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Non-Archimedean quantum mechanics

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Non-Archimedean quantum mechanics

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NON-ARCHIMEDEAN QUANTUM MECHANICS

Contents

§1. Introduction

It is well known that, traditionally, real numbers are used in theoretical and mathematical physics because length of segments, size of angles and etc. should be measured precisely from the Archimedean axioms. However, for instance, the real space-time $\mathbb{R}^4$ seem still to be not adequate to deal with certain microscopical and cosmological phenomena, and in quantum gravity and in string theory it was proved that a measurement of distance smaller than the Planck length (approximately $10^{-33}cm$) is impossible. Thus the non-Archimedean (n.a.) structure of space-time in quantum physics was considered by Blij and Monna [BM 68]. This paper did not find any response in physics and has been forgotten. Vladimirov and Volovich [VV 84] considered supersymmetric models on superspaces over the n.a. locally compact fields. The n.a. physical models provoked great interest in connection with string theory (see, for instance, [Vol 87a, 87b]). However, it proved to be very difficult to interpret physical models of a level as high as a p-adic string. Therefore, simpler n.a. models such as quantum mechanics and field theory were investigated: In [Vla 88] the theory of generalized functions, the p-adic Gaussian integrals based on the real valued Haar measure (see Appendix A), and the theory of Fourier transformations were studied over the field $\mathbb{Q}_p$ of p-adic numbers. Since these theories, two formalism of quantization over the space $\mathbb{Q}_p^n$ were proposed in [VV 89]. The first approach considers wave function $f : \mathbb{Q}_p^n \to \mathbb{C}$, and the second approach considers function $f : \mathbb{Q}_p^n \to \mathbb{Q}_p$.

To put it concretely, given a dynamical system of one particle moving in $\mathbb{R}^n$ with the position coordinate $x = (x_1, x_2, \cdots, x_n)$ and the momentum coordinate $k = (k_1, k_2, \cdots, k_n)$, the standard quantum mechanics starts with the family of self-adjoint operators $\mathbf{x}_j, \mathbf{k}_j$ ($j = 1, 2, \cdots, n$) in the Hilbert space $L^2(\mathbb{R}^n)$ of $\mathbb{C}$-valued Lebesgue square integrable functions $f(x)$ and $\hat{f}(k)$, which are related by the Fourier transform

$$(1.1) \qquad f(x) = h^{-n/2} \int_{\mathbb{R}^n} \hat{f}(k)\mathbf{e}\left(\frac{(k,x)}{h}\right) dk, \quad \hat{f}(k) = h^{-n/2} \int_{\mathbb{R}^n} f(x)\mathbf{e}\left(-\frac{(k,x)}{h}\right) dx,$$

where h is the Planck constant and $\mathbf{e}(t) = \exp(2\pi\sqrt{-1}\ t)$. The canonical conjugate pairs $\mathbf{x}_j, \mathbf{k}_j$ satisfy HCR (*Heisenberg commutation relations*)

$$(1.2) \qquad [\mathbf{x}_i, \mathbf{x}_j] = [\mathbf{k}_i, \mathbf{k}_j] = 0, \qquad [\mathbf{x}_i, \mathbf{k}_j] = \sqrt{-1}\hbar\delta_{ij},$$

where $\hbar = h/2\pi$, and δ_{ij} is 0 if $i \neq j$ and the identity operator Id if $i = j$. In the Schrödinger representation, the operators $\mathbf{x}_j, \mathbf{k}_j$ are realized by

$$(1.3) \qquad (\mathbf{x}_j f)(x) = x_j f(x),$$

$$(1.4) \qquad \left(\mathbf{k}_j \hat{f}\right)(k) = k_j \hat{f}(k) \quad \left(\text{or} \quad (\mathbf{k}_j f)(x) = \frac{\hbar}{\sqrt{-1}} \frac{\partial}{\partial x_j} f(x)\right).$$

Also, on the assumption that $\mathbf{x}_j, \mathbf{k}_j$ are bounded self-adjoint operators, unitary operators $U(x_j), V(k_j)$ given by

$$(1.5) \qquad \begin{cases} U(x_j) = \mathbf{e}(x_j \mathbf{k}_j / h) : f(s) \mapsto f(s + x_j), \\ V(k_j) = \mathbf{e}(k_j \mathbf{x}_j / h) : f(s) \mapsto \mathbf{e}(k_j s / h) f(s), \end{cases} \quad s, x_j, k_j \in \mathbb{R}, \ j = 1, 2, \cdots, n$$

satisfy WCR (*Weyl commutation relations*)

$$\begin{cases} U(x_i)U(x_j) = U(x_i + x_j), \quad V(k_i)V(k_j) = V(k_i + k_j), \\ U(x_j)V(k_j) = \mathbf{e}(k_j x_j/h)V(k_j)U(x_j). \end{cases} \tag{1.6}$$

The representation of the Heisenberg groups in $L^2(\mathbb{R}^n)$ is one of the cornerstones of ordinary quantum mechanics. I.e. for $t, h \in \mathbb{R}$ and $k, x, y \in \mathbb{R}^n$, the action of $W(x, k, t)$ on $f \in L^2(\mathbb{R}^n)$ given by

$$[W(x, k, t)f](y) = \mathbf{e}\left(\frac{t + (y + x, k)}{h}\right) f(y + x) \tag{1.7}$$

satisfy the Weyl relation

$$W(x, k, t)W(x', k', t') = W(x + x', k + k', t + t' - (k, x')). \tag{1.8}$$

Moreover, the group law

$$(x, k, t) \cdot (x', k', t') = (x + x', k + k', t + t' - (k, x')) \tag{1.9}$$

defines a real Lie group structure $G_{\mathbb{R}}$ on $\mathbb{R}^{2n+1}$. In 1930, Stone and Von Neumann proved the following uniqueness theorem: Any two irreducible unitary representations of the group $G_{\mathbb{R}}$ that map $(t, 0, 0)$ onto the operator $\mathbf{e}(t/h)\cdot\mathrm{Id}$ are unitary equivalent. See [Car 66], [LV 80] and [Yan 96] for informations on the real Heisenberg group, its irreducible unitary representation, the Maslov index and the application to the theory of theta-functions.

The equations (1.1) and (1.5) through (1.9) can be generalized to the p-adic case: For $k, x, y \in \mathbb{Q}_p^n$ and $h \in \mathbb{Q}_p$, the complex factor $\mathbf{e}(k_j y/h)$ and the Lebesgue measure on $\mathbb{R}^n$ are, respectively, replaced by the additive character $\chi_p(k_j y/h)$ of $\mathbb{Q}_p$ and the real valued Haar measure on $\mathbb{Q}_p^n$. In this case, the Hilbert space $L^2(\mathbb{Q}_p^n)$ of $\mathbb{C}$-valued square integrable functions on $\mathbb{Q}_p^n$ is used. The formalism of the p-adic quantum mechanics with $\mathbb{C}$-valued functions is based on a triple $\{L^2(\mathbb{Q}_p^n), W(z), U(t)\}$, where $W(z)$ is a unitary representation of the Heisenberg-Weyl group in $L^2(\mathbb{Q}_p^n)$ (i.e., the representation of WCR), z is a point in the classical phase space $\mathbb{Q}_p^n \times \mathbb{Q}_p^n$, and $U(t)$, $t \in \mathbb{Q}_p$, is a unitary representation of an additive subgroup of $\mathbb{Q}_p^n$ which defines dynamics (i.e., the time evolution operator). In particular, the quantum dynamics of the free particle and the harmonic oscillator were constructed. The above formalism was extended, by Zelenov [Zel 91, 92, 93, 94a, 94b], to the case of infinite-dimensional space, and the theory of representation of the p-adic Heisenberg group was studied. Zelenov's theory is based on a Weyl system (H, W) on the arbitrary dimensional p-adic sympletic space (V, B), where H is a complex Hilbert space and W is a continuous mapping from V to the family of unitary operators on H satisfying the condition (the Weyl relation) $W(x)W(y) = \chi_p(B(x, y)/2)W(x + y)$ (see §3, 3.1).

On the other hand, the equations (1.1) through (1.9) can not be easily generalized to the p-adic case with $\mathbb{Q}_p$-valued functions because an integration suitable for physical applications have not been constructed (although the theory of integration with respect to n.a. valued bounded measures have been considered by several authors (Monna-Springer-Schikhof)). In other words, the definitions of the Gaussian and the Feynman integrals in $\mathbb{R}$,

$$I_G = \int_{\mathbb{R}} \varphi(x) e^{-\frac{x^2}{2b}} \frac{dx}{\sqrt{2\pi b}}, \quad I_F = \int_{\mathbb{R}} \varphi(x) e^{-\frac{x^2}{2b\sqrt{-1}}} \frac{dx}{\sqrt{2\pi b\sqrt{-1}}},$$

are based on the Lebesgue measure dx on $\mathbb{R}$. A new difficulty which occurred in the n.a. theory is the absence of the Lebesgue measure with n.a. values. The solution of this problem was suggested by Khrennikov [Khr 90a, 90b, 91, 92, 95], in which for a n.a. valued field K he introduced the n.a. Gaussian distribution $\gamma_{a,B}$ (with mean value $a \in K^n$ and symmetric covariance matrix $B = (b_{ij})$, $b_{ij} \in K$, $\det(B) \neq 0$) on K^n as a continuous functional on the space $\mathcal{A}(K^n, Z_\tau)$, where Z_τ is one of the quadratic extensions of K, of entire analytic functions, and constructed the n.a. complex Hilbert space $L^2(K^n, \gamma_{0,b})$, $b \in K^\times$ (see §4, 4.1 through 4.3). In particular, the construction of $\gamma_{a,B}$ was based on the Parseval's equality and, by analogy with the theory of real functions, its n.a. Laplace transformation was assumed to be a quadratic exponent on K, i.e., $L'(\gamma_{a,B}) = e^{\frac{1}{2}(Bx,x)+(a,x)}$. By virtue of the Parseval's equality, in order to see how the distribution affects the test function, it is sufficient to know the n.a. Laplace transform of this distribution. Then the n.a. Lebesgue distribution dx was introduced on K^n as a distribution which acts on the test function $f(x) = \varphi(x)e^{-|x|^2}$, $\varphi(x) \in \mathcal{A}(K^n, Z_\tau)$, and which is absolutely continuous relative to the $\gamma_{a,B}$. Moreover, by means of the integral with respect to the above distributions $\gamma_{a,B}$ and dx, the Bargmann-Fock and the Schrödinger representations were constructed.

The purpose of this thesis is to stimulate the n.a. quantum mechanics by using mathematical apparatuses. Namely, the functional equation of the local zeta function, the definition of an l-sheaf on the l-space, and Mahler's theory of integration with respect to a ring $\mathbb{Z}_p$ of p-adic integers, Iwasawa isomorphism and the Morita p-adic Γ-function are, respectively, used in §2, §3 and §4:

The scalar propagators in $\mathbb{Q}_p^n$, which are the inverse Fourier transform of a kinetic operator $(\Box + m^2)$, $m \in \mathbb{R}$, are given by

$$\frac{1}{|(k,k)+m^2|_p}, \ \frac{1}{|(k,k)|_p + m^2}, \ \frac{1}{|k_1|_p^2 + \cdots + |k_n|_p^2 + m^2}, \ \frac{1}{|k|_p^2 + m^2}, \ \frac{1}{|k,m|_p^2},$$

where $k = (k_1, k_2, \cdots, k_n) \in \mathbb{Q}_p^n$ and $|k,m|_p = \max(|k|_p, |m|_p)$. In the one-dimensional case, the second, the third and the fourth propagators coincide; it was the version that was applied in quantum mechanics [Vla 88]. In the massless 2-dimensional case, the fourth version was proposed in [Par 88], in which another p-adic norm $|k|_p := |\sum_j k_j|_p$ was used. The fifth version was calculated in [Smi 91]. In particular, the second version was proposed for p-adic quantum field theory: Let Δ_p be Vladimirov's operator in [Vla 88], which is defined by

$$(\Delta_p \varphi)(x) = \int_{\mathbb{Q}_p^n} |(k,k)|_p \chi_p((k,x)) \widetilde{\varphi}(k) dk, \quad \varphi \in S(\mathbb{Q}_p^n), \tag{1.10}$$

where $S(\mathbb{Q}_p^n)$ is the space of Schwartz-Bruhat functions on $\mathbb{Q}_p^n$ and $\widetilde{\varphi}$ is the Fourier transform of φ,

$$\widetilde{\varphi}(k) = \int_{\mathbb{Q}_p^n} \varphi(x) \chi_p((k,x)) dx.$$

The p-adic Green function $G(x)$ that satisfies $(\Delta_p + m^2)G(x) = \delta(x)$, where $\delta(x)$ is the p-adic Dirac δ-distribution, was proposed in [VV 89]:

$$G(x) = \int_{\mathbb{Q}_p^n} \frac{\chi_p((k,x))}{|(k,k)|_p + m^2} dk, \quad m \in \mathbb{R}_{>0}. \tag{1.11}$$

Since

$$\frac{1}{|(k,k)|_p + m^2} = \lim_{\varepsilon \to \infty} \int_0^\varepsilon \exp\left(-m^2\theta - |(k,k)|_p \theta\right) d\theta, \quad \theta \in \mathbb{R}_{>0},$$

we have

$$G(x) = \lim_{N \to \infty} \int_{(p^{-N}\mathbb{Z}_p)^n} \chi_p((k,x)) \left(\lim_{\varepsilon \to \infty} \int_0^\varepsilon \exp\left(-m^2\theta - |(k,k)|_p \theta\right) d\theta \right) dk.$$

Since $\left|\chi_p((k,x)) \int_0^\varepsilon \exp\left(-m^2\theta - |(k,k)|_p\theta\right) d\theta\right| \leq 1/(|(k,k)|_p + m^2) \in L^1\left((p^{-N}\mathbb{Z}_p)^n\right)$, by Lebesgue's theorem and Fubini's theorem, we obtain

$$G(x) = \lim_{N, \varepsilon \to \infty} \int_0^\varepsilon \exp(-m^2\theta) \int_{(p^{-N}\mathbb{Z}_p)^n} \chi_p((k,x)) \exp(-|(k,k)|_p \theta) dk d\theta.$$

Expanding $\exp\left(-|(k,k)|_p\theta\right)$ into the Taylor series and using Weierstrass' criterion, $G(x)$ is expressed as

$$G(x) = \lim_{N, \varepsilon \to \infty} \int_0^\varepsilon \exp(-m^2\theta) \sum_{\alpha=0}^{\infty} \frac{(-\theta)^\alpha}{\alpha!} \left(\int_{(p^{-N}\mathbb{Z}_p)^n} |(k,k)|_p^\alpha \chi_p((k,x)) dk \right) d\theta. \tag{1.12}$$

For convenience, put

$$J = J(\alpha, n) = \int_{(p^{-N}\mathbb{Z}_p)^n} |(k,k)|_p^\alpha \chi_p((k,x)) dk. \tag{1.13}$$

The properties of $G(x)$ for $n=1$ were studied in [Vla 88]. For any odd prime p, the asymptotic expansions of $G(x)$ for $n=2$ and for $n=4$ were, respectively, given by Bikulov [Bik 91] and Kochubei [Koc 93].

In §2, more generally, we obtain an asymptotic expansion of the p-adic Green function $G(x)$ for any even dimension n and any odd prime p (resp. for any even dimension n and $p=2$) by calculating (1.13) in the functional equation of the local zeta function due to Rallis and Schiffmann [RS 73] (resp. in the t-representation due to Bikulov [Bik 91] and formulas of the p-adic Gaussian integral in Appendix A). At first, the author used only the method of t-representation. Then, Professor Fumihiro Sato at Rikkyo University suggested to simplify the proof by using the local functional equation of the prehomogeneous vector space. His advice gave a nice perspective and the possibility of a generalization. The author is very grateful to Professor F. Sato.

The main results of §2 are as follows:

Proposition 2.2.2. *For the trivial character χ, we have*

$$h_{-1}(t) = (t,-1)_H = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4 \\ -1 & \text{if } p \equiv 3 \pmod 4 \end{cases} \quad \text{for any } t \in \mathbb{Q}_p^\times, \tag{2.2.6}$$

$$\rho(h_{-1},\alpha) = \begin{cases} \Gamma_p(\alpha) & \text{if } p \equiv 1 \pmod 4 \\ -(1+p^{\alpha-1})/(1+p^{-\alpha}) & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{2.2.7}$$

Lemma 2.2.3. *Let Q be the standard quadratic form (x,x) on $\mathbb{Q}_p^n$. For $\alpha \in \mathbb{C}$ and any N sufficiently large,*

(a) *if either $n \equiv 0 \pmod 4$ or $[n \equiv 2 \pmod 4$ and $p \equiv 1 \pmod 4]$, then*

$$J(\alpha,n) = \overline{r(Q)}\Gamma_p(\alpha+1)\Gamma_p(\alpha+n/2)|(x,x)|_p^{-(\alpha+n/2)};$$

(b) *if $n \equiv 2 \pmod 4$ and $p \equiv 3 \pmod 4$, then*

$$J(\alpha,n) = \overline{r(Q)}\Gamma_p(\alpha+1)\frac{1+p^{\alpha+n/2-1}}{1+p^{-(\alpha+n/2)}}|(x,x)|_p^{-(\alpha+n/2)}.$$

For convenience, we denoted by Cond.1 the condition either $n \equiv 0 \pmod 4$ or $[n \equiv 2 \pmod 4$ and $p \equiv 1 \pmod 4]$; Cond.2 the condition $n \equiv 2 \pmod 4$ and $p \equiv 3 \pmod 4$.

Theorem 2.2.4. *For any even dimension n and any odd prime p, the Green function $G(x)$ defined by (1.11) has the following asymptotic expansion:*

$$G(x) \sim \begin{cases} \frac{-p^{n/2}(p^{n/2}-1)}{p(p+1)(p^{n/2+1}-1)}\left(\frac{p}{m}\right)^4 \frac{1}{|(x,x)|_p^{1+n/2}} & \text{if Cond.1} \\ \frac{p^{n/2}(p^{n/2}+1)}{p(p+1)(p^{n/2+1}+1)}\left(\frac{p}{m}\right)^4 \frac{1}{|(x,x)|_p^{1+n/2}} & \text{if Cond.2.} \end{cases}$$

Lemma 2.3.1. *For $p = 2$ and $\alpha \in \mathbb{C}$,*

(a) *if $n \equiv 0 \pmod 4$, then*

$$J(\alpha, n) = (-1)^{\frac{n}{4}} 2^{\frac{n}{2}-1} \Gamma_2(\alpha+1) \frac{2^{-(2\alpha+n)+1} - 2^{-(\alpha+n/2)}}{1 - 2^{-(\alpha+n/2)}} |(x,x)|_2^{-(\alpha+n/2)};$$

(b) *if $n \equiv 2 \pmod 4$, then*

$$J(\alpha, n) = (-1)^{-y_1+\frac{n}{4}+\frac{1}{2}} 2^{\frac{n}{2}-1} \Gamma_2(\alpha+1) |(x,x)|_2^{-(\alpha+n/2)},$$

where y_1 is the second digit of the canonical representation of $(x,x) \in \mathbb{Q}_2$, i.e., $(x,x) = 2^{-\beta}(1 + y_1 2 + \cdots)$, $0 \le y_j \le 1$, $\beta \in \mathbb{Z}$.

Theorem 2.3.2. *For any even dimension n and $p = 2$, the Green function $G(x)$ defined by (1.11) has the following asymptotic expansion:*

$$G(x) \sim \begin{cases} \frac{(-1)^{\frac{n}{4}+1} 2}{3m^4} \left(\frac{2^{n/2}-1}{2^{n/2+1}-1} \right) \frac{1}{|(x,x)|_2^{1+n/2}} & \text{if } n \equiv 0 \pmod 4 \\ \frac{(-1)^{-y_1+\frac{n}{4}+\frac{1}{2}} 2^{\frac{n}{2}+1}}{3m^4} \frac{1}{|(x,x)|_2^{1+n/2}} & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

In §3, we briefly review some results of a Weyl system (H, W) on the finite-dimensional p-adic sympletic space (V, B). In addition, in order to define an l-sheaf $(V, \mathcal{F})$, by Bernshtein and Zelevinskii [BZ 76, pp. 6–9], over (V, B), we introduce the concept of an algebraic Weyl system (H, W) on (V, B), and some necessary and sufficient conditions for a Weyl system (H, W) to be irreducible are investigated. As application, we give another proof of the Ston-Von Neumann Theorem of the p-adic Heisenberg group. From the Schrödinger representation associated to a selfdual $\mathbb{Z}_p$-lattice $\mathfrak{L}$ in (V, B), we construct a Weyl system $(H(\mathfrak{L}, \sigma), W_{\mathfrak{L},\sigma})$ depending a selfdual $\mathbb{Z}_p$-lattice $\mathfrak{L}$ and a $\mathbb{Q}_p$-valued function σ.

The main results of §3 are as follows:

Definition 3.2.2. a) A Weyl system (H, W) is said to be *algebraic* if there exists an open compact subgroup I of V such that $W(x)\varphi = \varphi$ for all $\varphi \in H$ and $x \in I$. For some open compact subgroup I of V, let $H^I = \{\varphi \in H : W(x)\varphi = \varphi \text{ for all } x \in I\}$ be the subspace of I-invariant vectors in H. Then it is clear that (H^I, W) is an algebraic Weyl system, called the *algebraic part* of W. In particular, $(H_0 = H^{V_0}, W)$ is the algebraic part of W.

b) Weyl system (H, W) is said to be *admissible* if it is algebraic and if for each open compact subgroup I of V, H^I is finite dimensional.

Theorem 3.2.3. *A Weyl system (H, W) is algebraic if and only if $S(V)H = H$.*

Theorem 3.2.5. *Let (H, W) be a Weyl system on (V, B). Then (H, W) is irreducible if and only if for any open compact subgroup $Y \in \mathcal{O}$ of V either $H^Y = 0$ or (H^Y, W) is irreducible.*

Proposition 3.2.6. *Let (H, W) be an admissible Weyl system on (V, B). Then we have:*

(i) *(H^*_{al}, W^*_{al}) is also admissible.*

(ii) *(H, W) is irreducible if and only if (H^*_{al}, W^*_{al}) is.*

Proposition 3.2.7 (Schur's Lemma). *If (H, W) is an irreducible Weyl system on (V, B), then $D = Hom_{S(V)}(H, H)$ is a division ring. In other words, $D = \mathbb{C}$, i.e., if $\phi : H \to H$ is a $S(V)$-module homomorphism, then ϕ is a scalar multiple of the identity morphism.*

Theorem 3.3.1 (The Stone-Von Neumann Theorem).

(a) *(H_ω, T_ω) is an irreducible unitary representation of N.*

(b) *For any Hilbert space H, every unitary representation (H, T) of N satisfying*

$$T(t, 0) = \chi_p(t) \cdot \mathrm{Id}_H \quad \textit{for } (t, 0) \in Im(\iota) \tag{3.3.5}$$

is a multiple of T_ω.

Theorem 3.4.3. *The Weyl system $(H(\mathfrak{L}, \sigma), W_{\mathfrak{L},\sigma})$ is irreducible if and only if $\sigma(\mathfrak{L}) \subset \mathbb{Z}_p$.*

In §4, we explain and supplement the articles [Khr 90a, 90b, 91, 92, 95] which are connected with the n.a. quantum mechanics with K-valued functions. In particular, we review the relation between Iwasawa isomorphism and unboundedness of the p-adic Gaussian distribution $\gamma_{0,b}$ ($b \in \mathbb{Q}_p^\times$) (cf. [KE 92]) and present some problems which are connected with the n.a. Gaussian distribution and the n.a. Hilbert space (cf. [Khr 95]). Using the Morita p-adic Γ-function $\Gamma_p^{(M)}(x)$ we construct a p-adic Hilbert space $H(\mathbb{Z}_p)$ and investigate the properties of a differential operator $\frac{d}{dz}$ and its adjoint $\left(\frac{d}{dz}\right)^*$ in $H(\mathbb{Z}_p)$. Moreover, as noted in the equations (1.2) through (1.8), we consider coordinate and momentum operators $\mathbf{x}_j, \mathbf{k}_j$ ($j = 1, 2, \cdots, n$) in the n.a. complex Hilbert space $L^2(K^n, \gamma_{0,b})$ and investigate the properties of operators $\mathbf{x}_j, \mathbf{k}_j$. Also, we construct a restricted n.a. Heisenberg group $\mathcal{N}_h$ depending $h \in K$.

The main results of §4 are as follows:

Theorem 4.2.4. *Every functional $\varphi \in \mathcal{A}(V_R, K)'$ (resp. $\varphi \in \mathcal{A}_1'$) is of the form*

$$\varphi(f) = \sum_{|\alpha|=0}^{\infty} f_\alpha \varphi_\alpha, \quad f = \sum_{|\alpha|=0}^{\infty} f_\alpha x^\alpha \in \mathcal{A}(V_R, K) \ (\textit{resp. } f \in \mathcal{A}_1), \tag{4.2.8}$$

where $\varphi_\alpha \in K$, if and only if $\{|\varphi_\alpha|_K/R^{|\alpha|}\}$ (resp. $\left\{|\varphi_\alpha|_K^{\frac{1}{|\alpha|}}\right\}$) is bounded as $|\alpha| \to \infty$. And we have $\mathcal{A}_1' = \bigcup_{R\in|K^\times|}\mathcal{A}(V_R,K)'$ as set.

Theorem 4.2.6. *Let $s(\mathcal{A}_i', \mathcal{A}_i)$, $i = 0, 1$, be the strong topology of $\mathcal{A}_i'$. Then we have:*

(1) *$\mathcal{A}_i$ are complete and reflexive.*

(2) *$(\mathcal{A}_1', s(\mathcal{A}_1', \mathcal{A}_1)) = (\mathcal{A}_0, T_I)$ and $(\mathcal{A}_0', s(\mathcal{A}_0', \mathcal{A}_0)) = (\mathcal{A}_1, T_P)$.*

Theorem 4.2.7. *The spaces $\mathcal{A}_i'$, $i = 0, 1$, can be explicitly described in the form of the spaces of infinite-order differential operators with coefficients from the field K.*

Proposition 4.3.12. *For $b \in K^\times$ and $R_1, R_2 \in |K^\times|$ with $R_1 \geq \sqrt{\mathfrak{r}(K)|b|_K/|2|_K}$ and $1 \leq R_2 \leq \sqrt{|b|_K/\mathfrak{r}(K)}$, we have*

$$\mathcal{A}(V_{R_1}, Z_\tau) \subset L^2(K^n, \gamma_{0,b}) \subset \mathcal{A}(V_{R_2}, Z_\tau).$$

Proposition 4.5.1. *The transform Z is an isomorphism of $C(\mathbb{Z}_p, \mathbb{Q}_p(\sqrt{\tau}))'$ and the function space*

$$F_\tau = \left\{f \in \mathcal{A}(B, \mathbb{Q}_p(\sqrt{\tau})) : ||f|| = \sup_n |f^{(n)}(1)/n!|_p < \infty\right\}.$$

Proposition 4.5.2.

$$\left(\frac{d}{dz}\right)^*(z^x) = \frac{(x+1)\Gamma_p^{(M)}(x+1)}{\Gamma_p^{(M)}(x+2)}z^{x+1}.$$

Proposition 4.6.1. *$\mathbf{x}_j, \mathbf{k}_j$ are bounded self-adjoint operators in $L^2(K^n, \gamma_{0,b})$.*

Theorem 4.6.2. *For any $y \in V_{R^*}$, $R^* = \left[\sqrt{|b|_K/\mathfrak{r}(K)}\right]_K$, the operator T defined by $(Tf)(x) = f(x+y)$ on the n.a. complex Hilbert space $L^2(K^n, \gamma_{0,b})$ is bounded and isometric.*

Theorem 4.6.3. *Let A be a bounded self-adjoint operator in a n.a. Hilbert space H over K^n. Then for $h, \tau \in K$ and $x \in V_{R_A}$, $R_A = \left[\frac{|h|_K}{\sqrt{|\tau|_K}}A(\mathfrak{r}(K))\right]_K$, $e^{\frac{\sqrt{\tau}}{h}xA}$ is an isometric unitary operator in H.*

Corollary 4.6.4. *Let $h, \tau \in K$ and*

$$x_j \in B_0\left(R_{\mathbf{k}_j}\right) \subset B_0\left(R^*\right), \quad k_j \in B_0\left(R_{\mathbf{x}_j}\right). \tag{4.6.6}$$

Then operators $U(x_j) = e^{\frac{\sqrt{\tau}}{h}x_j\mathbf{k}_j}$ and $V(k_j) = e^{\frac{\sqrt{\tau}}{h}k_j\mathbf{x}_j}$ in $L^2(K, \gamma_{0,b})$ are isometric unitary operators acting on $L^2(K, \gamma_{0,b})$ and satisfy WCR

$$\begin{cases} U(x_i)U(x_j) = U(x_i + x_j), \quad V(k_i)V(k_j) = V(k_i + k_j), \\ U(x_j)V(k_j) = e^{\frac{\sqrt{\tau}}{h}k_jx_j}V(k_j)U(x_j). \end{cases} \tag{4.6.7}$$

Corollary 4.6.5. *We have a n.a. analogue of Heisenberg uncertainty relations:*

$$R_{\mathbf{k}_j} \cdot R_{\mathbf{x}_j} = \left[\frac{|h|_K}{\sqrt{|\tau|_K}} \mathbf{k}_j(\mathfrak{r}(K)) \right]_K \left[\frac{|h|_K}{\sqrt{|\tau|_K}} \mathbf{x}_j(\mathfrak{r}(K)) \right]_K \geq \frac{|h|_K}{\sqrt{|\tau|_K}} \left(\frac{1}{\mathfrak{r}(K)} \right)^2 . \tag{4.6.8}$$

Proposition 4.6.6. *The set*

$$\mathcal{N}_h = \{W(x,k,t) : x \in V_{\mathbf{k}},\ k \in V_{\mathbf{x}} \text{ and } t \in K/\mathcal{D}_h\} \tag{4.6.13}$$

is a group for the product (4.6.12). We call $\mathcal{N}_h$ the restricted n.a. Heisenberg group depending $h \in K$.

Proposition 4.6.7. *For all $W, W' \in \mathcal{N}_h$ and $g, g' \in SL(2,K)$,*

(1) $[WW']_g = [W]_g[W']_g$,

(2) $[[W]_g]_{g'} = [W]_{gg'}$,

(3) $[W]_g = W$ *if and only if g =Id, (Id is the identity of* $SL(2,K)$*),*

(4) $\{[\cdot]_g : g \in SL(2,K)\}$ *is a group of automorphisms of* $\mathcal{N}_h$.

Notations. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{C}_1^\times$ be the set of positive integers, the ring of rational integers, the rational number field, the real number field, the complex number field and the set of complex numbers of modulus 1, respectively. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a field F, let $F_{>x} = \{y \in F : y > x\}$, $F^\times = \{x \in F : x \neq 0\}$ and denote its characteristic by char(F). Let p be any prime number. For any nonzero integer a, let ord$_p a$ be the highest power of p which divides a. (If $a = 0$, we agree to write ord$_p 0 = \infty$.) Note that ord$_p$ behaves a little like a logarithm: ord$_p(a_1 a_2) = $ ord$_p a_1 + $ ord$_p a_2$. So for any rational number $x = a/b \in \mathbb{Q}^\times$, define ord$_p x$ to be ord$_p a - $ ord$_p b$ and then we define the p-adic absolute value by the equality $|x|_p = p^{-\mathrm{ord}_p x}$, and $|0|_p = 0$. The p-adic absolute value has the following properties: (i) $|x|_p \geq 0$, and $|x|_p = 0 \Leftrightarrow x = 0$ (ii) $|xy|_p = |x|_p |y|_p$ (iii) $|x + y|_p \leq \max(|x|_p, |y|_p)$. The last property is called the ultrametric property. If $|x|_p > |y|_p$, then the preceding inequality becomes an equality:

$$|x + y|_p = \max(|x|_p, |y|_p) = |x|_p.$$

The field $\mathbb{Q}_p$ of p-adic numbers is defined to be the completion of the field $\mathbb{Q}$ with respect to the p-adic absolute value. Let $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ be the ring of p-adic integers.

To define the Fourier transform an additive character $\chi_p(x) = \exp(2\pi\sqrt{-1}\{x\}_p)$ of $\mathbb{Q}_p$ is used. Here $\{x\}_p$ is the fractional part of x. The n-dimensional p-adic space $\mathbb{Q}_p^n$ has the standard norm $|x|_p = \max_{1 \leq j \leq n} |x_j|_p$, $x = (x_1, x_2, \cdots, x_n) \in \mathbb{Q}_p^n$. The Fourier transform is defined with respect to the character $\chi_p((k,x)) = \prod_{j=1}^n \chi_p(k_j x_j)$, where $(k,x) = \sum_{j=1}^n k_j x_j$.

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§2. An asymptotic expansion of the p-adic Green function

2.1. The functional equation of the local zeta function. Rallis and Schiffmann [RS 73] investigated a distribution

$$\varphi \longmapsto Z_Q(\varphi, \chi, \alpha) = \int_E \varphi(x)\chi(Q(x))|Q(x)|^{\alpha - n/2} dx, \tag{2.1.1}$$

where $\alpha \in \mathbb{C}$, E is an n-dimensional vector space over the local field F of $\mathrm{char}(F) \neq 2$, Q is a nondegenerate quadratic form on E, and χ is a unitary character of $F^\times$.

In this section, we summarize well-known classical results of the local zeta function Z_Q attached to a quadratic form Q. For the proofs and more details, see [Car 64], [RS 73] and [Wei 64].

Let G be a locally compact abelian group and G^* the Pontrjagin dual of G. For $x \in G$ and $x^* \in G^*$, we write $\langle x, x^* \rangle = x^*(x)$. Let dx, dx^* denote Haar measures on G, G^* which are dual with each other. A continuous function $\varphi : G \to \mathbb{C}_1^\times$ is a *quadratic character* of G if the mapping

$$(x, y) \longmapsto \varphi(x+y)\varphi(x)^{-1}\varphi(y)^{-1} \quad \text{for } x, y \in G \tag{2.1.2}$$

is a bicharacter of $G \times G$. Then we can put

$$\varphi(x+y) = \varphi(x)\varphi(y)\langle x, \rho y \rangle, \tag{2.1.3}$$

where $\rho = \rho_\varphi$ is a symmetric continuous homomorphism from G to G^* and is called *associated with* φ. The quadratic character φ is *nondegenerate* if ρ is an isomorphism from G onto G^*. If φ is nondegenerate, the *modulus* $|\rho|$ of ρ is defined by the formula

$$|\rho| \int_G u(\rho x) dx = \int_{G^*} u(x^*) dx^* \quad \text{for } u \in L^1(G^*). \tag{2.1.4}$$

Note that $|\rho|$ depends on the choice of dx. Let $\Lambda(G)$ be the subspace of $L^1(G)$ consisting of all continuous L^1-functions u with the Fourier transform u^* in $L^1(G^*)$, where u^* is defined by $u^*(x^*) = \int_G u(x)\langle x, x^*\rangle dx$.

Theorem 2.1.1 (cf. [Wei 64, p. 161] and [Car 64, p. 95]). *If φ is a nondegenerate quadratic character of G, then there exists a constant $r(\varphi) \in \mathbb{C}_1^\times$ such that*

$$\int_G \varphi(x)u^*(\rho x)dx = r(\varphi)|\rho|^{-1/2}\int_G \overline{\varphi(x)}u(x)dx \quad \text{for } u \in \Lambda(G). \tag{2.1.5}$$

This means that the Fourier transformation of the nondegenerate quadratic character φ of G is $r(\varphi)|\rho|^{-1/2}\overline{\varphi(\rho^{-1}x^*)}$. From now, we choose the unique Haar measure dx on G satisfying $|\rho| = 1$; this measure is called *adapted for φ*. We identify G with G^* by means of ρ.

Proposition 2.1.2 (cf. [Wei 64, p. 169] and [Car 64, p. 96]). *Let φ be a nondegenerate quadratic character of G such that $\varphi(x) = 1$ for all x in a fixed subgroup H of G. Suppose that $\langle H, \rho x\rangle = 1$ implies $x \in H$. Then $r(\varphi) = 1$.*

Proposition 2.1.3 (cf. [Wei 64, p. 170]). *Let G_1 (resp. G_2) be a locally compact group and φ_1 (resp. φ_2) a nondegenerate quadratic character of G_1 (resp. G_2). Then the mapping*

$$\varphi_1 \otimes \varphi_2 \ : \ (x_1, x_2) \longmapsto \varphi_1(x_1)\varphi_2(x_2)$$

is a nondegenerate quadratic character of $G_1 \times G_2$, and $r(\varphi_1 \otimes \varphi_2) = r(\varphi_1)r(\varphi_2)$.

Now, let τ be a nontrivial additive character of F and E^* the algebraic dual of E. We identify E^* with the dual group of additive group of E. If Q is a nondegenerate quadratic form on E, then $\tau \circ Q$ is a nondegenerate quadratic character of the additive group of E. Let $B(x, y) = \{Q(x + y) - Q(x) - Q(y)\}$ be the nondegenerate symmetric bilinear form associated with Q. Then the isomorphism ρ from E onto E^* associated with $\tau \circ Q$ is defined by $\langle x, \rho y\rangle = \tau(B(x, y))$. We identify E^* with E by means of ρ and choose the Haar measure dx on E adapted for $\tau \circ Q$. Then for $u \in \Lambda(E)$ the Fourier transform $\hat{u}$ is defined by

$$\hat{u}(y) = \int_E u(x)\tau(B(x, y))dx. \tag{2.1.6}$$

By Theorem 2.1.1, there exists a constant $r(Q) = r(\tau \circ Q) \in \mathbb{C}_1^\times$ such that

$$\int_E \hat{u}(x)\tau(Q(x))dx = r(Q)\int_E u(x)\tau(-Q(x))dx. \tag{2.1.7}$$

The constant $r(Q)$ depends on the choice of τ and the formula (2.1.7) is valid for any $u \in \Lambda(E)$ and, in particular, for any $u \in S(E)$, where $S(E)$ is the space of Schwartz-Bruhat functions on E; this is a subspace of $\Lambda(E)$ and the Fourier transformation gives a bijection from $S(E)$ to $S(E^*)$. Let $(\ ,\ )_H$ be the Hilbert symbol defined on $F^\times/(F^\times)^2 \times F^\times/(F^\times)^2$. If we put $h_a(b) = (a,b)_H$, then $a \mapsto h_a$ is an isomorphism from the finite abelian group $F^\times/(F^\times)^2$ onto its dual.

We can find a coordinate system on E such that

$$Q(x) = a_1 x_1^2 + \cdots + a_n x_n^2 \quad (a_j \in F^\times,\ j = 1, \cdots, n). \tag{2.1.8}$$

Suppose F is ultrametric. Then the quadratic form Q is characterized by three invariant: The dimension n, the discriminant $D = a_1 \cdots a_n (F^\times)^2$ and the Hasse-Minkowski character $\prod_{k<j}(a_k, a_j)_H$. We put $\triangle = (-1)^{[n/2]} D$, where the symbol $[x]$ denote the greatest integer not exceeding x. By Proposition 2.1.3 and (2.1.7), we have the following proposition.

Proposition 2.1.4 (cf. [RS 73, pp. 499–504]). *Let $q(x) = x^2$ be a quadratic form on F; put $f(a) = r(aq)$ for $a \in F^\times$; and let Q be as in (2.1.8). Then we have:*

(i) *$\varphi(x) = f(x)/f(1)$ is a nondegenerate quadratic character of $F^\times/(F^\times)^2$ associated with the isomorphism $a \mapsto h_a$;*

(ii) *$r(\varphi)^{-1} = \sum_{a \in F^\times/(F^\times)^2} \overline{\varphi(a)}$;*

(iii) *$r(Q) = f(1)^{n-1} f(D) \prod_{k<j}(a_k, a_j)_H$.*

For $t \in F^\times$, we calculate the number $r(tQ)$. As a function of t, $r(tQ)$ is invariant under the subgroup $(F^\times)^2$ of $F^\times$. Thus we can put

$$r(tQ) = \sum_{a \in F^\times/(F^\times)^2} \beta_a(Q) h_a(t), \ (\beta_a(Q) \in \mathbb{C}). \tag{2.1.9}$$

Proposition 2.1.5 (cf. [RS 73, p. 505]). *If F is ultrametric, then we have*

$$r(tQ) = \begin{cases} r(Q) h_\triangle(t) & \text{if } n \text{ is even} \\ r(Q) r(\varphi) f(1) \sum_{a \in F^\times/(F^\times)^2} \overline{f(a\triangle)} h_a(t) & \text{if } n \text{ is odd.} \end{cases} \tag{2.1.10}$$

Let χ be a unitary character of $F^\times$ and $\alpha \in \mathbb{C}$. For $\varphi \in S(E)$, we define the local zeta function $Z_Q(\varphi, \chi, \alpha)$ by

$$Z_Q(\varphi, \chi, \alpha) = \int_E \varphi(x) \chi(Q(x)) |Q(x)|^{\alpha - n/2} dx. \tag{2.1.11}$$

Theorem 2.1.6 (cf. [RS 73, p. 521]). *The integral (2.1.11) is absolutely convergent for* $\mathrm{Re}(\alpha) > 0$ *(resp.* $\mathrm{Re}(\alpha) > n/2 - 1$*) if Q is anisotropic (resp. if Q is isotropic). Further,* *as a function of α, $Z_Q(\varphi,\chi,\alpha)$ has an analytic continuation to a meromorphic function on $\mathbb{C}$, and satisfies the functional equation*

(2.1.12)

$$Z_Q(\varphi,\chi,\alpha) = \rho(\chi,\alpha - n/2 + 1)\times \sum_{a\in F^\times/(F^\times)^2} \overline{\beta_a(Q)}h_a(-1)\rho(\chi h_a,\alpha)Z_Q(\hat{\varphi},\chi^{-1}h_a^{-1}, n/2-\alpha),$$

where $\beta_a(Q)$ is defined in (2.1.9) and $\rho(\chi,\alpha)$ is the gamma factor of Tate. Hence for all $\varphi \in S(F)$, we have

$$\int_{F^\times} \varphi(t)\chi(t)|t|^\alpha dt^\times = \rho(\chi,\alpha)\int_{F^\times} \hat{\varphi}(t)\chi^{-1}(t)|t|^{1-\alpha}dt^\times, \quad 0 < \mathrm{Re}(\alpha) < 1. \tag{2.1.13}$$

2.2. Calculation of $J = J(\alpha, n)$ for any odd prime p. In this section, we calculate $J(\alpha,n)$, given by (1.13), for any even dimension n and any odd prime p by using the functional equation of the local zeta function, and obtain an asymptotic expansion of the p-adic Green function.

From now on, we choose the standard quadratic form $Q(x) = (x, x)$ on $\mathbb{Q}_p^n$. For a unitary character χ of $\mathbb{Q}_p^\times$ and a test function $\varphi \in S(\mathbb{Q}_p^n)$, the local zeta function $Z_Q(\varphi,\chi,\alpha)$ is given by

$$Z_Q(\varphi,\chi,\alpha) = \int_{\{k\in\mathbb{Q}_p^n \mid (k,k)\neq 0\}} \varphi(k)\chi((k,k))|(k,k)|_p^{\alpha-n/2}dk. \tag{2.2.1}$$

When χ is trivial, we simply write $Z_Q(\varphi,\alpha)$. For any integer N and $y \in \mathbb{Q}_p^n$, let $\mathrm{ch}_{N,y}(k)$ denote the characteristic function of $y + (p^{-N}\mathbb{Z}_p)^n$. Fix an element $x \in \mathbb{Q}_p^n$ and put

$$\psi_{N,x}(k) = \chi_p((k,x))\mathrm{ch}_{N,0}(k). \tag{2.2.2}$$

Then $\psi_{N,x}(k)$ is in $S(\mathbb{Q}_p^n)$ and we have

$$J = J(\alpha,n) = Z_Q(\psi_{N,x},\alpha + n/2). \tag{2.2.3}$$

By the functional equation (2.1.12), we have

$$J = \rho(1,\alpha+1)\sum_{a\in\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2} \overline{\beta_a(Q)}h_a(-1)\rho(h_a,\alpha+n/2)Z_Q(\hat{\psi}_{N,x},h_a^{-1},-\alpha). \tag{2.2.4}$$

Note that

$$\hat{\psi}_{N,x}(k) = p^{nN} \times \mathrm{ch}_{-N,-x}(k).$$

Hence we have

$$\begin{aligned}
Z_Q(\hat{\psi}_{N,x}, h_a^{-1}, -\alpha) &= \int_{\{k\in\mathbb{Q}_p^n \mid (k,k)\neq 0\}} \hat{\psi}_{N,x}(k) h_a^{-1}((k,k))|(k,k)|_p^{-(\alpha+n/2)} dk \\
&= \int_{\{k\in\mathbb{Q}_p^n \mid (k,k)\neq 0\}} p^{nN} \mathrm{ch}_{-N,-x}(k) h_a^{-1}((k,k))|(k,k)|_p^{-(\alpha+n/2)} dk \\
&= p^{nN} \int_{\{k\in -x+(p^N\mathbb{Z}_p)^n\}} h_a^{-1}((k,k))|(k,k)|_p^{-(\alpha+n/2)} dk \\
&= h_a((x,x))|(x,x)|_p^{-(\alpha+n/2)} \quad \text{for any } N \text{ sufficiently large.}
\end{aligned}$$

On the other hand, by calculating (2.1.13) for the trivial character χ, we easily obtain $\rho(1, \alpha+1) = \Gamma_p(\alpha+1)$, where $\Gamma_p(\alpha) = (1-p^{\alpha-1})/(1-p^{-\alpha})$ is the Gelfand-Graev p-adic Γ-function (see (A.28) in Appendix A). Thus, for any N sufficiently large, we have

$$J = \Gamma_p(\alpha+1)|(x,x)|_p^{-(\alpha+n/2)} \sum_{a\in\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2} \overline{\beta_a(Q)} h_a(-(x,x))\rho(h_a, \alpha+n/2). \tag{2.2.5}$$

Proposition 2.2.1 (cf. [VVZ 94, p. 130]). *Let $p \neq 2$ and let ε be a unit, $\varepsilon \notin (\mathbb{Q}_p^\times)^2$. Then*

$$h_\varepsilon(x) = (x,\varepsilon)_H = 1 \textit{ if and only if } v(x) \textit{ is even,}$$

where $|x|_p = p^{v(x)}$, $v(x) \in \mathbb{Z}$.

Proposition 2.2.2. *For the trivial character χ, we have*

$$h_{-1}(t) = (t,-1)_H = \begin{cases} 1 & \textit{if } p \equiv 1 \pmod 4 \\ -1 & \textit{if } p \equiv 3 \pmod 4 \end{cases} \quad \textit{for any } t \in \mathbb{Q}_p^\times, \tag{2.2.6}$$

$$\rho(h_{-1}, \alpha) = \begin{cases} \Gamma_p(\alpha) & \textit{if } p \equiv 1 \pmod 4 \\ -(1+p^{\alpha-1})/(1+p^{-\alpha}) & \textit{if } p \equiv 3 \pmod 4). \end{cases} \tag{2.2.7}$$

Proof. Since $p \equiv 1 \pmod 4$ if and only if $-1 \in (\mathbb{Q}_p^\times)^2$, $h_{-1}(t) = 1$ and $\rho(h_{-1}, \alpha) = \Gamma_p(\alpha)$. Assume $p \equiv 3 \pmod 4$. If $h_{-1}(t) = 1$, then $t \in (\mathbb{Q}_p^\times)^2$ and $t = a^2 + b^2$ for some $a, b \in \mathbb{Q}_p^\times$. Thus $a_0^2 + b_0^2 \equiv 0 \pmod p$, i.e., -1 is a quadratic residue modulo p. Thus the Legendre symbol $(-1|p) = 1$. This is a contradiction. Hence $h_{-1}(t) = -1$. Next let $g_\alpha(t) = |t|_p^{\alpha-1} h_{-1}(t)$. Then $g_\alpha(t)$ is a multiplicative character of $\mathbb{Q}_p^\times$ and is a homogeneous generalized function of degree $g_\alpha(t)$. Since $\hat{g}_\alpha(tk) = |t|_p^{-1} g_\alpha(1/t)\hat{g}_\alpha(k) = |t|_p^{-\alpha} h_{-1}(t)\hat{g}_\alpha(k)$, the Fourier transform $\hat{g}_\alpha$ of g_α is a homogeneous generalized function of degree $|t|_p^{-\alpha} h_{-1}(t)$, i.e., $\hat{g}_\alpha(k)$ is proportional to degree $|k|_p^{-\alpha} h_{-1}(k)$. Hence we can write

$$\hat{g}_\alpha(k) = \Gamma_p(g_\alpha)|k|_p^{-\alpha} h_{-1}(k) \quad (\Gamma_p(g_\alpha) \in \mathbb{C}). \tag{2.2.8}$$

Putting $k = 1$ in (2.2.8), we obtain

$$\Gamma_p(g_\alpha) = -\hat{g}_\alpha(1) = -\int_{\mathbb{Q}_p} g_\alpha(t)\chi_p(t)dt.$$

Since $h_{-1}(t) = -1$ for all $t \in \mathbb{Q}_p^\times$, $g_1(t) \equiv -1$ and by Proposition 2.2.1, we can write $g_\alpha(t) = |t|_p^{\alpha-1+\pi\sqrt{-1}/\ln p}$. Therefore

$$\begin{aligned}\Gamma_p(g_\alpha) &= -\int_{\mathbb{Q}_p} |t|_p^{\alpha-1+\pi\sqrt{-1}/\ln p}\chi_p(t)dt \\ &= -\Gamma_p(\alpha + \pi\sqrt{-1}/\ln p) = -(1+p^{\alpha-1})/(1+p^{-\alpha}).\end{aligned}$$

In the formula (2.1.13), let $\varphi(t) = \chi_p(t) \in S(\mathbb{Q}_p)$. Then

$$\begin{aligned}\int_{\mathbb{Q}_p^\times} \hat{\chi}_p(t)h_{-1}(t)|t|_p^{1-\alpha}dt^\times &= \int_{\mathbb{Q}_p} \chi_p(t)\hat{g}_{-\alpha+1}(t)dt \\ &= \Gamma(g_{-\alpha+1})\int_{\mathbb{Q}_p} \chi_p(t)h_{-1}(t)|t|_p^{\alpha-1}dt \\ &= -(1+p^{-\alpha})/(1+p^{\alpha-1})\int_{\mathbb{Q}_p^\times} \chi_p(t)h_{-1}(t)|t|_p^{\alpha}dt^\times.\end{aligned}$$

Thus $\rho(h_{-1}, \alpha) = -(1+p^{\alpha-1})/(1+p^{-\alpha})$. ∎

Now we calculate $J = J(\alpha, n)$. From (2.1.9) and (2.1.10), we observe the following: If n is even, we have $\beta_a(Q) = 0$ if $a \neq \triangle$ and $\beta_\triangle(Q) = r(Q)$, where $\triangle = (-1)^{n/2}$; if n is odd, we have $\beta_a(Q) = r(Q)r(\varphi)f(1)\overline{f(a\triangle)}$, where $\triangle = (-1)^{[n/2]}$. Thus, for any N sufficiently large, (2.2.5) can be rewritten as follows: If n is even,

$$J = \overline{r(Q)}\Gamma_p(\alpha+1)h_\triangle(-(x,x))\rho(h_\triangle, \alpha+n/2)|(x,x)|_p^{-(\alpha+n/2)}; \tag{2.2.9}$$

if n is odd,

$$J = \overline{r(Q)}\cdot\overline{r(\varphi)}\Gamma_p(\alpha+1)h_\triangle(-(x,x))\rho(h_\triangle, \alpha+n/2)|(x,x)|_p^{-(\alpha+n/2)} + \Phi((x,x)), \tag{2.2.10}$$

where

$$\begin{aligned}\Phi((x,x)) = \overline{r(Q)}\cdot\overline{r(\varphi)}\cdot\overline{f(1)}\Gamma_p(\alpha+1)|(x,x)|_p^{-(\alpha+n/2)}&\times \\ \sum_{a\in\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2;\ a\neq\triangle} & f(a\triangle)h_a(-(x,x))\rho(h_a, \alpha+n/2).\end{aligned}$$

By Proposition 2.2.2, we obtain the following lemma.

Lemma 2.2.3. *Let Q be the standard quadratic form (x,x) on $\mathbb{Q}_p^n$. For $\alpha \in \mathbb{C}$ and any N sufficiently large,*

(a) *if either $n \equiv 0 \pmod 4$ or $[n \equiv 2 \pmod 4$ and $p \equiv 1 \pmod 4]$, then*

$$J(\alpha,n) = \overline{r(Q)}\Gamma_p(\alpha+1)\Gamma_p(\alpha+n/2)|(x,x)|_p^{-(\alpha+n/2)};$$

(b) *if $n \equiv 2 \pmod 4$ and $p \equiv 3 \pmod 4$, then*

$$J(\alpha,n) = \overline{r(Q)}\Gamma_p(\alpha+1)\frac{1+p^{\alpha+n/2-1}}{1+p^{-(\alpha+n/2)}}|(x,x)|_p^{-(\alpha+n/2)}.$$

For convenience, we denoted by Cond.1 the condition either $n \equiv 0 \pmod 4$ or $[n \equiv 2 \pmod 4$ and $p \equiv 1 \pmod 4]$; Cond.2 the condition $n \equiv 2 \pmod 4$ and $p \equiv 3 \pmod 4$.

Theorem 2.2.4. *For any even dimension n and $p \geq 3$, the Green function $G(x)$ defined by (1.11) has the following asymptotic expansion:*

$$G(x) \sim \begin{cases} \frac{-p^{n/2}(p^{n/2}-1)}{p(p+1)(p^{n/2+1}-1)}\left(\frac{p}{m}\right)^4 \frac{1}{|(x,x)|_p^{1+n/2}} & \textit{if Cond.1} \\ \\ \frac{p^{n/2}(p^{n/2}+1)}{p(p+1)(p^{n/2+1}+1)}\left(\frac{p}{m}\right)^4 \frac{1}{|(x,x)|_p^{1+n/2}} & \textit{if Cond.2.} \end{cases}$$

Proof. Suppose that n and p satisfy Cond.1. We substitute the formula (a) of Lemma 2.2.3 into the expression for the Green function (1.12):

$$\text{(2.2.11)}\quad G(x) = \overline{r(Q)}\int_0^\infty \exp(-m^2\theta)\sum_{\alpha=0}^{\infty}\frac{(-\theta)^\alpha}{\alpha!}\Gamma_p(\alpha+1)\Gamma_p(\alpha+n/2)|(x,x)|_p^{-(\alpha+n/2)}d\theta.$$

For further simplification of (2.2.11), we substitute the following expansion

$$\Gamma_p(\alpha+1)\Gamma_p(\alpha+n/2) = \sum_{r=0}^{\infty} a_r p^{-nr/2}\left[p^{-r\alpha} - (1+p^{n/2-1})p^{-(r-1)\alpha} + p^{n/2-1}p^{-(r-2)\alpha}\right],$$

where $a_r = \sum_{j=0}^{r} p^{(n/2-1)j}$, into the expression (2.2.11). We can change the order of summations because the double series of α and r are absolutely convergent. Thus we have

$$G(x) = \overline{r(Q)}|(x,x)|_p^{-n/2}\int_0^\infty \exp(-m^2\theta)\sum_{r=0}^{\infty}\frac{a_r}{p^{nr/2}}\times$$
$$\left[\exp\left(\frac{-\theta p^{-r}}{|(x,x)|_p}\right) - (1+p^{n/2-1})\exp\left(\frac{-\theta p^{-(r-1)}}{|(x,x)|_p}\right) + p^{n/2-1}\exp\left(\frac{-\theta p^{-(r-2)}}{|(x,x)|_p}\right)\right]d\theta.$$

The above series converges uniformly, so that by term by term integration and passage to limit, we obtain

$$
\begin{aligned}
G(x) &= \overline{r(Q)}|(x,x)|_p^{1-n/2} \sum_{r=0}^{\infty} a_r p^{-nr/2} \\
&\qquad \times \left[\frac{1}{m^2|(x,x)|_p + p^{-r}} - \frac{1+p^{n/2-1}}{m^2|(x,x)|_p + pp^{-r}} + \frac{p^{n/2-1}}{m^2|(x,x)|_p + p^2p^{-r}} \right] \\
&= \overline{r(Q)}|(x,x)|_p^{1-n/2}(p-1) \sum_{r=0}^{\infty} a_r p^{-(n/2+1)r} \\
&\qquad \times \frac{p^{-r}(p^2-p^{n/2}) - (p^{n/2}-1)m^2|(x,x)|_p}{(m^2|(x,x)|_p + p^{-r})(m^2|(x,x)|_p + pp^{-r})(m^2|(x,x)|_p + p^2p^{-r})}.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\lim_{|(x,x)|_p\to\infty} r(Q)|(x,x)|_p^{1+n/2} G(x) \\
&\qquad = \frac{(p-1)(1-p^{n/2})}{m^4} \sum_{r=0}^{\infty} a_r p^{-(n/2+1)r} \\
&\qquad = \frac{(p-1)(1-p^{n/2})}{(1-p^{n/2-1})m^4} \left(\sum_{r=0}^{\infty} p^{-(n/2+1)r} - p^{(n/2-1)} \sum_{r=0}^{\infty} p^{-2r} \right) \\
&\qquad = \frac{-p^{n/2}(p^{n/2}-1)}{p(p+1)(p^{n/2+1}-1)} \left(\frac{p}{m} \right)^4 .
\end{aligned}
$$

Similarly, in the case Cond.2, we obtain the desired result if we use the expansion

$$
\Gamma_p(\alpha+1) \frac{1+p^{\alpha+n/2-1}}{1+p^{-(\alpha+n/2)}} = \sum_{r=0}^{\infty} b_r p^{-nr/2} \left[p^{-r\alpha} - (1-p^{n/2-1})p^{-(r-1)\alpha} - p^{n/2-1}p^{-(r-2)\alpha} \right],
$$

where $b_r = \sum_{j=0}^{r} (-1)^{r-j} p^{(n/2-1)j}$. ∎

2.3. An alternative method for $p = 2$. In this section, we calculate $J(\alpha, n)$, given by (1.13), for any even dimension n and $p = 2$ by using the t-representation introduced by Bikulov [Bik 91] and obtain an asymptotic expansion of the p-adic Green function.

Bikulov [Bik 91] used the following formula (it is called the t-representation) to split the double integral $J(\alpha, n)$ into two one-dimensional p-adic Gaussian integrals (see Appendix A): For $\alpha > 0$ and $p^{-M} < |z|_p < p^m$ $(z \in \mathbb{Q}_p,\ m, M \in \mathbb{Z})$,

$$
|z|_p^{\alpha} = \Gamma_p(\alpha+1) \lim_{M,m\to\infty} \int_{p^{-m}\le|t|_p\le p^M} |t|_p^{-(\alpha+1)} (\chi_p(zt) - 1) dt. \tag{2.3.1}
$$

His method can be used for any prime number p. We use it to calculate $J(\alpha, n)$ for any even dimension n and $p = 2$. The results are given by the following lemma.

Lemma 2.3.1. *For $p = 2$ and $\alpha \in \mathbb{C}$,*

(a) *if $n \equiv 0 \pmod 4$, then*

$$J(\alpha, n) = (-1)^{\frac{n}{4}} 2^{\frac{n}{2}-1} \Gamma_2(\alpha+1) \frac{2^{-(2\alpha+n)+1} - 2^{-(\alpha+n/2)}}{1 - 2^{-(\alpha+n/2)}} |(x,x)|_2^{-(\alpha+n/2)};$$

(b) *if $n \equiv 2 \pmod 4$, then*

$$J(\alpha, n) = (-1)^{-y_1 + \frac{n}{4} + \frac{1}{2}} 2^{\frac{n}{2}-1} \Gamma_2(\alpha+1) |(x,x)|_2^{-(\alpha+n/2)},$$

where y_1 is the second digit of the canonical representation of $(x,x) \in \mathbb{Q}_2$, i.e., $(x,x) = 2^{-\beta}(1 + y_1 2 + \cdots)$, $0 \le y_j \le 1$, $\beta \in \mathbb{Z}$.

Proof. Let n be even and $p = 2$. In order to use the t-representation, setting $z = (k,k) \in \mathbb{Q}_2^{\times}$ in (2.3.1) and substituting it into (1.13), we obtain

$$\Gamma_2(\alpha+1) \int_{(2^{-N}\mathbb{Z}_2)^n} \left(\lim_{M,m\to\infty} \int_{2^{-m} \le |t|_2 \le 2^M} |t|_2^{-(\alpha+1)} \left(\chi_2(zt) - 1\right) dt \right) \chi_2((k,x)) dk, \tag{2.3.2}$$

for $2^{-M} < |z|_2 < 2^m$. Since $\chi_2((k,x)) \int_{2^{-m} \le |t|_2 \le 2^M} |t|_2^{-(\alpha+1)} \left(\chi_2(zt) - 1\right) dt$ (see (2.3.1)) converges uniformly as $M \to \infty$ for any $k \in (2^{-N}\mathbb{Z}_2)^n$, (2.3.2) can be rewritten in the form

$$\Gamma_2(\alpha+1) \lim_{M,m\to\infty} \int_{2^{-m} \le |t|_2 \le 2^M} |t|_2^{-(\alpha+1)} \times \left\{ \int_{|k_1|_2 \le 2^N} \cdots \int_{|k_n|_2 \le 2^N} \left(\prod_{j=1}^n \chi_2(tk_j^2 + x_j k_j) - \prod_{j=1}^n \chi_2(x_j k_j) \right) dk_1 \cdots dk_n \right\} dt.$$

Using the expressions (A.6) and (A.20) in Appendix A, and integrating it with respect to t, and taking the limit for $M \to \infty$, we obtain

$$\begin{aligned}
J = J(\alpha, n) &= \int_{(2^{-N} Z_2)^n} |(k,k)|_2^{\alpha} \chi_2((k,x)) dk \\
&= \Gamma_2(\alpha+1) \sum_{r \ge -2N+1} 2^{-(\alpha+1)r + n(1-r)/2} \\
&\quad \times \begin{cases} \prod_{j=1}^n \delta(|x_j|_2 - 2^{-N+1}) \int_{|t|_2 = 2^r} \lambda_2^n(t) \chi_2\left(\frac{(x,x)}{-4t}\right) dt, & |t|_2 = 2^{-2N+1} \\ \prod_{j=1}^n \Omega(2^N |x_j|_2) \int_{|t|_2 = 2^r} \lambda_2^n(t) \chi_2\left(\frac{(x,x)}{-4t}\right) dt, & |t|_2 = 2^{-2N+2} \\ \prod_{j=1}^n \Omega(2^{-N-r+1} |x_j|_2) \int_{|t|_2 = 2^r} \lambda_2^n(t) \chi_2\left(\frac{(x,x)}{-4t}\right) dt, & |t|_2 \ge 2^{-2N+3} \end{cases} \\
&\quad - 2^{nN} \Gamma_2(\alpha+1) \prod_{j=1}^n \Omega(2^N |x_j|_2) \sum_{r \ge -2N+1} 2^{-(\alpha+1)r} \int_{|t|_2 = 2^r} dt\ .
\end{aligned}$$

If $2^{-N+1} < |x|_2 = \max_{1\le j\le n}|x_j|_2 = 2^l \le 2^{N+r-1}$, we obtain

$$J = 2^{n/2}\Gamma_2(\alpha+1) \sum_{r\ge -N+l+1} 2^{-(\alpha+1+n/2)r} \int_{|t|_2=2^r} \lambda_2^n(t)\chi_2\left(\frac{(x,x)}{-4t}\right) dt. \tag{2.3.3}$$

Let $(x,x) = 2^{-\beta}(y,y)$, $|(y,y)|_2 = 1$. After the change of variable $t = -(y,y)/2^r s$ ($|s|_2 = 1$, $dt = 2^r ds$), we obtain

$$J = 2^{n/2}\Gamma_2(\alpha+1) \sum_{r\ge -N+l+1} 2^{-(\alpha+n/2)r} \int_{|s|_2=1} \lambda_2^n\left(\frac{(y,y)}{-2^r s}\right) \chi_2(2^{-\beta+r-2}s)ds. \tag{2.3.4}$$

Since $|-(y,y)/2^r s|_2 = 2^r$, we can write

$$\frac{(y,y)}{-2^r s} = 2^{-r}(1 + t_1 2 + t_2 2^2 + \cdots), \quad 0 \le t_j \le 1. \tag{2.3.5}$$

Since n is even, by the definition of λ_2 (see (A.17) in Appendix A), we have

$$\lambda_2^n\left(\frac{(y,y)}{-2^r s}\right) = \frac{(1+(-1)^{t_1+1/2})^n}{2^{n/2}}.$$

On the other hand, comparison of the second digits of the canonical representation on both sides in (2.3.5) gives $t_1 \equiv -(y_1+s_1) \pmod 2$, where s_1 and y_1 are the second digits of the canonical representation of s and (y,y), respectively. So we have

$$\lambda_2^n\left(\frac{(y,y)}{-2^r s}\right) = \frac{(1+(-1)^{t_1+1/2})^n}{2^{n/2}} = \frac{(1+(-1)^{-(y_1+s_1)+1/2})^n}{2^{n/2}}. \tag{2.3.6}$$

Substitution of the value (2.3.6) into (2.3.4) and the change of variable $s = 1 + 2s_1 + s'$ ($|s'|_2 \le 2^{-2}$, $0 \le s_1 \le 1$ and $ds' = ds$) gives

$$J = \Gamma_2(\alpha+1) \sum_{r\ge -N+l+1} 2^{-(\alpha+n/2)r} \left[(1+(-1)^{-y_1+1/2})^n C_1 + (1-(-1)^{-y_1+1/2})^n C_3\right] X, \tag{2.3.7}$$

where, by (A.6) in Appendix A,

$$X = \int_{|s'|_2\le 2^{-2}} \chi_2(2^{-\beta+r-2}s')ds' = \begin{cases} \frac{1}{4} & \text{for } r \ge \beta \\ 0 & \text{for } r < \beta, \end{cases}$$

$$C_1 = \chi_2(2^{-\beta+r-2}) = \exp\left(2\pi\sqrt{-1}\left\{2^{-\beta+r-2}\right\}_2\right) = \begin{cases} 1 & \text{for } r \ge \beta+2 \\ -1 & \text{for } r = \beta+1 \\ i & \text{for } r = \beta, \end{cases}$$

$$C_3 = \chi_2(2^{-\beta+r-2}3) = \exp\left(2\pi\sqrt{-1}\left\{2^{-\beta+r-2}3\right\}_2\right) = \begin{cases} 1 & \text{for } r \geq \beta+2 \\ -1 & \text{for } r = \beta+1 \\ -i & \text{for } r = \beta. \end{cases}$$

Consider the condition $-N+l+1 < \beta$ (since $-N+1 < l$, we have $2^{-2N} < |(x,x)|_2$). Substitution of the values X and C_j $(j=1,3)$ into (2.3.7) gives

$$J = \Gamma_2(\alpha+1)\left\{\left(\sum_{r\geq\beta+2} 2^{-(\alpha+n/2)r} - 2^{-(\alpha+n/2)(\beta+1)}\right)A + 2^{-(\alpha+n/2)\beta}B\right\}, \tag{2.3.8}$$

where

$$A = \frac{(1+(-1)^{-y_1+1/2})^n + (1-(-1)^{-y_1+1/2})^n}{4} = \begin{cases} (-1)^{\frac{n}{4}}2^{\frac{n}{2}-1}, & n \equiv 0 \pmod 4 \\ 0, & n \equiv 2 \pmod 4, \end{cases}$$

$$\begin{aligned} B &= \frac{(1+(-1)^{-y_1+1/2})^n - (1-(-1)^{-y_1+1/2})^n}{4}(-1)^{1/2} \\ &= \begin{cases} 0, & n \equiv 0 \pmod 4 \\ (-1)^{-y_1+\frac{n}{4}+\frac{1}{2}}2^{\frac{n}{2}-1}, & n \equiv 2 \pmod 4). \end{cases} \end{aligned}$$

Substitution of the values A and B into (2.3.8) gives the formulas (a) and (b). ∎

Theorem 2.3.2. *For any even dimension n and $p=2$, the Green function $G(x)$ defined by (1.11) has the following asymptotic expansion:*

$$G(x) \sim \begin{cases} \frac{(-1)^{\frac{n}{4}+1}2}{3m^4}\left(\frac{2^{n/2}-1}{2^{n/2+1}-1}\right)\frac{1}{|(x,x)|_2^{1+n/2}} & \textit{if } n \equiv 0 \pmod 4 \\ \frac{(-1)^{-y_1+\frac{n}{4}+\frac{1}{2}}2^{\frac{n}{2}+1}}{3m^4}\frac{1}{|(x,x)|_2^{1+n/2}} & \textit{if } n \equiv 2 \pmod 4). \end{cases}$$

Proof. We substitute the formulas (a) and (b) in Lemma 2.3.1 into the expression for the Green function (1.12) and use the expansions

$$\begin{aligned} &\Gamma_2(\alpha+1)\frac{2^{-(2\alpha+n)+1} - 2^{-(\alpha+n/2)}}{1-2^{-(\alpha+n/2)}} \\ &\qquad = 2^{-n/2}\sum_{r=0}^{\infty} c_r 2^{-nr/2}\left[2^{-r\alpha} - (1+2^{-n/2+1})2^{-(r+1)\alpha} + 2^{-n/2+1}2^{-(r+2)\alpha}\right], \end{aligned}$$

where $c_r = \sum_{j=0}^{r} 2^{(n/2-1)j}$; $\Gamma_2(\alpha+1) = \sum_{r=0}^{\infty} 2^{-r}\left(2^{-r\alpha} - 2^{-(r-1)\alpha}\right)$. Then the proof of the theorem follows the same process as in Theorem 2.2.4. ∎

§3. Algebraic Weyl system and application

3.1. Definitions and the general properties of Weyl systems. We review some of well-known results of a Weyl systems (H, W) on the finite dimensional p-adic symplectic space (V, B). For proofs and more details, see [Zel 91] and [VVZ 94] (cf. for the ∞-dimensional case, see [Zel 92] and [Zel 94b]).

By the definition of a p-adic symplectic space is the pair (V, B), where V is a finite dimensional $\mathbb{Q}_p$-vector space and B is a nondegenerate antisymmetric $\mathbb{Q}_p$-bilinear form on V. Then $\dim_{\mathbb{Q}_p} V = 2n$ is even (cf. for the ∞-dimensional case, see [Zel 94b, p. 423]).

Given $0 \neq e_1 \in V$, there must exist a $x \in V$ for which $B(e_1, x) \neq 0$, since B is nondegenerate. We choose a $a \in \mathbb{Q}_p$ so that $e_{n+1} = be_1 + ax$, and $B(e_1, e_{n+1}) = aB(e_1, x) = 1$. Then the hyperbolic plane $h_y = \text{span}\{e_1, e_{n+1}\}$ has matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with respect to the basis $\{e_1, e_{n+1}\}$. Since h_y is nondegenerate (i.e., the pair $(h_y, B|_{h_y \times h_y})$ is p-adic symplectic space), we have $V = h_y \oplus h_y^{\perp}$, where $h_y^{\perp}$ is also nondegenerate. Hence, we repeat the preceding construction in $h_y^{\perp}$, to obtain an orthogonal decomposition of V of the form

$$V = h_y^1 \oplus h_y^2 \oplus \cdots \oplus h_y^n,$$

where each h_y^j is a hyperbolic plane. Thus there is a basis $\{e_j : 1 \leq j \leq 2n\}$ of V for which the matrix of the form is $\begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$, where E_n is the $n \times n$ unit matrix, and 0 is the $n \times n$ null matrix. Also the map $P_j : V \to h_y^j,\ j = 1, 2, \cdots, n$, which is defined by the formula

$$P_j x = B(e_j, x) e_{n+j} + B(x, e_{n+j}) e_j,\ x \in V,$$

is the orthogonal projection map on h_y^j.

Let X_0 be a $\mathbb{Z}_p$-span of the symplectic basis $\{e_j : 1 \leq j \leq 2n\}$. Then X_0 is an open compact $\mathbb{Z}_p$-submodule of V and has the following properties:

$$B(x, y) \in \mathbb{Z}_p\ \forall x, y \in X_0; \quad \forall x \in V \backslash X_0,\ \exists y \in X_0 \text{ such that } B(x, y) \in \mathbb{Q}_p \backslash \mathbb{Z}_p. \tag{3.1.1}$$

Zelenov [Zel 91] defined a *Weyl system* on (V, B) as a pair (H, W) of a complex Hilbert space H and a continuous map $W : x \mapsto W(x)$ from V to the family of unitary operators on H satisfying the condition (the Weyl relation)

$$W(x) W(y) = \chi_p(B(x, y)/2) W(x + y). \tag{3.1.2}$$

Two Weyl systems (H, W) and (H', W') on (V, B) are *unitary equivalent* if there exists an intertwining unitary operator $U : H \to H'$. I.e. U is an unitary operator such that

$$U W(x) U^{-1} = W'(x),\ x \in V.$$

A Weyl system (H, W) is said to be *irreducible* if there exists no non-trivial subspace of H invariant under the $W(x)$, $x \in V$. We say that (H, W) can be *represented as a direct sum* $\oplus_{j\in I}(H_j, W)$ *of Weyl systems* (H_j, W) if H can be written as a direct sum $\oplus_{j\in I} H_j$ of subspaces H_j which are invariant under the action of operators $W(x)$.

We denote by $V_0 = \{x \in V : |x|_p \leq 1\}$ the open compact subgroup of V. Let (H, W) be a Weyl system on (V, B). A vector $\varphi_0 \in H$ is called a *vacuum vector* of (H, W) if the condition $W(x)\varphi_0 = \varphi_0$ is satisfied for all $x \in V_0$. The set of vacuum vectors of (H, W) forms the vacuum subspace H_0 of H.

Every Weyl system (H, W) on (V, B) is, in a certain sense, determined by its restriction $(H, W|_{V_0})$ on $(V_0, B|_{V_0 \times V_0})$. Since V_0 is a compact abelian group, all irreducible unitary representations of V_0 are one-dimensional. Let $V_0^* = \mathrm{Hom}(V_0, \mathbb{T})$ be the group of characters of V_0. Then we have

$$V^*/V_0^{\perp} \cong V/V_0 \ni \hat{\alpha} \overset{\sim}{\longmapsto} \hat{\alpha}^* \in V_0^*, \quad \hat{\alpha}^*(x) := \chi_p((\alpha, x)), \ x \in V_0,$$

where α is a representative of the coset $\hat{\alpha} \in V/V_0$. By the theory of unitary representations of compact groups, the representation space H can be expressed as an orthogonal sum

$$H = \oplus_{\hat{\alpha} \in V/V_0} H_{\hat{\alpha}},$$

where $H_{\hat{\alpha}}$ is the maximal subspace on which V_0 acts as a multiple of $\hat{\alpha}^*$.

Let us choose an element α from each coset $\hat{\alpha} \in V/V_0$ and denote the family of such elements by J_0. Let

$$\Sigma = \{\varphi_\alpha = W(\alpha)\varphi_0 \in H : \alpha \in J_0\}. \tag{3.1.3}$$

If α_1 and α_2 belong to the same coset $\hat{\alpha} \in V/V_0$, then, using (3.1.1), the definition of the vacuum vector and (3.1.3), we obtain

$$\varphi_{\alpha_1} = W(\alpha_1)\varphi_0 = W(\alpha_2 + (\alpha_1 - \alpha_2))\varphi_0 = \chi_p(B(\alpha_1, \alpha_2)/2)\varphi_{\alpha_2}.$$

Therefore, a change of the element α in $\hat{\alpha}$ induces only a scalar multiplication of φ_α. We call Σ *the system of coherent states* of (H, W).

The investigation of Weyl systems on p-adic symplectic spaces is essentially based on the notions of vacuum vector and the system of coherent states as follows:

Theorem 3.1.1 (cf. [Zel 91]). *Weyl systems has the following properties:*

(i) *For any Weyl system, there exists the vacuum vector.*

(ii) *A Weyl system (H, W) is irreducible if and only if its vacuum subspace H_0 is one-dimensional. Otherwise, if we choose some orthonormal basis $\{\varphi_0^i\}_{i\in\mathbb{N}}$ in H_0, (H, W) can* *be represented as a direct sum* $\oplus_{i\in\mathbb{N}}(H_i, W)$ *of irreducible Weyl systems* (H_i, W)*, where subspace* H_i *is the span of the vectors* $\{\varphi^i_\alpha = W(\alpha)\varphi^i_0 : \alpha \in J_0\}_{i\in\mathbb{N}}$.

(iii) *If* (H, W) *is irreducible Weyl system with the vacuum vector* φ_0*, then the system of coherent states* Σ *of* (H, W) *forms the orthonomal basis in* H.

(iv) *All irreducible Weyl systems are unitary equivalent.*

Example 3.1.2 (cf. [VV 89] and [Zel 89, in case of $p = 2$]). For 2-dimensional case, an irreducible Weyl system is constructed as a pair $(L^2(\mathbb{Q}_p), W)$ on $(\mathbb{Q}_p^2, B)$, where the unitary operator $W(z)$ is defined by

$$W(z)\varphi(x) = \chi_p(px + pq/2)\varphi(x + q), \quad z = (q, p) \in \mathbb{Q}_p^2,\ \varphi \in L^2(\mathbb{Q}_p),$$

the symplectic form $B : \mathbb{Q}_p^2 \times \mathbb{Q}_p^2 \to \mathbb{Q}_p$ is given by $B(z, z') = qp' - q'p$ $(z' = (q', p') \in \mathbb{Q}_p^2)$. The vacuum vector has the form $\varphi_0(x) = \Omega(|x|_p)$, where $\Omega(x)$ is 1 if $0 \le x \le 1$ and 0 if $x > 1$.

Example 3.1.3 (cf. [VVZ 94, p. 243]). We denote the tensor product $\otimes^n(L^2(\mathbb{Q}_p), W)$ of n irreducible Weyl systems of Example 3.1.2 by $(L^2(\mathbb{Q}_p^n), W^{(n)})$ on $(\mathbb{Q}_p^{2n}, B)$. Hence

$$W^{(n)}(z) = \otimes_{j=1}^n W(z_j),\ z = (z_1, z_2, \cdots, z_n) \in \mathbb{Q}_p^{2n}$$

and the vacuum vector has the form

$$\varphi_0^{(n)}(x) = \Omega(|x|_p) = \prod_{j=1}^n \Omega(|x_j|_p),\ x = (x_1, x_2, \cdots, x_n) \in \mathbb{Q}_p^n.$$

On the group V, we normalize the Haar measure dx by the condition $\int_{V_0} dx = 1$. In the Hilbert space $L^2(V)$ of $\mathbb{C}$-valued square integrable functions on V, the standard inner product and the norm are given by

$$(\psi, \varphi) = \int_{\mathbb{Q}_p} \psi(x)\overline{\varphi(x)}dx, \quad ||\psi||^2 = (\psi, \psi).$$

We define a Hilbert subspace $L^2(V_0)$ of $L^2(V)$ by

(3.1.4) $$L^2(V_0) = \left\{f \in L^2(V) : f(x + y) = \chi_p(B(x, y)/2)f(x)\ \forall y \in V_0\right\},$$

and an operator $\widetilde{W}(x)$ by

(3.1.5) $$\widetilde{W}(x)f(y) = \chi_p(B(x, y)/2)f(y - x) \ \ (x \in V,\ f \in L^2(V_0)).$$

Example 3.1.4 (cf. [Zel 91] and [VVZ 94, p. 243]). The pair $(L^2(V_0), \widetilde{W})$ is an irreducible Weyl system on (V, B), and the vacuum vector has the form $\varphi_0(x) = \Omega(|x|_p)$.

Let $Sp(2n, \mathbb{Q}_p)$ be the group of automorphisms of the space (V, B) that preserve the symplectic form B. We fix a basis of V and express any $g \in Sp(2n, \mathbb{Q}_p)$ by a matrix $(g_{jk}) \in M_{2n}(\mathbb{Q}_p)$. Then $G = \{g \in Sp(2n, \mathbb{Q}_p) : ||g||' = \max_{1\le j,k\le 2n}|g_{jk}|_p = 1\}$ forms a maximal compact subgroup of $Sp(2n, \mathbb{Q}_p)$.

Theorem 3.1.5 (cf. [Zel 91]). *Let (H, W) be an irreducible Weyl system on (V, B). Then a family of operators $\{U(g) : g \in G\}$ satisfying*

$$U(g)W(x) = W(gx)U(g), \quad x \in V$$

forms an unitary representation of G in H, and any vacuum vector of (H, W) is an eigenvector of the $U(g)$.

3.2. An algebraic Weyl system. First of all, we recall that the definition of *l-sheaf* on the *l-space* by Bernshtein and Zelevinskii [BZ 76, pp. 6–9]: A topological space X is said to be an l-space if it is Hausdorff, locally compact, and zero-dimensional. Denote by $C^\infty(X)$ and $S(X)$ the space of all locally constant $\mathbb{C}$-valued functions on X and the space of Schwartz-Bruhat functions on X, respectively. We say that an *l-sheaf is defined on X* if with each $x \in X$ there is associated a $\mathbb{C}$-vector space $\mathcal{F}_x$ and there is defined a family $\mathcal{F}$ of cross-sections (that is, mapping φ defined on X such that $\varphi(x) \in \mathcal{F}_x$ for each $x \in X$) such that the following conditions hold:

(i) $\mathcal{F}$ is invariant under addition and multiplication by functions in $C^\infty(X)$.

(ii) If φ is a cross-section that coincides with some cross-section in $\mathcal{F}$ in a neighbourhood of each point, then $\varphi \in \mathcal{F}$.

(iii) If $\varphi \in \mathcal{F}$, $x \in X$, and $\varphi(x) = 0$, then $\varphi = 0$ in some neighbourhood of x.

(iv) For any $x \in X$ and $\xi \in \mathcal{F}_x$ there exists a $\varphi \in \mathcal{F}$ such that $\varphi(x) = \xi$.

The l-sheaf itself is denoted by $(X, \mathcal{F})$. The spaces $\mathcal{F}_x$ are called *stalks*, and the elements of $\mathcal{F}$ *cross-sections of the sheaf*. We call the set $\operatorname{supp} \varphi = \{x \in X : \varphi(x) \neq 0\}$ the support of the cross-section $\varphi \in \mathcal{F}$. Condition (iii) guarantees that $\operatorname{supp} \varphi$ is closed.

A cross-section $\varphi \in \mathcal{F}$ is called *finite* if $\operatorname{supp} \varphi$ is compact. We denote the space of finite cross-sections of $(X, \mathcal{F})$ by $\mathcal{F}_c$. It is clear that $\mathcal{F}_c$ is a $S(X)$-module, and that $S(X)\mathcal{F}_c = \mathcal{F}_c$. It turns out that this property can be taken as the basis for the definition of an l-sheaf.

Proposition 3.2.1 (cf. [BZ 76, Proposition 1.14]). *Let M be a $S(X)$-module such that $S(X)M = M$. Then there exists one and up to isomorphism only one l-sheaf $(X, \mathcal{F})$ such that M is isomorphic as an $S(X)$-module to the space of finite cross-sections $\mathcal{F}_c$.*

Proposition 3.2.1 means that defining an l-sheaf on X is equivalent to defining an $S(X)$-module M such that $S(X)M = M$.

In this section, in order to define an l-sheaf $(V, \mathcal{F})$ on the p-adic symplectic space (V, B), we define an algebraic Weyl system, and we prove some necessary and sufficient conditions for Weyl systems (H, W) on (V, B) to be irreducible.

Let (H, W) be a Weyl system on (V, B) with a vacuum vector $\varphi_0 \in H_0$ and $S(V)$ the space of Schwartz-Bruhat functions on V. Then the convolution product

$$f \hat{*} g(x) = \int_V \chi_p(-B(x,y)/2) f(y) g(x-y) dy \tag{3.2.1}$$

makes $S(V)$ an associative $\mathbb{C}$-algebra without the unit element. For each $f \in S(V)$, we define a linear endomorphism $E_W(f)$ of H by

$$E_W(f)\varphi = \int_V f(x) W(x) \varphi \; dx, \quad \varphi \in H. \tag{3.2.2}$$

Since f is locally constant on V with compact support, this integral is well-defined. It is easy to see that $E_W(f \hat{*} g) = E_W(f)E_W(g)$. Indeed, for $f, g \in S(V)$ and $\varphi \in H$, using the Weyl relation (3.1.2) and (3.2.2), we get

$$\begin{aligned}
E_W(f \hat{*} g)\varphi &= \int_V \int_V \chi_p(-B(x,y)/2) f(y) g(x-y) W(x) \varphi \; dxdy \\
&= \int_V \int_V \chi_p(-B(t,y)/2) f(y) g(t) W(t+y) \varphi \; dtdy \\
&= \int_V g(t) \left(\int_V f(y) W(y) W(t) \varphi \; dy \right) dt \\
&= \int_V g(t) E_W(f) W(t) \varphi \; dt = E_W(f) E_W(g) \varphi.
\end{aligned}$$

Hence H is a $S(V)$-module.

Definition 3.2.2. a) Weyl system (H, W) is said to be *algebraic* if there exists an open compact subgroup I of V such that $W(x)\varphi = \varphi$ for all $\varphi \in H$ and $x \in I$. For some open compact subgroup I of V, let $H^I = \{\varphi \in H : W(x)\varphi = \varphi \text{ for all } x \in I\}$ be the subspace of I-invariant vectors in H. Then it is clear that (H^I, W) is an algebraic Weyl system, called the *algebraic part* of W. In particular, $(H_0 = H^{V_0}, W)$ is the algebraic part of W.

b) Weyl system (H, W) is said to be *admissible* if it is algebraic and if for each open compact subgroup I of V, H^I is finite dimensional.

Algebraic Weyl system gives an l-sheaf on (V, B) as follows:

Theorem 3.2.3. *Weyl system (H, W) is algebraic if and only if $S(V)H = H$.*

Proof. Let (H, W) be an algebraic. I.e. any $\varphi \in H$ is fixed by some open compact subgroup, say I, of V. Let $\xi_I = \text{volume}(I)^{-1}\times$ characteristic function of I. Then $E_W(\xi_I)\varphi = \varphi$. Conversely, let $S(V)H = H$. We can construct an open compact subgroup I of V such that $W(x)\varphi = \varphi$ for all $\varphi \in H$ and $x \in I$ as follows: Let $\varphi \in H$. Then

φ can be written as $\varphi = \sum_i E_W(f_i)\varphi_i$. Then, using the linearity of W, (3.2.2) and the Weyl relation (3.1.2), we have

$$\begin{aligned} W(x)\varphi &= W(x)\sum_i E_W(f_i)\varphi_i = \sum_i W(x)E_W(f_i)\varphi_i = \sum_i W(x)\int_V f_i(y)W(y)\varphi_i\, dy \\ &= \sum_i \int_V f_i(y)W(x)W(y)\varphi_i\, dy = \sum_i \int_V f_i(y)\chi_p(B(x,y)/2)W(x+y)\varphi_i\, dy \\ &= \sum_i \int_V f_i(t-x)\chi_p(B(x,t)/2)W(t)\varphi_i\, dt. \end{aligned}$$

Since f_i is Schwarz-Bruhat function, there exists a positive integer m sufficiently large such that $\mathrm{supp} f_i \subset p^{-m}X_0$ and for any $t \in \mathrm{supp} f_i$, $f_i(t-x) = f_i(t)$, $|x|_p \leq p^{-m}$, where X_0 is a $\mathbb{Z}_p$-span of the symplectic basis of V. Let $I = p^m X_0$. Then I is an open compact subgroup of V and by (3.1.1) $B(x,t) \in \mathbb{Z}_p$ for all t and x. Thus we have $W(x)\varphi = \varphi$ for all $\varphi \in H$ and $x \in I$. ∎

Let I be an open compact subgroup of V. Then $I \cap X_0$ is also an open compact subgroup of V. Let $\mathcal{O} = \{I \cap X_0 : I \text{ is an open compact subgroup of } V\}$ and $Y \in \mathcal{O}$, and let $\xi_Y = \text{volume}(Y)^{-1}\times$ characteristic function of Y. Then ξ_Y is an idempotent in $S(V)$ and $S(Y) = \xi_Y \hat{*} S(V) \hat{*} \xi_Y$ is an associative $\mathbb{C}$-algebra with the unit element ξ_Y. Let $l(x)$ and $r(x)$ be a translations of $S(V)$ given by

(3.2.3) $\qquad l(x)f(y) = \chi_p(B(x,y)/2)f(y-x), \quad r(x)f(y) = \chi_p(B(x,y)/2)f(y+x).$

The following is easily proved.

Proposition 3.2.4. *We have the following properties:*

(i) $W(x)E_W(f) = E_W(l(x)f)$ *for all* $x \in V$ *and* $f \in S(V)$.

(ii) $f\hat{*}(l(x)g) = (r(-x)f)\hat{*}g$, $(r(x)f)\hat{*}g = f\hat{*}(l(-x)g)$ *for all* $f, g \in S(V)$.

(iii) $l(x)(f\hat{*}g) = (l(x)f)\hat{*}g$, $r(x)(f\hat{*}g) = f\hat{*}(r(x)g)$ *for all* $f, g \in S(V)$.

(iv) $l(x)\xi_Y = r(x)\xi_Y = \xi_Y$ *for all* $x \in Y$.

By (iv) of Proposition 3.2.4, $S(Y)$ is the space of elements f of $S(V)$ such that $l(x)f = r(x)f = f$ for all $x \in Y$. Clearly, the image $E_W(\xi_Y)H$ coincides with the subspace H^Y of Y-invariant vectors in H. It follows, in particular, that for any exact sequence of $S(V)$-modules $0 \to H_1 \to H_2 \to H_3 \to 0$ the sequence $0 \to H_1^Y \to H_2^Y \to H_3^Y \to 0$ is also exact.

Let $H_W(Y) = \{\varphi \in H : W(-y)\varphi = \varphi\ \forall y \in Y\}$. Then the kernel of $E_W(\xi_Y)$ is the space $H(Y)$ spanned by all vectors of the form $W(y)\varphi - \varphi$, $y \in Y$, $\varphi \in H_W(Y)$. For it is clear that $E_W(\xi_Y)|_{H(Y)} = 0$ and that Y act trivially on $H/H(Y)$ since $H/H(Y) \cong \mathrm{Im}(E_W(\xi_Y)) = H^Y$, so that $E_W(\xi_Y)$ is the identity map on $H/H(Y)$.

Theorem 3.2.5. *Let (H, W) be a Weyl system on (V, B). Then (H, W) is irreducible if and only if for any open compact subgroup $Y \in \mathcal{O}$ of V either $H^Y = 0$ or (H^Y, W) is irreducible.*

Proof. Suppose that for any open compact subgroup $Y \in \mathcal{O}$ of V, $H^Y \neq 0$ and (H^Y, W) is reducible. Then there exists a non-trivial $S(Y)$-submodule $\mathfrak{a}$ of H^Y that is invariant with respect to the action of the operators $W(x)$, $x \in V$. Let $\mathfrak{b}$ be a $S(V)$-submodule of H generated by $\mathfrak{a}$. Since $\mathfrak{a} \subset \mathfrak{b}$, $\mathfrak{b}$ is non-trivial. Every $\varphi \in \mathfrak{b}$ is represented in the form

$$\varphi = \sum_{i=1}^{s} E_W(f_i)a_i + \sum_{j=1}^{t} n_j b_j \quad (s, t \in \mathbb{N},\ f_i \in S(V),\ a_i, b_j \in \mathfrak{a},\ n_j \in \mathbb{Z}).$$

Then, using (i) of Proposition 3.2.4, we have for $x \in V$,

$$W(x)\varphi = \sum_{i=1}^{s} W(x)E_W(f_i)a_i + \sum_{j=1}^{t} n_j W(x)b_j = \sum_{i=1}^{s} E_W(l(x)f_i)a_i + \sum_{j=1}^{t} n_j W(x)b_j.$$

Since $l(x)f_i \in S(V)$ and $W(x)b_j \in \mathfrak{a}$, $W(x)\varphi \in \mathfrak{b}$. Hence (H, W) is reducible. Conversely, suppose that $\mathfrak{b} \subset H$ is a non-trivial $S(V)$-submodule of (H, W). For any open compact subgroup $Y \in \mathcal{O}$ of V, the sequence $0 \to \mathfrak{b}^Y \to H^Y \to (H/\mathfrak{b})^Y \to 0$ is exact. Hence for $Y \in \mathcal{O}$ with $(H/\mathfrak{b})^Y \neq 0$, $\mathfrak{b}^Y$ is a non-trivial $S(Y)$-submodule of H^Y. ∎

Let (H, W) be an algebraic Weyl system on (V, B) and $H^* = \mathrm{Hom}_{\mathbb{C}}(H, \mathbb{C})$ the dual space of H. We define a Weyl system (H^*, W^*) on (V, B) by

$$\langle \varphi, W^*(x)\varphi^* \rangle = \langle W(-x)\varphi, \varphi^* \rangle, \tag{3.2.4}$$

for $x \in V$, $\varphi \in H$, $\varphi^* \in H^*$ and $\langle , \rangle$ is the natural pairing between H and H^*. This Weyl system (H^*, W^*) is not algebraic, so we take its algebraic part. More precisely, let $H^*_{al} = \bigcup_{Y \in \mathcal{O}} (H^*)^Y$ and $W^*_{al}(x) = W^*(x)|_{H^*_{al}}$, $x \in V$. Then (H^*_{al}, W^*_{al}) is an algebraic Weyl system on (V, B). Clearly, $\langle \varphi, E_{W^*_{al}}(f)\varphi^* \rangle = \langle E_W(\widetilde{f})\varphi, \varphi^* \rangle$ for $f \in S(V)$ where $\widetilde{f}(x) = f(-x)$.

Proposition 3.2.6. *Let (H, W) be an admissible Weyl system on (V, B). Then we have:*

(i) *(H^*_{al}, W^*_{al}) is also admissible.*

(ii) *(H, W) is irreducible if and only if (H^*_{al}, W^*_{al}) is.*

Proof. (i) We show that for all $Y \in \mathcal{O}$, $(H^*_{al})^Y$ is finite dimensional. Let $\varphi^* \in (H^*_{al})^Y$. Then we have for $\varphi \in H$,

$$\langle \varphi, \varphi^* \rangle = \langle \varphi, E_{W^*_{al}}(\xi_Y)\varphi^* \rangle = \langle E_W(\widetilde{\xi_Y})\varphi, \varphi^* \rangle = \langle E_W(\xi_Y)\varphi, \varphi^* \rangle.$$

This show that $(H_{al}^*)^Y = (H^Y)^*$. I.e. $(H_{al}^*)^Y$ is finite dimensional, since (H, W) is an admissible.
(ii) If $\mathfrak{a}$ is a non-trivial $S(V)$-submodule of H, then $\mathfrak{a}^\perp = \{\varphi^* \in H_{al}^* : \langle \mathfrak{a}, \varphi^* \rangle = 0\}$ is a non-trivial $S(V)$-submodule of H_{al}^*. Hence (H_{al}^*, W_{al}^*) is reducible. The converse follows from that $H \to (H_{al}^*)_{al}^*$ is an isomorphism, i.e., $(W_{al}^*)_{al}^* = W$. ∎

Proposition 3.2.7 (Schur's Lemma). *If (H, W) is an irreducible Weyl system on (V, B), then $D = Hom_{S(V)}(H, H)$ is a division ring. In other words, $D = \mathbb{C}$, i.e., if $\phi : H \to H$ is a $S(V)$-module homomorphism, then ϕ is a scalar multiple of the identity morphism.*

Proof. Clearly, any non-zero $\phi \in D$ is bijective, hence is invertible. Consequently, every non-zero element of D is a unit and thus D is a division ring. Also, if ϕ is not a scalar multiple of the identity map Id, then $\phi - \lambda \cdot \mathrm{Id}$ is invertible for any $\lambda \in \mathbb{C}$. Let $0 \neq \varphi \in H$. If a sequence $(\phi - \lambda_1 \cdot \mathrm{Id})^{-1}\varphi, (\phi - \lambda_2 \cdot \mathrm{Id})^{-1}\varphi, \cdots$, for $\lambda_i \in \mathbb{C}$ distinct, of elements in H is linearly dependent, then there exists a sequence $z_1, z_2, \cdots$ of elements in $\mathbb{C}$ such that not all the z_i, say z_1 and z_2, are equal to 0 and $z_1(\phi - \lambda_1 \cdot \mathrm{Id})^{-1}\varphi + z_2(\phi - \lambda_2 \cdot \mathrm{Id})^{-1}\varphi = 0$. It implies $(z_1 + z_2)\left(\phi - \frac{\lambda_1 + \lambda_2}{z_1 + z_2} \cdot \mathrm{Id}\right)\varphi = 0$, which contradicts the fact that $\phi - \lambda \cdot \mathrm{Id}$ is invertible for any $\lambda \in \mathbb{C}$. Thus $(\phi - \lambda \cdot \mathrm{Id})^{-1}\varphi$ $(\lambda \in \mathbb{C})$ are linearly independent. But, indeed, by (iii) of Theorem 3.1.1 H is spanned by $W(\alpha)\varphi_0,\ \alpha \in J_0$, which is countable. So H can not contain uncountable many linearly independent vectors. Therefore ϕ is a scalar multiple of the identity morphism. ∎

3.3. Application. In this section, using Theorem 3.2.5 we give another proof of the Stone-Von Neumann Theorem of p-adic Heisenberg group.

The Stone-Von Neumann Theorem. Heisenberg group $N = N(V, B)$ of a p-adic smplectic space (V, B) is the set of pairs $(t, x) \in \mathbb{Q}_p \times V$ with the multiplication law

$$(t, x) \cdot (t', x') = (t + t' + B(x, x')/2, x + x'),$$

and it satisfies an exact sequence $0 \longrightarrow \mathbb{Q}_p \xrightarrow{\iota} N(V, B) \xrightarrow{\kappa} V \longrightarrow 0$, where ι and κ are given by $\iota(t) = (t, 0)$ and $\kappa(t, x) = x$, respectively.

The group $\mathrm{Im}(\iota) = \{(t, 0) : t \in \mathbb{Q}_p\}$ is the center and the commutator subgroup of N. Thus a character η of $\mathrm{Im}(\iota)$ is given by the formula $\eta(\iota(t)) = \chi_p(\lambda t)$ for some $\lambda \in \mathbb{Q}_p$. From now we assume $\lambda = 1$. Let ℓ be a Lagrangian subspace of V. Then $L = \mathbb{Q}_p \times \ell$ is an abelian subgroup of N, and there exists a unique character ω of L such that ω induces η on $\mathrm{Im}(\iota)$ and ω induces the identity map on ℓ. Explicitly, such ω is given by

$$\omega(t, x) = \chi_p(t) \quad (t \in \mathbb{Q}_p,\ x \in \ell). \tag{3.3.1}$$

We denote by $(H_\omega, T_\omega) = \mathrm{Ind}(N, L; \omega)$ the unitary representation of N induced by the character ω of L. Hence, the Hilbert space H_ω is the completion of the space of all functions

$\varphi : N \to \mathbb{C}$ such that

$$\varphi(nh) = \omega^{-1}(h)\varphi(n) \quad (n \in N,\ h \in L), \tag{3.3.2}$$

$$\varphi \in L^2(N/L) \text{ for the Haar measure } d\dot{n} \text{ on } N/L, \tag{3.3.3}$$

and the unitary operator $T_\omega(n_0)$ on H_ω $(n_0 \in N)$ is the left multiplication of n_0^{-1}:

$$(T_\omega(n_0)\varphi)(n) = \varphi(n_0^{-1}n). \tag{3.3.4}$$

We have the Stone-Von Neumann Theorem in our case:

Theorem 3.3.1 (The Stone-Von Neumann Theorem).

(a) (H_ω, T_ω) *is an irreducible unitary representation of* N.

(b) *For any Hilbert space* H, *every unitary representation* (H, T) *of* N *satisfying*

$$T(t, 0) = \chi_p(t) \cdot \mathrm{Id}_H \quad \text{for } (t, 0) \in \mathit{Im}(\iota) \tag{3.3.5}$$

is a multiple of T_ω.

Remark. It can be seen from [Per 81, pp. 371–372] that the Stone-Von Neumann Theorem is connected with Weil [Wei 64] as follows: If $V = \ell \oplus \ell'$ is a decomposition of (V, B) into a sum of two Lagrangian subspaces, then we can define a map

$$H_\omega \ni \varphi \longmapsto \varphi|_{\ell'} \in L^2(\ell').$$

This is an intertwining unitary operator for the unitary representation (H_ω, T_ω) and the irreducible unitary representation $(L^2(\ell'), \Phi)$ of N, where Φ is given by

$$(\Phi(t, x)f)(v^*) = \eta(t + \langle u, v^*\rangle - \langle u, u^*\rangle/2)f(v^* - u^*)$$

for $x = (u, u^*) \in V$ and $f \in L^2(\ell')$ if $\ell' \cong \ell^*$ (Pontrjagin dual of ℓ) and $\langle\ ,\ \rangle$ is the canonical bilinear form on V,

Let us consider also the subgroup $\Delta = \{(t, 0) : t \in \mathbb{Z}_p\}$ of the center $\mathrm{Im}(\iota)$ of N. Thus all unitary representations of N satisfying (3.3.5) are trivial on Δ. Hence we may consider them as representations of $\widetilde{N} = N/\Delta$. We can identify $\widetilde{N}$ with $\mathbb{C}_1^\times \times V$ via $(t, x) \mapsto (\alpha, x)$. Then the multiplication law of $\widetilde{N}$ is given by

$$(\alpha, x) \cdot (\beta, y) = (\alpha\beta\chi_p(B(x, y)/2), x + y). \tag{3.3.6}$$

We call $\widetilde{N}$ the *p-adic Heisenberg group* of (V, B). The center $C(\widetilde{N})$ of $\widetilde{N}$ consists of the elements

$$C(\widetilde{N}) = \{(\alpha, 0) : \alpha \in \mathbb{C}_1^\times\} \simeq \mathbb{C}_1^\times$$

is a subgroup of the commutative group $\widetilde{L}$. Let ω' be a character of $\widetilde{L}$ extending the character $(\alpha, 0) \mapsto \alpha$ of the subgroup $C(\widetilde{N})$. We can consider ω' as a character of L satisfying (3.3.1).

Remark. If (H, W) is a Weyl system on (V, B), then the family of operators $T(t, x) = \chi_p(t)W(x)$, $(t, x) \in N$ (resp. $\widetilde{T}(\alpha, x) = \alpha W(x)$, $(\alpha, x) \in \widetilde{N}$), forms an unitary representation of N (resp. $\widetilde{N}$) on H. Conversely, if $T(t, x)$, $(t, x) \in N$ (resp. $\widetilde{T}(\alpha, x)$, $(\alpha, x) \in \widetilde{N}$), is some unitary representation of N (resp. $\widetilde{N}$) on the Hilbert space H satisfying the condition $T(t, 0) = \chi_p(t) \cdot \mathrm{Id}_H$ (resp. $\widetilde{T}(\alpha, 0) = \alpha \cdot \mathrm{Id}_H$), then the pair (H, W), $W(x) = \chi_p(-t)T(t, x)$ (resp. $W(x) = \widetilde{T}(1, x)$), $x \in V$, is a Weyl system on (V, B).

Another proof of the Stone-Von Neumann Theorem. Let (H_ω, T_ω) be the unitary representation of the Heisenberg group N. Each element of the Heisenberg group N is written uniquely as $(t, x) = (0, x) \cdot (t, 0)$. Hence, if $\varphi \in H_\omega$, (3.3.2) implies $\varphi(t, x) = \varphi((0, x) \cdot (t, 0)) = \chi_p(-t)\varphi(0, x)$. Thus φ is determined by its restriction to V. Hence the mapping

$$H_\omega \ni \varphi \overset{R}{\longmapsto} \varphi|_V \in L^2(V)$$

is an intertwining unitary operator. The unitary representation $\widetilde{T}(n) = RT_\omega(n)R^{-1}$ acts on $L^2(V)$ by the following formula: For $n = (t, x) \in N$,

$$\begin{aligned}
(\widetilde{T}(t, x)\varphi|_V)(y) &= RT_\omega(t, x)\varphi(0, y) = R\varphi((-t, -x) \cdot (0, y)) \\
&= R\varphi(-t - B(x, y)/2, y - x) \\
&= R\varphi((0, y - x) \cdot (-t - B(x, y)/2, 0)) \\
&= \chi_p(t)\chi_p(B(x, y)/2)\varphi|_V(y - x).
\end{aligned}$$

Let $W(x) = \chi_p(-t)\widetilde{T}(t, x)$. It does not depend on t and satisfies the Weyl relation (3.1.2). Thus $(L^2(V), W)$ is a Weyl system on (V, B). In particular, if $\varphi|_V \in L^2(V_0)$, then $(L^2(V), W) = (L^2(V_0), \widetilde{W})$ is an irreducible Weyl system on (V, B) (see Example 3.2.4). To complete proof, we must show that $(L^2(V), W)$ is an irreducible. Let $H = L^2(V)$ and $Y \in \mathcal{O}$. Then $H^Y = S(Y) \neq 0$, since $l(x)\varphi = \varphi = W(x)\varphi$ for all $x \in Y$. Let f_0 be a vaccum vector of $(S(Y), W)$. Then we have for $x \in V_0$ and $y \in V$,

$$f_0(y) = W(x)f_0(y) = \chi_p(B(x, y)/2)f_0(y - x).$$

Since $f_0 \in S(Y)$, for any point $y \in V$ there exists an integer m such that $f_0(y - x) = f_0(y)$, $|x|_p \leq p^m$. Thus $\mathrm{supp} f_0 \subset B_r = \{x \in V : |x|_p \leq p^r\}$, $r = \min\{0, m\}$ and

$f_0(y) \equiv$ constant, $y \in B_r$. Therefore $f_0(y) = c \cdot \Omega\left(p^{-r}|x|_p\right)$, $c \in \mathbb{C}$. It follows from (ii) of Theorem 3.1.1 and Theorem 3.2.5 that $(L^2(V), W)$ is an irreducible Weyl system on (V, B). ∎

3.4. A Weyl system depending a selfdual $\mathbb{Z}_p$-lattice and a $\mathbb{Q}_p$-valued function. Let $\mathfrak{L}$ be a lattice in (V, B). The dual lattice $\mathfrak{L}^*$ is defined by

$$\mathfrak{L}^* = \{x \in V \ : \ B(x, y) \in \mathbb{Z}_p \ \text{ for all } \ y \in \mathfrak{L}\}\,. \tag{3.4.1}$$

If $\mathfrak{L} = \mathfrak{L}^*$, then $\mathfrak{L}$ is called selfdual. From now we consider only case where $\mathfrak{L}$ is a selfdual lattice. We consider a commutative subgroup $\Gamma = \{(t, x) \ : \ t \in \mathbb{Q}_p, \ x \in \mathfrak{L}\}$ of N. Let $\widetilde{\Gamma}$ be the image of the group Γ in $\widetilde{N}$. The fact that the lattice $\mathfrak{L}$ is selfdual is equivalent to the fact that $\widetilde{\Gamma}$ is a maximal commutative subgroup of $\widetilde{N}$. Let τ be a character of $\widetilde{\Gamma}$ extending the character $(\alpha, 0) \mapsto \alpha$ of the subgroup $\mathbb{C}_1^\times$. By the Stone-Von Neumann Theorem, $(H_\tau, T_\tau) = \mathrm{Ind}(N, \Gamma; \tau)$ is an irreducible unitary representation of N.

Theorem 3.4.1 (cf. [LV 80, p. 143]). *There exists a canonical isomorphism $\Theta^\tau_{\mathfrak{L},\ell}$ between H_ω and H_τ intertwining the representations T_ω and T_τ:*

$$(\Theta^\tau_{\mathfrak{L},\ell}\varphi)(n) = \sum_{x \in \mathfrak{L}/(\mathfrak{L}\cap\ell)} \tau(x)\varphi(n \cdot (0, x)). \tag{3.4.2}$$

For any $\varphi \in H_\tau$, $x \in V$ and $y \in \mathfrak{L}$, we have

$$\begin{aligned}\varphi(0, x + y) &= \varphi((0, x) \cdot (-B(x, y)/2, y)) = \tau^{-1}(-B(x, y)/2, y)\varphi(0, x)\\ &= \chi_p(B(x, y)/2)\tau^{-1}(0, y)\varphi(0, x).\end{aligned}$$

Let $H(\mathfrak{L}, \sigma)$ be the Hilbert space obtained by completing the space of continuous functions $f : V \to \mathbb{C}$ satisfying the following two conditions

$$f(x + y) = \chi_p(B(x, y)/2 - \sigma(y))f(x) \quad (x \in V, \ y \in \mathfrak{L}), \tag{3.4.3}$$

where σ is a $\mathbb{Q}_p$-valued function on V satisfying $(\sigma(x + y) - \sigma(x) - \sigma(y))/p^n \in \mathbb{Z}_p$;

$$f \in L^2(V/\mathfrak{L}) \text{ for the Haar measure } d\dot{x} \text{ on } V/\mathfrak{L}. \tag{3.4.4}$$

We define a unitary operators $W_{\mathfrak{L},\sigma}(x)$ by

$$W_{\mathfrak{L},\sigma}(x)f(y) = \chi_p(B(x, y)/2 - \sigma(x))f(y - x), \quad y \in V, \ f \in H(\mathfrak{L}, \sigma). \tag{3.4.5}$$

Example 3.4.2. The pair $(H(\mathfrak{L}, \sigma), W_{\mathfrak{L},\sigma})$ is a Weyl system over (V, B). Indeed, the unitarity of the $W_{\mathfrak{L},\sigma}(x)$ is obvious. It is sufficient to check the Weyl relation. We have

$$\begin{aligned}W_{\mathfrak{L},\sigma}(x)W_{\mathfrak{L},\sigma}(y)f(z) &= W_{\mathfrak{L},\sigma}(x)\chi_p(B(y, z)/2 - \sigma(y))f(z - y)\\ &= \chi_p(B(x, z)/2 - \sigma(x))\chi_p(B(y, z - x)/2 - \sigma(y))f(z - x - y)\\ &= \chi_p(B(x, y)/2)\chi_p(B(x + y, z)/2 - \sigma(x) - \sigma(y))f(z - (x + y))\\ &= \chi_p(B(x, y)/2)\chi_p(B(x + y, z)/2 - \sigma(x + y))f(z - (x + y))\\ &= \chi_p(B(x, y)/2)W_{\mathfrak{L},\sigma}(x + y)f(z).\end{aligned}$$

Theorem 3.4.3. *The Weyl system* $(H(\mathfrak{L},\sigma), W_{\mathfrak{L},\sigma})$ *is irreducible if and only if* $\sigma(\mathfrak{L}) \subset \mathbb{Z}_p$.

Proof. Let H_0 be the vacuum subspace of $(H(\mathfrak{L},\sigma), W_{\mathfrak{L},\sigma})$, and let $f_0 \in H_0$ (by Theorem 3.1.1, such a vector exists). Then, using (3.1.3) and (3.4.5) we obtain

$$f_0(x) = W_{\mathfrak{L},\sigma}(z) f_0(x) = \chi_p(B(z,x)) f_0(x) \quad \text{for } z \in \mathfrak{L}.$$

Thus it satisfies supp $f_0 \subset \{x \in V \mid \chi_p(B(z,x)) = 1 \ \forall z \in \mathfrak{L}\} = \mathfrak{L}$. By (3.4.3), we have $f(x+y) = \chi_p(-\sigma(y)) f(x)$. Thus

$$\begin{aligned}
&(H(\mathfrak{L},\sigma), W_{\mathfrak{L},\sigma}) \text{ is irreducible} \\
&\qquad \iff \text{the vacuum subspace } H_0 \text{ is one-dimensional} \\
&\qquad \iff f(x) \equiv \text{constant}, \ x \in \mathfrak{L} \\
&\qquad \iff \sigma(\mathfrak{L}) \subset \mathbb{Z}_p. \quad \blacksquare
\end{aligned}$$

Remark. (i) We denote by $(H(\mathfrak{L}), W_{\mathfrak{L}})$ the irreducible Weyl system of Theorem 3.4.3. Then an irreducible unitary representation of the Heisenberg group $\widetilde{N} \cong \mathbb{C}_1^\times \times \mathbb{Q}_p^{2n}$ is defined as a pair $(H(\mathfrak{L}), T_{\mathfrak{L}})$, where $T_{\mathfrak{L}}(\alpha, x) = \alpha W_{\mathfrak{L}}(x)$. This representation is a p-adic analogue of the Cartier representation [Car 64] of the real Heisenberg group.

(ii) For the Heisenberg group $\widetilde{N} \cong \mathbb{C}_1^\times \times V$ of a p-adic symplectic space (V, B) of arbitrary dimension, $\mathfrak{L}$-representation corresponding Weyl $\mathfrak{L}$-system, which is analogues of Fock representations of commutation relations, was constructed by Zelenov [Zel 94b].

§4. Supplementary note on quantum mechanics with non-Archimedean number fields

4.1. Notations. Throughout this chapter, K is a complete field with a non-trivial n.a. valuation $|\cdot|_K$ and char$(K) = 0$. Let $a \in K$ and $r \in \mathbb{R}_{>0}$. The open disc (resp. closed disc) of radius r with centre a is

$$B_a(r^-) = \{x \in K : |x-a|_K < r\} \quad (\text{resp. } B_a(r) = \{x \in K : |x-a|_K \le r\}).$$

We denote by $r(K) = B_0(1)/B_0(1^-)$ and $|K^\times| = \{|a|_K : a \in K^\times\}$ the residue class field of K and the value group of K, respectively. The following elementary result of the n.a. analysis will be useful below: a series $\sum a_n,\ a_n \in K$, converges if and only if $|a_n|_K \to 0$ as $n \to 0$. Using the definition of the n.a. valuation, we get $|n|_K \le 1$ for $n \in \mathbb{N}$ and the sequence $|n!|_K$ is decreasing. Thus we assume that there is an estimate

$$1/|n!|_K \le \mathfrak{r}(K)^n \quad (n \in \mathbb{Z},\ \mathfrak{r}(K) \in \mathbb{R}_{>1}). \tag{4.1.1}$$

This estimate holds for the field $\mathbb{Q}_p$ (i.e., $\mathfrak{r}(\mathbb{Q}_p) = p^{\frac{1}{p-1}}$) and its finite extensions.

Let us suppose that quadratic equation $x^2 - \tau = 0,\ \tau \in K$, has no solution in the field K. Denote by Z_τ the quadratic extension $K(\sqrt{\tau})$ of K. The elements of Z_τ are represented as $z = x + \sqrt{\tau}y,\ x, y \in K$, the conjugate of z is given by $\overline{z} = x - \sqrt{\tau}y$ and the extension of the n.a. valuation to Z_τ is defined by $|z|_K = \sqrt{|\ |z|^2\ |_K}$, where $|z|^2 = z\overline{z} = x^2 - \tau y^2 \in K$ denotes the square of the length of $z \in Z_\tau$. We remark that it is impossible to define the length $|z|$ in K for all $z \in Z_\tau$.

Consider a n-dimensional space K^n of points $x = (x_1, x_2, \cdots, x_n),\ x_j \in K$. K^n is a n.a. K-Banach space by the pointwise operations and the n.a. norm $||x||_K = \max_{1\le j\le n}|x_j|_K$.

Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}_0^n$. We will signify by $D^\alpha f(x)$ the derivative of the function $f(x)$ of order $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$,

$$D^\alpha f(x) = f^{(\alpha)}(x) = \frac{\partial^\alpha f(x)}{\partial x^\alpha} = \frac{\partial^{|\alpha|} f(x_1, x_2, \cdots, x_n)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}},\quad D^0 f(x) = f(x).$$

We also use the following abbreviations: for $\beta = (\beta_1, \beta_2, \cdots, \beta_n) \in \mathbb{N}_0^n$,

$$\sum_{|\alpha|=0}^{\infty} = \sum_{\alpha_1=0}^{\infty}\cdots\sum_{\alpha_n=0}^{\infty},\ \sum_{|\alpha|=0}^{|\beta|} = \sum_{\alpha_1=0}^{\beta_1}\cdots\sum_{\alpha_n=0}^{\beta_n},\ \binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1}\cdots\binom{\alpha_n}{\beta_n},$$

$$f_\alpha = f_{\alpha_1,\cdots,\alpha_n},\quad x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n},\quad \alpha! = \alpha_1!\cdots\alpha_n!,$$

where $\binom{\alpha_j}{\beta_j} = \frac{\alpha_j(\alpha_j-1)\cdots(\alpha_j-\beta_j+1)}{\beta_j!}$.

4.2. The test function spaces and their duals. Raghunathan [Rag 68] investigated the topological properties of the space of all K-valued sequences $a = (a_0, a_1, \cdots)$ for which $\lim_{n\to\infty}|a_n|_K^{\frac{1}{n}} = 0$. This space correspond to the space of entire functions $x \mapsto \sum_{n=0}^{\infty} a_n x^n$ on K. Also he, in [Rag 69], showed that its dual space may be identified with the space of all germs of power series functions. Its ideal structure was studied in [Sri 73]. In this section, more generally, we shall focus our attention mainly on the space of entire functions with several variables. Since the proofs of [Rag 68, 69] can be translated in our case in the obvious manner, we stated same results without proof.

A function $f : K^n \to K$ is said to be an *entire* if the series

$$f = f(x) = \sum_{|\alpha|=0}^{\infty} f_\alpha x^\alpha \quad (f_\alpha \in K) \tag{4.2.1}$$

converges for all $x \in K^n$, or equivalently (cf. [Bru 63, p. 114]), if $\lim_{|\alpha|\to\infty}|f_\alpha|_K^{\frac{1}{|\alpha|}} = 0$. Let $\mathcal{A}_1 = \mathcal{A}(K^n, K)$ be the family of all entire functions f over K^n. Clearly, $\mathcal{A}_1$ is a K-vector space by $(f_1 + f_2)(x) = f_1(x) + f_2(x)$, $(af)(x) = af(x)$. For each $f \in \mathcal{A}_1$, we define $||f|| = \sup\left\{|f_0|_K,\ |f_\alpha|_K^{\frac{1}{|\alpha|}} : |\alpha| \geq 1\right\}$. This real valued function $||\cdot||$ has the following properties:

(4.2.2a) $||f|| \geq 0$ and $||f|| = 0$ if and only if $f = 0$, where

$$f = \sum_{|\alpha|=0}^{\infty} f_\alpha x^\alpha = 0 \Longleftrightarrow f_\alpha = 0 \ \text{ for every } |\alpha| \geq 0.$$

(4.2.2b) $||f + g|| \leq \max(||f||, ||g||)$.

(4.2.2c) $||af|| \leq \max(1, |a|_K)||f||$ $(a \in K)$.

It follows from (4.2.2) that $\rho(f, g) = ||f - g||$ defines a topology in $\mathcal{A}_1$. We denote by T_ρ the metric ρ-topology on $\mathcal{A}_1$. As is shown in [Rag 68], $(\mathcal{A}_1, T_\rho)$ is a complete, non-normable linear metric space over K and totally disconnected.

Let $R \in |K^\times|$. By $K_R\{X\}(= K_R\{X_1, \cdots, X_n\})$ we denote the set of all formal power series $f = \sum_{|\alpha|=0}^{\infty} f_\alpha X^\alpha$ for which $\lim_{|\alpha|\to\infty} |f_\alpha|_K R^{|\alpha|} = 0$. This $K_R\{X\}$ is a subring of $K[[X]]$. Define

$$(4.2.3) \qquad ||f||_R = \sup_\alpha |f_\alpha|_K R^{|\alpha|} \quad \left(f = \sum_{|\alpha|=0}^{\infty} f_\alpha X^\alpha \in K_R\{X\}\right).$$

For all $f, g \in K_R\{X\}$ we have $||f||_R > 0$ if and only if $f \neq 0$, $||f+g||_R \leq \max\{||f||_R, ||g||_R\}$ and $||fg||_R = ||f||_R ||g||_R$. Also, $K_R\{X\}$ is metrically complete. Thus $K_R\{X\}$ is the n.a. K-Banach algebra. Let $V_R = \{x \in K^n : ||x||_K \leq R\}$ be the ball in the space K^n. Every $f \in K_R\{X\}$ induces a bounded continuous function $G_f : \mathrm{Sp}K_R\{X\} \to K^n$ by

$$G_f(\varphi) = \varphi(f) \quad (\varphi \in \mathrm{Sp}K_R\{X\}),$$

where $\mathrm{Sp}K_R\{X\}$ is the spectrum of $K_R\{X\}$ (i.e., the set of all nonzero algebra homomorphisms $K_R\{X\} \to K^n$) and G (resp. G_f) is called the Gelfand transformation of $K_R\{X\}$ (resp. the Gelfand transform of f). Since $K[X]$ is a dence subset of $K_R\{X\}$, G_X is an injection. As $||X||_R = R$ we see that the range of G_X is contained in V_R. Conversely, every $x \in V_R$ determines an element φ of $\mathrm{Sp}K_R\{X\}$ for which $\varphi(X) = x$ by the formula $\varphi\left(\sum_{|\alpha|=0}^{\infty} f_\alpha X^\alpha\right) = \sum_{|\alpha|=0}^{\infty} f_\alpha x^\alpha$. Thus, we have a bijection $\mathrm{Sp}K_R\{X\} \to V_R$. It is easy to see that this bijection is a homeomorphism. In the sequel we identity φ with $\varphi(X)$, $\mathrm{Sp}K_R\{X\}$ with V_R and the Gelfand transform of $f = \sum_{|\alpha|=0}^{\infty} f_\alpha X^\alpha$ with the function $x \mapsto \sum_{|\alpha|=0}^{\infty} f_\alpha x^\alpha$ on V_R.

A function $f : V_R \to K$ is called *analytic* if there exist f_α's in K such that $f(x) = \sum_{|\alpha|=0}^{\infty} f_\alpha x^\alpha$ for all $x \in V_R$. Then clearly $\lim_{|\alpha|\to\infty} |f_\alpha|_K R^{|\alpha|} = 0$ and f is bounded and continuous. As in the Archimedean theory, the sum and product of analytic functions are analytic. Thus, the analytic functions on V_R form a subalgebra $\mathcal{A}(V_R, K)$ of the commutative n.a. K-Banach algebra $BC(V_R, K)$ of the bounded continuous functions $V_R \to K$ with respect to the sup-norm $||\cdot||_\infty$ given by

$$||f||_\infty = \sup\{|f(x)|_K : x \in V_R\}.$$

Theorem 4.2.1 (cf. [Van 78, Theorem 6.32]). *The Gelfand transformation*

$$K_R\{X\} \ni \sum_{|\alpha|=0}^{\infty} f_\alpha X^\alpha \to \sum_{|\alpha|=0}^{\infty} f_\alpha x^\alpha \in \mathcal{A}(V_R, K)$$

is an isometry if and only if K is not locally compact. Namely, if K is not locally compact, then $\mathcal{A}(V_R, K)$ is a commutative n.a. K-Banach algebra with respect to the n.a. norm $||\cdot||_R$ given by (4.2.3).

We have $\mathcal{A}_1 = \bigcap_{R\in|K^\times|} \mathcal{A}(V_R, K)$ as set. If $R < R_1$ $(R, R_1 \in |K^\times|)$, then $||f||_R \leq ||f||_{R_1}$, $\mathcal{A}(V_{R_1}, K) \subset \mathcal{A}(V_R, K)$ and the canonical injection $\mathcal{A}(V_{R_1}, K) \to \mathcal{A}(V_R, K)$ is linear continuous map. Hence the natural topology on $\mathcal{A}_1$ is the projective limit topology T_P. I.e. T_P is the weakest locally convex topology which makes all the canonical injections $\mathcal{A}_1 \to \mathcal{A}(V_R, K)$ continuous (for a general properties of the locally convex spaces we refer to [Mon 59]):

$$(\mathcal{A}_1 = \mathrm{projlim}_{R\to\infty}\mathcal{A}(V_R, K),\ T_P)\,. \tag{4.2.4}$$

It is also given by the invariant (n.a.) metric d defined by

$$d(f, g) = \sup_{j\geq 1}\frac{1}{2^j}\frac{||f-g||_{R_j}}{1+||f-g||_{R_j}} \quad (f, g \in \mathcal{A}_1 \text{ and } R_j \in |K^\times|), \tag{4.2.5}$$

where $\{R_j\}$ is an increasing sequence such that $\lim_{j\to\infty} R_j = \infty$ (cf. [Mon 70, pp. 32–33]).

Proposition 4.2.2. *The projective limit topology T_P and the metric ρ-topology T_ρ on $\mathcal{A}_1$ are equivalent.*

Proof. Let $\{f_i\}$, where f_i is given by

$$f_i(x) = \sum_{|\alpha|=0}^{\infty} f_{i,\alpha} x^\alpha \quad (f_{i,\alpha} \in K), \tag{4.2.6}$$

be any sequence in $\mathcal{A}_1$ such that $f_i \to f$ in T_ρ. If f is given by (4.2.1), then for each $\eta > 0$, there exists a $n_0 = n_0(\eta)$ such that $|f_{i,0} - f_0|_K \leq \eta$ and $|f_{i,\alpha} - f_\alpha|_K \leq \eta^{|\alpha|}$ for all $i \geq n_0$ and $|\alpha| \geq 1$. Therefore, for given $\epsilon > 0$ and each $R \in |K^\times|$ choosing η such that $\eta^{|\alpha|}R^{|\alpha|} < \epsilon$ and $0 < \eta \leq \epsilon$, we have $||f_i - f||_R \leq \sup_\alpha\{\eta,\ \eta^{|\alpha|}R^{|\alpha|}\} < \epsilon$ for all $i \geq n_0$ and each $R \in |K^\times|$. Thus $f_i \to f$ under each norm $||\cdot||_R$ and hence $f_i \to f$ in T_P. Conversely, let $f_i \to f$ in T_P. Then $f_i \to f$ under each norm $||\cdot||_R$. Thus, for each $R \in |K^\times|$, there exists a $N_0 = N_0(R)$ such that $||f_i - f||_R < 1$ for all $i \geq N_0$, which implies $|f_{i,\alpha} - f_\alpha|_K < R^{-|\alpha|}$ for all $i \geq N_0$ and $|\alpha| \geq 0$. Let $\epsilon > 0$ be given. Since K is non-trivial valued field, there exists $M \in |K^\times|$ such that $M \leq R$, $M^{-1} < \epsilon$. Then we have $||f_i - f|| < \epsilon$ for all $i \geq N_0$. Thus $f_i \to f$ in T_ρ. ∎

Every polynomial is of course an entire function and many *exotic* looking functions (from the Archimedean point of view) such as $f(x) = \sum_{n=0}^{\infty}(n!)^{n!}x^n$ and $f(x) = \sum_{n=0}^{\infty}\pi^{n^2}x^n$ ($\pi \in K$ with $|\pi|_K < 1$), are entire functions. But e^x, $\sin x$, $\cos x$, $\sinh x$ and $\cosh x$ converges for $x \in B_0(r^-)$, where $r = p^{-1/(p-1)}$ if $\mathrm{char}(r(K)) = p$ and $r = 1$ if $\mathrm{char}(r(K)) = 0$; and $\log(1+x)$, $\arctan x$ converges for $x \in B_0(1^-)$ (cf. [Sch 84, 25.6 and 25.7]). So, in contrast to the complex case, all these elementary functions are not entire but analytic at zero, where the definition of analytic at zero was introduced by Khrennikov [Khr 90a, 90b, 91, 92, 95] as follows:

Definition 4.2.3 (cf. [Khr 92, p. 763]). A function $f : K^n \to K$ is said to be *analytic at zero* if there exists $R \in |K^\times|$ such that $f \in \mathcal{A}(V_R, K)$. The space of functions analytic at zero, $\mathcal{A}_0 = \bigcup_{R\in|K^\times|}\mathcal{A}(V_R, K)$ (as set), is provided with the inductive limit topology T_I of the K-Banach spaces $\mathcal{A}(V_R, K)$:

$$(4.2.7)\qquad (\mathcal{A}_0 = \mathrm{indlim}_{R\to 0}\mathcal{A}(V_R, K),\ T_I)\,.$$

We denote by $\mathcal{A}_i'$, $i = 0, 1$, the set of K-linear continuous functionals on $\mathcal{A}_i$. $\mathcal{A}_i'$ is called the dual of the space $\mathcal{A}_i$.

Theorem 4.2.4. *Every functional $\varphi \in \mathcal{A}(V_R, K)'$ (resp. $\varphi \in \mathcal{A}_1'$) is of the form*

$$(4.2.8)\qquad \varphi(f) = \sum_{|\alpha|=0}^{\infty} f_\alpha\varphi_\alpha,\quad f = \sum_{|\alpha|=0}^{\infty} f_\alpha x^\alpha \in \mathcal{A}(V_R, K)\ (\text{resp. } f \in \mathcal{A}_1),$$

where $\varphi_\alpha \in K$, if and only if $\{|\varphi_\alpha|_K/R^{|\alpha|}\}$ (resp. $\left\{|\varphi_\alpha|_K^{\frac{1}{|\alpha|}}\right\}$) is bounded as $|\alpha| \to \infty$. And we have $\mathcal{A}_1' = \bigcup_{R\in|K^\times|}\mathcal{A}(V_R, K)'$ as set.

Proof. Suppose $\varphi \in \mathcal{A}(V_R, K)'$ (resp. $\varphi \in \mathcal{A}_1'$) is of the form (4.2.8). Then there exists a constant M such that $|\varphi(f)|_K \leq M||f||_R$ (cf. [Mon 46, Theorem 9 and 10]). Let $\varphi(x^\alpha) = \varphi_\alpha$. Then $|\varphi_\alpha|_K \leq MR^{|\alpha|}$ (resp. $|\varphi_\alpha|_K^{\frac{1}{|\alpha|}} \leq M^{\frac{1}{|\alpha|}}R$). Thus $\{|\varphi_\alpha|_K/R^{|\alpha|}\}$ (resp. $\left\{|\varphi_\alpha|_K^{\frac{1}{|\alpha|}}\right\}$)

is bounded sequence. Conversely, if $\left\{|\varphi_\alpha|_K/R^{|\alpha|}\right\}$ is bounded, then φ, given by (4.2.8), is well-defined and linear. Now $|\varphi(f)|_K \leq \sup_\alpha |f_\alpha|_K |\varphi_\alpha|_K \leq N||f||_R$ for some N and all $f \in \mathcal{A}(V_R, K)$. Hence φ is continuous on $\mathcal{A}(V_R, K)$ and thus $\varphi \in \mathcal{A}(V_R, K)'$.

On the other hand, suppose $\left\{|\varphi_\alpha|_K^{\frac{1}{|\alpha|}}\right\}$ is bounded and $f = \sum_{|\alpha|=0}^\infty f_\alpha x^\alpha \in \mathcal{A}_1$. Then we can find N_0 such that $|\varphi_0|_K \leq N_0$ and $|\varphi_\alpha|_K \leq N_0^{|\alpha|}$ for $|\alpha| \geq 1$. Since $|f_\alpha|_K^{\frac{1}{|\alpha|}} \to 0$, we can find n_0 such that $|f_\alpha|_K \leq (\frac{1}{2N_0})^{|\alpha|}$ for $|\alpha| \geq n_0$. Hence $|f_\alpha \varphi_\alpha|_K \leq (1/2)^{|\alpha|}$ for $|\alpha| \geq n_0$. Therefore φ, given by (4.2.8), is well-defined and linear. Let $f_i \to 0$ in $\mathcal{A}_1$ and f_i given by (4.2.6). Given $\epsilon > 0$ choose η such that $\eta^{|\alpha|} N_0^{|\alpha|} < \epsilon$. Since $f_i \to 0$, we can find j such that $||f_i|| \leq \eta$ for $i \geq j$. So $|\varphi(f_i)|_K \leq \sup_\alpha (\eta N_0)^{|\alpha|} \leq \epsilon$ for $i \geq j$. Thus φ is continuous on $\mathcal{A}_1$ and thus $\varphi \in \mathcal{A}_1'$.

Nextly, for each $R \in |K^\times|$, if φ is continuous on $(\mathcal{A}(V_R, K), T_R)$, where T_R is a topology induced by $||\cdot||_R$, then φ is continuous on $(\mathcal{A}_1, T_P)$. Thus $\bigcup_{R\in|K^\times|} \mathcal{A}(V_R, K)' \subset \mathcal{A}_1'$. To prove the reverse inclusion, let $\varphi \in \mathcal{A}_1'$. Then there exists a real number $M > 0$ such that $|\varphi_\alpha|_K \leq M^{|\alpha|}$, $|\alpha| \geq 1$. Since K is non-trivial, we can find $R \in |K^\times|$ such that $0 < M \leq R$. Thus $|\varphi_\alpha|_K \leq R^{|\alpha|}$, $|\alpha| \geq 1$. Hence $(|\varphi_\alpha|_K/R^{|\alpha|})$ is bounded and thus $\varphi \in \mathcal{A}(V_R, K)'$. Therefore $\mathcal{A}_1' \subset \bigcup_{R\in|K^\times|} \mathcal{A}(V_R, K)'$. ∎

For $R < R_1 < \cdots$ and $f = \sum_{|\alpha|=0}^\infty f_\alpha x^\alpha \in \mathcal{A}(V_R, K)$, we have

$$\begin{array}{ccccc} \mathcal{A}(V_R,K)' & \xleftarrow{\varphi_R} & B(R) = \{\lambda = (\lambda_\alpha) \in K^\infty : \sup|\lambda_\alpha|_K/R^{|\alpha|} < \infty\} & \subset B(R_1) & \subset \cdots \\ & & \cup & \cup & \cdots \\ \mathcal{A}(V_{\frac{1}{R}},K) & \xrightarrow{\phi_R} & A(R) = \{\lambda = (\lambda_\alpha) \in K^\infty : \lim\limits_{|\alpha|\to\infty} |\lambda_\alpha|_K/R^{|\alpha|} = 0\} & \subset A(R_1) & \subset \cdots, \end{array}$$

where φ_R and ϕ_R are isomorphisms of K-vector spaces which are, respectively, given by

$$\varphi_R(\lambda)(f) = \sum_{|\alpha|=0}^\infty f_\alpha \lambda_\alpha, \quad \phi_R(f) = (f_\alpha).$$

Proposition 4.2.5. $\bigcup_{R\in|K^\times|} B(R) = \bigcup_{R\in|K^\times|} A(R)$.

Proof. Choose R and take any $R_1 > R$. Then $B(R) \subset A(R_1)$ since $|f_\alpha|_K/R_1^{|\alpha|} = |f_\alpha|_K/R^{|\alpha|}(R/R_1)^{|\alpha|}$. Then, for $R < R_1 < R_2 < \cdots$ we obtain

$$B(R) \subset A(R_1) \subset B(R_1) \subset A(R_2) \subset B(R_2) \subset \cdots$$

and the result. ∎

Consequently, we have an one-to-one correspondence between $\mathcal{A}_1'$ and $\bigcup_{R\in|K^\times|} \mathcal{A}(V_{\frac{1}{R}}, K)$. I.e. we can regard $\mathcal{A}_1'$ as the space of all power series with positive radius of convergence at zero and it can be topologized by a metric induced by the valuation defined as

$$||\varphi|| = \sup\left\{|\varphi_0|_K,\ |\varphi_\alpha|_K^{\frac{1}{|\alpha|}} : |\alpha| \geq 1\right\} \quad (\varphi \in \mathcal{A}_1'),$$

which coincides with that of $\mathcal{A}_1$ when restricted to $\mathcal{A}_1$ because that $\mathcal{A}_1$ is the unique maximal metric vector subspace of $\mathcal{A}_1'$.

Theorem 4.2.6. *Let $s(\mathcal{A}_i', \mathcal{A}_i),\ i = 0, 1,$ be the strong topology of $\mathcal{A}_i'$. Then we have:*

(1) *$\mathcal{A}_i$ are complete and reflexive.*

(2) *$(\mathcal{A}_1', s(\mathcal{A}_1', \mathcal{A}_1)) = (\mathcal{A}_0, T_I)$ and $(\mathcal{A}_0', s(\mathcal{A}_0', \mathcal{A}_0)) = (\mathcal{A}_1, T_P)$.*

Proof. If K is spherically complete with respect to $|\cdot|_K$, then by [Mor 81, Lemma 3.5], the image by the canonical injection $\mathcal{A}(V_{R_1}, K) \to \mathcal{A}(V_R, K)$ $(R < R_1)$ of the unit ball of $\mathcal{A}(V_{R_1}, K)$ is c-compact. Hence it follows from [Mor 81, Theorem 3.3 and Theorem 3.4] that we have the theorem. On the other hand, let K be not spherically complete and $b \in K$ with $|b|_K = R$. Then the map

$$\sum_{|\alpha|=0}^{\infty} f_\alpha x^\alpha \longmapsto (f_0, f_{\alpha(1)}b^1, f_{\alpha(2)}b^2, f_{\alpha(3)}b^3, \cdots),$$

where $f_{\alpha(d)} \in K$ is the coefficient of the monomial x^α of degree d, induce isometry $\mathcal{A}(V_R, K) \xrightarrow{\sim} c_0$. Hence $\mathcal{A}(V_R, K)$ is of countable type and reflexive (cf. [Van 78, Corollary 4.18]). Since $\mathcal{A}_1$ contains all finite sums of the form $\sum_{|\alpha|=0}^{\infty} f_\alpha x^\alpha$, the image of the canonical injection $\mathcal{A}_1 \to \mathcal{A}(V_R, K)$ is dense for each R. Put

$$\left(\sum_{|\alpha|=0}^{\infty} f_\alpha x^\alpha, \sum_{|\beta|=0}^{\infty} g_\beta x^\beta\right)_R = \sum_{|\alpha|+|\beta|=0}^{\infty} f_\alpha g_\beta.$$

Then $(\ ,\ )_R : \mathcal{A}(V_R, K) \times \mathcal{A}(V_{\frac{1}{R}}, K) \to K$ is a nondegenerate bicontinuous bilinear form and $\mathcal{A}(V_R, K)$ and $\mathcal{A}(V_{\frac{1}{R}}, K)$ become mutually K-Banach dual spaces with respect to $(\ ,\ )_R$. Hence by [SM 86, 1. Duality Theorem] theorem holds. ∎

Here after, we consider the complete n.a. locally convex spaces $\mathcal{A}_i,\ i = 0, 1$, as the spaces of test functions (i.e., the spaces of the locally constant functions with compact support) and their dual spaces $\mathcal{A}_i'$ as the spaces of generalized functions (i.e., the linear sets with weak convergence). We denote by $\langle f, \varphi \rangle$ the effect of the generalized function φ over the test function f. An example of a member of $\mathcal{A}_i'$ is the (Dirac) delta functional at the point $y \in K^n$. It is denoted by $\delta(x - y)$ and defined on $\mathcal{A}_i$ by $\langle f(x), \delta(x - y) \rangle = f(y)$. The partial differential operator $\frac{\partial}{\partial x_j}$ is a continuous linear mapping of $\mathcal{A}(V_R, K)$ into $\mathcal{A}(V_R, K)$, as well as of $\mathcal{A}_i$ into $\mathcal{A}_i$. Indeed, for any $f \in \mathcal{A}(V_R, K)$, we have $||\frac{\partial f}{\partial x_j}||_R \leq R^{-1}||f||_R$. If $f \in \mathcal{A}_i$ then $f^{(\alpha)}$ is again in $\mathcal{A}_i$ (cf. [Van 78, p. 230]). Then for $\varphi \in \mathcal{A}_i'$ we define, as in the classical case, the $|\alpha|$th order partial derivative $\varphi^{(\alpha)}$ of φ by

$$\langle f, \varphi^{(\alpha)} \rangle = \langle f^{(\alpha)}, \varphi \rangle \quad (f \in \mathcal{A}_i). \tag{4.2.9}$$

Theorem 4.2.7. *The spaces $\mathcal{A}_i'$, $i = 0, 1$, can be explicitly described in the form of the spaces of infinite-order differential operators with coefficients from the field K.*

Proof. Since the sequence $\{x^\alpha\}$ is a Schauder basis for $\mathcal{A}_i$ with $||x^\alpha||_R = R^{|\alpha|}$ and with corresponding sequence of coefficient functional $\{\delta^{(\alpha)}/\alpha!\}$, if $\varphi \in \mathcal{A}_i'$, then for all $f = \sum_{|\alpha|=0}^{\infty} f_\alpha x^\alpha \in \mathcal{A}_i$ we have

$$\langle f, \varphi \rangle = \langle \sum_{|\alpha|=0}^{\infty} \langle f, \delta^{(\alpha)}/\alpha! \rangle x^\alpha, \varphi \rangle = \sum_{|\alpha|=0}^{\infty} \langle f, \delta^{(\alpha)}/\alpha! \rangle \langle x^\alpha, \varphi \rangle.$$

If the series $\sum_{|\alpha|=0}^{\infty} \langle x^\alpha, \varphi \rangle \delta^{(\alpha)}/\alpha!$ is $T_{\mathcal{A}_i'}$-convergent in $\mathcal{A}_i'$ where $T_{\mathcal{A}_i'}$ is a topology on $\mathcal{A}_i'$, then obviously $\varphi = \sum_{|\alpha|=0}^{\infty} \langle x^\alpha, \varphi \rangle \delta^{(\alpha)}/\alpha!$. Further it is easy to see that this representation of φ is unique and that the coefficient functional $\varphi \mapsto \langle x^\alpha, \varphi \rangle$ is continuous. Thus the sequence $\{\delta^{(\alpha)}/\alpha!\}$ is a $T_{\mathcal{A}_i'}$-Schauder basis in $\mathcal{A}_i'$. We give the topology $T_{\mathcal{A}_i'}$ on $\mathcal{A}_i'$ as follows: Let $P = \sum_{|\alpha|=0}^{\infty} P_\alpha \delta^{(\alpha)}$, $P_\alpha \in K$, be an infinite-order differential operator. Then we have

$$\begin{aligned} P = \sum_{|\alpha|=0}^{\infty} P_\alpha \delta^{(\alpha)} &\text{ converges in } \mathcal{A}(V_R, K)' \\ &\iff \langle f, P \rangle = \sum_{|\alpha|=0}^{\infty} f_\alpha P_\alpha \alpha! \text{ converges in } K \text{ for all } f = \sum_{|\alpha|=0}^{\infty} f_\alpha x^\alpha \in \mathcal{A}(V_R, K) \\ &\iff \sup_\alpha |P_\alpha|_K |\alpha!|_K / R^{|\alpha|} < \infty. \end{aligned}$$

Let

$$\mathcal{D}_q = \left\{ P = \sum_{|\alpha|=0}^{\infty} P_\alpha \delta^{(\alpha)} : P_\alpha \in K,\ ||P||_q = \sup_\alpha |P_\alpha|_K |\alpha!|_K q^{-|\alpha|} < \infty \right\}. \tag{4.2.10}$$

Then we have

$$\mathcal{A}_1' = \mathrm{indlim}_{q \to \infty} \mathcal{D}_q, \quad \mathcal{A}_0' = \mathrm{projlim}_{q \to 0} \mathcal{D}_q. \tag{4.2.11}$$ ∎

Corollary 4.2.8. *For all $R \in |K^\times|$, we have*

$$|\langle f, \varphi\rangle|_K \leq ||f||_R||\varphi||_R \quad (f \in \mathcal{A}_i,\ \varphi \in \mathcal{A}'_i,\ i = 0, 1).$$

Proof. Let $f = \sum_{|\alpha|=0}^{\infty} f_\alpha x^\alpha$ and $\varphi = \sum_{|\beta|=0}^{\infty} \varphi_\beta \delta^{(\beta)}$. Then, by using (4.2.9) and (4.2.10), we have $\langle f, \varphi\rangle = \sum_{|\alpha|=0}^{\infty} f_\alpha \varphi_\alpha \alpha!$. Thus

$$|\langle f, \varphi\rangle|_K \leq \sup_\alpha |f_\alpha \varphi_\alpha \alpha!|_K = \sup_\alpha |f_\alpha|_K R^{|\alpha|} |\varphi_\alpha|_K |\alpha!|_K R^{-|\alpha|} \leq ||f||_R||\varphi||_R. \ \blacksquare$$

4.3. The non-Archimedean Gaussian distributions and the Hilbert spaces. As usual, it is convenient to use the symbol of an integral to denote the action of a generalized function on a test function. Also, if not specified otherwise, $\int$ will always denote a $\int_{K^n}$. By the definition, we will use the symbol

$$\langle f, \varphi\rangle = \int f(x) d\varphi(x) \quad (f \in \mathcal{A}_i,\ \varphi \in \mathcal{A}'_i,\ i = 0, 1).$$

Definition 4.3.1 (cf. [Khr 92, Definition 1.1]). The (two sided) n.a. Laplace transform $L : \mathcal{A}'_0 \to \mathcal{A}_1$ of a generalized function $\mu \in \mathcal{A}'_0$ is defined to be the function

$$L(\mu)(y) = \langle e^{(\cdot, y)}, \mu\rangle = \int e^{(x,y)} d\mu(x), \tag{4.3.1}$$

where $(x, y) = \sum_{j=1}^{n} x_j y_j,\ x, y \in K^n$.

Theorem 4.3.2 (cf. [Khr 92, Theorem 1.2]). *The n.a. Laplace transform $L : \mathcal{A}'_0 \to \mathcal{A}_1$ and the adjoint operator $L' : \mathcal{A}'_1 \to \mathcal{A}_0$ defined by the Parseval's equality*

$$\int L'(\varphi)(x) d\mu(x) = \int L(\mu)(y) d\varphi(y) \quad (\mu \in \mathcal{A}'_0,\ \varphi \in \mathcal{A}'_1) \tag{4.3.2}$$

are linear isomorphisms. Moreover, $(L')^{-1} = (L^{-1})'$.

If $f(x) = \sum_{|\alpha|=0}^{\infty} f_\alpha x^\alpha \in \mathcal{A}_i,\ i = 0, 1$, then it is easy to see that for every $y \in K^n$ the function $f(x + y)$, considered as a function in x, is still in $\mathcal{A}_i$. Let $\varphi \in \mathcal{A}'_i$, then the convolution $\varphi * f$ of φ and f is defined by

$$(\varphi * f)(x) = \int f(x + y) d\varphi(y). \tag{4.3.3}$$

It is not hard to see that $\varphi * f \in \mathcal{A}_i$ whenever $f \in \mathcal{A}_i,\ \varphi \in \mathcal{A}'_i$. Note that in particular $(\delta * f)(x) = \int f(x + y) d\delta(y) = f(x)$. Hence $\delta * f = f$.

If φ and ψ are distributions, then the convolution $\varphi * \psi$ is defined by

$$\langle f, \varphi * \psi\rangle = \int f(x) d(\varphi * \psi)(x) = \int \left[\int f(x + y) d\varphi(x)\right] d\psi(y). \tag{4.3.4}$$

It is easy to see that $\varphi * \psi \in \mathcal{A}'_i$ whenever $\varphi, \psi \in \mathcal{A}'_i$. Also, as above, we derives that $\delta * \varphi = \varphi * \delta = \varphi$ for all φ.

The following properties of the n.a. Laplace transform are easy consequences of (4.3.1).

Proposition 4.3.3. *For $\mu, \mu_1, \mu_2 \in \mathcal{A}'_0$ and $\varphi, \varphi_1, \varphi_2 \in \mathcal{A}'_1$*

(4.3.5a) $L(\delta)(y) = 1$; $L'(\delta)(x) = 1$.

(4.3.5b) $L(\frac{\partial^\alpha}{\partial x^\alpha}\mu)(y) = y^\alpha L(\mu)(y)$; $L'(\frac{\partial^\alpha}{\partial y^\alpha}\varphi)(x) = x^\alpha L'(\varphi)(x)$.

(4.3.5c) $\frac{\partial^\alpha}{\partial y^\alpha} L(\mu)(y) = L(x^\alpha \mu)(y)$; $\frac{\partial^\alpha}{\partial x^\alpha} L'(\varphi)(x) = L'(y^\alpha \varphi)(x)$.

(4.3.5d) $L(\mu_1 * \mu_2)(y) = L(\mu_1)(y) L(\mu_2)(y)$; $L'(\varphi_1 * \varphi_2)(x) = L'(\varphi_1)(x) L'(\varphi_2)(x)$.

Application (cf. [DeK 96, chapter 5]). (Fundamental solutions of differential operators) Formula (4.3.5b) used to prove that the existence of solution of n.a. differential equation: Let $P_m(D) = \sum_{|\alpha|=0}^m P_\alpha D^\alpha$, $P_\alpha \in K$, be an arbitrary differential operator with constant coefficients. As usual, a solution of the equation

$$P_m(D)\varphi(x) = \delta(x) \quad (\delta, \varphi \in \mathcal{A}'_1) \tag{4.3.6}$$

is called a fundamental solution of the differential operator $P_m(D)$. Let $P_0 \neq 0$. With the aid of the n.a. Laplace transform L', we transform the equation (4.3.6) in the following equation in the space $\mathcal{A}_0$:

$$P_m(x) L'(\varphi)(x) = 1. \tag{4.3.7}$$

Since $P_0 \neq 0$, the function $\mu(x) = 1/P_m(x)$ belongs to the space $\mathcal{A}_0$ and the distribution $\varphi = (L')^{-1}(\mu) \in \mathcal{A}'_1$ is unique fundamental solution of the equation (4.3.7). Also, for $f \in \mathcal{A}_1$, using (4.3.5d) and (4.3.7) we have $L'(P_m(D)(\varphi * f))(x) = P_m(x) L'(\varphi * f)(x) = L'(f)(x)$. Thus $g = \varphi * f \in \mathcal{A}_1$ is the solution of the equation $P_m(D) g(x) = f(x)$.

Let $P(D)$ be an infinite-order differential operator. Assume that $P(x)$ belong to the space $\mathcal{A}_0$ and $P_0 \neq 0$. Then there exists a unique fundamental solution $\varphi \in \mathcal{A}'_1$ of the equation $P(x) L'(\varphi)(x) = 1$. Again $g = \varphi * f \in \mathcal{A}_1$ is the solution of the equation $P(D) g(x) = f(x)$. ∎

Definition 4.3.4 (cf. [Khr 92, Definition 1.2]). The n.a. Gaussian distribution on K^n (with mean value $a \in K^n$ and symmetric covariance matrix $B = (b_{ij})$, $b_{ij} \in K$, $\det B \neq 0$) is defined to be a generalized function $\gamma_{a,B} \in \mathcal{A}'_1$ with the Laplace transform

$$L'(\gamma_{a,B})(x) = \exp\left\{\frac{1}{2}(Bx, x) + (a, x)\right\}. \tag{4.3.8}$$

To calculate a integrals with the n.a. Gaussian distribution, we prove the following formula for integration by parts:

Proposition 4.3.5 (cf. [Khr 92, Theorem 3.1]). *If $f \in \mathcal{A}_1$ and $a \in K^n$, then*

$$\int f(x)(a,x)d\gamma_{0,B}(x) = \int \left(Ba, \frac{\partial}{\partial x}\right)(f)(x)d\gamma_{0,B}(x). \tag{4.3.9}$$

Proof. Let $\beta_a = (\delta', a) = \sum_{j=1}^n a_j \frac{\partial}{\partial y_j}\delta$. Using (4.3.5b), then $f(x)(a,x) = L(L^{-1}(f) * \beta_a)(x)$. Hence, using (4.3.2), (4.3.4) and the symmetry of $B = (b_{ij})$, we have

$$\begin{aligned}
I &= \int f(x)(a,x)d\gamma_{0,B}(x) = \int L'(\gamma_{0,B})(x)d(L^{-1}(f) * \beta_a)(x) \\
&= \int \left[\int L'(\gamma_{0,B})(x+y)dL^{-1}(f)(x)\right] d\beta_a(y) \\
&= \int \left(\sum_{j=1}^n a_j \frac{\partial}{\partial y_j}\left[\int L'(\gamma_{0,B})(x+y)dL^{-1}(f)(x)\right]\right) d\delta(y) \\
&= \left[\int L'(\gamma_{0,B})(x)\sum_{j=1}^n a_j \frac{\partial}{\partial y_j}\exp\left\{\frac{1}{2}(Bx,y) + \frac{1}{2}(By,x) + \frac{1}{2}(By,y)\right\} dL^{-1}(f)(x)\right]\Bigg|_{y=0} \\
&= \int L'(\gamma_{0,B})(x)\sum_{i=1}^n \left(\sum_{j=1}^n a_j b_{ij}\right) x_i dL^{-1}(f)(x) = \int L'(\gamma_{0,B})(x)(Ba,x)dL^{-1}(f)(x) \\
&= \int L\left((Ba,\cdot)L^{-1}(f)\right)(x)d\gamma_{0,B}(x) = \int \left(Ba, \frac{\partial}{\partial x}\right)(f)(x)d\gamma_{0,B}(x). \quad \blacksquare
\end{aligned}$$

Example 4.3.6. Using (4.3.5b) and (4.3.2), we have

$$\begin{aligned}
M_\alpha &= \int x^\alpha d\gamma_{0,B}(x) = \int L\left(\frac{\partial^\alpha}{\partial y^\alpha}\delta\right)(x)d\gamma_{0,B}(x) \\
&= \int L'(\gamma_{0,B})(y)d\left(\frac{\partial^\alpha \delta}{\partial y^\alpha}\right)(y) = \left[\frac{\partial^\alpha}{\partial y^\alpha}\exp\left\{\frac{1}{2}(By,y)\right\}\right]\Bigg|_{y=0} \\
&= \left[\prod_{i=1}^n \frac{\partial^{\alpha_i}}{\partial y_i^{\alpha_i}}\exp\left\{\frac{1}{2}\sum_{j=1}^n b_{ij}y_j y_i\right\}\right]\Bigg|_{y=0} \\
&= \left[\prod_{i=1}^n \frac{\partial^{\alpha_i}}{\partial y_i^{\alpha_i}}\left(\exp\left\{\frac{1}{2}\sum_{j=1, j\neq i}^n b_{ij}y_j y_i\right\}\exp\left\{\frac{1}{2}b_{ii}y_i^2\right\}\right)\right]\Bigg|_{y=0} \\
&= \left[\prod_{i=1}^n \frac{\partial^{\alpha_i}}{\partial y_i^{\alpha_i}}\exp\left\{\frac{1}{2}b_{ii}y_i^2\right\}\right]\Bigg|_{y=0}.
\end{aligned}$$

If $\alpha = (2\beta_1, 2\beta_2, \cdots, 2\beta_n)$, then

$$\begin{aligned} M_\alpha &= (1\cdot 3\cdot 5\cdots(2\beta_1-1))\cdots(1\cdot 3\cdot 5\cdots(2\beta_n-1))b_{11}^{\beta_1}\cdots b_{nn}^{\beta_n} \\ &= \left(\frac{(2\beta_1)!}{\beta_1!}\left(\frac{b_{11}}{2}\right)^{\beta_1}\right)\cdots\left(\frac{(2\beta_n)!}{\beta_n!}\left(\frac{b_{nn}}{2}\right)^{\beta_n}\right) = \frac{(2\beta)!}{\beta!}\left(\frac{\mathrm{d}(B)}{2}\right)^{\beta}, \end{aligned}$$

where $\beta = (\beta_1, \beta_2, \cdots, \beta_n) \in \mathbb{N}_0^n$ and $\mathrm{d}(B) = (b_{11}, b_{22}, \cdots, b_{nn}))$. Thus for any entire function $f(x) = \sum_{|\alpha|=0}^{\infty} f_\alpha x^\alpha \in \mathcal{A}_1$, we have

$$\int f(x)d\gamma_{0,B}(x) = \sum_{|\alpha|=0}^{\infty} f_{2\alpha}\frac{(2\alpha)!}{\alpha!}\left(\frac{\mathrm{d}(B)}{2}\right)^{\alpha}. \tag{4.3.10}$$

Remark. All the previous consideration can be extended to the case of the analytic functions $f : K^n \to Z_\tau$ (resp. $f : Z_\tau^n \to Z_\tau$) and the n.a. Gaussian distributions $\gamma_{a,B}$, $a \in K^n$ (resp. $a \in Z_\tau^n$), $B = (b_{ij})$, $b_{ij} \in Z_\tau$. Then we have the space $\mathcal{A}(K^n, Z_\tau)$ (resp. $\mathcal{A}(Z_\tau^n, Z_\tau)$) of entire functions, the space $\mathcal{A}_0(K^n, Z_\tau)$ (resp. $\mathcal{A}_0(Z_\tau^n, Z_\tau)$) of functions analytic at zero, and the spaces of distributions $\mathcal{A}(K^n, Z_\tau)'$ (resp. $\mathcal{A}(Z_\tau^n, Z_\tau)'$) and $\mathcal{A}_0(K^n, Z_\tau)'$ (resp. $\mathcal{A}_0(Z_\tau^n, Z_\tau)'$).

Example 4.3.7. We denote by z and $\bar{z}$ independent variables on Z_τ^n; $w = (z, \bar{z})$ is a variable on Z_τ^{2n}, $z\bar{z} = \frac{1}{2}(Bw, w)$, where B has the form $\begin{pmatrix} 0 & E_n \\ E_n & 0 \end{pmatrix}$. Using (4.3.5b) and (4.3.2), we have

$$\begin{aligned} \int_{Z_\tau^{2n}} z^\alpha \bar{z}^\beta d\gamma_{0,B}(w) &= \int_{Z_\tau^{2n}} L\left(\frac{\partial^{\alpha+\beta}}{\partial z^\beta \partial \bar{z}^\alpha}\delta\right)(w)d\gamma_{0,B}(w) \\ &= \int_{Z_\tau^{2n}} L'(\gamma_{0,B})(w)d\left(\frac{\partial^{\alpha+\beta}\delta}{\partial z^\beta \partial \bar{z}^\alpha}\right)(w) = \left.\frac{\partial^{\alpha+\beta}}{\partial z^\beta \partial \bar{z}^\alpha}e^{z\bar{z}}\right|_{w=0} = \delta_{\alpha\beta}\alpha!. \end{aligned}$$

Thus for any entire function $f(w) = \sum_{|\alpha|=0}^{\infty} f_\alpha z^\alpha \bar{z}^\alpha \in \mathcal{A}(Z_\tau^{2n}, Z_\tau)$, we have

$$\int_{Z_\tau^{2n}} f(w)d\gamma_{0,B}(w) = \sum_{|\alpha|=0}^{\infty} f_\alpha \alpha!. \tag{4.3.11}$$

Example 4.3.8. We introduce the Hermitian polynomials $H_{m,b}(t)$ on the field K:

$$H_{m,b}(t) = (-1)^m e^{\frac{t^2}{2b}}\frac{d^m}{dt^m}e^{-\frac{t^2}{2b}} \quad (m \in \mathbb{N}_0,\ b \in K^\times,\ t \in K), \tag{4.3.12}$$

i.e., $H_{0,b}(t) = 1$, $H_{1,b}(t) = \frac{t}{b}$, $H_{2,b}(t) = -\frac{1}{b} + \frac{t^2}{b^2}$, $H_{3,b}(t) = -\frac{3t}{b^2} + \frac{t^3}{b^3}$, $H_{4,b}(t) = \frac{3}{b^2} - \frac{6t^2}{b^3} + \frac{t^4}{b^4}$, $H_{5,b}(t) = \frac{15t}{b^3} - \frac{10t^3}{b^4} + \frac{t^5}{b^5}, \cdots$. Then we have the following properties:

(4.3.13a) $H_{2m,b}(0) = (-1)^m \frac{(2m)!}{m!2^m b^m}, \quad H_{2m+1,b}(0) = 0,$

(4.3.13b) $H_{m+1,b}(t) = \frac{t}{b} H_{m,b}(t) - \frac{m}{b} H_{m-1,b}(t),$

(4.3.13c) $H_{m,b}(t+s) = \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} \left(\frac{s}{b}\right)^j H_{m-j,b}(t)$ for $s \in K,$

(4.3.13d) $\frac{d}{dt} H_{m,b}(t) = \frac{t}{b} H_{m,b}(t) - H_{m+1,b}(t) = \frac{m}{b} H_{m-1,b}(t).$

(4.3.13e) $\int_K H_{m,b}(t) H_{m',b}(t) d\gamma_{0,b}(t) = \delta_{mm'} \frac{m!}{b^m}.$

In particular, to compute the integral (4.3.13e) we used the fact that the product of the Hermitian polynomial $H_{m,b}$ by the distribution $\gamma_{0,b}$ is equal to the generalized derivative of $\gamma_{0,b}$:

$$H_{m,b}(t) d\gamma_{0,b}(t) = d\left(\frac{d^m \gamma_{0,b}}{dt^m}\right)(t). \tag{4.3.14}$$

Kalish [Kal 47] investigated the properties of a p-adic Hilbert space $(E, ||\cdot||, (\cdot,\cdot))$, where $(E, ||\cdot||)$ is a n.a. K-Banach space and $(\cdot,\cdot) : E \times E \to K$ is a symmetric bilinear inner product satisfying the Cauchy-Schwarz inequality

$$|(x,y)|_K \leq ||x|| ||y||. \tag{4.3.15}$$

The orthogonality was defined by the inner product $(\cdot,\cdot)$. On the other hand, the orthogonality based not on an inner product but on a norm defined by Monna [Mon 70] as follows: A system of vectors $(e_j)_{j\in J}$ in a n.a. K-Banach space $(E, ||\cdot||)$ is said to be orthogonal if

$$||\sum_{j\in S} x_j e_j|| = \max_{j\in S} |x_j|_K ||e_j||, \tag{4.3.16}$$

for every finite subset $S \subset J$ and for all $x_j \in K$. An orthogonal system $(e_j)_{j\in J}$ is called an orthogonal basis in E if $x = \sum x_j e_j$ for every vector $x \in E$. If a n.a. K-Banach space E has an orthogonal basis (e_j), then E is called an orthogonal n.a. K-Banach space.

There is no canonical way of defining the inner product $(\cdot,\cdot)$ in the orthogonal n.a. K-Banach space $(E, ||\cdot||)$. Khrennikov [Khr 90a, 90b, 91, 92, 95] gave a new definition of a n.a. Hilbert space that connect two definition of orthogonality as follows: Let $(E, ||\cdot||)$ be an orthogonal n.a. K-Banach space, (e_j) an orthogonal basis in E and $(\cdot,\cdot)$ a symmetric bilinear inner product satisfying (4.3.15). Assume that $(e_i, e_j) = 0$, $i \neq j$. Then we have $(x,x) = \sum \lambda_j x_j^2$, where $\lambda_j = (e_j, e_j) \neq 0$. This series converges if and only if $\lim_{j\to\infty} |x_j|_K \sqrt{|\lambda_j|_K} = 0$. But the n.a. K-Banach space E consists of these x for which $\lim_{j\to\infty} |x_j|_K ||e_j|| = 0$. If $||e_j||^2 \in |K^\times|$, then we can take as λ_j any elements of the field K such that $|\lambda_j|_K = ||e_j||^2$. Now if $||e_j||^2$ is not in $|K^\times|$, then it is impossible, in general, to find $\lambda_j \in K$. Thus it is natural to include the numbers λ_j into the definition of a n.a. Hilbert space.

For the sequence $\lambda = (\lambda_n) \in K^\infty,\ \lambda_n \neq 0$ for all n, we set

$$H_\lambda = \left\{ f = (f_n) \in K^\infty : \text{the series } \sum f_n^2 \lambda_n \text{ converges in } K \right\}. \tag{4.3.17}$$

$$\left(\text{resp. } H_\lambda = \left\{ f = (f_n) \in Z_\tau^\infty : \text{the series } \sum |f_n|^2 \lambda_n \text{ converges in } K \right\} \right). \tag{4.3.18}$$

In the space H_λ we introduce a norm $||\cdot||_\lambda$ and an inner product $(\cdot,\cdot)_\lambda$ relative to which the base vectors $e_j = (\delta_{ij})$ are orthogonal as follows:

$$||f||_\lambda = \max_n |f_n|_K \sqrt{|\lambda_n|_K}; \tag{4.3.19a}$$

$$(f,g)_\lambda = \sum f_n g_n \lambda_n \ \left(\text{resp. } (f,g)_\lambda = \sum f_n \overline{g_n} \lambda_n \right). \tag{4.3.19b}$$

Then the space H_λ is a n.a. K-Banach space and $(\cdot,\cdot)_\lambda$ is continuous on $H_\lambda \times H_\lambda$ and satisfy the Cauchy-Schwarz inequality (4.3.15).

Definition 4.3.9 (cf. [Khr 92, Definition 2.1]). The triplet $(H_\lambda, ||\cdot||_\lambda, (\cdot,\cdot)_\lambda)$ is called a n.a. coordinate (resp. complex) Hilbert space.

An inner product on the n.a. K-Banach space E is an arbitrary nondegenerated symmetric bilinear form. It is evidently impossible to introduce an analog of the positive definiteness of a bilinear form.

The triplets $(E_i, ||\cdot||_i, (\cdot,\cdot)_i),\ i = 1, 2$, where E_i are the n.a. K-Banach spaces, $||\cdot||_i$ are norms, and $(\cdot,\cdot)_i$ are inner products satisfying (4.3.15), are isomorphic if there exists an isometric and unitary isomorphism $I : E_1 \to E_2$.

Definition 4.3.10 (cf. [Khr 92, Definition 2.2]). The triplet $(E, ||\cdot||, (\cdot,\cdot))$ is said to be a n.a. (resp. complex) Hilbert space if it is isomorphic to a n.a. coordinate (resp. complex) Hilbert space $(H_\lambda, ||\cdot||_\lambda, (\cdot,\cdot)_\lambda)$ for some λ.

Example 4.3.11. This example illustrates how the n.a. Gaussian distribution $\gamma_{0,b}$ can be used to construct a n.a. complex Hilbert space $L^2(K^n, \gamma_{0,b})$, according to the following three step:
Step 1. On the space $\mathcal{A}(K^n, Z_\tau)$ we consider a n.a. Gaussian distribution $\gamma_{0,b},\ b \in K^\times$. Using the fact that the space $\mathcal{A}(K^n, Z_\tau)$ is a topological algebra, we introduce on $\mathcal{A}(K^n, Z_\tau)$ the inner product as a continuous map $(\cdot,\cdot) : \mathcal{A}(K^n, Z_\tau) \times \mathcal{A}(K^n, Z_\tau) \to Z_\tau$ defined by

$$(f,g) = \int f(x)\overline{g(x)} d\gamma_{0,b}(x). \tag{4.3.20}$$

Step 2. We denote by $H_{\alpha,b}(x),\ b \in K^\times$, the Hermitian polynomials corresponding to the n.a. Gaussian distribution $\gamma_{0,b}$:

$$H_{\alpha,b}(x) = H_{(\alpha_1,\alpha_2,\cdots,\alpha_n),b}(x_1,x_2,\cdots,x_n) = \prod_{j=1}^{n} H_{\alpha_j,b}(x_j),$$

where $H_{\alpha_j,b}(x_j)$ is given by (4.3.12). By (4.3.13e), we see that $H_{\alpha,b}$ is orthogonal to $H_{\beta,b}$ for $\alpha \neq \beta$ with respect to (4.3.20) and $(H_{\alpha,b}, H_{\alpha,b}) = \alpha!/b^{|\alpha|}$. Then we can prove that for any $f \in \mathcal{A}(K^n, Z_\tau)$, we have the orthonormal series expansion of f with respect to $\{H_{\alpha,b}/(H_{\alpha,b}, H_{\alpha,b})\}$. I.e.

$$f(x) = \sum_{|\alpha|=0}^{\infty} \widetilde{f_\alpha} H_{\alpha,b}(x), \text{ where } \widetilde{f_\alpha} = \frac{(f, H_{\alpha,b})}{(H_{\alpha,b}, H_{\alpha,b})} \in Z_\tau. \tag{4.3.21}$$

Indeed, it is sufficient to show that for $n = 1$ the series (4.3.21) converges to f in $\mathcal{A}(K, Z_\tau)$. Let $f(t) = \sum_{m=0}^{\infty} f_m t^m \in \mathcal{A}(K, Z_\tau)$. By the recurrence formula (4.3.13b), we have for $x \in B_0(R)$, where $R \in |K^\times|$ with $R \geq 1$,

$$||H_{m+1,b}||_R \leq \left(\frac{R}{|b|_K}\right) \max\left(||H_{m,b}||_R, ||H_{m-1,b}||_R\right) \leq \cdots \leq \left(\frac{R}{|b|_K}\right)^{m+1}. \tag{4.3.22}$$

Also, using (4.3.20) and Example 4.3.6, we have

$$(f, H_{l,b}) = \sum_{j=0}^{\infty} \frac{f_{2j+l}(2j+l)!\ b^j}{j!\ 2^j}, \tag{4.3.23}$$

and using hypothesis (4.1.1), we obtain for $R_1 \geq \sqrt{\mathfrak{r}(K)|b|_K/|2|_K}$,

$$|(f, H_{l,b})|_K \leq \sup_j |f_{2j+l}|_K R_1^{2j+l} \left(\frac{\mathfrak{r}(K)|b|_K}{R_1^2|2|_K}\right)^j \frac{1}{R_1^l} \leq \frac{||f||_{R_1}}{R_1^l}. \tag{4.3.24}$$

Thus we have for $R \geq 1$ and $R_1 \geq \sqrt{\mathfrak{r}(K)|b|_K/|2|_K}$,

$$|\widetilde{f_m}|_K ||H_{m,b}||_R \leq ||f||_{R_1} \left(\frac{\mathfrak{r}(K)R}{R_1}\right)^m \to 0 \text{ as } m \to \infty$$

if $R_1 > \mathfrak{r}(K)R$. Therefore $\sum_{m=0}^{\infty} \widetilde{f_m} H_{m,b}(t) \in \mathcal{A}(K, Z_\tau)$ for all $f \in \mathcal{A}(K, Z_\tau)$. Next we show that $\sum_{m=0}^{\infty} \widetilde{f_m} H_{m,b}(t)$ converges to the function f. For this, it is sufficient to show that $\frac{d^n}{dt^n} f(0) = \frac{d^n}{dt^n} g(0)$ for all $n \in \mathbb{N}_0$, where $g(t) = \sum_{m=0}^{\infty} \widetilde{f_m} H_{m,b}(t)$. Indeed, using (4.3.13a) and (4.3.13d), we have

$$\frac{d^n}{dt^n} g(0) = \sum_{j=0}^{\infty} (-1)^j \frac{(f, H_{2j+n,b})}{(2j+n)!} \frac{(2j+n)! b^j}{j! 2^j}.$$

Thus if we take $f_{2j+n} = (-1)^j (f, H_{2j+n,b})/(2j+n)!$, then by (4.3.23), we have

$$\frac{d^n}{dt^n} g(0) = (f, H_{n.b}) = \frac{d^n}{dt^n} f(0) \text{ for all } n \in \mathbb{N}_0.$$

Step 3. The formula (4.3.21) gives the coordinate representation for (4.3.20):

$$(f,g) = \sum_{|\alpha|=0}^{\infty} \widetilde{f_\alpha} \overline{\widetilde{g_\alpha}} \; \alpha!/b^{|\alpha|}. \tag{4.3.25}$$

We introduce on $\mathcal{A}(K^n, Z_\tau)$ a n.a. norm relative to which the Hermitian polynomials are orthogonal as follows:

$$||f|| = \max_\alpha |\widetilde{f_\alpha}|_K \sqrt{|\alpha!|_K / |b|_K^{|\alpha|}}. \tag{4.3.26}$$

The completion of the space $\mathcal{A}(K^n, Z_\tau)$ in this norm is called *the space of square-integrable functions with respect to the n.a. Gaussian distribution* $\gamma_{0,b}$, and is denoted by $L^2(K^n, \gamma_{0,b})$:

$$L^2(K^n, \gamma_{0,b}) = \left\{ f = \sum_{|\alpha|=0}^{\infty} \widetilde{f_\alpha} H_{\alpha,b}(x) : \lim_{|\alpha| \to \infty} |\widetilde{f_\alpha}|_K \sqrt{|\alpha!|_K / |b|_K^{|\alpha|}} = 0 \right\}. \tag{4.3.27}$$

The inner product (4.3.25) is continuous on $L^2(K^n, \gamma_{0,b})$ and satisfy the Cauchy-Schwarz inequality (4.3.15). The triple $(L^2(K^n, \gamma_{0,b}), ||\cdot||, (\cdot,\cdot))$ is a n.a. complex Hilbert space of class $H_{(\alpha!/b^{|\alpha|})}$. ∎

Since the dual space to a n.a. (complex) Hilbert space does not coincide with it, we have the following nested Hilbert space

$$\mathcal{A}(K^n, Z_\tau) \subset L^2(K^n, \gamma_{0,b}) \subset L^2(K^n, \gamma_{0,b})' \subset \mathcal{A}(K^n, Z_\tau)'.$$

Proposition 4.3.12. *For $b \in K^\times$ and $R_1, R_2 \in |K^\times|$ with $R_1 \geq \sqrt{\mathfrak{r}(K)|b|_K/|2|_K}$ and $1 \leq R_2 \leq \sqrt{|b|_K/\mathfrak{r}(K)}$, we have*

$$\mathcal{A}(V_{R_1}, Z_\tau) \subset L^2(K^n, \gamma_{0,b}) \subset \mathcal{A}(V_{R_2}, Z_\tau).$$

Proof. Let $n=1$ and $f(t) = \sum_{m=0}^{\infty} \widetilde{f_m} H_{m,b}(t) \in L^2(K, \gamma_{0,b})$. Using (4.3.22), (4.3.26) and the hypothesis (4.1.1), we have

$$|\widetilde{f_m}|_K ||H_{m,b}||_{R_2} \leq |\widetilde{f_m}|_K \left(\frac{R_2}{|b|_K} \right)^m \leq |\widetilde{f_m}|_K \sqrt{\frac{|m!|_K}{|b|_K^m}} \left(R_2 \sqrt{\frac{\mathfrak{r}(K)}{|b|_K}} \right)^m \leq ||f||$$

for $1 \leq R_2 \leq \sqrt{|b|_K/\mathfrak{r}(K)}$. Thus $||f||_{R_2} \leq \max_m|\widetilde{f_m}|_K||H_{m,b}||_{R_2} \leq ||f||$. Next let $f(t) = \sum_{m=0}^{\infty} f_m t^m \in \mathcal{A}(V_{R_1}, Z_\tau)$ and $\widetilde{f_m} = (f, H_{m,b})/(H_{m,b}, H_{m,b}) \in Z_\tau$. Using (4.3.13e) and (4.3.24), we get that for $R_1 \geq \sqrt{\mathfrak{r}(K)|b|_K/|2|_K}$,

$$|\widetilde{f_m}|_K = \left|\frac{(f, H_{m,b})}{(H_{m,b}, H_{m,b})}\right|_K = \left|\frac{b^m}{m!}\right|_K |(f, H_{m,b})|_K \leq \frac{||f||_{R_1}}{|m!|_K}\left(\frac{|b|_K}{R_1}\right)^m. \tag{4.3.28}$$

Consider the function $g(t) = \sum_{m=0}^{\infty} \widetilde{f_m} H_{m,b}(t)$. Then $g \in L^2(K^n, \gamma_{0,b})$, since using (4.3.27) and the hypothesis (4.1.1), we have

$$|\widetilde{f_m}|_K \sqrt{\frac{|m!|_K}{|b|_K^m}} \leq ||f||_{R_1}\left(\frac{\sqrt{r(K)|b|_K}}{R_1}\right)^m \to 0 \quad (m \to \infty). \quad \blacksquare$$

4.4. Iwasawa isomorphism and unboundedness, and problems. In 1990, the question of whether the p-adic Gaussian distribution $\gamma_{0,b},\ b \in \mathbb{Q}_p^\times$, can be extended up to the measure on $\mathbb{Z}_p$ was considered by Khrennikov. In [KE 92] the answer was given by negative for a wide class of correlations b, $p \neq 2$. In the case of $p = 2$ the question still remains open. In this section, we explain and supplement the article [KE 92] and present the open problems.

Let us restrict to the case $K = \mathbb{Q}_p$. Denote by $\mathbb{C}_p$ the completion of the algebraic closure of $\mathbb{Q}_p$. The absolute value $|\cdot|_p$ extends uniquely to $\mathbb{C}_p$ and we use for it the same symbol $|\cdot|_p$. Let $\mathfrak{O} = \{x \in \mathbb{C}_p : |x|_p \leq 1\}$ be the integer ring of the field $\mathbb{C}_p$.

A set $\mu = \{\mu_n\}$ of functions is called a measure on $\mathbb{Z}_p$ if for each integer $n \geq 0$ the function μ_n is defined on $\mathbb{Z}_p/p^n\mathbb{Z}_p$, takes values in $\mathfrak{O}$, and satisfy the condition

$$\mu_n(a + p^n\mathbb{Z}_p) = \sum_{b=0}^{p-1} \mu_{n+1}(a + bp^n + p^{n+1}\mathbb{Z}_p). \tag{4.4.1}$$

We denote by $M(\mathbb{Z}_p, \mathfrak{O})$ the set of the measures on $\mathbb{Z}_p$. Then, Iwasawa isomorphism say that there is a one-to-one $\mathfrak{O}$-linear correspondence of the formal power series ring $\mathfrak{O}[[X]]$ onto $M(\mathbb{Z}_p, \mathfrak{O})$

$$\mathfrak{O}[[X]] \ni f(X) \mapsto \mu = \mu(f) = \{\mu_n\} \in M(\mathbb{Z}_p, \mathfrak{O})$$

such that

$$\mu_n(a + p^n\mathbb{Z}_p) = \frac{1}{p^n}\sum_{\zeta}^{(n)} \zeta^{-a} f(\zeta - 1), \tag{4.4.2}$$

where, in the sum $\sum_{\zeta}^{(n)}$, ζ runs over all the p^nth roots of unity in $\mathbb{C}_p$. Conversely, if $f(X) = \sum_{l=0}^{\infty} c_l X^l$, then the coefficients c_l are given by the formula

$$c_l = \lim_{n\to\infty} \sum_{k=0}^{p^n-1} \binom{k}{l} \mu_n(k). \tag{4.4.3}$$

Let $F(C(\mathbb{Z}_p, \mathfrak{O}), \mathfrak{O})$ be the set of bounded linear functionals λ defined on the set $C(\mathbb{Z}_p, \mathfrak{O})$ of continuous functions with values in $\mathfrak{O}$. Then, there is a one-to-one $\mathfrak{O}$-linear isomorphism

$$M(\mathbb{Z}_p, \mathfrak{O}) \ni \mu = \{\mu_n\} \mapsto \lambda \in F(C(\mathbb{Z}_p, \mathfrak{O}), \mathfrak{O})$$

such that

$$\lambda(\varphi) = \int_{\mathbb{Z}_p} \varphi(x) d\mu(x) = \lim_{n\to\infty} \sum_{k=0}^{p^n-1} \varphi(k)\mu_n(k). \tag{4.4.4}$$

If we denote by $B\mathbb{C}_p[[X]]$, $BD(\mathbb{Z}_p, \mathbb{C}_p)$, and $BF(C(\mathbb{Z}_p, \mathbb{C}_p), \mathbb{C}_p)$, respectively, the set of bounded power series, the set of bounded distributions and the set of bounded $\mathbb{C}_p$-linear functionals, we can easily extend the above correspondence $f(X) \leftrightarrow \mu \leftrightarrow \lambda$ to the one-to-one $\mathbb{C}_p$-linear correspondence of

$$B\mathbb{C}_p[[X]] \longleftrightarrow BD(\mathbb{Z}_p, \mathbb{C}_p) \longleftrightarrow BF(C(\mathbb{Z}_p, \mathbb{C}_p), \mathbb{C}_p).$$

But we can not extend the correspondences to the one-to-one $\mathbb{C}_p$-linear mapping on

$$\mathbb{C}_p[[X]] \longleftrightarrow D(\mathbb{Z}_p, \mathbb{C}_p) \longleftrightarrow F(C(\mathbb{Z}_p, \mathbb{C}_p), \mathbb{C}_p).$$

Endo [End 83] gave a one-to-one $\mathbb{C}_p$-linear correspondence between $F(\mathbb{C}_p[X], \mathbb{C}_p)$ and $\mathbb{C}_p[[X]]$ as follows: Let $f(X) = \sum_{l=0}^{\infty} c_l X^l \in \mathbb{C}_p[[X]]$. We put $f_M = f_M(X) = \sum_{l=0}^{M} c_l X^l$ (partial sum of $f(X)$). Since $f_M(X)$ is a polynomial and so is bounded, we can define the measure $\mu_{f_M} = \{\mu_{f_M,n}\}$ on $\mathbb{Z}_p$ and define for any $\varphi \in C(\mathbb{Z}_p, \mathbb{C}_p)$ the integral

$$\lambda_{f_M}(\varphi) = \int_{\mathbb{Z}_p} \varphi(x) d\mu_{f_M}(x).$$

If the p-adic limit $\lim_{M\to\infty} \lambda_{f_M}(\varphi)$ exists, we define the limit as the integral of the continuous function φ by the power series $f(X)$ and we denote it by

$$\lambda(\varphi) = \int_{\mathbb{Z}_p} \varphi(x) d\mu_f(x). \tag{4.4.5}$$

If the power series $f(X)$ is bounded, the new integral coincides with the original one.

Let $\varphi(x) = \sum_{l=0}^{\infty} a_l \binom{x}{l} \in C(\mathbb{Z}_p, \mathbb{C}_p)$ be the Mahler expansion [Mah 58]. Then we see that

$$a_l = \int_{\mathbb{Z}_p} \varphi(x) d\mu_{X^l}(x) = \sum_{k=0}^{l} \binom{l}{k} (-1)^{l-k} \varphi(k) \text{ and } |a_l|_p \to 0 \text{ as } l \to \infty$$

and that

$$\lambda_{f_M}(\varphi) = \int_{\mathbb{Z}_p} \varphi(x) d\mu_{f_M}(x) = \sum_{l=0}^{M} c_l a_l.$$

Thus the integral $\int_{\mathbb{Z}_p} \varphi(x) d\mu_f(x)$ exists if and only if the infinite sum $\sum_{l=0}^{\infty} c_l a_l$ converges. In particular, since $x^n = \sum_{l=0}^{n} \binom{x}{l} \sum_{k=0}^{l} \binom{l}{k} (-1)^{l-k} k^n$, if $M \geq n$, we get

$$\int_{\mathbb{Z}_p} x^n d\mu_{f_M}(x) = \sum_{l=0}^{n} c_l \sum_{k=0}^{l} \binom{l}{k} (-1)^{l-k} k^n. \tag{4.4.6}$$

This shows that, if $M \geq n$, the right hand side is independent on the choice of M, so that the integral $\lambda(x^n) = \int_{\mathbb{Z}_p} x^n d\mu_f(x)$ always exists. By linearlity the integral $\int_{\mathbb{Z}_p} \varphi(x) d\mu_f(x)$ always exists for arbitrary polynomial $\varphi(x)$ in $\mathbb{C}_p[x]$ with degree less than M. Thus we have the one-to-one and invertible correspondence

$$\mathbb{C}_p[[X]] \ni f \longleftrightarrow \lambda \in F(\mathbb{C}_p[X], \mathbb{C}_p)$$

such that

$$\lambda(\varphi) = \int_{\mathbb{Z}_p} \varphi(x) d\mu_f(x), \quad f(X) = \sum_{l=0}^{\infty} c_l X^l \quad \text{and} \quad c_l = \lambda\left(\binom{x}{l}\right).$$

From now, we suppose that the series $\lambda(\varphi) = \sum_{n=0}^{\infty} \varphi_n \lambda(x^n)$ converges for any analytic function $\varphi(x) = \sum_{n=0}^{\infty} \varphi_n x^n$ defined on $\mathbb{Z}_p$. In that case, the distribution λ extends to a linear functional on the space of analytic functions. I.e. it is a p-adic Gaussian distribution $\gamma_{0,b}$ ($b \in \mathbb{Q}_p^{\times}$). By (4.3.10) in Example 4.3.6, for any analytic function $\varphi(x) = \sum_{n=0}^{\infty} \varphi_n x^n$ defined on $\mathbb{Z}_p$ if $\frac{b}{2} \in \mathbb{Z}_p$, then the sum

$$\int_{\mathbb{Z}_p} \varphi(x) d\gamma_{0,b}(x) = \sum_{k=0}^{\infty} \frac{(2k)!}{k!} \left(\frac{b}{2}\right)^k \varphi_{2k} \tag{4.4.7}$$

converges. Let $f(X) = \sum_{l=0}^{\infty} c_l X^l \in \mathbb{C}_p[[X]]$ and $c_l = \int_{\mathbb{Z}_p} \binom{x}{l} d\gamma_{0,b}(x)$. Then using (4.4.6) and (4.4.7), we have

$$g(Z) = f(e^Z - 1) = \sum_{l=0}^{\infty} c_l (e^Z - 1)^l = \sum_{n=0}^{\infty} \frac{Z^n}{n!} \left(\int_{\mathbb{Z}_p} x^n d\gamma_{0,b}(x) \right) = e^{\frac{b}{2} Z^2}.$$

On the other hand,

$$g(Z) = \sum_{n=0}^{\infty} \frac{Z^n}{n!} \left(\int_{\mathbb{Z}_p} x^n d\gamma_{0,b}(x) \right) = \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \frac{(xZ)^n}{n!} d\gamma_{0,b}(x) = \int_{\mathbb{Z}_p} e^{xZ} d\gamma_{0,b}(x).$$

Thus we have the following formulas:

$$\int_{\mathbb{Z}_p} e^{xZ} d\gamma_{0,b}(x) = e^{\frac{b}{2}Z^2} \ ; \tag{4.4.8}$$

$$\begin{aligned} f(X) = e^{\frac{b}{2}(\log(1+X))^2} &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{b}{2}\right)^n (\log(1+X))^{2n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{b}{2}\right)^n \left(\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} X^m\right)^{2n} = 1 + \sum_{l=2}^{\infty} c_l X^l, \end{aligned}$$

where

$$c_l = \sum_{1 \le m \le \frac{l}{2}} \frac{(-1)^l}{m!}\left(\frac{b}{2}\right)^m \sum_{x_1+\cdots+x_{2m}=l, x_j \in \mathbb{N}} \frac{1}{x_1 x_2 \cdots x_{2m}}. \tag{4.4.9}$$

In contrast to the real theory, we have the following theorem:

Theorem 4.4.1 (cf. [KE 92, Theorem 1]). *A p-adic Gaussian distribution $\gamma_{0,b}$ ($b \in \mathbb{Q}_p^\times$) is not a measure on $\mathbb{Z}_p$. I.e. the coefficients $\{c_l\}$ of the function $f(X)$ are unbounded: Let $\frac{b}{2} = p^a u$, where u is a p-adic unit and $a \in \mathbb{Z}$.*

(1) If a is even, then for all primes p the coefficients $\{c_l\}$ are unbounded.

(2) If a is odd, then for all odd primes p the coefficients $\{c_l\}$ are unbounded.

For $l = 2np^{[a/2]+1}$, where $n = p^b$ (a power of p), the equality

$$|c_l|_p = p^{2n(1-\delta)+(n-1)/(p-1)}$$

is valid, where $\delta = 0$ if a is even, and $\delta = \frac{1}{2}$ if a is odd.

Problem 1. Is it possible to prove (2) in Theorem 4.4.1 for $p = 2$?

Problem 2. Is there a method, called regularization, for turning the p-adic Gaussian distribution $\gamma_{0,b}$ into measure on $\mathbb{Z}_p$?

Problem 3. Is it possible to extend $\gamma_{0,b}$ on the space of $C^1(\mathbb{Q}_p)$ or $C^\infty(\mathbb{Q}_p)$?

Problem 4. Is it possible to prove the problem of boundedness or unboundedness of a n.a. Gaussian distribution for any n.a. valued field K ?

The analytical function $\varphi(x)$ defined on $\mathbb{Z}_p$ is said to be $\gamma_{0,b}$-negligible (or annihilator of the p-adic Gaussian distribution $\gamma_{0,b}$) if

$$\int_{\mathbb{Z}_p} f(x)\varphi(x) d\gamma_{0,b}(x) = 0 \quad \left(\text{resp. } \int_{\mathbb{Z}_p} x^n \varphi(x) d\gamma_{0,b}(x) = 0\right) \tag{4.4.10}$$

for every analytical function $f(x)$ (resp. for all $n = 0, 1, 2, \cdots$). Let N be the set of all $\gamma_{0,b}$-negligible functions. It is easy to see that N is a $\mathbb{Q}_p[x]$-module. For the p-adic Gaussian distribution $\gamma_{0,b}$ ($b \in \mathbb{Q}_p^\times$), we have the same property of the usual Gaussian distribution:

Theorem 4.4.2 (cf. [EK 95]). *If p is odd prime and $\frac{b}{2}$ is a p-adic integer (i.e., $b \in \mathbb{Z}_p$), then there are no non-zero $\gamma_{0,b}$-negligible functions. I.e. $N = \{0\}$.*

Problem 5. Is it possible to prove Theorem 4.4.2 for $p = 2$?

Problem 6. Find the class of $\gamma_{0,b}$-negligible functions for any n.a. valued field K.

The isomorphicity relation (see Definition 4.3.10) decomposes the set of the n.a. Hilbert spaces into equivalence classes. An equivalence class is characterized by some coordinate representative H_λ, for instance, $H_{(1)}$ and $H_{(2^n)}$ belong to the same equivalence class for the field $K = \mathbb{Q}_p,\ p \neq 2$, and to different classes for the field $K = \mathbb{Q}_2$.

Problem 7. What are the restrictions on the weight sequence $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ for the unitary isomorphism of H_λ and H_μ ?

Problem 8. Let us consider a linear operator $U : H_\lambda \to H_\mu$. It can be realized as an infinite matrix. What are the restrictions on the coefficients to be a unitary isometry ? It would be interesting to study not only the canonical basis but an arbitrary orthogonal basis.

Problem 9. Does there exist some kind of the coordinate representation of the norm on H_λ in an arbitrary orthogonal basis ?

Problem 10. Is it possible to define the Hilbert space topology with aid of only the inner product ?

4.5. A p-adic Hilbert space associated with the Morita p-adic Γ-function. In this section, we construct a p-adic Hilbert space using the Morita p-adic Γ-function.

We set $B = \{z \in \mathbb{Q}_p(\sqrt{\tau}) : |z-1|_p < 1\}$. For any $x \in \mathbb{Z}_p$, the function

$$B \ni z \longmapsto z^x = \sum_{n=0}^{\infty} \binom{x}{n} (z-1)^n \in \mathbb{Q}_p(\sqrt{\tau})$$

is an analytic because the mapping $x \mapsto \binom{x}{n}$ carries $\mathbb{Z}_p$ to $\mathbb{Z}_p$. I.e. $z^x \in \mathcal{A}(B, \mathbb{Q}_p(\sqrt{\tau}))$ for $x \in \mathbb{Z}_p$. Also, if $z \in B$, then $|z-1|_p^n \to 0$ as $n \to \infty$, and thus the function

$$\mathbb{Z}_p \ni x \mapsto z^x \in \mathbb{Q}_p(\sqrt{\tau})$$

is continuous, i.e., $z^x \in C(\mathbb{Z}_p, \mathbb{Q}_p(\sqrt{\tau}))$ for $z \in B$, where $C(\mathbb{Z}_p, \mathbb{Q}_p(\sqrt{\tau}))$ is the space of continuous functions $f : \mathbb{Z}_p \to \mathbb{Q}_p(\sqrt{\tau})$ equipped with the topology induced by the sup-norm

$$||f||_\infty = \sup\{|f(x)|_p : x \in \mathbb{Z}_p\}.$$

Here after, we consider the space $C(\mathbb{Z}_p, \mathbb{Q}_p(\sqrt{\tau}))$ as the space of test functions and its dual space $C(\mathbb{Z}_p, \mathbb{Q}_p(\sqrt{\tau}))'$ as the space of generalized functions. Let $f(x) = \sum_{n=0}^{\infty} f_n \begin{pmatrix} x \\ n \end{pmatrix} \in C(\mathbb{Z}_p, \mathbb{Q}_p(\sqrt{\tau}))$ be the Mahler expansion. Then we see that

$$f_n = \sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} f(n-k) \text{ and } |f_n|_p \to 0 \text{ as } n \to \infty.$$

The system of polynomial $\begin{pmatrix} x \\ n \end{pmatrix}$ is an orthogonal basis with respect to the norm $|| \cdot ||_\infty$ in $C(\mathbb{Z}_p, \mathbb{Q}_p(\sqrt{\tau}))$ in the sense of the theory of n.a. Banach space. Thus we have

$$C(\mathbb{Z}_p, \mathbb{Q}_p(\sqrt{\tau}))' = \{\varphi = (\varphi_n) \in \left(\mathbb{Q}_p(\sqrt{\tau})\right)^\infty : ||\varphi||_\infty = \sup_n |\varphi_n|_p < \infty\} \tag{4.5.1}$$

and we can define a transform $Z : C(\mathbb{Z}_p, \mathbb{Q}_p(\sqrt{\tau}))' \to C(\mathbb{Z}_p, \mathbb{Q}_p(\sqrt{\tau}))'$ by

$$(Z\varphi)(z) = \int_{\mathbb{Z}_p} z^x d\varphi(x). \tag{4.5.2}$$

Proposition 4.5.1. *The transform Z is an isomorphism of $C(\mathbb{Z}_p, \mathbb{Q}_p(\sqrt{\tau}))'$ and the function space*

$$F_\tau = \{f \in \mathcal{A}(B, \mathbb{Q}_p(\sqrt{\tau})) : ||f|| = sup_n |f^{(n)}(1)/n!|_p < \infty\}.$$

Let $(z^x, z^x) = |z^x|^2 = \Gamma_p^{(M)}(x+1)$, where $\Gamma_p^{(M)}(x)$ is the Morita p-adic Γ-function (see Appendix B), and

$$H(\mathbb{Z}_p) = \left\{ f(z) = \sum_{x \in \mathbb{Z}_p} f_x z^x : \text{the series } |f|^2 = \sum_{x \in \mathbb{Z}_p} \Gamma_p^{(M)}(x+1) |f_x|^2 \text{ converges in } \mathbb{Q}_p \right\}.$$

The scalar product is $(f, g) = \sum_{x \in \mathbb{Z}_p} \Gamma_p^{(M)}(x+1) f_x \overline{g_x}$ and the norm is $||f|| = \sup\{|f_x|_p : x \in \mathbb{Z}_p\}$, which satisfying the Cauchy-Schwarz inequality $|(f,g)|_p \le ||f|| ||g||$. The triple $(H(\mathbb{Z}_p), ||\cdot||, (\cdot,\cdot))$ is a p-adic Hilbert space of class $H_{(\Gamma_p^{(M)}(x+1))}$.

The differential operator $\frac{d}{dz}$ is continuous linear mapping from $H(\mathbb{Z}_p)$ to $H(\mathbb{Z}_p)$. Indeed, for any $f \in H(\mathbb{Z}_p)$, we have $||df/dz|| = ||f||$. Let $\left(\frac{d}{dz}\right)^*$ be the adjoint in $H(\mathbb{Z}_p)$ of the operator $\frac{d}{dz}$. Then

$$\left(\frac{d}{dz} f, g\right) = \sum_{x \in \mathbb{Z}_p} \sum_{y \in \mathbb{Z}_p} f_x \overline{g_y} x (z^{x-1}, z^y) = \sum_{x \in \mathbb{Z}_p} f_x \overline{g_{x-1}} x \Gamma_p^{(M)}(x).$$

Let A be a linear operator in $H(\mathbb{Z}_p)$ such that $A(z^x) = a(x) z^{x+1}$. Then

$$(f, Ag) = \sum_{x \in \mathbb{Z}_p} \sum_{y \in \mathbb{Z}_p} f_x \overline{g_y} a(y) (z^x, z^{y+1}) = \sum_{x \in \mathbb{Z}_p} f_x \overline{g_{x-1}} a(x-1) \Gamma_p^{(M)}(x+1).$$

Therefore, for the adjoint operator $\left(\frac{d}{dz}\right)^*$, we have

Proposition 4.5.2.
$$\left(\frac{d}{dz}\right)^*(z^x) = \frac{(x+1)\Gamma_p^{(M)}(x+1)}{\Gamma_p^{(M)}(x+2)} z^{x+1}.$$

By (B.16) in Appendix B, if $|x+1|_p = 1$, then $\left(\frac{d}{dz}\right)^*(z^x) = -z^{x+1}$ and if $|x+1|_p < 1$, then $\left(\frac{d}{dz}\right)^*(z^x) = -(x+1)z^{x+1}$. Thus

$$\left(\frac{d}{dz}\right)^* f(z) = -\left\{\sum_{|x+1|_p=1} f_x z^{x+1} + \sum_{|x+1|_p<1} (x+1) f_x z^{x+1}\right\}. \tag{4.5.3}$$

Also, we have the commutator $[\frac{d}{dz}, \left(\frac{d}{dz}\right)^*] = \lambda(x+1) - \lambda(x)$ of the operators $\frac{d}{dz}$ and $\left(\frac{d}{dz}\right)^*$, where

$$\lambda(x) = \frac{x^2\Gamma_p^{(M)}(x)}{\Gamma_p^{(M)}(x+1)}. \tag{4.5.4}$$

4.6. A restricted non-Archimedean Heisenberg group $\mathcal{N}_h$ depending $h \in K$. Let $r \in \mathbb{R}_{>0}$. Set $[r]_K = \sup\{R \in |K^\times| : R < r\}$. For a operator A in a n.a. Hilbert space H over K^n and the real number $\mathfrak{r}(K) \in \mathbb{R}_{>1}$ in the hypothesis (4.1.1), we set $A(\mathfrak{r}(K)) = 1/||A||\mathfrak{r}(K)$.

On the n.a. complex Hilbert space $L^2(K^n, \gamma_{0,b})$ coordinate and momentum operators $\mathbf{x}_j, \mathbf{k}_j$, $j = 1, 2, \cdots, n$, are introduced by usual formulas:

$$(\mathbf{x}_j f)(x) = x_j f(x), \tag{4.6.1}$$

$$(\mathbf{k}_j f)(x) = \frac{h}{\sqrt{\tau}} \frac{\partial}{\partial x_j} f(x) \quad (\tau, h \in K). \tag{4.6.2}$$

Then these operators satisfy HCR

$$[\mathbf{x}_i, \mathbf{x}_j] = [\mathbf{k}_i, \mathbf{k}_j] = 0, \quad [\mathbf{x}_i, \mathbf{k}_j] = \frac{h}{\sqrt{\tau}} \delta_{ij}. \tag{4.6.3}$$

Proposition 4.6.1. $\mathbf{x}_j, \mathbf{k}_j$ *are bounded self-adjoint operators in* $L^2(K^n, \gamma_{0,b})$.

Proof. Since $\mathbf{x}_j, \mathbf{k}_j$ are symmetry for the inner product (4.3.20), it is sufficient to show that $\mathbf{x}_j, \mathbf{k}_j$ are bounded. Let $f(x) = \sum_{|\alpha|=0}^\infty \widetilde{f_\alpha} H_{\alpha,b}(x) \in L^2(K^n, \gamma_{0,b})$. Then we have $f(x) = \sum_{|\hat{\alpha}|=0}^\infty H_{\hat{\alpha},b}(\hat{x}) \sum_{\alpha_j=0}^\infty \widetilde{f_\alpha} H_{\alpha_j,b}(x_j)$, where $\hat{\alpha}$ and $\hat{x}$ denote respectively α and x dropped jth element. Set $g(\hat{x}) = \sum_{|\hat{\alpha}|=0}^\infty H_{\hat{\alpha},b}(\hat{x})$. Then $||g||^2 = \max_{\hat{\alpha}} |\hat{\alpha}!|_K / |b|_K^{|\hat{\alpha}|}$. Using (4.3.13b) and (4.3.13d), we get

$$\begin{cases} (\mathbf{x}_j f)(x) = g(\hat{x}) \left(\sum_{\alpha_j=0}^\infty b\widetilde{f_\alpha} H_{\alpha_j+1,b}(x_j) + \sum_{\alpha_j=0}^\infty \alpha_j \widetilde{f_\alpha} H_{\alpha_j-1,b}(x_j)\right), \\ (\mathbf{k}_j f)(x) = \frac{h}{\sqrt{\tau}} g(\hat{x}) \sum_{\alpha_j=0}^\infty \frac{\alpha_j}{b} \widetilde{f_\alpha} H_{\alpha_j-1,b}(x_j). \end{cases}$$

Using (4.3.26) and the strong triangle inequality, we have

$$||\mathbf{x}_j f||^2 \leq |b|_K ||f||^2, \quad ||\mathbf{k}_j f||^2 \leq \frac{|h|_K^2}{|b\tau|_K} ||f||^2.$$

Thus we get

$$||\mathbf{x}_j|| \leq \sqrt{|b|_K}, \quad ||\mathbf{k}_j|| \leq \frac{|h|_K}{\sqrt{|b\tau|_K}}. \tag{4.6.4}$$ ∎

Theorem 4.6.2. *For any $y \in V_{R^*}$, $R^* = \left[\sqrt{|b|_K/\mathfrak{r}(K)}\right]_K$, the operator T defined by $(Tf)(x) = f(x+y)$ on the n.a. complex Hilbert space $L^2(K^n, \gamma_{0,b})$ is bounded and isometric.*

Proof. For the Hermitian polynomials $H_{\alpha,b}$ ($b \in K^{\times}$), using the formula (4.3.13c), we have

$$(TH_{\alpha,b})(x) = \sum_{|\beta|=0}^{|\alpha|} \binom{\alpha}{\beta} \frac{y^\beta}{b^{|\beta|}} H_{\alpha-\beta,b}(x) \quad (\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{N}_0^n). \tag{4.6.5}$$

Let $f(x) = \sum_{|\alpha|=0}^{\infty} \widetilde{f_\alpha} H_{\alpha,b}(x) \in L^2(K^n, \gamma_{\alpha,b})$. Using (4.6.5), we have

$$(Tf)(x) = \left(\mathrm{Id} + \sum_{|\beta|=1}^{\infty} y^\beta A_\beta\right)(f)(x),$$

where the operators A_β are defined by

$$(A_\beta f)(x) = \frac{1}{b^{|\beta|}} \sum_{|\xi|=0}^{\infty} \widetilde{f_{\beta+\xi}} \binom{\beta+\xi}{\beta} H_{\xi,b}(x) \in L^2(K^n, \gamma_{0,b}).$$

Using (4.3.26), we have

$$||A_\beta f||^2 = \frac{1}{|b^{|\beta|}\beta!|_K} \max\left\{ |f_{\beta+\xi}|_K^2 \left|\frac{(\beta+\xi)!}{b^{|\beta+\xi|}}\right|_K \left|\frac{(\beta+\xi)!}{\xi!\beta!}\right|_K \right\} \leq ||f||^2 \left(\frac{\mathfrak{r}(K)}{|b|_K}\right)^{|\beta|}.$$

Thus we get $||A_\beta|| \leq \left(\sqrt{\mathfrak{r}(K)/|b|_K}\right)^{|\beta|}$. Let $y \in V_{R^*}$, $R^* = \left[\sqrt{|b|_K/\mathfrak{r}(K)}\right]_K$. Then we have $||y^\beta||_K ||A_\beta|| < 1$ and

$$||Tf|| = \max\left\{ ||f||, ||\left(\sum_{|\beta|=1}^{\infty} y^\beta A_\beta\right) f|| \right\} = ||f||.$$ ∎

Theorem 4.6.3. *Let A be a bounded self-adjoint operator in a n.a. Hilbert space H over K^n. Then for $h, \tau \in K$ and $x \in V_{R_A}$, $R_A = \left[\frac{|h|_K}{\sqrt{|\tau|_K}} A(\mathfrak{r}(K))\right]_K$, $e^{\frac{\sqrt{\tau}}{h}xA}$ is an isometric unitary operator in H.*

Proof. Since

$$e^{\frac{\sqrt{\tau}}{h}xA} = \mathrm{Id} + \sum_{m=1}^{\infty} \frac{(\sqrt{\tau}xA)^m}{h^m m!} = \mathrm{Id} + \sum_{m=1}^{\infty} A_m$$

and $||A_m|| < 1$ for $m = 1, 2, \cdots$, we have

$$||e^{\frac{\sqrt{\tau}}{h}xA} f|| = \max(||f||, ||A_m f||) = ||f||.$$

Thus the isometric property is proved. Also, a simple algebraic computations dependent only on the fact that $(\sqrt{\tau})^2 = \tau$ gives unitarity. ∎

Corollary 4.6.4. *Let $h, \tau \in K$ and*

$$x_j \in B_0\left(R_{\mathbf{k}_j}\right) \subset B_0\left(R^*\right), \quad k_j \in B_0\left(R_{\mathbf{x}_j}\right). \tag{4.6.6}$$

Then operators $U(x_j) = e^{\frac{\sqrt{\tau}}{h}x_j\mathbf{k}_j}$ and $V(k_j) = e^{\frac{\sqrt{\tau}}{h}k_j\mathbf{x}_j}$ in $L^2(K, \gamma_{0,b})$ are isometric unitary operators acting on $L^2(K, \gamma_{0,b})$ and satisfy WCR

$$\begin{cases} U(x_i)U(x_j) = U(x_i + x_j), \quad V(k_i)V(k_j) = V(k_i + k_j), \\ U(x_j)V(k_j) = e^{\frac{\sqrt{\tau}}{h}k_j x_j} V(k_j)U(x_j). \end{cases} \tag{4.6.7}$$

Corollary 4.6.5. *We have a n.a. analogue of Heisenberg uncertainty relations:*

$$R_{\mathbf{k}_j} \cdot R_{\mathbf{x}_j} = \left[\frac{|h|_K}{\sqrt{|\tau|_K}} \mathbf{k}_j(\mathfrak{r}(K))\right]_K \left[\frac{|h|_K}{\sqrt{|\tau|_K}} \mathbf{x}_j(\mathfrak{r}(K))\right]_K \geq \frac{|h|_K}{\sqrt{|\tau|_K}} \left(\frac{1}{\mathfrak{r}(K)}\right)^2. \tag{4.6.8}$$

Proof. It is easy consequence of the inequality (4.6.4). ∎

Let $f \in L^2(K^n, \gamma_{0,b})$, $t \in K$ and

$$x = (x_1, \cdots, x_n) \in V_{\mathbf{k}} = \prod_{j=1}^{n} B_0\left(R_{\mathbf{k}_j}\right), \quad k = (k_1, \cdots, k_n) \in V_{\mathbf{x}} = \prod_{j=1}^{n} B_0\left(R_{\mathbf{x}_j}\right).$$

We define the action of $W(x, k, t)$ on f as

$$[W(x, k, t)f](y) = \exp\left\{\frac{\sqrt{\tau}}{h}(t + (y + x, k))\right\} f(y + x) \quad (\tau, h \in K). \tag{4.6.9}$$

$$\text{(or } W(x,k,t) = e^{\frac{\sqrt{\tau}}{h}t}U(x_1)U(x_2)\cdots U(x_n)V(k_1)V(k_2)\cdots V(k_n)).$$

Let $\mathcal{D}_h$ be the region of convergence of the exponential function $e^{\frac{\sqrt{\tau}}{h}t}$ for fixed $\tau \in K$ satisfying $|\tau|_K = p^{\frac{2}{1-p}}$ if $\mathrm{char}(r(K)) = p$ and $|\tau|_K = 1$ if $\mathrm{char}(r(K)) = 0$. Then by [Sch 84, Theorem 25.6], $\mathcal{D}_h$ is given by

$$\mathcal{D}_h = \{t \in K : |t|_K < |h|_K\}. \tag{4.6.10}$$

Since $e^{\frac{\sqrt{\tau}}{h}t} \equiv e^{\frac{\sqrt{\tau}}{h}s} \pmod{\mathcal{D}_h}$ if and only if $t \equiv s \pmod{\mathcal{D}_h}$, we have

$$W(x,k,t) \equiv W(x,k,s) \pmod{\mathcal{D}_h} \text{ if and only if } t \equiv s \pmod{\mathcal{D}_h}. \tag{4.6.11}$$

Thus we obtain a one-to-one parameterization if t is chosen among representatives of $K/\mathcal{D}_h$.

The product of two transformations (4.6.9) given by

$$W(x,k,t)W(x',k',t') = W(x+x', k+k', t+t'-(x',k)). \tag{4.6.12}$$

We see that $W(0,0,0)$ is the identity and that $W(-x,-k,-t-(x,k))$ is the inverse of $W(x,k,t)$. Thus we have

Proposition 4.6.6. *The set*

$$\mathcal{N}_h = \{W(x,k,t) : x \in V_{\mathbf{k}},\ k \in V_{\mathbf{x}} \text{ and } t \in K/\mathcal{D}_h\} \tag{4.6.13}$$

is a group for the product (4.6.12). We call $\mathcal{N}_h$ the restricted n.a. Heisenberg group depending $h \in K$.

The center of $\mathcal{N}_h$ consists in the elements $W(0,0,t)$ and we have

$$W(x,k,t)W(x',k',t') = W(0,0,(x-k')-(x',k))W(x',k',t')W(x,k,t). \tag{4.6.14}$$

This shows that (4.6.9) defines a projective representation of the abelian group $V_{\mathbf{k}} \times V_{\mathbf{x}}$.

In particular, for the one-dimensional case, let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,K)$ and $[\cdot]_g$ the mapping of $\mathcal{N}_h$ into itself defined by

$$[W(x,k,t)]_g = W\left(ax+ck, bx+dk, t - \frac{abx^2}{2} - \frac{cdk^2}{2} - bcxk\right). \tag{4.6.15}$$

The following properties of the mapping $[\cdot]_g$ are easy consequences of (4.6.12), (4.6.15) and the multiplication of 2 by 2 matrices.

Proposition 4.6.7. *For all $W, W' \in \mathcal{N}_h$ and $g, g' \in SL(2, K)$*

(1) $[WW']_g = [W]_g[W']_g$,

(2) $[[W]_g]_{g'} = [W]_{gg'}$,

(3) $[W]_g = W$ *if and only if g =Id, (Id is the identity of* $SL(2, K)$*),*

(4) $\{[\cdot]_g : g \in SL(2, K)\}$ *is a group of automorphisms of* $\mathcal{N}_h$.

Proof of (4). By (1) in Proposition 4.6.7, $[\cdot]_g$ is an homomorphism of $\mathcal{N}_h$. Under this homomorphism, $W \in \mathcal{N}_h$ is the image of an unique element $[W]_{g^{-1}}$ of $\mathcal{N}_h$ as a consequence of (2) and (3) in Proposition 4.6.7. This shows that $[\cdot]_g$ is an automorphism of $\mathcal{N}_h$ and by (2) in Proposition 4.6.7 the above set of transformations inherits the group properties of $\mathrm{SL}(2, K)$. ∎

Appendix A: The p-adic Gaussian integrals and the Gelfand-Graev p-adic Γ-function. The purpose of this appendix is to explain how to compute p-adic Gaussian integrals and to find the p-adic counterparts of the Gelfand-Graev Γ- and B-functions, which are defined by integrals of suitable combinations of additive and multiplicative characters of the field $\mathbb{R}$.

Being locally compact, $\mathbb{Q}_p$ has a real-valued Haar measure, i.e., the unique translation invariant measure dx with the properties

$$d(ax) = |a|_p dx \ (a \neq 0); \text{ the measure of } \mathbb{Z}_p = \mu(\mathbb{Z}_p) = \int_{|x|_p \leq 1} dx = 1. \tag{A.1}$$

We calculate the measure of any ball $p^{-r}\mathbb{Z}_p$ for $r \in \mathbb{Z}$. Let us start with $r = 1$. Since $p^{-1}\mathbb{Z}_p = \bigcup_{b=0}^{p-1}(bp^{-1} + \mathbb{Z}_p)$, we have $\mu(p^{-1}\mathbb{Z}_p) = \sum_{b=0}^{p-1} \mu(bp^{-1} + \mathbb{Z}_p) = p$. Repeating the reasoning we get

$$\mu(p^{-r}\mathbb{Z}_p) = \int_{|x|_p \leq p^r} dx = p^r. \tag{A.2}$$

Using (A.2), clearly

$$\int_{|x|_p = p^r} dx = \int_{|x|_p \leq p^r} dx - \int_{|x|_p \leq p^{r-1}} dx = p^r(1 - p^{-1}). \tag{A.3}$$

Since the set $\{|x|_p : x \in \mathbb{Q}_p\}$ is discrete and takes the countable set of numbers $|x|_p = p^r$ for $r \in \mathbb{Z}$, we have $\mathbb{Q}_p = \coprod_{r \in \mathbb{Z}}\{x : |x|_p = p^r\} \cup \{0\}$ and so for a sufficiently well-defined function $f : \mathbb{Q}_p \to \mathbb{C}$

$$\int_{\mathbb{Q}_p} f(x)dx = \sum_{r=-\infty}^{\infty} \int_{|x|_p = p^r} f(x)dx. \tag{A.4}$$

One important property of the p-adic integration is its behavior under $GL(2,\mathbb{Q}_p)$ transformations of the integration variables. $GL(2,\mathbb{Q}_p)$ acts on a p-adic variable x by

$$x \to x' = \frac{ax+b}{cx+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Q}_p),$$

and the change of measure is given by

$$dx' = \frac{|ad-bc|_p}{|cx+d|_p^2}dx. \tag{A.5}$$

Integrals of the form $\int_{\mathbb{Q}_p} \chi_p(ax^2+bx)dx$ are called the p-adic Gaussian integrals. Here χ_p is the usual character on $\mathbb{Q}_p$. Using (A.2) and (A.3), for $b \neq 0$ we get

$$\int_{|x|_p \le p^r} \chi_p(bx)dx = p^r \Omega(p^r|b|_p), \tag{A.6}$$

where $\Omega(x)$ is 1 if $0 \le x \le 1$ and 0 if $x > 1$;

$$\int_{|x|_p = p^r} \chi_p(bx)dx = \begin{cases} p^r(1-p^{-1}) & \text{for } |b|_p \le p^{-r} \\ -p^{r-1} & \text{for } |b|_p = p^{-r+1} \\ 0 & \text{for } |b|_p > p^{-r+1}. \end{cases} \tag{A.7}$$

For $a \neq 0$, using (A.4) we write, with $t = x + b/2a$ and $dx = dt$,

$$\int_{\mathbb{Q}_p} \chi_p(ax^2+bx)dx = \chi_p\left(-\frac{b^2}{4a}\right) \sum_{m=-\infty}^{\infty} G_m(a) = \chi_p\left(-\frac{b^2}{4a}\right) \sum_{m=-\infty}^{\infty} \int_{|t|_p=p^m} \chi_p(at^2)dt.$$

To calculate $G_m(a)$, it should be noted that: Using (A.2) and the necessary and sufficient condition for the equation $x^2 = a$ has a solution $x \in \mathbb{Q}_p$ $(p \neq 2)$, we have

$$\int_{\mathbb{Z}_p^2} dy = \sum_{m=0}^{\infty} \sum_{(y_0|p)=1, y_0=1}^{p-1} \int_{\{p^{2m}(y_0+\cdots)\}} dy = \frac{p-1}{2} \sum_{m=0}^{\infty} p^{-2m-1} = \frac{1}{2(1+p^{-1})},$$

where $(y_0|p)$ is the Legendre symbol. Also, for all prime number p, we use the quadratic changes variables $dx^2 = \frac{1}{2}|2x|_p dx$, then using (A.3), we have for $p \neq 2$

$$\int_{\mathbb{Z}_p^2} dy = \frac{1}{2}\int_{\mathbb{Z}_p} |2x|_p dx = \frac{1}{2}\sum_{m=0}^{\infty} p^{-m} \int_{|x|_p = p^{-m}} dx = \frac{1}{2(1+p^{-1})}.$$

Thus, changing variables and focusing on the case $p \neq 2$, we have, with $x = at^2$, $dx = \frac{1}{2}|a|_p|t|_p dt$, and $\mathrm{ord}_p x = -k = \mathrm{ord}_p a - 2m$,

$$G_m(a) = \frac{2}{|a|_p} p^{-m} \int_{(x_0|p)=(a_0|p), |x|_p=p^k} \chi_p(x)dx,$$

where x_0 and a_0 are the first digit of x and a, respectively. If $k \leq 0$, then $\chi_p(x) = 1$ and using (A.3), we have

$$G_m(a) = p^m(1 - p^{-1}) \quad \text{for } 2m \leq \mathrm{ord}_p a. \tag{A.8}$$

If $k = 1$, then we have

$$G_m(a) = \frac{2}{\sqrt{p|a|_p}} \sum_{(x_0|p)=(a_0|p),x_0=1}^{p-1} e^{2\pi\sqrt{-1}x_0/p} = \frac{G(x_0^{-1};p) - 1}{\sqrt{p|a|_p}},$$

where $G(n;m) = \sum_{r=1}^{m} e^{2\pi\sqrt{-1}nr^2/m}$ is the quadratic Gauss sum. Since p is an odd prime and $p \nmid x_0$, we have the formula (cf. [Apo 76, p. 195])

$$G(x_0^{-1};p) = (x_0|p)G(1;p) = \begin{cases} (x_0|p)\sqrt{p} & \text{for } p \equiv 1 \pmod 4 \\ \sqrt{-1}(x_0|p)\sqrt{p} & \text{for } p \equiv 3 \pmod 4). \end{cases} \tag{A.9}$$

We use an arithmetic function $\lambda_p : \mathbb{Q}_p^\times \to \mathbb{C}$ defined by, for $p \neq 2$,

$$\lambda_p(x) = \begin{cases} 1 & \text{if } k \text{ is even} \\ (x_0|p) & \text{if } k \text{ is odd and } p \equiv 1 \pmod 4 \\ \sqrt{-1}(x_0|p) & \text{if } k \text{ is odd and } p \equiv 3 \pmod 4), \end{cases} \tag{A.10}$$

where $x = p^{-k}(x_0 + x_1 p + \cdots)$. Then we obtain

$$G_m(a) = \frac{\lambda_p(a)}{\sqrt{|a|_p}} - \frac{1}{\sqrt{p|a|_p}} \quad \text{for } 2m = 1 + \mathrm{ord}_p a. \tag{A.11}$$

Finally, if $k > 1$, then we have

$$G_m(a) = \frac{2}{\sqrt{p|a|_p}} \sum_{(x_0|p)=(a_0|p),x_0=1}^{p-1} e^{2\pi\sqrt{-1}x_0/p} \sum_{x_1=0}^{p-1} \cdots \sum_{x_{k-1}=0}^{p-1} e^{2\pi\sqrt{-1}x_{k-1}/p} = 0 \tag{A.12}$$

for $2m > 1 + \mathrm{ord}_p a$, since $\sum_{x=0}^{p-1} e^{2\pi\sqrt{-1}x/p} = 0$.

On the other hand, for $p = 2$, we have easily the following formulas: Let $\alpha = 1 + \alpha_1 2 + \alpha_2 2^2 + \cdots$, then

$$\int_{|y|_2=2^r} \chi_2(\alpha y^2)dy = \begin{cases} 2^{r-1} & \text{for } r \leq 0 \\ (-1)^{\alpha_1}\sqrt{-1} & \text{for } r = 1 \\ 0 & \text{for } r \geq 2, \end{cases} \tag{A.13}$$

$$\text{(A.14)} \qquad \int_{|y|_2=2^r} \chi_2(2\alpha y^2)dy = \begin{cases} 2^{r-1} & \text{for } r \le 0 \\ -1 & \text{for } r = 1 \\ \sqrt{2}(-1)^{\alpha_1+\alpha_2}(1+(-1)^{\alpha_1}\sqrt{-1}) & \text{for } r = 2 \\ 0 & \text{for } r \ge 3. \end{cases}$$

From (A.13) and (A.14) we obtain the values of the integrals

$$\text{(A.15a)} \qquad \int_{\mathbb{Q}_2} \chi_2(\alpha y^2)dy = 1 + (-1)^{\alpha_1}\sqrt{-1};$$

$$\text{(A.15b)} \qquad \int_{\mathbb{Q}_2} \chi_2(2\alpha y^2)dy = \sqrt{2}(-1)^{\alpha_1+\alpha_2}(1+(-1)^{\alpha_1}\sqrt{-1}).$$

Let $a = 2^{-m}(1 + a_1 2 + a_2 2^2 + \cdots)$. If m is even (resp. m is odd), then changing variable $y = 2^{-m/2}t$ (resp. $y = 2^{-(m+1)/2}t$) and using (A.15), we obtain

$$\int_{\mathbb{Q}_2} \chi_2(at^2)dt = \begin{cases} \frac{1+(-1)^{a_1}\sqrt{-1}}{\sqrt{|a|_2}} & \text{for } m \text{ is even} \\ \frac{(-1)^{a_1+a_2}(1+(-1)^{a_1}\sqrt{-1})}{\sqrt{|a|_2}} & \text{for } m \text{ is odd.} \end{cases}$$

Thus, summing up (A.9) and (A.11), and using an arithmetic function $\lambda_2 : \mathbb{Q}_2^\times \to \mathbb{C}$ defined by

$$\text{(A.17)} \qquad \lambda_2(a) = \begin{cases} \frac{1}{\sqrt{2}}(1+(-1)^{a_1}\sqrt{-1}) & \text{if } m \text{ is even} \\ \frac{1}{\sqrt{2}}(-1)^{a_1+a_2}(1+(-1)^{a_1}\sqrt{-1}) & \text{if } m \text{ is odd} \end{cases}$$

gives

$$\text{(A.18)} \qquad \int_{\mathbb{Q}_p} \chi_p(ax^2+bx)dx = \lambda_p(a)|2a|_p^{-1/2}\chi_p\left(-\frac{b^2}{4a}\right) \quad (a \ne 0).$$

The symbol $\lambda_p(a)$ has the following properties: For $a, b \in \mathbb{Q}_p^\times$,

(i) $|\lambda_p(a)| = 1$ and $\lambda_p(a)\lambda_p(-a) = 1$; (ii) $\lambda_p(a^2 b) = \lambda_p(a)$;

(iii) $\lambda_p(a)\lambda_p(b) = \lambda_p(a+b)\lambda_p(\frac{1}{a}+\frac{1}{b})$; (iv) $\prod_{p=2}^{\infty} \lambda_p(a) = 1$.

Remark. A function similar to $\lambda_p(a)$ was considered by Weil [Wei 64] for locally compact fields, and the function $\lambda_p(a)$ is connected with the Hilbert symbol $(\ ,\ )_H$ by

$$\lambda_p(a)\lambda_p(b) = (a,b)_H \lambda_p(ab) \quad \text{for } a, b \in Q_p^\times,\ p \ne 2.$$

Here by definition the Hilbert symbol $(a,b)_H,\ a,b \in \mathbb{Q}_p^\times$, is equal to $+1$ or -1 subject to if the form $ax^2 + by^2 - z^2$ represents 0 in the field $\mathbb{Q}_p$ or not.

The p-adic Gaussian integrals on the disc $|x|_p \le p^r$ is given as follows: If $p \neq 2$ and $a \neq 0$,
(A.19)

$$\int_{|x|_p \le p^r} \chi_p(ax^2 + bx)dx = \begin{cases} p^r \Omega(p^r |b|_p) & \text{for } |a|_p \le p^{-2r} \\ \lambda_p(a)|2a|_p^{-1/2} \chi_p\left(-\frac{b^2}{4a}\right) \Omega\left(p^{-r}\left|\frac{b}{2a}\right|_p\right) & \text{for } |4a|_p > p^{-2r}; \end{cases}$$

if $p = 2$ and $a \neq 0$,
(A.20)

$$\int_{|x|_2 \le 2^r} \chi_2(ax^2 + bx)dx = \begin{cases} 2^r \Omega(2^r |b|_2) & \text{for } |a|_2 \le 2^{-2r} \\ \lambda_2(a)|2a|_2^{-1/2} \chi_2\left(-\frac{b^2}{4a}\right) \delta(|b|_2 - 2^{1-r}) & \text{for } |a|_2 = 2^{-2r+1} \\ \lambda_2(a)|2a|_2^{-1/2} \chi_2\left(-\frac{b^2}{4a}\right) \Omega(2^r|b|_2) & \text{for } |a|_2 = 2^{-2r+2} \\ \lambda_2(a)|2a|_2^{-1/2} \chi_2\left(-\frac{b^2}{4a}\right) \Omega\left(2^{-2r}\left|\frac{b}{2a}\right|_2\right) & \text{for } |a|_2 \ge 2^{-2r+3}, \end{cases}$$

where $\delta(|b|_p - p^r)$ is 1 if $|b|_p = p^r$ and 0 if $|b|_p \neq p^r$.

The Gelfand-Graev Γ- and B-functions, $\Gamma_\infty(\alpha)$ and $B_\infty(\alpha,\beta)$ are defined by: for any $\alpha, \beta \in \mathbb{R}$

$$\Gamma_\infty(\alpha) = \int_{\mathbb{R}} |x|^{\alpha-1} e^{-2\pi\sqrt{-1}x} dx = 2(2\pi)^{-\alpha}\Gamma(\alpha)\cos\frac{\pi\alpha}{2}, \tag{A.21}$$

$$B_\infty(\alpha,\beta) = \int_{\mathbb{R}} |x|^{\alpha-1}|1-x|^{\beta-1}dx = \frac{\Gamma_\infty(\alpha)\Gamma_\infty(\beta)}{\Gamma_\infty(\alpha+\beta)}. \tag{A.22}$$

The Γ_∞-function satisfies the relation

$$\Gamma_\infty(\alpha)\Gamma_\infty(1-\alpha) = 1. \tag{A.23}$$

According to (A.22) and (A.23), the B_∞-function can be represented in the following form

$$B_\infty(\alpha,\beta) = \Gamma_\infty(\alpha)\Gamma_\infty(\beta)\Gamma_\infty(1-\alpha-\beta). \tag{A.24}$$

We can show that other symmetric representations of the B_∞-functions are given by

$$B_\infty(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)} + \frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}, \tag{A.25a}$$

$$B_\infty(\alpha,\beta) = \frac{4}{\pi}\cos\frac{\pi\alpha}{2}\cos\frac{\pi\beta}{2}\cos\frac{\pi\gamma}{2}\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma), \tag{A.25b}$$

(A.25c) $$B_\infty(\alpha,\beta)=\frac{\zeta(1-\alpha)}{\zeta(\alpha)}\frac{\zeta(1-\beta)}{\zeta(\beta)}\frac{\zeta(1-\gamma)}{\zeta(\gamma)},$$

where $\gamma = 1-\alpha-\beta$ and $\zeta(s)$ is the Riemann zeta function. By the well-known functional equation

(A.26) $$\zeta(1-\alpha)=2(2\pi)^{-\alpha}\cos\frac{\pi\alpha}{2}\Gamma(\alpha)\zeta(\alpha)$$

and the relation (A.21), we have

(A.27) $$\Gamma_\infty(\alpha)=\frac{\zeta(1-\alpha)}{\zeta(\alpha)}.$$

The Gelfand-Graev p-adic Γ- and B-functions are defined analogously by changing $\mathbb{R}$ by $\mathbb{Q}_p$. We define

(A.28) $$\Gamma_p(\alpha)=\int_{\mathbb{Q}_p}|x|_p^{\alpha-1}\chi_p(x)dx=\frac{1-p^{\alpha-1}}{1-p^{-\alpha}},\ \alpha\in\mathbb{R}$$

and

(A.29) $$B_p(\alpha,\beta)=\int_{\mathbb{Q}_p}|x|_p^{\alpha-1}|1-x|_p^{\beta-1}dx=\frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)},\ \alpha,\ \beta\in\mathbb{R}.$$

Then we have the following:

(A.30) $$\Gamma_p(\alpha)\Gamma_p(1-\alpha)=1,$$

(A.31) $$B_p(\alpha,\beta)=\Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(1-\alpha-\beta).$$

Appendix B: The Γ-function and the Morita p-adic Γ-function. Let $s=\sigma+it\in\mathbb{C}$, where $i=\sqrt{-1}$. There are several equivalent definitions for the Γ-function as follows:

(B.1;Euler's formula) $$\Gamma(s)=\frac{1}{s}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^s\left(1+\frac{s}{n}\right)^{-1}.$$

(B.2;Euler's integral) $$\Gamma(s)=\int_0^\infty e^{-z}z^{s-1}dz \ \text{ for } \sigma>0.$$

This integral is convergent only when $\sigma>0$. To show that, we remark that for a fixed $\epsilon\in\mathbb{R}_{>0}$ and for any $\sigma\in\mathbb{R}$, there exists $M>0$ such that if $z>\epsilon$, then $Mz^{-2}>e^{-z}z^{\sigma-1}$. Thus we know that

$$\left|\int_\epsilon^\infty e^{-z}z^{s-1}dz\right|\le\int_\epsilon^\infty e^{-z}z^{\sigma-1}dz\le M\int_\epsilon^\infty z^{-2}dx=\frac{M}{\epsilon}.$$

Thus integral $\int_\epsilon^\infty e^{-z}z^{s-1}dz$ converges for all s and gives an analytic function of s. The problem of divergence lies in the other integral:

$$\left|\int_0^\epsilon e^{-z}z^{s-1}dz\right| \le \int_0^\epsilon e^{-z}z^{\sigma-1}dz \le \int_0^\epsilon z^{\sigma-1}dz = \frac{\epsilon^\sigma}{\sigma} \text{ if } \sigma > 0.$$

To find the analytic continuation of $\Gamma(s)$, we consider $e^z z^{s-1}$ as a function of $z \in \mathbb{C}$ and then it is multivalued with a branch point at $z = 0$. Make a branch cut along the negative real axis and let $P(\epsilon)$ be the contour which is a loop around the negative real axis, and is composed of three parts $C_1, C_2(\epsilon), C_3$. $C_2(\epsilon)$ is a counterclockwise oriented circle of radius $\epsilon < 2\pi$ with center 0 starting from $\epsilon = |\epsilon|e^{-\pi i}$, and C_1, C_3 are, respectively, the lower and upper edges of a cut in the z-plane along the negative real axis (i.e., $z = re^{-\pi i}$ on C_1 and $z = re^{\pi i}$ on C_2 where r varies from ϵ to ∞). Then we have

$$\int_{P(\epsilon)} e^z z^{s-1}dz = \lim_{\epsilon\to 0}\left\{\left(e^{\pi i s} - e^{-\pi i s}\right)\int_\epsilon^\infty e^{-r}r^{s-1}dr + i\epsilon^s\int_{C_2(\epsilon)} e^{\epsilon e^{i\theta}}(e^{i\theta})^{s-1}e^{i\theta}d\theta\right\}$$

For $\sigma > 0$ the second integral on the right vanishes as $\epsilon \to 0$. After taking the limit, we have an analytic continuation of $\Gamma(s)$ to the left-hand half plane,

(B.3) $$\Gamma(s) = \frac{1}{2i\sin(\pi s)}\int_{P(\epsilon)} e^z z^{s-1}dz \text{ for } \sigma > 0 \text{ and for } s \ne \text{integer}.$$

By writing the integral (B.2) as two piece,

$$\Gamma(s) = \int_0^1 e^{-z}z^{s-1}dz + \int_1^\infty e^{-z}z^{s-1}dz,$$

we see that the second term is analytic everywhere because its derivative exists and the range of integration does not include any singularities of the integrand. Now expand e^{-z} and reverse the order of summation and integration. This gives,

$$\int_0^1 e^{-z}z^{s-1}dz = \sum_{n=0}^\infty \frac{(-1)^n}{n!}\int_0^1 z^{s+n-1}dz = \sum_{n=0}^\infty \frac{(-1)^n}{n!}\frac{1}{s+n}.$$

Thus we have

(B.4) the singularities of $\Gamma(s)$ are simple poles on the real axis at $s = 0, -1, -2, \cdots$, with residue $(-1)^n/n!$ at $s = -n$.

(B.5;Euler) $$\Gamma(s) = \lim_{n\to\infty}\frac{n^s n!}{s(s+1)\cdots(s+n)} \quad \text{for } s \ne 0, -1, -2, \cdots.$$

(B.6;Weierstrass) $$\frac{1}{\Gamma(s)} = se^{Cs}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-s/n} \text{ for all } s,$$

where $C = \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right) = 0.5772157\cdots$ is Euler's constant.

(B.7) $\Gamma(s)$ has an essential singularity at $s=\infty$.

(B.8) $\Gamma(s)$ has no zeros because the product converges for all s in (B.6).

(B.9) $\Gamma(1)=1$, $\Gamma(\frac{1}{2})=\pi$, $\Gamma(n)=(n-1)!$, $\lim_{n\to\infty}\frac{\Gamma(s+n)}{\Gamma(n)n^s}=1$ and $C=-\Gamma'(1)$.

(B.10) (Two functional equations) $\Gamma(s+1)=s\Gamma(s)$ and $\Gamma(s)\Gamma(1-s)=\pi/\sin(\pi s)$ for all s.

(B.11) (Legendre's duplication formula) $\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)=2\sqrt{\pi}2^{-2s}\Gamma(2s)$.

(B.12) (Gauss' multiplication formula)

$$\Gamma\left(\frac{s}{n}\right)\Gamma\left(\frac{s+1}{n}\right)\cdots\Gamma\left(\frac{s+n-1}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}\Gamma(s)}{n^{s-\frac{1}{2}}} \text{ for all } s \text{ and all } n\in\mathbb{N}.$$

(B.13) $$\Gamma(s) = \int_0^1\left(\ln\frac{1}{t}\right)^{s-1}dt \text{ for } \sigma>0.$$

(B.14) $$B(\alpha,\beta)=B(\beta,\alpha)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

where $B(\alpha,\beta)$ is the B-function defined by

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt,\ \mathrm{Re}\alpha>0,\ \mathrm{Re}\beta>0.$$

Let p be an odd prime. Many authors have considered p-adic analogs of the classical Γ-function. Morita defined the following function in [Mor 75]

(B.15) $$\Gamma_p^{(M)}(z)=\lim_{m\to z}(-1)^m\prod_{(p,j)=1,j=1}^{m-1} j,$$

where m approaches z through positive integers. $\Gamma_p^{(M)}(z)$ is called the Morita p-adic Γ-function. This gives a continuous function $\Gamma_p^{(M)}:\mathbb{Z}_p\to\mathbb{Z}_p^\times$, where $\mathbb{Z}_p^\times$ is the group of p-adic units, which satisfies the functional equation:

(B.16) $$\Gamma_p^{(M)}(z+1)=\begin{cases}-z\Gamma_p^{(M)}(z) & \text{if } z\in\mathbb{Z}_p^\times\\ -\Gamma_p^{(M)}(z) & \text{if } z\in p\mathbb{Z}_p.\end{cases}$$

Morita shows that $\Gamma_p^{(M)}$ is analytic on $p\mathbb{Z}_p$ and gives a uniform analytic function on the set $\{x \in \mathbb{C}_p : |x|_p \leq p\}$. It is not analytic on the closed unit disc, or else the functional equation $\Gamma_p^{(M)}(z+1) = -\Gamma_p^{(M)}(z)$ would hold on all of $\mathbb{Z}_p$.

$\Gamma_p^{(M)}(z)$ satisfies the following properties:

(B.17) $$\Gamma_p^{(M)}(z)\Gamma_p^{(M)}(1-z) = (-1)^{z_0} \text{ if } z_0 \equiv z \pmod{p},\ z_0 \in \mathbb{Z}_p^\times.$$

(B.18) $$\lim_{p\to\infty} \Gamma_p^{(M)}(n) = (-1)^n \Gamma(n),\ n \in \mathbb{N}.$$

(B.19) $$\Gamma_p^{(M)}(mz) = c_m m^{mz-1-[(mz-1)/p]} \prod_{(m,p)=1,a=0}^{m-1} \Gamma_p^{(M)}(z+a/m),$$

where $m \in \mathbb{N}$ and c_m is a constant depending on m.

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