$$

Weakly harmonic maps

Let

$$H^{1,2}(M;N)=\{u\in H^{1,2}(M;\mathbb{R}^{\hat{D}});u(x)\in N \quad \text{for almost every } x\in M\},$$

where $H^{1,2}(M;\mathbb{R}^{\hat{D}})$ is the standard Sobolev space of L^2-maps $u\colon M\to\mathbb{R}^{\hat{D}}$ with distributional derivative $\nabla u\in L^2$. That is, $H^{1,2}(M;N)$ is the space of maps $u\colon M\to N$ with finite energy $E(u)$. It was observed by Schoen-Uhlenbeck [**98**] that in general $H^{1,2}(M;N)$ as defined above is larger than the weak closure of $C^\infty(M;N)$ in the $H^{1,2}$-norm

$$\|u\|^2_{H^{1,2}}=\int_M (|u|^2+|\nabla u|^2)\,dvol_M,$$

which in turn is larger than the strong closure of $C^\infty(M;N)$ in $H^{1,2}(M;N)$. However, if $\dim M=2$, these spaces all coincide. By a result of Bethuel [**4**], the same is true if $\pi_2(N)=0$. The relations between Sobolev spaces whose elements are maps from M to N and it's weak- and strong- closures were analyzed by Bethuel-Zheng [**9**] and Bethuel [**4**].

Definition 1.2. A map $u\in H^{1,2}(M;N)$ is called weakly harmonic if u satisfies (1.3) in the distribution sense.

Example 1.2. The map $u\colon B_1^d(0)\subset\mathbb{R}^d\to S^{d-1}$ given by

$$u(x)=\frac{x}{|x|},$$

belongs to $H^{1,2}(B_1^d(0);S^{d-1})$ for $d\geq 3$ and weakly solves (1.3), that is, this map u satisfies

$$-\triangle u=|\nabla u|^2u \quad\text{in}\quad \big(\mathcal{D}(B_1^d(0);\mathbb{R}^d)\big)'.$$

Existence of harmonic maps

As in Hodge theory, where one seeks to realize a de Rham cohomology class by a harmonic differential form, a basic existence problem for harmonic maps is the following:

Homotopy problem: *Given a map $u_0\colon M \to N$, is there a harmonic map u homotopic to u_0?*

This question, as we shall see below, has an affirmative answer if the sectional curvature K^N of N is non-positive (Eells-Sampson [**36**]) or if $d=2$ and $\pi_2(N)=0$ (Lemaire [**76**], Sacks-Uhlenbeck [**95**]). However, for $N=S^2$ and $d=2$, we have the following counter-examples:

EXAMPLE 1.3. (Lemaire [**76**], Wente [**119**]): If $u\colon B_1^2(0)\subset\mathbb{R}^2\to S^2$ is harmonic and $u|_{\partial B_1^2(0)}\equiv$const, then $u\equiv$const.

EXAMPLE 1.4. (Eells-Wood [**37**]): If $u\colon T^2\to S^2$ is harmonic, then $\deg u\neq\pm 1$.

In higher dimensions ($d\geq 3$), hardly any result is known for the homotopy problem unless $K^N\leq 0$. However, there are various existence results for the Dirichlet problem.

Dirichlet problem and variational methods

The Dirichlet problem can be formulated as follows: *Suppose $\partial M\neq\emptyset$ and let $u_0\colon M\to N$ be any given map belonging to $H^{1,2}(M;N)$. Is there a harmonic map $u\colon M\to N$ such that $u=u_0$ on ∂M?*

The Dirichlet problem can be attacked by using variational methods. Once we can seek a map which minimizes E among the class

$$H^{1,2}_{u_0}(M;N)=\left\{u\in H^{1,2}(M;N); u=u_0 \text{ on } \partial M\right\},$$

we obtain a weakly harmonic map u satisfying the desired boundary condition because the harmonic maps are the critical points of the energy (1.1) in the class above.

In other words, the variational methods are to find a map $u_{\min}\in H^{1,2}_{u_0}(M;N)$ with

$$E[u_{\min}]=\inf_{u\in H^{1,2}_{u_0}(M;N)} E[u]. \tag{1.8}$$

The map satisfying (1.8) is usually called "minimizer." It is easy to check that (1.3) is the Euler-Lagrange equations of the energy (1.1) among $H^{1,2}_{u_0}(M;N)$. That is, to show the existence of a harmonic map, we have only to search a smooth minimizer. If $\dim M=2$, Morrey [**83**] has proved:

THEOREM 1.1. *Every minimizer $u_{\min}\colon M\to N$ with $\dim M=2$, is smooth.*

On the other hand, in $\dim M\geq 3$, by the topological obstruction, we have nohope to prove the everywhere regularity. For instance the equator map $x\to x/|x|$

: $B_1^3(0)\to S^2$ is the unique absolute minimizer of $E[\cdot]$ in $H_x^{1,2}(B_1^3(0);S^2)$. (Brezis-Coron-Lieb [**11**], Lin [**79**]). Schoen-Uhlenbeck [**97**] and [**98**] have established:

THEOREM 1.2. *A minimizer* $u_{\min}$ *of* $E[\cdot]$ *in* $H_{u_0}^{1,2}(M;N)$ *with* $\dim M\geq 3$ *is smooth on an open set whose compliment has at most* $(\dim M-3)$*-dimensional Hausdorff dimension.*

This result is best possible. For example, even if a map $u_0|_{\partial B_1^d(0)}:\partial B_1^d(0)=S^d\to S^d$ is smooth with 0-degree, a minimizer of $E[\cdot]$ in $H_{u_0}^{1,2}(B_1^d(0);S^{d-1})$ is not always smooth, nevertheless a smooth extention of $u_0|_{\partial B_1^d(0)}$ inside $B_1^d(0)$ exists. In addition if we regard the map $u(x)=x/|x|$ as a map $u\colon B_1^d(0)\subset\mathbb{R}^d\to S^{d-1}\subset S^d\subset\mathbb{R}^{d+1}$, there is no topological reason for a singularity. However the maps u above are still minimizer in $H_x^{1,2}(B_1^d(0);S^d)$ if $d\geq 7$ (Jäger-Kaul [**68**], Baldes [**2**]). Related to liquid crystal theory, a property of $E[\cdot]$ in $H_{u_0}^{1,2}(B_1^3(0);S^2)$ has been much drawn into concerns from 80' (Ericksen-Kinderlehrer [**38**]). Basic facts to the minimizers of $E[\cdot]$ in $H_{u_0}^{1,2}(B_1^3(0);S^2)$ are the following:

(i) *Monotonicity for the scaled energy and the hybrid inequality*: The fundamental technical properties of minimizers may be monotonicity and the hybrid inequality: Let $u_{\min}$ be minimizers of $E[\cdot]$ in $H_{u_0}^{1,2}(B_1^3(0);S^2)$ and set $x_0\in B_1^3(0)$, any ball $B_{r_1}(x_0)\subset B_{r_2}(x_0)\subset\subset B_1^3(0)$; Then monotonicity denotes
$$\frac{1}{r_1}\int_{B_{r_1}}|\nabla u|^2\,dx\leq\frac{1}{r_2}\int_{B_{r_2}}|\nabla u|^2\,dx,$$
while the hybrid inequality is given by
$$\int_{B_r}|\nabla u|^2\,dx\leq C\int_{\partial B_r}|\nabla u||u-a|\,d\mathcal{H}^2,$$
for any vector $a\in\mathbb{R}^3$ and any ball $B_r(x_0)\subset\subset B_1^3(0)$ with some positive number C depending only on the dimension 3. For the proof, see Schoen-Uhlenbeck [**97**] and Hardt-Kinderlehrer-Lin [**59**].

(ii) *Tangential approximation*: A rich development occurred in a paper of Simon [**102**] which provides with the problem the existence of the unique tangential approximating maps. At regular points, the tangential approximating map is constant. For a singular point y of $u_{\min}$ in $B_1^3(0)$, in the light of Simon's result [**102**], there exists the unique harmonic map $f\colon S^2\to S^2$ such that $u(y+r\omega_2)\to f(\omega_2)$ as $r\searrow 0$ uniformly with $\omega_2=(x-y)/|x-y|$.

Similarly, one can try to solve the homotopy problem by minimizing $E[\cdot]$ in a given homotopy class. However, Lemaire's example shows that in general homotopy classes are not weakly closed in $H^{1,2}(M;N)$.

This is made explicit by the following example, whose construction relies on the fact that the conformal group on S^2 acts non-compactly on $H^{1,2}(S^2;S^2)$.

EXAMPLE 1.5. When we set $\pi_p\colon S^2\setminus\{p\}\to\mathbb{R}^2$ which denotes stereographic projection and $D_\lambda x=\lambda x$ which is dilation by a factor λ, the maps

$$u_\lambda=\pi_p^{-1}\circ D_\lambda\circ\pi_p\colon S^2\to S^2,$$

are homotopic to the identity $id=u_1\colon S^2\to S^2$. But

$$u_\lambda\rightharpoonup u_\infty(x)\equiv p\quad(\lambda\to\infty),$$

weakly in $H^{1,2}(S^2;S^2)$. (Accidentally, by conformal invariance of E in dimension $d=2$, all u_λ are harmonic!)

Non-minimizing harmonic maps: By the lack of information, the study of non-minimizing harmonic maps is quite challenging. If $\dim M=2$, Grüter [**55**] proved smoothness of conformal weakly harmonic maps. This result was extended by Schoen [**96**] to harmonic maps that are stationary with respect to variations of parameters in the domain: He showed that the harmonic maps above have a holomorphic Hopf differential

$$(\partial u)^2dz^2:=\big(|u_x|^2-|u_y|^2-2i\langle u_x,u_y\rangle\big)dz^2$$

in suitable conformal parameters $z=x+iy$ on M. Finally Hélein [**63**] recently has shown the regularity of weakly harmonic maps in general.

THEOREM 1.3 (Hélein [**63**]). *Assume* $\dim M=2$ *and let* $u\in H^{1,2}(M,N)$ *be weakly harmonic. Then* $u\in C^\infty(M;N)$.

Proof. For $N=S^{\hat{D}-1}$ and $M=B_1^2(0)$, his proof invokes a trick: the equivalence of (1.3) and

$$-\triangle u^i=\sum_{j=1}^{\hat{D}}\langle\nabla u^j,u^i\nabla u^j-u^j\nabla u^i\rangle\,(i=1,\dots,\hat{D}),$$

in the light of $0=\nabla|u|^2=2\sum_j u^j\nabla u^j$. He then observes that for any $i,j=1,2,\dots,\hat{D}$, there holds

$$\mathrm{div}(u^i\nabla u^j-u^j\nabla u^i)=u^i\triangle u^j-u^j\triangle u^i=0. \tag{1.9}$$

Thus there is a potential $a^{ij}\in H^{1,2}$ such that

$$\mathrm{rot}\,a^{ij}=u^i\nabla u^j-u^j\nabla u^i$$

and (1.3) takes the form

$$-\triangle u^i=\sum_{j=1}^{\hat{D}}(u^j_{x_1}a^{ij}_{x_2}-u^j_{x_2}a^{ij}_{x_1})\quad\text{for}\quad x=(x_1,x_2)\in B_1^2(0).$$

Here the right-hand side is the sum of Jacobians of $H^{1,2}$-maps. Continuity of u (and hence smoothness) then follows from results of Wente [**118**] and Brezis-Coron [**10**].

Notice that (1.9) is a consequence of Noether's principle and the symmetries of $S^{\hat{D}-1}$. Hélein then generalized this simple and beautiful idea to arbitrary target manifolds by an ingenious choice of rotated frame fields on $u^{-1}TN$. $\square$

Inspired by Hélein's result, Evans [**40**] has obtained a partial regularity result for "stationary" weakly harmonic maps from B^d to S^{D-1} with $d\geq 3$ and $D\geq 3$. A drastic difference between $d=2$ and $d\geq 3$ actually emerges to the smoothness result on weakly harmonic maps. Hardt-Lin-Poon [**61**] has constructed examples of harmonic maps $u\colon B_1^3(0)\to S^2$ with the cylindrical symmetry whose boundary data $u|_{\partial B_1^3(0)} : \partial B_1(0)\cong S^2\to S^2$ have degree 0 so that u possesses an arbitrarily large number of singular points on the axis of symmetry and Rivière [**93**] has exhibited weakly harmonic maps $u\colon \mathbb{R}^3\to S^2$ with line singularities.

2. Evolutional harmonic maps

The Eells-Sampson result

By Examples 1.3, 1.4 and 1.5 above, we know that it may be difficult to solve the homotopy problem for harmonic maps by variational methods. To overcome these difficulties, Eells - Sampson [**36**] proposed to study the evolution problem

$$u_t-\triangle_M u=A(u)(\nabla u,\nabla u)_M \quad \text{on } [0,\infty)\times M \tag{1.10}$$

with initial and boundary data

$$u=u_0 \quad \text{at } t=0 \text{ and on } [0,\infty)\times\partial M \tag{1.11}$$

for maps $u\colon [0,\infty)\times M\to N\subset\mathbb{R}^{\hat{D}}$, the idea behind this strategy of course being that a continuous deformation $u(t,\cdot)$ of u_0 will remain in the given homotopy class. Moreover, the "energy inequality" (see Lemma 1.1 below) shows that (1.10) is the (L^2)-gradient flow for E, whence one can hope that the solution $u(t,\cdot)$ for $t\to\infty$ will converge to a critical point (a harmonic map) of E, if we can show that the (L^2)-gradient flow for E is smooth; for suitable targets, this program is successfully solved.

THEOREM 1.4 (Eells-Sampson [**36**]). *Suppose that M is compact without boundary and that the sectional curvature K^N of N is non-positive. Then for any $u_0\in C^\infty(M;N)$, the Cauchy problem* (1.10) *and* (1.11) *admits the unique, global, smooth solution $u\colon M\times[0,\infty)\to N$ which, as $t\to\infty$ suitably, converges smoothly to a harmonic map $u_\infty\in C^\infty(M;N)$ homotopic to u_0.*

The proof uses three ingredients.

LEMMA 1.1 (Energy inequality). *Set $E(u(t)) = \int_M e(u)(t,\cdot)dvol_M$ and let T be any positive number For a smooth solution u of* (1.10) *and* (1.11), *it follows*

$$\int_0^T\int_M |u_t|^2\,dt\,dvol_M+E\big(u(T)\big)\leq E(u_0).$$

Proof. Recall that $A(u)(\nabla u,\nabla u)\perp T_uN$. Multiplying (1.10) by u_t and integrating by parts, we obtain

$$\int_M |u_t|^2 dvol_M + \frac{dE}{dt}(u(t))=0$$

for any $t\geq 0$ and the desired estimate (in fact, with equality) follows from integrating in t. □

Lemma 1.2 (Bochner inequality). *If $K^N\leq 0$, then for any smooth solution u of* (1.10) *with energy density $e(u)$, there holds*

$$\left(\frac{\partial}{\partial t}-\triangle_M\right)e(u)\leq Ce(u) \tag{1.12}$$

with a constant C depending only on the Ricci curvature of M.

Proof. To derive this estimate we use the representation of u in suitable local coordinates on N: $u=(u^1,\ldots,u^{\hat{D}})$. As in deriving (1.4) from (1.3) in the stationary case, we can conclude that (in normal coordinates around x_0 on M)

$$\left(\frac{\partial}{\partial t}-\triangle_M\right)e(u)+|\nabla du|^2+R_{\alpha\beta}u^i_{x_\alpha}u^i_{x_\beta}=\tilde{R}_{ikjl}u^i_{x_\alpha}u^j_{x_\alpha}u^k_{x_\beta}u^l_{x_\beta}$$

at (t,x_0), where ∇, $R_{\alpha\beta}$, $\tilde{R}_{ikjl}$, respectively, denotes the pull-back covariant derivative on $T^*M\otimes u^{-1}TN$, the Ricci curvature on M and the Riemann curvature tensor on N. From this identity, the claim follows. □

From (2), we obtain

$$\left(\frac{\partial}{\partial t}-\triangle_M\right)e(u)+|\nabla^2 u|^2\leq C_M e(u)+C_N\big(e(u)\big)^2, \tag{1.13}$$

where $|\nabla^2 u|^2 = \gamma^{\alpha\beta}\gamma^{\mu\nu}\ u^i_{x_\alpha x_\mu}u^i_{x_\beta x_\nu}$ and C_M, C_N respectively denotes constant depending only on the Ricci curvature of M and the second fundamental form of N.

The final ingredient is Moser's [**84**] sup-estimate for sub-solutions of parabolic equations. Set

$$\mathcal{L}=\frac{\partial}{\partial t}-\triangle_M.$$

By $Q_R(z_0)$, denote the cylinder

$$Q_R(z_0)=\{z=(t,x);t_0-R^2<t<t_0,|x-x_0|<R\}$$

in local coordinates on M, where $z_0=(t_0,x_0)\in M$ and $R>0$. Note that $\gamma^{\alpha\beta}\in C^\infty(Q_R(z_0))$ satisfies the uniform ellipticity condition: there exist positive numbers $0<\lambda\leq\Lambda$ such that for any vector $\xi\in\mathbb{R}^d$

$$\lambda|\xi|^2\leq\sqrt{|\gamma|}\gamma^{\alpha\beta}(z)\xi_\alpha\xi_\beta\leq\Lambda|\xi|^2$$

holds in $z\in Q_R(z_0)$.

LEMMA 1.3. *Suppose that any smooth nonnegative function v on $Q_R(z_0)$ satisfies $\mathcal{L}v \le C_1 v$ where C_1 is a positive constant. Then for some positive constant $C_{\mathrm{H}}=C_{\mathrm{H}}(\lambda,\Lambda,C_1)$ the following holds*

$$\sup_{Q_{R/2(0)}} e(v) \le C_{\mathrm{H}} R^{-(d+2)} \int_{Q_R(z_0)} e(v)\,dz.$$

Proof of Theorem 1.4. Local existence can be inferred from the a-priori estimates for uniformly parabolic equations Ladyženskaya-Solonnikov-Ural'ceva [**75**]) and the standard implicit function theorem; see Hamilton [**57**].

Let T be a maximal time of a local existence of the solution of (1.10) and (1.11) and $u\in C^\infty([0,T)\times M;N)$ the solution of (1.10). By Lemma 1.2, the energy density is a subsolution to the equation

$$(\partial_t - \triangle_M)e(u) \le Ce(u).$$

Let ι_M be the convexity radius on M. Choose $R<\min\{1,\sqrt{T},\iota_M\}$ and apply Lemma 1.3 to conclude that

$$\begin{aligned} e(u)(z_0) &\le CR^{-(d+2)} \int_{Q_R(z_0)} e(u)\,dz \\ &\le CR^{-(d+2)} \int_{t_0-R^2}^{t_0} E(u(t))\,dt \\ &\le CR^{-d} E(u_0) \end{aligned}$$

for any $z_0=(t_0,x_0)$, $t_0\ge R^2$, where $C=C(M)$. Hence $|\nabla u|$ is uniformly bounded on any compact set on $[0,T]\times M$. By a boot-strap argument, we obtain uniform bounds for all derivatives of u on the same region as above; there exists a positive constant C depending only on $E(u_0)$, M and N such that $\sup_K|\nabla^k u| \le C$ for any compact set K in $[0,T]\times M$ and for any positive integer k. The solution thus can be continued as a smooth solution to (1.10) on $[0,\infty)\times M$. The preceding argument, then gives the uniform a-priori bounds for u and its derivatives on $[1,\infty)\times M$ depending only on M, N and $E(u_0)$. Hence, by Ascoli and Arzéla's theorem, the flow $(u(t,\cdot))_{t\ge1}$ is relatively compact in any C^k-Topology. Finally, by Lemma 1.1, for a sequence $t_k\to\infty$ we have $u_t(t_k,\cdot)\to0$ in $L^2(M)$, therefore a sub-sequence $(u(t_k,\cdot))$ converges smoothly to a harmonic map u_∞. As $u(t_k,\cdot)$ is homotopic to u_0 through the flow $(u(t,\cdot))_{0\le t\le t_k}$ for any k, so is u_∞. If $K^N<0$, the uniqueness of u_∞ and the convergencity of the flow $u(t,\cdot)\to u_\infty$ follows from a maximum principle due to Jäger-Kaul [**67**]. □

The Eells-Sampson result was extended to harmonic maps with boundary by Hamilton [**57**]. The condition $K^N\le0$ can be replaced by the condition that the image of u_0 (and hence of u) should support a uniformly strictly convex function; see Jost [**69**] and von Wahl [**116**]. However, none of these results can be applied in case $u_0\colon T^2\to S^2$ has degree 1 and indeed Example 1.4 above shows that the flow (1.10) and (1.11) in this case cannot exist for all time and does not converge smoothly as $t\to\infty$.

3. Finite time blow-up

It is remained to prove the question whether the heat flow (1.10) will develop singularities in finite or infinite time. Reversing the order of the historical developments, we first state this problem and later present the existence results for weak solutions that are the answers to these questions.

The heat flow (1.10) is a quasi-linear parabolic system and therefore hard to deal with explicitly. However when $N=S^d$, by enough symmetry, (1.10) can be reduced to a scalar equation in only two variables. Consider equivariant maps

$$u_0(x)=\left(\frac{x}{|x|}\sin h_0(r),\cos h_0(r)\right)$$

of $B_1^d(0)$ into S^d, where $r=|x|$, $h_0(0)=0$ and let $u\colon\ [0,T)\times B_1^d(0)\to S^d\subset\mathbb{R}^{d+1}$ be the corresponding smooth solution of (1.10) and (1.11), defined on a maximal time interval $[0,T)$. By uniqueness, also u is equivariant and can be written

$$u(t,x)=\left(\frac{x}{|x|}\sin h(t,r),\cos h(t,r)\right) \tag{1.14}$$

in terms of a smooth map $h\colon [0,T)\times[0,1]\to\mathbb{R}$ satisfying the initial and boundary conditions

$$\begin{aligned} &h(0,r)=h_0(r) && \text{for} \quad 0\le r\le 1,\\ &h(t,0)=h_0(0)=h_{0,rr}(0)=0 && \text{for} \quad 0\le t\le T,\\ &h(t,1)=h_0(1)(=:b) && \text{for} \quad 0\le t\le T. \end{aligned} \tag{1.15}$$

In terms of h, the equation (1.10)

$$u_t-\triangle u=|\nabla u|^2u,$$

becomes

$$h_t-h_{rr}-\frac{d-1}{r}h_r+\frac{\sin 2h}{2r^2}=0. \tag{1.16}$$

If $d\ge 3$, it was shown by Coron-Ghidaglia [**28**] that for suitable h_0, the solution h of (1.15) and (1.16)—and therefore the solution u of (1.10) and (1.11)—cannot be smoothly continued beyond some finite time. We will later see a deeper reason for this; see Theorem 1.11.

The case $d=2$ is more interesting. In dimension $d=2$, a family of stationary solutions h of (1.16) with $h(0)=0$ is obtained by stereographic projection from the south pole; see Example 1.5. In terms of $r=|x|$ and the polar angle $\theta=\phi(r)$ the stereographic projection is given by

$$\frac{\sin\theta}{1+\cos\theta}=r$$

we can write ϕ as

$$\phi(r)=\arccos\left(\frac{1-r^2}{1+r^2}\right).$$

From a dilation $r\to r\lambda$, we obtain the family

$$\phi_\lambda(r)=\phi\left(\frac{r}{\lambda}\right)=\arccos\left(\frac{\lambda^2-r^2}{\lambda^2+r^2}\right),\qquad \lambda>0,$$

of stationary solutions of (1.16) with $\phi_\lambda(0)=0$.

THEOREM 1.5 (Grayson-Hamilton [**50**], Chang-Ding [**17**]). *Let $d=2$. Suppose $|h_0|\le\pi$. Then the solution h of* (1.15) *and* (1.16) *exists for all time.*

Proof. The idea is to construct a barrier, preventing h from becoming discontinuous in finite time. Smoothness of h in $(0,1)$ then follows from general results on quasilinear parabolic systems; see Ladyženskaya-Solonnikov-Ural'ceva [**75**].

First note that Lemma 1.1 translates into the uniform energy bound

$$\pi\int_0^1|h_r|^2r\,dr\le E(u(t))\le E(u_0).$$

Hence, by Sobolev's embedding theorem, $h(t,\cdot)$ is locally Hölder continuous on $(0,1]$ uniformly in t, and a singularity can only develop at the origin.

We assume $0\le h_0\le\pi$ for simplicity. Let $h_1\ge h_0$ satisfy $\pi/2<h_1\le\pi$, $h_1(0)=\pi$ and $h_1(a)<\pi$ for some $a\in(0,1]$. That is, h_1 maps into the (convex) lower hemisphere. Therefore, equation (1.10) and hence (1.16) with initial data $h(0,\cdot)=h_1$ possesses a global, smooth solution $\tilde h$. Moreover by the maximum principle (See Jäger-Kaul [**67**]), $\tilde h(t,r)<\pi$ for $0<r<1$ and $t>0$. Choose a strictly increasing function $\lambda(t)$ such that $\phi_{\lambda(0)}>h_0$ on $(0,a]$ and $\phi_{\lambda(t)}(a)>\tilde h(t,a)$ for all $t\ge0$. Let

$$\bar h(r,t)=\begin{cases}\inf\{\phi_{\lambda(t)}(r),\tilde h(t,r)\} & 0\le r\le a,\\ \tilde h(t,r) & a\le r\le 1,\end{cases}$$

be our barrier. Note that $\bar h$ is a supersolution to (1.16). Similarly, $\underline{h}\equiv0$ is a subsolution. Hence $0\le h(t,r)\le\bar h(t,r)$ on $[0,T]\times[0,1]$ by the maximum principle and $h(t,\cdot)$ is continuous. As a result, the smoothness of h follows form Chang-Ding [**17**, Lemma 3.1] Thus u is smooth at $r=0$ for any $t\ge0$. ☐

Quite surprisingly, the initial condition that $|b_0|\le\pi$ in Theorem 1.5 is sharp.

THEOREM 1.6 (Chang-Ding-Ye [**18**]). *Suppose $|b|>\pi$. Then the solution h to* (1.15) *and* (1.16) *blows up at finite time.*

Proof. Let $b>\pi$. We show the existence of a sub-solution f to (1.16) with $f(t,0)=0\le f\le f(t,1)=b$ such that $f_r(t,0)\to\infty$ as $t\to T$ for some $T<\infty$. Set $h_0=f(0,\cdot)$ and let h be the corresponding solution of (1.15) and (1.16), the maximum principle then implies that $h\ge f$ on $[0,1]\times[0,T)$. Consequently, h must blow up before time T. (For general initial data the proof is somewhat more complicated.)

We make the following ansatz for f:

$$f(t,r)=\phi_{\lambda(t)}(r)+\phi_\mu(r^{1+\epsilon}),$$

where $\epsilon>0,\mu>0$ and $\lambda=\lambda(t)$ will be suitably chosen later. Note that for any $\epsilon>0$ we have

$$\phi_\mu(r^{1+\epsilon})=\arccos\left(\frac{\mu^2-r^{2+2\epsilon}}{\mu^2+r^{2+2\epsilon}}\right)\to 0 \quad (\mu\to\infty)$$

uniformly in $r\in[0,1]$. Hence, for any given $\epsilon>0$, taking $\mu^2\ge 2/\epsilon+1$, we obtain

$$\cos\phi_\mu(r^{1+\epsilon})\ge\frac{1}{1+\epsilon}\quad \text{for } r\in[0,1].$$

Next observe that for any μ and $\epsilon>0$ the function $\theta(r)=\phi_\mu(r^{1+\epsilon})$ satisfies

$$-\theta_{rr}-\frac{1}{r}\theta_r+\frac{(1+\epsilon)^2\sin 2\theta}{2r^2}=0.$$

Therefore

$$\begin{aligned}
\tau(f):=&f_{rr}+\frac{1}{r}f_r-\frac{\sin 2f}{2r^2}\\
=&\frac{\big(-\sin 2(\phi_\lambda+\theta)+\sin 2\phi_\lambda+(1+\epsilon)^2\sin 2\theta\big)}{2r^2}\\
=&\frac{\Big(\sin\big((2\phi_\lambda+\theta)-\theta\big)-\sin\big((2\phi_\lambda+\theta)+\theta\big)+(1+\epsilon)^2\sin 2\theta\Big)}{2r^2}\\
=&\frac{\big(-2\cos(2\phi_\lambda+\theta)\sin\theta+2(1+\epsilon)^2\cos\theta\sin\theta\big)}{2r^2}\\
\ge&\frac{(1+\epsilon)-\cos(2\phi_\lambda+\theta)}{r^2}\sin\theta\\
\ge&\frac{\epsilon}{r^2}\sin\theta=\frac{\epsilon}{r^2}\frac{2\mu r^{1+\epsilon}}{\mu^2+r^{2+2\epsilon}}\ge\epsilon_1 r^{\epsilon-1}\quad\text{as long as } \epsilon_1=\frac{2\mu\epsilon}{\mu^2+1}>0.
\end{aligned}$$

On the other hand, we have

$$\frac{\partial f}{\partial t}(t,r)=\frac{\partial\phi_{\lambda(t)}(r)}{\partial t}=-\frac{2r}{\lambda^2+r^2}\frac{d\lambda(t)}{dt}.$$

Let $\lambda(t)$ solve $d\lambda/dt=-\delta\lambda^\epsilon$ where $\delta>0$ to be determined:

$$\lambda(t)=[\lambda_0^{1-\epsilon}-(1-\epsilon)\delta t]^{1/(1-\epsilon)}.$$

Then $f_r(t,0)\to+\infty$ as $t\nearrow T=\lambda_0^{1-\epsilon}/(1-\epsilon)\delta$. Finally

$$\frac{\partial f}{\partial t}-\tau(f)\leq\frac{2\delta\lambda^\epsilon r}{\lambda^2+r^2}-\epsilon_1 r^{\epsilon-1}=\left[\frac{2\delta\lambda^\epsilon r^{2-\epsilon}}{\lambda^2+r^2}-\epsilon_1\right]r^{\epsilon-1}.$$

But by Young's inequality

$$\lambda^\epsilon r^{2-\epsilon}\leq C(\epsilon)(\lambda^2+r^2),$$

and hence f is a sub-solution of (1.16) for sufficiently small $\delta=\delta(\epsilon)>0$ as desired. Since $\epsilon>0$ is arbitrary, $\sup|f(0,\cdot)-\pi|$ can be taken as small as we wish □

4. Global existence and uniqueness of partially regular weak solutions on the surface

Theorem 1.6 shows that in general, smooth and global solutions to (1.10) and (1.11) do not exist. However, the following question comes to my mind: Is there a "weak" analogue of Theorem 1.4 that will still provide a satisfactory meanings towards deciding the homotopy problem? We state an exact definition on "a global weak solution of (1.10) and (1.11):" Set $V((0,\infty)\times M;N)=L^\infty((0,\infty);H^{1,2}(M;N))\cap H^{1,2}((0,\infty);L^2(M:N))$.

Definition 1.3. Let $u_0\in H^{1,2}(M;N)$. If we say a map $u\in V((0,\infty)\times M;N)$ to be "weakly evolutional harmonic map", u satisfy (1.10) in $\mathcal{D}'([0,\infty)\times M)$, $\lim_{t\to+0}u(t,\cdot)=u_0(\cdot)$ in the sense of $L^2(M)$-norm.

The following result is mainly due to Struwe [**107**]; the result was extended to the case $\partial M\neq\emptyset$ by Chang [**16**].

Theorem 1.7. *Suppose that M is a compact Riemann surface and $N\subset\mathbb{R}^{\hat{D}}$ is compact Riemannian manifold without boundary. Then for any $u_0\in H^{1,2}(M;N)$, there exists a global weak solution $u\colon[0,\infty)\times M\to N$ of (1.10) and (1.11) satisfying the energy inequality and belonging to class C^∞ on $(0,\infty)\times M$ away from finitely many points $(t_{s,j},x_{s,j})$ $(j=1,\ldots,\mathcal{J})$. The solution u is unique in this class.*

At a singularity (t_s,x_s), a (non-constant) *harmonic sphere $\bar{u}\colon S^2\to N$ separates in the sense that for suitable sequences $t_k\nearrow t_s$, $x_k\to x_s$, $R_k\searrow 0$, the rescaled maps*

$$u_k(x)=u(t_k,x_k+R_k\bar{x})\colon B_1^2(0)\to N$$

(in a local conformal chart around x_s) *converge to a non-constant harmonic limit $\tilde{u}\colon\mathbb{R}^2\to N$ in $H^{2,2}_{loc}(\mathbb{R}^2;N)$. $\tilde{u}$ has finite energy and extends to a smooth harmonic map $\bar{u}\colon S^2\cong\overline{\mathbb{R}^2}\to N$.*

Finally, for a suitable sequence $t_k\to\infty$, the sequence $(u(t_k,\cdot))$ converges weakly to a smooth harmonic map $u_\infty\colon M\to N$ in $H^{1,2}(M;N)$.

The convergence is strong away from finitely many points x_l^∞ $(l=1,\ldots,L)$ and there holds $K+L\leq\epsilon_0^{-1}E(u_0)$ where

$$\epsilon_0=\inf\{E(\bar{u});\bar{u}\colon S^2\to N\textit{ is a non-constant smooth harmonic map}\}>0$$

is a constant depending only on N.

The proof of this result is based on the following inequality:

LEMMA 1.4 (Ladyženskaya [**74**]). *For any $v\in H^{1,2}(\mathbb{R}^2)$, $v\in L^4(\mathbb{R}^2)$ and*

$$\|v\|_{L^4}^4\leq 2\|v\|_{L^2}^2\|\nabla v\|_{L^2}^2$$

hold.

See also Friedman [**45**] or Struwe [**112**, Lemma 3.5.7].

We now proceed with the proof of Theorem 1.7.

Positivity of ϵ_0: We apply Lemma 1.4 to $v=|\nabla u|\phi$, where u solves (1.3) on S^2 and $\phi\in C^\infty(S^2)$ is a smooth cut-off function $0\leq\phi\leq 1$ with support in a coordinate neighborhood on S^2 to give

$$\begin{aligned}(1.17)\qquad &\int_{S^2}|\nabla u|^4\phi^2\,dvol_M\\ &\leq C\int_{\operatorname{supp}\phi}|\nabla u|^2\,dvol_M\left(\int_{S^2}|\nabla^2 u|^2\phi^2\,dvol_M+\int_{S^2}|\nabla u|^2|\nabla\phi|^2\,dvol_M\right).\end{aligned}$$

In particular, if (ϕ_j^2) is a smooth partition of unity subordinate to a finite cover (U_j) of S^2 by local coordinate charts $U_j\,(j=1,\dots,\mathcal{J})$, we obtain from (1.3), (1.17) and the Calderón-Zygmund inequality that

$$\begin{aligned}\int_{S^2}|\nabla^2 u|^2\,dvol_{S^2}&\leq C\int_{S^2}|\triangle u|^2\,dvol_{S^2}+C\int_{S^2}|u|^2\,dvol_{S^2}\\ &\leq C\int_{S^2}|\nabla u|^4\,dvol_{S^2}+C\\ &\leq CE(u)\Big(\int_{S^2}|\nabla^2 u|^2\,dvol_{S^2}+E(u)\Big)+C\end{aligned}$$

with a constant $C=C(N)$. Hence the number ϵ_0 defined in Theorem 1.7, is strictly positive. Indeed, if $E(u_k)\to 0$ for a sequence of harmonic maps $u_k\colon S^2\to N$, we obtain that $u_k\to const$ in $H^{2,2}(S^2;N)\hookrightarrow C^0(S^2;N)$. In particular, $u_k(S^2)$ lies in a convex geodesic ball on N for large k. Thus u_k must be constant, which provides a contradiction.

L^2-estimates for $\nabla^2 u$: By (1.17), it is important to control the energy locally. For $\Omega\subset M$, denote

$$E(u;\Omega)=\int_\Omega e(u)\,dvol_M.$$

LEMMA 1.5. *Set $R_0=\min\{1,\iota_M\}$ where ι_M is the convex radius of M. Denote $B_R(x_0)$ by the ball in a local conformal chart around x_0 with $0<2R\leq R_0$. Then for any positive number R with $0<2R\leq R_0$, a smooth solution $u\colon[0,T]\times M\to N$ of* (1.10) *and* (1.11) *satisfies*

$$E\big(u(T);B_R(x_0)\big)\leq E\big(u_0;B_{2R}(x_0)\big)+\frac{CT}{R^2}E(u_0)$$

with $C{=}C(M{,}N)$.

Proof. Let $\phi{\in}\mathcal{D}(B_{2R}(x_0))$ satisfy $0{\le}\phi{\le}1$, $\phi{\equiv}1$ on $B_R(x_0)$, $|\nabla\phi|{\le}2/R$ and test (1.10) with $u_t\phi^2$ to obtain

$$\int_M |u_t|^2\phi^2\,dvol_M+\frac{d}{dt}\left(\int_M e(u)\phi^2\,dvol_M\right)\le C\int_M |\nabla u||u_t||\nabla\phi|\phi\,dvol_M$$
$$\le\int_M |u_t|^2\phi^2\,dvol_M+C\int_M |\nabla u|^2|\nabla\phi|^2\,dvol_M.$$

Hence

$$\frac{d}{dt}\left(\int_M e(u)\phi^2\,dvol_M\right)\le CR^{-2}E(u(t))\le CR^{-2}E(u_0)$$

and the lemma follows by integration with respect to a time variable from 0 to T. □

In particular, for any given $\epsilon_1{>}0$ and $u_0{\in}H^{1,2}(M;N)$, there exists a number $T_1{>}0$ depending only on a maximal positive number R_1 with

$$\sup_{x_0\in M} E(u_0;B_{2R_1}(x_0))<\epsilon_1,$$

the geometry of M, N and $E(u_0)$, such that any smooth solution u of (1.10) and (1.11) satisfies

$$\sup_{0\le t\le T_1,x_0\in M} E(u(t);B_{R_1}(x_0))<2\epsilon_1.$$

Actually, we can set $T_1{=}(\epsilon_1R_1^2)/(CE(u_0))$ where C is the constant in Lemma 1.5.

Since M is compact, we can choose finite points (x_j) $(j{=}1{,}2{,}\ldots\mathcal{J})$ so that $M{\subset}\cup_{j=1}^{\mathcal{J}}B_{R_j}(x_j)$. Thus for any given $\epsilon_1{>}0$, let $R_1{>}0$ be determined as above and ϕ_i smooth cut-off functions subordinate to a cover of M by balls $B_{2R_1}(x_i)$ satisfying $0{\le}\phi_i{\le}1$, $|\nabla\phi_i|{\le}2/R_1$ and $\sum_i\phi_i^2{=}1$. Then by (1.17) we have

$$\int_M |\nabla u|^4\,dvol_M=\sum_i\int_M |\nabla u|^4\phi_i^2\,dvol_M$$
$$\le C\sup_i E\Big(u(t);B_{2R_1}(x_i)\Big)\left(\int_M |\nabla^2u(t)|^2\,dvol_M+R_1^{-2}E(u_0)\right)$$
$$\le C\epsilon_1\left(\int_M |\nabla^2u(t)|^2\,dvol_M+R_1^{-2}E(u_0)\right)$$

for any t. Moreover, similar to our Bochner-type estimate (1.13), multiplying (1.10) by $\triangle_M u$ and integrating by parts, we have

$$\frac{d}{dt}(E(u(t)))+\int_M |\triangle_M u|^2\,dvol_M\le C\int_M |\nabla u|^4\,dvol_M.$$

By integrating over $(0,T_1)$ and combining the above estimates with the Calderón-Zygmund inequality, we obtain that

$$\int_0^{T_1}\int_M |\nabla^2 u|^2 dtdvol_M \leq C\int_0^{T_1}\int_M |\triangle_M u|^2 dtdvol_M + CT_1$$

$$\leq C\epsilon_1 \int_0^{T_1}\int_M |\nabla^2 u|^2 dtdvol_M + \frac{CT_1}{R_1^2}E(u_0) + CT_1,$$

with a constant $C=C(M,N)$. Thus, for sufficiently small $\epsilon_1>0$, we obtain an a-priori bound for u in the norm

$$\|u\|^2_{W((0,T)\times M)} = \sup_{0<t<T} E(u(t)) + \int_0^T\int_M \left(|\nabla^2 u|^2 + |u_t|^2\right) dtdvol_M$$

with $T=T_1$ of the form

$$\|u\|^2_{W((0,T_1)\times M)} \leq C(1+\frac{T_1}{R_1^2})E(u_0) + CT_1.$$

Here we use the fact that $W((0,T)\times M)$ is an admissible class for (1.10) in the sense that if $u\in W((0,T)\times M)$ solves (1.10) with finite energy initial data $u_0\in H^{1,2}(M;N)$, then $u\in C^\infty((0,T]\times M;N)$.

Local existence: A maximal positive number R_1 can be chosen uniformly for a set of initial data which is compact in $\{u_0\}+\overset{\circ}{H}{}^{1,2}(M;N)$. In particular, if $u_{0m}\in C^\infty(M;N)$ converges to u_0 in $H^{1,2}(M;N)$ and if $\{u_m\}$ $(m=1,2,\ldots)$ is the corresponding sequence of local solutions (1.10) for initial $\{u_{0m}\}$ $(m=1,2,\ldots)$ by the above a-priori estimate, we have $\|u_m\|^2_{W((0,T_1)\times M)} \leq C$ $(1+T_1/R_1^2)$ $E(u_0)+$ CT_1 where $T_1=(\epsilon_1 R_1^2)/(CE(u_0))$ and the sequence $\{u_m\}$ $(m=1,2,\ldots)$, weakly converges weakly to $u\in W((0,T_1)\times M)$.

$\nabla u_m \to \nabla u$ in $L^2(M)$ for almost every t and then it is easy to pass to the limit in equation (1.10); u then solves (1.10) classically in $(0,T_1)\times M$. Finally, by Lemma 1.1, u achieves its initial data continuously in $H^{1,2}(M;N)$.

Uniqueness: The space of functions with bounded $W((0,T)\times M)$-norm is a uniqueness class. Indeed, if u and $v\in W((0,T)\times M)$ weakly solve (1.10) with $u(0)=u_0=v(0)$, their difference $w=u-v$ satisfies

$$|w_t - \triangle_M w| \leq C|w|(|\nabla u|^2+|\nabla v|^2) + C|\nabla w|(|\nabla u|+|\nabla v|).$$

Multiply the above inequality by w and perform the integrate by parts to verify for almost every $t\geq 0$

$$\frac{1}{2}\int_M |w(t)|^2 dvol_M + \int_0^t\int_M |\nabla w|^2 dsdvol_M$$

$$\leq C\int_0^t\int_M |w|^2(|\nabla u|^2+|\nabla v|^2) dsdvol_M$$

$$
\begin{aligned}
&+C\int_0^t\int_M |w||\nabla w|(|\nabla u|+|\nabla v|)\,ds\,dvol_M\\
&\le C\left(\int_0^t\int_M |w|^4 ds\,dvol_M\right)^{1/2}\left(\int_0^t\int_M \left(|\nabla u|^4+|\nabla v|^4\right) ds\,dvol_M\right)^{1/2}\\
&+C\left(\int_0^t\int_M |w|^4 ds\,dvol_M\right)^{1/4}\left(\int_0^t\int_M |\nabla w|^2 ds\,dvol_M\right)^{1/2}\\
&\quad\times\left(\int_0^t\int_M \left(|\nabla u|^4+|\nabla v|^4\right) ds\,dvol_M\right)^{1/4}\\
&\le C\epsilon(t)\left[\sup_{0\le s\le T_1}\int_M |w(s)|^2 ds+\int_0^t\int_M |\nabla w|^2 ds\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\epsilon(t)=&\left(\int_0^t\int_M \left(|\nabla u|^4+|\nabla v|^4\right) ds\,dvol_M\right)^{1/4}\\
&\times\left(1+\left(\int_0^t\int_M \left(|\nabla u|^4+|\nabla v|^4\right) ds\,dvol_M\right)^{1/4}\right).
\end{aligned}
$$

Here we applied Lemma 1.4 to the term of $\int_0^t\int_M |w|^4 ds\,dvol_M$ in the estimates above. As in the same way as before, on account of Lemma 1.5 we estimate

$$\epsilon(t)\to 0 \quad \text{as} \quad t\to +0.$$

Choosing $t>0$ small enough with $\|w(t)\|_{L^2(M)} = \sup_{0\le s\le T_1}\|w(s)\|_{L^2(M)}$, the uniqueness follows.

Global continuation: The local solution u constructed above can be extended until the first time $T=t_{\mathrm{s},1}$ such that

$$\limsup_{t\nearrow T}\left(\sup_{x_0} E\big(u(t);B_R(x_0)\big)\right)\ge\epsilon_1$$

for all $R>0$. By Lemma 1.1, $u_t\in L^2([0,T]\times M)$. Hence the L^2-limit $u_1=\lim_{t\nearrow T}u(t)$ exists. Let v be the local solution of (1.10) with initial data $v=u_1$ at time T. The composed function

$$w(t)=\begin{cases} u(t) & 0\le t\le T,\\ v(t) & T\le t,\end{cases}$$

then is a weak solution of (1.10). By iteration we obtain a weak solution u on a maximal time interval $[0,\bar{T})$. If $\bar{T}<\infty$, the above arguments again permit us to extend u beyond $\bar{T}$, contradicting our assumption that $\bar{T}$ was maximal. Hence $\bar{T}=\infty$.

Finiteness of the singular set: Let $t_{\mathrm{s}}=t_{\mathrm{s},1}>0$ be the first singular time and set

$$\mathrm{Sing}(t_{\mathrm{s}})=\{x_0\in M;\limsup_{t\nearrow t_{\mathrm{s}}} E(u(t);B_R(x_0))>\epsilon_1\,\text{for any}\,R>0\}.$$

$\mathrm{Sing}(t_{\mathrm{s}})$ is finite. Indeed, let $x_1,\ldots,x_J\in\mathrm{Sing}(t_{\mathrm{s}})$. Choose $R>0$ such that $B_{2R}(x_i)\cap B_{2R}(x_j)=\emptyset$ $(i\neq j)$. Then by Lemma 1.5 for some $\tau\in[t_{\mathrm{s}}-(\epsilon_1 R^2)\ /(2CE(u_0)),t_{\mathrm{s}})$ where C is the constant in Lemma 1.5,we obtain

$$\begin{aligned}K\epsilon_1 &\leq \sum_{i=1}^{K}\limsup_{t\nearrow t_{\mathrm{s}}} E(u(t);B_R(x_i))\\ &\leq \sum_{i=1}^{K}\Big(E(u(\tau);B_{2R}(x_i))+\frac{\epsilon_1}{2}\Big)\leq E(u_0)+\frac{K\epsilon_1}{2},\end{aligned}$$

and $K=\#\mathrm{Sing}(t_{\mathrm{s},1})\leq 2E(u_0)\epsilon_1^{-1}$. Set K be K_1. Moreover, for $u_1=\lim_{t\nearrow t_{\mathrm{s},1}}u(t)$ we have

$$\begin{aligned}E(u_1) &= \lim_{R\searrow 0} E\left(u_1;M\setminus\bigcup_{i=1}^{K_1}B_{2R}(x_i)\right)\\ &\leq \lim_{R\searrow 0}\liminf_{t\nearrow t_{\mathrm{s},1}} E\left(u(t);M\setminus\bigcup_{i=1}^{K_1}B_{2R}(x_i)\right)\\ &\leq \lim_{R\searrow 0}\liminf_{t\nearrow t_{\mathrm{s},1}}\left(E(u(t))-\sum_{i=1}^{K_1}E\big(u(t);B_{2R}(x_i)\big)\right)\\ &\leq \lim_{R\searrow 0}\left(E(u_0)-\limsup_{t\nearrow t_{\mathrm{s},1}}\sum_{i=1}^{K_1}E\big(u(t);B_{2R}(x_i)\big)\right)\\ &\leq E(u_0)-\sum_{i=1}^{K_1}\lim_{R\searrow 0}\limsup_{t\nearrow t_{\mathrm{s},1}}E\big(u(t);B_R(x_i)\big)\\ &\leq E(u_0)-K_1\epsilon_1.\end{aligned}$$

Similarly, let K_2, K_3, ... be the number of concentration points at consecutive times $t_{\mathrm{s},2}<t_{\mathrm{s},3}<\ldots$ and let $u_j=\lim_{t\nearrow t_{\mathrm{s},j}}u(t)$ for $j=2,3,\ldots$. Then by induction we obtain

$$E(u_j)\leq E(u_{j-1})-K_j\epsilon_1\leq\cdots\leq E(u_0)-(K_1+\cdots+K_j)\epsilon_1.$$

Thus it follows that the total number K_Σ of concentration points and at same time the number of concentration times t_j are finite; $K_\Sigma\leq E(u_0)\epsilon_1^{-1}$.

Smoothness: Let $t_{\mathrm{s}}=t_{\mathrm{s},j}$ for some j. To see that u is smooth up to time t_{s} away from $\mathrm{Sing}(t_{\mathrm{s}})$, we present an argument based on scaling, as proposed by Schoen [**96**] in the stationary case. For a simplicity, we discuss the smoothness in $M=\mathbb{R}^2$. In addition, by scaling we can assume $t_{\mathrm{s}}\geq 1$ and we shift time so that $t_{\mathrm{s}}=0$. The solution u then is defined on a domain containing $[-1,0]\times M$. For any $R>0$ and $z_0=(t_0,x_0)$, denote

$$Q_R(z_0)=\big\{z=(t,x);t_0-R^2<t<t_0,|x-x_0|<R\big\},$$

and shorten $Q_R(0)$ to Q_R and Q_1 to Q when no confusion may arise.

Proposition 1.3. *Suppose that $u \in C^\infty(Q;N)$ solves (1.10). There exist constants C and $\epsilon_2 > 0$ depending only on N such that if*

$$\sup_{-1 \le t \le 0} E\big(u(t); B_1(0)\big) < \epsilon_2,$$

then

$$\sup_{Q_{1/2}(0)} |\nabla u(t,x)| \le C$$

(*and corresponding bounds for higher derivatives*), *holds.*

Proof. Choose $0 \le \rho < 1$ so that

$$(1-\rho)^2 \sup_{Q_\rho(0)} e(u) = \max_{0 \le \sigma \le 1} \left\{ (1-\sigma)^2 \sup_{Q_\sigma(0)} e(u) \right\}$$

and let $z_{\max} = (t_{\max}, x_{\max}) \in Q_{\rho(0)}$ satisfy

$$e(u)(z_{\max}) = \sup_{Q_\rho(0)} e(u) =: e_0.$$

Then either $e_0(1-\rho)^2 < 4$ in which case

$$\sup_{Q_{1/2}(0)} e(u) \le 4(1-\rho)^2 e_0 < 16,$$

or $e_0^{-\frac{1}{2}} \le (1-\rho)/2$. In the latter case, the scaled function

$$v(\bar{t}, \bar{x}) = u(t_{\max} + e_0^{-1}\bar{t}, x_{\max} + e_0^{-1/2}\bar{x})$$

is well defined on $Q_1(0)$. Moreover, $e(v)(0) = 1$ and

$$\sup_{Q_1(0)} e(v) \le e_0^{-1} \sup_{Q_{(1+\rho)/2}(0)} e(u) \le e_0^{-1} \frac{(1-\rho)^2 \sup_{Q_\rho(0)} e(u)}{((1-\rho)/2)^2} = 4.$$

Thus $e(v)$ satisfies a linear differential inequality

$$e(v)_{\bar{t}} - \triangle e(v) \le C e(v) \quad \text{on} \quad Q_1(0).$$

By Lemma 1.3, we have

$$\begin{aligned} 1 = e(v)(0,0) = \le \sup_{Q_{1/2}(0)} e(v) &\le C_{\mathrm{H}} \int_{Q_1(0)} e(v)\, d\bar{t} d\bar{x} \le e_0^2 \int_{Q_1(0)} e(u)\, dt dx \\ &\le C \sup_{-1 < t < 0} \Big(E\big(u(t); B_1(0)\big)\Big) \le C\epsilon_2. \end{aligned}$$

This is a contradiction if we take $C_{\mathrm{H}}\epsilon_2 < 1/2$. Thus we conclude our claim. $\square$

Blow-up of singularities: As in the same way in Struwe [**107**], we use the scaling technique to analyze the singularities in more detail. Let $(t_{\mathrm{s}}, x_{\mathrm{s}})$ be a singular point of the solution u constructed above. Shift time so that $(t_{\mathrm{s}}, x_{\mathrm{s}}) = (0,0)$. Moreover after scaling we can assume that $u \in C^\infty(Q_1(0) \setminus \{0\}; N)$. Let $\{R_k\}$ $(k = 1, 2, \ldots)$ be a sequence of numbers $R_k\in(0,1), R_k\searrow 0$ $(k\to\infty)$. Set $\epsilon_1>0$ be as above and take sequences $\{t_k\}$ and $\{x_k\}$ $(k=1,2,\dots)$ satisfying $t_k\searrow 0$ and $x_k\to 0$ and

$$E\big(u(t_k);B_{R_k}(x_k)\big)=\sup_{-1\le t\le t_k;(t,x)\in Q_1(0)} E(u(t);B_{R_k}(x))=\frac{\epsilon_1}{L},$$

where L is the number of unit discs needed to cover $B_2(0)$. We may assume $t_k-4R_k^2\ge -1$. Rescale

$$\bar t=\frac{t-t_k}{R_k^2},\ \bar x=\frac{x-x_k}{R_k},$$

and set

$$v_k(\bar t,\bar x)=u\big(t_k+R_k^2\bar t, x_k+R_k\bar x\big),$$

$v_k\colon Q_k=\big\{(\bar t,\bar x)\in(-4,0]\times\mathbb{R}^2; (t_k+R_k^2\bar t, x_k+R_k\bar x)\in Q_1(0)\big\}\to N$. Note that by Lemma 1.1 we have

$$\int_{Q_k}|(v_k)_{\bar t}|^2\,d\bar t\,dvol_M\le\int_{t_k-4R_k^2}^{t_k}\int_M|u_t|^2\,dt\,dvol_M\to 0\qquad(k\to\infty),$$

$$E\big(v_k(t)\big)\le E(u_0),\qquad -4\le\bar t\le 0,\qquad k\in N.$$

Moreover, we obtain

$$\begin{aligned}\sup_{(\bar t,\bar x)\in Q_k}E\big(v_k(\bar t);B_2(\bar x)\big)&\le L\sup_{(\bar t,\bar x)\in Q_k}E\big(v_k(\bar t);B_1(\bar x)\big)\\&\le L\sup_{(t,x)\in D_k}E\big(u(t);B_{R_k}(x)\big)\le LE(u(t_k);B_{R_k}(x_k))=\epsilon_1.\end{aligned}$$

Thus, by Proposition 1.3, the sequence $\{v_k\}$ $(k=1,2,\dots)$ is locally a priori bounded in C^l for any $l\in\mathbb{N}$ and a sub-sequence converges strongly in $H^{1,2}_{\mathrm{loc}}((-1,0)\times\mathbb{R}^2;N)$ to a smooth solution v of (1.10) on $[-1,0]\times\mathbb{R}^2$. Moreover, from $v_{\bar t}=0$, it follows that $v(\bar t,\cdot)=:\tilde u$ is harmonic. Finally

$$E(\tilde u;B_1(0))=\lim_{k\to\infty}E\big(v_k(0);B_1(0)\big)=\lim_{k\to\infty}E\big(u(t_k);B_{R_k}(x)\big)=\frac{\epsilon_1}{L},$$

and then $\tilde u$ is non-constant. If $\tilde u\colon\mathbb{R}^2\to N$, by conformal equivalence $\mathbb{R}^2\cong S^2\setminus\{p\}$, $\tilde u$ induces a weakly harmonic map $\bar u\colon S^2\to N$. By Hélein's result, $\bar u$ is smooth. Thus, singularities at small scales can be seen as harmonic spheres. In particular we obtain the estimate $\epsilon_1\ge\epsilon_0$.

A similar analysis is possible at concentration points at "$t_{\mathrm{s}}=\infty$;"

see Struwe [**112**, Lecture III.5] for details. Moreover, it seems possible to iterate the above procedure and decompose u in the limit $t\nearrow t_{\mathrm{s}}$ into its weak limit $u(t_{\mathrm{s}})$ and a finite sum of harmonic spheres $\bar u_1,\dots,\bar u_M$ for some positive integer M, similar to Struwe [**106**] to a related problem, we arrive at

$$E\big(u(t_{\mathrm{s}})\big)+\sum_{l=1}^{M}E(\bar u_l)=\lim_{t\nearrow t_{\mathrm{s}}}E\big(u(t)\big).\tag{1.18}$$

This analysis has been done by Ding-Tian [**30**]. See also Qing-Tian [**91**] and Wang [**117**]. They call (1.18) "the energy identity."

Extensions and generalizations

Theorem 1.7 has been extended to target manifolds N with boundary by Chen-Musina [**23**]. The same technique can be used to study the evolution problems related to other two-dimensional variational problems. For instance, in Struwe [**108**] and Rey [**92**], the evolution problem for surfaces of prescribed mean curvature is investigated; Li Ma [**77**] has studied the evolution of harmonic maps with free boundaries.

5. Existence of global, partially regular weak solutions in higher dimensional manifold

Earlier we observed that singularities must do emerge even for energy-minimizing weakly harmonic maps if $d=\dim M\geq 3$ and therefore for the evolution problem (1.10). The following result was obtained by Chen-Struwe [**24**].

THEOREM 1.8. *Suppose that M is a d-dimensional compact manifold with $d\geq 3$ and $\partial M=\emptyset$. For any $u_0\in H^{1,2}(M;N)$, there exists a distribution solution $u\colon [0,\infty)\times M\to N$ of* (1.10) *and* (1.11) *satisfying the energy inequality and being smooth away from a closed set Σ such that for each t the slice $\Sigma(t):=\Sigma\cap(\{t\}\times M)$ is of co-dimension more than or equal to* 2. *As $t\to\infty$ suitably, $u(t)$ converges weakly to a weakly harmonic limit u_∞ which is smooth away from a closed set $\Sigma(\infty)$ of co-dimension more than or equal to* 2.

Originally, the estimate on the co-dimension of Σ was obtained in space-time, the above improvement is due to X. Cheng [**25**]. For manifolds M with boundary $\partial M\neq\emptyset$, a similar existence and an interior partial regularity result hold; see Chen [**20**]. Boundary regularity is open. The proof of Theorem 1.8 rests on two pillars: A penalty approximation scheme for (1.10) is developed independently by Chen [**19**], Keller-Rubinstein-Sternberg [**73**] and Shatah [**100**]; while a monotonicity estimate for (1.10) is due to Struwe [**109**].

Penalty approximation

We discuss the case $N=S^{D-1}\subset\mathbb{R}^D$. Given $u_0\in H^{1,2}(M;S^{D-1})$ and $\lambda\in\mathbb{R}_+$; consider the Cauchy problem

$$u_t-\triangle_M u+\lambda(|u|^2-1)u=0, \tag{1.19}$$

$$u|_{t=0}=u_0 \tag{1.20}$$

for maps $u\colon [0,\infty)\times M\to\mathbb{R}^D$. This scheme enables us to "forget" the target constraint. We can regard all map $u\colon M\to\mathbb{R}^D$ as admissible. However, we "penalize" violation of the constraint $|u|^2=1$ more and more severely as $\lambda\to\infty$. (1.19) is the L^2-gradient flow for the functional

$$E_\lambda(u)=E(u)+\lambda\int_M\frac{(|u|^2-1)^2}{4}dvol_M.$$

Indeed, we have

LEMMA 1.6. *If* $u_\lambda\in C^\infty([0,T)\times M;\mathbb{R}^D)$ *solves* (1.19) *and* (1.20), *then we have the following identity*:

$$\int_0^T\int_M|\partial_t u_\lambda|^2\,dtdvol_M+E_\lambda(u_\lambda(T))=E_\lambda(u_0)=E(u_0)$$

holds; in particular, u attains its initial data continuously in $H^{1,2}(M;\mathbb{R}^D)$

Proof. Multiply (1.19) by $\partial_t u_\lambda$ and integrate to obtain the energy estimate. Since $\partial_t u_\lambda\in L^2([0,T)\times M)$, clearly $u_\lambda(t)\to u_0$ in $L^2(M)$ and weakly in $H^{1,2}(M,N)$ as $t\to+0$. Since also

$$\limsup_{t\to+0}E\big(u_\lambda(t)\big)\le\limsup_{t\to+0}E_\lambda\big(u_\lambda(t)\big)\le E(u_0),$$

we also have strong $H^{1,2}$-convergence. □

Moreover, we have an L^∞ a-priori bound.

LEMMA 1.7. *If* $u_\lambda\in C^\infty([0,T)\times M;\mathbb{R}^D)$ *solves* (1.19) *and* (1.20), *then*

$$\|u_\lambda\|_{L^\infty}\le 1.$$

Proof. Multiply (1.19) by u_λ to obtain

$$\left(\frac{d}{dt}-\triangle\right)\frac{|u_\lambda|^2}{2}+|\nabla u_\lambda|^2+\lambda(|u_\lambda|^2-1)|u_\lambda|^2=0;$$

in particular,

$$\left(\frac{d}{dt}-\triangle+2\lambda|u_\lambda|^2\right)\big(|u_\lambda|^2-1\big)\le 0.$$

The claim now follows from the parabolic maximum principle, since $|u(0)|=|u_0|\le 1$. □

Thus for any $\lambda\in\mathbb{R}_+$ we have the unique, global, smooth solution u_λ of (1.19) and (1.20). Moreover,

$$\|\partial_t u_\lambda\|^2_{L^2([0,\infty)\times M)}\le E(u_0),\quad \sup_t E\big(u_\lambda(t)\big)\le E(u_0),$$

$$|u_\lambda|\le 1,\quad \sup_t\||u_\lambda(t)|^2-1\|^2_{L^2(M)}\le\frac{4E(u_0)}{\lambda}\to 0.$$

Thus, passing to a sub-sequence, if necessary, we can claim that for a map $u \in L^2((0,\infty);H^{1,2}(M)) \cap H^{1,2}((0,\infty);L^2(M))$

$$\begin{aligned} u_\lambda &\to u \quad \text{in } L^2_{\mathrm{loc}}([0,\infty)\times M;\mathbb{R}^D),\\ \nabla u_\lambda &\to \nabla u \quad \text{weakly-}* \text{ in } L^\infty([0,\infty);L^2(M)),\\ \partial_t u_\lambda &\to \partial_t u \quad \text{weakly in } L^2([0,\infty)\times M) \end{aligned} \tag{1.21}$$

and $|u|=1$. Relations (1.21) are not sufficient to pass to the limit $\lambda\to\infty$ in (1.19). In case $N=S^{D-1}$, however, a clever manipulation of (1.19) will do the trick.

Lemma 1.8. *Suppose $N=S^{D-1}\subset\mathbb{R}^D$. A map $u\in V((0,\infty)\times M;N)$ satisfies*

$$|u|=1 \quad \text{a.e} \quad z\in(0,\infty)\times M, \tag{1.22}$$

$$u_t\wedge u-\operatorname{div}(\nabla u\wedge u)=0, \quad \text{in the sense of} \quad \mathcal{D}'((0,\infty)\times M;\mathbb{R}^D), \tag{1.23}$$

where "$\wedge$" denotes the wedge product between $\mathbb{R}^D$-vectors, if and only if the map $u\in V((0,\infty)\times M;N)$ is a weakly evolutional harmonic map.

Proof. If u is a weakly evolutional harmonic map, an approximation argument justifies taking an exterior product between (1.10) and ψ, where $\psi\in \mathcal{D}((0,\infty)\times M;\wedge^2(\mathbb{R}^D))$, then we obtain (1.23). Conversely, suppose taht u weakly solves (1.22) and (1.23); (1.23) is written by

$$(u^i_t u^j-u^j_t u^i)-\operatorname{div}(\nabla u^i u^j-\nabla u^j u^i)=0, \qquad (i,j=1,2,\dots,D). \tag{1.24}$$

Multiply (1.24) by $u^j\eta^i$ where $\eta=(\eta^i)$ $(i=1,2,\dots,D)\in\mathcal{D}((0,\infty)\times M;\mathbb{R}^D)$, summing up it from 1 to D with respect to j we deduce that u is a weakly evolutional harmonic map. □

Now, taking the wedge product of (1.19) with u_λ, the nonlinear term of (1.19) vanishes. Because of the divergence structure of this equation and since by (1.21), we have

$$\begin{aligned} \partial_t u_\lambda\wedge u_\lambda &\to \partial_t u\wedge u,\\ \nabla u_\lambda\wedge u_\lambda &\to \nabla u\wedge u \end{aligned}$$

weakly in $L^2_{\mathrm{loc}}([0,\infty)\times M)$, we may pass to the limit $\lambda\to\infty$ suitably and find that also u satisfies (1.23). That is, by Lemma 1.6 and Lemma 1.8, u is a weak solution of (1.10) and (1.11).

Note that this method extended to a homogeneous space by Hélein [**64**] and the regularity of the approximating maps u_λ may be lost in the limit.

The monotonicity formula

Monotonicity first were introduced as a tool in the regularity study for minimal hypersurfaces. After that, Schoen-Uhlenbeck [**97**] and Giaquinta-Giusti [**48**] observed that similar estimates hold for the energy-minimizing harmonic maps and can be used to obtain a partial regularity result for such a minimizer. We review the monotonicity for energy-minimizing harmonic maps $u\colon B{=}B_1(0){\to}N$.

THEOREM 1.9. *If* u *is energy-minimizing* (*for its boundary values*), *then for any* $0{<}\rho{<}r{<}1$ *with* $B_\rho{\subset}B_r \subset\subset B_1(0)$, *there holds*

$$\rho^{2-d}\int_{B_\rho}|\nabla u|^2dx\leq r^{2-d}\int_{B_r}|\nabla u|^2dx.$$

Proof. Note that the quantity

$$\Phi(\rho):=\Phi(\rho;u)=\frac{\rho^{2-d}}{2}\int_{B_\rho}|\nabla u|^2dx$$

is invariant under scaling $u(x)\to u_R(\bar{x})=u(R\bar{x})$, which (in case $M{=}B_1(0)$) also leaves (1.3) invariant. This observation allows us to give a simple proof of Theorem 1.9 for *smooth* harmonic maps as follows: Note that $\Phi(\rho){=}\Phi(\rho;u){=}\Phi(1;u_\rho)$. Hence, for instance at $\rho{=}1$, we have

$$\begin{aligned}\frac{d\Phi(\rho)}{d\rho}&=\frac{d\Phi}{d\rho}(1;u_\rho)=\int_{B_1(0)}\langle\nabla u_\rho,\nabla\left(\frac{\partial u_\rho}{\partial\rho}\right)\rangle dx\\&=\int_{\partial B_1(0)}\langle\langle x,\nabla\rangle u_\rho,\frac{\partial u_\rho}{\partial\rho}\rangle d\omega_{d-1}-\int_{B_1(0)}\langle\triangle u_\rho,\frac{\partial u_\rho}{\partial\rho}\rangle dx.\end{aligned}$$

Since $\partial u_\rho/\partial\rho=\langle x/\rho,\nabla\rangle u_\rho\in T_{u_\rho}N$, by (1.2), the last term vanishes and the boundary integral simply becomes

$$\left.\frac{d\Phi(\rho)}{d\rho}\right|_{\rho=1}=\int_{\partial B_1(0)}|\langle x,\nabla\rangle u_\rho|^2d\omega_{d-1}\geq 0,$$

proving the monotonicity for smooth u. □

Moreover, since $\Phi(\rho)$ scales as a dimension-less quantity, smallness of $\Phi(\rho)$ for some $\rho{>}0$ yields a-priori bounds for u near the origin.

A similar result holds in the time-dependent setting. For simplicity we consider smooth solutions $u\in C^\infty([-1,0)\times\mathbb{R}^d;N)$ of (1.10) with $E(u(t))\leq E_0<\infty$ in $-1\leq t<0$. Denote by

$$G(t,x)=\frac{1}{\left(\sqrt{4\pi|t|}\right)^d}\exp\left(-\frac{|x|^2}{4|t|}\right),\quad t<0,$$

the fundamental solution to the backward heat equation on $\mathbb{R}_- \times \mathbb{R}^d$. Using G as a weight function, we define

$$\Phi(\rho) = \Phi(\rho; u) = \frac{\rho^2}{2} \int_{\{-\rho^2\} \times \mathbb{R}^d} |\nabla u|^2 G dx,$$

We then obtain:

THEOREM 1.10 (Struwe [**109**]). *For u as above and any $0 < \rho < r \leq 1$, the following holds*

$$\Phi(\rho) \leq \Phi(r). \tag{1.25}$$

Proof. As in the stationary case, we use invariance of (1.10) under scaling $u \to u_R(t,x) = u(R^2 t, Rx)$. Note that $\Phi(\rho) = \Phi(\rho; u) = \Phi(1; u_\rho)$. Hence, at $\rho = 1$, we compute

$$\frac{d\Phi(\rho)}{d\rho} = \frac{d\Phi}{d\rho}(1; u_\rho) = \int_{\{-1\} \times \mathbb{R}^d} \langle \nabla u_\rho, \nabla \left(\frac{\partial u_\rho}{\partial \rho} \right) \rangle G d\bar{x}.$$

Integrate by parts; use (1.10) and the relation $\nabla G(\bar{t}, \bar{x}) = \bar{x}/2\bar{t} G$ to obtain

$$\begin{aligned} \frac{d\Phi(\rho)}{d\rho} &= \int_{\{-1\} \times \mathbb{R}^d} \langle \left(-\triangle u_\rho - \frac{\langle \bar{x}, \nabla \rangle u_\rho}{2\bar{t}} \right), \frac{\partial u_\rho}{\partial \rho} \rangle G d\bar{x} \\ &= -\int_{\{-1\} \times \mathbb{R}^d} \langle \frac{2\bar{t} \partial_{\bar{t}} u_\rho + \langle \bar{x}, \nabla \rangle u_\rho}{2\bar{t}}, \frac{\partial u_\rho}{\partial \rho} \rangle G d\bar{x} \\ &= \frac{1}{2} \int_{\{-1\} \times \mathbb{R}^d} |2\bar{t} \partial_{\bar{t}} u_\rho + \langle \bar{x}, \nabla \rangle u_\rho|^2 G d\bar{x} \geq 0. \end{aligned}$$

Since $E(u(t)) \leq E_0 < \infty$, no boundary terms appear. □

REMARK 1.1. (1.25) is the energy inequality for (1.10) in similarity coordinates $s = -\log|t|$ and $y = x/\sqrt{|t|}$, as introduced by Giga-Kohn [**49**] in a different problem.

By a scaling argument as in the proof of Proposition 1.3, smallness of $\Phi(\rho)$ can be turned into an a-priori gradient bound for u.

PROPOSITION 1.4 (Struwe [**109**]). *Let us suppose that a solution u of* (1.10) *belongs to $C^\infty([-1,0) \times \mathbb{R}^d; N)$. There exists $\epsilon_0 = \epsilon_0(d, N) > 0$ such that for some positive number $R > 0$ if $\Phi(R) < \epsilon_0$ then*

$$\sup_{Q_{\delta R}(0)} |\nabla u| \leq \frac{C}{R}$$

holds with constants $\delta = \delta(d, N, E_0) > 0$ and $C = C(d, N, E_0)$.

Proof. Scaling with R, we may assume $R = 1$. For any $\delta > 0$, choose $\rho \in (0, \delta)$ and $z_{\max} = (t_{\max}, x_{\max}) \in Q_\rho$ satisfying

$$(\delta - \rho)^2 \sup_{Q_\rho} e(u) = \max_{0 \leq \sigma \leq \delta} \{ (\delta - \sigma)^2 \sup_{Q_\sigma} e(u) \},$$

$$e(u)(z_{\max})=\sup_{Q_\rho} e(u)=e_0.$$

First assume $e_0^{-1}\leq((\delta-\rho)/2)^2$ and scale $v(\bar t,\bar x)=u(t_{\max}+e_0^{-1}\bar t, x_{\max}+e_0^{-1/2}\bar x)$. Note that $v\in C^\infty(Q_1;N)$ and

$$\begin{aligned}\sup_{Q_1} e(v)(0)&=e_0^{-1}\sup_{Q_{e_0^{-1/2}}(z_{\max})} e(u)\leq e_0^{-1}\sup_{Q_{(\delta+\rho)/2}} e(u)\\ &\leq 4e_0^{-1}\sup_{Q_\rho} e(u)=4,\end{aligned}$$

while

$$e(v)(0)=1.$$

Therefore, by the Bochner inequality (1.13), we have

$$\left(\frac{d}{d\bar t}-\triangle\right)e(v)\leq Ce(v)\quad\text{in}\quad Q_1(0).$$

Lemma 1.3 gives

$$1=e(v)(0)\leq C_{\mathrm H}\int_{Q_1} e(v)\,d\bar t d\bar x=C_{\mathrm H}e_0^{d/2}\int_{Q_{e_0^{-1/2}}(z_{\max})} e(u)\,dtdx.$$

Set $G_{z_{\mathrm s}}(z)=G(z-z_{\mathrm s})$ with $z_{\mathrm s}=(t_{\mathrm s},x_{\mathrm s})$ and choose $z_{\mathrm s}=z_{\max}+(0,e_0^{-1})$. Then

$$1\leq C\int_{Q_{e_0^{-1/2}}(z_{\max})} e(u)G_{z_{\mathrm s}}\,dtdx,$$

and by applying Theorem 1.10 for each $t\in[t_{\max}-e_0^{-1},t_{\max}]$, we proceed to estimate

$$1\leq C\int_{\{-1\}\times\mathbb{R}^d} e(u)G_{z_{\mathrm s}}\,dx.$$

Now $|x_{\mathrm s}|\leq\delta$ and $|t_{\mathrm s}|\leq\delta^2$. Thus at $t=-1$ we can estimate

$$\begin{aligned}|G_{z_{\mathrm s}}-G|&\leq\frac{1}{\sqrt{4\pi}^d}\left(\left|1-\frac{1}{\sqrt{|1+t_{\mathrm s}|}^d}\right|+\left|\exp\left(-\frac{|x|^2}{4}\right)-\exp\left(-\frac{|x-x_{\mathrm s}|^2}{4|1+t_{\mathrm s}|}\right)\right|\right)\\ &\leq C\delta,\end{aligned}$$

and therefore we obtain that

$$1\leq C\delta\int_{\{-1\}\times\mathbb{R}^d} e(u)\,dx+C\int_{\{-1\}\times\mathbb{R}^d} e(u)G\,dx\leq C_1\delta E_0+C_1\Phi(1),$$

with a uniform constant $C_1=C_1(d,N)$. Choosing $\delta=1/(2C_1E_0)$ and $\epsilon_0=1/(2C_1)$, the above inequality will lead to a contradiction. Thus $(\delta-\rho)^2e_0\leq4$: we deduce

$$\sup_{Q_{\delta/2}} e(u)\leq16\delta^{-2}.$$

The proof is complete. $\square$

As an application we establish the following result from Struwe [**109**].

Proposition 1.5. *Suppose that $u_k \in C^\infty([-1,0)\times\mathbb{R}^d;N)$ is a sequence of solutions to* (1.10) *with $E(u_k(t))\le E_0<\infty$ for any t and for all $k\in\mathbb{N}$. Moreover, suppose $u_k(-1)\to u_0$ in $H^{1,2}_{loc}(\mathbb{R}^d;N)$ $(k\to\infty)$ and*

$$\begin{aligned} u_k &\longrightarrow u \quad in\ L^2_{loc}([-1,0]\times\mathbb{R}^d),\\ \partial_t u_k &\longrightarrow \partial_t u \quad in\ L^2_{loc}([-1,0]\times\mathbb{R}^d),\\ \nabla u_k &\longrightarrow \nabla u \quad in\ L^2_{loc}([-1,0]\times\mathbb{R}^d). \end{aligned}$$

Then u weakly solves (1.10) *and* (1.11) *and u is smooth away from a closed set Σ of co-dimension more than or equal to* 2; moreover, for any $R>0$ we have

$$\Phi(R;u)+\int_{-1}^{-R^2}\int_{\mathbb{R}^d}\frac{|2tu_t+\langle x,\nabla\rangle u|^2}{2|t|}G\,dt\,dx\le\Phi(-1;u). \tag{1.26}$$

Proof. Set

$$\Sigma=\bigcap_{R>0}\{z;\liminf_{k\to\infty}\Phi_z(R;u_k)\ge\epsilon_0\},$$

where for any points $z_0=(t_0,x_0)$,

$$\Phi_{z_0}(r;u_k)=\frac{r^2}{2}\int_{\{t_0-r^2\}\times\mathbb{R}^d}|\nabla u_k|^2G_{z_0}\,dx \qquad \text{in} \qquad t_0-r^2\ge-1,$$

Σ is relatively closed. Indeed, if $z_\infty\in\overline{\Sigma}$, there exists a sequence of points $z_l=(t_l,x_l)$ $\in\Sigma$ $(l=1,2,\dots)$ such that $z_l\to z_\infty$. By definition of Σ and the monotonicity in Theorem 1.10, we have

$$\liminf_{l\to\infty}\liminf_{k\to\infty}\left(\frac{R^2}{2}\int_{\{t_l-R^2\}\times\mathbb{R}^d}|\nabla u_k|^2G_{z_l}\,dx\right)\ge\epsilon_0$$

for any $R>0$. Since $G_{z_l}\to G_{z_\infty}$ uniformly on any compact set away from z_∞ and since $E(u_k(t))\le E_0<\infty$, the limits $l\to\infty$ and $k\to\infty$ may be interchanged for any fixed $R>0$, thus we obtain

$$\liminf_{k\to\infty}\Phi^k_{z_\infty}(R)\ge\epsilon_0,$$

for all $R>0$: $z_\infty\in\Sigma$.

Next observe that for $z_0\notin\Sigma$ there is a sequence $\{u_k\}$ $(k=1,2,\dots)$ and some $R>0$ such that

$$\Phi_{z_0}(R;u_k)<\epsilon_0.$$

Proposition 1.4 implies that

$$\sup_{Q_{\delta R}(z_0)}|\nabla u_k|\le\frac{C}{R}$$

hold for a positive constant $\delta=\delta(d,N,E_0)>0$ uniformly in k and similar bounds for higher derivatives. Thus we may pass to the limit $k\to\infty$ in (1.10) and find that u is a smooth solution of (1.10) on any compact set outside Σ. In order to assert that u extends to a weak solution beyond Σ, we need to estimate the d-dimensional Hausdorff measure with respect to the parabolic metric

$$d_{\mathrm{P}}((t,x),(s,y))=|t-s|^{1/2}+|x-y|.$$

For a set $S\subset\mathbb{R}_+\times\mathbb{R}^d$, the latter is defined as

$$\mathcal{H}^d(S;d_P)=c(d)\sup_{R>0}\left\{\inf_{\text{covering}}\left\{\sum_i r_i^d;S\subset\bigcup_i P_{r_i}(z_i),z_i\in S,r_i<R\right\}\right\},$$

where $c(d)$ is a normalizing constant and

$$P_r(z_0)=\{z=(t,x);|t-t_0|<r^2,|x-x_0|<r\}\quad\text{with}\quad z_0=(t_0,x_0).$$

Fix a compact set $P\subset[-1,0)\times\mathbb{R}^d$ and let $S=P\cap\Sigma$. Fix $R>0$ and let $P_{r_i}(z_i)(r_i<R)$ be a cover of S. Since S is compact, we may assume that the cover is finite. Moreover, a simple variant of Vitali's covering lemma shows that there is a disjoint sub-family $P_{r_i}(z_i)$ $(i\in\mathcal{J})$ such that $S\subset\cup_{i\in\mathcal{J}}P_{5r_i}(z_i)$. Let $\bar{z}_i=z_i+(0,r_i^2)$ $(i\in\mathcal{J})$. Since $\mathcal{J}$ is finite, there exists $k\in\mathbb{N}$ such that

$$\begin{aligned}
\epsilon_0\le\Phi_{z_i}(\theta_0r_i;u_k)&\le C\int_{t_i-4\theta_0^2r_i^2}^{t_i-\theta_0^2r_i^2}\int_{\mathbb{R}^d}|\nabla u_k|^2G_{z_i}\,dtdx\\
&\le C(\theta_0)r_i^{-d}\int_{Q_{r_i}(z_i)}|\nabla u_k|^2\,dtdx\\
&+C\theta_0^{-d}\exp\left(-\frac{1}{16(1+4\theta_0)\theta_0}\right)\int_{t_i-4\theta_0^2r_i^2}^{t_i-\theta_0^2r_i^2}\int_{\mathbb{R}^d\setminus B_{r_i}}|\nabla u_k|^2G_{\bar{z}_i}\,dtdx\\
&\le C(\theta_0)r_i^{-d}\int_{Q_{r_i}(z_i)}|\nabla u_k|^2\,dtdx+C\theta_0^{2-d}\exp\left(-\frac{1}{16(1+4\theta_0)\theta_0}\right)E_0
\end{aligned}$$

for all $i\in\mathcal{J}$ where we used the fact that

$$G_{z_i}\le\theta_0^{-d}\exp\left(-\frac{1}{16(1+4\theta_0)\theta_0}\right)G_{\bar{z}_i}\qquad\text{on }[t_i-4\theta_0^2r_i^2,t_i-\theta_0^2r_i^2]\times\mathbb{R}^d\setminus Q_{r_i}(z_i)$$

and Theorem 1.10 to derive the 1st inequality. Take $\theta_0>0$ satisfying

$$C\theta_0^{2-d}\exp\left(-\frac{1}{16(1+4\theta_0)\theta_0}\right)E_0<\frac{\epsilon_0}{2}$$

to verify

$$r_i^d\le C\int_{P_{r_i}(z_i)}|\nabla u_k|^2\,dtdx.$$

Summing over $i\in\mathcal{J}$, we obtain

$$\sum_{i\in\mathcal{J}}r_i^d\le C\sum_{i\in\mathcal{J}}\int_{P_{r_i}(z_i)}|\nabla u_k|^2\,dtdx$$

$$=C\int_{\cup_{i\in\mathcal{J}}P_{r_i}(z_i)}|\nabla u_k|^2\,dt\,dx\le C(\mathcal{J})E_0$$

with constants C independent of $R>0$. Thus the d-dimensional Hausdorff measure of Σ is locally finite.

In particular, for a suitable cover $(Q_{r_i}(z_i))_{i\in\mathcal{J}}$ of $S(r_i<R)$, we can achieve that

$$\mathcal{L}^{2+d}\left(\bigcup_{i\in\mathcal{J}}P_{r_i}(z_i)\right)\to 0 \qquad \text{as } R\searrow 0,$$

where $\mathcal{L}^{2+d}$ denotes the Lebesgue measure on (t,x) in $\mathbb{R}_+\times\mathbb{R}^d$ with respect to the parabolic metric. Now let $\phi\in\mathcal{D}(P_2(0))$ satisfy $0\le\phi\le 1,\phi\equiv 1$ on $P_1(0)$ and scale $\phi_i(z)=\phi((t-t_i)/r_i^2,(x-x_i)/r_i)\in\mathcal{D}(P_{2r_i}(z_i))$. Given $\psi\in\mathcal{D}(P_1;\mathbb{R}^D)$, then $\tau=\psi\inf_i(1-\phi_i)$ is a Lipschitz function and $\tau(z)\to\psi(z)$ a.e. as $R\searrow 0$. Multiplying (1.10) by τ, we obtain

$$\begin{aligned}
&\int_{-1}^{0}\int_{\mathbb{R}^d}(u_t-\triangle u-A(u)(\nabla u,\nabla u))\psi\,dt\,dx\\
&\le C\int_{-1}^{0}\int_{\mathbb{R}^d}|\nabla u||\nabla\inf_i(1-\phi_i)|\,dt\,dx+o(1)\\
&\le C\|\nabla u\|_{L^2\left(\bigcup_{i\in\mathcal{J}}P_{r_i}(z_i)\right)}\left(\int_{-1}^{0}\int_{\mathbb{R}^d}|\nabla\inf_i(1-\phi_i)|^2\,dt\,dx\right)^{1/2}+o(1)\\
&\le o(1)\left(\sum_i\int_{P_{r_i}(z_i)}r_i^{-2}\,dt\,dx\right)^{1/2}+o(1)\\
&=o(1)\left(\sum_i r_i^d\right)^{1/2}+o(1)\to 0 \qquad (R\searrow 0),
\end{aligned}$$

where $o(1)\to 0$ as $R\searrow 0$ and u weakly solves (1.10). Finally (1.26) follows from (1.25) and the fact that $\Phi(R;u)\le\liminf_{k\to\infty}\Phi(R;u_k)$, $\Phi(1;u)=\Phi(1;u_0)=\lim_{k\to\infty}\Phi(1;u_k)$. $\square$

The proof of Theorem 1.8 uses the fact that results similar to Theorem 1.10 and Proposition 1.4 hold for solutions u to (1.10) on a compact manifold M, where Φ is defined with reference to a local coordinate chart V and where we truncate the integrand with a smooth cut-off function $\tau\in\mathcal{D}(V)$. Set $\lambda\in\mathbb{R}_+$. If U_δ is 3δ-tubular neighborhood on N in $\mathbb{R}^{\hat{D}}$ and if for maps $u\colon M\to\mathbb{R}^{\hat{D}}$, E_λ is

$$E_\lambda(u):=E(u)+\lambda\int_M\chi(\operatorname{dist}^2(u,N))\,dx,$$

where $\chi(s)=s$ for $s\le\delta$, $\chi'(s)\ge 0,\chi(s)\equiv 2\delta$ for $s\ge 3\delta$, then the sequence of approximate solutions $\{u_\lambda\}$ $(\lambda>0)$ to (1.10) defined by the gradient flow of E_λ again satisfies an analogue of Theorem 1.10 and Proposition 1.4. Similar to Proposition

1.5, we then establish that a sub-sequence $\{u_\lambda\}$ $(\lambda>0)$ converges weakly to a partially regular weak solution u of (1.10) and (1.11). Moreover, inequality (1.25) holds. See Chen-Struwe [**24**] for details.

Let us now turn to some further consequences of the monotonicity formula.

Nonuniqueness

Coron [**27**] observed that for certain weakly harmonic maps $u_0{:}B^3\to S^2$, the stationary weak solution $u(t,x)=u_0(x)$ of (1.10) does not satisfy (1.26), hence must be different from the solution constructed in Theorem 1.8.

We repeat his construction: Suppose $u_0\in H^{1,2}_{\mathrm{loc}}(\mathbb{R}^3;S^2)$ is weakly harmonic, $u_0(x)=u_0(x/|x|)$ and consider $u(t,x)=u_0(x)$. Then u weakly solves (1.10) and

$$\Phi_{\bar{z}}(\rho)=\frac{1}{2\sqrt{4\pi}^3\rho}\int_{\mathbb{R}^3}|\nabla u_0|^2\exp\left(-\frac{|x-\bar{x}|^2}{4\rho^2}\right)dx<\infty$$

for any $\bar{z}=(\bar{t},\bar{x})\in\mathbb{R}_+\times\mathbb{R}^3$ and any $\rho>0$. Suppose that u satisfies (1.25). This implies

$$\frac{1}{\rho}\int_{\mathbb{R}^3}|\nabla u_0|^2\exp\left(-\frac{|x-\bar{x}|^2}{4\rho^2}\right)dx\leq\frac{1}{r}\int_{\mathbb{R}^3}|\nabla u_0|^2\exp\left(-\frac{|x-\bar{x}|^2}{4r^2}\right)dx \tag{1.27}$$

for any $0<\rho<r<\infty$. We show that (1.27) does not hold for a suitable map u_0. This ill-behaved map u_0 is obtained as follows. Let $\pi{:}\ S^2\setminus\{(0,0,1)\}\to\mathbb{R}^2\cong\mathbb{C}$ be the stereographic projection from the north pole $(0,0,1)$ of S^2 and let $g{:}\mathbb{C}\to\mathbb{C}$ be a rational map. Composing the weakly harmonic map $u{:}x\to x/|x|$ from Example 1.2 with π and g we obtain a map

$$u_0(x)=\pi^{-1}\left(g\left(\pi\left(\frac{x}{|x|}\right)\right)\right).$$

Regarding $u_0(x)=u_0(x/|x|)$ as a map $u_0{:}S^2\to S^2$, by conformal invariance u_0 is harmonic; hence $u_0{:}\mathbb{R}^3\to S^2$ is weakly harmonic. By suitable choice of g (for instance, $g(z)=\lambda z$ with $\lambda\in\mathbb{R}$ and $\lambda>1$), we can achieve that the center of mass

$$q=\int_{S^2}|\nabla u_0(\frac{x}{|x|})|^2\frac{x}{|x|}dvol_{S^2}\neq 0.$$

(Hence the map u_0 is not minimizing for its boundary values on $B^3_1(0)$; see Brezis-Coron-Lieb [**11**, Remark 7.6].)

Denote

$$\phi(\rho,\bar{x})=\frac{1}{\rho}\int_{\mathbb{R}^3}|\nabla u_0|^2\exp\left(-\frac{|x-\bar{x}|^2}{4\rho^2}\right)dx$$

for simplicity. Note that

$$\phi(\rho,0)=\int_0^\infty\left(\int_{S^2}|\nabla u_0(\frac{x}{|x|})|^2dvol_{S^2}\right)\exp\left(-\frac{|x|^2}{4\rho^2}\right)\frac{d|x|}{\rho}(:=a_0)$$

is independent of $\rho>0$. Moreover, compute

$$\nabla_{\bar{x}}\phi(\rho,0)=\int_{\mathbb{R}^3}|\nabla_{S^2}u_0(\frac{x}{|x|})|^2\frac{x}{2\rho^3}\exp\left(-\frac{|x|^2}{4\rho^2}\right)dx$$
$$=\int_0^\infty\left(\int_{S^2}|\nabla_{S^2}u_0(\frac{x}{|x|})|^2\frac{x}{|x|}dvol_{S^2}\right)\cdot\frac{\exp\left(-\frac{|x|^2}{4\rho^2}\right)}{2\rho^3}|x|d|x|$$
$$=q\int_0^\infty\frac{\exp(-\sigma)d\sigma}{\rho}=\frac{q}{\rho}.$$

Hence for $\bar{x}=tq$ and $0<\rho<r$, if $t>0$ is sufficiently small, we then obtain

$$\phi(\rho,\bar{x})=a_0+t\frac{|q|^2}{\rho}+O(t^2)>\phi(r,\bar{x})=a_0+t\frac{|q|^2}{r}+O(t^2),$$

contradicting (1.27). On the other hand, as in Theorem 1.8 we can construct weak solutions $\tilde{u}$ to (1.10) for initial data u_0 satisfying (1.25): This reads that $u\neq\tilde{u}$ and therefore we show nonuniqueness in the energy class of weak solutions to (1.10) and (1.11).

Note that we look at spontaneous symmetry breaking, since u cannot be of the form $u(t,x)=v(t,x/|x|)$. The latter map v would solve (1.10) and (1.11) on $[0,\infty)\times S^2$. Since $u_0{:}S^2\to S^2$ is smooth and harmonic, by local unique solvability of (1.10) and (1.11) on $[0,\infty)\times S^2$ for smooth data, this would imply $v(t)\equiv u_0$.

It is remained to discuss a class of functions satisfying (1.10) and (1.11) which possesses a unique solution. Struwe, M suggests that the class of solutions satisfying the strong monotonicity formula

$$\Phi_{\bar{z}}(\rho)\leq\Phi_{\bar{z}}(r)$$

for all $\bar{z}$ and all $0<\rho<r\leq\sqrt{t}$ is a likely candidate.

Development of singularities

The most surprising aspect of the monotonicity formula is that it may be used to prove that (1.10) and (1.11) in general will develop singularities in arbitrarily short time. The existence of singularities was first established by Coron-Ghidaglia [**28**]; see also Grayson-Hamilton [**50**]. These results were based on comparison principles for the reduced harmonic map evolution problem (1.16) in the equivariant setting. A deeper reason for the formation of singularities was worked out by Chen-Ding [**21**]. This is related to a result by White [**120**].

THEOREM 1.11. *Let M and N be compact Riemannian manifolds and consider a smooth map $u_0\colon M\to N$. Then*

$$\inf\{E(u);u\in C^\infty(M;N),\quad u\text{ is homotopic to }u_0\}>0$$

if and only if the restriction of u_0 to a 2-skeleton of M is not homotopic to a constant.

REMARK 1.2. In particular, there are examples of non-trivial homotopy classes of maps $u_0\colon M\to N$ such that

$$\inf\{E(u); u \text{ is homotopic to } u_0\}=0.$$

EXAMPLE 1.6. Let $u_1=\mathrm{id}\colon S^3\to S^3$. Let $\pi\colon S^3\setminus\{(0,0,0,1)\}\to\mathbb{R}^3$ be the stereographic projection and let $D_\lambda\colon \mathbb{R}^3\to\mathbb{R}^3, D_\lambda(x)=\lambda x$ be dilation with $\lambda>0$. Then define

$$u_\lambda=\pi^{-1}\circ D_\lambda\circ\pi\colon S^3\to S^3.$$

Clearly, $u_\lambda\sim u_1=id$ for all $\lambda>0$ and $E(u_\lambda)\to 0 \quad (\lambda\to\infty)$.

Singularities of first and second kind

Let $u\in C^\infty([-1,0)\times\mathbb{R}^d;N)$ be a solution to (1.10) with an isolated singularity at the origin and satisfying (1.25). If

$$|\nabla u(t,x)|^2\leq\frac{C}{|t|}, \tag{1.28}$$

the rescaled sequence

$$u_R(\bar t,\bar x)=u(R^2\bar t,R\bar x),\quad R>0$$

satisfies the same estimate and hence a sub-sequence converges smoothly locally on $(-\infty,0)\times\mathbb{R}^d$ to a smooth limit $\bar u$ as $R\searrow 0$. $\bar u$ satisfies (1.10). Moreover, $\bar u\not\equiv$const; otherwise $\Phi(R;u)=\Phi(1,u_R)<\epsilon_0$ for some $R>0$ and u is regular at 0. Since by (1.26) there holds

$$\begin{aligned}\int_{-TR^2}^{-\tau R^2}\int_{\mathbb{R}^d}\frac{|2t\partial_t u+\langle x,\nabla\rangle u|^2}{2|t|}G\,dt\,dx&\leq\lim_{R\searrow 0}\int_{-T}^{-\tau}\int_{\mathbb{R}^d}\frac{|2\bar t\partial_{\bar t}u_R+\langle\bar x,\nabla\rangle u_R|^2}{2|\bar t|}G\,d\bar t\,d\bar x\\&=\int_{-T}^{-\tau}\int_{\mathbb{R}^d}\frac{|2\bar t\partial_{\bar t}\bar u+\langle\bar x,\nabla\rangle\bar u|^2}{2|\bar t|}G\,d\bar t\,d\bar x=0\end{aligned}$$

for any $0<\tau<T<\infty$, it follows that $\bar u$ satisfies

$$2\bar t\bar u_{\bar t}+\bar x\nabla\bar u\equiv 0;$$

that is possibly

$$\bar u(\bar t,\bar x)=v\left(\frac{\bar x}{\sqrt{|\bar t|}}\right)$$

or

$$\bar u(\bar t,\bar x)=w\left(\frac{\bar x}{|\bar x|}\right).$$

Struwe, M calls the singular point satisfying (1.28) "singular point of the first kind." It is not known whether self-similar solutions $\bar u(t,x)=v\left(x/\sqrt{|t|}\right)$ actually may exist. All other singular points are said to be "singularities of second kind." Since singularities in $d=2$ by Theorem 1.7 are related to time-independent harmonic maps $\bar u\colon S^2\to N$ of finite energy and since a non-constant, radially homogeneous map $\bar{u}(x)=w(x/|x|)$ in $d=2$ has infinite energy, Theorem 1.6 of Chang-Ding-Ye [**18**] above shows that singularities of second kind exist for the evolution problem (1.10).

Extensions and generalizations

Further current developments include the evolution problem to harmonic maps on general complete, non-compact manifolds (Li-Tam [**78**]) and to harmonic maps with symmetry (see Grotowsky [**51**], [**52**], [**53**] and [**54**]).

6. Notation

In this section, we collect abridged notation and function spaces used in the following chapter. Let T be a positive number or $+\infty$. Set Ω be a bounded domain in $\mathbb{R}^d$ with a smooth boundary.

Abridged Notation

(i) $B_1^d(0) = \{x=(x_1,x_2,\dots,x_d);\ |x|=\sqrt{\sum_{\alpha=1}^d (x_\alpha)^2}<1\}$, we abbreviate $B_1^d(0)$ to B_1 or B when no ambiguity may occur.

(ii) Q_T is $(0,T)\times\Omega$ and ∂Q_T (the boundary of Q_T) means $\{0\}\times\Omega\cup[0,T)\times\partial\Omega$.

(iii) The parabolic metric function $d(z,z')$ is given by

$$d_{\mathrm{P}}(z,z')=|t-t'|^{1/2}+|x-x'|$$

whenever $z=(t,x)$ and $z'=(t',x')\in Q_T$, and we set

$$\begin{aligned} d_{\mathrm{P}}(z,\partial Q_T) &= \inf_{z'\in\partial Q_T} d_{\mathrm{P}}(z,z'),\\ \operatorname{diam}(\Omega) &= \sup_{x,y\in\Omega}|x-y|. \end{aligned}$$

(iv) ν_x denotes the outward normal unit vector to Ω at $x\in\partial\Omega$. When no confusion may arise, we shorten ν_x to ν.

(v) For a point $x_0=(x_{0,\alpha})$ $(\alpha=1,\dots,d)\in\mathbb{R}^d$ or $z_0=(t_0,x_0)\in\mathbb{R}^{1+d}$ and positive numbers ρ,τ, we set

$$\begin{cases} B_\rho(x_0) & =\{x\in\mathbb{R}^d : |x-x_0|<\rho\},\\ C_\rho(x_0) & =\{x\in\mathbb{R}^d : |x_\alpha-x_{0,\alpha}|<\rho\,(\alpha=1,\dots,d)\},\\ P_{\rho,\tau}(z_0) & =\{(t,x)\in\mathbb{R}^{1+d} : t_0-\tau^2<t\le t_0+\tau^2, |x-x_0|<\rho\},\\ Q_{\rho,\tau}(z_0) & =\{(t,x)\in\mathbb{R}^{1+d} : t_0-\tau^2<t\le t_0, |x-x_0|<\rho\},\\ D_{\rho,\tau}(z_0) & =\{(t,x)\in\mathbb{R}^{1+d} : t_0-\tau^2<t\le t_0, |x_\alpha-x_{0,\alpha}|<\rho\\ & \quad (\alpha=1,\dots,d)\},\\ P_\rho(z_0) & =P_{\rho,\rho^2}(z_0),\quad Q_\rho(z_0)=Q_{\rho,\rho^2}(z_0)\ \text{and}\ D_\rho(z_0)=D_{\rho,\rho^2}(z_0). \end{cases}$$

We call $B_\rho(x_0)$, $C_\rho(x_0)$, $P_\rho(z_0)$, $Q_\rho(z_0)$ and $D_\rho(z_0)$, a ball, a cube, a full cylinder, a cylinder and a semi-cube and moreover when no confusion may arise, we respectively abbreviate $B_\rho(x_0)$, $C_\rho(x_0)$, $P_\rho(z_0)$, $Q_\rho(z_0)$ and $D_\rho(z_0)$ to B,C,P,Q and D.

(vi) For vectors $x=(x_\alpha)$, $y=(y_\alpha)\in\mathbb{R}^d$, $u=(u^i)$, $v=(v^i)\in\mathbb{R}^D$ and any matrices $A=(A^i_\alpha)$, $B=(B^i_\alpha)\in\mathbb{R}^{d\times D}$ $(\alpha=1,\dots,d;i=1,\dots,D)$, $\langle\cdot,\cdot\rangle$ respectively designates the inner product given by $\langle x,y\rangle=\sum_{\alpha=1}^d x_\alpha y_\alpha$, $\langle u,v\rangle=\sum_{i=1}^D u^i v^i$ and $\langle A,B\rangle=\sum_{\alpha=1}^d\sum_{i=1}^D A^i_\alpha B^i_\alpha$. The respective norm is also given by $|u|=\sqrt{\sum_{i=1}^D (u^i)^2}$ and $|A|=\sqrt{\sum_{\alpha=1}^d\sum_{i=1}^D (A^i_\alpha)^2}$. Moreover, for any vector $x=(x_\alpha)$ $(\alpha=1,2,\dots,d)$, we use the symbol $\langle x,\nabla\rangle$ as $\sum_{\alpha=1}^d x_\alpha\partial/\partial x_\alpha$.

(vii) For a map $v:\mathbb{R}^d\to\mathbb{R}^D$, we denote ∇v and $\nabla^2 v$ by

$$\nabla u=\left(\frac{\partial u^i}{\partial x_\alpha}\right)\qquad(\alpha=1,2,\dots,d;i=1,2,\dots,D);$$

$$\nabla^2 u=\left(\frac{\partial^2 u^i}{\partial x_\alpha\partial x_\beta}\right)\qquad(\alpha,\beta=1,2,\dots,d;i=1,2,\dots,D).$$

In addition, a normal derivative $\partial u/\partial\nu$ is given by $\langle\nu,\nabla\rangle u$, while a tangential derivative $\nabla_{\tan}$ is by $\nabla u-\nu\langle\nu,\nabla\rangle u$.

(viii) Let κ be a positive number. For any function u on Q_T, we write the truncation function of u by

$$u^{(\kappa)}=\max(u-\kappa,0).$$

(ix) $A^{(\kappa)}_R(z_0)$ is the set of $Q_R(z_0)$ at which $e_\lambda(u_\lambda)>\kappa$.

(x) $\mathcal{L}^{d_i}(A)$ expresses the Lebesgue measure of a measurable set A in $\mathbb{R}^{d_i}$ with respect to the canonical metric in $d_i=d-1,d$ and with respect to the parabolic metric in $d_i=d+1$.

(xi) $\overline{A}$ is the closure of A where A is a set in $\mathbb{R}^{d_i}$, $(d_i=d,d+1)$.

(xii) The letters C, $C_{\text{Alphabets}}$ and $C^{\text{Number}}_{\text{Alphabets}}$ denote generic constants.

Function spaces

(i) If $1\le p<\infty$, $L^p(\Omega)$ and $L^p(Q_T)$ is respectively the Banach space consisting of all pth summable functions on Ω and Q_T with the norm of $||f||_{L^p(\Omega)}=\left(\int_\Omega|f(x)|^p dx\right)^{1/p}$ and $||f||_{L^p(Q_T)}=\left(\int_{Q_T}|f(z)|^p dz\right)^{1/p}$.

(ii) $L^\infty(\Omega)$ is the space of essentially bounded functions on Ω with the norm of $||f||_{L^\infty(\Omega)}=\operatorname*{ess\cdot sup}_{x\in\Omega}|f(x)|$.

(iii) $C^\infty[0,T)$, $C^\infty(\Omega)$ and $C^\infty(Q_T)$ respectively means the space of infinite differentiable functions on $[0,T)$, Ω and Q_T.

(iv) $\mathcal{D}(\Omega)$ and $\mathcal{D}(Q_T)$ respectively denotes the space of infinite differentiable functions with a compact support on Ω and Q_T and $\mathcal{D}'(Q_T)$ does the dual of $\mathcal{D}(Q_T)$.

(v) $H^{1,2}(\Omega)=\{f\in L^2(\Omega)|\ \partial f/\partial x_\alpha\in L^2(\Omega),\ (\alpha=1,\dots,d)\}$.

(vi) $\overset{\circ}{H}{}^{1,2}(\Omega) \;=\; H^{1,2}(\Omega) \cap \{f\!=\!0\,\mathrm{on}\,\partial\Omega \text{ in the trace sense}\}$.

In the following, $X(\Omega)$ denotes a Banach space on Ω endowed with the norm $||\cdot||_{X(\Omega)}$.

(v) $X_{\mathrm{loc}}(\Omega) \;=\; \{f\!\in\! X(K) \text{ for any open set } \overline{K}\!\subset\subset\!\Omega\}$.

(vi) $X(\Omega;\mathbb{R}^D) \;=\; \{u\!=\!(u^i)\;(i\!=\!1,\ldots,D) \mid u^i\!\in\! X(\Omega)\}$.

(vii) $X(\Omega;S^{D-1}) = \{u\!=\!(u^i)(i\!=\!1,\ldots,D)\!\in\! X(\Omega;\mathbb{R}^D) \mid |u|\!=\!1\,\mathrm{for\,a.e}\,x\!\in\!\Omega\}$.

(viii) $L^\infty(Q_T;\mathbb{R}^D) = \{f|f$ is measurable from $Q_T\!\to\!\mathbb{R}^D$ with the norm of $||f||_{L^\infty(Q_T)} = \underset{z\in Q_T}{\mathrm{ess\cdot sup}}|f|\}$.

(ix) $C^0(0,T;H^{1,2}(\Omega;\mathbb{R}^D)) \;=\; \{f|f$ is continuous from $[0,T)\!\to\! H^{1,2}(\Omega;\mathbb{R}^D)\}$.

(x) $H^{1,2}(0,T;X(\Omega;\mathbb{R}^D))$
$= \{f|f$ and $\partial f/\partial t$ are measurable of $[0,T)\!\to\! X(\Omega;\mathbb{R}^D)$ satisfying
$\left(\int_0^T ||f(t)||^2_{X(\Omega)}dt\right)^{1/2} + \left(\int_0^T \;||\partial f/\partial t(t)||^2_{X(\Omega)}dt\right)^{1/2} <\infty\}$.

(xi) $H^{1,2}(Q_T;\mathbb{R}^D) = H^{1,2}(0,T;L^2(\Omega;\mathbb{R}^D)) \cap L^2(0,T;H^{1,2}(\Omega;\mathbb{R}^D))$.

(xii) $\overset{\circ}{H}{}^{1,2}(Q_T;\mathbb{R}^D) = H^{1,2}(0,T;L^2(\Omega;\mathbb{R}^D)) \cap L^2(0,T;\overset{\circ}{H}{}^{1,2}(\Omega;\mathbb{R}^D))$.

(xiii) $V(Q_T;S^{D-1}) = H^{1,2}(0,T;L^2(\Omega;S^{D-1})) \cap L^\infty(0,T;H^{1,2}(\Omega;S^{D-1}))$.

CHAPTER 2

Modified strong evolutional harmonic maps

1. Introduction and theorem

In chapter 1, we have reviewed various results on harmonic maps and evolutional harmonic maps. In this chapter, to discuss the well-behaved solutions of the weakly evolutional harmonic maps from $(0,+\infty)\times B_1^3(0)\to S^{D-1}$, we propose a certain function class and discuss an evolutional harmonic map belonging to this function class, which will be called "modified strong evolutional harmonic maps." The similar class of solutions of evolutional harmonic maps can be seen in Feldman [**43**] and Chen-Li-Lin [**22**]. We begin with formulating "modified strong evolutional harmonic maps" from $(0,+\infty)\times B_1^3(0)$ to S^{D-1}. Here $B_1^3(0)$ is the open unit ball in $\mathbb{R}^3$ and S^{D-1} denotes the $(D-1)$-dimensional unit sphere in $\mathbb{R}^D$ where D is a positive integer with $D\geq 2$. In the following we abbreviate $B_1^3(0)$ to B_1. Consider the Sobolev space $H^{1,2}(B_1;S^{D-1}) := \{u\in H^{1,2}(B_1;\mathbb{R}^D)\ ;|u|=1 \ \text{for a.e.}\ x\in B_1\}$. When we set $Q_\infty=[0,+\infty)\times B_1$ and fix $u_0 \in H^{1,2}(B_1;S^{D-1})$ being any given maps, "weakly evolutional harmonic maps u" are to satisfy the following:

$$u:Q_\infty\to S^{D-1}, \tag{2.1}$$

$$\frac{\partial u}{\partial t}=\triangle u+|\nabla u|^2u \qquad \text{in} \quad Q_\infty, \tag{2.2}$$

$$u(0,x)=u_0(x) \qquad \text{in} \quad \{0\}\times B_1, \tag{2.3}$$

$$u(t,x)=u_0(x) \qquad \text{in} \quad [0,T)\times\partial B_1. \tag{2.4}$$

This parabolic system strictly holds in the following weak sense:

$$u\in V(Q_\infty;S^{D-1}), \tag{2.5}$$

$$\int_Q [\langle\frac{\partial u}{\partial t},\phi\rangle+\langle\nabla u,\nabla\phi\rangle-\langle u,\phi\rangle|\nabla u|^2]dz=0$$
$$\text{for} \quad \text{any} \quad \phi\in\mathcal{D}(Q_\infty;\mathbb{R}^D), \tag{2.6}$$

$$u(t,x)-u_0(x)\in \overset{\circ}{H}{}^{1,2}(B_1;\mathbb{R}^D) \quad \text{for almost every}\ t\in(0,+\infty), \tag{2.7}$$

$$\lim_{t\to+0} u(t,\cdot)=u_0(\cdot) \quad \text{in} \quad L^2(B_1;\mathbb{R}^D), \tag{2.8}$$

where the definition of various function spaces V,$\mathcal{D}$, $\overset{\circ}{H}{}^{1,2}$ and L^2 will be given in p.42.

We call a map u with (2.5),(2.6),(2.7) and (2.8) weakly evolutional harmonic maps; as we saw in the previous chapter, weakly evolutional harmonic maps are still not unique. To distinguish a well-behaved solution among many weakly evolutional harmonic maps, we give a new notion of solutions to the problem (2.1),(2.2), (2.3) and (2.4); we name such a solution as "modified strong evolutional harmonic map." The exact definition of modified strong evolutional harmonic map from Q_∞ to S^{D-1} is as follows: For any unit vector $\mathbf{e}\in\mathbb{R}^{\mathbf{3}}$, and for any compact set $Q\subset\subset Q_\infty$, if weakly evolutional harmonic maps satisfy

$$\lim_{h\searrow 0}\frac{1}{h}\int_Q |\langle x+h\mathbf{e},\nabla\rangle u(t,x+h\mathbf{e})-\langle x,\nabla\rangle u(z)|^2\,dz=0, \tag{2.9}$$

$$\lim_{h\searrow 0}\frac{1}{h}\int_Q |\nabla u(t+h,x)-\nabla u(z)|^2\,dz=0, \tag{2.10}$$

then this heat flows u i.e. map u satisfying (2.5), (2.6), (2.7), (2.8), (2.9) and (2.10) is said to be "modified strong evolutional harmonic map."

In this chapter, we discuss a partial regularity result on the modified strong evolutional harmonic maps. The further imposition, i.e. (2.9) and (2.10) on modified strong evolutional harmonic map u, is required to show that u satisfies a fundamental energy estimate and a monotonicity for the scaled energy on the whole domain.

We must recall that the monotonicity for the scaled energy is a one of the key ingredients to investigate the structure of the singular set of solutions to various elliptic systems or variational problems.

We close this section by introducing a partial regularity result, which is our main theorem of this chapter. The proof of theorem is directly derived from the following two theorems: Former theorem proves a Hausdorff dimensional estimate for a singular set to modified strong evolutional harmonic map. This sort of theorem is founded in Caffarelli-Kohn-Nirenberg [**14**] and Giaquinta-Giusti [**47**]. We state this theorem without proof. On the other hand, the latter crucially is modeled on Evans [**40**]:

THEOREM 2.1. *Let u be the modified strong heat flows. When we set*

$$\Sigma_u:=\left\{z_0=(t_0,x_0)\in Q_\infty;\liminf_{r\searrow 0} r^{-3}\int_{Q_r(z_0)}|\nabla u(z)|^2 dz>0\right\},$$

then Σ_u is a relatively closed set having a property of $\mathcal{H}^3(\Sigma_u)=0$ with respect to the parabolic metric.

THEOREM 2.2. *There exist constants $0<\epsilon_0$, $\tau_0<1$ such that for any $Q_r(z_0)\subset\subset Q$*

$$
\begin{aligned}
&\frac{r^{-3}}{2}\int_{Q_r(z_0)}|\nabla u(z)|^2dz<\epsilon_0 \quad \text{implies}\\
&\frac{(\tau_0 r)^{-3}}{2}\int_{Q_{\tau_0 r}(z_0)}|\nabla u(z)|^2dz\le\frac{1}{2}\frac{r^{-3}}{2}\int_{Q_r(z_0)}|\nabla u(z)|^2dz.
\end{aligned}
\tag{2.11}
$$

Now we can state our main theorem in this chapter:

THEOREM 2.3 (Main theorem of this chapter). *Modified strong evolutional harmonic maps are Hölder continuous on an open set in Q_∞ whose compliment has zero 3-dimensional Hausdorff measure with respect to the parabolic metric.*

Preparation

We introduce a glossary of notation, function spaces and various results from functional analysis used only in this chapter.

Abridged Notation. Suppose X be a Banach space. Let $\{v_k\}$ $(k=1,2,3,\ldots)$ be a sequence of functions in X and v a function in X.

$$
\begin{aligned}
v_k &\overset{k\to\infty}{\rightharpoonup} v \quad \text{in} \quad X,\\
v_k &\overset{k\to\infty}{\longrightarrow} v \quad \text{in} \quad X,
\end{aligned}
$$

respectively means that $\{v_k\}$ $(k=1,2,3,\ldots)$ weakly and strongly converges to v in X.

Let h be a positive number sufficiently small. For a map $u:Q_\infty\to\mathbb{R}^D$, various forward and backward difference operators are given by

$$
\begin{aligned}
&\frac{\partial^+ u}{\partial t}(z)=\frac{1}{h}(u(t+h,x)-u(t,x)),\\
&\nabla^- u(z)=\begin{cases} 1/h \quad (u(t,x)-u(t,|x|-h,x/|x|)) & h\le|x|,\\ 1/h \quad (u(t,x)-u(t,h-|x|,x/|x|)) & |x|<h,\end{cases}
\end{aligned}
$$

with $x=(|x|,\dfrac{x}{|x|})$.

For a map $f:Q_\infty\to\mathbb{R}^D$, we use the following notation:

$$
\fint_A f(z)dz=\frac{1}{\mathcal{L}^{3+1}(A)}\int_A f(z)dz, \quad \fint_B f(t,y)dy=\frac{1}{\mathcal{L}^3(B)}\int_B f(t,y)dy,
$$

where A and B is respectively a measurable set on $\mathbb{R}^{1+3}$ and $\mathbb{R}^3$.

Preliminaries Lemmas. We review a few technical lemmas which will play a crucial role on the proof of our result. First we state the well-known decomposition lemma on L^2- maps (See O.A.Ladyženskaja [**74**].).

LEMMA 2.1 (Weyl Decomposition Lemma). *Let* $\overset{\circ}{\boldsymbol{J}}(Q_1;\mathbb{R}^3)$ *be a closure of* $\{v\in \mathcal{D}(Q_1);\ \mathrm{div} v=0\}$ *in* $L^2(Q_1;\mathbb{R}^3)$*. Then for a map* $f\in L^2(Q_1;\mathbb{R}^3)$*, there exists a map* $g\in\overset{\circ}{\boldsymbol{J}}(Q_1;\mathbb{R}^3)$ *and a function* $h\in L^2(0,1;H^{1,2}(B_1))$*, such that* f *is uniquely decomposed to* $f=g+\nabla h$*.*

To state the second lemma, we first introduce the definition and a few properties on B.M.O and the Hardy space $\mathcal{H}^1$. These function spaces are used to control the nonlinear term of our parabolic systems. Let f be a function belonging to $L^1_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3)$. Set

$$[f]_{\mathrm{B.M.O}}:=\sup_{x\in\mathbb{R}^3,r\in\mathbb{R}_+}⨍_{B_r(x)}|f-f_{x,r}|dy,$$

$$\text{with}\, f_{x,r}=⨍_{B_r(x)}f(y)dy.$$

Then we say that f has bounded mean oscillation (B.M.O) provided $[f]_{\mathrm{B.M.O}}<\infty$. Also ϕ is any smooth function with a support in the unit ball and $\int_{\mathbb{R}^3}\phi(x)dx=1$. Say that f belongs to the Hardy space $\mathcal{H}^1$ if $f^*\in L^1(\mathbb{R}^3)$ where f^* is defined by

$$f^*(x):=\sup_{r>0}\left|⨍_{B_r(x)}f(y)\phi\left(\frac{x-y}{r}\right)dy\right|.$$

Also, its norm is given by $||f||_{\mathcal{H}^1}:=||f^*||_{L^1(\mathbb{R}^3)}$. For the facts above, we refer to Fefferman-Stein [**42**], Stein [**105**] and Torchinsky [**115**]. The well-known result states that $(\mathcal{H}^1)'=$ B.M.O and

$$\left|\int_{\mathbb{R}^3}f(y)g(y)dy\right|\le C_{FS}[f]_{B.M.O}||g||_{\mathcal{H}^1} \tag{2.12}$$

holds for any $f\in$ B.M.O and any $g\in\mathcal{H}^1$.

We successively introduce the following lemma by Coifman-Lions-Meyer-Sem mes [**26**].

LEMMA 2.2 (Coifman-Lions-Meyer-Semmes [**26**]). *Assume* $f\in H^{1,2}(\mathbb{R}^3)$ *and* $g\in L^2(\mathbb{R}^3;\mathbb{R}^3)$ *with* $\mathit{div} g=0$ *in the distribution sense. Then* $\langle\nabla f,g\rangle\in\mathcal{H}^1$ *with the following inequality:*

$$||\langle\nabla f,g\rangle||_{\mathcal{H}^1}\le C_{CLMS}||\nabla f||_{L^2(\mathbb{R}^3)}||g||_{L^2(\mathbb{R}^3;\mathbb{R}^3)}. \tag{2.13}$$

Next, we prepare the fundamental energy inequality on modified strong evolutional harmonic maps. Let $z_0=(t_0,x_0)$ be a point in Q_∞ and positive numbers t_1,t_2 and r_1,r_2 with $0<t_0-t_2<t_0-t_1\le t_0$ and all balls $B_{r_1}(x_0)$ and $B_{r_2}(x_0)$ satisfying

$B_{r_1}(x_0) \subset\subset B_{r_2}(x_0) \subset\subset B$. We also choose two cut-off functions $\eta \in \mathcal{D}(B_{r_2}(x_0))$ and $\chi \in C^\infty(0,t_0]$ given by

$$\eta(|x|) = \begin{cases} 1 & \text{in} \quad B_{r_1}, \\ 0 & \text{outside} \quad B_{r_2}, \end{cases}$$
$$0 \le \eta(|x|) \le 1, \quad |\nabla \eta(|x|)| \le 2/(r_2 - r_1),$$

$$\chi(t) = \begin{cases} 1 & t_0 - t_1 < t \le t_0, \\ 0 & t \le t_0 - t_2, \end{cases}$$
$$0 \le \chi(t) \le 1, \quad |d\chi/dt(t)| \le 2/(t_2 - t_1).$$

Lemma 2.3 (An energy inequality). *For modified strong evolutional harmonic map u, there holds*

$$\begin{aligned} &\frac{1}{2}\int_{t_0-t_1}^{t_0} dt \int_{B_{r_1}} \left| \frac{\partial u}{\partial t}(z) \right|^2 dx + \frac{1}{2}\int_{B_{r_1}} |\nabla u(t_0,x)|^2 dx \\ &\le \left[\frac{8}{(r_2 - r_1)^2} + \frac{1}{(t_2 - t_1)} \right] \int_{t_0-t_2}^{t_0} dt \int_{B_{r_2}} |\nabla u(z)|^2 dx. \end{aligned} \tag{2.14}$$

Finally, we show the monotonicity inequalities for the scaled energies of modified strong evolutional harmonic map.

Lemma 2.4 (A monotonicity formula). *The modified strong evolutional harmonic maps u satisfy the following monotonicity formula*:

$$\begin{aligned} &\frac{1}{r_2^2}\int_{t_0-r_2^2}^{t_0} \frac{dt}{r_1} \int_{B_{r_1}(x_0)} |\nabla u(z)|^2 dx \\ &\qquad \le \frac{C_M}{(2r_2)^2}\int_{t_0-(2r_2)^2}^{t_0} \frac{dt}{2r_2} \int_{B_{2r_2}(x_0)} |\nabla u(z)|^2 dx \end{aligned} \tag{2.15}$$

holds for any concentric cylinders $(t_0, t_0 - r_2^2) \times B_{r_1}(x_0) \subset (t_0, t_0 - (2r_2)^2) \times B_{2r_2}(x_0) \subset\subset Q_\infty$ and C_M is a positive constant independent of r_1, r_2, z_0, u.

Proof of lemma 2.3

First, fix $0 < h$ and recall the weak formula (2.6); We test $\partial^+ u/\partial t \ \eta^2 \chi$ into ϕ in (2.6). This is always possible by taking the mollifier of $\partial^+ u/\partial t$. Then we obtain

$$\int_{t_0-t_2}^{t_0} dt \int_{B_{r_2}} \langle \frac{\partial u}{\partial t}(z), \frac{\partial^+ u}{\partial t}(z) \rangle \eta^2(|x|) \chi(t) dx$$

$$
\begin{aligned}
&+\int_{t_0-t_2}^{t_0} dt \int_{B_{r_2}} \langle \frac{\partial^+}{\partial t}\nabla u(z), \nabla u(z)\rangle \eta^2(|x|)\chi(t)dx \\
&+2\int_{t_0-t_2}^{t_0} dt \int_{B_{r_2}} \langle \nabla u(z), \frac{\partial^+ u}{\partial t}(z)\nabla\eta(|x|)\rangle \eta(|x|)\chi(t)dx \\
&=\int_{t_0-t_2}^{t_0} dt \int_{B_{r_2}} |\nabla u(z)|^2 \langle u(z), \frac{\partial^+ u}{\partial t}(z)\rangle \eta^2(|x|)\chi(t)dx.
\end{aligned}
$$

Here, a non-negativity of the right-hand side is derived from $\langle u(z),u(t+h,x)-u(z)\rangle \leq 0$ because of $|u(z)|=1$, a.e $\quad z\in Q_\infty$.

Thus, by applying Schwarz inequality to the 3rd term on the left-hand side, we infer

$$
\begin{aligned}
&\int_{t_0-t_2}^{t_0} dt \int_{B_{r_2}} \langle \frac{\partial u}{\partial t}(z), \frac{\partial^+ u}{\partial t}(z)\rangle \eta^2(|x|)\chi(t)dx - \frac{1}{2}\int_{t_0-t_2}^{t_0} dt \int_{B_{r_2}} \left|\frac{\partial^+ u}{\partial t}(z)\right|^2 \eta^2(|x|)\chi(t)dx \\
&\quad -\frac{1}{2h}\int_{t_0-t_2}^{t_0} dt \int_{B_{r_2}} |\nabla u(t+h,x)-\nabla u(t,x)|^2 \eta^2(|x|)\chi(t)dx \\
&\quad +\frac{1}{2h}\int_{t_0-t_2}^{t_0} dt \int_{B_{r_2}} \left[|\nabla u(t+h,x)|^2-|\nabla u(t,x)|^2\right] \eta^2(|x|)\chi(t)dx \\
&\quad \leq 2\int_{t_0-t_2}^{t_0} dt \int_{B_{r_2}} |\nabla u(z)|^2 |\nabla\eta(|x|)|^2 \chi(t)dx,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\int_{t_0-t_2}^{t_0} dt \int_{B_{r_2}} \langle \frac{\partial u}{\partial t}(z), \frac{\partial^+ u}{\partial t}(z)\rangle \eta^2(|x|)\chi(t)dx \\
&\quad -\frac{1}{2}\int_{t_0-t_2}^{t_0} dt \int_{B_{r_2}} \left|\frac{\partial^+ u}{\partial t}(z)\right|^2 \eta^2(|x|)\chi(t)dx \\
&\quad -\frac{1}{2h}\int_{t_0-t_2}^{t_0} dt \int_{B_{r_2}} |\nabla u(t+h,x)-\nabla u(t,x)|^2 \eta^2(|x|)\chi(t)dx \\
&\quad +\frac{1}{2h}\int_{t_0}^{t_0+h} dt \int_{B_{r_2}} |\nabla u(z)|^2 \eta^2(|x|)\chi(t)dx \\
&-\frac{1}{2h}\int_{t_0-t_2}^{t_0-t_2+h} dt \int_{B_{r_2}} |\nabla u(z)|^2 \eta^2(|x|)\chi(t)dx \\
&\quad \leq \frac{1}{2}\int_{t_0-t_2+h}^{t_0+h} dt \int_{B_{r_2}} |\nabla u(z)|^2 \eta^2(|x|)\left(\frac{\chi(t)-\chi(t-h)}{h}\right)dx
\end{aligned}
$$

$$+\frac{8}{(r_2-r_2)^2}\int_{t_0-t_2}^{t_0}dt\int_{B_{r_2}}|\nabla u(z)|^2\chi(t)dx.$$

By a few properties of definition on modified strong evolutional harmonic map, i.e. (2.5) and (2.10), we can pass to the limit $h\searrow 0$. Thus we can conclude the claim of this lemma.

Proof of lemma 2.4

First of all, let us remark that our system (2.6) is invariant under translation $z\to z-z_0$, hence we may shift $z_0\to 0$. For any positive number δ sufficiently small and any positive number $r<r_2/2$, we define three support functions given by

$$\eta_\delta(x)\equiv\eta_\delta(|x|)=\begin{cases}1 & |x|\le r,\\ -\dfrac{1}{\delta}(|x|-(r+\delta)) & r<|x|\le r+\delta,\\ 0 & r+\delta<|x|,\end{cases}$$

$$\Psi_\delta(x)\equiv\Psi_\delta(|x|)=\begin{cases}\int_{|x|}^{r+\delta}t\eta_\delta(t)dt & |x|<r+\delta,\\ 0 & r+\delta\le|x|,\end{cases}$$

$$\Psi_0(x)\equiv\Psi_0(|x|)=\begin{cases}\frac{1}{2}(r^2-|x|^2) & |x|<r,\\ 0 & r\le|x|.\end{cases}$$

Here note that $\eta_\delta(|x|)$ and $\Psi_\delta(|x|)$ have the following relations:

$$\frac{d\Psi_\delta}{d\rho}(\rho)+\rho\eta_\delta(\rho)=0 \qquad \text{in} \quad \rho>0. \tag{2.16}$$

After the preparation above, we start the proof of our monotonicity for the scaled energy. To this end, recall the weak formula: The evolutional evolutional harmonic maps u satisfy

$$\begin{aligned}\int_{Q_{r_2}}\langle\frac{\partial u}{\partial t}(z),\phi(z)\rangle dz+\int_{Q_{r_2}}\langle\nabla u(z),\nabla\phi(z)\rangle dz&\\ =\int_{Q_{r_2}}|\nabla u(z)|^2\langle u(z),\phi(z)\rangle dz&\end{aligned} \tag{2.17}$$

for all maps $\phi\in\mathcal{D}\ (Q_{r_2};\mathbb{R}^D)$.

As before, set a positive number h with $0<h<|t_0-T|$. By approximations, we can take ϕ as follows:

$$\phi(z)=\frac{\partial^+ u}{\partial t}(z)\Psi_\delta(|x|)+|x|\nabla^- u(z)\eta_\delta(|x|).$$

We calculate the space gradients of ϕ:

$$\nabla\phi(z)=\frac{\partial^+}{\partial t}\nabla u(z)\Psi_\delta(|x|)+\frac{\partial^+ u}{\partial t}(z)\nabla\Psi_\delta(|x|)$$
$$+\nabla[|x|\nabla^- u(z)]\eta_\delta(|x|)+|x|\nabla^- u(z)\nabla\eta_\delta(|x|).$$

By testing ϕ chosen above into (2.17), we obtain

$$\begin{aligned}
&\int_{Q_{r_2}}\langle\frac{\partial u}{\partial t}(z),\frac{\partial^+ u}{\partial t}(z)\rangle\Psi_\delta(|x|)dz+\int_{Q_{r_2}}\langle\frac{\partial u}{\partial t}(z),|x|\nabla^- u(z)\rangle\eta_\delta(|x|)dz\\
&+\int_{Q_{r_2}}\langle\nabla u(z),\frac{\partial^+}{\partial t}\nabla u(z)\rangle\Psi_\delta(|x|)dz+\int_{Q_{r_2}}\langle\nabla u(z),\frac{\partial u^+}{\partial t}(z)\nabla\Psi_\delta(|x|)\rangle dz\\
&+\int_{Q_{r_2}}\langle\nabla u(z),\nabla\left[|x|\nabla^- u(z)\right]\rangle\eta_\delta(|x|)dz\\
&+\int_{Q_{r_2}}\langle\nabla u(z),|x|\nabla^- u(z)\nabla\eta_\delta(|x|)\rangle dz\\
&=\int_{Q_{r_2}}|\nabla u(z)|^2\langle u(z),\frac{\partial^+ u}{\partial t}(z)\Psi_\delta(|x|)+|x|\nabla^- u(z)\eta_\delta(|x|)\rangle dz. \qquad (2.18)
\end{aligned}$$

We successively perform the estimates of the 3rd term and the 5th term on the left-hand side:

$$\begin{aligned}
&\int_{Q_{r_2}}\langle\nabla u(z),\frac{\partial^+}{\partial t}\nabla u(z)\rangle\Psi_\delta(|x|)dz\\
&=\frac{1}{h}\int_{Q_{r_2}}\langle\nabla u(z),\nabla u(t+h,x)-\nabla u(z)\rangle\Psi_\delta(|x|)dz\\
&=-\frac{1}{2h}\int_{Q_{r_2}}|\nabla u(t+h,x)-\nabla u(z)|^2\Psi_\delta(|x|)dz\\
&+\frac{1}{2h}\left(\int_{-r_2^2}^0 dt\int_{B_{r_2}}|\nabla u(t+h,x)|^2-|\nabla u(t,x)|^2\right)\Psi_\delta(|x|)dx\\
&=-\frac{1}{2h}\int_{Q_{r_2}}|\nabla u(t+h,x)-\nabla u(z)|^2\Psi_\delta(|x|)dz\\
&+\frac{1}{2h}\int_0^h dt\int_{B_{r_2}}|\nabla u(z)|^2\Psi_\delta(|x|)dx\\
&-\frac{1}{2h}\int_{-r_2^2}^{-r_2^2+h}dt\int_{B_{r_2}}|\nabla u(z)|^2\Psi_\delta(|x|)dx. \qquad (2.19)
\end{aligned}$$

$$
\begin{aligned}
&\int_{Q_{r_2}} \langle \nabla u(z), \nabla \left[|x| \nabla^- u(z)\right] \rangle \eta_\delta(|x|) dz \\
&= \int_{Q_{r_2}} \langle \nabla u(z), \nabla |x| \nabla^- u(z) \rangle \eta_\delta(|x|) dz \\
&+ \int_{Q_{r_2}} \langle \nabla u(z), |x| \nabla \nabla^- u(z) \rangle \eta_\delta(|x|) dz \\
&= \int_{Q_{r_2}} \langle \nabla u(z), \nabla |x| \nabla^- u(z) \rangle \eta_\delta(|x|) dz \\
&+ \frac{1}{2h} \int_{-r_2^2}^0 \int_{B_{r_2} \setminus B_h} |x| \left| \nabla u(z) - \nabla u(t, |x| - h, \frac{x}{|x|}) \right|^2 \eta_\delta(|x|) dz \\
&+ \frac{1}{2h} \int_{-r_2^2}^0 dt \int_{B_{r_2} \setminus B_h} |x| |\nabla u(z)|^2 \eta_\delta(|x|) dx \\
&- \frac{1}{2h} \int_{-r_2^2}^0 dt \int_{B_{r_2} \setminus B_h} |x| |\nabla u(t, |x| - h, \frac{x}{|x|})|^2 \eta_\delta(|x|) dx \\
&- \frac{1}{h} \int_{-r_2^2}^0 dt \int_{B_h} |x| \langle \nabla u(z), \nabla u(z) - \nabla u(t, h - |x|, \frac{x}{|x|}) \rangle \eta_\delta(|x|) dx \\
&= \int_{Q_{r_2}} \langle \nabla u(z), \nabla |x| \nabla^- u(z) \rangle \eta_\delta(|x|) dz \\
&+ \frac{1}{2h} \int_{-r_2^2}^0 dt \int_{B_{r_2} \setminus B_h} |x| \left| \nabla u(z) - \nabla u(t, |x| - h, \frac{x}{|x|}) \right|^2 \eta_\delta(|x|) dz \\
&- \frac{1}{2} \int_{-r_2^2}^0 dt \int_{B_{r_2} \setminus B_h} |\nabla u(t, |x| - h, \frac{x}{|x|})|^2 \nabla^- \left(\eta_\delta(|x|) |x|^3 \right) \frac{dx}{|x|^2} \\
&+ \frac{1}{2h} \int_{-r_2^2}^0 dt \int_{B_{r_2} \setminus B_{r_2 - h}} |x| |\nabla u(z)|^2 \eta_\delta(|x|) dx \\
&- \frac{1}{2h} \int_{-r_2^2}^0 dt \int_{B_h} |x| |\nabla u(z)|^2 \eta_\delta(|x|) dx \\
&+ \frac{1}{h} \int_{-r_2^2}^0 dt \int_{B_h} |x| \langle \nabla u(z), \nabla u(z) - \nabla u(t, h - |x|, \frac{x}{|x|}) \rangle \eta_\delta(|x|) dx. \qquad (2.20)
\end{aligned}
$$

Combining (2.18) with (2.19) and (2.20), we infer

$$
\frac{(r+\delta)^2}{2} \int_{Q_{r_2}} \left| \frac{\partial u}{\partial t}(z) \right|^2 \eta_\delta(|x|) dz
$$

$$
\begin{aligned}
&+\int_{Q_{r_2}}\langle\frac{\partial u}{\partial t}(z),\frac{\partial u^+}{\partial t}(z)-\frac{\partial u}{\partial t}(z)\rangle\Psi_\delta(|x|)dz\\
&-\frac{1}{2h}\int_{Q_{r_2}}|\nabla u(t+h,x)-\nabla u(z)|^2\Psi_\delta(|x|)dz\\
&+\frac{1}{2h}\int_{-r_2^2}^0 dt\int_{B_{r_2}\setminus B_h}|x|\left|\nabla u(z)-\nabla u(t,|x|-h,\frac{x}{|x|})\right|^2\eta_\delta(|x|)dx\\
&+\int_{Q_{r_2}}\left(\langle\frac{\partial u}{\partial t}(z),|x|\nabla^- u(z)\rangle\eta_\delta(|x|)+\langle\langle x,\nabla\rangle u(z),\frac{\partial^+ u}{\partial t}(z)\rangle\frac{\Psi_\delta'(|x|)}{|x|}\right)dz\\
&+\frac{1}{2h}\int_0^h dt\int_{B_{r_2}}|\nabla u(z)|^2\Psi_\delta(|x|)dx\\
&-\frac{1}{2h}\int_{-r_2^2}^{-r_2^2+h} dt\int_{B_{r_2}}|\nabla u(z)|^2\Psi_\delta(|x|)dx\\
&+\int_{Q_{r_2}}\langle\nabla u(z),\nabla|x|\nabla^- u(z)\rangle\eta_\delta(|x|)dz\\
&+\int_{Q_{r_2}}\langle\nabla u(z),|x|\nabla^- u(z)\nabla\eta_\delta(|x|)\rangle dz\\
&-\frac{1}{2}\int_{-r_2^2}^0 dt\int_{B_{r_2}\setminus B_h}|\nabla u(t,|x|-h,\frac{x}{|x|})|^2\nabla^-\left(\eta_\delta(|x|)|x|^3\right)\frac{dx}{|x|^2}\\
&+\frac{1}{2h}\int_{-r_2^2}^0 dt\int_{B_{r_2}\setminus B_{r_2-h}}|x||\nabla u(z)|^2\eta_\delta(|x|)dx\\
&-\frac{1}{2h}\int_{-r_2^2}^0 dt\int_{B_h}|x||\nabla u(z)|^2\eta_\delta(|x|)dx\\
&+\frac{1}{h}\int_{-r_2^2}^0 dt\int_{B_h}|x|\langle\nabla u(z),\nabla u(z)-\nabla u(t,h-|x|,\frac{x}{|x|})\rangle\eta_\delta(|x|)dx\\
&\geq\int_{Q_{r_2}}|\nabla u(z)|^2\langle u(z),\frac{\partial^+ u}{\partial t}(z)\Psi_\delta(|x|)+|x|\nabla^- u(z)\eta_\delta(|x|)\rangle dz,
\end{aligned}
\tag{2.21}
$$

where we used $\Psi_\delta(|x|)\leq\ (r+\delta)^2/2\cdot\eta_\delta(|x|)$ in the 1st term on the left-hand side.

We here estimate the 1st term on the left-side hand in (2.21) as follows: First, recall the weak formula (2.6):

$$
\int_{Q_{r_2}}\langle\frac{\partial u}{\partial t}(z),\phi(z)\rangle dz+\int_{Q_{r_2}}\langle\nabla u(z),\nabla\phi(z)\rangle dz=\int_{Q_{r_2}}|\nabla u(z)|^2\langle u(z),\phi(z)\rangle dz
$$

for any mapping $\phi\in\ \mathcal{D}\ (Q;\mathbb{R}^D)$.

As before, by the approximation, we can substitute $\partial^+ u/\partial t\ \eta_\delta$ for ϕ to obtain

$$\begin{aligned}
&\int_{Q_{r_2}} \left|\frac{\partial u}{\partial t}(z)\right|^2 \eta_\delta(|x|)dz + \int_{Q_{r_2}} \langle \frac{\partial u}{\partial t}(z), \frac{\partial^+ u}{\partial t}(z) - \frac{\partial u}{\partial t}(z)\rangle \eta_\delta(|x|)dz \\
&+ \int_{Q_{r_2}} \langle \nabla u(z), \frac{\partial^+}{\partial t}\nabla u(z)\rangle \eta_\delta(|x|)dz + \int_{Q_{r_2}} \langle \langle x, \nabla\rangle u(z), \frac{\partial^+ u}{\partial t}(z)\rangle \frac{\eta_\delta'(|x|)}{|x|}dz \\
&= \int_{Q_{r_2}} |\nabla u(z)|^2 \langle u(z), \frac{\partial^+ u}{\partial t}(z)\rangle \eta_\delta(|x|)dz.
\end{aligned}$$

A similar calculation modifies the term above to

$$\begin{aligned}
&\int_{Q_{r_2}} \left|\frac{\partial u}{\partial t}(z)\right|^2 \eta_\delta(|x|)dz = -\int_{Q_{r_2}} \langle \frac{\partial u}{\partial t}(z), \frac{\partial^+ u}{\partial t}(z) - \frac{\partial u}{\partial t}(z)\rangle \eta_\delta(|x|)dz \\
&+ \frac{1}{2h}\int_{Q_{r_2}} |\nabla u(t+h,x) - \nabla u(z)|^2 \eta_\delta(|x|)dz \\
&- \frac{1}{2h}\int_{Q_{r_2}} \left(|\nabla u(t+h,x)|^2 - |\nabla u(z)|^2\right)\eta_\delta(|x|)dz \\
&- \int_{Q_{r_2}} \langle \langle x, \nabla\rangle u(z), \frac{\partial^+ u}{\partial t}(z)\rangle \frac{\eta_\delta'(|x|)}{|x|}dz \\
&+ \int_{Q_{r_2}} |\nabla u(z)|^2 \langle u(z), \frac{\partial^+ u}{\partial t}(z)\rangle \eta_\delta(|x|)dz \\
&= -\int_{Q_{r_2}} \langle \frac{\partial u}{\partial t}(z), \frac{\partial^+ u}{\partial t}(z) - \frac{\partial u}{\partial t}(z)\rangle \eta_\delta(|x|)dz \\
&+ \frac{1}{2h}\int_{Q_{r_2}} |\nabla u(t+h,x) - \nabla u(z)|^2 \eta_\delta(|x|)dz \\
&- \frac{1}{2h}\int_0^h dt \int_{B_{r_2}} |\nabla u(z)|^2 \eta_\delta(|x|)dx \\
&+ \frac{1}{2h}\int_{-r_2^2}^{-r_2^2+h} dt \int_{B_{r_2}} |\nabla u(z)|^2 \eta_\delta(|x|)dx \\
&- \int_{Q_{r_2}} \langle \langle x, \nabla\rangle u(z), \frac{\partial^+ u}{\partial t}(z)\rangle \frac{\eta_\delta'(|x|)}{|x|}dz \\
&+ \int_{Q_{r_2}} |\nabla u(z)|^2 \langle u(z), \frac{\partial^+ u}{\partial t}(z)\rangle \eta_\delta(|x|)dz. \qquad (2.22)
\end{aligned}$$

We substitute (2.22) for (2.21), which leads to

$$
\begin{aligned}
&\int_{Q_{r_2}}\langle\frac{\partial u}{\partial t}(z),\frac{\partial^+u}{\partial t}(z)-\frac{\partial u}{\partial t}(z)\rangle\left(\Psi_\delta(|x|)-\frac{(r+\delta)^2}{2}\eta_\delta(|x|)\right)dz\\
&-\frac{1}{2h}\int_{Q_{r_2}}|\nabla u(t+h,x)-\nabla u(z)|^2\left(\Psi_\delta(|x|)-\frac{(r+\delta)^2}{2}\eta_\delta(|x|)\right)dz\\
&+\frac{1}{2h}\int_{-r_2^2}^0 dt\int_{B_{r_2}\setminus B_h}|x||\nabla u(z)-\nabla u(t,|x|-h,\frac{x}{|x|})|^2\eta_\delta(|x|)dx\\
&+\int_{Q_{r_2}}\left(\langle\frac{\partial u}{\partial t}(z),|x|\nabla^-u(z)\rangle\eta_\delta(|x|)dz\right.\\
&\qquad\left.+\langle\langle x,\nabla\rangle u(z),\frac{\partial^+u}{\partial t}(z)\rangle\frac{\Psi_\delta'(|x|)}{|x|}\right)dz\\
&+\frac{1}{2h}\int_0^h dt\int_{B_{r_2}}|\nabla u(z)|^2\left(\Psi_\delta(|x|)-\frac{(r+\delta)^2}{2}\eta_\delta(|x|)\right)dx\\
&-\frac{1}{2h}\int_{-r_2^2}^{-r_2^2+h}dt\int_{B_{r_2}}|\nabla u(z)|^2\left(\Psi_\delta(|x|)-\frac{(r+\delta)^2}{2}\eta_\delta(|x|)\right)dx\\
&+\int_{Q_{r_2}}\langle\nabla u(z),\nabla|x|\nabla^-u(z)\rangle\eta_\delta(|x|)dz\\
&+\int_{Q_{r_2}}\langle\nabla u(z),|x|\nabla^-u(z)\rangle\nabla\eta_\delta(|x|)dz\\
&-\frac{1}{2}\int_{-r_2^2}^0 dt\int_{B_{r_2}\setminus B_h}|\nabla u(t,|x|-h,\frac{x}{|x|})|^2\nabla^-\left(\eta_\delta(|x|)|x|^3\right)\frac{dx}{|x|^2}\\
&+\frac{1}{2h}\int_{-r_2^2}^0 dt\int_{B_{r_2}\setminus B_{r_2-h}}|x||\nabla u(z)|^2\eta_\delta(|x|)dx\\
&-\frac{(r+\delta)^2}{2}\int_{Q_{r_2}}\langle\langle x,\nabla\rangle u(z),\frac{\partial^+u}{\partial t}(z)\rangle\frac{\eta_\delta'(|x|)}{|x|}dz\\
&+\frac{1}{h}\int_{-r_2^2}^0 dt\int_{B_h}|x|\langle\nabla u(z),\nabla u(z)-\nabla u(t,h-|x|,\frac{x}{|x|})\rangle\eta_\delta(|x|)dx\\
&\geq\int_{Q_{r_2}}|\nabla u(z)|^2\langle u(z),\frac{\partial^+u}{\partial t}(z)\rangle\left(\Psi_\delta(|x|)-\frac{(r+\delta)^2}{2}\eta_\delta(|x|)\right)dz\\
&+\int_{Q_{r_2}}|\nabla u(z)|^2\langle u(z),|x|\nabla^-u(z)\rangle\eta_\delta(|x|)dz\geq 0. \qquad (2.23)
\end{aligned}
$$

A non-negativity on the last 2-terms in (2.23) follows from $\langle u(z),u(t+h,x)-u(z)\rangle\leq 0$, $\langle u(z),u(z)-u(t,\pm(|x|-h),x/|x|)\rangle\geq 0$, and $\Psi_\delta(|x|)\leq(r+\delta)^2/2\cdot\eta_\delta(|x|)$.

We now pass to the limit $h\searrow 0$ in (2.23): By recalling a few properties of definition of modified strong evolutional harmonic map: (2.5), (2.9), (2.10) and by a property of cut-functions, we infer

$$
\begin{aligned}
&+\frac{1}{2}\int_{B_{r_2}}|\nabla u(0,x)|^2\left(\Psi_\delta(|x|)-\frac{(r+\delta)^2}{2}\eta_\delta(|x|)\right)dx\\
&-\frac{1}{2}\int_{B_{r_2}}|\nabla u(-r_2^2,x)|^2\left(\Psi_\delta(|x|)-\frac{(r+\delta)^2}{2}\eta_\delta(|x|)\right)dx\\
&+\int_{Q_{r_2}}|\nabla u(z)|^2\eta_\delta(|x|)dz\\
&-\frac{3}{2}\int_{Q_{r_2}}|\nabla u(z)|^2\eta_\delta(|x|)dz-\frac{1}{2}\int_{Q_{r_2}}|\nabla u(z)|^2|x|\eta_\delta'(|x|)dz\\
&+\int_{Q_{r_2}}|\langle x,\nabla\rangle u(z)|^2\frac{\eta_\delta'(|x|)}{|x|}dz\\
&-\frac{(r+\delta)^2}{2}\int_{Q_{r_2}}\langle\langle x,\nabla\rangle u(z),\frac{\partial u}{\partial t}(z)\rangle\frac{\eta_\delta'(|x|)}{|x|}dz\geq 0.
\end{aligned}
$$

Finally, taking the limit $\delta\searrow 0$, we conclude

$$
\begin{aligned}
&\frac{1}{2}\int_{B_r}|\nabla u(0,x)|^2\left(\Psi_0(|x|)-\frac{r^2}{2}\right)dx-\frac{1}{2}\int_{B_r}|\nabla u(-r_2^2,x)|^2\left(\Psi_0(|x|)-\frac{r^2}{2}\right)dx\\
&-\frac{1}{2}\int_{Q_{r,r_2}}|\nabla u(z)|^2dz+\frac{r}{2}\int_{-r_2^2}^0dt\int_{\partial B_r}|\nabla u(z)|^2d\mathcal{H}^2\\
&+\frac{r}{2}\int_{-r_2^2}^0dt\int_{\partial B_r}\langle\langle x,\nabla\rangle u(z),\frac{\partial u}{\partial t}(z)\rangle d\mathcal{H}^2-\frac{1}{r}\int_{-r_2^2}^0dt\int_{\partial B_r}|\langle x,\nabla\rangle u(z)|^2d\mathcal{H}^2\\
&=\frac{1}{2}\int_{B_r}|\nabla u(0,x)|^2\left(\Psi_0(|x|)-\frac{r^2}{2}\right)dx-\frac{1}{2}\int_{B_r}|\nabla u(-r_2^2,x)|^2\left(\Psi_0(|x|)-\frac{r^2}{2}\right)dx\\
&-\frac{1}{2}\int_{Q_{r,r_2}}|\nabla u(z)|^2dz+\frac{r}{2}\int_{-r_2^2}^0dt\int_{\partial B_r}|\nabla u(z)|^2d\mathcal{H}^2\\
&+\frac{r^3}{16}\int_{-r_2^2}^0dt\int_{\partial B_r}\left|\frac{\partial u}{\partial t}(z)\right|^2d\mathcal{H}^2\\
&-\frac{1}{r}\int_{-r_2^2}^0dt\int_{\partial B_r}|\langle x,\nabla\rangle u(z)-\frac{r^2}{4}\frac{\partial u}{\partial t}(z)|^2d\mathcal{H}^2\geq 0.
\end{aligned}
$$

Multiplying the above by r^{-2}, we deduce

$$
\begin{aligned}
&\frac{1}{2}\int_{B_r}|\nabla u(-r_2^2,x)|^2\left(-\frac{\Psi_0(|x|)}{r^2}+\frac{1}{2}\right)dx+\frac{r}{16}\int_{-r_2^2}^0 dt\int_{\partial B_r}\left|\frac{\partial u}{\partial t}(z)\right|^2 d\mathcal{H}^2\\
&-\frac{1}{2r^2}\int_{Q_{r,r_2}}|\nabla u(z)|^2dz+\frac{1}{2r}\int_{-r_2^2}^0 dt\int_{\partial B_r}|\nabla u(z)|^2 d\mathcal{H}^2\\
&-\frac{1}{r^3}\int_{-r_2^2}^0 dt\int_{\partial B_r}|\langle x,\nabla\rangle u(z)-\frac{r^2}{4}\frac{\partial u}{\partial t}(z)|^2 d\mathcal{H}^2\geq 0.
\end{aligned}
$$

We thus show that for a.e. $r>0$ with $0<r<r_2$,

$$
\begin{aligned}
&-\frac{1}{2}\int_{B_r}|\nabla u(-r_2^2,x)|^2\left(\frac{\Psi_0(|x|)}{r^2}-\frac{1}{2}\right)dx\\
&+\frac{r}{16}\int_{-r_2^2}^0\int_{\partial B_r}\left|\frac{\partial u}{\partial t}(z)\right|^2 dz+\frac{1}{2}\frac{d}{dr}\left(\frac{1}{r}\int_{Q_{r,r_2}}|\nabla u(z)|^2dz\right)\\
&\geq\frac{1}{r^3}\int_{-r_2^2}^0 dt\int_{\partial B_r}|\langle x,\nabla\rangle u(z)-\frac{r^2}{4}\frac{\partial u}{\partial t}(z)|^2 d\mathcal{H}^2\geq 0,
\end{aligned}
\tag{2.24}
$$

holds.

We integrate (2.24) from r_1 to r_2 with respect to r, which obtains

$$
\begin{aligned}
&\frac{r_2}{4}\int_{B_{r_2}}|\nabla u(-r_2^2,x)|^2dx+\frac{r_2}{16}\int_{Q_{r_2}}|\frac{\partial u}{\partial t}(z)|^2dz\\
&+\frac{1}{2r_2}\int_{Q_{r_2}}|\nabla u(z)|^2dz\geq\frac{1}{2r_1}\int_{Q_{r_1,r_2}}|\nabla u(z)|^2dz.
\end{aligned}
$$

We apply Lemma 2.3 to the 1st and the the 2nd term on the left-hand side above to complete our proof. □

2. Proof of theorem

2.1. Proof of theorem 2.2

We prove the claim of this theorem by contradiction: Were our statement false, there would exist a sequence of cubes $\{Q_{R_k}(z_k)\}$ $(k=1,2,\ldots)$ in Q_∞ such that

$$
\begin{aligned}
&\frac{R_k^{-3}}{2}\int_{Q_{R_k}(z_k)}|\nabla u(z)|^2dz<\frac{1}{k}\quad\text{whereas}\\
&\frac{(\tau_0R_k)^{-3}}{2}\int_{Q_{\tau_0R_k}(z_k)}|\nabla u(z)|^2dz
\end{aligned}
$$

$$> \frac{{R_k}^{-3}}{2} \int_{Q_{R_k}(z_k)} |\nabla u(z)|^2 dz \quad \text{would hold.} \tag{2.25}$$

We rescale the variables $z = (t,x)$ to the unit cubes $Q_1(0) \subset \mathbb{R}^{1+3}$ as follows:

$$\bar{t} = \frac{1}{R_k^2}(t - t_k), \quad \bar{x} = \frac{1}{R_k}(x - x_k), \quad \bar{z} = (\bar{t}, \bar{x}),$$

$$\lambda_k = \sqrt{\frac{{R_k}^{-3}}{2} \int_{Q_{R_k}(z_k)} |\nabla u(z)|^2 dz},$$

$$u_{Q_{6/8R_k}(z_k)} = \fint_{Q_{6/8R_k}(z_k)} u(z')dz', \qquad v_k(\bar{z}) = \frac{1}{\lambda_k}\left[u(z) - u_{Q_{6/8R_k}(z_k)}\right].$$

From the change of variables, (2.25) becomes

$$\lambda_k^2 \leq \frac{1}{k}, \qquad \frac{\tau_0^{-3}}{2} \int_{Q_{\tau_0}(0)} |\nabla v_k(\bar{z})|^2 d\bar{z} > 1. \tag{2.26}$$

Also, choose $w \in \mathcal{D}(Q_{7/8}(0);\mathbb{R}^D)$ and set

$$w_k(z) = w\left(\frac{t - t_k}{R_k^2}, \frac{x - x_k}{R_k}\right) \in \mathcal{D}(Q_{7R_k/8}(z_k);\mathbb{R}^D).$$

We substitute w_k for ϕ in (2.6), which gives

$$\begin{aligned} &\int_{Q_{7/8R_k}(z_k)} \langle \frac{\partial u}{\partial t}(z), w_k(z)\rangle dz + \int_{Q_{7/8R_k}(z_k)} \langle \nabla u(z), \nabla w_k(z)\rangle dz \\ &= \int_{Q_{7/8R_k}(z_k)} |\nabla u(z)|^2 \langle u(z), w_k(z)\rangle dz. \end{aligned} \tag{2.27}$$

From definition of v_k: $u(z) = \lambda_k v_k(\bar{z}) + u_{Q_{6/8R_k}(z_k)}$,
we find that v_k satisfies

$$\begin{aligned} &\int_{Q_{7/8}(0)} \langle \frac{\partial v_k}{\partial \bar{t}}(\bar{z}), w(\bar{z})\rangle d\bar{z} + \int_{Q_{7/8}(0)} \langle \nabla v_k(\bar{z}), \nabla w(\bar{z})\rangle d\bar{z} \\ &= \lambda_k \int_{Q_{7/8}(0)} |\nabla v_k(\bar{z})|^2 \langle u(\bar{z}), w(\bar{z})\rangle d\bar{z}. \end{aligned} \tag{2.28}$$

From lemma 2.3, we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_{Q_{6/8R_k}(z_k)}\left|\frac{\partial u}{\partial t}(z)\right|^2 dz+\frac{1}{2}\int_{B_{6/8R_k}(x_k)}|\nabla u(t,x)|^2 dx \\
&\leq C\int_{Q_{7/8R_k}(z_k)}|\nabla u(z)|^2 dz,
\end{aligned}
\tag{2.29}
$$

for almost all t with $t_k-(6/8R_k)^2< t\leq t_k$.

By transferring the variables $z=(t,x)$ to $\bar{z}=(\bar{t},\bar{x})$, we find that the above (2.29) becomes

$$
\begin{aligned}
&\frac{1}{2}\int_{Q_{6/8}(0)}\left|\frac{\partial v_k}{\partial \bar{t}}(\bar{z})\right|^2 d\bar{z}+\frac{1}{2}\int_{B_{6/8}(0)}|\nabla v_k(\bar{t},\bar{x})|^2 d\bar{x} \\
&\leq C\int_{Q_{7/8}(0)}|\nabla v_k(\bar{z})|^2 d\bar{z}\leq C,
\end{aligned}
\tag{2.30}
$$

with a.e $\bar{t}$ in $(-(6/8)^2,0]$.

Thus, there exists a subsequence $\{v_{k(i)}\}$ of $\{v_k\}\in V(Q_{6/8}(0);\mathbb{R}^D)$ $(i=1,2,\ldots)$ and a mapping $v\in V(Q_{6/8}(0);\mathbb{R}^D)$ such that

$$
\begin{cases}
\nabla v_{k(i)}(\bar{z}) \quad \overset{k\to\infty}{\rightharpoonup} \quad \nabla v(\bar{z}) \quad \text{in}\, L^2(Q_{6/8}(0);\mathbb{R}^D), \\
\dfrac{\partial v_{k(i)}}{\partial t}(\bar{z}) \quad \overset{k\to\infty}{\rightharpoonup} \quad \dfrac{\partial v}{\partial t}(\bar{z}) \quad \text{in}\, L^2(Q_{6/8}(0);\mathbb{R}^D).
\end{cases}
\tag{2.31}
$$

In view of these facts (2.28), (2.30) and (2.31), we deduce that

$$
\int_{Q_{6/8}(0)}\langle\frac{\partial v}{\partial \bar{t}}(\bar{z}),w(\bar{z})\rangle d\bar{z}+\int_{Q_{6/8}(0)}\langle\nabla v(\bar{z}),\nabla w(\bar{z})\rangle d\bar{z}=0
$$

holds for all $w\in \mathcal{D}(Q_{6/8}(0);\mathbb{R}^D)$. This shows that the map above v is the solution of the linear heat equations, i.e. v satisfies

$$
\frac{\partial v}{\partial t}(\bar{z})-\triangle v(\bar{z})=0 \qquad \text{locally} \quad \text{in} \quad Q_{6/8}(0).
\tag{2.32}
$$

Thus Lemma 1.3 implies that for any positive number $\tau_0<1/2$,

$$
||\nabla v||_{L^\infty(Q_{\tau_0}(0))}\leq C_H\left[⨍_{Q_{6/8}(0)}|\nabla v(\bar{z})|^2 d\bar{z}\right]^{1/2}
$$

holds where C_H is a universal positive constant.

Now, we assume

$$\nabla v_k(\bar z) \overset{k\to\infty}{\longrightarrow} \nabla v(\bar z) \qquad \text{in} \quad L^2(Q_{1/2}(0);\mathbb{R}^{3\times D}). \tag{2.33}$$

Thus, from assumption (2.26), we deduce that

$$\frac{1}{2}<\frac{1}{2\tau_0^3}\int_{Q_{\tau_0}(0)}|\nabla v(\bar z)|^2 d\bar z \qquad \text{implies}$$

$$\begin{aligned}\frac{1}{2}&<\frac{1}{2\tau_0^3}\int_{Q_{\tau_0}(0)}|\nabla v(\bar z)|^2 d\bar z\leq\frac{1}{2\tau_0^3}|Q_{\tau_0}(0)|||\nabla v||^2_{L^\infty(Q_{\tau_0}(0))}\\ &\leq C_H^2\frac{|Q_1|}{|Q_{6/8}|}\frac{\tau_0^2}{2}\int_{Q_{6/8}(0)}|\nabla v(\bar z)|^2 d\bar z<\frac{1}{4},\end{aligned}$$

as long as $0<\tau_0<\dfrac{1}{2}\left(\dfrac{4}{3}\right)^{5/2}\dfrac{1}{C_H}$.

Since τ_0 is any positive number with $\tau_0<1/2$, then the above choice of τ_0 gives a contradiction.

2.2. Proof of theorem 2.3

Since $r^{-3}\int_{Q_r(z_0)}|\nabla u(z)|^2dz$ is continuous function with respect to z_0, if $z_0\in Q/\Sigma_u$, there exists an open cube $Q_{r_0}(z_0)$ with $Q_{2r_0}(z_0)\subset\subset Q_\infty$ such that for any $z\in Q_{r_0}(z_0)$,

$$r_0^{-3}\int_{Q_{r_0}(z)}|\nabla u(\bar z)|^2 d\bar z\leq\epsilon_0.$$

By using (2.11), a standard iteration technique implies that

$$\int_{Q_r(z)}|\nabla u(\bar z)|^2 d\bar z\leq\frac{1}{\tau_0^3}\left(\frac{r}{r_0}\right)^{3+\log 2/\log(1/\tau_0)}\int_{Q_{r_0}(z)}|\nabla u(\bar z)|^2 d\bar z$$

holds for any $r<r_0/2$.

By using Lemma 2.3, we then infer

$$\begin{aligned}&r^2\int_{Q_{r/2}(z)}\left|\frac{\partial u}{\partial t}(\bar z)\right|^2 d\bar z+\int_{Q_{r/2}(z)}|\nabla u(\bar z)|^2 d\bar z\\ &\leq C\left(\frac{r}{r_0}\right)^{3+\log 2/\log(1/\tau_0)}\int_{Q_{2r_0}(z_0)}|\nabla u(\bar z)|^2 d\bar z.\end{aligned}$$

Then it consequently follows from Campanato [**15**] that $u\in C^\alpha(Q/\Sigma_u;\mathbb{R}^3)$ with $\alpha=1/2\log 2/\log(1/\tau_0)$.

3. Compactness of the blow-up sequence

The reminder of this paper is to show the compactness of a blow-up sequence. This compactness is our main technical result. To this end, we first prepare the inequality of Poincaré type:

LEMMA 2.5. *In any* $\bar{t}$ *in* $(-(6/8)^2,0]$, v_k *satisfies*

$$||v_k(\bar{t},\cdot)||_{L^2(B_{6/8}(0))} \leq C, \tag{2.34}$$

where C *is a positive constant depending only on* D.

Proof. Recall definition of v_k:

$$\begin{aligned}
&||v_k(\bar{t},\cdot)||_{L^2(B_{6/8(0)})}\\
&= \frac{1}{\lambda_k}\left[\int_{B_{6/8}(0)} |u(\bar{t},\bar{x}) - \fint_{Q_{6/8}(0)} u(\bar{s},\bar{y})d\bar{s}d\bar{y}|^2 d\bar{x}\right]^{1/2}\\
&\leq \frac{1}{\lambda_k}\left[\int_{B_{6/8}(0)} |u(\bar{t},\bar{x}) - \fint_{B_{6/8}(0)} u(\bar{t},\bar{y})d\bar{y}|^2 d\bar{x}\right]^{1/2}\\
&+ \frac{1}{\lambda_k}\left[\int_{B_{6/8}(0)} |\fint_{Q_{6/8}(0)} (u(\bar{t},\bar{y}) - u(\bar{s},\bar{y}))d\bar{s}d\bar{y}|^2 d\bar{x}\right]^{1/2}\\
&\leq C||\nabla v_k(\bar{t},\cdot)||_{L^2(B_{6/8}(0))} + C||\partial_t v_k||_{L^2(Q_{6/8}(0))} \leq C,
\end{aligned} \tag{2.35}$$

where we used Lemma 2.3 and $1/2\int_{Q_1}|\nabla v_k|^2 d\bar{z} = 1$. $\square$

Now we are in the position to state main lemma:

LEMMA 2.6. *Let* v_k $(k=1,2,\ldots)$ *and* v *be maps appearing in the previous section.*

$$\text{Then} \qquad \nabla v_k(\bar{z}) \overset{k\to\infty}{\longrightarrow} \nabla v(\bar{z}) \qquad \text{in} \quad L^2(Q_{1/2}(0);\mathbb{R}^{3\times D}).$$

Proof. To prove the strong convergence of $\{\nabla v_k\}$ $(k=1,2,\ldots)$, first recall (2.28), (2.32):

$$\begin{aligned}
&\int_{Q_{6/8}(0)} \langle \frac{\partial v_k}{\partial \bar{t}}(\bar{z}), w(\bar{z})\rangle d\bar{z} + \int_{Q_{6/8}(0)} \langle \nabla v_k(\bar{z}), \nabla w(\bar{z})\rangle d\bar{z}\\
&= \lambda_k \int_{Q_{6/8}(0)} |\nabla v_k(\bar{z})|^2 \langle u(\bar{z}), w(\bar{z})\rangle d\bar{z},
\end{aligned} \tag{2.36}$$

and

$$\int_{Q_{6/8}(0)}\langle\frac{\partial v}{\partial t}(\bar z),w(\bar z)\rangle d\bar z+\int_{Q_{6/8}(0)}\langle\nabla v(\bar z),\nabla w(\bar z)\rangle d\bar z=0, \tag{2.37}$$

where w is any mapping of $\mathcal{D}(Q_{6/8}(0);\mathbb{R}^D)$.

After subtracted (2.37) from (2.36), we have

$$\begin{aligned}
&\int_{Q_{6/8}(0)}\langle\frac{\partial}{\partial t}(v_k(\bar z)-v(\bar z)),w(\bar z)\rangle d\bar z\\
&+\int_{Q_{6/8}(0)}\langle\nabla(v_k(\bar z)-v(\bar z)),\nabla w(\bar z)\rangle d\bar z\\
&=\lambda_k\int_{Q_{6/8}(0)}|\nabla v_k(\bar z)|^2\langle u(\bar z),w(\bar z)\rangle d\bar z.
\end{aligned} \tag{2.38}$$

By the approximation, the same identity obtains for $w\in L^2(-(6/8)^2,0;\ \overset{\circ}{H}{}^{1,2}(B_{6/8}(0);\ \mathbb{R}^D)\big)\cap L^\infty(Q_{6/8}(0);\mathbb{R}^D)$. Fix $\phi\in C^\infty(-(6/8)^2,0;\ C^\infty(B_{6/8}(0);\mathbb{R}_+)\big)$ satisfying

$$\begin{aligned}
&\phi(\bar z)=\begin{cases}1 & \text{in}\quad Q_{4/8}(0),\\ 0 & \text{outside}\quad \mathbb{R}^{1+3}/Q_{5/8}(0),\end{cases}\\
&0\le\phi(\bar z)\le 1,|\nabla\phi(\bar z)|\le 16,\\
&|\nabla^2\phi(\bar z)|\le 128,|\partial_t\phi(\bar z)|\le 128.
\end{aligned}$$

Substituting $(v_k-v)\phi^3$ for w in the identity (2.38), we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{B_{6/8}(0)}|v_k(\bar z)-v(\bar z)|^2\phi^3(\bar z)d\bar x\Big|_{t=-(6/8)^2}^{0}\\
&-\frac{3}{2}\int_{Q_{6/8}(0)}|v_k(\bar z)-v(\bar z)|^2\phi^2(\bar z)\frac{\partial\phi}{\partial t}(\bar z)d\bar z\\
&+\int_{Q_{6/8}(0)}|\nabla(v_k(\bar z)-v(\bar z))|^2\phi^3(\bar z)d\bar z\\
&+3\int_{Q_{6/8}(0)}\langle\nabla(v_k(\bar z)-v(\bar z))\phi^{3/2}(\bar z),(v_k(\bar z)-v(\bar z))\phi^{1/2}(\bar z)\nabla\phi(\bar z)\rangle d\bar z\\
&=\lambda_k\int_{Q_{6/8}(0)}|\nabla v_k(\bar z)|^2\langle u(\bar z),(v_k(\bar z)-v(\bar z))\phi^3(\bar z)\rangle d\bar z.
\end{aligned}$$

We here apply Schwartz inequality to the 4th term on the left-hand side above and invoke Hélein's trick (See Hélein [**64**].) on the right-hand side; We obtain

$$
\begin{aligned}
&\frac{1}{2}\int_{Q_{6/8}(0)}|\nabla(v_k(\bar z)-v(\bar z))|^2\phi^3(\bar z)d\bar z\\
&\le \frac{1}{2}\int_{Q_{6/8}(0)}|v_k(\bar z)-v(\bar z)|^2\left(3\phi^2(\bar z)\frac{\partial\phi}{\partial\bar t}(\bar z)+9\phi(\bar z)|\nabla\phi(\bar z)|^2\right)d\bar z\\
&+\lambda_k\int_{Q_{6/8}(0)}|\nabla v_k(\bar z)|^2\langle u(\bar z),(v_k(\bar z)-v(\bar z))\phi^3(\bar z)\rangle d\bar z\\
&=\frac{1}{2}\int_{Q_{6/8}(0)}|v_k(\bar z)-v(\bar z)|^2\left(3\phi^2(\bar z)\frac{\partial\phi}{\partial\bar t}(\bar z)+9\phi(\bar z)|\nabla\phi(\bar z)|^2\right)d\bar z\\
&-\lambda_k\sum_{i,j=1}^{D}\int_{Q_{6/8}(0)}\langle v_k^i(\bar z)\nabla\phi(\bar z),(v_k^j(\bar z)-v^j(\bar z))\phi(\bar z)\\
&\qquad\times\phi(\bar z)\left(u^j(\bar z)\nabla v_k^i(\bar z)-u^i(\bar z)\nabla v_k^j(\bar z)\right)\rangle d\bar z\\
&+\lambda_k\sum_{i,j=1}^{D}\int_{Q_{6/8}(0)}\langle\nabla(v_k^i(\bar z)\phi(\bar z)),(v_k^j(\bar z)-v^j(\bar z))\phi(\bar z)\\
&\qquad\times\phi(\bar z)\left(u^j(\bar z)\nabla v_k^i(\bar z)-u^i(\bar z)\nabla v_k^j(\bar z)\right)\rangle d\bar z. \qquad (2.39)
\end{aligned}
$$

We proceed to estimate (2.39). For this purpose, we implement Lemma 2.1 to the 3rd term on the right-hand side in (2.39): When we set $\mathbb{B}_k^{i,j}=\left(B_{\alpha,k}^{i,j}\right)$ $(\alpha=1,2,3)$ as $B_{\alpha,k}^{i,j}(\bar z)=\left(u^j\nabla_\alpha v_k^i-u^i\nabla_\alpha v_k^j\right)\phi$, there exists a map $\mathbb{C}_k^{i,j}\in L^2(Q_{6/8}(0);\mathbb{R}^3)$ and a function $\phi_k^{i,j}\in L^2\left(-(6/8)^2,0;\overset{\circ}{H}{}^{1,2}(B_{6/8}(0);\mathbb{R}^3)\right)$ such that

$$\mathbb{B}_k^{i,j}=\mathbb{C}_k^{i,j}+\nabla\phi_k^{i,j}\quad\text{in}\,Q_{6/8}(0),$$

$$\text{with}\qquad \mathbb{C}_k^{i,j}\in\overset{\circ}{\boldsymbol{J}},$$
$$\begin{cases}\triangle\phi_k^{i,j} &=\operatorname{div}\mathbb{B}_k^{i,j}\quad\text{in}\quad L^2(Q_{6/8}(0)),\\ \partial\phi_k^{i,j}/\partial n &=0\quad\text{on}\quad\partial B_{6/8}(0),\quad\text{a.e}t.\end{cases}$$

Here we must remark that $\operatorname{div}\mathbb{B}_k^{i,j}$ does belong to $L^2(Q_{6/8}(0))$. Indeed, since v_k satisfies (2.36),

$$
\begin{aligned}
\operatorname{div}\mathbb{B}_k^{i,j}&=\sum_{\alpha=1}^{3}\nabla_\alpha\left[\left(u^j\nabla_\alpha v_k^i-u^i\nabla_\alpha v_k^j\right)\phi\right]\\
&=\left(u^j\triangle v_k^i-u^i\triangle v_k^j\right)\phi+\langle u^j\nabla v_k^i-u^i\nabla v_k^j,\nabla\phi\rangle\\
&=\left(u^j\partial_{\bar t}v_k^i-u^i\partial_{\bar t}v_k^j\right)\phi+\langle u^j\nabla v_k^i-u^i\nabla v_k^j,\nabla\phi\rangle\ \left(:=D_k^{i,j}\right)\quad\text{in}\quad\mathcal{D}'(Q_{6/8}(0))
\end{aligned}
$$

holds. In the following, we write $\phi_k^{i,j}=\triangle^{-1}D_k^{i,j}$.

Thus, (2.39) becomes

$$
\begin{aligned}
&\frac{1}{2}\int_{Q_{6/8}(0)}|\nabla(v_k(\bar z)-v(\bar z))|^2\phi^3(\bar z)d\bar z\\
&\leq \frac{1}{2}\int_{Q_{6/8}(0)}|v_k(\bar z)-v(\bar z)|^2\left(3\phi^2(\bar z)\frac{\partial\phi}{\partial t}(\bar z)+9\phi(\bar z)|\nabla\phi(\bar z)|^2\right)d\bar z\\
&-\lambda_k\sum_{i,j=1}^{D}\int_{Q_{6/8}(0)}\langle v_k^i(\bar z)\nabla\phi(\bar z),(v_k^j(\bar z)-v^j(\bar z))\phi(\bar z)\mathbb{B}_k^{i,j}(\bar z)\rangle d\bar z\\
&+\lambda_k\sum_{i,j=1}^{D}\int_{Q_{6/8}(0)}\langle\nabla\left(v_k^i(\bar z)\phi(\bar z)\right),(v_k^j(\bar z)-v^j(\bar z))\phi(\bar z)\mathbb{C}_k^{i,j}(\bar z)\rangle d\bar z\\
&+\lambda_k\sum_{i,j=1}^{D}\int_{-(6/8)^2}^{0}d\bar t\int_{\mathbb{R}^3}\langle\nabla\left(v_k^i(\bar z)\phi(\bar z)\right),(v_k^j(\bar z)-v^j(\bar z))\phi(\bar z)\nabla(\triangle^{-1}D_k^{i,j}(\bar z))\rangle d\bar x\\
&=(R_1^k)+(R_2^k)+(R_3^k)+(R_4^k).
\end{aligned}
$$

From now we begin with estimating (R_j^k), $(j=1,2,3,4)$. First, since $||\nabla v_k||_{L^2(Q_{6/8}(0))}$ and $||\partial v_k/\partial\bar t||_{L^2(Q_{6/8}(0))}$ are uniform bounded with respect to k, Rellich-Kondrachov theorem reads

$$
\lim_{k\to\infty}(R_1^k)=0. \tag{2.40}
$$

Next, we perform the estimates of (R_2^k) as follows: By using Hölder inequality and Sobolev imbedding theorem $\overset{\circ}{H}{}^{1,2}(Q_{6/8}(0))\hookrightarrow L^4(Q_{6/8}(0))$, we infer

$$
\begin{aligned}
(R_2^k)&\leq\lambda_k\sum_{i,j=1}^{D}\left(\int_{Q_{6/8}(0)}|v_k^i(\bar z)|^4|\nabla\phi(\bar z)|^4d\bar z\right)^{1/4}\\
&\quad\times\left(\int_{Q_{6/8}(0)}|v_k^j(\bar z)-v_j(\bar z)|^4d\bar z\right)^{1/4}\left(\int_{Q_{6/8}(0)}|\mathbb{B}_k^{i,j}(\bar z)|^2d\bar z\right)^{1/2}\\
&=C\lambda_k||v_k||_{L^4(Q_{6/8}(0))}\\
&\quad\times\left(||v_k||_{L^4(Q_{6/8}(0))}+\liminf_{k\to\infty}||v_k||_{L^4(Q_{6/8}(0))}\right)||\nabla v_k||_{L^2(Q_{6/8}(0))}\\
&\leq C\lambda_k\left(||\nabla v_k||_{L^2(Q_{6/8}(0))}+||\partial_{\bar t}v_k||_{L^2(Q_{6/8}(0))}\right)\\
&\quad\times\Big[\left(||\nabla v_k||_{L^2(Q_{6/8}(0))}+\liminf_{k\to\infty}||\nabla v_k||_{L^2(Q_{6/8}(0))}\right)
\end{aligned}
$$

$$+\left(||\partial_{\bar t}v_k||_{L^2(Q_{6/8}(0))}+\liminf_{k\to\infty}||\partial_{\bar t}v_k||_{L^2(Q_{6/8}(0))}\right)\Big]$$

$$\times||\nabla v_k||_{L^2(Q_{6/8}(0))}\le C\lambda_k. \tag{2.41}$$

The last evaluations directly follow from Lemma 2.3, definition of v_k and $1/2\int_{Q_1(0)}|\nabla v_k|^2\ \ dz\ =1$.

We next perform the estimates of (R_3^k) and (R_4^k) (k=1,2,...): We apply Lemma 2.2 to $\nabla v_k^i\phi\ \mathbb{C}_k^{i,j}$. This is possible because of $v_k^i\phi\in H^{1,2}(B_{6/8}(0))$ for a.e t and $\mathbb{C}_k^{i,j}\in L^2(B_{6/8}(0))$ with $\mathrm{div}\mathbb{C}_k^{i,j}=0$ in $\mathcal{D}'\ (B_{6/8}(0);\mathbb{R}^3)$ for a.e t. Also note that ϕ vanishes outside $B_{5/8}(0)$, and v,v_k belong to $L^\infty(Q_{5/8}(0);\mathbb{R}^D)$. Then by using $(\mathcal{H}^1)'=$B.M.O, i.e. (2.12), we infer the following inequality:

$$\begin{aligned}
(R_3^k)&:=\lambda_k\sum_{i,j=1}^D\int_{-(6/8)^2}^0 d\bar t\int_{\mathbb{R}^3}\langle\nabla\left(v_k^i(\bar z)\phi(\bar z)\right),(v_k^j(\bar z)-v^j(\bar z))\phi(\bar z)\mathbb{C}_k^{i,j}(\bar z)\rangle d\bar x\\
&\le\lambda_k C_{\mathrm{FS}}\sum_{i,j=1}^D\int_{-(6/8)^2}^0||\nabla\left(v_k^i\phi\right)\mathbb{C}_k^{i,j}||_{\mathcal{H}^1}[(v_k^j-v^j)\phi]_{\mathrm{B.M.O}}d\bar t.\\
&\le\lambda_k C_{\mathrm{FS}}C_{\mathrm{CLMS}}\sum_{i,j=1}^D\int_{-(6/8)^2}^0||\nabla\left(v_k^i\phi\right)||_{L^2(\mathbb{R}^3)}||\mathbb{B}_k^{i,j}-\nabla(\triangle^{-1}D_k^{i,j})||_{L^2(\mathbb{R}^3)}\\
&\qquad\times[(v_k^j-v^j)\phi]_{\mathrm{B.M.O}}d\bar t\\
&\le 2\lambda_k C_{\mathrm{FS}}C_{\mathrm{CLMS}}\operatorname*{ess\cdot sup}_{-(6/8)^2<\bar t\le 0}||\nabla(v_k\phi)||_{L^2(\mathbb{R}^3)}\\
&\qquad\times\sum_{i,j=1}^D\left(\int_{-(6/8)^2}^0\left(||B_k^{i,j}||^2_{L^2(\mathbb{R}^3)}+||\nabla(\triangle^{-1}D_k^{i,j})||^2_{L^2(\mathbb{R}^3)}\right)d\bar t\right)^{1/2}\\
&\qquad\times\left(\int_{-(6/8)^2}^0[(v_k-v)\phi]^2_{\mathrm{B.M.O}}d\bar t\right)^{1/2}\\
&=2\lambda_k C_{\mathrm{FS}}C_{\mathrm{CLMS}}\times(R_{3.1}^k)\times(R_{3.2}^k)\times(R_{3.3}^k).
\end{aligned}$$

First, by using Lemma 2.3 and lemma 2.5, we obtain

$$(R_{3.1}^k)\le\operatorname*{ess\cdot sup}_{-(6/8)^2<\bar t\le 0}||\nabla v_k\phi||_{L^2(\mathbb{R}^3)}+\operatorname*{ess\cdot sup}_{-(6/8)^2<\bar t\le 0}||v_k\nabla\phi||_{L^2(\mathbb{R}^3)}\le C.$$

Next, we estimate $(R_{3.2}^k)$: We invoke a continuity of $\nabla\triangle^{-1}$ from L^2 to L^2. Then we obtain

$$(R_{3.2}^k)=\sum_{i,j=1}^D\left(\int_{-(6/8)^2}^0 d\bar t\int_{B_{6/8}(0)}|u^i(\bar z)\nabla v_k^j(\bar z)-u^j(\bar z)\nabla v_k^i(\bar z)|^2 d\bar x\right.$$

$$+\int_{-(6/8)^2}^{0} d\bar{t}\int_{B_{6/8}(0)} |\nabla(\triangle^{-1}D_k^{i,j})(\bar{z})|^2 d\bar{x}\Bigg)^{1/2}$$

$$\leq \sum_{i,j=1}^{D}\left(\int_{-(6/8)^2}^{0} d\bar{t}\int_{B_{6/8}(0)} |u^i(\bar{z})\nabla v_k^j(\bar{z}) - u^j(\bar{z})\nabla v_k^i(\bar{z})|^2 d\bar{x}\right)^{1/2}$$

$$+C_{RH}\sum_{i,j=1}^{D}\left(\int_{-(6/8)^2}^{0} d\bar{t}\int_{B_{6/8}(0)} |D_k^{i,j}(\bar{z})|^2 d\bar{x}\right)^{1/2}.$$

Once we recall definition of $D_k^{i,j}$, we then proceed to estimate $(R_{3.2}^k)$ as follows:

$$(R_{3.2}^k)\leq \sum_{i,j=1}^{D}\left(\int_{-(6/8)^2}^{0} d\bar{t}\int_{B_{6/8}(0)} |u^j(\bar{z})\nabla v_k^i(\bar{z}) - u^i(\bar{z})\nabla v_k^j(\bar{z})|^2 d\bar{x}\right)^{1/2}$$

$$+C\sum_{i,j=1}^{D}\left(\int_{-(6/8)^2}^{0} d\bar{t}\int_{B_{6/8}(0)} |u^j(\bar{z})\partial_t v_k^i(\bar{z}) - u^i(\bar{z})\partial_t v_k^j(\bar{z})|^2 d\bar{x}\right)^{1/2}$$

$$+C\sum_{i,j=1}^{D}\left(\int_{-(6/8)^2}^{0} d\bar{t}\int_{B_{6/8}(0)} |\langle u^j(\bar{z})\nabla v_k^i(\bar{z}) - u^i(\bar{z})\nabla v_k^j(\bar{z}),\nabla\phi(\bar{z})\rangle|^2 d\bar{x}\right)^{1/2}$$

$$\leq C||\nabla v_k||^2_{L^2(Q_{6/8}(0))} + C||\partial_t v_k||^2_{L^2(Q_{6/8}(0))}\leq C. \tag{2.42}$$

Finally, $(R_{3.3}^k)$ will be estimated from above. By definition of B.M.O, we can always assume that there exist sequences of positive numbers $\{r_\nu\}$ with $0< r_\nu <1/8$ and points $\{x_\nu\}$ in $B_{6/8}(0)$ $(\nu=1,2,\dots)$ such that

$$(R_{3.3}^k)=\left(\int_{-(6/8)^2}^{0} d\bar{t}\lim_{\nu\to\infty}\left(\fint_{B_{r_\nu}(x_\nu)}\Big|(v_k(\bar{z})-v(\bar{z}))\phi(\bar{z}) - ((v_k-v)\phi)_{B_{r_\nu}(x_\nu)}\Big| d\bar{x}\right)^2\right)^{1/2}$$

$$\leq \liminf_{\nu\to\infty}\left(\int_{-(6/8)^2}^{0} d\bar{t}\fint_{B_{r_\nu}(x_\nu)}\Big|(v_k(\bar{z})-v(\bar{z}))\phi(\bar{z}) - ((v_k-v)\phi)_{B_{r_\nu}(x_\nu)}\Big|^2 d\bar{x}\right)^{1/2}$$

$$\leq C\liminf_{\nu\to\infty}\left(\int_{-(6/8)^2}^{0} \frac{d\bar{t}}{r_\nu}\int_{B_{r_\nu}(x_\nu)} |\nabla((v_k(\bar{z})-v(\bar{z}))\phi(\bar{z}))|^2 d\bar{x}\right)^{1/2}$$

$$\leq C\liminf_{\nu\to\infty}\left(\int_{-(6/8)^2}^{0} \frac{d\bar{t}}{r_\nu}\int_{B_{r_\nu}(x_\nu)} |\nabla(v_k(\bar{z})-v(\bar{z}))\phi(\bar{z})|^2 d\bar{x}\right)^{1/2}$$

$$+C\liminf_{\nu\to\infty}\left(\int_{-(6/8)^2}^{0} \frac{d\bar{t}}{r_\nu}\int_{B_{r_\nu}(x_\nu)} |v_k(\bar{z})-v(\bar{z})|^2|\nabla\phi(\bar{z})|^2 d\bar{x}\right)^{1/2}.$$

Successively, using Lemma 2.4 to the 1st term and Hölder inequality to the 2nd term, we infer

$$
\begin{aligned}
(R^k_{3.3}) \leq & C \liminf_{\nu\to\infty}\left(\int_{-(6/8)^2}^{0}\frac{d\bar{t}}{r_\nu}\int_{B_{r_\nu}(x_\nu)}|\nabla v_k(\bar{z})|^2 d\bar{x}\right)^{1/2}\\
&+C\liminf_{\nu\to\infty}\left(\int_{-(6/8)^2}^{0}\frac{d\bar{t}}{r_\nu}\int_{B_{r_\nu}(x_\nu)}|\nabla v(\bar{z})|^2 d\bar{x}\right)^{1/2}\\
&+C\liminf_{\nu\to\infty}\left(\int_{-(6/8)^2}^{0}d\bar{t}\left(\int_{B_{r_\nu}(x_\nu)}|v_k(\bar{z})-v(\bar{z})|^3|\nabla\phi(\bar{z})|^3 d\bar{x}\right)^{2/3}\right)^{1/2}\\
\leq & C+C\left(\int_{-(6/8)^2}^{0}d\bar{t}\left(\int_{B_{6/8}(0)}|v_k(\bar{z})-v(\bar{z})|^6|\nabla\phi(\bar{z})|^6 d\bar{x}\right)^{1/3}\right)^{1/2}.
\end{aligned}
$$

We apply Sobolev imbedding theorem $\overset{\circ}{H}{}^{1,2}(B_{6/8}(0))\hookrightarrow L^6(B_{6/8}(0))$ to the 2nd term; We can continue to estimate $(R^k_{3.3})$ as follows.

$$
\begin{aligned}
(R^k_{3.3}) = & C+C\left(\int_{Q_{6/8}(0)}|\nabla(v_k(\bar{z})-v(\bar{z}))|^2|\nabla\phi(\bar{z})|^2 d\bar{z}\right)^{1/2}\\
&+C\left(\int_{Q_{6/8}(0)}|v_k(\bar{z})-v(\bar{z})|^2|\nabla^2\phi(\bar{z})|^2 d\bar{z}\right)^{1/2}.
\end{aligned}
$$

From Lemma 2.5, it consequently follows

$$
(\mathrm{R}_{3.i}) \leq C \quad (i=1,2,3). \tag{2.43}
$$

Finally, we perform the estimates of (R^k_4): For this purpose, note that

$$
\left|\nabla\triangle^{-1}D^{i,j}_k(\bar{t},\bar{x})\right| \leq \frac{1}{2\pi}\int_{B_{6/8}(0)}\left|\bar{x}-\bar{y}\right|^{-2}|D^{i,j}_k(\bar{t},\bar{y})\big|d\bar{y}.
$$

In addition, Riesz-Hölder inequality implies

$$
||\nabla\triangle^{-1}D^{i,j}_k||_{L^3(B_{6/8}(0))} \leq C||D^{i,j}_k||_{L^2(B_{6/8}(0))}
$$

for a.e. t.

Then we implement Hölder inequality, Sobolev inequality and Riesz-Hölder inequality to obtain

$$
\begin{aligned}
(R_4^k) \le & \lambda_k \sum_{i,j=1}^{D} \int_{-(6/8)^2}^{0} d\bar{t} ||\nabla(v_k^i \phi)||_{L^2(B_{6/8}(0))} ||(v_k^j - v^j)\phi||_{L^6(B_{6/8}(0))} \\
& \times ||\nabla(\triangle^{-1} D_k^{i,j})||_{L^3(B_{6/8}(0))} \\
\le & C\lambda_k \sum_{i,j=1}^{D} \int_{-(6/8)^2}^{0} d\bar{t} ||\nabla(v_k^i \phi)||_{L^2(B_{6/8}(0))} ||\nabla((v_k^j - v_j)\phi)||_{L^2(B_{6/8}(0))} ||D_k^{i,j}||_{L^2(B_{6/8}(0))}.
\end{aligned}
$$

As in the same way as in the estimates of (R_3^k), we conclude the estimates of (R_4^k):

$$
\begin{aligned}
(R_4^k) \le & C\lambda_k \operatorname*{ess\cdot sup}_{-(6/8)^2 < t \le 0} \Big(||\nabla v_k \phi||_{L^2(B_{6/8}(0))} + ||v_k \nabla \phi||_{L^2(B_{6/8}(0))} \Big) \\
& \times \operatorname*{ess\cdot sup}_{-(6/8)^2 < t \le 0} \Big(||\nabla v_k \phi||_{L^2(B_{6/8}(0))} + ||v_k \nabla \phi||_{L^2(B_{6/8}(0))} \\
& \quad + ||\nabla v \phi||_{L^2(B_{6/8}(0))} + ||v \nabla \phi||_{L^2(B_{6/8}(0))} \Big) \\
& \times \sum_{i,j=1}^{D} ||D_k^{i,j}||_{L^2(Q_{6/8}(0))} \le C\lambda_k. \qquad (2.44)
\end{aligned}
$$

From (2.40), (2.41), (2.43) and (2.44), we conclude

$$
\int_{Q_{4/8}(0)} |\nabla(v_k(\bar{z}) - v(\bar{z}))|^2 d\bar{z} = O(\lambda_k),
$$

which completes the proof of Lemma 2.6. ☐

CHAPTER 3

Evolutional Ginzburg-Landau mappings

1. Introduction

In chapter I, section 5, we observed that to construct "a weakly evolutional harmonic map into a sphere," Chen [**19**] plied a penalty scheme. The corresponding elliptic system:

$$\triangle u+\lambda(|u|^2-1)u=0 \qquad u\colon B_1(0)\to\mathbb{R}^{\hat{D}},\lambda\in\mathbb{R}_+$$

are called "Ginzburg-Landau systems," which stem from superconductors and superfluids such as helium II in physics. For the physical background, ask to Bethuel-Brezis-Hélein [**5**] and Neu [**88**]. Main interest of Ginzburg-Landau systems in physics, is to investigate the zero points of it because zero points are translated into the normal state in the superconductivity state. In this chapter we mainly study the relation between zero sets of solutions of Ginzburg-Landau systems and the singular set of weakly evolutional harmonic maps into a sphere that are constructed by passing to the limit of a subsequence of $\lambda\to\infty$ of u_λ which is a solution to (3.1), (3.2) and (3.3). We start this chapter by stating our problem exactly:

Let d and D be positive integers greater than or equal to 3 and λ a positive integer. Suppose that $\Omega\subset\mathbb{R}^d$ is a domain. For any fixed positive number T, the parabolic cylinder Q_T is defined by $Q_T=(0,T)\times\Omega\subset\mathbb{R}^{1+d}$.

The evolutional Ginzburg-Landau mappings $u_\lambda=(u_\lambda^1,\cdots,u_\lambda^D):Q_T\to\mathbb{R}^D$ are given as the solutions to the following parabolic systems:

$$\frac{\partial u_\lambda}{\partial t}=\triangle u_\lambda-\lambda(|u_\lambda|^2-1)u_\lambda \quad \text{for} \quad z=(t,x)\in Q_T, \tag{3.1}$$

$$u_\lambda(0,x)=u_0(0,x) \qquad \text{for} \quad t=0, \quad x\in\Omega, \tag{3.2}$$

$$u_\lambda(t,x)=u_0(t,x) \qquad \text{for} \quad t\in[0,T), \quad x\in\partial\Omega \tag{3.3}$$

where $u_0=(u_0^1,\cdots,u_0^D):\overline{Q_T}\to\mathbb{R}^D$ with $|u_0|=\sqrt{\sum_{i=1}^D|u_0^i|^2}=1$ for a.e. $z=(t,x)\in\partial Q_T$.

This system (3.1) is the L^2-gradient flow for the Ginzburg-Landau energy

$$E_\lambda(u) := \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \frac{\lambda}{2}\left(|u|^2 - 1\right)^2\right) dx. \tag{3.4}$$

In the following, we call $e_\lambda(u_\lambda)$ the energy-density of the evolutional Ginzburg-Landau mappings defined by

$$e_\lambda(u_\lambda) = \frac{1}{2}(|\nabla u_\lambda|^2 + \frac{\lambda}{2}\left(|u_\lambda|^2 - 1\right)^2). \tag{3.5}$$

This chapter studies the evolutional Ginzburg-Landau mappings on $Q_T = (0,T)\times\Omega$ when Ω is a bounded star-shaped domain with respect to the origin in $\mathbb{R}^d$ $(d \geq 3)$ with the following properties:

(i) $\partial\Omega$ is of C^4-boundary.
(ii) $\gamma(x) := \inf_{\lambda>0, \lambda x \notin \partial\Omega} \lambda x$ belongs to $C^1(\overline{\Omega})$.
(iii) The inner product between each point $x \in \partial\Omega$ and the outward normal unit vector ν_x at $x \in \partial\Omega$, denoted by $\langle x, \nu_x\rangle$, is greater than 2.

We prove that the energy density $e_\lambda(u_\lambda)$ of the evolutional Ginzburg-Landau mappings u_λ is uniformly bounded with respect to the parameter λ if the scaled energy of the evolutional Ginzburg-Landau mappings is uniformly small. After it, we indicate that these energy estimates much concern the behavior of the evolutional Ginzburg-Landau mappings near the so-called cluster set of zero points of the evolutional Ginzburg-Landau mappings u_λ. The uniform bounded estimates to an energy density $e_\lambda(u_\lambda)$ may be firstly obtained by Chen-Struwe [**24**] in case of $\Omega = \mathbb{R}^d$ with $d \geq 2$ and $D \geq 2$.

This chapter is organized as follows: In section 2, we mention existence and uniqueness results to our initial - boundary value problem. Section 3 collects the technical auxiliary lemmas: We establish a fundamental energy estimate, a Bochner type formula, the monotonicity inequality for the energy density and a Fefferman-Phong type inequality; Section 4 first applies ϵ-regularity theory to the scaled energy of the evolutional Ginzburg-Landau mappings. The proof of the theorem above is performed by reductio ad absurdum. Successively we discuss a certain estimate related to a behavior near zero sets of the evolutional Ginzburg-Landau mappings. Finally, we study an alternative approach to prove the uniform boundedness of $e_\lambda(u_\lambda)$ on the cylinder where the scaled energy is uniformly small with respect to λ. The benefit of this method is the point that we can explicitly estimate a certain constant appeared in the statement of the theorem. We state this result as the third theorem.

2. Existence and uniqueness

In this section we first formulates our problem exactly and we next establish the existence and uniqueness theorems to the problem. Set d and D be positive numbers greater than 2. Let u_0 be a map belonging to $C^0(0,T;W^{1,2}(\Omega;\mathbb{R}^D))$ with

$u_0|_{\partial\Omega}$ (A trace to $\partial\Omega$) $\in C^\infty([0,T)\times\partial\Omega;\mathbb{R}^D)$ and $|u_0| = \sqrt{\sum_{i=1}^D (u_0^i)^2} = 1$ on ∂Q_T.

DEFINITION 3.1. If we say $u_\lambda : Q_T \to \mathbb{R}^D$ to be evolutional Ginzburg-Landau system. a smooth map u_λ satisfies (3.1) classically, (3.2) and (3.3) in the sense that $\lim_{t\to+0}||u_\lambda(t,\cdot)-u_0(t,\cdot)||_{L^2(\Omega)}=0$ and $u_\lambda(t,\cdot)-u_0(t,\cdot)\in \overset{\circ}{H}{}^{1,2}(\Omega;\mathbb{R}^D)$ for a.e. $t\in(0,T)$.

To the initial-boundary value problems (3.1), (3.2) and (3.3), we show

THEOREM 3.1. *There uniquely exists the evolutional Ginzburg-Landau system u_λ such that*

$$u_\lambda \in C^\infty([0,T)\times\overline{\Omega}), \tag{3.6}$$

$$|u_\lambda| \le 1 \quad \text{in} \quad Q_T, \tag{3.7}$$

$$\begin{aligned} &\frac{1}{2}\int_{Q_T}\left|\frac{\partial u_\lambda}{\partial t}(z)\right|^2 dz + \frac{1}{4}\operatorname*{ess\cdot sup}_{0\le t\le T}\int_\Omega e_\lambda(u_\lambda)(t,x)\,dx \\ &\le \frac{\epsilon_0}{2}\int_0^T dt\int_{\partial\Omega}\big(\langle x,\nu\rangle + |x|^2\big)\,|\nabla_{\tan}u_0(z)|^2\, d\mathcal{H}^{d-1} \\ &+\frac{1}{\epsilon_0}\int_0^T dt\int_{\partial\Omega}\left|\frac{\partial u_0}{\partial t}(z)\right|^2 d\mathcal{H}^{d-1} + \int_\Omega |\nabla u_0(0,x)|^2\,dx \end{aligned} \tag{3.8}$$

with $\epsilon_0 = \min(1/(2\mathrm{diam}(\Omega)^2),\ 1/(1+||\nabla\gamma||^2_{L^\infty}\ \mathrm{diam}(\Omega)^2))$.

Proof of Theorem 3.1

For the time being, suppose that evolutional Ginzburg-Landau mappings exist and these satisfy

$$\lim_{h\searrow 0}\sup_{0<t\le h}\int_\Omega |\nabla u_\lambda(t,x)|^2\,dx = \int_\Omega |\nabla u_0(0,x)|^2\,dx. \tag{3.9}$$

Then we preliminarily verify the uniqueness, the boundedness (3.7) and the fundamental energy inequalities (3.8). To this end, assume the existences of two evolutional Ginzburg-Landau mappings u_λ and v_λ: these maps u_λ and v_λ satisfy

$$\frac{\partial u_\lambda}{\partial t}(z) = \triangle u_\lambda(z) - \lambda(|u_\lambda(z)|^2-1)u_\lambda(z) \qquad \text{in} \quad Q_T, \tag{3.10}$$

$$\frac{\partial v_\lambda}{\partial t}(z) = \triangle v_\lambda(z) - \lambda(|v_\lambda(z)|^2-1)v_\lambda(z) \qquad \text{in} \quad Q_T, \tag{3.11}$$

$$u_\lambda(0,x) = v_\lambda(0,x) = u_0(0,x) \qquad \text{in} \quad \{0\}\times\Omega, \tag{3.12}$$

$$u_\lambda(z)=v_\lambda(z)=u_0(t,x) \quad \text{in} \quad (0,T)\times\partial\Omega. \tag{3.13}$$

Subtracting (3.11) from (3.10), multiplying $\phi\in C_0^\infty(Q_T;\mathbb{R}^D)$ and integrating it over Q_T, we have

$$\begin{aligned}&\int_{Q_T}\langle\frac{\partial}{\partial t}(u_\lambda(z)-v_\lambda(z)),\phi(z)\rangle dz+\int_{Q_T}\langle\nabla(u_\lambda(z)-v_\lambda(z)),\nabla\phi(z)\rangle dz\\ &=-\lambda\int_{Q_T}\langle(|u_\lambda(z)|^2-1)u_\lambda(z)-(|v_\lambda(z)|^2-1)v_\lambda(z),\phi(z)\rangle dz.\end{aligned}\tag{3.14}$$

Noting $u_\lambda-v_\lambda=0$ on ∂Q_T, we substitute ϕ in (3.14) for $(u_\lambda-v_\lambda)\ e^{-2\lambda t}$ to obtain

$$\begin{aligned}&\frac{1}{2}\int_{Q_T}\frac{\partial}{\partial t}|u_\lambda(z)-v_\lambda(z)|^2e^{-2\lambda t}dz\\ &+\int_{Q_T}|\nabla(u_\lambda(z)-v_\lambda(z))|^2e^{-2\lambda t}dz\\ &=-\lambda\int_{Q_T}\big(|u_\lambda(z)|^4+|v_\lambda(z)|^4\\ &\qquad-(|u_\lambda(z)|^2+|v_\lambda(z)|^2)\langle u_\lambda(z),v_\lambda(z)\rangle\big)e^{-2\lambda t}dz\\ &+\lambda\int_{Q_T}|u_\lambda(z)-v_\lambda(z)|^2e^{-2\lambda t}dz.\end{aligned}\tag{3.15}$$

By using Schwarz inequality to the 1st term on the right-hand side, we arrive at

$$\begin{aligned}&\frac{1}{2}\int_{\Omega}|u_\lambda(T,x)-v_\lambda(T,x)|^2e^{-2\lambda T}dx+\int_{Q_T}|\nabla(u_\lambda(z)-v_\lambda(z))|^2e^{-2\lambda t}dz\\ &-\frac{\lambda}{2}\int_{Q_T}\big||u_\lambda(z)|^2-|v_\lambda(z)|^2\big|^2e^{-2\lambda t}dz\le 0.\end{aligned}\tag{3.16}$$

Thus, we deduce $u_\lambda\equiv v_\lambda$ in Q_T.

Second, we show (3.7): We multiply (3.1) by $(|u_\lambda|^2-1)^{(0)}\ u_\lambda$ and integrate it on Q_T with respect to z. Then by recalling $|u_0|=1$ on ∂Q_T, we obtain

$$\begin{aligned}&\frac{1}{4}\int_{\Omega\cap\{|u_\lambda|\ge 1\}}\big(|u_\lambda(T,x)|^2-1\big)^2dx\\ &+\int_{Q_T\cap\{|u_\lambda|\ge 1\}}|\nabla u_\lambda(z)|^2(|u_\lambda(z)|^2-1)^{(0)}dz\\ &+\frac{1}{2}\int_{Q_T\cap\{|u_\lambda|\ge 1\}}\big|\nabla|u_\lambda(z)|^2\big|^2dz\end{aligned}$$

$$+\lambda\int_{Q_T\cap\{|u_\lambda|\geq 1\}}\left(|u_\lambda(z)|^2-1\right)^2|u_\lambda(z)|^2\,dz=0.$$

Thus we conclude $\mathcal{L}^{d+1}\left(\{z\in Q_T;|u_\lambda|\geq 1\}\right)=0$: $|u_\lambda|\leq 1$ for a.e. $z\in Q_T$. Since $u_\lambda\in C^2(Q_T)$, $|u_\lambda|\leq 1$ on $z\in Q_T$.

Next, we prove (3.8). For this purpose, we first estimate $\int_0^T dt\int_{\partial\Omega}|\partial u_\lambda/\partial\nu(t,x)|^2\,d\mathcal{H}^{d-1}$: We multiply (3.1) by $\partial u_\lambda/\partial t\ (\gamma^2(x)-|x|^2)/2+\langle x,\nabla\rangle\ u_\lambda$, integrating it on $(h,t)\times\Omega\subset Q_T$ over z to imply

$$\begin{aligned}
0=&\frac{1}{2}\int_h^t dt\int_\Omega\left|\frac{\partial u_\lambda}{\partial t}(z)\right|^2(\gamma^2(x)-|x|^2)\,dx\\
&-\frac{1}{2}\int_h^t dt\int_\Omega\left\langle\triangle u_\lambda(z),\frac{\partial u_\lambda}{\partial t}(z)\right\rangle(\gamma^2(x)-|x|^2)\,dx\\
&+\frac{\lambda}{8}\int_h^t dt\int_\Omega\frac{\partial}{\partial t}\left(|u_\lambda(z)|^2-1\right)^2(\gamma^2(x)-|x|^2)\,dx\\
&-\int_h^t dt\int_\Omega\langle\triangle u_\lambda(z),\langle x,\nabla\rangle u_\lambda(z)\rangle\,dx\\
&+\frac{\lambda}{4}\int_h^t dt\int_\Omega\langle x,\nabla\rangle\left(|u_\lambda(z)|^2-1\right)^2dx.
\end{aligned}$$

Repeating the integral by parts and taking the limit $h\searrow 0$ and noting $\nabla=\nu\partial/\partial\nu+\nabla_{\tan}$, we have

$$\begin{aligned}
0=&\frac{1}{2}\int_{Q_t}\left|\frac{\partial u_\lambda}{\partial t}(z)\right|^2(\gamma^2(x)-|x|^2)\,dz\\
&+\frac{1}{2}\int_\Omega e_\lambda(u_\lambda)(z)(\gamma^2(x)-|x|^2)\,dx-\frac{1}{4}\int_\Omega|\nabla u_0(h,x)|^2(\gamma^2(x)-|x|^2)\,dx\\
&+\int_{Q_t}\left\langle\langle\nabla\gamma(x),\nabla\rangle u_\lambda(z),\frac{\partial u_\lambda}{\partial t}(z)\right\rangle\gamma(x)\,dz\\
&+\frac{2-d}{2}\int_{Q_t}|\nabla u_\lambda(z)|^2dz-\frac{\lambda d}{4}\int_{Q_t}\left(|u_\lambda(z)|^2-1\right)^2dz\\
&-\int_0^t dt\int_{\partial\Omega}\langle x,\nu\rangle\left|\frac{\partial u_\lambda}{\partial\nu}(z)\right|^2d\mathcal{H}^{d-1}-\int_0^t dt\int_{\partial\Omega}\left\langle\frac{\partial u_\lambda}{\partial\nu}(z),\langle x,\nabla_{\tan}\rangle u_0(z)\right\rangle d\mathcal{H}^{d-1}\\
&+\frac{1}{2}\int_0^t dt\int_{\partial\Omega}\langle x,\nu\rangle|\nabla u_\lambda(z)|^2d\mathcal{H}^{d-1}.
\end{aligned}$$

Thus we infer

$$
\begin{aligned}
&\frac{1}{2}\int_0^t dt \int_{\partial\Omega} \left|\frac{\partial u_\lambda}{\partial \nu}(z)\right|^2 d\mathcal{H}^{d-1} \\
&\leq \frac{\mathrm{diam}(\Omega)^2}{2}\int_\Omega e_\lambda(u_\lambda)(t,x)\,dx \\
&+\frac{1+||\nabla\gamma||_{L^\infty}^2}{2}\mathrm{diam}(\Omega)^2\int_{Q_t}\left|\frac{\partial u_\lambda}{\partial t}(z)\right|dz \\
&+\frac{3-d}{2}\int_{Q_t}|\nabla u_\lambda(z)|^2 dz - \frac{\lambda d}{4}\int_{Q_t}\left(|u_\lambda(z)|^2-1\right)^2 dz \\
&+\frac{1}{2}\int_0^t dt\int_{\partial\Omega}\left(\langle x,\nu\rangle + |x|^2\right)|\nabla_{\tan} u_0(z)|^2 d\mathcal{H}^{d-1}.
\end{aligned} \tag{3.17}
$$

By using (3.17), we complete (3.8). Similarly as above, a multiplier of (3.1) by $\partial u_\lambda/\partial t$, an integration of it over $(h,t)\times\Omega\subset Q_T$, the integral by parts and the limit $h\searrow 0$ read

$$
\begin{aligned}
&\int_{Q_t}\left|\frac{\partial u_\lambda}{\partial t}(z)\right|^2 dz + \frac{1}{2}\int_\Omega e_\lambda(u_\lambda)(t,x)\,dx \\
&=\int_0^t dt\int_{\partial\Omega}\left\langle \frac{\partial u_\lambda}{\partial\nu}(z), \frac{\partial u_0}{\partial t}(z)\right\rangle dx + \frac{1}{2}\int_\Omega |\nabla u_0(0,x)|^2 dx \\
&\leq \frac{\epsilon_0}{2}\int_0^t dt\int_{\partial\Omega}\left|\frac{\partial u_\lambda}{\partial\nu}(z)\right|^2 d\mathcal{H}^{d-1} + \frac{1}{2\epsilon_0}\int_0^t dt\int_{\partial\Omega}\left|\frac{\partial u_0}{\partial t}(z)\right|^2 d\mathcal{H}^{d-1} \\
&+\frac{1}{2}\int_\Omega|\nabla u_0(0,x)|^2 dx
\end{aligned} \tag{3.18}
$$

for any positive ϵ_0.

By substituting (3.17) for (3.18) and taking ϵ_0 as

$$
\epsilon_0 = \frac{1}{2(1+||\nabla\gamma||_{L^\infty}^2)\mathrm{diam}(\Omega)^2},
$$

we have

$$
\begin{aligned}
&\frac{1}{2}\int_{Q_t}\left|\frac{\partial u_\lambda}{\partial t}(z)\right|^2 dz + \frac{1}{2}\int_\Omega e_\lambda(u_\lambda)(t,x)\,dx \\
&\leq \frac{\epsilon_0}{2}\left(\mathrm{diam}(\Omega)+\mathrm{diam}(\Omega)^2\right)\int_0^t dt\int_{\partial\Omega}|\nabla_{\tan}u_0(z)|\,d\mathcal{H}^{d-1} \\
&+\frac{1}{2\epsilon_0}\int_0^t dt\int_{\partial\Omega}\left|\frac{\partial u_0}{\partial t}(z)\right|^2 d\mathcal{H}^{d-1} + \frac{1}{2}\int_\Omega |\nabla u_0(0,x)|^2 dx
\end{aligned}
$$

for any $t \in (0,T)$.

From now on, we prove the existence of the evolutional Ginzburg-Landau system: We begin with constructing an approximation map of the evolutional Ginzburg-Landau system: Let $v^{(0)}$ be the solution of

$$\begin{aligned}
&\frac{\partial v^{(0)}}{\partial t}(z) - \triangle v^{(0)}(z) = 0 \qquad \text{in} \quad Q_T, \\
&v^{(0)}(0,x) = u_0(0,x) \qquad \text{at} \quad \{0\} \times \Omega, \\
&v^{(0)}(t,x) = u_0(t,x) \qquad \text{on} \quad (0,T) \times \partial\Omega.
\end{aligned}$$

Set $X = \{v \in C^\infty \ ((0,t_\lambda) \times \Omega) \ ; ||v||_{L^\infty((0,t_\lambda)\times\Omega)} \leq 1, \ v(z)\big|_{\partial\Omega} = 0, \ v(0,x) = 0\}$ with a certain positive number t_λ depending only on λ. Assume that $v^{(j)}$ $(j = 0,1,\ldots,k-1)$ exists in X.

Then $v^{(k)}$ is decided by

$$\begin{aligned}
v^{(k)}(z) = \mathcal{A}v^{(k-1)}(z) = -\lambda \int_0^t ds \int_\Omega U(t-s;x,y)(|v^{(k-1)}(s,y) + v^{(0)}(s,y)|^2 - 1) \\
\times \left(v^{(k-1)}(s,y) + v^{(0)}(s,y)\right) dy,
\end{aligned}$$

where $U(t;x,y)$ $(t>0, x,y \in \bar{\Omega})$ is the fundamental solution of

$$\begin{aligned}
&\frac{\partial w}{\partial t}(z) - \triangle w(z) = 0 \qquad (t>0, x \in \Omega), \\
&w(t,x) = 0 \qquad \text{on} \quad (0,T) \times \partial\Omega.
\end{aligned}$$

By employing Ladyzhenskaya-Solonnikov-Uralceva [**75**] and Friedman [**45**, Theorem 6 in Chap III], a choice of t_λ implies that $\mathcal{A}$ is a contraction operator from X to X: If we define u_λ by $v^{(\infty)} + v^{(0)}$, then we easily show that the mapping u_λ satisfies (3.1), (3.2), (3.3) (3.6) and (3.9). By virtue of an energy inequality (3.8), we can extend u_λ constructed above in $(0,t_\lambda) \times \Omega$ to Q_T. $\square$

3. Theorems

After the preparation in the former chapter, we can state our main theorems of this chapter. To this end, we need to introduce more symbols:

(i) $\Phi^1_R(z_0)$ and $\Phi^2_R(z_0)$ will be given in Corollary 3.1.
(ii) For any positive number t, $\delta(t) := \sqrt{(12(d-2)+1)|\log t|}$.
(iii) $\mathbf{N} := \cup_{\lambda=1}^\infty \mathbf{N}_\lambda$ with $\mathbf{N}_\lambda := \{z_0 \in Q_T; u_\lambda(z_0) = 0\}$.
(iv) If we say a point z_0 to belong to $\mathbf{N}^{\mathsf{CL}}$, it means the following: there exist sequences of positive integers $\lambda(j)$ and of points $z_{\lambda(j)} \in \mathbf{N}_{\lambda(j)}$ with $\lambda(1) < \lambda(2) < \cdots < \infty$ such that $z_{\lambda(j)} \to z_0$ as $j \to \infty$.

(v) $\mathbf{Reg} := \bigcup_{\lambda_0=1}^{\infty} \bigcup_{R>0} \bigcap_{\lambda=\lambda_0}^{\infty} \{z_0 \in Q_T;\ \Phi_R^1(t_0+2(R/4)^2, x_0) < \epsilon_1$ with $(t_0+2(R/4)^2 - 4R^2, t_0+2(R/4)^2+4R^2) \times B_{(1+\delta(R))R}(x_0) \subset\subset Q_T\}$ for a positive number ϵ_1.

THEOREM 3.2. *Then there exists a positive constant ϵ_1 depending only on d such that if for a point $z_0 = (t_0, x_0)$ and a cylinder $(t_0+(R/4)^2-4R^2, t_0+(R/4)^2+4R^2) \times B_{(1+\delta(R))R}(x_0) \subset\subset Q_T$, the evolutional Ginzburg-Landau system u_λ satisfies*

$$\Phi_R^1(t_0+(R/4)^2, x_0) < \epsilon_1 \quad \text{then,} \quad \sup_{Q_{R/4}(z_0)} e_\lambda(u_\lambda) < 64R^{-2}. \tag{3.19}$$

Next theorem is

THEOREM 3.3.

$$\mathbf{N}^{\mathsf{CL}} \bigcap \mathbf{Reg} = \emptyset, \qquad \text{i.e.} \quad \mathbf{N}^{\mathsf{CL}} \subset \complement\mathbf{Reg}. \tag{3.20}$$

REMARK 3.1. By using (3.20), as in the similar way as in the proof of Chen-Struwe [**24**], we find

$$\mathcal{H}^d(\mathbf{N}^{\mathsf{CL}}) \leq \mathcal{H}^d(\complement\mathbf{Reg}) < \infty.$$

REMARK 3.2. Let B^d be the d-dimensional unit ball and u_0 an initial-boundary mapping of the evolutional Ginzburg-Landau system satisfying

$$\begin{aligned}
&u_0 \in W^{1,2}(B^d; S^{d-1}),\\
&u_0\,|_{\partial B^d} \in C^\infty(\partial B^d; S^{d-1}),\\
&u_0(\partial B^d)\,\text{is not homotopic to a constant map.}
\end{aligned}$$

Here note that by Bethuel-Zheng [**9**], such u_0 exists. From the theory of degree of a mapping (See for example, Nirenberg [**89**].), the evolutional Ginzburg-Landau system u_λ must have at least one zero point in B^d at each $t \in (0,T)$ if $u_\lambda|_{\partial B^d} = u_0|_{\partial B^d}$ has a non-zero degree. This shows $\mathbf{N}_\lambda \neq \emptyset$ and furthermore if u_λ has a finite zero point in B^d at each t $(0<t<T)$, $\mathbf{N}_\lambda$ may be finite in 2-dimensional Hausdorff dimension with respect to the parabolic metric. While, as showing in the previous remark, $\mathbf{N}^{\mathsf{CL}}$ has at most finite d-dimensional Hausdorff measure. These observations draw me to the question how sharply the Hausdorff measure of $\mathbf{N}^{\mathsf{CL}}$ depends on the Hausdorff dimension of the zero set of the evolutional Ginzburg-Landau system with the initial-boundary value. Recently several literatures study a zero set of solutions of various equations. For instance, to second-order elliptic equations, it is studied by Caffarelli-Friedman [**13**] and Hardt-Simon [**62**] and to the heat equations on an analytic compact Riemann manifold, it by Lin [**80**].

From Lemma 3.4 stated below, Giaquinta [**46**] and Ladyzhenskaya-Solonnikov-Uralceva [**75**], the 3rd theorem follows

THEOREM 3.4. *Let u_λ be the evolutional Ginzburg-Landau system. Then for any point $z_0 = (t_0, x_0)$ and a cylinder $(t_0 + (R/4)^2 - 4R^2, t_0 + (R/4)^2 + 4R^2) \times B_{(1+\delta(R))R}(x_0)$,*

$$\Phi_R^2(z_0) \le \min\left(\frac{1}{64\cdot 24\cdot C_{\mathrm{FP}}}\cdot\frac{d}{d+2}, (\frac{1}{C_{\mathrm{RH}}^3})^{d/\delta_0(d+2)}(\frac{1}{4})^{(d+2)/(d+1)(1+\delta_0)/\delta_0^2}\frac{1}{C_{\mathrm{RH}}^1}\right)$$

where

$$C_{\mathrm{RH}}^1 = \frac{C_{\mathrm{SO}}}{\tau^2(1-\tau^2)},$$
$$C_{\mathrm{RH}}^2 = \frac{C_{\mathrm{SO}}}{d(2\tau^2-1)}\left(\frac{3d}{d+2}\cdot\frac{8\tau}{1-\tau} + 4d\frac{16\tau^2}{(1-\tau^2)} + \frac{8d}{(d+2)^2}\cdot\frac{16\tau^2}{(1-\tau)^2}\right.$$
$$\left. + \frac{d}{(d+2)^2}\cdot\frac{16\tau^2}{(1-\tau)^2} + \frac{64}{d+2}\right),$$
$$C_{\mathrm{RH}}^3 = 4^{1/d}\big(4^4 C_{\mathrm{RH}}^2 + \frac{(64d)}{(d+2)}3^{2(1+2/d)}\big),$$

$\tau = 1/2(1+\sqrt{2})$, *C_{FP} is the nonlinear Fefferman-Phong's constant appears in Lemma 3.4 and C_{SO} is the best Sobolev constant,*
implies

$$\sup_{Q_{R/4}(z_0)} e_\lambda(u_\lambda)(z) \le 2/R^2. \tag{3.21}$$

4. Technical lemmas

We review a few technical lemmas which will play a crucial role on the proof of our results. The first lemma is as follows:

LEMMA 3.1. *Let u_λ be the evolutional Ginzburg-Landau mappings. Then we have*

$$\frac{\partial |u_\lambda|^2}{\partial t} - \triangle |u_\lambda|^2 + 2\lambda(|u_\lambda|^2 - 1)|u_\lambda|^2 \le 0, \tag{3.22}$$
$$\left(\frac{\partial}{\partial t} - \triangle\right) e_\lambda(u_\lambda) + f_\lambda(u_\lambda) \le 16 e_\lambda^2(u_\lambda) \tag{3.23}$$
$$\text{with} \quad f_\lambda(u_\lambda) = \frac{1}{2}\left(|\nabla^2 u_\lambda|^2 + 4\lambda |u_\lambda|^2 \big|\nabla |u_\lambda|\big|^2\right).$$

We successively present the second lemma:

LEMMA 3.2 (A local energy inequality). *Let $z_0 = (t_0, x_0)$ be a point in Q_T and set positive numbers t_1, t_2 and r_1, r_2 with $0 < t_0 - t_2 < t_0 - t_1 \le t_0$ and $r_1 < r_2$ with $C_{2r_2}(x_0) \subset\subset \Omega$. Then the evolutional Ginzburg-Landau system u_λ satisfies*

$$\tag{3.24} \frac{1}{2}\int_{t_0-t_1}^{t_0} dt\int_{B_{r_1}} \left|\frac{\partial u_\lambda}{\partial t}(z)\right|^2 dx + \int_{B_{r_1}} e_\lambda(u_\lambda)(t,x)\,dx \le \left[\frac{32}{(r_2-r_1)^2}+\frac{4}{(t_2-t_1)}\right]\int_{t_0-t_2}^{t_0} dt\int_{B_{r_2}} e_\lambda(u_\lambda)(z)\,dx,$$

$$\tag{3.25} \frac{1}{2}\int_{t_0-t_1}^{t_0} dt\int_{C_{r_1}} \left|\frac{\partial u_\lambda}{\partial t}(z)\right|^2 dx + \int_{C_{r_1}} e_\lambda(u_\lambda)(t,x)\,dx \le \left[\frac{32}{(r_2-r_1)^2}+\frac{4}{(t_2-t_1)}\right]\int_{t_0-t_2}^{t_0} dt\int_{C_{r_2}} e_\lambda(u_\lambda)(z)\,dx$$

for any time t with $t_0-t_1 < t \le t_0$.

Next, we state the monotonicity inequality for the scaled energies of the evolutional Ginzburg-Landau mappings

LEMMA 3.3 (A monotonicity formula). *Let u_λ be the evolutional Ginzburg-Landau mappings Then the following inequality holds*:

$$\tag{3.26} \begin{aligned} &\frac{1}{{r_1}^d}\int_{t_0-r_1^2}^{t_0} dt\int_\Omega e_\lambda(u_\lambda)(z)e^{\frac{|x-x_0|^2}{4(t-t_0)}}\,dx \\ &\le \frac{1}{{r_2}^d}\int_{t_0-r_2^2}^{t_0} dt\int_\Omega e_\lambda(u_\lambda)(z)e^{\frac{|x-x_0|^2}{4(t-t_0)}}\,dx \\ &+\frac{r_2}{d+1}\left(\sup_{x\in\partial\Omega}\left(|x|+\frac{|x|^2}{\epsilon_0}\right)\sup_{z\in\partial Q_T}|\nabla_{\tan}u_0(z)|^2 \right.\\ &\qquad \left. +2\sup_{z\in\partial Q_T}\left|\frac{\partial u_0}{\partial t}(z)\right|^2\right)\mathcal{L}^{d-1}(\partial\Omega) \\ &\qquad \times\sup_{0\le s<\infty}\max(s^{1-d},s^{5-d})\exp\left(-\frac{1}{4s^2}\inf_{x\in\partial\Omega}|x|^2\right) \end{aligned}$$

for any point $z_0{=}(t_0,x_0) \in Q_T$ and any positive numbers r_1, r_2 with $r_1{<}r_2$. Here $\epsilon_0 = \min_{x\in\partial\Omega}\,(\langle x,\nu_x\rangle/2-1)$.

To prove the next lemma and Theorem 3.4, we prepare the following corollary; this corollary is easily derived from Lemma 3.2 and Lemma 3.3:

COROLLARY 3.1. *Give two parabolic cylinders $Q_R(z_0) \subset (t_0-4R^2,t_0+4R^2) \times B_{(1+\delta(R))R}(x_0) \subset\subset Q_T$ with $R<e^{-d/(d+2)}$. For any point $\bar z_0 = (\bar t_0,\bar x_0) \in Q_R(z_0)$ and any positive number $r<R$, we have*

$$r^{-d}\int_{Q_r(\bar z_0)} e_\lambda(u_\lambda)(z)\,dz$$

$$\leq\left(8+\frac{2^{1-d}}{3}\right)e^{1/4}R^{-d}\int_{t_0-3R^2}^{t_0+R^2}dt\int_{B_{(1+\delta(R))R}(x_0)}e_\lambda(u_\lambda)(z)\,dx$$
$$+\left(8+\frac{2^{1-d}}{3}\right)e^{1/4}\frac{R^{2-d}}{2}\exp\left(-\frac{\delta^2(R)}{4}\right)\operatorname*{ess\cdot sup}_{0\leq t\leq T}\int_\Omega e_\lambda(u_\lambda)(t,x)\,dx$$
$$+\frac{R}{2}\left(\sup_{x\in\partial\Omega}\left(|x|+\frac{|x|^2}{\epsilon_0}\right)\sup_{z\in\partial Q_T}|\nabla_{\tan}u_0(z)|^2\right.$$
$$\left.+2\sup_{z\in\partial Q_T}\left|\frac{\partial u_0}{\partial t}(z)\right|^2\right)\mathcal{L}^{d-1}(\partial\Omega)$$
$$\times\sup_{0\leq s<\infty}\max(s^{1-d},s^{5-d})\exp\left(-\frac{1}{4s^2}\inf_{x\in\partial\Omega}|x|^2\right)\left(=:\Phi^1_R(z_0)\right),\tag{3.27}$$

$$\operatorname*{ess\cdot sup}_{\bar t_0-r^2<t\leq\bar t_0}r^{2-d}\int_{C_r(\bar x_0)}e_\lambda(u_\lambda)(t,x)\,dx$$
$$\leq 36\sqrt{3}e^{1/4}R^{-d}\int_{t_0-4R^2}^{t_0+R^2}dt\int_{C_{(1+\delta(R))R}(x_0)}e_\lambda(u_\lambda)(z)\,dx$$
$$+10\sqrt{3}R^{2-d}\exp\left(-\frac{\delta^2(R)}{12}\right)\operatorname*{ess\cdot sup}_{0\leq t\leq T}\int_\Omega e_\lambda(u_\lambda)(t,x)\,dx$$
$$+5\sqrt{3}R\left(\sup_{x\in\partial\Omega}\left(|x|+\frac{|x|^2}{\epsilon_0}\right)\sup_{z\in\partial Q_T}|\nabla_{\tan}u_0(z)|^2\right.$$
$$\left.+2\sup_{z\in\partial Q_T}\left|\frac{\partial u_0}{\partial t}(z)\right|^2\right)\mathcal{L}^{d-1}(\partial\Omega)$$
$$\times\sup_{0\leq s\leq\infty}\max(s^{1-d},s^{5-d})\exp(-\frac{1}{4s^2}\inf_{x\in\partial\Omega}|x|^2)\left(:=\overline{\Phi}^2_R(z_0)\right)\tag{3.28}$$

for the number $\epsilon_0=\min_{x\in\partial\Omega}\left(\langle x,\nu_x\rangle/2-1\right)$.

By employing Corollary 3.1, we can prove the following inequality. This inequality can be regarded as a sort of the weighted Poincaré inequality. In the following convenience, we set $\Phi^2_R(z_0)=:\overline{\Phi}^2_R(z_0)+1/R^d\int_{D_R(z_0)}e_\lambda(u_\lambda)(z)\,dz$:

LEMMA 3.4 (Nonlinear Fefferman-Phong inequality). *Let* η *be a non-negative smooth function with a compact support on a cube* $D_R(z_0)$ *and* $\sup_{D_R(z_0)}\eta(z)\leq 1$. *Then, for the Ginzburg-Landau energy density* $e_\lambda(u_\lambda)$,

$$\int_{D_R(z_0)}\left|\left(\sqrt{e^{(\kappa)}_\lambda(u_\lambda)(z)}\right)^p\eta(z)\right|^{2+2/p}dz\leq C_{FP}\Phi^2_R(z_0)$$
$$\times\int_{D_R(z_0)}\left|\nabla\left(\left(\sqrt{e^{(\kappa)}_\lambda(u_\lambda)(z)}\right)^p\eta(z)\right)\right|^2dz\tag{3.29}$$

holds where $p=1{,}1+2/d$ *and* C_{FP} *is a positive constant independent of* κ, R, z_0, u_λ *and* η.

Proof of lemma 3.1

The inequality (3.22) directly follows from multiplying (3.1) by u_λ.

On the other hand, (3.23) can be proved as follows: Taking the gradients of the both sides of (3.1) and multiplying it by ∇u_λ, we have

$$\begin{aligned}&\frac{1}{2}\left(\frac{\partial}{\partial t}-\triangle\right)|\nabla u_\lambda|^2\\&+|\nabla^2 u_\lambda|^2+2\lambda|u_\lambda|^2\big|\nabla|u_\lambda|\big|^2+\lambda\left(|u_\lambda|^2-1\right)|\nabla u_\lambda|^2=0\quad\text{in}\,Q_T.\end{aligned}\tag{3.30}$$

The similar way as above, namely a multiplier of (3.1) by $\lambda(|u_\lambda|^2-1)\,u_\lambda$ leads to

$$\begin{aligned}&\frac{\lambda}{4}\left(\frac{\partial}{\partial t}-\triangle\right)\left(|u_\lambda|^2-1\right)^2+2\lambda|u_\lambda|^2\big|\nabla|u_\lambda|\big|^2\\&+\lambda\left(|u_\lambda|^2-1\right)|\nabla u_\lambda|^2+\lambda^2\left(|u_\lambda|^2-1\right)^2|u_\lambda|^2=0.\end{aligned}\tag{3.31}$$

Adding (3.30) with (3.31), we infer

$$\begin{aligned}&\frac{1}{2}\left(\frac{\partial}{\partial t}-\triangle\right)\left(|\nabla u_\lambda|^2+\frac{\lambda}{2}\left(|u_\lambda|^2-1\right)^2\right)\\&+|\nabla^2 u_\lambda|^2+4\lambda|u_\lambda|^2|\nabla|u_\lambda||^2\\&=-2\lambda\left(|u_\lambda|^2-1\right)|\nabla u_\lambda|^2-\lambda^2\left(|u_\lambda|^2-1\right)^2|u_\lambda|^2\\&\le\left(|u_\lambda|^2+\left(|u_\lambda|^2-1\right)^2\right)^{-1}|\nabla u_\lambda|^4+\lambda^2\left(|u_\lambda|^2-1\right)^4\\&\le 16\left(\frac{1}{2}|\nabla u_\lambda|^2+\frac{\lambda}{4}\left(|u_\lambda|^2-1\right)^2\right)^2.\end{aligned}\tag{3.32}$$

□

Proof of lemma 3.2

We multiply (3.1) by $\partial_t u_\lambda\eta^2\chi$, integrate it over $(t_0-t_2,t_0)\times B_{r_2}$ and $(t_0-t_2,t)\times B_{r_2}$ for any $t\in(t_0-t_2,t_0)$ with respect to z and perform the integral by parts to conclude (3.24). Here two smooth cut-off functions η and χ are given by

$$
\eta(|x|)=\begin{cases}1 & \text{in} \quad B_{r_1},\\ 0 & \text{outside} \quad B_{r_2},\end{cases}
$$
$$
0\le\eta(|x|)\le 1, \quad |\nabla\eta(|x|)|\le 2/(r_2-r_1),
$$
$$
\chi(t)=\begin{cases}1 & t_0-t_1<t\le t_0,\\ 0 & t\le t_0-t_2,\end{cases}
$$
$$
0\le\chi(t)\le 1, \quad |d\chi/dt(t)|\le 2/(t_2-t_1).
$$

We can prove the desired inequality (3.25) as in the same way as above. □

Proof of lemma 3.3

First of all, let us remark that our system (3.1) is invariant under translation $z\to z-z_0$, hence we may shift $z_0\to 0$.

We mean ϕ by

$$
\phi(z)=\frac{\partial u_\lambda}{\partial t}(z)2te^{\frac{|x|^2}{4t}}+\langle x,\nabla\rangle u_\lambda(z)e^{\frac{|x|^2}{4t}}.
$$

multiplier of (3.1) by ϕ implies

$$
\begin{aligned}
0=&\left|\frac{\partial u_\lambda}{\partial t}\right|^2 2te^{\frac{|x|^2}{4t}}+\left\langle\frac{\partial u_\lambda}{\partial t},\langle x,\nabla\rangle u_\lambda\right\rangle e^{\frac{|x|^2}{4t}}\\
&-\left\langle\triangle u_\lambda,\frac{\partial u_\lambda}{\partial t}\right\rangle 2te^{\frac{|x|^2}{4t}}-\langle\triangle u_\lambda,\langle x,\nabla\rangle u_\lambda\rangle e^{\frac{|x|^2}{4t}}\\
&+\frac{\lambda}{4}\frac{\partial}{\partial t}\left(|u_\lambda|^2-1\right)^2 2te^{\frac{|x|^2}{4t}}+\frac{\lambda}{4}\langle x,\nabla\rangle\left(|u_\lambda|^2-1\right)^2 2te^{\frac{|x|^2}{4t}}.
\end{aligned}\tag{3.33}
$$

We integrate (3.33) on $(-r^2,0)\times\Omega$ to obtain

$$
\begin{aligned}
0=&\int_{-r^2}^0 dt\int_\Omega\left|\frac{\partial u_\lambda}{\partial t}(z)\right|^2 2te^{\frac{|x|^2}{4t}}dz\\
&+\int_{-r^2}^0 dt\int_\Omega\langle\frac{\partial u_\lambda}{\partial t}(z),\langle x,\nabla\rangle u_\lambda(z)\rangle e^{\frac{|x|^2}{4t}}dx\\
&-\int_{-r^2}^0 dt\int_{\partial\Omega}\langle\frac{\partial u_\lambda}{\partial\nu}(z),\frac{\partial u_\lambda}{\partial t}(z)\rangle 2te^{\frac{|x|^2}{4t}}d\mathcal{H}^{d-1}\\
&+\frac{1}{2}\int_{-r^2}^0 dt\int_\Omega\frac{\partial}{\partial t}|\nabla u_\lambda(z)|^2 2te^{\frac{|x|^2}{4t}}dx
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{-r^2}^{0}dt\int_{\Omega}\langle\langle\nabla e^{\frac{|x|^2}{4t}},\nabla u_\lambda(z)\rangle,\frac{\partial u_\lambda}{\partial t}(z)\rangle 2t\,dx\\
&-\int_{-r^2}^{0}dt\int_{\partial\Omega}\langle\frac{\partial u_\lambda}{\partial \nu}(z),\langle x,\nabla\rangle u_\lambda(z)\rangle e^{\frac{|x|^2}{4t}}\,d\mathcal{H}^{d-1}\\
&+\frac{2-d}{2}\int_{-r^2}^{0}dt\int_{\Omega}|\nabla u_\lambda(z)|^2 e^{\frac{|x|^2}{4t}}\,dx\\
&-\frac{1}{2}\int_{-r^2}^{0}dt\int_{\Omega}\langle x,\nabla\rangle e^{\frac{|x|^2}{4t}}|\nabla u_\lambda(z)|^2 dx\\
&+\frac{1}{2}\int_{-r^2}^{0}dt\int_{\partial\Omega}\langle x,\nu_x\rangle|\nabla u_\lambda(z)|^2 e^{\frac{|x|^2}{4t}}\,d\mathcal{H}^{d-1}\\
&+\int_{-r^2}^{0}dt\int_{\Omega}\langle\langle\nabla e^{\frac{|x|^2}{4t}},\nabla u_\lambda(z)\rangle,\langle x,\nabla\rangle u_\lambda(z)\rangle dx\\
&+\frac{\lambda}{4}\int_{-r^2}^{0}dt\int_{\Omega}\frac{\partial}{\partial t}\left(|u_\lambda(z)|^2-1\right)^2 2te^{\frac{|x|^2}{4t}}\,dx\\
&-\frac{d\lambda}{4}\int_{-r^2}^{0}dt\int_{\Omega}\left(|u_\lambda(z)|^2-1\right)^2 e^{\frac{|x|^2}{4t}}\,dx\\
&-\frac{\lambda}{4}\int_{-r^2}^{0}dt\int_{\Omega}\langle x,\nabla\rangle e^{\frac{|x|^2}{4t}}\left(|u_\lambda(z)|^2-1\right)^2 dx\\
&=-2\int_{-r^2}^{0}dt\int_{\Omega}\left|\frac{\partial u_\lambda}{\partial t}(z)\sqrt{|t|}-\frac{\langle x,\nabla\rangle}{2\sqrt{|t|}}u_\lambda(z)\right|^2 e^{\frac{|x|^2}{4t}}\,dx\\
&-\int_{-r^2}^{0}dt\int_{\partial\Omega}\left|\frac{\partial u_0}{\partial t}(z)|t|-\frac{\partial u_\lambda}{\partial \nu}(z)\right|^2 e^{\frac{|x|^2}{4t}}\,d\mathcal{H}^{d-1}\\
&+\int_{-r^2}^{0}dt\int_{\partial\Omega}\left\langle\frac{\partial u_\lambda}{\partial \nu}(z),\frac{\partial u_\lambda}{\partial \nu}(z)-\langle x,\nabla\rangle u_\lambda(z)\right\rangle e^{\frac{|x|^2}{4t}}\,d\mathcal{H}^{d-1}\\
&+\frac{1}{2}\int_{-r^2}^{0}dt\int_{\partial\Omega}\langle x,\nu_x\rangle|\nabla u_\lambda(z)|^2 e^{\frac{|x|^2}{4t}}\,d\mathcal{H}^{d-1}\\
&+\int_{-r^2}^{0}dt\int_{\partial\Omega}\left|\frac{\partial u_0}{\partial t}(z)\right|^2 t^2 e^{\frac{|x|^2}{4t}}\,d\mathcal{H}^{d-1}\\
&+\int_{-r^2}^{0}dt\int_{\Omega}\frac{\partial e_\lambda}{\partial t}(u_\lambda)(z)\,2te^{\frac{|x|^2}{4t}}\,dx\\
&+(2-d)\int_{-r^2}^{0}dt\int_{\Omega}e_\lambda(u_\lambda)(z)e^{\frac{|x|^2}{4t}}\,dx\\
&\quad-\int_{-r^2}^{0}dt\int_{\Omega}\langle x,\nabla\rangle e^{\frac{|x|^2}{4t}}e_\lambda(u_\lambda)(z)\,dx
\end{aligned}
$$

$$-\frac{\lambda}{4}\int_{-r^2}^{0}dt\int_{\Omega}\left(|u_\lambda|^2-1\right)^2 e^{\frac{|x|^2}{4t}}\,dx. \tag{3.34}$$

Since

$$-\frac{\partial e^{\frac{|x|^2}{4t}}}{\partial t}2t-\langle x,\nabla\rangle e^{\frac{|x|^2}{4t}}=0,$$
$$\nabla u_\lambda=\nu_x\langle\nu_x,\nabla\rangle u_\lambda+\nabla_{\tan}u_\lambda,$$

using Schwarz's inequality, we arrive at

$$\begin{aligned}
&-2\int_{-r^2}^{0}dt\int_{\Omega}\left|\frac{\partial u_\lambda}{\partial t}(z)-\frac{\langle x,\nabla\rangle}{2|t|}u_\lambda(z)\right|^2|t|e^{\frac{|x|^2}{4t}}\,dx\\
&-\int_{-r^2}^{0}dt\int_{\partial\Omega}\left|\frac{\partial u_0}{\partial t}(z)|t|-\frac{\partial u_\lambda}{\partial\nu}(z)\right|^2 e^{\frac{|x|^2}{4t}}\,d\mathcal{H}^{d-1}\\
&+\int_{-r^2}^{0}dt\int_{\partial\Omega}\left(1+\frac{\epsilon_0}{2}-\frac{\langle x,\nu_x\rangle}{2}\right)\left|\frac{\partial u_\lambda}{\partial\nu}(z)\right|^2 e^{\frac{|x|^2}{4t}}\,d\mathcal{H}^{d-1}\\
&+\int_{-r^2}^{0}dt\int_{\partial\Omega}\left|\frac{\partial u_0}{\partial t}(z)\right|^2 t^2 e^{\frac{|x|^2}{4t}}\,d\mathcal{H}^{d-1}\\
&+\frac{1}{2}\int_{-r^2}^{0}dt\int_{\partial\Omega}\left(\langle x,\nu_x\rangle+\frac{1}{\epsilon_0}|x|^2\right)|\nabla_{\tan}u_0(z)|^2 e^{\frac{|x|^2}{4t}}\,d\mathcal{H}^{d-1}\\
&+\int_{\Omega}e_\lambda(u_\lambda)(z)2te^{\frac{|x|^2}{4t}}\,dx\Big|_{t=-r^2}^{0}-d\int_{-r^2}^{0}dt\int_{\Omega}e_\lambda(u_\lambda)(z)e^{\frac{|x|^2}{4t}}\,dx\geq 0
\end{aligned}\tag{3.35}$$

with any positive number ϵ_0.

Dividing (3.35) by r^{-d-1} and recalling a property of $\partial\Omega$, i.e. $1+\epsilon_0/2<\langle x,\nu_x\rangle/2$ on $x\in\partial\Omega$ as long as $\epsilon_0<2\min_{x\in\partial\Omega}\left(\langle x,\nu_x\rangle/2-1\right)$, we conclude

$$\begin{aligned}
&\frac{d}{dr}\left(r^{-d}\int_{-r^2}^{0}dt\int_{\Omega}e_\lambda(u_\lambda)(z)e^{\frac{|x|^2}{4t}}\,dx\right)\\
&+\frac{r^{-d-1}}{2}\int_{-r^2}^{0}dt\int_{\partial\Omega}\left(\langle x,\nu\rangle+\frac{1}{\epsilon_0}|x|^2\right)|\nabla_{\tan}u_0(z)|^2 e^{\frac{|x|^2}{4t}}\,d\mathcal{H}^{d-1}\\
&+r^{-d-1}\int_{-r^2}^{0}dt\int_{\partial\Omega}\left|\frac{\partial u_0}{\partial t}(z)\right|^2 t^2 e^{\frac{|x|^2}{4t}}\,d\mathcal{H}^{d-1}\geq 0.
\end{aligned}\tag{3.36}$$

We integrate (3.36) from r_1 to r_2 with respect to r to deduce the desired estimates. □

Proof of Corollary 3.1

Substituting respectively $\sqrt{2}r$, $\sqrt{2}R$ and $\bar{t}_0+r^2$ for r_1, r_2 and t_0 in Lemma 3.3 and noting $e^{-1/4} \le \exp\big(|x-\bar{x}_0|^2 \,/\, (4(t-(\bar{t}_0+r^2)))\big)$ in $\bar{t}_0-r^2 < t \le \bar{t}_0$, we have

$$
\begin{aligned}
& r^{-d}\int_{\bar{t}_0-r^2}^{\bar{t}_0} dt \int_{B_r(\bar{x}_0)} e_\lambda(u_\lambda)(z)\,dx \\
\le{}& e^{1/4} 2d/2 r^{-d}\int_{\bar{t}_0+r^2-(\sqrt{2}r)^2}^{\bar{t}_0+r^2} dt \int_{B_r(\bar{x}_0)} e_\lambda(u_\lambda)(z)\exp\left(\frac{|x-\bar{x}_0|^2}{4(t-(\bar{t}_0+r^2))}\right)dx \\
\le{}& e^{1/4} 2^{d/2} (\sqrt{2}R)^{-d}\int_{\bar{t}_0+r^2-(\sqrt{2}R)^2}^{\bar{t}_0+r^2} dt \int_{\Omega} e_\lambda(u_\lambda)(z)\exp\left(\frac{|x-\bar{x}_0|^2}{4(t-(\bar{t}_0+r^2))}\right)dx \\
&+\frac{2^{d/2}R}{d+1}\left(\sup_{x\in\partial\Omega}\left(|x|+\frac{|x|^2}{\epsilon_0}\right)\sup_{z\in\partial Q_T}|\nabla_{\tan}u_0(z)|^2+2\sup_{z\in\partial Q_T}\left|\frac{\partial u_0}{\partial t}(z)\right|^2\right) \\
&\qquad\times\mathcal{L}^{d-1}(\partial\Omega)\sup_{0\le s\le\infty}\max(s^{1-d},s^{5-d})\exp\left(-\frac{1}{4s^2}\inf_{x\in\partial\Omega}|x|^2\right) \\
\le{}& e^{1/4}R^{-d}\int_{t_0-3R^2}^{t_0+R^2} dt\int_{B_{(1+\delta(R))R}(x_0)} e_\lambda(u_\lambda)(z)\,dx \\
&+2e^{1/4}R^{2-d}\exp\left(-\frac{\delta^2(R)}{8}\right)\operatorname*{ess\cdot sup}_{0\le t\le T}\int_\Omega e_\lambda(u_\lambda)(t,x)\,dx \\
&+\frac{R}{d+1}\left(\sup_{x\in\partial\Omega}\left(|x|+\frac{|x|^2}{\epsilon_0}\right)\sup_{z\in\partial Q_T}|\nabla_{\tan}u_0(z)|^2+2\sup_{z\in\partial Q_T}\left|\frac{\partial u_0}{\partial t}(z)\right|^2\right) \\
&\qquad\times\mathcal{L}^{d-1}(\partial\Omega)\sup_{0\le s\le\infty}\max(s^{1-d},s^{5-d})\exp\left(-\frac{1}{4s^2}\inf_{x\in\partial\Omega}|x|^2\right)
\end{aligned}
$$

$$\big(=:\Phi^1_R(z_0)\big). \tag{3.37}$$

Here note $r \le R$ and $\bar{t}_0 \in (t_0-R^2, t_0)$.

On the other hand, we use $e^{-1/4} \le \exp\big(|x-\bar{x}_0|^2/(4(t-(\bar{t}_0+r^2)))\big)$ for each $t \in (\bar{t}_0-r^2, \bar{t}_0)$ and we respectively take $\sqrt{3}r$, $\sqrt{3}R$ and $\bar{t}_0+r^2$ as r_1, r_2 and t_0 in Lemma 3.3. Then we can estimate $r^{-d}\int_{\bar{t}_0-2r^2}^{\bar{t}_0} dt \int_{C_{2r}(\bar{x}_0)} e_\lambda(u_\lambda)(z)\,dx$ as follows:

$$
\begin{aligned}
& r^{-d}\int_{\bar{t}_0-2r^2}^{\bar{t}_0} dt \int_{C_r(\bar{x}_0)} e_\lambda(u_\lambda)(z)\,dx \\
\le{}& e^{1/4}3^{d/2}(\sqrt{3}r)^{-d}\int_{\bar{t}_0+r^2-(\sqrt{3}r)^2}^{\bar{t}_0+r^2} dt\int_{C_r(\bar{x}_0)} e_\lambda(u_\lambda)(z)\exp\left(\frac{|x-\bar{x}_0|^2}{4(t-(\bar{t}_0+r^2))}\right)dx \\
\le{}& e^{1/4}3^{d/2}(\sqrt{3}R)^{-d}\int_{\bar{t}_0+r^2-(\sqrt{3}R)^2}^{\bar{t}_0+r^2} dt\int_{\Omega} e_\lambda(u_\lambda)(z)\exp\left(\frac{|x-\bar{x}_0|^2}{4(t-(\bar{t}_0+r^2))}\right)dx
\end{aligned}
$$

$$+\frac{\sqrt{3}R}{d+1}\left(\sup_{x\in\partial\Omega}\left(|x|+\frac{|x|^2}{\epsilon_0}\right)\sup_{z\in\partial Q_T}|\nabla_{\tan}u_0(z)|^2+2\sup_{z\in\partial Q_T}\left|\frac{\partial u_0}{\partial t}(z)\right|^2\right)\mathcal{L}^{d-1}(\partial\Omega)$$
$$\times\sup_{0\le s\le\infty}\max(s^{1-d},s^{5-d})\exp\left(-\frac{1}{4s^2}\inf_{x\in\partial\Omega}|x|^2\right)$$
$$\le e^{1/4}R^{-d}\int_{\bar t_0+r^2-(\sqrt{3}R)^2}^{\bar t_0+r^2}dt\int_{B_{\delta(R)R}(\bar x_0)}e_\lambda(u_\lambda)(z)\exp\left(\frac{|x-\bar x_0|^2}{4(t-(\bar t_0+r^2))}\right)dx$$
$$+e^{1/4}3R^{2-d}\exp\left(-\frac{\delta^2(R)}{12}\right)\operatorname{ess\cdot sup}_{0\le t\le T}\int_\Omega e_\lambda(u_\lambda)(t,x)\,dx$$
$$+\frac{3R}{d+1}\left(\sup_{x\in\partial\Omega}\left(|x|+\frac{|x|^2}{\epsilon_0}\right)\sup_{z\in\partial Q_T}|\nabla_{\tan}u_0(z)|^2+2\sup_{z\in\partial Q_T}\left|\frac{\partial u_0}{\partial t}(z)\right|^2\right)\mathcal{L}^{d-1}(\partial\Omega)$$
$$\times\sup_{0\le s\le\infty}\max(s^{1-d},s^{5-d})\exp\left(-\frac{1}{4s^2}\inf_{x\in\partial\Omega}|x|^2\right).$$

We must remark that $r\le R$, $\bar t_0\in(t_0-R^2,t_0)$ and $\bar x_0\in B_R(x_0)$; Then we obtain

$$r^{-d}\int_{\bar t_0-2r^2}^{\bar t_0}dt\int_{C_{2r}(\bar x_0)}e_\lambda(u_\lambda)(z)\,dx$$
$$\le e^{1/4}R^{-d}\int_{t_0-4R^2}^{t_0+R^2}dt\int_{B_{(1+\delta(R))R}(x_0)}e_\lambda(u_\lambda)(z)\,dx$$
$$+e^{1/4}3R^{2-d}\exp\left(-\frac{\delta^2(R)}{12}\right)\operatorname{ess\cdot sup}_{0\le t\le T}\int_\Omega e_\lambda(u_\lambda)(t,x)\,dx$$
$$+\frac{3R}{d+1}\left(\sup_{x\in\partial\Omega}\left(|x|+\frac{|x|^2}{\epsilon_0}\right)\sup_{z\in\partial Q_T}|\nabla_{\tan}u_0(z)|^2\right.$$
$$\left.+2\sup_{z\in\partial Q_T}\left|\frac{\partial u_0}{\partial t}(z)\right|^2\right)\mathcal{L}^{d-1}(\partial\Omega)$$
$$\times\sup_{0\le s\le\infty}\max(s^{1-d},s^{5-d})\exp\left(-\frac{1}{4s^2}\inf_{x\in\partial\Omega}|x|^2\right)\quad\left(=:\overline{\Phi}_R^2(z_0)\right).\tag{3.38}$$

We next set $r_1=r$, $r_2=2r$, $t_1=r^2$, $t_2=2r^2$ and $\bar z_0=z_0$ in Lemma 3.2; Then the left-hand side of (3.38) can be estimated from below as follows:

$$\operatorname{ess\cdot sup}_{\bar t_0-r^2<t\le\bar t_0}r^{-d}\int_{C_r(\bar x_0)}e_\lambda(u_\lambda)(t,x)\,dx$$
$$\le 36r^{-d}\int_{\bar t_0-2r^2}^{\bar t_0}dt\int_{C_{2r}(\bar x_0)}e_\lambda(u_\lambda)(z)\,dx.\tag{3.39}$$

Combining (3.38) with (3.39), we can conclude our second claim. $\square$

Proof of lemma 3.4

First of all, we introduce a few symbols: For any cube C and a function f on $\mathbb{R}^{1+d}$, we define a few averaging functions of f by

$$\begin{aligned}
&f_C(t):=\frac{1}{\mathcal{L}^d(C)}\int_C f(t,\bar{x})d\bar{x},\\
&\overline{f}_C(t):=f(t,x)-f_C(t),\\
&f^*(z):=f^*(t,x)=\sup_{x\in C}\frac{1}{\mathcal{L}^d(C)}\int_C|f(t,\bar{x})|d\bar{x},\\
&f^{\#}(z):=f^{\#}(t,x)=\sup_{x\in C}\frac{1}{\mathcal{L}^d(C)}\int_C|f(t,\bar{x})-f_C(t)|d\bar{x}.
\end{aligned}$$

We must remark that without a loss of generality, it suffices to prove this inequality as $((\sqrt{e_\lambda^{(\kappa)}(u_\lambda)})^p\eta)_{C_R(x_0)}$ instead of $((\sqrt{e_\lambda^{(\kappa)}(u_\lambda)})^p\eta)$.

We divide our proof into two steps; the first claim is to show the following: Recall $D_R(z_0)=(t_0-R^2,t_0]\times C_R(x_0)$. For any function $f\in\mathcal{D}(D_R(z_0))$ with $f_{C_R(x_0)}(t)=0$ in each $t\in(t_0-R^2,t_0]$,

$$\int_{C_R(x_0)}|f^*(z)|^{2+2/p}dx\le C^1_{\mathrm{FP}}\int_{C_R(x_0)}|f^{\#}(z)|^{2/p}|f^*(z)|^2\,dx \tag{3.40}$$

holds where C^1_{FP} is a positive constant depending only on d.

$\because$) Let $\alpha_0=1/\mathcal{L}^d(C_R)\int_{C_R(x_0)}|f(t,\bar{x})|d\bar{x}$. Suppose that K be a certain positive number sufficiently large. A Calderón-Zygmund stopping process yields for each $\alpha>\alpha_0$ the following: There exist two sequences of cubes $\{C_k\}$, $\{C_{k,l}\}$ $(k,l=1,2,\dots)$ satisfying

$$\begin{aligned}
&C_{k,l}\subset C_k,\\
&\alpha\le\frac{1}{\mathcal{L}^d(C_k)}\int_{C_k}|f(t,\bar{x})|d\bar{x}<2^{d+3}\alpha,\\
&K\alpha\le\frac{1}{\mathcal{L}^d(C_{k,l})}\int_{C_{k,l}}|f(t,\bar{x})|d\bar{x}<2^{d+3}K\alpha.
\end{aligned}\tag{3.41}$$

Here give a positive number δ sufficiently small. Note that obviously $f^{\#}>\delta\alpha$ holds on any cubes $C_k\ni x$ with

$$\frac{1}{\mathcal{L}^d(C_k)}\int_{C_k}|f(t,\bar{x})-f_{C_k}(t)|d\bar{x}>\delta\alpha.$$

On the remaining cubes, we have

$$\delta\alpha\mathcal{L}^d(C_k)\geq\int_{C_k}|f(t,\bar{x})-f_{C_k}(t)|d\bar{x}$$
$$\geq\sum_l\int_{C_{k,l}}|f(t,\bar{x})-f_{C_k}(t)|d\bar{x}$$
$$\geq\frac{K\alpha}{2}\sum_l\mathcal{L}^d(C_{k,l})\quad\text{as long as}\quad K\geq 2^{d+4}.$$

Thus, we infer

$$\mathcal{L}^d(\{(t,x);f^*>K\alpha\})=\sum_{k,l}\mathcal{L}^d(C_{k,l})$$
$$=\sum_{k,l}\mathcal{L}^d(C_{k,l}\cap\{(t,x);f^{\#}\leq\delta\alpha\})$$
$$+\sum_{k,l}\mathcal{L}^d(C_{k,l}\cap\{(t,x);f^{\#}>\delta\alpha\})$$
$$\leq\frac{2\delta}{K}\sum_k\mathcal{L}^d(C_k)+\mathcal{L}^d(\{(t,x);f^*>K\alpha\}\cap\{(t,x);f^{\#}>\delta\alpha\})$$
$$\leq\frac{2\delta}{K}\mathcal{L}^d(\{(t,x);f^*>\alpha\})$$
$$+\mathcal{L}^d(\{(t,x);f^*>K\alpha\}\cap\{(t,x);f^{\#}>\delta\alpha\})\tag{3.42}$$

for $\alpha>0$.
By noting

$$\{(t,x);f^*>K\alpha\}\cap\{(t,x);f^{\#}>\delta\alpha\}$$
$$\subset\{(t,x);(f^{\#})^{p/(p+1)}(f^*)^{1/(p+1)}>\delta^{p/(p+1)}K^{1/(p+1)}\alpha\},\tag{3.43}$$

we then multiply the above (3.43) by $\alpha^{1+2/p}d\alpha$ and integrate it from α_0 to ∞ over α to verify

$$\int_{\alpha_0}^{\infty}\alpha^{1+2/p}d\alpha\mathcal{L}^d(\{(t,x);f^*>K\alpha\})$$
$$\leq\frac{2\delta}{K}\int_{\alpha_0}^{\infty}\alpha^{1+2/p}d\alpha\mathcal{L}^d(\{(t,x);f^*>\alpha\})$$

$$+\int_{\alpha_0}^{\infty}\alpha^{1+2/p}d\alpha\mathcal{L}^d(\{(t,x);(f^{\#})^{p/(p+1)}(f^*)^{1/(p+1)}>\delta^{p/(p+1)}K^{1/(p+1)}\alpha\}).$$

Namely, we obtain

$$\begin{aligned}&\left(\frac{1}{K}\right)^{2(1+1/p)}\int_{K\alpha_0}^{\infty}\alpha^{1+2/p}d\alpha\mathcal{L}^d(\{(t,x);f^*>\alpha\})\\&\le\frac{2\delta}{K}\int_0^{\infty}\alpha^{1+2/p}d\alpha\mathcal{L}^d(\{(t,x);f^*>\alpha\})\\&+\left(\frac{1}{\delta}\right)^2\left(\frac{1}{K}\right)^{2/p}\int_0^{\infty}\alpha^{1+2/p}d\alpha\mathcal{L}^d(\{(t,x);(f^{\#})^{p/(p+1)}(f^*)^{1/(p+1)}>\alpha\}),\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\int_{\{x\in C_R(x_0);f^*>\alpha_0\}}|f^*(z)|^{2+2/p}dx\le 2\delta K^{1+2/p}\int_{C_R(x_0)}|f^*(z)|^{2+2/p}dx\\&+\frac{p}{p+2}\left(\frac{1}{\delta}\right)^2K^{2(1+2/p)}\int_{C_R(x_0)}(f^{\#}(z))^2(f^*(z))^{2/p}dx.\end{aligned}\tag{3.44}$$

While, noting $f_{C_R}(t)=0$, $\int_{\{x\in C_R(x_0);f^*\le\alpha_0\}}|f^*(z)|^{2+2/p}dx$ is estimated as follows:

$$\begin{aligned}&\int_{\{x\in C_R(x_0);f^*\le K\alpha_0\}}|f^*(z)|^{2+2/p}dx\\&\le\mathcal{L}^d(C_R(x_0))K^{2+2/p}\alpha_0^{2+2/p}\le K^{2+2/p}\int_{C_R(x_0)}(f^{\#}(z))^2(f^*(z))^{2/p}dx.\end{aligned}\tag{3.45}$$

If we take $\delta=1/(4K^{1+2/p})$, we thus complete the first claim with $C^1_{\mathrm{FP}}=2p/(p+2)(1/\delta)^2\ K^2+K^{2+2/p}=32\ K^{2+6/p}+K^{2+2/p}$. $\square$

Substituting f for $\left(\overline{(e_\lambda^{(\kappa)}(u_\lambda))^{p/2}\eta}\right)_{C_R(x_0)}$ with any $\eta\in C_0^\infty(C_R(x_0))$, $\sup_{C_R(x_0)}\eta\le 1$, $p=1$ or $1+2/d$ and any positive number $\kappa>0$, gives

$$\begin{aligned}&\int_{C_R(x_0)}\left|\left(\overline{\left(e_\lambda^{(\kappa)}(u_\lambda)(z)\right)^{p/2}\eta(z)}\right)_{C_R(x_0)}\right|^{2+2/p}dx\\&\le C^1_{\mathrm{FP}}\int_{C_R(x_0)}\left(\left(\left(\overline{(e_\lambda^{(\kappa)}(u_\lambda)(z))^{p/2}\eta(z)}\right)_{C_R(x_0)}\right)^{\#}\right)^2\\&\qquad\times\left(\left(\left(\overline{(e_\lambda^{(\kappa)}(u_\lambda)(z))^{p/2}\eta(z)}\right)_{C_R(x_0)}\right)^{*}\right)^{2/p}dx.\end{aligned}$$

Thus, to establish the result of our lemma, it suffices to show

$$\int_{C_R(x_0)} \left(\left(\overline{((e_\lambda^{(\kappa)}(u_\lambda)(z))^{p/2}\eta(z))}_{C_R(x_0)} \right)^{\#} \right)^2 \left(\left(\overline{((e_\lambda^{(\kappa)}(u_\lambda)(z))^{p/2}\eta(z))}_{C_R(x_0)} \right)^{*} \right)^{2/p} dx$$

(3.46)

$$\leq \frac{p\ 2^{2/p}\ 5^d}{2-p} \Phi_R^2(z_0) \int_{C_R(x_0)} \left|\nabla\Big(\big(\sqrt{e_\lambda^{(k)}(u_\lambda)(z)}\big)^p \eta(z)\Big)\right|^2 dx.$$

$\because$) Put

$$\alpha_0 = \frac{1}{\mathcal{L}^d(C_R)} \int_{C_R(z_0)} |f(t,\bar{x}) - f_{C_R(x_0)}(t)|\, d\bar{x}$$

and set $K = 2^{d+5}$.

For any $\alpha > \alpha_0$ and each $t \in (t_0 - R^2, t_0]$, we apply the Calderón-Zygmund stopping process; $\{x \in C_R(x_0); f^{\#}(z) > \alpha\}$ and $\{x \in C_R(x_0); f^{\#}(z) > K\alpha\}$ are divided into cylinders C_k, and $C_{k,l}$ with the following properties:

$$\{x \in C_R(x_0); f^{\#}(t,x) > \alpha\} = \bigcup_k C_k,$$
$$\{x \in C_R(x_0); f^{\#}(t,x) > K\alpha\} = \bigcup_{k,l} C_{k,l}.$$

$$C_{k,l} \subset C_k,$$
$$\alpha \leq \frac{1}{\mathcal{L}^d(C_k)} \int_{C_k} |f(t,\bar{x}) - f_{C_k}(t)|\, d\bar{x} < 2^{d+3}\alpha,$$
$$K\alpha \leq \frac{1}{\mathcal{L}^d(C_{k,l})} \int_{C_{k,l}} |f(t,\bar{x}) - f_{C_{k,l}}(t)|\, d\bar{x} < 2^{d+3}K\alpha.$$

Now observe that

$$(\nabla f)^* \geq \alpha (\mathrm{diam} C_k)^{-1} \qquad \text{throughout} \quad C_k, \tag{3.47}$$

$$\mathcal{L}^d(C_k) \geq 2\mathcal{L}^d(\cup_l C_{k,l}) \tag{3.48}$$

Indeed, (3.47) follows from

$$\frac{1}{\mathcal{L}^d(C_k)} \int_{C_k} |f(t,\bar{x}) - f_{C_k}(t)|\, d\bar{x} \leq (\mathrm{diam} C_k) \frac{1}{\mathcal{L}^d(C_k)} \int_{C_k} |\nabla f(t,\bar{x})|\, d\bar{x}.$$

While (3.48) is a consequence of

$$2^{d+3}\alpha\mathcal{L}^d(C_k)>\int_{C_k}|f(t,\bar{x})-f_{C_k}(t)|d\bar{x}\geq\sum_l\int_{C_{k,l}}|f(t,\bar{x})-f_{C_k}(t)|d\bar{x}$$
$$\geq\frac{1}{2}\sum_l\int_{C_{k,l}}|f(t,\bar{x})-f_{C_{k,l}}(t)|d\bar{x}\geq\frac{K\alpha}{2}\sum_l\mathcal{L}^d(C_{k,l})$$

and the fact that $K = 2^{d+5}$. We thus write

$$\int_{C_k\setminus\cup_l C_{k,l}}|(\nabla f)^*(z)|^2dx\geq\alpha^2(\mathrm{diam}C_k)^{-2}\mathcal{L}^d(C_k/\bigcup_l C_{k,l})$$
$$\geq\frac{\alpha^2}{2}(\mathrm{diam}C_k)^{-2}\mathcal{L}^d(C_k). \tag{3.49}$$

Also from Corollary 3.1 and the maximal function theory (See Ziemer [**122**].), we obtain the following:

$$\mu(C_k):=\int_{C_k}\left(\left(\overline{((\sqrt{e_\lambda^{(\kappa)}(u_\lambda)(z)})^p\eta(z))_{C_R(x_0)}}\right)^*\right)^{2/p}dx$$
$$\leq\frac{p\ 2^{2/p}\ 5^d}{2-p}\int_{C_k}\left(\overline{((\sqrt{e_\lambda^{(\kappa)}(u_\lambda)(z)})^p\eta(z))_{C_R(x_0)}}\right)^{2/p}dx$$
$$=\frac{p\ 2^{2/p}\ 5^d}{2-p}\int_{C_k}\Bigg|(\sqrt{e_\lambda^{(\kappa)}(u_\lambda)(z)})^p\eta(z)$$
$$-\frac{1}{\mathcal{L}^d(C_R)}\int_{C_R(x_0)}(\sqrt{e_\lambda^{(\kappa)}(u_\lambda)(t,\bar{x})})^p\eta(t,\bar{x})d\bar{x}\Bigg|^{2/p}dx \tag{3.50}$$
$$\leq\frac{p\ 2^{4/p-1}\ 5^d}{2-p}\int_{C_k}\left|(\sqrt{e_\lambda^{(\kappa)}(u_\lambda)(z)})^p\eta(z)\right|^{2/p}dx$$
$$+\frac{p\ 2^{4/p-1}\ 5^d}{2-p}\left(\frac{1}{\mathcal{L}^d(C_R)}\right)^{2/p}\mathcal{L}^d(C_k)\left|\int_{C_R(x_0)}(\sqrt{e_\lambda^{(\kappa)}(u_\lambda)(z)})^p\eta(z)dx\right|^{2/p}$$
$$\leq\frac{p\ 2^{4/p-1}\ 5^d}{2-p}(\mathrm{diam}C_k)^{d-2}\Phi_R^2(z_0).$$

Combining (3.49) with (3.50), we infer

$$\int_{C_k/\cup_l C_{k,l}}|(\nabla f)^*(z)|^2dx\geq\frac{2-p}{24\ p\ 2^{4/p-1}\ 5^d}(\Phi_R^2(z_0))^{-1}\frac{\alpha^2}{2}\mu(C_k), \tag{3.51}$$

which implies

$$\int_{\{\alpha<f^{\#}\leq K\alpha\}}|(\nabla f)^*(z)|^2dx\geq\frac{2-p}{24\ p\ 2^{4/p-1}\ 5^d}(\Phi_R^2(z_0))^{-1}$$
$$\times\frac{\alpha^2}{2}\mu(f^{\#}>\alpha).$$

Let now α range over $2K^n\alpha_0$ $(n=0,1,2,\ldots)$ and sum over n to imply

$$\begin{aligned}&\int_{C_R(z_0)}|(\nabla f)^*(z)|^2dx\\&\quad\geq\frac{1}{K^2}\cdot\frac{2-p}{p\ 2^{4/p-1}\ 5^d}\frac{(\Phi_R^2(z_0))^{-1}}{2}\int_{\{f^{\#}>2\alpha_0\}}|f^{\#}(z)|^2d\mu.\end{aligned}\tag{3.52}$$

On the other hand, $\int_{\{f^{\#}\leq 2\alpha_0\}}|f^{\#}(z)|^2\ d\mu$ is estimated as follows:

$$\begin{aligned}&\int_{\{f^{\#}\leq 2\alpha_0\}}|f^{\#}(z)|^2d\mu\leq 4\alpha_0^2\mu(C_R(x_0))\\&\leq\frac{4\ 24\ p\ 2^{4/p-1}\ 5^d}{2-p}\left(\frac{1}{\mathcal{L}^d(C_R)}\int_{C_R(x_0)}|f(t,\bar{x})-f_{C_R(x_0)}(t)|d\bar{x}\right)^2\Phi_R^2(z_0)R^{d-2}\\&\quad\leq\frac{4\ 24\ p\ 2^{4/p-1}\ 5^d\ C_{\mathrm{PO}}}{2-p}\Phi_R^2(z_0)\left(\int_{C_R(x_0)}|\nabla f(t,\bar{x})|^2d\bar{x}\right).\end{aligned}\tag{3.53}$$

Combining (3.52) with (3.53), applying the maximal function theory and integrating it with respect to t on $(t_0-R^2,t_0]$, establish our claim. $\square$

5. Proof of theorems

5.1. Proof of Theorem 3.2

Set $\sigma\in(0,R/4)$. Also fix $\bar{z}_0=(\bar{t}_0,\bar{x}_0)\in Q_{R/4}(z_0)$.

Since u_λ is smooth, there exists $\sigma_0\in[0,R/4]$ such that

$$\left(\frac{R}{4}-\sigma_0\right)^2\sup_{Q_{\sigma_0}(\bar{z}_0)}e_\lambda(u_\lambda)=\max_{0\leq\sigma\leq R/4}\left(\frac{R}{4}-\sigma\right)^2\sup_{Q_\sigma(\bar{z}_0)}e_\lambda(u_\lambda).\tag{3.54}$$

Moreover, there exists a point $z_{\max}=(t_{\max},x_{\max})\in\overline{Q}_{\sigma_0}(\bar{z}_0)$ such that

$$\sup_{Q_{\sigma_0}(\bar{z}_0)}e_\lambda(u_\lambda)=e_\lambda(u_\lambda)(z_{\max}):=e_0.\tag{3.55}$$

We must note that $\sigma_0=R/4$ implies $\sup_{Q_{R/4}(z_0)}e_\lambda(u)\equiv 0$. Thus we can always assume $R/4-\sigma_0>0$.

If we set $\rho_0=1/2(R/4-\sigma_0)$, by choice of σ_0 and $z_{\max}$, we obtain

$$\sup_{Q_{\rho_0}(z_{\max})} e_\lambda(u_\lambda) \le \sup_{Q_{\sigma_0+\rho_0}(\bar{z}_0)} e_\lambda(u_\lambda) \le 4e_0. \tag{3.56}$$

Now we define

$$r_0 := \sqrt{e_0}\cdot\rho_0,\ \bar{\lambda} := \lambda/e_0,$$
$$v_\lambda(\bar{t},\bar{x}) := u_\lambda\left(\frac{\bar{t}}{e_0}+t_{\max}, \frac{\bar{x}}{\sqrt{e_0}}+x_{\max}\right).$$

Assume $r_0 \ge 1$. Note that v_λ also solves (3.1) in Q_{r_0} and moreover v_λ satisfies

$$e_{\bar{\lambda}}(v_\lambda)(0,0) = 1, \quad \sup_{Q_{r_0}(0)} e_{\bar{\lambda}}(v_\lambda) \le 4. \tag{3.57}$$

By (3.23) in Lemma 3.1 and (3.57), $e_{\bar{\lambda}}(v_\lambda)$ satisfies

$$\left(\frac{\partial}{\partial \bar{t}} - \triangle\right) e_{\bar{\lambda}}(v_\lambda) \le 64 e_{\bar{\lambda}}(v_\lambda) \quad \text{in} \quad Q_{r_0}. \tag{3.58}$$

Thus Lemma 1.3 implies

$$1 = e_{\bar{\lambda}}(v_\lambda)(0,0) \le C_{\mathrm{H}} \int_{-1}^{+1} d\bar{t} \int_{B_1(0)} e_{\bar{\lambda}}(v_\lambda)(\bar{z})\, d\bar{x}. \tag{3.59}$$

But scaling back, by means of $1/\sqrt{e_0}+\sigma_0 \le \rho_0 < R/4$, and the monotonicity (3.27) in Corollary 3.1, we have

$$\begin{aligned} e_{\bar{\lambda}}(v_\lambda)(0,0) &\le C(\sqrt{e_0})^d \int_{t_{\max}-1/e_0}^{t_{\max}+1/e_0} dt \int_{B_{1/\sqrt{e_0}}(x_{\max})} e_\lambda(u_\lambda)(z)\, dx \\ &\le C\varPhi^1_R(z_0)(t_0+(R/4)^2, x_0) \le C\epsilon_1. \end{aligned} \tag{3.60}$$

We arrive at a contradiction if we take $\epsilon_1 < 1/C$. Consequently we can assert that $r_0 \le 1$, which implies our claim. □

5.2. Proof of Theorem 3.3

Suppose that $\mathbf{N}^{\mathsf{CL}} \bigcap \mathbf{Reg} \ni z_0$ for a point $z_0 \in Q_T$. Then from definition of $\mathbf{N}^{\mathsf{CL}}$ and **Reg**, there exist a sequence of λ_j and positive numbers λ_0 and R_0 such that

$$\{z_{\lambda(j)}\}\,(\lambda(1) < \lambda(2) < \cdots < \infty)\ \text{with}\ z_{\lambda(j)} \in \mathbf{N}_{\lambda(j)} \quad \text{s.t.} \quad z_{\lambda(j)} \to z_0, \tag{3.61}$$
$$\lambda_0, R_0 > 0 \qquad \text{s.t.} \qquad \varPhi^1_{R_0}(t_0+2(R_0/4)^2, x_0) \le \epsilon_1 \qquad \text{for} \quad \text{any}\ \lambda \ge \lambda_0. \tag{3.62}$$

Theorem 3.2 implies the following:

$$\sup_{Q_{R_0/4}(t_0+(R_0/4)^2,x_0)} |\nabla u_{\lambda(j)}| \leq \frac{8\sqrt{2}}{R_0} \quad (\lambda(1)<\lambda(2)<\cdots<\infty). \tag{3.63}$$

Since $z_{\lambda(j)} \to z_0$ as $j\to\infty$, there exists $j_0 \in \mathbb{N}$ such that for any $j \geq j_0$, $z_{\lambda(j)} \in Q_{R_0/4}\,(t_0+(R_0/4)^2,x_0)$; and $\{u_{\lambda(j)}\}$ $(j=j_0,j_0+1,j_0+2,\ldots)$ is uniform bounded and equi-continuous. Then Ascoli-Arzelà's theorem asserts that for a sub-sequence $\{\lambda(j')\}$ $(j'=1,2,\ldots)$ of $\{\lambda(j)\}$ $(j=j_0,j_0+1,\ldots)$, $u_{\lambda(j')}$ uniformly converges to u_∞ on $Q_{R_0/4}(t_0+(R/4)^2,x_0)$ as $j'\to\infty$ and u_∞ is continuous mapping in $Q_{R_0/4}(t_0+(R/4)^2,x_0)$. From (3.8), $|u_\infty| = 1$ on $Q_{R_0/4}(t_0+(R/4)^2,x_0)$. However this forces a contradiction because the following holds:

$$\begin{aligned}
|u_\infty(z_0)| &\leq |u_\infty(z_0)-u_{\lambda(j')}(z_0)| + |u_{\lambda(j')}(z_0)-u_{\lambda(j')}(z_{\lambda(j')})| \\
&\leq |u_\infty(z_0)-u_{\lambda(j')}(z_0)| + \frac{8}{R_0}|z_0-z_{\lambda(j')}| \to 0 \\
&\text{as}\, j'\to\infty. \quad \square
\end{aligned} \tag{3.64}$$

5.3. Proof of Theorem 3.4

First of all, we prove that

$$\left(\frac{1}{R^{d+2}}\int_{Q_{R/2}(z_0)} e_\lambda^{1+2/d}(u_\lambda)(z)\,dz\right)^{d/(d+2)} \leq \frac{C_{RH}^1}{R^{d+2}}\int_{Q_R(z_0)} e_\lambda(u_\lambda)(z)\,dz \tag{3.65}$$

holds for any cylinders $Q_{R/2}(z_0) \subset Q_R(z_0) \subset\subset Q_T$ and a positive constant C_{RH}^1 depending only on d.

$\because$) Define a sequence of numbers R_j by

$$R_j = \begin{cases} R/2 & (j=0), \\ R/2+(1-\tau)/\tau\sum_{i=1}^{j}\tau^i R/2 & (j=1,2,\ldots). \end{cases} \tag{3.66}$$

Furthermore, give a smooth cut-off function η with

$$\eta(z) = \begin{cases} 1 & \text{on} \quad Q_{R_{j-1}}(z_0), \\ 0 & \text{outside} \quad Q_{R_j}(z_0) \end{cases} \quad \text{for} \quad t\leq t_0 \tag{3.67}$$

and

$$\begin{aligned}
&0\leq \eta(z)\leq 1, \qquad |\nabla\eta(z)| \leq \frac{4\tau}{(1-\tau)\tau^j}\frac{1}{R}, \\
&|\triangle\eta(z)| \leq \frac{8\tau^2}{(1-\tau)^2\tau^{2j}}\frac{1}{R^2}, \left|\frac{\partial\eta}{\partial t}(z)\right| \leq \frac{4\tau}{(1-\tau)\tau^j}\frac{1}{R^2}.
\end{aligned}$$

Then, we multiply (3.23) by η^4 and successively integrate it on $Q_R(z_0)$ to imply

$$\begin{aligned}&\int_{Q_R(z_0)}\frac{\partial e_\lambda}{\partial t}(u_\lambda)(z)\eta^4(z)\,dz-\int_{Q_R(z_0)}\triangle e_\lambda(u_\lambda)(z)\eta^4(z)\,dz\\ &+\int_{Q_R(z_0)}f_\lambda(u_\lambda)(z)\eta^4(z)\,dz\le 16\int_{Q_R(z_0)}e_\lambda^2(u_\lambda)(z)\eta^4(z)\,dz.\end{aligned}\tag{3.68}$$

From the integral by parts in the 1st and the 2nd term on the left-hand side in (3.68), we obtain

$$\begin{aligned}&\int_{Q_R(z_0)}f_\lambda(u_\lambda)(z)\eta^4(z)\,dz\\ &\le\int_{Q_R(z_0)}e_\lambda(u_\lambda)(z)\Bigg(4\eta^3(z)\frac{\partial\eta}{\partial t}(z)+12\eta^2(z)|\nabla\eta(z)|^2\\ &\qquad\times\ 4\eta^3(z)\triangle\eta(z)\Bigg)\,dz\\ &\quad+16\int_{Q_R(z_0)}(e_\lambda(u_\lambda)(z))^2\eta^4(z)\,dz.\end{aligned}\tag{3.69}$$

By recalling a property of η and applying Lemma 3.4 as $p=1$ and $\kappa=0$ to the 1st term on the right-hand side in (3.69), we infer

$$\begin{aligned}&\int_{Q_R(z_0)}f_\lambda(u_\lambda)(z)\eta^4(z)\,dz\\ &\le\left(\frac{14\cdot 16\tau^2}{(1-\tau)^2}+\frac{16\tau}{(1-\tau)}\right)\frac{1}{\tau^{2j}R^2}\int_{Q_R(z_0)}e_\lambda(u_\lambda)(z)\,dz\\ &\quad+16C_{\mathrm{FP}}\varPhi_R^2(z_0)\int_{Q_R(z_0)}\left|\nabla\left(\sqrt{e_\lambda(u_\lambda)(z)}\,\eta(z)\right)\right|^2dz.\end{aligned}\tag{3.70}$$

Noting $|\nabla e_\lambda(u_\lambda)|\le 4\sqrt{f_\lambda(u_\lambda)}\,\sqrt{e_\lambda(u_\lambda)}$, we proceed to estimate

$$\begin{aligned}&\int_{Q_{R_j}(z_0)}f_\lambda(u_\lambda)(z)\eta^4(z)\,dz\\ &\le\left((14+32C_{\mathrm{FP}}\varPhi_R^2(z_0))\frac{16\tau^2}{(1-\tau)^2}+\frac{16\tau}{(1-\tau)}\right)\\ &\quad\times\frac{1}{\tau^{2j}R^2}\int_{Q_R(z_0)}e_\lambda(u_\lambda)(z)\,dz\end{aligned}$$

$$+16\cdot 8C_{\mathrm{FP}}\Phi_R^2(z_0)\int_{Q_{R_j}(z_0)} f_\lambda(u_\lambda)(z)\,dz. \tag{3.71}$$

As in the similar calculation as above, we obtain

$$\begin{aligned}
&\operatorname*{ess\cdot sup}_{t_0-R_j^2\le t\le t_0}\int_{B_{R_j}(x_0)} e_\lambda(u_\lambda)(t,x)\eta^4(t,x)\,dx\\
&\le\left((14+32C_{\mathrm{FP}}\Phi_R^2(z_0))\frac{16\tau^2}{(1-\tau)^2}+\frac{16\tau}{(1-\tau)}\right)\\
&\qquad\times\frac{1}{\tau^{2j}R^2}\int_{Q_R(z_0)} e_\lambda(u_\lambda)(z)\,dz\\
&\quad+16\cdot 8C_{\mathrm{FP}}\Phi_R^2(z_0)\int_{Q_{R_j}(z_0)} f_\lambda(u_\lambda)(z)\,dz.
\end{aligned}\tag{3.72}$$

We again invoke $|\nabla e_\lambda(u_\lambda)|\le 4\sqrt{f_\lambda(u_\lambda)}\ \sqrt{e_\lambda(u_\lambda)}$ and (3.71) to estimate as follows:

$$\begin{aligned}
&\int_{Q_{R_j}(z_0)}\left|\nabla\left(e_\lambda(u_\lambda)(z)\eta^4(z)\right)\right|dz\\
&\le\int_{Q_{R_j}(z_0)} f_\lambda(u_\lambda)(z)\eta^4(z)\,dz+4\int_{Q_{R_j}(z_0)} e_\lambda(u_\lambda)(z)\eta^4(z)\,dz\\
&+4\int_{Q_{R_j}(z_0)} e_\lambda(u_\lambda)(z)\eta^3(z)|\nabla\eta(z)|\,dz\\
&\le 4\int_{Q_R(z_0)} e_\lambda(u_\lambda)(z)\,dz+16\cdot 8C_{\mathrm{FP}}\Phi_R^2(z_0)\int_{Q_{R_j}(z_0)} f_\lambda(u_\lambda)(z)\,dz\\
&+\left((14+32C_{\mathrm{FP}}\Phi_R^2(z_0))\frac{16\tau^2}{(1-\tau)^2}+\frac{32\tau}{(1-\tau)}\right)\\
&\quad\times\frac{1}{\tau^{2j}R^2}\int_{Q_R(z_0)} e_\lambda(u_\lambda)(z)\,dz.
\end{aligned}\tag{3.73}$$

Thus, from (3.71), (3.72) and (3.73), the Sobolev imbedding theorem of the parabolic type (We refer to Ladyženskaya-Solonnikov-Ural'ceva, N.N [**[75]**, Chapter II, p75].) and definition of η, it follows that

$$C_{\mathrm{SO}}^{-1}\left(\int_{Q_{R/2}(z_0)}\left(e_\lambda(u_\lambda)(z)\right)^{1+2/d}dz\right)^{d/(d+2)}+\int_{Q_{R_{j-1}}(z_0)} f_\lambda(u_\lambda)(z)\,dz$$

$$\le 24\cdot 16 C_{\mathrm{FP}}\Phi_R^2(z_0)\int_{Q_{R_j}(z_0)} f_\lambda(u_\lambda)(z)\,dz$$
$$+\left(\left(14+16C_{\mathrm{FP}}\Phi_R^2(z_0)\right)\frac{48\tau^2}{(1-\tau)^2}+\frac{64\tau}{(1-\tau)}+4\right)\frac{1}{\tau^{2j}R^2}\int_{Q_R(z_0)} e_\lambda(u_\lambda)(z)\,dz$$

where C_{SO} is the best Sobolev constant.
Since $24\cdot 32\ C_{\mathrm{FP}}\Phi_R^2(z_0) < 1/2$, recalling definition of η, we have

$$C_{\mathrm{SO}}^{-1}\left(\int_{Q_{R/2}(z_0)}\left(e_\lambda^{1+2/d}(u_\lambda)(z)\right)^{1+2/d}dz\right)^{d/(d+2)}+\int_{Q_{R_{j-1}}(z_0)} f_\lambda(u_\lambda)(z)\,dz$$
$$\le\frac{1}{2}\int_{Q_{R_j}(z_0)} f_\lambda(u_\lambda)(z)\,dz$$
$$+\left(\frac{(1+14\cdot 48)\ \tau^2}{(1-\tau)^2}+\frac{64\tau}{(1-\tau)}+4\right)\frac{1}{\tau^{2j}R^2}\int_{Q_R(z_0)} e_\lambda(u_\lambda)(z)\,dz.$$

If we employ the usual iteration technique by Giaquinta [**46**, Theorem 3.1, p. 159 and Lemma 3.1, p. 161] to the inequality above, we then conclude

$$C_{\mathrm{SO}}^{-1}\sum_{i=0}^{j-1}\left(\frac{1}{2}\right)^i\left(\int_{Q_{R/2}(z_0)}(e_\lambda(u_\lambda)(z))^{1+2/d}dz\right)^{d/(d+2)}$$
$$+\int_{Q_{R/2}(z_0)} f_\lambda(u_\lambda)(z)\,dz$$
$$\le\left(\frac{1}{2}\right)^j\int_{Q_{R_j}(z_0)} f_\lambda(u_\lambda)(z)\,dz+\left(\frac{(1+14\cdot 48)\ \tau^2}{(1-\tau)^2}+\frac{64\tau}{(1-\tau)}+4\right)$$
$$\times\sum_{i=0}^{j-1}\left(\frac{1}{2\tau^2}\right)^i\frac{1}{\tau^2R^2}\int_{Q_R(z_0)} e_\lambda(u_\lambda)(z)\,dz. \tag{3.74}$$

Let $j\to\infty$ to obtain the conclusion (3.65) because of $1/\sqrt{2} < \tau < 1$, i.e. $\tau = 1/2(1+\sqrt{2})$. □

We proceed to the second step: Set σ be any number with $0<\sigma<1/2$. Then we establish a higher integrability of the truncation function of $e_\lambda(u_\lambda)$ as follows:

$$\left(\int_{Q_{(1-\sigma)R/2}(z_0)}\left(e_\lambda^{(\kappa)}(u_\lambda)(z)\right)^{(1+2/d)^2}dz\right)^{d/(d+2)}$$
$$\le\frac{C_{RH}^2}{(\sigma R)^2}\int_{Q_{R/2}(z_0)}\left(e_\lambda^{(\kappa)}(u_\lambda)(z)\right)^{1+2/d}dz$$

$$+32(d+2)R^{4/d}\ \kappa^{2/d(d+2)}\ \mathcal{L}^{d+2}(A^{(\kappa)}_{R/2}(z_0)) \tag{3.75}$$

holds for any cylinders $Q_{(1-\sigma)R/2}(z_0) \subset Q_{R/2}(z_0) \subset\subset Q_T$ where C^2_{RH} is a positive constant independent of σ, R, z_0, κ and $e_\lambda(u_\lambda)$.

$\because$) We define a sequence of numbers R_j by

$$R_j = \begin{cases} (1-\sigma)R/2 & (j=0), \\ (1-\sigma)R/2 + (1-\tau)/\tau \sum_{i=1}^{j} \tau^i (\sigma R)/2 & (j=1,2,\ldots). \end{cases} \tag{3.76}$$

Also choose a smooth cut-off function η satisfying

$$\eta(z) = \begin{cases} 1 & \text{on} \quad Q_{R_{j-1}}(z_0), \\ 0 & \text{outside} \quad Q_{R_j}(z_0) \end{cases} \quad \text{in} \quad t<t_0, \tag{3.77}$$

with

$$\begin{aligned} &0 \le \eta(z) \le 1, \quad |\nabla \eta(z)| \le \frac{4\tau}{\sigma(1-\tau)\tau^j}\frac{1}{R}, \\ &|\triangle \eta(z)| \le \frac{8\tau^2}{\sigma^2(1-\tau)^2\tau^{2j}}\frac{1}{R^2}, \quad |\frac{\partial \eta}{\partial t}(z)| \le \frac{4\tau}{\sigma(1-\sigma)(1-\tau)\tau^j}\frac{1}{R^2}. \end{aligned}$$

Set $p = 1+2/d$; A multiplier of (3.23) by $(e^{(\kappa)}_\lambda(u_\lambda))^{2/d}\ \eta^{2+2/p}$ and an integration of it on $Q_R(z_0)$ yield

$$\begin{aligned} &\int_{Q_R(z_0)} \frac{\partial e_\lambda}{\partial t}(u_\lambda)(z)\big(e^{(\kappa)}_\lambda(u_\lambda)(z)\big)^{2/d}\eta^{2+2/p}(z)\,dz \\ &+\frac{2}{d}\int_{Q_R(z_0)} \big((e^{(\kappa)}_\lambda(u_\lambda)(z)\big)^{2/d-1}\Big|\nabla e^{(\kappa)}_\lambda(u_\lambda)(z)\Big|^2\eta^{2+2/p}(z)\,dz \\ &+\int_{Q_R(z_0)} \langle \nabla e^{(\kappa)}_\lambda(u_\lambda)(z), \big(e^{(\kappa)}_\lambda(u_\lambda)(z)\big)^{2/d}\nabla \eta^{2+2/p}(z)\rangle\,dz \\ &\le 32\int_{Q_R(z_0)} \big(e^{(\kappa)}_\lambda(u_\lambda)(z)\big)^{2+2/d}\eta^{2+2/p}(z)\,dz \\ &+32\ \kappa^2\int_{Q_R(z_0)} \big(e^{(\kappa)}_\lambda(u_\lambda)(z)\big)^{2/d}\eta^{2+2/p}(z)\,dz. \end{aligned} \tag{3.78}$$

By using the integral by parts in the 1st term, Schwarz's inequality in the 3rd term on the left-hand side and applying Lemma 3.4 to the 1st term on the right-hand side in (3.78), we obtain

$$
\begin{aligned}
&\frac{d}{d+2}\int_{B_R(x_0)}\left(e_\lambda^{(\kappa)}(u_\lambda)(t_0,x)\right)^{1+2/d}\eta^{2+2/p}(t_0,x)\,dx\\
&+\frac{4d}{(d+2)^2}\int_{Q_R(z_0)}\left|\nabla\left(e_\lambda^{(\kappa)}(u_\lambda)(z)\right)^{1/2(1+2/d)}\right|^2\eta^{2+2/p}(z)\,dz\\
&\leq\frac{d}{d+2}\left(2+\frac{2}{p}\right)\int_{Q_R(z_0)}\left(e_\lambda^{(\kappa)}(u_\lambda)(z)\right)^{(1+2/d)}\eta^{1+2/p}(z)\frac{\partial\eta}{\partial t}(z)\,dz\\
&+d(1+\frac{1}{p})^2\int_{Q_R(z_0)}\left(e_\lambda^{(\kappa)}(u_\lambda)(z)\right)^{(1+2/d)}|\nabla\eta(z)|^2\,dz\\
&+32C_{\mathrm{FP}}\Phi_R^2(z_0)\int_{Q_R(z_0)}\left|\nabla\left((\sqrt{e_\lambda^{(\kappa)}(u_\lambda)(z)})^p\eta(z)\right)\right|^2dz\\
&+32\,\kappa^2\int_{Q_R(z_0)}\left(e_\lambda^{(\kappa)}(u_\lambda)(z)\right)^{2/d}\eta^{2+2/p}(z)\,dz. \qquad (3.79)
\end{aligned}
$$

Namely, we infer

$$
\begin{aligned}
&\frac{d}{(d+2)^2}\int_{Q_R(z_0)}\left|\nabla\left[\left(e_\lambda^{(\kappa)}(u_\lambda)(z)\right)^{1/2(1+2/d)}\eta^{1+1/p}(z)\right]\right|^2dz\\
&+\frac{2d}{(d+2)^2}\int_{Q_R(z_0)}\left|\nabla\left(e_\lambda^{(\kappa)}(u_\lambda)(z)\right)^{1/2(1+2/d)}\right|^2\eta^{2+2/p}(z)\,dz\\
&\leq 64C_{\mathrm{FP}}\Phi_R^2(z_0)\int_{Q_R(z_0)}\left|\nabla\left(e_\lambda^{(\kappa)}(u_\lambda)(z)\right)^{1/2(1+2/d)}\right|^2\eta^2(z)\,dz\\
&+\int_{Q_R(z_0)}\left(e_\lambda^{(\kappa)}(u_\lambda)(z)\right)^{(1+2/d)}\left(\frac{d}{d+2}\left(2+\frac{2}{p}\right)\eta^{1+2/p}(z)\frac{\partial\eta}{\partial t}(z)\right.\\
&\qquad+d(1+\frac{1}{p})^2|\nabla\eta(z)|^2+\frac{2d}{(d+2)^2}(1+\frac{1}{p})^2|\nabla\eta(z)|^2\\
&\qquad+64C_{\mathrm{FP}}\Phi_R^2(z_0)|\nabla\eta(z)|^2\\
&\qquad\left.+\frac{64}{d+2}\frac{1}{R^2}\right)dz+\frac{32\,d}{d+2}R^{4/d}\kappa^{2/d(d+2)}\,\mathcal{L}^{d+2}(A_R^{(\kappa)}(z_0)). \qquad (3.80)
\end{aligned}
$$

As in the similar calculation above, we infer

$$
\begin{aligned}
&\frac{d}{(d+2)^2}\operatorname*{ess\cdot sup}_{t_0-R^2<t\leq t_0}\int_{B_R(z_0)}\left|\left(e_\lambda^{(\kappa)}(u_\lambda)(z)\right)^{1/2(1+2/d)}\right|^2\eta^{2+2/p}(z)\,dz\\
&+\frac{2d}{(d+2)^2}\int_{Q_R(z_0)}\left|\nabla\left(e_\lambda^{(\kappa)}(u_\lambda)(z)\right)^{1/2(1+2/d)}\right|^2\eta^{2+2/p}(z)\,dz
\end{aligned}
$$

$$
\begin{aligned}
&\leq 64 C_{\mathrm{FP}} \Phi_R^2(z_0) \int_{Q_R(z_0)} \left| \nabla \left(e_\lambda^{(\kappa)}(u_\lambda)(z) \right)^{1/2(1+2/d)} \right|^2 \eta^2(z)\, dz \\
&+ \int_{Q_R(z_0)} \left(e_\lambda^{(\kappa)}(u_\lambda)(z) \right)^{(1+2/d)} \left(\frac{d}{d+2} \left(1 + \frac{2}{p} \right) \eta^{1+2/p}(z) \frac{\partial \eta}{\partial t}(z) \right. \\
&\quad + d(1+\frac{2}{p})^2 |\nabla \eta(z)|^2 + \frac{2d}{(d+2)^2} \left(1 + \frac{1}{p} \right)^2 |\nabla \eta(z)|^2 \\
&\quad \left. + 64 C_{\mathrm{FP}} \Phi_R^2(z_0) |\nabla \eta(z)|^2 + \frac{64}{d+2} \frac{1}{R^2} \right) dz \\
&\quad + \frac{32\, d}{d+2} R^{4/d} \kappa^{2/d(d+2)}\ \mathcal{L}^{d+2}(A_R^{(\kappa)}(z_0)).
\end{aligned}
\tag{3.81}
$$

Thus, summing up (3.80) with (3.81) and applying the Sobolev imbedding theorem of the parabolic type as before, we arrive at

$$
\begin{aligned}
&\frac{C_{\mathrm{SO}}^{-1} d}{(d+2)^2} \left(\int_{Q_R(z_0)} \left(e_\lambda^{(\kappa)}(u_\lambda)(z) \right)^{(1+2/d)^2} \eta^{2(1+1/p)(1+2/d)}(z)\, dz \right)^{d/d+2} \\
&+ \frac{4d}{(d+2)^2} \int_{Q_R(z_0)} \left| \nabla \left(e_\lambda^{(\kappa)}(u_\lambda)(z) \right)^{1/2(1+2/d)} \right|^2 \eta^{2+2/p}(z)\, dz \\
&\leq 128\ C_{\mathrm{FP}} \Phi_R^2(z_0) \int_{Q_R(z_0)} \left| \nabla \left(e_\lambda^{(\kappa)}(u_\lambda)(z) \right)^{1/2(1+2/d)} \right|^2 dz \\
&+ 2 \int_{Q_R(z_0)} \left(e_\lambda^{(\kappa)}(u_\lambda)(z) \right)^{(1+2/d)} \left(\frac{d}{d+2} \left(1 + \frac{2}{p} \right) \frac{\partial \eta}{\partial t}(z) \eta^{1+2/p}(z) \right. \\
&\quad + d(1+\frac{1}{p})^2 |\nabla \eta(z)|^2 + \frac{2d}{(d+2)^2} \left(1 + \frac{1}{p} \right)^2 |\nabla \eta(z)|^2 \\
&\quad + 64 C_{\mathrm{FP}} \Phi_R^2(z_0) |\nabla \eta(z)|^2 \\
&\quad \left. + \frac{64}{d+2} \frac{1}{R^2} \right) dz + \frac{64\, d}{d+2} R^{4/d} \kappa^{2/d(d+2)}\ \mathcal{L}^{d+2}(A_R^{(\kappa)}(z_0)).
\end{aligned}
\tag{3.82}
$$

Since $\Phi_R^2(z_0) < 1/(64\ C_{\mathrm{FP}})\ d/(d+2)^2$, we then conclude

$$
\begin{aligned}
&\frac{C_{\mathrm{SO}}^{-1} d}{(d+2)^2} \left(\int_{Q_{(1-\sigma)R/2}(z_0)} \left(e_\lambda^{(\kappa)}(u_\lambda)(z) \right)^{(1+2/d)^2} dz \right)^{d/d+2} \\
&+ \frac{4d}{(d+2)^2} \int_{Q_{R_{j-1}}(z_0)} \left| \nabla \left(e_\lambda^{(\kappa)}(u_\lambda)(z) \right)^{1/2(1+2/d)} \right|^2 dz
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{2d}{(d+2)^2}\int_{Q_{R_j}(z_0)}\left|\nabla\left(e_\lambda^{(\kappa)}(u_\lambda)(z)\right)^{1/2(1+2/d)}\right|^2 dz\\
&\quad+\left(\frac{d}{d+2}\left(1+\frac{2}{p}\right)\cdot\frac{4\tau}{(1-\sigma)(1-\tau)}+d\left(1+\frac{1}{p}\right)^2\frac{16\tau^2}{(1-\tau)^2}\right.\\
&\quad\left.+\frac{2d}{(d+2)^2}\left(1+\frac{1}{p}\right)^2\frac{16\tau^2}{(1-\tau)^2}+\frac{d}{(d+2)^2}\frac{16\ \tau^2}{(1-\tau)^2}+\frac{64}{d+2}\right)\\
&\quad\times\frac{2}{(\sigma R)^2}\frac{1}{\tau^{2j}}\int_{Q_R(z_0)}\left(e_\lambda^{(\kappa)}(u_\lambda)(z)\right)^{(1+2/d)}dz\\
&\quad+\frac{64\ d}{d+2}R^{4/d}\kappa^{2/d(d+2)}\ \mathcal{L}^{d+2}(A_R^{(\kappa)}(z_0)).
\end{aligned}
\tag{3.83}
$$

We here used the iteration technique from Giaquinta [**46**, Theorem 3.1, p. 159 and Lemma 3.1, p. 161] again; a choice of τ i.e. $1/\sqrt{2}<\tau<1$ and the limit for $j\to\infty$ proves our second step. □

Now we are in the position of completing our theorem: To this end, take a sequence of decreasing cylinders:

$$
Q_{R_j(z_0)}\qquad\text{with}\qquad R_j=\frac{R}{4}+\frac{R}{2^{j+2}}\quad(j=0,1,\ldots)
$$

and a sequence of increasing levels:

$$
k_j=M+M\left(1-\frac{1}{2^j}\right)\quad(j=0,1,\ldots)
$$

where M is a certain positive number selected below.
Here in (3.75), choose

$$
\begin{aligned}
&(1-\sigma)R/2=R_{j+1},\quad R/2=R_j,\quad \sigma R/2=R/2^{j+3},\\
&\kappa=k_{j+1}.
\end{aligned}
$$

Then recalling (3.75), we can estimate $1/(M^{1+2/d}R^{d+2})\ \int_{A_{R_{j+1}}^{(k_{j+1})}(z_0)}\ (e_\lambda^{(k_{j+1})}(u_\lambda)(z))^{1+2/d}\ dz$ as follows:

$$
\begin{aligned}
&\frac{1}{M^{1+2/d}R^{d+2}}\int_{A_{R_{j+1}}^{(k_{j+1})}(z_0)}\left(e_\lambda^{(k_{j+1})}(u_\lambda)(z)\right)^{1+2/d}dz\\
&\leq\frac{1}{M^{1+2/d}R^{d+2}}\mathcal{L}^{d+2}(A_{R_{j+1}}^{(k_{j+1})}(z_0))^{2/(d+2)}
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\int_{A^{(k_{j+1})}_{R_{j+1}}(z_0)}\left(e^{(k_{j+1})}_{\lambda}(u_\lambda)(z)\right)^{(1+2/d)^2}dz\right)^{d/(d+2)}\\
&\le\frac{1}{M^{1+2/d}R^{d+2}}\mathcal{L}^{d+2}(A^{(k_{j+1})}_{R_j}(z_0))^{2/(d+2)}\\
&\quad\times\left(\frac{C^2_{\mathrm{RH}}4^{j+4}}{R^2}\int_{A^{(k_{j+1})}_{R_j}(z_0)}\left(e^{(k_{j+1})}_{\lambda}(u_\lambda)(z)\right)^{(1+2/d)}dz\right.\\
&\quad+\left.\frac{64\ d}{d+2}\ R^{4/d}(2M)^{2/d(d+2)}\ \mathcal{L}^{d+2}(A^{(k_{j+1})}_{R_j}(z_0))\right).
\end{aligned}
$$

By means of

$$
\begin{aligned}
&\mathcal{L}^{d+2}(A^{(k_{j+1})}_{R_j}(z_0))\\
&\le(k_{j+1}-k_j)^{-(1+2/d)}\int_{A^{(k_j)}_{R_j}(z_0)}\left(e^{(k_j)}_{\lambda}(u_\lambda)(z)\right)^{(1+2/d)}dz,
\end{aligned}
$$

we can proceed to estimate

$$
\begin{aligned}
&\frac{1}{M^{1+2/d}R^{d+2}}\int_{A^{(k_{j+1})}_{R_{j+1}}(z_0)}\left(e^{(k_{j+1})}_{\lambda}(u_\lambda)(z)\right)^{1+2/d}dz\\
&\le 4^{1/d}4^{(1+1/d)j}\left(C^2_{\mathrm{RH}}\ 4^4+\frac{64\ d}{d+2}\ 2^{3(1+2/d)}M^{1+2/d}R^{2(1+2/d)}\right)\\
&\quad\times\left(\frac{1}{M^{1+2/d}R^{d+2}}\int_{A^{(k_j)}_{R_j}(z_0)}\left(e^{(k_j)}_{\lambda}(u_\lambda)(z)\right)^{1+2/d}dz\right)^{1+\delta_0}\\
&=C^3_{\mathrm{RH}}\cdot 4^{(1/d+1)j}\left(\frac{1}{M^{1+2/d}R^{d+2}}\int_{A^{(k_j)}_{R_j}(z_0)}\left(e^{(k_j)}_{\lambda}(u_\lambda)(z)\right)^{1+2/d}dz\right)^{1+\delta_0}
\end{aligned}
\tag{3.84}
$$

with $C^3_{\mathrm{RH}}=4^{1/d}\,(4^4C^2_{\mathrm{RH}}+(64\ d)/(d+2)3^{2(1+2/d)}(MR^2)^{1+2/d}\,)$ and $\delta_0=2/(d+2)$. Thus, an induction for j leads to

$$
\begin{aligned}
&\frac{1}{M^{1+2/d}R^{d+2}}\int_{A^{(k_j)}_{R_j}(z_0)}\left(e^{(k_j)}_{\lambda}(u_\lambda)(z)\right)^{1+2/d}dz\\
&\le(C^3_{\mathrm{RH}})^{-1/\delta_0}4^{-(1+1/d)(1+\delta_0)/\delta_0}\\
&\quad\times\left((C^3_{\mathrm{RH}})^{1/\delta_0}4^{(1+1/d)(1+\delta_0)/\delta_0^2}\frac{1}{M^{1+2/d}R^{d+2}}\int_{Q_{R/2}(z_0)}\left(e^{(k_0)}_{\lambda}(u_\lambda)(z)\right)^{1+2/d}dz\right)^{(1+\delta_0)j}.
\end{aligned}
$$

Here recall (3.65); We infer

$$
\begin{aligned}
&\frac{1}{M^{1+2/d}R^{d+2}}\int_{A_{R_j}^{(k_j)}(z_0)}\left(e_\lambda^{(k_j)}(u_\lambda)(z)\right)^{1+2/d}dz\\
&\leq (C_{\mathrm{RH}}^3)^{-1/\delta_0}4^{-(1+1/d)(1+\delta_0)/\delta_0}\\
&\times\left((C_{\mathrm{RH}}^3)^{1/\delta_0}4^{(1+1/d)(1+\delta_0)/\delta_0^2}\frac{1}{M^{1+2/d}}\left(\frac{C_{\mathrm{RH}}^1}{R^{d+2}}\int_{Q_{R/2}(z_0)}e_\lambda(u_\lambda)(z)dz\right)^{1+2/d}\right)^{(1+\delta_0)j}\\
&\leq (C_{\mathrm{RH}}^3)^{-1/\delta_0}4^{-(1+1/d)(1+\delta_0)/\delta_0}\\
&\times\left((C_{\mathrm{RH}}^3)^{1/\delta_0}4^{(1+1/d)(1+\delta_0)/\delta_0^2}\left(\frac{1}{MR^2}\right)^{1+2/d}\left(C_{\mathrm{RH}}^1\ \Phi_R^2(z_0)\right)^{1+2/d}\right)^{(1+\delta_0)j}
\end{aligned}
$$

where we used Corollary 3.1 and

If we choose $M = 1/R^2$, since

$$
\Phi_R^2 < (C_{\mathrm{RH}}^3)^{-d/(\delta_0(d+2))}4^{-(d+2)/(d+1)(1+\delta_0)/\delta^2}(C_{\mathrm{RH}}^1)^{-1},
$$

we can pass to the limit $j\to\infty$; We conclude

$$
\int_{A_{R/4}^{(k_\infty)}(z_0)} e_\lambda^{(k_\infty)}(u_\lambda)(z)dz = 0 \qquad \text{which} \quad \text{deduces}
$$

$$
\sup_{Q_{R/4}(z_0)} e_\lambda(u_\lambda)(z) \leq \frac{2}{R^2}.
$$

This provides us the assertion of this theorem. □

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