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Some prehomogeneous representations defined by cubic forms

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Hisatoshi Ikai

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Hisatoshi Ikai

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Preface

This work is based on my thesis at Tohoku University in 1996. Though the present work is mostly concerned with preliminaries to define some class of prehomogeneous vector spaces over an arbitrary commutative ring, the initial problem which motivates me is a number-theoretic one to study some quotient sets and associated zeta functions. Therefore, it will be appropriate to mention on the place from which this work arises.

Let k be a commutative ring. Consider a triple (G, θ, M) consisting of a k-group G and its linear representation $\theta : G \to \mathbf{GL}(M)$ in a finitely generated projective k-module M. In the case where k is a field and G is reductive, there is a notion of *prehomogeneous vector spaces* introduced by M. Sato. Recall that such a triple (G, θ, M) is said to be *prehomogeneous* if we have, after tensoring with an algebraic closure $\bar{k}$ of k, a *Zariski open* $G(\bar{k})$*-orbit* in $M \otimes_k \bar{k}$. A *classification* is given by M. Sato and Kimura in [S-K], for the irreducible ones over the field of complex numbers. One of the motivations to consider prehomogeneous vector spaces can be seen in a number-theoretic situation, where k is the *ring of rational integers* and the triple in question becomes a prehomogeneous vector space at the geometric fiber over the infinite prime. Then, one seeks a systematic way of constructing Drichlet series with nice properties such as functional equations, and is lead to considering certain generating functions called *zeta functions*. On zeta functions associated with prehomogeneous vector spaces, there are basic researches of Sato-Shintani [S-S] and F. Sato [Sf]. In general, however, it is a difficult problem to determine the principal part of the zeta function. Together with precise estimates for the convergences of integrals, we need to know the orbit structures. In fact, it sometimes turns out that the open orbit counts interesting arithmetic objects such as field extensions and ideal classes, and that their asymptotic properties are derived from information on the zeta function. Examples of such a successful case can be seen in works of Shintani, where he considered the spaces of binary cubic forms in [Sh 1] and of binary quadratic forms in [Sh 2], to improve the results of Davenport [Dav] and of Siegel [Siegel], respectively. On the other hand, many of the number-theoretic problems can also be formulated in the language of *adeles*, where k is taken to be a *global field*. The adelic versions of Shintani's works are given by D. J. Wright [W] and Yukie [Y 1]. Yukie also handled in [Y 2] the zeta function associated to the more complicated case, namely the space of pairs of ternary quadratic forms.

Inspired by these works, I tried to investigate some other prehomogeneous vector spaces in the number-theoretic setting, which covers not only the adelic ones but also the ones over the rational integers. The examples which are chosen in my work are related to the Jordan algebras of degree three, and the "cubic forms "in the title means the so called generic norms of such algebras. The materials go back to the work of Freudenthal

[Freu] on the geometries of some exceptional Lie groups, and many investigations have been done by many authors from various points of view. Among them, I would like to mention on the Baily's one, which treats an arithmetic quotient of the bounded symmetric domain of type VI. In fact, the attempt to understand [Baily] motivates me to work over an arbitrary commutative ring.

I wish to thank many individuals who have stimulated and encouraged me during this work. Especially, my deep appreciation goes to Professor Yasuo Morita without whose constant support and encouragement this work would probably not have been done.

H. Ikai

Contents

Introduction

1. Let k be a *field*. Consider a triple (G, θ, M) consisting of a reductive k-group G and its finite-dimensional linear representation $\theta : G \to \mathbf{GL}(M)$ such that we have, after tensoring with an algebraic closure $\bar{k}$ of k, the following condition $(\star)$: Denote by $\tilde{G}$ the simply connected covering group of $G \otimes_k \bar{k}$ and fix a pair $(\tilde{T}, \tilde{B})$ of a split maximal torus $\tilde{T}$ of $\tilde{G}$ and a Borel group $\tilde{B}$ containing $\tilde{T}$. Denote also by $\mathcal{L}(\chi)$ the invertible sheaf on $\tilde{G}/\tilde{B}$ associated to a character χ of $\tilde{T}$, and by w_0 the symmetry with respect to $\tilde{B}$, which is an element of the Weyl group of $(\tilde{G}, \tilde{T})$. Then:

$(\star)$ The Dynkin diagram Γ of $(\tilde{G}, \tilde{T}, \tilde{B})$ is C_3, A_5, D_6, or E_7, and we have an isomorphism

$$M \otimes_k \bar{k} \xrightarrow{\sim} H^0(\tilde{G}/\tilde{B}, \mathcal{L}(w_0(\lambda)))$$

of $\tilde{G}$-modules, where λ is the weight defined by the table:

Γ	C_3	A_5	D_6	E_7
λ	ϖ_3	ϖ_3	ϖ_6	ϖ_7

(notation is of [Bou, Lie, VI, §4]).

2. It is known that those (G, θ, M) as in **1** are related to *Jordan algebras* $J = H_3(\mathcal{C})$ of 3×3 Hermitian matrices with coefficients in various composition algebras $\mathcal{C}$ (cf. [Freu, VIII]). Also we call attention to the fact that such a triple gives rise to a *prehomogeneous vector space*, which admits a certain *quartic form* as a relative invariant (cf. [S-K]). More precisely, if $\bar{k}$ is taken to be the field of the complex numbers, the pair $(\mathbf{G}_{\mathbf{m}\bar{k}} \times \tilde{G}, M \otimes_k \bar{k})$ is the prehomogeneous vector space referred to, in the table in [S-K, §7, I)], as

$$\begin{aligned}
(14) &= (GL(1) \times Sp(3),\ V(1) \otimes V(14)) \quad \text{if} \quad \Gamma = C_3, \\
(5) &= (GL(6),\ V(20)) \quad \text{if} \quad \Gamma = A_5, \\
(23) &= (GL(1) \times Spin(12),\ V(1) \otimes V(32)) \quad \text{if} \quad \Gamma = D_6, \quad \text{and} \\
(29) &= (GL(1) \times E_7,\ V(1) \otimes V(56)) \quad \text{if} \quad \Gamma = E_7.
\end{aligned}$$

(Here we omit the symbol of the representation. For example, (5) is the third exterior power of the six-dimensional standard representation of $GL(6)$, and (14) is obtained from (5) via the inclusion $Sp(3) \to GL(6)$.)

To apply the theory of prehomogeneous vector spaces to number theory, we assume k to be a global field. Then one of the main problems is to determine the quotient set

$(k^* \times G(k))\backslash M$ and the corresponding stabilizers in $\mathbf{G}_{\mathbf{m}k} \times G$ (cf. [W-Y]). This article arouses out of an attempt to solve this problem along the following line:

a) Since k is a field, we have the canonical bijection

$$(k^* \times G(k))\backslash(M - \{0\}) \xrightarrow{\sim} G(k)\backslash \mathbf{P}(M)(k)$$

of sets and, for each $m \in M - \{0\}$, the canonical isomorphism $\mathbf{Cent}_{\mu_k \times G}(m) \xrightarrow{\sim} \mathbf{Cent}_G(k \cdot m)$ of stabilizers. Therefore, considering the projective representation $G \to \mathbf{Aut}(\mathbf{P}(M))$ associated to θ, it suffices to determine the quotient set $G(k)\backslash \mathbf{P}(M)(k)$ together with the corresponding stabilizers in G.

b) Choose points

$$u_0, u_1, \ldots \in \mathbf{P}(M)(k)$$

so that, if possible, the set $\mathbf{P}(M)(\bar{k})$ becomes the *union* of the $O_G(u_i)(\bar{k})$'s, where $O_G(u_i)$ denotes the *orbit-sheaf* (with respect to the fppf topology, cf. [D-G, III, §3, 1.6]) of u_i under G.

c) For each u_i in **b)**, determine the stabilizer

$$\mathbf{Cent}_G(u_i)$$

and the set Ker $[H^1(k, \mathbf{Cent}_G(u_i)) \to H^1(k, G)]$, which is identified with the quotient set $G(k)\backslash O_G(u_i)(k)$ (cf. [D-G, III, §4, 4.7]).

3. Among many works concerned with such (G, θ, M) as in **1**, [Igusa 1] and [Baily] are closely related our research. For (G, θ, M) of type C_3, D_6, or E_7 in **1**, Igusa determined the quotient set $G(k)\backslash M$ and the corresponding stabilizers in G, under the assumption that k is an *algebraically closed field of characteristic different from two and three* (cf. [Igusa 1, §7]). On the other hand, using a $\mathbf{Z}$-form of the real octonion division algebra, Baily treated a triple (G, θ, M) over the *ring of rational integers* such that the associated analytic group $G(\mathbf{R})$ is a Lie group of type E_7, whose identity component modulo center is the automorphism group of a bounded symmetric domain in $\mathbf{C}^{27}$ (cf. [Baily]). In general, the property stated in **1** does not determine one triple (G, θ, M), and hence such a problem as in **2** depends on the *choice* of one triple. Such a triple may be called a *form.* Instead of considering *all* the forms of a special representation, we are trying to define a triple (G, θ, M) over an arbitrary commutative ring such that some (or all, if possible,) of the fibers have the property stated in **1**. The construction of (G, θ, M) and almost all of the calculations work over an arbitrary commutative ring. Also our construction contains Baily's case, which is not of split type and causes special difficulties in characteristic two. In fact, the desire to handle such a case leads us to considering schemes over $\mathbf{Z}$, and hence we are obliged to construct everything without assumption on the base ring. In particular, we need to include the case of characteristic two and three, which are avoided in [Igusa 1]. In this article, we give an axiomatic construction of (G, θ, M) and determine two stabilizers and one orbit-sheaf. More precise description is as follows:

4. Let k be an arbitrary *commutative ring.* Consider a quadruple $(J; N, \sharp, T)$ as the data, where J is a *finitely generated projective* k-module, N a *cubic form* on J, i.e., N is a homogeneous element of degree three of the symmetric algebra $\mathbf{S}({}^tJ)$ of the k-module ${}^tJ$ dual to J, $\sharp$ a *quadratic map* in J, which is a certain endomorphism of the k-scheme $\operatorname{Spec}\mathbf{S}({}^tJ)$, and T a *symmetric bilinear form* on J, satisfying some conditions (cf. §1). Then:

a) We define a k-group *sheaf* G with respect to the fppf topology and its *linear representation* $\theta : G \to \mathbf{G}(M)$ in the k-module $M := k \oplus J \oplus k \oplus J$.

b) We choose two k-valued points u_0 and u_1 of the projective space $\mathbf{P}(M) :=$ $\operatorname{Proj}\mathbf{S}({}^tM)$, and determine their *stabilizers* $\mathbf{Cent}_G(u_i)$ $(i = 0, 1)$ in G and the *orbit-sheaf* $O_G(u_1)$ of u_1 under G (cf. §3).

c) We choose a *quartic form* $f \in \mathbf{S}^4({}^tM)$ and prove that G stabilizes f (cf. §4). Then the action of G is induced on the open subscheme $D_+(f)$ of $\mathbf{P}(M)$. The point u_0 in **b)** belongs to $D_+(f)(k)$.

d) Under some additional condition on $(J; N, \sharp, T)$, we prove that, for any k-algebra $k \to K$ with an algebraically closed field K, the action of $G(K)$ on $D_+(f)(K)$ is *transitive.*

5. Main tools of our construction are the notion of *Jordan pairs* (cf. [LJP]) and the general theory of associated algebraic groups (cf. [LAG]), both due to Loos. Nowaday, the theory of Jordan algebras is generalized as follows:

a) If k is a *field* of *characteristic different from two*, the category $\mathbf{LJA}_k$ of *linear* Jordan algebras over k is defined, whose objects are classical Jordan algebras (cf. [Jac 3]).

b) For an *arbitrary commutative ring* k, the category $\mathbf{QJA}_k$ of *quadratic* Jordan algebras over k is defined (McCrimmon, cf. [Jac 1]). If k is a field of characteristic different from two, the *equivalence* $\mathbf{LJA}_k \xrightarrow{\approx} \mathbf{QJA}_k$ is defined.

c) For an *arbitrary commutative ring* k, the category $\mathbf{JP}_k$ of *Jordan pairs* over k and a functor $\mathbf{QJA}_k \to \mathbf{JP}_k$ is defined (cf. [LJP]). Under this functor, two objects in $\mathbf{QJA}_k$ are identified in $\mathbf{JP}_k$ modulo *isotopism*, which is weaker than isomorphism.

The restriction on the base k in [Igusa 1, 2] comes from the fact that he uses the theory of *linear* Jordan algebras. However there is an axiomatic construction of Jordan algebras of 3×3 Hermitian matrices from the viewpoint of *quadratic* Jordan algebras by McCrimmon (cf. [Mc]). This allows us to handle the object $J = H_3(\mathcal{C})$ over an arbitrary commutative ring. On the other hand, the theory of algebraic groups associated to Jordan pairs is developed by Loos over an arbitrary commutative ring (cf. [LAG]). Hence the classical construction as in [Igusa 1] can be generalized to arbitrary base ring. Though k suffices to be a field in many problems concerned with prehomogeneous vector spaces, so far as this article, we thus generalize k to be an arbitrary commutative ring.

6. Here is a short description of this article. We take an arbitrary commutative ring as a base, and consider in the category of *sheaves* over k with respect to the fppf topology. Following [Roby], we use the term "polynomial laws" instead of "polynomial functions". They are certain *k-scheme morphisms* and cubic form, quartic form,...etc. are all special polynomial laws (cf. **0.4**).

In §1, a quadruple $(J; N, \sharp, T)$ is introduced. Under the condition $(*)$ in **1.3**, which is always assumed after **1.6**, this quadruple defines a *Jordan pair* denoted by (J, J), and basic identities are proved. The quadruple $(J; N, \sharp, T)$ is modeled on the Jordan algebra $J = H_3(\mathcal{C})$ of 3×3 Hermitian matrices with coefficients in a composition algebra $\mathcal{C}$, for which there is an axiomatic construction by McCrimmon (cf. [Mc]). Since we use the theory of Jordan pairs instead of Jordan algebras, we slightly modified McCrimmon's construction. His starting data may be considered as a couple $((J; N, \sharp, T), c)$ of our quadruple $(J; N, \sharp, T)$ and a special element c of J (cf. Appendix A).

In §2, the k-group sheaf G and the linear representation $\theta : G \to \mathbf{GL}(M)$ is constructed. The machinery being based on [LAG], our task are to set up materials and to prove necessary identities. Namely, to construct G, it suffice to construct a "Jordan system ", which is given in **2.1**–**2.3**. To construct θ, we must prove some fundamental relations through complicated calculation (cf. **2.5**–**2.8**). G turns out to be the following k-sheaf: G contains the k-group scheme H, whose points $H(R)$ with value in a k-algebra R is the group of all $h = (\chi(h), h_+, h_-) \in R^* \times GL(J \otimes_k R) \times GL(J \otimes_k R)$ satisfying

$$\text{(H1)} \qquad T(h_+x, h_-y) = T(x, y),$$

$$\text{(H2)} \qquad (h_+x)^\sharp = \chi(h)^{-1} h_- x^\sharp, \quad (h_-x)^\sharp = \chi(h) h_+ x^\sharp,$$

$$\text{(H3)} \qquad N(h_+^{-1}x) = N(h_-x) = \chi(h) N(x),$$

for all scalar extensions. G also contains two subgroups U^+, U^- isomorphic to the vector group associated to J. They are normalized by H and the morphism $U^- \times U^+ \times U^- \times H \to G$ induced by the multiplication is an epimorphism of k-sheaves. M is the direct sum $k \oplus J \oplus k \oplus J$ whose elements are written in the form

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}$$

with $\alpha, \beta \in k$, and $a, b \in J$. M is imbedded into a k-module consisting of pairs of polynomial laws on J (cf. **2.7**). In general, there are functors associating to any k-Jordan pair V an adjoint k-group $\mathbf{PG}(V)$ (cf. [LAG, 5.14, 5.15]) and a smooth quasi-projective

k-scheme $X(V)$ (cf. [LHA, 2.1, 2.2]), whose automorphism group coincides with $\mathbf{PG}(V)$ if V is separable (cf. [LHA, 4.6]). In some special situation where $k = \mathbf{C}$ and V carries a "positive Hermitian involution ", V determines a bounded symmetric domain whose compact dual is $X(V)^{\mathrm{an}}$ (cf. [LBSD]). In our situation, we have a morphism $G \to \mathbf{PG}(J, J)$ whose kernel is the group of square roots of the unit (cf. **2.3**, [LAG, 5.15]).

In §3, two points u_0 and u_1 in $\mathbf{P}(M)(k)$ are chosen to be the ones corresponding to the elements

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

in M, respectively. Then their stabilizers $\mathbf{Cent}_G(u_i)$ and the orbit-sheaf $O_G(u_1)$ of u_1 are determined. $\mathbf{Cent}_G(u_0)$ is complicated and the result is stated in **3.3** after introducing notation in **3.2**. On the other hand, $\mathbf{Cent}_G(u_1)$ is the semi-direct product of H and U^- (cf. **3.9**) and $O_G(u_1)$ is isomorphic to $X(J, J)$, the quasi-projective scheme associated to the Jordan pair (J, J) (cf. **3.10**).

In §4, a quartic form f and an alternating form $\{,\}$ on M, which are taken from [Freu, I], are introduced. They are stabilized by G. Following [Igusa 1], we also call f the *Freudenthal quartic*. Once f and $\{,\}$ are given, the fact that they are stabilized by G can be verified by calculation, which is simplified by the notion of scheme-theoretical density (cf. [EGA, I, 5.4]). After §4, G acts on the principal open subscheme $D_+(f)$ of $\mathbf{P}(M)$.

In §5, under some additional assumption on the quadruple $(J; N, \sharp, T)$ (cf. **5.4** $(**)$), the transitivity of the action of $G(K)$ on $D_+(f)(K)$ is proved for any k-algebra $k \to K$ with an algebraically closed field K. The proof is precisely the same as that in [Igusa 1, pp.427-428].

These are the contents of the main part §§1–5. Each section is ended with NOTE which mentions on some remarks and open questions. Also there are two appendices. In Appendix A, the relation between our construction in §1 and that of McCrimmon is discussed. In Appendix B, a class of examples of the quadruple $(J; N, \sharp, T)$ as in §1 is given.

Terminology

0.1 Throughout this article, k is an arbitrary commutative associative unitary ring. k-**alg** stands for the category of commutative associative unitary k-algebras. We denote by k-**alg**$^\wedge$ the category of *k-functors*, whose objects are set-valued covariant functors on k-**alg** and morphisms are natural transformations of functors. We follow the general conventions of [D-G]. In particular, the category of *k-schemes* is a full-subcategory of k-**alg**$^\wedge$. By a *k-sheaf*, we understand an fppf k-sheaf. $\mathbf{O}_k \in k$-**alg**$^\wedge$ stands for the affine line, i.e., the forgetful functor. For $X \in k$-**alg**$^\wedge$, we define $\mathcal{O}(X) \in k$-**alg** by $\mathcal{O}(X) := k$-**alg**$^\wedge(X, \mathbf{O}_k)$, the set of morphisms $X \to \mathbf{O}_k$ with natural k-algebra structure, and call an element of it a *function* on X. $\mu_k \in k$-**alg**$^\wedge$ stands for the functor $R \mapsto R^* := \{\text{invertible elements of } R\}$, which is an open subfunctor of $\mathbf{O}_k$. For any integer $n \geq 0$, we denote by ${}_n\mu_k$ the functor $R \mapsto \{t \in R^* \mid t^n = 1\}$. The *affine* k-schemes are, by definition, the representable functors on k-**alg**. If **Spec** $A \in k$-**alg**$^\wedge$ is the functor represented by $A \in k$-**alg**, we have, for any $X \in k$-**alg**$^\wedge$, a canonical morphism $\psi_X : X \to \mathbf{Spec}\ \mathcal{O}(X)$ and an equivalence X affine $\Leftrightarrow$ ψ_X invertible. For example, $\mathbf{O}_k$, μ_k, and ${}_n\mu_k$ are all affine, and we have $\mathcal{O}(\mathbf{O}_k) = k[T]$ (the polynomial ring in one variable), $\mathcal{O}(\mu_k) = k[T, T^{-1}]$ (the localization of $k[T]$ at T), and $\mathcal{O}({}_n\mu_k) = k[T]/(T^n - 1)$.

0.2 Following [LAG, 1.4], we use the notion of *dense* subfunctors. Namely, for $X \in k$-**alg**$^\wedge$ and a subfunctor $U \subset X$, U is said to be *dense* in X if the following property holds for any scalar extensions: any open subfunctor V of X has no closed subfunctor $Z \subset V$ containing $U \cap V$ other than V. The next lemma is cited from [LAG, 1.5]. It is based on [EGA IV, 11.10.10] and [SGA3, Exp. XVIII, Prop. 1.2].

Lemma (cf. [LAG, 1.5]): *Let X be a smooth separated algebraic k-scheme with non-empty connected fibers, and U an open subscheme of X. Then the following conditions are equivalent:*

(i) *U is dense in X.*

(ii) *There exist an fppf extension R of k such that $U(R) \neq \emptyset$.*

(iii) *$U(K) \neq \emptyset$ for any algebraically closed field $K \in k$-**alg**.*

0.3 Let M be a k-module. We define k-functors $M_a, \mathbf{P}(M)$ and M_m by setting

$$M_a(R) := M_R := M \otimes_k R,$$

$$\mathbf{P}(M)(R) := \{L \mid \text{direct factor of } M_R \text{ and invertible as an } R\text{-module}\},$$

$$M_m(R) := \{x \in M_R \mid \lambda(x) = 1, \exists \lambda \in {}^t(M_R)\},$$

for $R \in k$-**alg**, where ${}^t(M_R)$ stands for the R-module dual to M_R. Denote by $p_M : M_m \to \mathbf{P}(M)$ the morphism of k-functors sending $x \in M_m(R), R \in k$-**alg**, to $p_M(x) := R \cdot x$, the R-submodule of M_R spanned by x. We abbreviate $\mathcal{O}(M) := \mathcal{O}(M_a) \in k$-**alg** (cf. **0.1**, see also **0.4** below). If M is finitely generated and projective, then:

a) M_a is an affine algebraic k-scheme represented by the symmetric algebra $\mathbf{S}({}^tM)$ (cf. [D-G, II, §1, 2.1]). In particular, M_a is *smooth* (cf. [D-G, I, §1, 4.6]) with irreducible fibers. Also $\mathcal{O}(M)$ is canonically identified with $\mathbf{S}({}^tM)$ (cf. **0.1**). In the case where M is an *invertible* k-module L, the tensor algebra $\mathbf{T}({}^tL)$ coincides with $\mathbf{S}({}^tL)$, and we have an isomorphism

$$\bigoplus_{p\geq 0} {}^t(L^{\otimes p}) \xrightarrow{\sim} \mathcal{O}(L) \tag{1}$$

of k-algebras.

b) For any $m \in M = M_a(k)$, the corresponding morphism $m^\sharp$: Spec $k \to M_a$ is a *finitely presented* closed immersion. Indeed, since $x \mapsto x - m$ is an automorphism of M_a, it suffices to verify for the zero-section Spec $k \to M_a$. Then the corresponding k-algebra homomorphism is the augmentation $\mathbf{S}({}^tM) \to k$ whose kernel I is generated by ${}^tM$. However ${}^tM$ is finitely generated by our assumption. Hence so is I.

c) $\mathbf{P}(M)$ is a k-scheme represented by the geometric space Proj $\mathbf{S}({}^tM)$ (cf. [EGA, II, 4.2.3]), hence by Proj $\mathcal{O}(M)$ (cf. **a)**). We recall the identification of $\mathbf{P}(M)$ and Proj $\mathcal{O}(M)$ after [EGA, II, 4.2.1]. For any $L \in \mathbf{P}(M)(R), R \in k$-**alg**, the corresponding morphism r_L : Spec $R \to$ Proj $\mathcal{O}(M)$ is described as follows: Define a homomorphism of graded R-algebras

$$\psi_L : \mathcal{O}(M) \otimes_k R \longrightarrow \bigoplus_{p\geq 0} {}^t(L^{\otimes p})$$

by the composite of canonical maps $\mathcal{O}(M) \otimes_k R \xrightarrow{\sim} \mathcal{O}(M_R) \to \mathcal{O}(L) \xrightarrow{\sim} \bigoplus_{p\geq 0} {}^t(L^{\otimes p})$ (cf. (1)). Thus we have

$$\psi_L(f \otimes 1)(x \otimes \cdots \otimes x) = f(x), \tag{2}$$

for $f \in \mathcal{O}^p(M), p \geq 0$, and $x \in L$. Since the transpose ${}^tM_R \to {}^tL$ of the inclusion is *surjective*, the morphism $r_{\mathcal{L},\psi}$ associated to the invertible sheaf $\mathcal{L} := ({}^tL)^\sim$ and the homomorphism $\psi := (\psi_L)^\sim$ is everywhere defined on Spec R ([EGA, II, 3.7.1, 3.7.4]), which is r_L by definition.

d) M_m is an *open* subscheme of M_a which admits the structure of μ_k-*torsor* over $\mathbf{P}(M)$ (cf. [D-G, III, §4, 1.3]) with the action of μ_k by the scalar multiplications and with the structural morphism p_M. Note that, if $K \in k$-**alg** is a *field*, we have $M_m(K) = M_K - \{0\}$

and the map $M_m(K) \to \mathbf{P}(M)(K)$ sending x to $K \cdot x$ is *surjective.* Hence, we have a bijection

$$K^*\backslash(M_K - \{0\}) \xrightarrow{\sim} \mathbf{P}(M)(K). \tag{3}$$

0.4 Let M, N be k-modules. By a *polynomial law* on the couple (M, N), we understand a morphism of k-functors $M_a \to N_a$ (cf. [Roby, p.219]). Let $\mathbf{f} : M_a \to N_a$ be a polynomial law and p an integer ≥ 0. We say that $\mathbf{f}$ is *homogeneous of degree* p if $\mathbf{f}(tx) = t^p\mathbf{f}(x)$ for all $t \in R, x \in M_R, R \in k$-**alg**. Denote by $\mathcal{O}(M, N)$ (resp. by $\mathcal{O}^p(M, N)$) the k-module of the polynomial laws (resp. those which are homogeneous of degree p) on (M, N). For $N = k$, we write $\mathcal{O}(M) := \mathcal{O}(M, k)$ and $\mathcal{O}^p(M) := \mathcal{O}^p(M, k)$. Denote by N^M the k-module of the maps from the underlying set of M to that of N. We say that a map $Q \in N^M$ is *quadratic* if $Q(tx) = t^2Q(x)$ for $t \in k$ and $x \in M$, and if the map $M \times M \to N$ sending (x, y) to $Q(x + y) - Q(x) - Q(y)$ is bilinear. In this case, we write

$$Q(x, y) := Q(x + y) - Q(x) - Q(y).$$

By definition, we have the natural map from $\mathcal{O}^p(M, N)$ to N^M which is not injective in general. However, this is the case if $p \leq 2$. More precisely, $\mathcal{O}^0(M, N)$ (resp. $\mathcal{O}^1(M, N)$ $\mathcal{O}^2(M, N)$) is identified with the *constant* (resp. *linear, quadratic*) maps from M to N. We refer to [Roby, Prop. I.5, Cor. of Prop. I.6, Prop. II.1] and [Bou, Alg. IV, §5, Exercices] for details. For this reason, of $\mathcal{O}^2(M, N)$ is called also a *quadratic map.* Similarly, an element of $\mathcal{O}^2(M)$ is called a *quadratic form.* By a *cubic* (resp. *quartic, ...*) *form* on M, we understand an element of $\mathcal{O}^p(M)$ for $p = 3$ (resp. $p = 4, ...$).

0.5 Let M, N be k-modules and f a polynomial law on (M, N). (cf. **0.4**). For any $x, y \in M_R, R \in k$-**alg**, we set

$$f(x + \epsilon y) =: f(x) + \epsilon\partial_y f(x) \in N_{R[\epsilon]},$$

where $R[\epsilon]$ is the ring of *dual numbers*, to obtain a polynomial law $\partial_y f \in \mathcal{O}(M_R, N_R)$. This definition may be read as follows: the *tangent bundle* T_{M_a} of the k-functor M_a can be identified with $M_a \times M_a$ by means of $T_{M_a}(R) := M_{R[\epsilon]} \ni a + \epsilon b \mapsto (a, b) \in M_R \times M_R$. f is a morphism $M_a \to N_a$ (cf. **0.4**), from which the morphism $T_f : T_{M_a} \to T_{N_a}$ is induced. Then we have

$$T_f(x, y) = (f(x), \partial_y f(x))$$

for all $x, y \in M_R, R \in k$-**alg**.

0.6 Let M be a finitely generated projective k-module. For any $p \geq 0$ and $f \in \mathcal{O}^p(M)$, we define a subfunctor $\mathbf{D}_+(f)$ of $\mathbf{P}(M)$ by setting

$$\mathbf{D}_+(f)(R) := \{L \in \mathbf{P}(M)(R) \mid \psi_L(f \otimes 1)(u) = 1, \exists u \in L^{\otimes p}\},$$

for $R \in k$-**alg** (cf. **0.3 c)**). Then we have the following lemma, in which $D_+(f)$ stands for the open subspace of Proj $\mathcal{O}(M)$ whose points are the homogeneous prime ideals of $\mathcal{O}(M)$ not containing f, $(M_a)_f$ for the principal open subscheme of M_a defined by the section f, and p_M for the morphism $M_m \to \mathbf{P}(M)$ defined in **0.3**.

Lemma: a) $\mathbf{D}_+(f)$ *is an open subscheme of* $\mathbf{P}(M)$ *with geometric realization* $D_+(f)$; *in particular,* $\mathbf{D}_+(f)$ *is an affine smooth* k*-scheme.*

b) *We have* $p_M^{-1}(\mathbf{D}_+(f)) = M_m \cap (M_a)_f$.

Proof. The last assertion of a) follows from the former and from [EGA, II, 2.3.6, IV, 17.3.9]. To prove the rest, fix $R \in k$-**alg** arbitrarily.

a): For $L \in \mathbf{P}(M)(R)$, we have equivalences [r_L : Spec $R \to$ Proj $\mathcal{O}(M)$ factors through $D_+(f)$] (cf. **0.3 c)**) $\Leftrightarrow$ [$r_L^{-1}(D_+(f)) =$ Spec R] $\Leftrightarrow$ [(Spec $R)_s =$ Spec R for $s := \psi_L(f \otimes 1) \in \Gamma(\text{Spec } R, ({}^tL^{\otimes p})^\sim)$] (cf. [EGA, 3.7.3.1]) $\Leftrightarrow$ [the R-linear form $\psi_L(f \otimes 1) : L^{\otimes p} \to R$ is not zero on all fibers] $\Leftrightarrow$ [the R-linear form $\psi_L(f \otimes 1) : L^{\otimes p} \to R$ is surjective] (cf. [Bou, Alg.Commm. II, §3, no.3, Th.1]) $\Leftrightarrow$ [$L \in \mathbf{D}_+(f)(R)$], from which a) follows.

b): The assertion amounts to saying that $\{x \in M_m(R) \mid R \cdot x \in \mathbf{D}_+(f)(R)\} = \{x \in M_m(R) \mid f(x) \in R^*\}$ (cf. **0.3**). Let $x \in M_m(R)$. If $f(x) \in R^*$, then we have $\psi_{R \cdot x}(f \otimes 1)(u) = 1$ for $u := f(x)^{-1}x \otimes \cdots \otimes x \in (R \cdot x)^{\otimes p}$ (cf. **0.3** (2)), and hence $R \cdot x \in \mathbf{D}_+(f)(R)$. Conversely if $R \cdot x \in \mathbf{D}_+(f)(R)$, then we have $\psi_{R \cdot x}(f \otimes 1)(u) = 1$ for some $u \in (R \cdot x)^{\otimes p}$, which must be of the form $\lambda x \otimes \cdots \otimes x$ with $\lambda \in R$, and hence $1 = \psi_{R.x}(f \otimes 1)(u) = \lambda f(x)$ (cf. **0.3** (2)), i.e., $f(x) \in R^*$. This completes the proof.

In particular, for a *field* $K \in k$-**alg**, the map $x \mapsto K \cdot x$ induces a bijection

$$\{x \in M_K | f(x) \in K^*\}/K^* \xrightarrow{\sim} \mathbf{D}_+(f)(K), \tag{1}$$

as is seen from part **b)** of the lemma and **0.3** (3).

Finally, we look at the relation to group actions. Let G be a k-group functor acting on M by a linear representation $\theta : G \to \mathbf{GL}(M)$. Then we have the *right* action $\tilde{\theta}$ of G on $\mathcal{O}(M)$ such that

$$(\tilde{\theta}(g) \cdot \varphi)(m) := \varphi(\theta(g) \cdot m), \tag{2}$$

for $g \in G(R), \varphi \in \mathcal{O}(M) \otimes_k R \simeq \mathcal{O}(M_R), R \in k$-**alg**, and $m \in M_S, S \in R$-**alg** (cf. [D-G, II, §2, 1.2 c)]). On the other hand, G acts on $\mathbf{P}(M)$ by setting $g \cdot L :=$

the image of L under the automorphism $\theta(g) \in GL(M_R)$. Note that the diagram

$$\begin{tikzcd}
\mathcal{O}(M) \otimes_k R \arrow[r, "\psi_L"] & \bigoplus_{p\geq 0} {}^t(L^{\otimes p}) \\
\mathcal{O}(M) \otimes_k R \arrow[u, "\tilde{\theta}(g)"] \arrow[r, "\psi_{g\cdot L}"] & \bigoplus_{p\geq 0} {}^t((g\cdot L)^{\otimes p}) \arrow[u, "\oplus_{p\geq 0} {}^t(\theta(g)^{\otimes p})"']
\end{tikzcd}$$

is *commutative.* Indeed, it suffices to show the formula

$$\psi_L(\tilde{\theta}(g)\cdot\varphi)(u) = \psi_{g\cdot L}(\varphi)(\theta(g)^{\otimes p}\cdot u), \tag{3}$$

for all $p \geq 0, \varphi \in \mathcal{O}^p(M) \otimes_k R$ and $u \in L^{\otimes p}$, which may be verified after any fppf extension of R. Hence we may assume u to be of the form $\lambda x \otimes \cdots \otimes x$ with $\lambda \in R, x \in L$. Then the assertion becomes a consequence of (2) and **0.3** (2). Note also that, if $f \in \mathcal{O}^p(M), p \geq 0$, and if G *stabilizes* f, then the subscheme $\mathbf{D}_+(f)$ of $\mathbf{P}(M)$ is *stable* under G. This follows from the definition and (3).

1 Basic Jordan identities

1.1 Consider a quadruple $(J; N, \sharp, T)$, where J is a k-module, N a cubic form (cf. **0.4**) on J, $?^\sharp : x \mapsto x^\sharp$ a quadratic map (cf. **0.4**) from J to J, and T a symmetric k-bilinear form on J, satisfying the following two identities (CJ1) and (CJ2): for all $x, y \in J_R, R \in k$-**alg** (cf. **0.3**), we have

(CJ1) $$x^{\sharp\sharp} = N(x)x,$$

(CJ2) $$\partial_y N(x) = T(x^\sharp, y).$$

The identities (CJ1) and (CJ2) should be read as commutative diagrams

(CJ1bis)
$$\begin{tikzcd} J_a \arrow[rr, "\sharp"] & & J_a \\ J_a \arrow[u, "\sharp"] \arrow[rr, "(N,\mathrm{Id})"] & & \mathbf{O}_k \times J_a \arrow[u, "\varphi"'] \end{tikzcd}$$

and

(CJ2bis)
$$\begin{tikzcd} T_{J_a} \arrow[rr, "T_N"] & & T_{\mathbf{O}_k} \\ J_a \times J_a \arrow[u, "\mathrm{can.}\wr"] \arrow[rr, "(N\circ \mathrm{pr}_1, \psi)"] & & \mathbf{O}_k \times \mathbf{O}_k \arrow[u, "\wr\mathrm{can.}"'] \end{tikzcd}$$

of *k-functors*, where $\varphi : \mathbf{O}_k \times J_a \to J_a$ is the scalar multiplication and $\psi : J_a \times J_a \to \mathbf{O}_k$ is the morphism sending (x, y) to $T(x^\sharp, y)$ (cf. **0.5**).

In the following, we fix such a quadruple $(J; N, \sharp, T)$ and set

(1) $$x \times y := (x+y)^\sharp - x^\sharp - y^\sharp,$$

(2) $$Q(x)y := T(x, y)x - x^\sharp \times y,$$

(3) $$N(x, y) := 1 - T(x, y) + T(x^\sharp, y^\sharp) - N(x)N(y),$$

(4) $$P(x, y) := x - x^\sharp \times y + N(x)y^\sharp,$$

for all $x, y \in J_R$, $R \in k$-**alg**, to obtain a bilinear product $\times$ in J and polynomial laws $Q \in \mathcal{O}^2(J, \mathrm{End}(J))$, $N(\,,\,) \in \mathcal{O}(J \times J)$, $P(\,,\,) \in \mathcal{O}(J \times J, J)$ (cf. **0.4**). By definition, $(x, y, z) \mapsto Q(x, z)y$ (cf. **0.4**) is a trilinear product in J. Denote any scalar extension of it by $\{\quad\}$ and let $D(x, y)z := \{xyz\}$. Hence we have

(5) $$D(x, y)z := \{xyz\} := Q(x, z)y = T(x, y)z + T(y, z)x - (z \times x) \times y,$$

for all $x, y, z \in J_R$. Finally we set

$$B(x, y)z := z - \{xyz\} + Q(x)Q(y)z, \tag{6}$$

to obtain a polynomial law $B \in \mathcal{O}(J \times J, \mathrm{End}(J))$.

1.2 Let $R \in k$-**alg**, and $x, y, z, u, v \in J_R$. Since N is a cubic form, there exists an R-linear form $\tilde{N}$ on the degree 3 component $\Gamma_3(J_R)$ of the Γ-algebra $\Gamma(J_R)$ of J_R such that $N(x) = \langle \gamma_3(x), \tilde{N} \rangle$ (cf. [Bou, IV, §5, exerc.10)]). $\gamma_p : J_R \to \Gamma(J_R)$ $(p \geq 0)$ satisfy

$$\gamma_3(x + y) = \gamma_3(x) + \gamma_2(x)\gamma_1(y) + \gamma_1(x)\gamma_2(y) + \gamma_3(y),$$

$$\gamma_2(x + y)\gamma_1(z) = \gamma_2(x)\gamma_1(z) + \gamma_2(y)\gamma_1(z) + \gamma_1(x)\gamma_1(y)\gamma_1(z),$$

$$\gamma_2(x)\gamma_1(x) = 3\gamma_3(x),$$

(cf. [Bou, IV, §5, exerc.2)]), and $T(u^\sharp, v) = \partial_v N(u) = \langle \gamma_2(u)\gamma_1(v), \tilde{N} \rangle$ by (CJ2). Hence, applying $\langle ?, \tilde{N} \rangle$ to the above identities, we get

$$N(x + y) = N(x) + T(x^\sharp, y) + T(x, y^\sharp) + N(y), \tag{CJ3}$$

$$T(x \times y, z) = N(x, y, z) = T(x, y \times z), \tag{CJ4}$$

$$T(x^\sharp, x) = 3N(x), \tag{CJ5}$$

where $N(x, y, z) := \langle \gamma_1(x)\gamma_1(y)\gamma_1(z), \tilde{N} \rangle = N(x + y + z) - N(x + y) - N(y + z) - N(z + x) + N(x) + N(y) + N(z)$. Since $N(x, y, z)$ is symmetric, the latter equality of (CJ4) follows from the former. Next, taking the scalar extension $R \to R[t]$ to the polynomial ring in one variable t, replacing x by $x + ty$ in (CJ1), expanding the result by using **1.1**(1) and (CJ3), and comparing the terms in t, t^2, we get

$$x^\sharp \times (x \times y) = N(x)y + T(x^\sharp, y)x, \tag{CJ6}$$

$$x^\sharp \times y^\sharp + (x \times y)^\sharp = T(x^\sharp, y)y + T(x, y^\sharp)x. \tag{CJ7}$$

Linearization of (CJ7) with respect to y yields

$$x^\sharp \times (y \times z) + (x \times y) \times (x \times z) = T(x^\sharp, y)z + T(x^\sharp, z)y + T(x, y \times z)x. \tag{CJ8}$$

Applying $T(?, z)$ to (CJ6) with (CJ4) in mind, we get

$$N(x^\sharp, x \times y, z) = N(x)T(y, z) + T(x^\sharp, y)T(x, z). \tag{CJ9}$$

If we interchange y and z in (CJ9), and calculate the left-hand side using (CJ4) and the symmetry of $N(\ ,\ ,\ )$, then the result is

$$N(x, x^\sharp \times y, z) = N(x)T(y, z) + T(x^\sharp, z)T(x, y). \tag{CJ10}$$

Applying $T(x^\sharp, ?)$ to (CJ7) with (CJ4), (CJ5) and (CJ1) in mind, we get

$$T(x^\sharp, (x \times y)^\sharp) = T(x^\sharp, y)^2 + N(x)T(x, y^\sharp). \tag{CJ11}$$

On the other hand, we have

$$T(Q(x)u, v) = T(x, u)T(x, v) - T(x^\sharp, u \times v), \tag{1}$$

by **1.1**(2) and (CJ4), which gives $T(Q(x)u, v) = T(Q(x)v, u)$, since the right-hand side of (1) is symmetric in u and v. However, T is also symmetric by assumption. This gives

$$T(Q(x)u, v) = T(u, Q(x)v). \tag{2}$$

Similarly, we have $T(D(x, y)u, v) = T(x, y)T(u, v) + T(y, u)T(x, v) - T(u \times x, v \times y)$ by **1.1**(5) and (CJ4), which gives

$$T(D(x, y)u, v) = T(u, D(y, x)v). \tag{3}$$

Finally, using (2), (3) and **1.1** (6), we get

$$T(B(x, y)u, v) = T(u, B(y, x)v). \tag{4}$$

1.3 Consider the following condition on a quadruple $(J; N, \sharp, T)$ in **1.1**:

$(*)$ J is a finitely generated projective k-module and there exist $c_1, c_2 \in J$ and a linear form λ on J such that $N(c_1) \in k^*, \lambda(c_2^\sharp) = 1$.

Assume that $(*)$ is satisfied. Then we have:

a) The k-functor $J_a : R \mapsto J_R$ becomes a smooth separated algebraic k-scheme with non-empty connected fibers.

b) The inverse image of $J_m \subset J_a$ under the morphism $?^\sharp : J_a \to J_a$ and the principal open subscheme defined by the section $N \in \mathcal{O}(J)$ are both *dense* in J_a.

c) The morphism $?^\sharp : J_a \to J_a$ is *scheme-theoretically dominant* (cf. [EGA I, 5.4.2]).

Indeed, **a)** (resp. **b)**) follows from **0.3** (resp. **0.2**), and **c)** amounts to saying that the corresponding ring homomorphism $\mathcal{O}(J) \to \mathcal{O}(J)$, say φ, is *injective* (cf. [EGA I, 5.4.1]).

This can be verified as follows: since $?^\sharp$ is quadratic, we have $\varphi(\mathcal{O}^p(J)) \subset \mathcal{O}^{2p}(J)$ for all $p \geq 0$. This implies that $\ker\varphi$ is a *homogeneous* ideal. Thus it suffices to show that any *homogeneous* element f of $\ker\varphi$ is zero. Indeed, choosing $p \geq 0$ so that $f \in \mathcal{O}^p(J)$, we have $0 = \varphi(f)(x^\sharp) = f(x^{\sharp\sharp}) = f(N(x)x)$ (by (CJ1)) $= N(x)^p f(x)$. Hence $f : J_a \to \mathbf{O}_k$ vanishes on the principal open subscheme of J_a defined by the section $N \in \mathcal{O}(J)$. This implies $f = 0$, in view of **b)**.

Using **b)**, we get

$$N(x^\sharp) = N(x)^2, \tag{CJ12}$$

$$x \times (x^\sharp \times y) = N(x)y + T(x,y)x^\sharp, \tag{CJ13}$$

for all $x, y \in J_R, R \in k$-**alg**. Indeed, we have $N(x^\sharp)x^\sharp = x^{\sharp\sharp\sharp} = N(x)^2 x^\sharp$ by (CJ1). Hence the morphisms $N \circ \sharp : J_a \to \mathbf{O}_k$ and $N^2 : J_a \to \mathbf{O}_k$ coincide on the inverse image of $J_m \subset J_a$ under the morphism $?^\sharp : J_a \to J_a$. In view of **b)**, this implies $N \circ \sharp = N^2$, namely (CJ12). As for (CJ13), we fix y and consider the morphisms $f : J_a \otimes_k R \to J_a \otimes_k R$ sending x to $x \times (x^\sharp \times y)$ and $g : J_a \otimes_k R \to J_a \otimes_k R$ sending x to $N(x)y + T(x,y)x^\sharp$. Replacing x by $x^\sharp$ in (CJ6) and using (CJ1) and (CJ12), we get $N(x)x \times (x^\sharp \times y) = N(x)^2 y + N(x)T(x,y)x^\sharp$, namely $N(x)f(x) = N(x)g(x)$. Hence the morphisms f and g coincide on the $\otimes_k R$ of the principal open subscheme of J_a defined by the section $N \in \mathcal{O}(J)$. In view of **b)**, this implies $f = g$, namely (CJ13).

1.4 Theorem (a modification of McCrimmon [Mc, Th. 1]): *Under the assumption* $(*)$*, the data* $(V^\pm, Q_\pm)$ *with* $V^+ = V^- := J, Q_+ = Q_- := Q$ *is a Jordan pair over* k*, which has an invertible element.*

1.5 The proof of the theorem requires long calculations. Here we indicate its outlines with some additional identities for later use. We first recall that (cf. [LJP, 1.2]) a *Jordan pair* over k is a pair of k-modules (V^+, V^-) together with a pair (Q_+, Q_-) of quadratic maps $Q_\sigma : V^\sigma \to \mathrm{Hom}(V^{-\sigma}, V^\sigma)$, $\sigma = \pm$, satisfying

$$D_\sigma(x,y)Q_\sigma(x) = Q_\sigma(x)D_{-\sigma}(y,x), \tag{JP1}$$

$$D_\sigma(Q_\sigma(x)y, y) = D_\sigma(x, Q_{-\sigma}(y)x), \tag{JP2}$$

$$Q_\sigma(Q_\sigma(x)y) = Q_\sigma(x)Q_{-\sigma}(y)Q_\sigma(x), \tag{JP3}$$

for all $\sigma = \pm, x \in V_R^\sigma, y \in V_R^{-\sigma}, R \in k$-**alg**. Here we set $D_\sigma(x,y)z := Q_\sigma(x,z)y$. An element x in V^σ is said to be *invertible* if the linear map $Q_\sigma(x) : V^{-\sigma} \to V^\sigma$ is invertible (cf. [LJP, 1.10]). Returning to the situation in **1.1** and assume the condition $(*)$ in **1.3**. Let $R \in k$-**alg**, and $x, y, z \in J_R$. By direct calculations using **1.2**, we have $Q(x)Q(x^\sharp) = N(x)^2\mathrm{Id}$ and $Q(x^\sharp)Q(x) = N(x^\sharp)\mathrm{Id}$ which become

$$\text{(CJ14)} \qquad Q(x)Q(x^\sharp) = Q(x^\sharp)Q(x) = N(x)^2\mathrm{Id}$$

by (CJ12). Hence $Q(c_1) \in \mathrm{End}(J)$ is invertible for c_1 in **1.3** $(*)$. Thus it remains to check the defining identities of Jordan pairs. Start with taking the scalar extension $R \to R[t]$ to the polynomial ring in one variable t, replace x by $x + tz$ in (CJ13), expand the result by using **1.1** (1), (CJ3), and compare the terms in t. Then we get

$$\text{(CJ15)} \qquad x \times ((x \times z) \times y) + z \times (x^\sharp \times y) = T(x^\sharp, z)y + T(x,y)x \times z + T(y,z)x^\sharp.$$

Moreover, by direct calculations using **1.1** (1), (2) and (CJ13, 1, 7), we have

$$\text{(CJ16)} \qquad (Q(x)y)^\sharp = Q(x^\sharp)y^\sharp.$$

We can now verify (JP1), (JP2) and (JP3) by straightforward calculations using (CJ6, 3), (CJ15, 8, 4) and (CJ16, 6, 15, 10), respectively. Thus the proof of theorem is complete.

Let us introduce some more identities. Add $2N(x)y$ to (CJ6) (resp. (CJ13)), use (CJ5), and subtract $x^\sharp \times (x \times y)$, (resp. $x \times (x^\sharp \times y)$). Then we get

$$2N(x)y = T(x^\sharp, x)y + T(x^\sharp, y)x - x^\sharp \times (x \times y)$$

$$(\text{resp. } 2N(x)y = T(x^\sharp, x)y + T(x,y)x^\sharp - x \times (x^\sharp \times y)),$$

which in operator forms become

$$\text{(CJ17)} \qquad D(x, x^\sharp) = D(x^\sharp, x) = 2N(x)\mathrm{Id},$$

whose linearization yields

$$\text{(CJ18)} \qquad D(x, x \times y) + D(y, x^\sharp) = D(x \times y, x) + D(x^\sharp, y) = 2T(x^\sharp, y)\mathrm{Id}.$$

1.6 From now on, we apply the notion of Jordan pair (cf, [LJP]) to $(V^\pm, Q_\pm)$ with $V^+ = V^- := J, Q_+ = Q_- := Q$. Recall that an element x of J is said to be *invertible* if $Q(x) \in \mathrm{End}(J)$ is invertible (cf. [LJP, 1.10]). In this case, $x^{-1} := Q(x)^{-1}x$ is the *inverse* of x (cf. [LJP, 1.10]). If $N(x) \in k^*$, then x is invertible by (CJ14), and we have $x^{-1} = N(x)^{-2}Q(x^\sharp)x = N(x)^{-1}x^\sharp$ by **1.1** (1), (2), and (CJ1, 5).

Proposition: *An element x of J is invertible if and only if the scalar $N(x)$ is invertible; if that is the case, we have*

$$x^{-1} = N(x)^{-1} x^\sharp, \tag{1}$$

and, for any $y \in J$,

$$N(x, y) = N(x) N(x^{-1} - y). \tag{2}$$

Indeed, we have $N(x^\sharp - N(x)y) = N(x)^2 N(x, y)$ by (CJ3, 12, 1). Hence (2) follows from (1). It remains to prove the implication: x invertible $\Rightarrow N(x) \in k^*$; this is a consequence of (CJ20) in the following lemma, since there exist $c_1, y \in J$ such that $Q(x)y = c_1, N(c_1) \in k^*$ (cf. **1.3**$(*)$).

1.7 Lemma: *For all $R \in k$-**alg**, and $x, y, z \in J_R$, we have*

$$N(x \times y) = T(x^\sharp, y) T(x, y^\sharp) - N(x) N(y), \tag{CJ19}$$

$$N(Q(x)y) = N(x)^2 N(y), \tag{CJ20}$$

$$N(B(x, y)z) = N(x, y)^2 N(z). \tag{CJ21}$$

Proof. Taking the scalar extension $R \to R[t]$ to the polynomial ring in one variable t, replacing x by $x + ty$ in (CJ12), expanding the result, and comparing the terms in t^3, we get $N(x \times y) + N(x^\sharp, x \times y, y^\sharp) = 2N(x)N(y) + T(x^\sharp, y)T(x, y^\sharp)$, which becomes (CJ19) by (CJ9) and (CJ5). (CJ20) follows from the expansion of the left-hand side using **1.1** (2), (CJ3), and (CJ19). For (CJ21) we may assume $N(x)$ to be invertible, since the principal open subscheme defined by the section $(x, y) \mapsto N(x)$ is dense in $J_a \times J_a$ (cf. **0.2**). Then x is invertible by the remark at the beginning of **1.6** (which is independent of the proposition) and we have $B(x, y) = Q(x)Q(x^{-1} - y)$ by [LJP, 2.12]. Thus the assertion follows from (CJ20) and **1.6** (2).

1.8 Recall that a pair (x, y) of elements of J is said to be *quasi-invertible* if $B(x, y) \in \mathrm{End}(J)$ is invertible (cf. [LJP, 3.2]). In this case, $x^y := B(x, y)^{-1}(x - Q(x)y)$ is the *quasi-inverse* of (x, y) (cf. [LJP, 3.2]).

Corollary: *A pair (x, y) of elements of J is quasi-invertible if and only if the scalar $N(x, y)$ is invertible; if that is the case, we have*

(CJ22)
$$x^y = N(x, y)^{-1}P(x, y),$$

(CJ23)
$$(x^y)^\sharp = N(x, y)^{-1}(x^\sharp - N(x)y),$$

(CJ24)
$$N(x^y) = N(x, y)^{-1}N(x),$$

and, for any $z, w \in J$,

(CJ25)
$$(B(x, y)z)^\sharp = N(x, y)^2B(y, x)^{-1}z^\sharp,$$

(CJ26)
$$(B(x, y)z) \times (B(x, y)w) = N(x, y)^2B(y, x)^{-1}(z \times w).$$

The quasi-invertibility of (x, y) implies the invertibility of $N(x, y)$ by (CJ21), since there exist $c_1, z \in J$ such that $B(x, y)z = c_1, N(c_1) \in k^*$. Conversely if $N(x, y)$ is invertible, then we have $B(x, y)z = x - Q(x)y$ and $B(x, y)Q(z)y = Q(x)y$ for $z := N(x, y)^{-1}P(x, y)$ by the following **1.9** (3), (4). This implies the quasi-invertibility of (x, y) together with (CJ22) by [LJP, 3.2 (ii)]. We have

(CJ23bis)
$$P(x, y)^\sharp = N(x, y)(x^\sharp - N(x)y),$$

by direct calculation using (CJ1, 6, 7, 13). Hence (CJ23) follows from (CJ22). Since (CJ26) is a linearization of (CJ25), it remains to verify (CJ24) and (CJ25). We may assume x to be invertible, since the principal open subscheme defined by the section $(x, y) \mapsto N(x)$ is dense in $J_a \times J_a$ (cf. **0.2**). Then, we have $x^y = (x^{-1} - y)^{-1}, B(x, y) = Q(x)Q(x^{-1} - y)$, and $B(y, x) = Q(x^{-1} - y)Q(x)$ (cf. [LJP, 2.12, 3.13]). Hence (CJ24) follows from **1.6** (2), and (CJ25) can be verified as follows:

$$\begin{aligned}
(B(x, y)z)^\sharp &= (Q(x)Q(x^{-1} - y)z)^\sharp \\
&= Q(x^\sharp)Q((x^{-1} - y)^\sharp)z^\sharp \quad \text{(by (CJ16))} \\
&= N(x)^2N(x^{-1} - y)^2Q(x)^{-1}Q(x^{-1} - y)^{-1}z^\sharp \quad \text{(by (CJ14))} \\
&= N(x, y)^2B(y, x)^{-1}z^\sharp \quad \text{(by } \mathbf{1.6} \text{ (2))}.
\end{aligned}$$

1.9 Lemma: *For any $x, y, z \in J_R, R \in k$-**alg**, we have*

(1)
$$B(x, y)y^\sharp = y^\sharp - N(y)x - N(y)(x - Q(x)y),$$

(2) $$B(x,y)(z \times y) = z \times P(y,x) + T(z,x)(y^\sharp - N(y)x) - T(z,y^\sharp)(x - Q(x)y),$$

(3) $$B(x,y)P(x,y) = x - Q(x)y,$$

(4) $$B(x,y)Q(P(x,y)) = N(x,y)^2 Q(x)y,$$

(5) $$B(x,y)(z - (z \times x) \times y + T(z,x^\sharp)y^\sharp) = N(x,y)z - T(z,P(y,x))(x - Q(x)y).$$

This lemma was used in the proof of **1.8** (also will be used in **2.7**). All the formulas can be proved independently of **1.8** by direct calculation.

1.10 Lemma: *For any* $x, y, z \in J_R,\ t \in R,\ R \in k$-**alg** *such that* (x,y) *is quasi-invertible, we have*

(1) $$N(tx,z) = N(x,tz),$$

(2) $$P(tx,z) = P(x,tz),$$

(3) $$N(x,y)N(x^y,z) = N(x,y+z),$$

(4) $$N(x,y)P(x^y,z) = P(x,y+z).$$

Proof. (1), (2): Direct consequences of the definitions **1.1** (3), (4).

(3): We may assume x to be invertible, since the principal open subscheme defined by the section $(x,y,z) \mapsto N(x)$ is dense in $J_a \times J_a \times J_a$. Then, by **1.6** (2) and (CJ24), we have $N(x^y,z) = N(x^y)N((x^y)^{-1} - z) = N(x,y)^{-1}N(x)N((x^{-1} - y) - z) = N(x,y)^{-1}N(x)N(x^{-1} - (y+z)) = N(x,y)^{-1}N(x,y+z)$.

(4): We may assume $(x, y+z)$ to be quasi-invertible, since the principal open subscheme defined by the section $(x,y,z) \mapsto N(x,y+z)$ is dense in $J_a \times J_a \times J_a$. Then, by (CJ22), (3), and [LJP, 3.7 (a)], we have $P(x,y+z) = N(x,y+z)x^{y+z} =$
$N(x,y)N(x^y,z)(x^y)^z = N(x,y)P(x^y,z)$.

NOTE

Using the density arguments, we may replace (CJ25) and (CJ26) by $B(y,x)\cdot(B(x,y)z)^\sharp = N(x,y)^2 z^\sharp$ and $B(y,x)\cdot(B(x,y)z)\times(B(x,y)w) = N(x,y)^2 z\times w$, respectively, which are valid for arbitrary (x,y). Our quadruple $(J;N,\sharp,T)$ is modeled on the McCrimmon's axiomatic construction of the Jordan algebra $H_3(\mathcal{C})$ of 3×3 Hermitian matrices with coefficients in a composition algebra $\mathcal{C}$ (cf. [Mc]). We refer to Appendices A and B for details.

2 Representation

2.0 In this section, let $(J; N, \sharp, T)$ be a quadruple as in **1.1** satisfying the condition $(*)$ in **1.3**, and we use the following notation:

(J, J): the associated Jordan pair (cf. **1.4**), i.e., the Jordan pair $V = (V^{\pm}, Q_{\pm})$ with $V^+ = V^- := J$, $Q_+ = Q_- := Q$, $Q(x)y := T(x,y)x - x^\sharp \times y$ (cf. **1.1** (2)).

$\mathcal{W}$: the scheme of quasi-invertible pairs in (J, J). This is precisely the principal open subscheme of $J_a \times J_a$ defined by the section $(x, y) \mapsto N(x, y)$ (cf. **1.8**), which is dense in $J_a \times J_a$.

Recall that the *automorphism group* $\mathrm{Aut}(V)$ of a Jordan pair $V = (V^{\pm}, Q_{\pm})$ is the group of all $(h_+, h_-) \in GL(V^+) \times GL(V^-)$ such that $h_\sigma Q_\sigma(x) = Q(h_\sigma(x))h_{-\sigma}$ for $\sigma = \pm, x \in V_R^\sigma, R \in k$-**alg** (cf. [LJP, 1.3]). The k-group functor $R \mapsto \mathrm{Aut}(V_R)$ is denoted by $\mathbf{Aut}(V)$, which is an affine algebraic k-group scheme (cf. [LAG, 2.3]).

2.1 Consider the k-group scheme $\mu_k \times \mathbf{GL}(J)^2$ and denote any R-valued point h of it in the form

$$h =: (\chi(h), h_+, h_-), \tag{1}$$

where $\chi(h) \in R^*$ and $h_+, h_- \in GL(J_R)$. Denote by H the subgroup scheme of $\mu_k \times \mathbf{GL}(J)^2$ whose R-valued points is the group of h's satisfying

$$T(h_+x, h_-y) = T(x, y), \tag{H1}$$

$$(h_+x)^\sharp = \chi(h)^{-1}h_-x^\sharp, (h_-x)^\sharp = \chi(h)h_+x^\sharp, \tag{H2}$$

$$N(h_+^{-1}x) = N(h_-x) = \chi(h)N(x), \tag{H3}$$

for all $x, y \in J_S, S \in R$-**alg**. Note that we have

$$\begin{cases} (h_+x)^\sharp = \chi(h)^{-1}h_-x^\sharp, & (h_+x) \times (h_+y) = \chi(h)^{-1}h_-(x \times y), \\ (h_-x)^\sharp = \chi(h)h_+x^\sharp, & (h_-x) \times (h_-y) = \chi(h)h_+(x \times y), \end{cases} \tag{H2bis}$$

which is the linearization of (H2). Note also that the inclusion $H \to \mu_k \times \mathbf{GL}(J)^2$ is a *finitely presented closed immersion*. Indeed, our definition amounts to saying that the diagram

$$\begin{array}{ccc} H & \xrightarrow{\text{incl.}} & \mu_k \times \mathbf{GL}(J)^2 \\ \downarrow & & \downarrow \mathbf{d} \\ \mathrm{Spec}\, k & \xrightarrow{\mathbf{s}} & E_a \end{array}$$

is *Cartesian*, where $E := {}^t(J \otimes J) \times \mathcal{O}^2(J,J)^2 \times \mathcal{O}^3(J)^2$, $\mathbf{s} :=$ the section corresponding to $(T, \sharp, \sharp, N, N) \in E = E_a(k)$, and $\mathbf{d}(\lambda, h_+, h_-) := (T \circ (h_+ \otimes h_-),\ \lambda h_-^{-1} \circ \sharp \circ h_+,\ \lambda^{-1} h_+^{-1} \circ \sharp \circ h_-,\ \lambda^{-1} N \circ h_+^{-1},\ \lambda^{-1} N \circ h_-)$ for $\lambda \in R^*, h_+, h_- \in GL(J_R), R \in k$-**alg**. However the section $\mathbf{s}$ is a finitely presented closed immersion, since E is a finitely generated projective k-module (cf. **0.3 b)**).

In particular, H is an *affine algebraic* k-group scheme. If $h \in H(R)$, then

(2) $$h^\vee := (\chi(h)^{-1}, h_-, h_+)$$

also belongs to $H(R)$ and $h \mapsto h^\vee$ becomes an automorphism of H of period two. If $t \in R^*$, then

(3) $$\mathbf{z}(t) := (t^{-3},\ t\mathrm{Id},\ t^{-1}\mathrm{Id})$$

belongs to $H(R)$ and, varying R, we get an inclusion $\mathbf{z} : \mu_k \to H$, which factors through the center of H. Define $-1 \in H(k)$ to be $\mathbf{z}(-1)$ and set

(4) $$-h := \mathbf{z}(-1)h \in H(R)$$

for $h \in H(R), R \in k$-**alg**. Then $h \mapsto -h$ becomes an automorphism of H of period two. Since we have $\chi(\mathbf{z}(t)) = t^{-3}$ by (3), the character $\chi : H \to \mu_k$ is an epimorphism of k-sheaves. By **1.2** (2), (CJ4), (CJ16), and **1.6** (1),

(5) $$b(x) := (N(x),\ N(x)^{-1}Q(x),\ N(x)Q(x^{-1}))$$

belongs to $H(R)$ for invertible $x \in J_R$.

2.2 We see from (H2bis) and **2.0** that $\mathrm{pr}_2 : H \to \mathbf{GL}(J)^2$ factors through $\mathbf{Aut}(J,J)$. Since there exists $x \in J$ such that $N(x) \in k^*$ (cf. **1.3** $(*)$), we see from (H3) that the morphism $H \to \mathbf{Aut}(J,J)$ sending h to (h_+, h_-) is a monomorphism. If $h \in H(R)$ and $(x,y) \in \mathcal{W}(R)$, then

(1) $$\rho(h) = (\rho_+(h), \rho_-(h)) := (\chi(h)h_+,\ \chi(h)^{-1}h_-)$$

as well as (h_+, h_-) belongs to $\mathbf{Aut}(J,J)(R)$, while

(2) $$b(x,y) := (N(x,y),\ N(x,y)^{-1}B(x,y),\ N(x,y)B(y,x)^{-1})$$

belongs to $H(R)$ by **1.2** (4), (CJ25), and (CJ21). Varying R, we get a homomorphism $\rho : H \to \mathbf{Aut}(J,J)$ of k-groups and a morphism $b : \mathcal{W} \to H$ of k-schemes. Note that we have

(3) $$\rho_+(h^\vee) = \rho_-(h), \qquad \rho_-(h^\vee) = \rho_+(h),$$

$$(4) \qquad b(x,y)^\vee = b(y,x)^{-1},$$

by definition and **2.1** (2).

2.3 Lemma: *$((J,J),H,\rho,b)$ is a Jordan system and the kernel of $\rho : H \to \mathbf{Aut}(J,J)$ is the functor-image of ${}_2\mu_k \subset \mu_k$ under $\mathbf{z} : \mu_k \to H$.*

Proof. We first recall that (cf. [LAG, 5.1]) a *Jordan system* over k is a quadruple (V,H,ρ,b) where 1) $V = (V^\pm, Q_\pm)$ is a Jordan pair with $V^\pm$ finitely generated projective k-modules, 2) H is a separated k-group sheaf, 3) $\rho = (\rho_+,\rho_-)$ is a homomorphism $H \to \mathbf{Aut}(V)$ of k-groups, 4) b is a morphism $\mathcal{W} \to H$, with $\mathcal{W}$ the scheme of quasi-invertible pairs of V, satisfying

$$(\text{JS1}) \qquad \rho(b(x,y)) = (B_+(x,y), B_-(y,x)^{-1}),$$

$$(\text{JS2}) \qquad hb(x,y)h^{-1} = b(\rho_+(h)x, \rho_-(h)y),$$

$$(\text{JS3}) \qquad b(tx, t^{-1}y) = b(x,y),$$

$$(\text{JS4}) \qquad b(x,y)b(x^y,w) = b(x,y+w),$$

$$(\text{JS5}) \qquad b(z,y^x)b(x,y) = b(z+x,y),$$

for all $R \in k$-**alg**, $t \in R^*$, $h \in H(R)$, and $x,z \in V_R^+$, $y,w \in V_R^-$ such that (x,y), $(x,y+w)$, $(x+z,y) \in \mathcal{W}(R)$. Here we set, in the notation of **1.5**, $B_\sigma(u,v) := \mathrm{Id} - D_\sigma(u,v) + Q_\sigma(u)Q_{-\sigma}(v)$. In our situation, (JS1) follows from the definitions (1) and (2) in **2.2**, and (JS2–5) from [LJP, 3.9] and **1.10**. As for the last assertion, we have $\rho(\mathbf{z}(t)) = (t^{-2}\mathrm{Id}, t^2\mathrm{Id})$ by **2.1** (3) and **2.2** (1). Hence $\rho\mathbf{z}$ is trivial on ${}_2\mu_k$ (cf. **0.1**). Conversely if $\rho(h) = 1$, then we have $h_+ = t^{-1}\mathrm{Id}$ and $h_- = t\mathrm{Id}$ with $t := \chi(h)$, from which we get $t^2 = 1$ by (H3). This shows $h = \mathbf{z}(t)$ by **2.1** (3).

2.4 Denote by (G,ψ) the elementary system associated to the Jordan system $((J,J),H,\rho,b)$ in **2.2** (cf. [LAG, 5.2]). By definition, G is a separated k-group sheaf, ψ is an action $\mu_k \times G \to G$ of μ_k on G, and we have a diagram

$$J_a \overset{\exp_+,\exp_-}{\Longrightarrow} G \longleftarrow H$$

of k-group sheaves whose arrows are all monomorphic ([LAG, 3.1, 3.3]). Hence, H can be identified with its image, which coincides with the subgroup sheaf G^{ψ}, the set of fixed points of G under ψ (cf. [LAG, 4.9]). Denote by $U^{\sigma}(\sigma = \pm)$ the functor-image of $\exp_{\sigma}$. Then, H normalizes U^{σ} and the multiplication $U^{+} \times U^{-} \times H \times U^{+} \to G$ is an epimorphism of k-sheaves (cf. [LAG, 3.6, 3.8]). The multiplication $U^{-} \times H \times U^{+} \to G$ is an open immersion (cf. [LAG, 3.4]) whose functor-image Ω is dense in G (cf. [LAG, 3.8]). We have

$$\exp_{+}(x)\exp_{-}(y) = \exp_{-}(y^{x})b(x,y)\exp_{+}(x^{y})$$

for $(x,y) \in \mathcal{W}(R), R \in k$-**alg**, and $\mathcal{W} \subset J_a \times J_a$ coincides with the inverse image of $\Omega \subset G$ under the morphism $J_a \times J_a \to G$ sending (x,y) to $\exp_{+}(x)\exp_{-}(y)$ (cf. [LAG, 4.1]).

2.5 Consider the k-module

$$M := k \oplus J \oplus k \oplus J =: \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \,\middle|\, \alpha, \beta \in k, a, b \in J \right\}, \tag{1}$$

and set

$$\theta_0(h) \cdot \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} := \begin{pmatrix} \chi(h)^{-1}\alpha & h_{+}a \\ h_{-}b & \chi(h)\beta \end{pmatrix}, \tag{2}$$

$$\theta_{+}(x) \cdot \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} := \begin{pmatrix} \alpha & a + \alpha x \\ b + a \times x + \alpha x^{\sharp} & \beta + T(b,x) + T(a,x^{\sharp}) + \alpha N(x) \end{pmatrix}, \tag{3}$$

$$\theta_{-}(y) \cdot \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} := \begin{pmatrix} \alpha - T(a,y) + T(b,y^{\sharp}) - \beta N(y) & a - b \times y + \beta y^{\sharp} \\ b - \beta y & \beta \end{pmatrix}, \tag{4}$$

$$\phi(t) \cdot \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} := \begin{pmatrix} t^{-1}\alpha & a \\ tb & t^{2}\beta \end{pmatrix}, \tag{5}$$

$$\varepsilon \cdot \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} := \begin{pmatrix} \beta & -b \\ a & -\alpha \end{pmatrix}, \tag{6}$$

for all $R \in k$-**alg**$, h \in H(R), t \in R^{*}, x, y, a, b \in J_R$, and $\alpha, \beta \in R$. Thus we have

$$\varepsilon^{2} = -\mathrm{Id}, \tag{7}$$

$$\varepsilon\theta_{+}(x)\varepsilon^{-1} = \theta_{-}(x), \tag{8}$$

$$\varepsilon\theta_0(h)\varepsilon^{-1} = \theta_0(h^\vee). \tag{9}$$

By (2), the endomorphisms $\theta_0(h)$ of the R-module M_R are invertible and $h \mapsto \theta_0(h)$ is a homomorphism. Since $\theta_+(0) = \mathrm{Id}$ and $\theta_+(x)\theta_+(y) = \theta_+(x+y)$ by (3) and (CJ3,4), it follows, in view of (7) and (8), that the endmorphisms $\theta_\sigma(x)$ $(\sigma = \pm)$ are also invertible and $x \mapsto \theta_\sigma(x)$ are homomorphisms. Varying R, we get a diagram

$$J_a \overset{\theta_+,\theta_-}{\Longrightarrow} \mathbf{GL}(M) \overset{\theta_0}{\longleftarrow} H$$

of k-group schemes. Note that we have

$$(\mathrm{Int}\phi(t)) \cdot (\theta_0(h)) = \theta_0(h), \tag{10}$$

$$(\mathrm{Int}\phi(t)) \cdot (\theta_+(x)) = \theta_+(tx), \tag{11}$$

$$(\mathrm{Int}\phi(t)) \cdot (\theta_-(y)) = \theta_-(t^{-1}y), \tag{12}$$

for $t \in R^*, h \in H(R), x, y \in J_R$. In addition, we have

$$\theta_0(-h) = -\theta_0(h), \tag{13}$$

$$\theta_+(x)\theta_-(x^{-1})\theta_+(x) = -\theta_0(b(x))\varepsilon, \tag{14}$$

for invertible $x \in J_R$, by (2), (3), (4), **2.1** (4), (5), and straightforward calculation.

2.6 Theorem: *There exists a unique homomorphism $\theta : G \to \mathbf{GL}(M)$ of k-group sheaves extending θ_0, θ_+, and θ_-; moreover, θ is a monomorphism.*

To prove the first assertion, it suffices to verify

$$(\mathrm{Int}\theta_0(h)) \cdot (\theta_+(x)) = \theta_+(\rho_+(h).x), \tag{1}$$

$$(\mathrm{Int}\theta_0(h)) \cdot (\theta_-(y)) = \theta_-(\rho_-(h).y), \tag{2}$$

$$\theta_+(x)\theta_-(y) = \theta_-(y^x)\theta_0(b(x,y))\theta_+(x^y), \tag{3}$$

for all $R \in k$-**alg**$, h \in H(R), (x,y) \in \mathcal{W}(R)$ (cf. [LAG, 4.14]). Direct calculation using $(\mathrm{H1}, 2\mathrm{bis}, 3)$ shows (1) and, in view of **2.5** (8), (9) and **2.2** (3), we see that (2) follows from (1). We prove (3) in **2.8** after introducing some formulas. We now prove the last assertion of the theorem on which the following **2.7** and **2.8** do not depend. Let $g \in G(R), R \in k$-**alg**, such that $\theta(g) = \mathrm{Id}$. Then there exist an fppf extension S of R and $x, y, z \in J_S, h \in H(S)$ such that

$$g_S = \exp_+(x)\exp_-(y)\exp_+(z)h$$

(cf. **2.4**). Hence we have $\mathrm{Id} = \theta(g_S) = \theta_+(x)\theta_-(y)\theta_+(z)\theta_0(h)$. In particular

$$\theta_+(x)\theta_-(y)\theta_+(z) \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -N(y) & y^\sharp - N(y)x \\ -P(y,x) & N(y,x) \end{pmatrix}$$

and

$$\theta_0(h)^{-1} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \chi(h)^{-1} \end{pmatrix}$$

are equal and we get $N(y) = 0, y^\sharp - N(y)x = 0$, i.e., $y^\sharp = 0$, and $P(x,y) = 0$ (cf. **1.1** (4)), i.e., $y = 0$, successively. Hence $g_S = \exp_+(x+z)h$. Thus we have $\mathrm{Id} = \theta(g_S) = \theta_+(x+z)\theta_0(h)$. In particular

$$\theta_+(x+z) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+z \\ (x+z)^\sharp & 1 + N(x+z) \end{pmatrix}$$

and

$$\theta_0(h)^{-1} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \chi(h) & 0 \\ 0 & \chi(h)^{-1} \end{pmatrix}$$

are equal and we get $x + z = 0$. Hence $g_S = h$. Now $\mathrm{Id} = \theta(g_S) = \theta_0(h)$ implies $h = 1$ by **2.5** (2) and, since $R \to S$ is fppf and G is a sheaf, $g_S = h = 1$ implies $g = 1$. This shows the last assertion.

2.7 For the proof of **2.6** (3), we introduce some formulas. For any $m \in M$ with entries α, β, a, b (cf. **2.5** (1)), we define polynomial laws $m^\delta \in \mathcal{O}(J)$, and $m^\nu \in \mathcal{O}(J,J)$ by setting

(1) $$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}^\delta (w) := \alpha - T(a,w) + T(b, w^\sharp) - \beta N(w),$$

(2) $$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}^\nu (w) := a - b \times w + \beta w^\sharp,$$

for all $w \in J_R, R \in k$-**alg** (cf. **0.4**).

Lemma: *The map $M \to \mathcal{O}(J) \times \mathcal{O}(J,J)$ sending m to (m^δ, m^ν) is injective k-linear, and the following formulas hold for all $m \in M, h \in H(k), x, y \in J, t \in k^*$ and $w \in J_R, R \in$ k-**alg** such that (w,x) is quasi-invertible:*

$$(\theta_-(y)\cdot m)^\delta(w) = m^\delta(y+w), \tag{3}$$

$$(\theta_-(y)\cdot m)^\nu(w) = m^\nu(y+w), \tag{4}$$

$$(\phi(t)\cdot m)^\delta(w) = t^{-1}m^\delta(tw), \tag{5}$$

$$(\phi(t)\cdot m)^\nu(w) = m^\nu(tw), \tag{6}$$

$$(\theta_0(h)\cdot m)^\delta(w) = \chi(h)^{-1}m^\delta(\rho_-(h)^{-1}w), \tag{7}$$

$$(\theta_0(h)\cdot m)^\nu(w) = h_+m^\nu(\rho_-(h)^{-1}w), \tag{8}$$

$$(\theta_+(x)\cdot m)^\delta(w) = N(x,w)m^\delta(w^x), \tag{9}$$

$$(\theta_+(x)\cdot m)^\nu(w) = (\theta_+(x)\cdot m)^\delta(w)x^w + N(x,w)B(x,w)^{-1}m^\nu(w^x). \tag{10}$$

Proof. $m \mapsto (m^\delta, m^\nu)$ is k-linear by the definitions (1) and (2). To show the injectivity, let $m^\delta = 0$ and $m^\nu = 0$ for $m \in M$ with entries α, β, a, b. Then we have, by (1) and (2), $\alpha = m^\delta(0) = 0$ and $a = m^\nu(0) = 0$. Moreover, if t is a variable over k and $w \in J$, then $\beta N(w) \in k$ is the coefficient of t^3 in $m^\delta(tw) \in k[t]$ and there exists $w \in J$ such that $N(w) \in k^*$ (cf. **1.3** $(*)$). This shows $\beta = 0$. There remain relations $T(b, w^\sharp) = 0$ and $b \times w = 0$, which yield $b = 0$, since $0 = (b\times w)\times w^\sharp = N(w)b + T(w^\sharp, b)w = N(w)b$ by (CJ6). Thus we get $m = 0$, which shows the injectivity. Let us show the latter half of the proposition. By the definitions (1), (2) and **2.5** (4), we have

$$\theta_-(w)\cdot m = \begin{pmatrix} m^\delta(w) & m^\nu \\ b - \beta w & \beta \end{pmatrix}$$

(notation as in (1), (2)). Hence (3) and (4) follow from the fact that θ_- is a homomorphism (cf. **2.5**). On the other hand, (5) and (6) follow from

$$\theta_-(w)\phi(t)\cdot m = \phi(t)\theta_-(tw)\cdot m = \begin{pmatrix} t^{-1}m^\delta(w) & m^\nu(tw) \\ t(b-\beta tw) & t^2\beta \end{pmatrix}$$

(cf. **2.5** (5), (12)). Moreover we have, by (1), (2) and **2.5** (2),

$$(\theta_0(h)\cdot m)^\delta(w) = \chi(h)^{-1}\alpha - T(h_+a, w) + T(h_-b, w^\sharp) - \chi(h)\beta N(w),$$

$$(\theta_0(h)\cdot m)^\nu(w) = h_+a - (h_-b)\times w + \chi(h)\beta w^\sharp,$$

from which (7) and (8) follow, in view of **2.1** (H1), (H2bis), (H3). Finally, we have

(9bis) $$(\theta_+(x)\cdot m)^\delta(w) = \alpha N(x,w) - T(a, P(w,x)) + T(b, w^\sharp - N(w)x) - \beta N(w)$$

and

(10bis) $$\begin{aligned}&(\theta_+(x)\cdot m)^\nu(w)\\ &= \alpha P(x,w) + a - (a\times x)\times w + T(a, x^\sharp)w^\sharp - b\times w + T(b,x)w^\sharp + \beta w^\sharp\end{aligned}$$

by (1), (2), **2.5** (3) and (CJ6). Thus (9) follows from (9bis) and (CJ22, 23, 24). (10) acted on by $B(x,w)$ becomes

$$B(x,w)(\theta_+(x)\cdot m)^\nu(w) = (\theta_+(x)\cdot m)^\delta(w)(x - Q(x)w) + N(x,w)m^\nu(w^x),$$

which we prove by acting $B(x,w)$ on (10bis) and by using **1.9** (1), (2), (3), (5), (CJ22, 23, 24) and the above (9bis).

2.8 *Verification of* **2.6** (3). Now we show the identity $\theta_-(y)\theta_+(x) = \theta_+(x^y)\theta_0(b(x,y)^{-1})\theta_-(y^x)$ which becomes **2.6** (3) when we take inverses and replace (x,y) by $(-x,-y)$. After taking scalar extension and applying the first part of the lemma in **2.7**, we are reduced to verifying the equalities of *polynomial laws*

(1) $$(\theta_-(y)\theta_+(x)\cdot m)^\delta = (\theta_+(x^y)\theta_0(b(x,y)^{-1})\theta_-(y^x)\cdot m)^\delta,$$

(2) $$(\theta_-(y)\theta_+(x)\cdot m)^\nu = (\theta_+(x^y)\theta_0(b(x,y)^{-1})\theta_-(y^x)\cdot m)^\nu,$$

for arbitrary $(x,y)\in\mathcal{W}(k)$ and $m\in M$. For this, it suffices to verify the equalities of the values at $w\in J_R, R\in k$-**alg** such that (w, x^y) is quasi-invertible, since such w's form a dense subscheme of J_a. Then the following calculations work:

$$\begin{aligned}&(\theta_+(x^y)\theta_0(b(x,y)^{-1})\theta_-(y^x)\cdot m)^\delta(w)\\
&= N(x^y,w)(\theta_0(b(x,y)^{-1})\theta_-(y^x)\cdot m)^\delta(w^{(x^y)}) \quad (\text{by } \mathbf{2.7}\ (9))\\
&= N(x^y,w)N(x,y)(\theta_-(y^x)\cdot m)^\delta(B(y,x)^{-1}(w^{(x^y)})) \quad (\text{by } \mathbf{2.7}\ (7))\\
&= N(x,y+w)m^\delta(y^x + B(y,x)^{-1}(w^{(x^y)})) \quad (\text{by } \mathbf{2.7}\ (3),\ \mathbf{1.9}\ (3))\\
&= N(x,y+w)m^\delta((y+w)^x) \quad (\text{by [LJP, 3.7 (2)]})\\
&= (\theta_+(x)\cdot m)^\delta(y+w) \quad (\text{by } \mathbf{2.7}\ (9))\\
&= (\theta_-(y)\theta_+(x)\cdot m)^\delta(w) \quad (\text{by}\mathbf{2.7}\ (3)),\end{aligned}$$

from which (1) follows. Moreover

$$
\begin{aligned}
&(\theta_+(x^y)\theta_0(b(x,y)^{-1})\theta_-(y^x)\cdot m)^\nu(w)\\
&\quad=(\theta_+(x^y)\theta_0(b(x,y)^{-1})\theta_-(y^x)\cdot m)^\delta(w)(x^y)^w\\
&\qquad\quad+N(x^y,w)B(x^y,w)^{-1}(\theta_0(b(x,y)^{-1})\theta_-(y^x)\cdot m)^\nu(w^{(x^y)})\quad(\text{by }\mathbf{2.7}\ (10))\\
&\quad=(\theta_-(y)\theta_+(x)\cdot m)^\delta(w)x^{y+w}\\
&\qquad\quad+N(x^y,w)B(x^y,w)^{-1}N(x,y)B(x,y)^{-1}(\theta_-(y^x)\cdot m)^\nu(B(y,x)^{-1}w^{(x^y)})\\
&\qquad\qquad(\text{by (1) above, }\mathbf{2.7}\ (8),\ [\text{LJP, 3.7 (1)}])\\
&\quad=(\theta_+(x)\cdot m)^\delta(y+w)x^{y+w}+N(x,y+w)B(x,y+w)^{-1}m^\nu((y+w)^x)\\
&\qquad\qquad(\text{by }\mathbf{2.7}\ (3),\ (4),\ \mathbf{1.10}\ (3),\ [\text{LJP, 3.6 (JP33), 3.7 (2)}])\\
&\quad=(\theta_+(x)\cdot m)^\nu(y+w)\quad(\text{by }\mathbf{2.7}\ (10))\\
&\quad=(\theta_-(y)\theta_+(x)\cdot m)^\nu(w)\quad(\text{by }\mathbf{2.7}\ (4)),
\end{aligned}
$$

from which (2) follows. This completes the verification of **2.6** (3).

NOTE

The matrix notation for an element of the k-module $M = k\oplus J\oplus k\oplus J$ (cf. **2.5**) is taken from [Fau]. The idea of imbedding M into $\mathcal{O}(J)\times\mathcal{O}(J,J)$ (cf. **2.7**) is inspired by [LHA]. To determine the *type* of the geometric fiber of the representation $\theta : G \to \mathbf{GL}(M)$, it seems necessary to define some object, say *split data*, for a quadruple $(J; N, \sharp, T)$, so that the following two conditions are satisfied: a) If k is an algebraically closed field, a split data exists; b) If a split data exists, G is a "splitable reductive k-group"(cf. [SGA3, Exp. XXII, 1.13]), and one split data define one "spliting"(cf. [SGA3, Exp. XXII, 1.13]) of G.

3 Stabilizers

3.0 We keep the notation in §2. The representation θ induces linear and projective representations
$$\mu_k \times G \to \mathbf{GL}(M).$$
and
$$G \to \mathbf{Aut}(\mathbf{P}(M)),$$
respectively. Note that, if $m \in M_m(k)$ (cf. **0.3**), then $\mathrm{pr}_2 : \mu_k \times G \to G$ induces an *isomorphism*
$$\mathbf{Cent}_{\mu_k \times G}(m) \simeq \mathbf{Cent}_G(p_M(m)),$$
since M_m is a μ_k-*torsor* with structure morphism $p_M : M_m \to \mathbf{P}(M)$ (cf. **0.3**). We consider the two elements
$$u_0 := p_M(m_0) \quad \text{and} \quad u_1 := p_M(m_1) \quad \in \mathbf{P}(M)(k),$$
where
$$m_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad m_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$
The propose of this section is to determine their stabilizers in G and the orbit-sheaf of u_1 under G. The stabilizers are canonically isomorphic to those of m_0 and m_1 in $\mu_k \times G$, respectively.

3.1 We first introduce a notational convention. Recall that θ is a monomorphism (cf. **2.6**) and the image of $G(k)$ under $\theta(k)$ contains $\varepsilon \in GL(M)$ (cf. **2.5** (14)). We regard ε as an element of $G(k)$ via θ. Thus we have

(1) $$\varepsilon^2 = -1,$$

(2) $$\varepsilon h \varepsilon^{-1} = h^\vee,$$

(3) $$\exp_+(x)\exp_-(x^{-1})\exp_+(x) = -b(x)\varepsilon,$$

for $h \in H(R)$, and invertible $x \in J_R, R \in k$-**alg** (cf. **2.5** (7), (9), (13), (14)).

3.2 Let $H' \subset H$ be the kernel of the character $h \mapsto \chi(h)^4$ of H (cf. **2.1**). If $h = (\chi(h), h_+, h_-) \in H'(R), R \in k$-**alg**, then
$$s(h) := (\lambda, \lambda^2 h_-, \lambda^2 h_+) \qquad \text{with} \qquad \lambda := \chi(h)$$

belongs to $H'(R)$. Indeed, since $\lambda^4 = 1$, we have $\mathbf{z}(\lambda^2) = (\lambda^2, \lambda^2\mathrm{Id}, \lambda^2\mathrm{Id})$ (cf. **2.1** (3)), and hence $s(h) = \mathbf{z}(\lambda^2)h^\vee \in H(R)$ (cf. **2.1** (2)). In particular, we have

$$(1) \qquad \chi(s(h)) = \chi(h),$$

from which $s(h) \in H'(R)$ follows. Thus we get an *automorphism* $s : h \mapsto s(h)$ of the k-group H' of period two. Let the constant k-group $(\mathbf{Z}/2\mathbf{Z})_k$ act on H' via s (cf. [D-G, II, §1, 3.3 a)]), and construct the semi-direct product $H' \times_s (\mathbf{Z}/2\mathbf{Z})_k$. Hence we have

$$(2) \qquad (h, f) \cdot (h', f') = (hs_f(h'), f * f'),$$

for all $h, h' \in H'(R)$, and $f, f' \in (\mathbf{Z}/2\mathbf{Z})_k(R), R \in k$-**alg**, where we regard $(\mathbf{Z}/2\mathbf{Z})_k(R)$ as the group of idempotents in R with operation $f * f' := f + f' - 2ff'$ (cf. [D-G, III, §5, 2.4]), and $s_f(h') \in H'(R)$ is the element corresponding to $(h', s(h')) \in H'(R_{1-f}) \times H'(R_f)$ under the decomposition $R \simeq R_{1-f} \times R_f$ of R with respect to the idempotent f. Therefore, in view of (1),

$$(h, f) \mapsto \chi(h) : H' \times_s (\mathbf{Z}/2\mathbf{Z})_k \longrightarrow {}_4\mu_k$$

is a character. Moreover, since the morphism $(\mathbf{Z}/2\mathbf{Z})_k \to {}_2\mu_k$ sending f to $1 - 2f$ is also a character, we can define a character $\chi' : H' \times_s (\mathbf{Z}/2\mathbf{Z})_k \to {}_2\mu_k$ by setting

$$(3) \qquad \chi'(h, f) := \chi(h)^2(1 - 2f),$$

for $h \in H'(R)$ and $f \in (\mathbf{Z}/2\mathbf{Z})_k(R), R \in k$-**alg**. Let $H'' \subset H' \times_s (\mathbf{Z}/2\mathbf{Z})_k$ be the kernel of χ'. Hence we have

$$(4) \qquad H''(R) = \{(h, f) \in H(R) \times R \mid f^2 = f, \chi(h)^2 = 1 - 2f\},$$

for all $R \in k$-**alg**. For any $(h, f) \in H''(R), R \in k$-**alg**, define $\mathbf{f}(h, f) \in G(R)$ to be the element with components $(h, h\varepsilon) \in G(R_{1-f}) \times G(R_f)$ under the decomposition $R \simeq R_{1-f} \times R_f$. Varying R, we get a morphism

$$\mathbf{f} : H'' \to G$$

of k-sheaves.

3.3 Theorem: $\mathbf{f}$ *is a homomorphism of k-group sheaves and factors into the composite*

$$\mathbf{f} : H'' \xrightarrow{\sim} \mathbf{Cent}_G(u_0) \xrightarrow{\text{incl.}} G,$$

whose first arrow is an isomorphism.

3.4 First, we show that $\mathbf{f}$ is a homomorphism. Consider $(h,f),(h',f') \in H''(R), R \in k$-**alg**, and describe any element in $G(R)$ in terms of four components with respect to the decomposition

$$R \simeq R_{(1-f)(1-f')} \times R_{(1-f)f'} \times R_{f(1-f')} \times R_{ff'} \tag{1}$$

of R. Then we have

$$\mathbf{f}(h,f) = (h, h, h\varepsilon, h\varepsilon) \quad \text{and} \quad \mathbf{f}(h',f') = (h', h'\varepsilon, h', h'\varepsilon)$$

by definition, so that we have

$$\mathbf{f}(h,f)\mathbf{f}(h',f') = (hh', hh'\varepsilon, h(h')^\vee\varepsilon, -h(h')^\vee) \tag{2}$$

by **3.1** (1), (2). On the other hand, we have

$$\mathbf{f}((h,f)(h',f')) = (hs_f(h'), hs_f(h')\varepsilon, hs_f(h')\varepsilon, hs_f(h')) \tag{3}$$

by **3.2** (2). However, by **3.2** (4) and the formula $s(h) = \mathbf{z}(\chi(h)^2)h^\vee$, the components of $s_f(h') \in H'(R)$ with respect to $R \simeq R_{1-f} \times R_f$ is $(h', s(h')) = (h', \mathbf{z}(1-2f')h'^\vee) \in H'(R_{1-f}) \times H'(R_f)$, and that of $1-2f' \in R$ with respect to $R \simeq R_{1-f'} \times R'_f$ is $(1,-1)$. Thus we have

$$s_f(h') = (h', h', (h')^\vee, -(h')^\vee) \tag{4}$$

with respect to (1). Hence, by (2), (3), and (4), we get $\mathbf{f}((h,f)(h',f')) = \mathbf{f}(h,f)\mathbf{f}(h',f')$.

3.5 Next, we show that $\mathbf{f}$ is a *monomorphism*. Consider $(h,f) \in H''(R), R \in k$-**alg** such that $\mathbf{f}(h,f) = 1_{G(R)}$. Then we have $h = 1$ in $G(R_{1-f})$ and $h\varepsilon = 1$ in $G(R_f)$. Hence we have $h = -\varepsilon$ in $G(R_f)$ (cf. **3.1** (1)), which yields $h = 1$ and $-\varepsilon = 1$ in $G(R_f)$, since $H \cap U^+U^-U^+$ is trivial (cf. [LAG, 3.6(c)]). In particular, we have $h = 1$ in $H(R)$. Moreover, in view of **2.5** (6), $-\varepsilon = 1$ occurs only when $R_f = 0$. Thus we have $f = 0$.

3.6 For any $(h,f) \in H''(R), R \in k$-**alg**, we have

$$\theta(\mathbf{f}(h,f)) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \chi(h)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Indeed, since $\mathbf{f}(h,f) = (h, h\varepsilon)$ and $\chi(h)^2 = 1 - 2f = (1,-1)$ with respect to the decomposition $R \simeq R_{1-f} \times R_f$ (cf. **3.2** (3) (4)), we have

$$\theta(\mathbf{f}(h,f)) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{R_{1-f}} = \theta_0(h) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \chi(h)^{-1} & 0 \\ 0 & \chi(h) \end{pmatrix} = \chi(h)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\theta(\mathbf{f}(h,f)) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{R_f} = \theta_0(h\varepsilon) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \chi(h)^{-1} & \\ 0 & -\chi(h) \end{pmatrix} = \chi(h)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

from which the assertion follows. Thus the morphism $\mathbf{f} : H'' \to G$ factors through $\mathbf{Cent}_G(u_0)$. The resulting morphism $\mathbf{f}' : H'' \to \mathbf{Cent}_G(u_0)$ is a *monomorphism*, since so is $\mathbf{f}$ (cf. **3.5**). To complete the proof of Theorem **3.3**, it remains to show that $\mathbf{f}'$ is an *epimorphism* (cf. [D-G, III, §1, 2.1]). In view of **2.4**, the question is reduced to the following lemma:

3.7 Lemma: *Let $R \in k$-**alg**, $\nu \in R^*, x, y, z \in J_R$ and $h \in H(R)$ such that*

(1) $$\nu\theta_+(x)\theta_-(y)\theta_+(z)\theta_0(h) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, there exists an idempotent $f \in R$ and an element $h' \in H(R)$ with the following properties:

(i) $\chi(h')^2 = 1 - 2f$,

(ii) $\chi(h') = \nu$,

(iii) *the components of $g := \exp_+(x)\exp_-(y)\exp_+(z)h \in G(R)$ with respect to the decomposition $R \simeq R_{1-f} \times R_f$ are $(h', h'\varepsilon) \in G(R_{1-f}) \times G(R_f)$.*

Proof. We define $\alpha, t \in R$ and $a \in J_R$, depending on (ν, x, y, z, h), by

(2) $$t := \chi(h)^2,$$

(3) $$\alpha := N(z, y) - tN(y),$$

(4) $$a := z - z^\sharp \times y + (N(z) + t)y^\sharp.$$

Then direct calculation shows that (1) is equivalent to the four conditions:

(5) $$\alpha = \chi(h)\nu^{-1},$$

(6) $$\alpha^3 = t^2 N(y)(T(z,y) - z) - tN(z,y)(T(z,y) - 1),$$

(7) $$y = Q(y)z,$$

$$(8) \qquad x = -\alpha^{-1}a.$$

By (2), (3) and (5), we have

$$(9) \qquad \chi(h)^{-1}\nu N(z,y) - \chi(h)\nu N(y) = 1.$$

Moreover, by (7), we have

$$(10) \qquad N(z,y)N(y) = 0.$$

Indeed, the left-hand side equals $N(y) - N(y)^2N(z) - T(y,z)N(y) + T(y^\sharp, z^\sharp)N(y)$ (cf. **1.1** (4)). Acting $N(?)$ and $Q(y^\sharp)$ on (7) with (CJ20, 1, 5, 14) in mind, we get $N(y) = N(y)^2N(z)$ and $N(y)y^\sharp = N(y)^2z$. Hence we have $T(y,z)N(y) = T(y,z)N(y)^2N(z) = T(y, N(y)y^\sharp)N(z) = 3N(y)^2N(z) = 3N(y)$, and $T(y^\sharp, z^\sharp)N(y) = T(N(y)^2z, z^\sharp)$ $= 3N(y)^2N(z) = 3N(y)$ (cf. (CJ5)). Thus (10) holds. By (9) and (10), the element

$$f := -\chi(h)\nu N(y)$$

of R is an idempotent. Since y becomes invertible after the scalar extension $R \to R_f$ (cf. **1.6**), we can define $h' \in H(R)$ to be the element with components $(h, -b(y^{-1})h^\vee) \in H(R_{1-f}) \times H(R_f)$ (cf. **2.1**) with respect to the decomposition $R \simeq R_{1-f} \times R_f$. We claim that (f, h') is what we want. Namely:

a) After the scalar extension $R \to R_{1-f}$, we have $\chi(h')^2 = 1-2f, \chi(h') = \nu$, and $g = h'$. Indeed, (y,z) becomes quasi-invertible by (9) and **1.7**, and we have $B(y,z)y = B(y,z)Q(y)z = Q(y - Q(y)z) = 0$ by (7) and [LJP, 2.11 (JP23)], from which we get $y = 0$. Then we have $1 = \chi(h)\nu^{-1}$ (by (3), (5)), $1 = \chi(h)^2$ (by (2), (3), (6)), $x = -z$ (by (3), (4), (8)), and $g = \exp_+(x)\exp_-(y)\exp_+(z)h = h$. Thus the assertion follows, since we have $h' = h$ and $1 - 2f = (1-2f)(1-f)/(1-f) = 1$ after our scalar extension.

b) After the scalar extension $R \to R_f$, we have $\chi(h')^2 = 1-2f, \chi(h') = \nu$, and $g = h'\varepsilon$. Indeed, y becomes invertible by **1.6**, and we have $z = y^{-1}$ by (7), from which we get $N(z,y) = 0,\ T(z,y) = 3,\ z^\sharp \times y = 2z$, and $y^\sharp = N(z)^{-1}z$, in view of **1.6** (2), (CJ5), and (CJ1). Then we have $\nu = -N(y)^{-1}\chi(h)^{-1}$ (by (2), (3), (5)), $N(y)^2\chi(h)^2 = -1$ (by (2), (3), (6)), $x = y^{-1}$ (by (2), (3), (4), (8)), and $g = \exp_+(x)\exp_-(y)\exp_+(z)h = \exp_+(x)\exp_-(x^{-1})\exp_+(x)h = -b(x)\varepsilon h = -b(y^{-1})h^\vee\varepsilon$ (by **3.1** (2), (3)). Thus the assertion follows, since we have $h' = -b(y^{-1})h^\vee, 1-2f = (1-2f)f/f = -1, \chi(h') = -N(y)^{-1}\chi(h)^{-1}$ (by **2.1** (1), (4)) after our scalar extension.

3.8 Recall that H normalizes U^+ and U^- (cf. **2.4**). Consider the semi-direct product $U^- \times H$ and the *homomorphism*

$$\mathbf{m} : U^- \times H \longrightarrow G$$

induced by the multiplication, which is a *monomorphism* since the multiplication induces an open immersion $U^- \times H \times U^+ \to G$ (cf. **2.4**).

Theorem: $\mathbf{m}$ *factors into the composite*

$$\mathbf{m} : U^- \times H \xrightarrow{\sim} \mathbf{Cent}_G(u_1) \xrightarrow{\text{incl.}} G$$

whose first arrow is an isomorphism.

Indeed, for any $y \in J_R, h \in H(R), R \in k$-**alg**, we have

$$\theta_-(y) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\theta_0(h) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \chi(h)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

by **2.5** (2), (4). Hence the morphism $\mathbf{m}$ factors through $\mathbf{Cent}_G(u_1)$. The resulting morphism $\mathbf{m}' : U^- \times H \to \mathbf{Cent}_G(u_1)$ is a *monomorphism*, since so is $\mathbf{m}$. To complete the proof of the theorem, it remains to show that $\mathbf{m}'$ is an *epimorphism* (cf. [D-G, III, §1, 2.1]). In view of **2.4**, the question is reduced to the following lemma:

3.9 Lemma: *Let $R \in k$-**alg**, $\nu \in R^*, x, y, z \in J_R$ and $h \in H(R)$ such that*

$$\nu\theta_-(y)\theta_+(x)\theta_-(z)\theta_0(h). \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then we have $x = 0$ and $\nu = \chi(h)$.

Proof. Indeed, we have, by **2.5** (2), (3), (4),

$$\theta_-(y)\theta_+(x)\theta_-(z)\theta_0(h). \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \chi(h)^{-1} \begin{pmatrix} N(x,y) & P(x,y) \\ x^\sharp - N(x)y & N(x) \end{pmatrix},$$

where $N(x,y) := 1 - T(x,y) + T(x^\sharp, y^\sharp) - N(x)N(y)$, and $P(x,y) := x - x^\sharp \times y + N(x)y^\sharp$ (cf. **1.1** (3), (4)). This shows our assertion.

3.10 Consider the *orbit-sheaf* (cf. [D-G, III, §3, 1.6]) of u_1 (cf. **3.0**) under G, which we denote by $O_G(u_1)$. Recall that (cf. [LHA, 2.1, 2.2]) the Jordan pair (J, J) (cf. **1.6**) defines a quasi-projective algebraic k-scheme $X(J,J)$ which is the *quotient-sheaf* of $J_a \times J_a$ by the equivalence relation:

$$(1) \qquad (x,y) \sim (x',y') \Leftrightarrow (x, y-y') \quad \text{is quasi-invertible and} \quad x' = x^{y-y'},$$

for (x,y) and (x',y') in $J_R \times J_R, R \in k$-**alg**. On the other hand, in view of **2.4** and **3.8**, we have an *epimorphism* $\pi' : J_a \times J_a \to O_G(u_1)$ such that

$$\pi'(x,y) = p_M \begin{pmatrix} N(x,y) & P(x,y) \\ x^\sharp - N(x)y & N(x) \end{pmatrix}.$$

This is precisely the composite of the morphism $J_a \times J_a \to G$ sending (x,y) to $\exp_-(y)\exp_+(x)$ and the orbit-morphism $G \to O_G(u_1)$ of u_1 under G.

Corollary: *The morphism π' factors into the composite*

$$\pi' : J_a \times J_a \xrightarrow{\text{can.}} X(J,J) \xrightarrow{\sim} O_G(u_1),$$

whose second arrow is an isomorphism.

This follows from **3.8** and the fact (cf. [LHA, 4.3]) that the equivalence relation (1) coincides with the fibration by the morphism $J_a \times J_a \to G/U^-H$ sending (x,y) to $\exp_-(y)\exp_+(x) \bmod U^-H$.

NOTE

As for the orbit-sheaf $O_G(u_0)$ of u_0, if k is a field and some condition on the quadruple $(J; N, \sharp, T)$ is satisfied, this becomes the principal open subscheme defined by the Freudenthal quartic. For this, see NOTE in §5.

4 Freudenthal quartic

4.1 Recall that $(J; N, \sharp, T)$ is a quadruple as in **1.1** satisfying the condition $(*)$ in **1.3**, that G is the k-group sheaf defined in **2.4**, and that M is the k-module $k \oplus J \oplus k \oplus J$ (cf. **2.5**) on which G acts via the representation θ defined in **2.6**. Consider the quartic form (cf. **0.4**) $f \in \mathcal{O}^4(M)$ and the alternating form $\{,\} \in {}^t(\wedge^2 M)$ such that

$$(1) \qquad f \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} := (T(a,b) - \alpha\beta)^2 + 4N(a)\beta + 4N(b)\alpha - 4T(a^\sharp, b^\sharp),$$

$$(2) \qquad \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}, \begin{pmatrix} \alpha' & a' \\ b' & \beta' \end{pmatrix} \right\} := T(a, b') - T(b, a') + \beta\alpha' - \alpha\beta',$$

for all $\alpha, \alpha', \beta, \beta' \in R$, and $a, b, a', b' \in J_R, R \in k$-**alg**.

Proposition: *G stabilizes f and* $\{,\}$.

Proof. The assertion amounts to saying that $f(gm) = f(m)$ and $\{gm, gm'\} = \{m, m'\}$ for all $g \in G(R), R \in k$-**alg**$, m, m' \in M_S$, and $S \in R$-**alg**, which may be read as identities of elements in $\mathcal{O}^4(M_R)$ and ${}^t(\wedge^2 M_R)$. Since the k-functors $R \mapsto \mathcal{O}^4(M_R)$ and $R \mapsto {}^t(\wedge^2 M_R)$ are schemes, in particular sheaves, these identities may be verified after any fppf extension of R, with which everything is compatible. Thus we may take R to be k and assume that g is of the form $\exp_+(x)\exp_-(y)h\exp_+(z)$ with $x, y, z \in J, h \in H(k)$ (cf. **2.4**). Then, in view of **2.5** (8), the problem reduces to the verifications of

$$(3) \qquad f(\theta_0(h) \cdot m) = f(m), \quad \{\theta_0(h) \cdot m, \theta_0(h).m'\} = \{m, m'\},$$

$$(4) \qquad f(\varepsilon \cdot m) = f(m), \quad \{\varepsilon \cdot m, \varepsilon.m'\} = \{m, m'\},$$

$$(5) \qquad f(\theta_+(x) \cdot m) = f(m), \quad \{\theta_+(x) \cdot m, \theta_+(x) \cdot m'\} = \{m, m'\},$$

for $h \in H(k), x \in J$ and $m, m' \in M_R, R \in k$-**alg**. Indeed, (3), (4), and the last part of (5) are direct consequences of calculation using the definitions in **2.1**, **2.5** combined with (CJ4, 5). To prove the first part of (5), we introduce some formulas:

4.2 Define polynomial laws (cf. **0.4**) $\gamma, \eta \in \mathcal{O}^3(M)$ and $c \in \mathcal{O}^2(M, J)$ by setting

$$(1) \qquad \gamma \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} := \alpha^2\beta - \alpha T(a,b) + 2N(a),$$

$$\eta\begin{pmatrix}\alpha & a\\ b & \beta\end{pmatrix} := N(a)-\alpha^2\beta, \tag{2}$$

$$c\begin{pmatrix}\alpha & a\\ b & \beta\end{pmatrix} := \alpha b - a^\sharp, \tag{3}$$

for $\alpha,\beta\in R, a,b\in J_R, R\in k$-**alg**.

Lemma: *For any $x\in J_R, h\in H(R)$, and any*

$$m=\begin{pmatrix}\alpha & a\\ b & \beta\end{pmatrix}\in M_R,$$

*$R\in k$-**alg**, the following formulas hold:*

$$\gamma(\theta_+(x)\cdot m)=\gamma(m), \tag{4}$$

$$\eta(\theta_+(x)\cdot m)=\eta(m)-\alpha T(c(m),x), \tag{5}$$

$$c(\theta_+(x)\cdot m)=c(m), \tag{6}$$

$$\gamma(\theta_0(h)\cdot m)=\chi(h)^{-1}\gamma(m), \tag{7}$$

$$\eta(\theta_0(h)\cdot m)=\chi(h)^{-1}\eta(m), \tag{8}$$

$$c(\theta_0(h)\cdot m)=\chi(h)^{-1}h_-c(m), \tag{9}$$

$$\gamma(m)^2+4N(c(m))=\alpha^2 f(m). \tag{10}$$

Proof. (7), (8), (9) are clear from the definitions (1), (2), (3), **2.5** (3), and **2.1** (H1), (H2), (H3). As for (4), (5), (6), we calculate

$$\begin{aligned}
\gamma(\theta_+(x)\cdot m) &= \alpha^2(\beta+T(b,x)+T(a,x^\sharp)+\alpha N(x))\\
&\quad -\alpha T(a+\alpha x, b+a\times x+\alpha x^\sharp)\\
&\quad +2N(a+\alpha x) \qquad \text{(by (1) and \textbf{2.5} (3))}\\
&= \alpha^2(\beta+T(b,x)+T(a,x^\sharp)+\alpha N(x))\\
&\quad -\alpha(T(a,b)+\alpha T(x,b)+2T(a^\sharp,x)+3\alpha T(a,x^\sharp)+3\alpha^2N(x))\\
&\quad +2(N(a)+\alpha T(a^\sharp,x)+\alpha^2T(a,x^\sharp)+\alpha^3N(x))\\
&\quad \text{(by (CJ4, 5, 3))}\\
&= \alpha^2\beta-\alpha T(a,b)+2N(a)\\
&= \gamma(m) \qquad \text{(by (1))},
\end{aligned}$$

$$
\begin{aligned}
\eta(\theta_+(x)\cdot m) &= N(a+\alpha x) - \alpha^2(\beta + T(b,x) + T(a,x^\sharp) + \alpha N(x)) \\
&\quad \text{(by (2) and } \mathbf{2.5}\text{ (3))} \\
&= N(a) + \alpha T(a^\sharp, x) + \alpha^2 T(a, x^\sharp) + \alpha^3 N(x)) \\
&\quad - \alpha^2(\beta + T(b,x) + T(a,x^\sharp) + \alpha N(x)) \qquad \text{(by (CJ3))} \\
&= N(a) + \alpha T(a^\sharp, x) - \alpha^2\beta - \alpha^2 T(b,x) \\
&= \eta(m) - \alpha T(c(m), x) \qquad \text{(by (2), (3))},
\end{aligned}
$$

and

$$
\begin{aligned}
&c(\theta_+(x)\cdot m) \\
&\quad = \alpha(b + a\times x + \alpha x^\sharp) - (a+\alpha x)^\sharp \quad \text{(by (2) and } \mathbf{2.5}\text{ (3))} \\
&\quad = \alpha b + \alpha a \times x + \alpha^2 x^\sharp) - (a^\sharp + \alpha a\times x + \alpha^2 x^\sharp) \quad \text{(by } \mathbf{1.1}\text{ (1))} \\
&\quad = \alpha b - a^\sharp \\
&\quad = c(m) \qquad \text{((by (3))}.
\end{aligned}
$$

Finally we have

$$
\begin{aligned}
&\gamma(m)^2 + 4N(c(m)) \\
&\quad = \alpha^2(\alpha\beta - T(a,b))^2 + 4N(a)\alpha(\alpha\beta - T(a,b)) + 4N(a)^2 + \\
&\qquad + 4(\alpha^3 N(b) - \alpha^2 T(b^\sharp, a^\sharp) + \alpha N(a) T(b,a) - N(a)^2) \\
&\qquad \text{(by (1), (2) and (CJ3, 1, 12))} \\
&\quad = \alpha^2(\alpha\beta - T(a,b))^2 + 4\alpha^2 N(a)\beta + 4\alpha^3 N(b) - 4\alpha^2 T(b^\sharp, a^\sharp) \\
&\quad = -\alpha^2 f(m) \qquad \text{(by } \mathbf{4.1}\text{ (1))},
\end{aligned}
$$

which is (10).

4.3 *Proof of the first part of* **4.1** (5). This may be read as an identity of two *polynomial laws* f and $f' := (m \mapsto f(\theta_+(x).m))$ in $\mathcal{O}^4(M)$. Thus we consider the principal open subscheme M_m^+ of M_a defined by the section

$$
\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \mapsto \alpha
$$

(in fact M_m^+ is contained in M_m). Since M_m^+ is *dense* in M_a, we are reduced to showing that $f|M_m^+ = f'|M_m^+$ in $\mathcal{O}(M_m^+)$, which follows from **4.2** (4), (6), (10). This completes the verification of the first part of **4.1** (5).

f is the (-4)-times of the quartic form defined by Freudenthal in [Freu, I (4.9)], and $\{,\}$ is precisely the alternating form in [Freu, I (4.5)]. Since $\theta : G \to \mathbf{GL}(M)$ is a *monomorphism* (cf. **2.6**), we may consider G to be a *subgroup sheaf* of $\mathbf{GL}(M)$ via θ. Then, **4.1** amounts to saying that $G \subset \mathbf{Cent}(f, \{\})$, the stabilizer of the quartic form f and the alternating form $\{,\}$ in $\mathbf{GL}(M)$. For the present, we do not know whether $G = \mathbf{Cent}(f, \{\})$ or not. It is known, however, that this is the case if k is an *algebraically closed field of characteristic different from two and three* and if $(J; N, \sharp, T)$ is derived from the Jordan algebra $H_3(\mathcal{C})$ with $\mathcal{C}$ a composition algebras (cf. [Igusa 1, 2]).

5 Transitivity

5.1 We keep the notation in §4. Denote by $(M_a)_f$ (resp. $\mathbf{D}_+(f)$) the open subscheme of M_a (resp. $\mathbf{P}(M)$) defined by the section $f \in \mathcal{O}(M_a)$ (cf. **4.1** (1), **0.6**), and define subschemes $(M_a)_f^+, (M_a)_f^{++}$ of $(M_a)_f$ by

$$(M_a)_f^+(R) := \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \in (M_a)_f(R) \;\middle|\; \alpha \in R^* \right\},$$

$$(M_a)_f^{++}(R) := \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \;\middle|\; \alpha, \beta \in R^* \right\},$$

for $R \in k$-**alg**. Since G stabilizes f, the subscheme $(M_a)_f$ is stable under G, and so is $D_+(f)$ (cf. **0.6**). Recall that we have two vector subgroups $U^\sigma (\sigma = \pm)$ of G together with isomorphisms $J_a \simeq U^\sigma$ (cf. **2.4**), and that the composite $J_a \simeq U^\sigma \subset G \xrightarrow{\theta} \mathbf{GL}(M)$ coincides with $\theta_\sigma : J_a \to \mathbf{GL}(M)$ described in **2.5** (3), (4).

5.2 Proposition: *If $K \in k$-**alg** is an algebraically closed field, then we have*

$$(M_a)_f^+(K) = U^+(K)U^-(K) \cdot (M_a)_f^{++}(K).$$

Proof. In fact, for any $R \in k$-**alg** and $m \in (M_a)_f(R)$, we have

$$m_{R(m)} \in U^+(R(m))U^-(R(m)).(M_a)_f^{++}(R(m)),$$

where $R(m)$ is the quotient of the polynomial ring $R[\omega]$ in one variable ω, by the principal ideal generated by the polynomial $N(c(m))\omega^2 + \gamma(m)\omega - 1$ (cf. **4.2** (1), (2)). Indeed, writing

$$m = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}$$

with $\alpha, \beta \in R, a, b \in J_R$, we have

$$\theta_+(-\alpha^{-1}a) \cdot m = \phi(\alpha^{-1}). \begin{pmatrix} 1 & 0 \\ c(m) & \gamma(m) \end{pmatrix},$$

$$\gamma(m)^2 + 4N(c(m)) = \alpha^2 f(m),$$

by **2.5** (3), (5), **4.2** (1), (2), (3). Hence the problem reduces to the following lemma:

5.3 Lemma: *Assume $c \in J$ and $\gamma \in k$ to be given so that the condition*

(1) $$\gamma^2 + 4N(c) \in k^*$$

holds. Consider the polynomial

$$q(\omega) := -1 + \gamma\omega + N(c)\omega^2 \in k[\omega] \tag{2}$$

*(ω a variable over k), and let $R := k[\omega]/(q) \in k$-**alg**. Then, there exist $x, y \in J_R$ such that*

$$\theta_-(y)\theta_+(x) \cdot \begin{pmatrix} 1 & 0 \\ c & \gamma \end{pmatrix}_R \in (M_a)_f^{++}(R).$$

Proof. Consider one more variable ρ over k, and define polynomials $\eta \in k[\omega], \zeta, \epsilon, \xi \in k[\omega, \rho]$ by

$$\eta(\omega) := \gamma + 3N(c)\omega + N(c)^2\omega^3, \tag{3}$$

$$\zeta(\omega, \rho) := 1 + N(c)\omega^2 - \eta(\omega)\rho, \tag{4}$$

$$\epsilon(\omega, \rho) := \omega - (1 + N(c)\omega^2)\rho, \tag{5}$$

$$\xi(\omega, \rho) := 1 - 3N(c)\epsilon(\omega, \rho)\rho - N(c)\eta(\omega)\rho^3, \tag{6}$$

to obtain

$$\theta_-(\rho c)\theta_+(\omega c^\sharp) \cdot \begin{pmatrix} 1 & 0 \\ c & \gamma \end{pmatrix} = \begin{pmatrix} \xi(\omega, \rho) & (\epsilon(\omega, \rho) - \zeta(\omega, \rho)\rho)c^\sharp \\ \zeta(\omega, \rho)c & \eta(\omega) \end{pmatrix}, \tag{7}$$

$$\eta(\omega) = q(\omega)(N(c)\omega - \gamma) + (\gamma^2 + 4N(c))\omega, \tag{8}$$

$$\eta(\omega)\epsilon(\omega, \rho) = q(\omega) + (1 + N(c)\omega^2)\zeta(\omega, \rho). \tag{9}$$

Let $\omega_0 := \omega \bmod q \in R$. Then we have $q(\omega_0) = 0$ by the definition of R. Also (2) tells us that ω_0 is invertible with inverse $\gamma + N(c)\omega_0$. Therefore so is $\eta(\omega_0)$ by (1) and (8). Let $\rho_0 := \eta(\omega_0)^{-1}(1 + N(c)\omega_0^2)$. Then we have $\zeta(\omega_0, \rho_0) = \epsilon(\omega_0, \rho_0) = 0$ by (4) and (9), and (7) tells us that $x := \omega_0 c^\sharp$ and $y := \rho_0 c$ are what we want.

5.4 Consider the following condition on a quadruple $(J; N, \sharp, T)$:

$(**)$ For any field $K \in k$-**alg** of characteristic different from two, the symmetric bilinear form $(x, y) \mapsto T(x, y)$ on J_K is non-degenerate.

Proposition: *Under the assumption* $(**)$, *we have*

$$(M_a)_f(K) = U^-(K) \cdot (M_a)_f^+(K),$$

for any infinite field $K \in k$-**alg**.

Proof. In view of **2.5** (3) and **2.7** (1), it suffices to show that for any $m \in (M_a)_f(K)$ there exists $w \in J_K$ such that $m^\delta(w) \in K^*$. Then we are reduced to showing that the *polynomial law* $m^\delta \in \mathcal{O}(J_K)$ (cf. **0.4**) is not zero, since K is an infinite field (cf. [Bou, IV, §2, no 3, Cor. 2 of Prop. 9]). In general, we have

$$(1) \qquad \{m \in M \mid m^\delta = 0\} = \left\{ \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \middle| \; a, b \in J, T(a, ?) = T(b, ?) = 0 \in \mathcal{O}^1(J) \right\}.$$

Indeed, the left-hand side contains the right-hand side by the definition **2.7** (1). To see the converse, let $m^\delta = 0$ for $m \in M$ with entries α, β, a, b. Then, equating the homogeneous components of the polynomial m^δ to zero, we get $\alpha = 0, T(a, ?) = 0 \in \mathcal{O}^1(J), T(b, ?^\sharp) = 0 \in \mathcal{O}^2(J)$, and $\beta N = 0 \in \mathcal{O}^3(J)$. However this also implies $T(b, ?) = 0$ and $\beta = 0$, since the morphism $?^\sharp$ is *scheme-theoretically dominant* (cf. **1.3 c)**) and there exists $c_1 \in J$ such that $N(c_1) \in k^*$ (cf. **1.3** $(*)$). This shows (1). Now apply (1) after the scalar extension $k \to K$. If $\mathrm{char}(K) \neq 2$, the right-hand side of (1) is $\{0\}$ by our assumption $(**)$, and if $\mathrm{char}K = 2$, we have $f(m) = 0$ for all m in the right-hand side of (1). In all cases, we have $\{m \in M_K \mid m^\delta = 0\} = \{m \in M_K \mid f(m) = 0\}$, i.e., $m^\delta \in \mathcal{O}(J_K)$ is not zero if $m \in (M_a)_f(K)$.

5.5 Let us assume the condition $(**)$ in **5.4**.

Corollary 1: *For any algebraically closed field* $K \in k$-**alg**, *the action of* $G(K)$ *on* $D_+(f)(K)$ *is transitive.*

In view of the canonical bijection $\{x \in M_K | f(x) \in K^*\}/K^* \xrightarrow{\sim} \mathbf{D}_+(f)(K)$ (cf. **0.6** (1)), this follows from:

Corollary 2: *For any algebraically closed field* $K \in k$-**alg**, *the set* $(M_a)_f(K)$ *is a single orbit under* $K^* \times G(K)$.

Indeed, if $m, m' \in (M_a)_f(K)$, there exists $t \in K^*$ such that $t^4 = f(m)^{-1} f(m')$, since K is algebraically closed. For such t, we have $f(tm) = f(m')$, since f is a quartic form (cf. **0.4**). Now the assertion follows from:

Corollary 3: *For any algebraically closed field $K \in k$-**alg** and $i \in K^*$, the set $\{m \in M_K \mid f(m) = i\}$ is a single orbit under $G(K)$.*

Proof. By **5.2** and **5.4**, we are reduced to verifying that two elements

$$m = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad \text{and} \quad m' = \begin{pmatrix} \alpha' & 0 \\ 0 & \beta' \end{pmatrix}$$

of M_K are conjugate under $G(K)$ if $(\alpha\beta)^2 = (\alpha'\beta')^2$, or, in view of the action of ε, if $\alpha\beta = \alpha'\beta'$. Since $\chi : H \to \mu_k$ is an *epimorphism* of k-sheaves (cf. **2.1**), there exists $h \in H(K)$ such that $\chi(h) = \beta'^{-1}\beta$. For such h, we have $\theta_0(h) \cdot m = m'$ (cf. **2.5** (1)).

This corollary was proved by Igusa in [Igusa 1, p.428] in the case where $\mathrm{char}(K) \neq 2, 3$.

NOTE

We expect that the orbit-sheaf $O_G(u_0)$ of u_0 (cf. **3.0**) coincides with the principal open subscheme $\mathbf{D}_+(f)$ defined by the Freudenthal quartic f (cf. **4.1**). This is the case if the condition $(**)$ in **5.4** and the following two conditions are satisfied: a) H is *flat* over k; b) G is a *scheme*. Indeed, under these conditions, G becomes algebraic and flat over k (cf. **2.1**, [LAG, 5.11 (b)]) and $G(K)$ is transitive on $\mathbf{D}_+(f)(K)$ for any algebraically closed field $K \in k$-**alg** (cf. **5.5**). Also the k-scheme $\mathbf{D}_+(f)$ is smooth (cf. **0.6**). Thus the assertion follows from [D-G, III, §3, 2.1]. Note that the above conditions a) and b) are always satisfied if k is a *field* (cf. [LAG, 5.13 (b)]).

Appendix

A Relations to McCrimmon's construction

A.1 Consider a triple $(J; N, \sharp)$ consisting of a k-module J, a cubic form $N \in \mathcal{O}^3(J)$, and a quadratic map $\sharp \in \mathcal{O}^2(J, J)$ satisfying (CJ1) in **1.1**. Let T be a symmetric bilinear form on T. We say that an element c of J is a *basepoint* of $(J; N, \sharp, T)$ if the following formulas hold:

$$\text{(BP1)} \qquad N(c) = 1,$$

$$\text{(BP2)} \qquad c^\sharp = c,$$

$$\text{(BP3)} \qquad Q(c) = \mathrm{Id},$$

where Q is the quadratic map $J \to \mathrm{End}(J)$ defined by the formula (2) in **1.1**. On the other hand, if $c \in J$, we set

$$T_c(x, y) := \partial_x N(c) \partial_y N(c) - \partial_{x \times y} N(c)$$

(cf. **0.6**) to obtain a symmetric bilinear form T_c on J. Now we consider the following two sets:

$$\mathbf{T}(J; N, \sharp) := \{T \mid \text{symmetric bilinear form on } J \text{ satisfying (CJ2)}\},$$
$$\mathbf{Mc}(J; N, \sharp) := \{c \in J \mid \text{ basepoint of } (J; N, \sharp, T) \text{ and } T_c \text{ satisfies (CJ2)}\}$$

(cf. **1.1**). Then our starting data is precisely a quadruple $(J; N, \sharp, T)$ consisting of $(J; N, \sharp)$ and $T \in \mathbf{T}(J; N, \sharp)$. On the other hand, that of McCrimmon is a quadruple $(J; N, \sharp, c)$ consisting of $(J; N, \sharp)$ and $c \in \mathbf{Mc}(J; N, \sharp)$ (cf. [Mc]). Using (BP2), (BP3), and **1.2** (1) for $x := c$, it can be seen that the map $c \mapsto (T_c, c)$ induces the *injection*

$$\mathbf{Mc}(J; N, \sharp) \hookrightarrow \mathbf{T}(J; N, \sharp) \times J,$$

whose *image* consists of (T, c)'s such that c becomes a basepoint of $(J; N, \sharp, T)$. Therefore, McCrimmon's construction may be considered as a couple of our construction and a choice of a basepoint. However the condition $(*)$ in **1.3**, which has been assumed after **1.6**, *almost* requires the *existence* of a basepoint in the following sense:

A.2 Lemma: *Let* $(J; N, \sharp, T)$ *be a quadruple as in* **1.1** *satisfying* (CJ1, 2) *and* **1.3** $(*)$. *Let* c *be an element of* J *satisfying* $N(c) \in k^*$. *Define a quadruple* $(J; N', \sharp', T')$ *by setting*

$$\text{(1)} \qquad N'(x) := N(c)N(x),$$

$$(2) \qquad x^{\sharp'} := N(c)Q(c)^{-1}x^{\sharp},$$

$$(3) \qquad T'(x,y) := T(x, Q(c)y),$$

for $x, y \in J_R, R \in k$-**alg**. *Then,* $(J; N', \sharp', T')$ *satisfies* (CJ1, 2) *and* **1.3** $(*)$*, and admits* $c^{-1} = N(c)^{-1}c^{\sharp}$ (cf. **1.6** (1)) *as a basepoint; moreover, we have*

$$(4) \qquad Q'(x) = Q(x)Q(c)$$

for $Q' : J \to \mathrm{End}(J)$ *defined by* **1.1** (2).

Proof. T is a *symmetric* bilinear form by **1.2** (2). **1.3** $(*)$ for $(J; N', \sharp', T')$ is clear from definitions, and (BP3) for c^{-1} is a consequence of (4). Thus it remains the verifications of (CJ1, 2), (BP1, 2) and (4). Take $x, y \in J$ arbitrarily. We have $Q(c)^{-1} = N(c)^{-2}Q(c^{\sharp})$ by (CJ14), so that $x^{\sharp'} = N(c)^{-1}Q(c^{\sharp})x^{\sharp} = N(c)^{-1}(Q(c)x)^{\sharp}$ by (CJ16), and hence $(x^{\sharp'})^{\sharp} = N(c)^{-2}N(Q(c)x)Q(c)x$ (by (CJ1)) $= N(x)Q(c)x$(by (CJ20)), from which (CJ1) follows. Also we have $T'(x^{\sharp'}, y) = T(N(c)Q(c)^{-1}x^{\sharp}, Q(c)y)$ (by (2), (3)) $= N(c)T(x^{\sharp}, y)$ (by **1.2** (2)) $= N(c)\partial_y N(x)$ (by (CJ2)) $= \partial_y N'(x)$ (by (1)), which is (CJ2). As for (BP1, 2), we have $N'(c^{-1}) = N(C)N(N(c)^{-1}c^{\sharp})$ (by (1) and **1.6** (1))$= N(c)N(c)^{-3}N(c)^2$ (by (CJ12))$= 1$, and $(c^{-1})^{\sharp'} = N(c)Q(C)^{-1}(N(c)^{-1}c^{\sharp})$ (by (1) and **1.6** (1)) $= N(c)Q(c)^{-1}N(c)^{-2}N(c)c$ (by (CJ1)) $= Q(c)^{-1}c = c$ (by the definition of inverse, cf. **1.6**), from which the assertion follows. To prove (4), denote by $\times'$ the linearization of $\sharp'$. Then we have

$$(2\mathrm{lin}) \qquad x \times' y = N(c)Q(c)^{-1}(x^{\sharp} \times y),$$

so that $Q'(x)y = T'(x,y)x - x^{\sharp'} \times' y$ (by **1.1** (2)) $= T(x, Q(c)y)x - N(c)Q(c)^{-1}x^{\sharp} \times' y$ (by (2), (3)) $= T(x, Q(c)y)x - N(c)^2Q(c)^{-1}((Q(c)^{-1}x^{\sharp}) \times y)$ (by (2lin)). However we have $(Q(c)^{-1}x^{\sharp}) \times y = (Q(c)^{-1}x^{\sharp}) \times (Q(c)^{-1}Q(c)y) = N(c)^{-4}(Q(c^{\sharp})x^{\sharp}) \times (Q(c^{\sharp})Q(c)y)$ (by (CJ14)) $= N(c)^{-4}Q(c^{\sharp\sharp})(x^{\sharp} \times Q(c)y)$ (by the linearized (CJ16)) $= N(c)^{-2}Q(c)(x^{\sharp} \times Q(c)y)$, from which the assertion follows.

A.3 Our modification aims to translate the McCrimmon 's result into the terminology of Jordan pairs. This makes it easy to quote theorems from [LAG]. In fact, Jordan pairs are natural objects to have many advantages; we refer to [LJP] for details.

B A class of examples

B.1 As is mentioned in Introduction, our quadruple $(J; N, \sharp, T)$ is modeled on the Jordan algebra of 3×3 *Hermitian matrices* with coefficients in a *composition algebra.* Here we give a class of examples of $(J; N, \sharp, T)$ along this line. We start with a data

$$(J_0,\ J_1;\ q,\ \overline{?},\ T_1,\ \circ,\ \sharp) \tag{1}$$

where J_0 and J_1 are k-modules, q a *quadratic form* on J_0, $\overline{?}$ an *involutive automorphism* $v \mapsto \bar{v}$ of J_0 keeping q invariant, T_1 a *symmetric bilinear form* on J_1, $\circ$ a *bilinear map* $(a, v) \mapsto a \circ v$ from $J_1 \times J_0$ to J_1, and $\sharp$ a *quadratic map* $a \mapsto a^\sharp$ from J_1 to J_0.
We linearize $\sharp$

$$a \times b := (a + b)^\sharp - a^\sharp - b^\sharp \tag{2}$$

to obtain a *commutative bilinear composition* $\times$ on J_1 with value in J_0. We construct a quadruple $(J; N, \sharp, T)$ by setting

$$J := k \oplus J_1 \oplus J_0 =: \left\{ \begin{pmatrix} \xi & a \\ * & v \end{pmatrix} \middle|\ \xi \in k,\ a \in J_1,\ v \in J_0 \right\}, \tag{3}$$

and

$$N \begin{pmatrix} \xi & a \\ * & v \end{pmatrix} := q(a^\sharp, \bar{v}) + \xi q(v), \tag{4}$$

$$\begin{pmatrix} \xi & a \\ * & v \end{pmatrix}^\sharp := \begin{pmatrix} q(v) & -a \circ \overline{v} \\ * & a^\sharp + \xi \overline{v} \end{pmatrix}, \tag{5}$$

$$T(\begin{pmatrix} \xi & a \\ * & v \end{pmatrix}, \begin{pmatrix} \eta & b \\ * & w \end{pmatrix}) := \xi\eta + T_1(a, b) + q(v, \bar{w}), \tag{6}$$

for all $R \in k$-**alg**, $\xi, \eta \in R$, $a, b \in J_{1R}$, and $v, w \in J_{0R}$. Consider the following condition on the data (1): we have

$$q(a^\sharp) = 0, \tag{MJ1}$$

$$(a \circ \overline{v}) \circ v = q(v)a, \tag{MJ2}$$

$$(a \circ \overline{v}) \circ \overline{a^\sharp} = q(a^\sharp, \overline{v})a, \tag{MJ3}$$

(MJ4) $$(a \circ \overline{v})^\sharp + q(v)\overline{a^\sharp} = q(a^\sharp, \overline{v})v,$$

(MJ5) $$q(a \times b, \overline{v}) = -T_1(a \circ \overline{v}, b),$$

for all $a, b \in J_{1R},\ v \in J_{0R},\ R \in k$-**alg**, and

($\star$) J_0 and J_1 are finitely generated projective k-modules and there exists $e_0 \in J_0$ such that $q(e_0) = 1$.

Proposition: *If the data* (1) *satisfies the conditions* (MJ1–5) *and* ($\star$), *then the quadruple* $(J; N, \sharp, T)$ *defined by* (3–6) *satisfies* (CJ1), (CJ2), *and* ($*$) *in* **1.3**.

Proof. We calculate
$$N\begin{pmatrix} 1 & 0 \\ * & e_0 \end{pmatrix} = q(e_0) = 1$$
(cf. (4), ($\star$)), and
$$\begin{pmatrix} 0 & 0 \\ * & e_0 \end{pmatrix}^\sharp = \begin{pmatrix} q(e_0) & 0 \\ * & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ * & 0 \end{pmatrix}$$
(cf. (5), ($\star$)) to obtain the condition ($*$) in **1.3**. To see the conditions (CJ1) and (CJ2), set
$$x := \begin{pmatrix} \xi & a \\ * & v \end{pmatrix}, \quad y := \begin{pmatrix} \eta & b \\ * & w \end{pmatrix},$$
with $\xi,\ \eta,\ a,\ b,\ v,\ w$ as above, and denote by $R[\varepsilon]$ the ring of dual numbers. Then we have

$$\begin{aligned}
x^{\sharp\sharp} &= \begin{pmatrix} q(v) & -a \circ \bar{v} \\ * & a^\sharp + \xi\bar{v} \end{pmatrix}^\sharp \quad \text{(by (5))} \\
&= \begin{pmatrix} q(a^\sharp + \xi\bar{v}) & (a \circ \bar{v}) \circ \overline{(a^\sharp + \xi\bar{v})} \\ * & (a \circ \bar{v})^\sharp + q(v)\overline{(a^\sharp + \xi\bar{v})} \end{pmatrix} \quad \text{(by (5))} \\
&= \begin{pmatrix} \xi q(a^\sharp, \bar{v}) + \xi^2 q(v) & q(a^\sharp, \bar{v})a + \xi q(v)a \\ * & q(a^\sharp, \bar{v})v + \xi q(v)v \end{pmatrix} \quad \text{(by (MJ1–4))} \\
&= N(x)x \quad \text{(by (4))},
\end{aligned}$$

which is (CJ1). Also we have

$$\begin{aligned}
N(x + \varepsilon y) &= N\begin{pmatrix} \xi + \varepsilon\eta & a + \varepsilon b \\ * & v + \varepsilon w \end{pmatrix} \\
&= q((a + \varepsilon b)^\sharp, \bar{v} + \varepsilon\bar{w}) + (\xi + \varepsilon\eta)q(v + \varepsilon w) \quad \text{(by (4))}
\end{aligned}$$

$$
\begin{aligned}
&= q(a^\sharp + \varepsilon a \times b, \bar v + \varepsilon \bar w) + (\xi + \varepsilon\eta)(q(v) + \varepsilon q(v,w)) \\
&= q(a^\sharp, \bar v) + \xi q(v) + \varepsilon(q(a^\sharp, \bar w) + q(a \times b, \bar v) + \xi q(v,w) + \eta q(v)) \\
&= N(x) + \\
&\qquad \varepsilon(q(a^\sharp, \bar w) - T_1(a \circ \bar v, b) + \xi q(v,w) + \eta q(v)) \quad \text{(by (4) and (MJ5))},
\end{aligned}
$$

and

$$
\begin{aligned}
T(x^\sharp, y) &= T(\begin{pmatrix} q(v) & -a \circ \bar v \\ * & a^\sharp + \xi \bar v \end{pmatrix}, \begin{pmatrix} \eta & b \\ * & w \end{pmatrix}) \quad \text{(by (5))} \\
&= \eta q(v) - T_1(a \circ \bar v, b) + q(a^\sharp + \xi \bar v, \bar w) \quad \text{(by (6))} \\
&= \eta q(v) - T_1(a \circ \bar v, b) + q(a^\sharp, \bar w) + \xi q(v,w).
\end{aligned}
$$

Hence we have $N(x + \varepsilon y) = N(x) + \varepsilon T(x^\sharp, y)$, which is (CJ2).

B.2 Consider a data $(J_0, J_1;\ q,\ \overline{?},\ T_1,\ \circ,\ \sharp)$ with the properties (MJ1–5) and $(\star)$ in **B.1**, and set

(1) $$U(v)w := q(v, \bar w)v - q(v)\bar w,$$

(2) $$Q_1(a)b := T_1(a,b)a + b \circ \overline{a^\sharp},$$

(3) $$B_0(a,b)v := (1 - T_1(a,b))v - (a \circ \bar v) \times b + q(a^\sharp, v)\overline{b^\sharp},$$

(4) $$B_1(a,b)c := c - Q_1(a,c)b + Q_1(a)Q_1(b)c,$$

(5) $$N_1(a,b) := 1 - T_1(a,b) + q(a^\sharp, \overline{b^\sharp}),$$

(6) $$P_1(a,b) := a + b \circ \overline{a^\sharp},$$

(7) $$V(v)a := a \circ v,$$

to obtain polynomial laws $U \in \mathcal{O}^2(J_0, \mathrm{End}(J_0))$, $Q_1 \in \mathcal{O}^2(J_1, \mathrm{End}(J_1))$, $B_i \in \mathcal{O}(J_1 \times J_1, \mathrm{End}(J_i))$ for $i = 0, 1$, $N_1 \in \mathcal{O}(J_1 \times J_1)$, $P_1 \in \mathcal{O}(J_1 \times J_1, J_1)$, and $V \in \mathcal{O}^1(J_0, \mathrm{End}(J_1))$. We construct a quadruple $(J; N, \sharp, T)$ as in **B.1** (3–6), which satisfies the conditions (CJ1), (CJ2), and $(*)$ in **1.3**. Also we set

(8) $$e_1 := \begin{pmatrix} 1 & 0 \\ * & 0 \end{pmatrix} \quad \text{and} \quad J_2 := \text{the submodule of } J \text{ spanned by } e_1.$$

Note that we have

$$e_1^\sharp = 0 \quad \text{and} \quad N(e_1) = 0 \tag{9}$$

by **B.1** (4) and (5). We use the symbols $x \times y$, $Q(x)y$, $N(x,y)$, $\{xyz\}$, $D(x,y)$, and $B(x,y)$, for $x, y, z \in J$, introduced in **1.1**. In particular, we linearize **B.1** (5) to obtain

$$\begin{pmatrix} \xi & a \\ * & v \end{pmatrix} \times \begin{pmatrix} \eta & b \\ * & w \end{pmatrix} = \begin{pmatrix} q(v,w) & -a \circ \bar{w} - b \circ \bar{v} \\ * & a \times b + \xi \bar{w} + \eta \bar{v} \end{pmatrix}. \tag{10}$$

Recall that we have $Q(x)y = T(x,y)x - x^\sharp \times y$ and that (J, J) together with (Q, Q) becomes a *Jordan pair* (cf. **1.4**). We often identify J_0 (resp. J_1) with the image in J. Note that this identification causes no confusion to the symbol $a^\sharp$ (cf. **B.1** (5)). In this notation, we have

$$v^\sharp = q(v)e_1, \tag{11}$$

$$e_1 \times v = \bar{v}, \tag{12}$$

$$a \times v = -a \circ \bar{v}, \tag{13}$$

$$e_1 \times a = 0, \tag{14}$$

by (10) and **B.1** (5). Also we have

$$N(a,b) = N_1(a,b), \qquad P(a,b) = P_1(a,b) \tag{15}$$

by (5), (6), (13), **B.1** (6), and **1.1** (3), (4). On the other hand, we have

$$Q(e_1)\begin{pmatrix} \xi & a \\ * & v \end{pmatrix} = \begin{pmatrix} \xi & 0 \\ * & 0 \end{pmatrix} \quad \text{and} \quad D(e_1, e_1)\begin{pmatrix} \xi & a \\ * & v \end{pmatrix} = \begin{pmatrix} 2\xi & a \\ * & 0 \end{pmatrix}$$

by straightforward calculations. In particular, we have $Q(e_1)e_1 = e_1$, $\operatorname{im} Q(e_1) = J_2$, $\ker[1 - D(e_1, e_1)] = J_1$, and $\ker Q(e_1) \cap \ker D(e_1, e_1) = J_0$. In the terminology of Jordan pairs, it can be said that (e_1, e_1) is an *idempotent* of the Jordan pair (J, J) and that (J_i, J_i) is the *Peirce-i-space* with respect to this idempotent (cf. [LJP, §5]). Thus we have the following *Peirce relations* (cf. [LJP, 5.4]):

$$\{e_1 e_1 e_1\} = 2e_1, \quad \{e_1 e_1 a\} = a, \quad \{e_1 e_1 v\} = 0, \tag{16}$$

$$\{e_1 v x\} = \{x v e_1\} = \{v e_1 x\} = 0, \tag{17}$$

$$Q(J_i)J_j \subset J_{2i-j}, \tag{18}$$

$$\{J_i\ J_j\ J_l\} \subset J_{i-j+l}, \tag{19}$$

where $a \in J_1$, $v \in J_0$, $x \in J$, and $J_n := 0$ for $n \neq 0, 1, 2$. In particular, each (J_i, J_i) inherits from (J, J) the structure of *Jordan pair*. Note that the induced quadratic operators $J_i \to \mathrm{End}(J_i)$, for $i = 0$ and 1, are precisely U and Q_1 defined in (1) and (2), respectively. In fact, using the foregoing definitions and formulas, we have the following formulas for $a, b, c \in J_1$ and $v, w \in J_0$:

$$Q(v)w = U(v)w, \tag{20}$$

$$Q(a)b = Q_1(a)b, \tag{21}$$

$$\{abe_1\} = T_1(a, b)e_1, \tag{22}$$

$$\{abv\} = T_1(a, b)v + (a \circ \bar{v}) \times b, \tag{23}$$

$$Q(a)e_1 = -\overline{a^\sharp}, \tag{24}$$

$$Q(a)v = -q(a^\sharp, v)e_1, \tag{25}$$

$$B(a, b) \begin{pmatrix} \xi & c \\ * & w \end{pmatrix} = \begin{pmatrix} N_1(a, b)\xi & B_1(a, b)c \\ * & B_0(a, b)w \end{pmatrix}, \tag{26}$$

$$Q(e_1 + v) \begin{pmatrix} \xi & c \\ * & w \end{pmatrix} = \begin{pmatrix} \xi & c \circ v \\ * & U(v)w \end{pmatrix}. \tag{27}$$

In the following, we consider (J_0, J_0) and (J_1, J_1), as well as (J, J), to be *Jordan pairs* by means of the compositions $U(v)w$ and $Q_1(a)b$, respectively.

B.3 Let us consider the Jordan pair (J_0, J_0). Note first that we have

$$U(\bar{v})U(v)w = q(v)^2 w, \tag{1}$$

$$q(U(v)w) = q(v)^2 q(w), \tag{2}$$

by **B.2** (1). These applied to $w = e_0$ in **B.1** ($\star$) tell us that v is *invertible*, i.e., the endomorphism $U(v) \in \mathrm{End}(J_0)$ is invertible ([LJP, 1.10]) if and only if v is *non-singular*, i.e., the scalar $q(v)$ is invertible. For such v, denote by s_v the *reflection* along v. By definition, this is the automorphism $w \mapsto w - q(v)^{-1}q(v,w)v$ of J_0, which is also described as

$$s_v(w) = -q(v)^{-1}U(v)\bar{w} \tag{3}$$

by **B.2** (1). On the other hand, we set $\overline{A}(w) := \overline{A(\overline{w})}$ to obtain a map $A \mapsto \overline{A}$ in $\mathrm{End}(J_0)$, which is in fact an *involutive automorphism* with respect to the k-algebra structure. By **B.2** (1), we have

$$\overline{U(v)} = U(\overline{v}). \tag{4}$$

Also we see that he orthogonal group $O(q)$ is stable under this involution, and that the pair $(h, \bar{h})$, for any $h \in O(q)$, is an automorphism of the Jordan pair (J_0, J_0) (cf. [LJP, 1.3]). Namely, we have $U(hv)\bar{h}w = hU(v)w$ and $U(\bar{h}v)hw = \bar{h}U(v)w$. In this way, we obtain a homomorphism

$$h \mapsto (h, \bar{h}) : O(q) \to \mathrm{Aut}(J_0, J_0) \tag{5}$$

of groups. Finally we observe the formula

$$q(U(v)w, w') = q(w, U(\bar{v})w'), \tag{6}$$

which can be verified by straightforward calculation using the definition **B.2** (1).

B.4 As for the Jordan pair (J_1, J_1), note first that we have

$$T_1(B_1(a,b)c, c') = T_1(c, B_1(b,a)c') \tag{1}$$

by **B.2** (26), **B.1** (6), and **1.2** (4). Also we have

$$a \circ a^\sharp = 0, \tag{2}$$

$$a \circ b^\sharp = -b \circ (a \times b), \tag{3}$$

since $a \circ a^\sharp = -a \times (a^\sharp \times e_1)$ (by **B.2** (12), (13)) $= N(a)e_1 + T(a, e_1)a^\sharp$ (by (CJ13)) $= 0$ (by **B.1** (4), (6)), and $-b \circ (a \times b) = b \times ((a \times b) \times e_1)$ (by **B.2** (12), (13)) $= T(b^\sharp, a)e_1 + T(b, e_1)b \times a + T(a, e_1)b^\sharp - (b^\sharp \times e_1) \times a$ (by (CJ15)) $= -(b^\sharp \times e_1) \times a$ (by **B.1** (6)) $= a \circ b^\sharp$ (by **B.2** (12), (13)). We denote by $\mathcal{W}_1$ the scheme of quasi-invertible pairs of

(J_1, J_1) (cf, [LJP, 3.2]). From the property [LJP, 3.2 (ii)] of quasi-inverses, it follows that $\mathcal{W}_1 \subset \mathcal{W}$, the scheme of quasi-invertible pairs of (J, J), and that quasi-inverses in (J_1, J_1) can be calculated in (J, J). The converse also holds to yield

$$\mathcal{W}_1 = (J_{1a} \times J_{1a}) \cap \mathcal{W}, \tag{4}$$

since **B.2** (26) tells us that $B(a, b) \in \mathrm{End}(J_R)$ is invertible if and only if so are $N_1(a, b) \in R$, $B_1(a, b) \in \mathrm{End}(J_{1R})$, and $B_0(a, b) \in \mathrm{End}(J_{0R})$. In view of **B.2** (15), **1.8**, and [LJP, 3.6 (JP33)], we have shown the following lemma except for the last four formulas:

Lemma: *A pair (a, b) of elements of J_1 is quasi-invertible if and only if the scalar $N_1(a, b)$ is invertible; if that is the case, the endomorphism $B_0(a, b) \in \mathrm{End}(J_0)$ is also invertible, and we have the following formulas for all $c \in J_1$ and $v \in J_0$:*

$$a^b = N_1(a, b)^{-1} P_1(a, b), \tag{5}$$

$$N_1(a, b) N_1(a^b, c) = N_1(a, b + c), \tag{6}$$

$$B_0(a, b) B_0(a^b, c) = B_0(a, b + c), \tag{7}$$

$$B_1(a, b) B_1(a^b, c) = B_1(a, b + c), \tag{8}$$

$$\overline{B_0(a, b)v} = N_1(a, b) B_0(b, a)^{-1} \cdot \overline{v}, \tag{9}$$

$$q(B_0(a, b)v) = N_1(a, b) q(v), \tag{8}$$

$$(B_1(a, b)c) \circ \overline{(B_0(a, b)v)} = N_1(a, b)^2 B_1(b, a)^{-1} \cdot (c \circ \overline{v}), \tag{11}$$

$$(B_1(a, b)c)^\sharp = N_1(a, b)^2 B_0(b, a)^{-1} \cdot c^\sharp. \tag{12}$$

As for the formulas (9–12), we apply $\sharp$ to **B.2** (26). After expanding the right-hand side (resp. left-hand side) by means of **B.1** (5) (resp. (CJ25), **B.1** (5), **B.2** (26), (15)), and comparing the each components with suitable specializations, we get the assertions.

B.5 *Basic identities.* We deduce some identities. Assume that all the conditions in **B.1** are satisfied, and let $a, b \in J_{1R}, v, w \in J_{0R}, R \in k$-**alg**. We linearize (MJ2) to obtain

$$(a \circ \bar{v}) \circ w + (a \circ \bar{w}) \circ v = q(v, w)a. \tag{MJ6}$$

Then we calculate $(a \circ \bar{v}) \circ (U(v)w) = q(v, \bar{w})(a \circ \bar{v}) \circ v - q(v)(a \circ \bar{v}) \circ \bar{w}$ (by **B.2** (1)) $= q(v)(q(v, \bar{w})a - (a \circ \bar{v}) \circ \bar{w})$ (by (MJ2)) $= q(v)(a \circ w) \circ v$ (by (MJ6)). Namely

$$\text{(MJ7)} \qquad (a \circ \bar{v}) \circ (U(v)w) = q(v)(a \circ w) \circ v.$$

Replacing a by $a \circ v$ in (MJ7) with (MJ2) in mind, we get $q(v)a \circ (U(v)w) = q(v)((a \circ v) \circ w) \circ v$. However, by **0.2** and **B.1** ($\star$), the principal open subscheme of J_{0a} defined by the section $q : J_{0a} \to \mathbf{O}_k$ is *dense* in J_{0a}. Thus we conclude

$$\text{(MJ8)} \qquad a \circ (U(v)w) = ((a \circ v) \circ w) \circ v.$$

In terms of **B.2** (1), the identity (MJ4) can be read as

$$\text{(MJ4bis)} \qquad (a \circ \bar{v})^\sharp = U(v) \cdot a^\sharp,$$

whose linearization yields

$$\text{(MJ9)} \qquad (a \circ \bar{v}) \times (b \circ \bar{v}) = U(v)(a \times b).$$

Now we calculate $q(v)^2 T_1(a, b) = T_1((a \circ v) \circ \bar{v}, (b \circ \bar{v}) \circ v))$ (by (MJ2)) $= -q((a \circ v) \times ((b \circ \bar{v}) \circ v), \bar{v})$ (by (MJ5)) $= -q(U(\bar{v})(a \times (b \circ \bar{v})), \bar{v})$ (by (MJ9)) $= -q(a \times (b \circ \bar{v}), U(v)\bar{v})$ (by **B.3** (6)) $= -q(v)q(a \times (b \circ \bar{v}), v)$ (by **B.2** (1)) $= q(v)T_1(a \circ v, b \circ \bar{v})$ (by (MJ5)), from which we get

$$\text{(MJ10)} \qquad T_1(a \circ v, b \circ \bar{v}) = q(v)T_1(a, b).$$

Similarly, we calculate $q(v)a \times (b \circ v) = ((a \circ \bar{v}) \circ v) \times (b \circ v)$ (by (MJ2)) $= U(\bar{v})((a \circ \bar{v}) \times b)$ (by (MJ9)) $= q((a \circ \overline{v}) \times b, v)\overline{v} - q(v)\overline{(a \circ \overline{v}) \times b}$ (by (MJ5)) $= -q(v)(T_1(a, b)\overline{v} + \overline{(a \circ \overline{v}) \times b})$ (by (MJ2)), from which we get

$$\text{(MJ11)} \qquad a \times (b \circ v) + \overline{(a \circ \overline{v}) \times b} = -T_1(a, b)\overline{v}.$$

Finally we observe the formula

$$\text{(MJ12)} \qquad (a - (a \circ w) \circ \bar{v})^\sharp = B(v, \bar{w}) \cdot a^\sharp.$$

Indeed, the left-hand side equals $a^\sharp - a \times ((a \circ w) \circ \bar{v}) + ((a \circ w) \circ \bar{v})^\sharp$ by **B.1** (1), and we have $((a \circ w) \circ \bar{v})^\sharp = U(v)U(\bar{w}) \cdot a^\sharp = Q(v)Q(\bar{w}) \cdot a^\sharp$ by (MJ4bis) and **B.2** (20). As for $a \times ((a \circ w) \circ \bar{v})$, we calculate $a \times ((a \circ w) \circ \bar{v}) = a \times ((a \times \bar{w}) \times v)$ (by **B.2** (13)) $= T(a^\sharp, \bar{w})v + T(a, v)a \times \bar{w} + T(v, \bar{w})a^\sharp - \bar{w} \times (a^\sharp \times v)$ (by (CJ15)) $= T(a^\sharp, \bar{w})v + T(v, \bar{w})a^\sharp - \bar{w} \times (a^\sharp \times v)$ (by **B.1** (6)) $= \{v\bar{w}a^\sharp\}$ (by **1.1** (5)). This shows our assertion, in view of **1.1** (6).

B.6 Assume that the following condition $(\star\star)$, which is stronger than $(\star)$ in **B.1**, is satisfied:

$(\star\star)$ The k-modules J_0 and J_1 are finitely generated projective and there exists $e_0 \in J_0$ such that $q(e_0) = 1, \bar{v} = q(e_0, v)e_0 - v$, and $a \circ e_0 = a$, for $v \in J_0, a \in J_1$.

Then, we have

$$U(e_0)v = v, \tag{1}$$

$$e_0^\sharp = e_1, \quad \text{namely} \quad \begin{pmatrix} 0 & 0 \\ * & e_0 \end{pmatrix}^\sharp = \begin{pmatrix} 1 & 0 \\ * & 0 \end{pmatrix}, \tag{2}$$

$$\begin{pmatrix} \xi & a \\ * & v \end{pmatrix} \times e_0 = \begin{pmatrix} q(v, e_0) & -a \\ * & \xi e_0 \end{pmatrix}, \tag{3}$$

$$Q(e_0)\begin{pmatrix} \xi & a \\ * & v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ * & v \end{pmatrix} \quad \text{and} \quad D(e_0, e_0)\begin{pmatrix} \xi & a \\ * & v \end{pmatrix} = \begin{pmatrix} 0 & a \\ * & 2v \end{pmatrix}, \tag{4}$$

$$\{ae_0v\} = a \circ v = V(v)a, \tag{5}$$

by straightforward calculations. In the terminology of Jordan pairs, it can be said that:

a) e_0 is an *invertible element* of the Jordan pair (J_0, J_0) and the associated Jordan algebra (cf. [LJP, 1.9]) is precisely the *Jordan algebra of the quadratic form q with the base point* e_0 (cf. [Jac 1, I-§5]).

b) (e_0, e_0) is an *idempotent* of the Jordan pair (J, J) with *Peirce-i-space* (J_{2-i}, J_{2-i}) (cf. [LJP, 5.4]), and the Peirce-2-space, considered as a Jordan algebra (cf. [LJP, 5.5]), is precisely J_0 with the structure mentioned in **a)**.

Also, by setting

$$e := e_1 + e_0 = \begin{pmatrix} 1 & 0 \\ * & e_0 \end{pmatrix} \in J, \tag{6}$$

we have

$$N(e) = 1, \tag{7}$$

$$e^\sharp = e, \tag{8}$$

$$(9) \qquad Q(e) = \mathrm{Id},$$

$$(10) \qquad a \circ v = \{aev\},$$

by straightforward calculation. In the terminology of Jordan pairs, it can be said that $e = e_1 + e_0$ is an *invertible element* of the Jordan pair (J, J) and that the associated Jordan algebra (cf. [LJP, 1.9]) is the J with U-operator $U_x y := Q(x)y$, with squaring $x^2 := Q(x)e$, and with unit e. Recall the *circle product* in a *Jordan algebra* which is the linearization $x \circ y := (x+y)^2 - x^2 - y^2$ of the squaring. In our situation, we have $x \circ y = \{xey\}$. Thus (10) tells us that our circle product $(a, v) \mapsto a \circ v : J_1 \times J_0 \to J_1$ coincides with the one induced by this Jordan algebra structure.

B.7 An example of such a data $(J_0, J_1, ; q, \overline{?}, T_1, \circ, \sharp)$ as in **B.1** can be constructed from an *alternative Cayley algebra* $\mathcal{C}$, in the sense of [Bou, Alg. III, App., no 2], whose underlying k-module is *faithfully projective*, in the sense of [Bass, II, §5]. We first recall some basic facts on such algebras. Recall that a *Cayley algebra* over k is an *unitary* k-algebra equipped with an *anti-automorphism* $x \mapsto \tilde{x}$, called the *conjugation*, such that all the $x + \tilde{x}$ and $x\tilde{x}$ belong to the image of k (cf. [Bou, Alg. III, §2, no 4]). Also recall that a k-algebra is said to be *alternative* if every subalgebra having *two* generators is *associative* (cf. [Bou, Alg. III, App., no 1]), and that a k-module is said to be *faithfully projective* if it is finitely generated projective and faithful (cf. [Bass, II-5.10]).

Let $\mathcal{C}$ be an alternative Cayley algebra which is faithfully projective as a k-module. Since $x + \tilde{x}$ belongs to the *center* of $\mathcal{C}$, we have $x\tilde{x} = \tilde{x}x$. Also the structural homomorphism $k \to \mathcal{C}$ sending λ to $\lambda \cdot 1_{\mathcal{C}}$ is an *isomorphism* onto a *direct factor* (cf. [Bass, III-2.17], whose proof works regardless of associativity). Therefore we obtain two k-valued functions t and n, called *trace* and *norm*, respectively, by writing

$$(1) \qquad x + \tilde{x} = t(x) \cdot 1_{\mathcal{C}} \quad \text{and} \quad x\tilde{x} = \tilde{x}x = n(x) \cdot 1_{\mathcal{C}}.$$

By constructions, t is a *linear form* and n a *quadratic form.* We have

$$(2) \qquad n(x, y) = t(x\tilde{y}) = x\tilde{y} + y\tilde{x}$$

by (1). Also we have

$$(3) \qquad t(xy) = t(yx)$$

by the general property of Cayley algebras (cf. [Bou, Alg. III, §2, no 4, (17)]). On the other hand, x, y and $1_{\mathcal{C}}$ generate an *associative* subalgebra, since $\mathcal{C}$ is *alternative* (see the proof of [Bou, Alg. III, App., no 2, Prop. 2]). Hence we have

$$(4) \qquad \tilde{x}(xy) = (y\tilde{x})x = n(x)y,$$

$$n(xy) = n(x)n(y). \tag{5}$$

Also we recall the formula

$$t(x(yz)) = t((xy)z) \tag{6}$$

(cf. [Bou, Alg. III, App., Exercice 1]) and the *Moufang's identity*

$$(xy)(zx) = x(yz)x. \tag{7}$$

The proof of (7) can be seen, for example, in [Jac 3, p.16]. We deduce some formulas for later use. We linearize (4) to obtain

$$\tilde{x}(zy) + \tilde{z}(xy) = (y\tilde{x})z + (y\tilde{z})x = n(x,z)y. \tag{8}$$

Using (2), (3), and (6), we have

$$n(x, yz) = n(x\tilde{z}, y) = n(\tilde{y}x, z). \tag{9}$$

Finally we have

$$n(x)yz + (x\tilde{z})(\tilde{y}x) = n(x, yz)x, \tag{10}$$

since $n(x)yz + (x\tilde{z})(\tilde{y}x) = ((yz)\tilde{x})x + (x(yz)^{\sim})x$ (by (4) and (7)) $= n(x, yz)x$ (by (2)).

B.8 Let $\mathcal{C}$ be an alternative Cayley algebra which is faithfully projective as a k-module. We use the symbols $\tilde{x}$, $t(x)$, and $n(x)$ as in **B.7**. Also we denote by $M_{p,q}(\mathcal{C})$ the k-module of $p \times q$ matrices with coefficients in $\mathcal{C}$, and by $H_p(\mathcal{C})$ the submodule of $M_{p,p}(\mathcal{C})$ consisting of *Hermitian* matrices with diagonal entries in k. We construct a data

$$(J_0, J_1;\ q,\ \overline{?},\ T_1, \circ, \sharp) \tag{1}$$

as in **B.1** by setting

$$J_0 := H_2(\mathcal{C}) =: \left\{ v = \begin{pmatrix} \xi_2 & x_3 \\ \tilde{x}_3 & \xi_3 \end{pmatrix} \;\middle|\; \xi_2, \xi_3 \in k,\ x_3 \in \mathcal{C} \right\}, \tag{2}$$

$$J_1 := M_{1,2}(\mathcal{C}) =: \{a = (x_1, x_2) \mid x_1, x_2 \in \mathcal{C}\}, \tag{3}$$

$$q \begin{pmatrix} \xi_2 & x_3 \\ \tilde{x}_3 & \xi_3 \end{pmatrix} := \xi_2\xi_3 - n(x_3), \tag{4}$$

$$\begin{pmatrix} \xi_2 & x_3 \\ \tilde{x}_3 & \xi_3 \end{pmatrix}^{-} := \begin{pmatrix} \xi_3 & -x_3 \\ -\tilde{x}_3 & \xi_2 \end{pmatrix}, \tag{5}$$

$$T_1((x_1, x_2),\, (y_1, y_2)) := n(x_1, y_1) + n(x_2, y_2), \tag{6}$$

$$(x_1, x_2) \circ \begin{pmatrix} \xi_2 & x_3 \\ \tilde{x}_3 & \xi_3 \end{pmatrix} := (\xi_2 x_1 + x_2 \tilde{x}_3, x_1 x_3 + \xi_3 x_2), \tag{7}$$

$$(x_1, x_2)^\sharp := \begin{pmatrix} -n(x_2) & \tilde{x}_1 x_2 \\ \tilde{x}_2 x_1 & -n(x_1) \end{pmatrix}. \tag{8}$$

By definitions, we have

$$q\left(\begin{pmatrix} \xi_2 & x_3 \\ \tilde{x}_3 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_2 & y_3 \\ \tilde{y}_3 & \eta_3 \end{pmatrix}^{-}\right) = \xi_2 \eta_2 + \xi_3 \eta_3 + n(x_3, y_3), \tag{9}$$

$$(x_1, x_2) \times (y_1, y_2) = \begin{pmatrix} -n(x_2, y_2) & \tilde{x}_1 y_2 + \tilde{y}_1 x_2 \\ \tilde{x}_2 y_1 + \tilde{y}_2 x_1 & -n(x_1, y_1) \end{pmatrix}. \tag{10}$$

Proposition: *The data* (1) *satisfies the conditions* (MJ1–5) *in* **B.1** *and* ($\star\star$) *in* **B.6**; *moreover this satisfies the following condition:*

($\star\star\star$) *The set* $\{a^\sharp \mid a \in J_1\}$ *generates the* k*-module* J_0.

Proof. Indeed, the element

$$e_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in J_0 \tag{11}$$

satisfies the condition ($\star\star$) in **B.6**, and ($\star\star\star$) follows from the formula

$$\begin{pmatrix} \xi_2 & x_3 \\ \tilde{x}_3 & \xi_3 \end{pmatrix} = -\xi_2 (0, 1_{\mathcal{C}})^\sharp - \xi_3 (1_{\mathcal{C}}, 0)^\sharp + (1_{\mathcal{C}}, x_3)^\sharp - (0, x_3)^\sharp - (1_{\mathcal{C}}, 0)^\sharp,$$

which is a consequence of the definition (8). To see (MJ1–5), let

$$v = \begin{pmatrix} \xi_2 & x_3 \\ \tilde{x}_3 & \xi_3 \end{pmatrix}, \quad a = (x_1, x_2), \quad b = (y_1, y_2)$$

with $\xi_i \in R$ and $x_i, y_i \in \mathcal{C}_R,\ R \in k$-**alg**. Then we have

$$\begin{aligned} q(a^\sharp) &= q\begin{pmatrix} -n(x_2) & \tilde{x}_1x_2 \\ \tilde{x}_2x_1 & -n(x_1) \end{pmatrix} \quad \text{(by (8))} \\ &= n(x_2)n(x_1) - n(\tilde{x}_1x_2) \quad \text{(by (4))} \\ &= 0 \quad \text{((by }\mathbf{B.7}\text{ (5))},\end{aligned}$$

and

$$\begin{aligned} (a\circ\bar{v})\circ v &= ((x_1,x_2)\circ\begin{pmatrix} \xi_3 & -x_3 \\ -\tilde{x}_3 & \xi_2 \end{pmatrix})\circ\begin{pmatrix} \xi_2 & x_3 \\ \tilde{x}_3 & \xi_3 \end{pmatrix} \quad \text{(by (5))} \\ &= (\xi_3x_1 - x_2\tilde{x}_3, -x_1x_3+\xi_2x_2)\circ\begin{pmatrix} \xi_2 & x_3 \\ \tilde{x}_3 & \xi_3 \end{pmatrix} \quad \text{(by (7))} \\ &= (\xi_2\xi_3x_1 - n(x_3)x_1, -n(x_3)x_2+\xi_2\xi_3x_2) \quad \text{(by (7) and }\mathbf{B.7}\text{ (4))} \\ &= q(v)a \quad \text{(by (3))},\end{aligned}$$

which are (MJ1) and (MJ2), respectively. To prove (MJ3) and (MJ4), we compute

$$\begin{aligned} q(a^\sharp,\bar{v}) &= q(\begin{pmatrix} -n(x_2) & \tilde{x}_1x_2 \\ \tilde{x}_2x_1 & -n(x_1) \end{pmatrix}, \overline{\begin{pmatrix} \xi_2 & x_3 \\ \tilde{x}_3 & \xi_3 \end{pmatrix}}) \quad \text{(by (8))} \\ &= -n(x_2)\xi_2 - n(x_1)\xi_3 + n(\tilde{x}_1x_2, x_3) \quad \text{(by (9))},\end{aligned}$$

$$\begin{aligned} &(a\circ\bar{v})\circ\overline{a^\sharp} \\ &= ((x_1,x_2)\circ\begin{pmatrix} \xi_3 & -x_3 \\ -\tilde{x}_3 & \xi_2 \end{pmatrix}\circ\begin{pmatrix} -n(x_1) & -\tilde{x}_1x_2 \\ -\tilde{x}_2x_1 & -n(x_2) \end{pmatrix} \quad \text{(by (5) and (8))} \\ &= (\xi_3x_1 - x_2\tilde{x}_3, -x_1x_3+\xi_2x_2)\circ\begin{pmatrix} -n(x_1) & -\tilde{x}_1x_2 \\ -\tilde{x}_2x_1 & -n(x_2) \end{pmatrix} \quad \text{(by (7))} \\ &= \begin{matrix} (-\xi_3n(x_1)x_1 + n(x_1)x_2\tilde{x}_3 + (x_1x_3)(\tilde{x}_2x_1) - \xi_2n(x_2)x_1, \\ -\xi_3n(x_1)x_2 + (x_2\tilde{x}_3)(\tilde{x}_1x_2) + n(x_2)x_1x_3 - \xi_2n(x_2)x_2) \end{matrix} \quad \text{(by (7) and }\mathbf{B.7}\text{ (4))} \\ &= \begin{matrix} (-\xi_3n(x_1)x_1 + n(x_1,x_2\tilde{x}_3)x_1 - \xi_2n(x_2)x_1, \\ -\xi_3n(x_1)x_2 + n(x_2,x_1x_3)x_2 - \xi_2n(x_2)x_2), \end{matrix} \quad \text{(by }\mathbf{B.7}\text{ (10))}\end{aligned}$$

and

$$\begin{aligned} &(a\circ\bar{v})^\sharp + q(v)\overline{a^\sharp} \\ &= (\xi_3x_1 - x_2\tilde{x}_3, -x_1x_3+\xi_2x_2)^\sharp + (\xi_2\xi_3 - n(x_3))\begin{pmatrix} -n(x_1) & -\tilde{x}_1x_2 \\ -\tilde{x}_2x_1 & -n(x_2) \end{pmatrix} \\ &\qquad \text{(by (7), (4), (8) and (5))} \\ &= \begin{pmatrix} \xi_2' & x_3' \\ \tilde{x}'_3 & \xi_3' \end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}
\xi_2' &= -n(-x_1x_3+\xi_2x_2)-(\xi_2\xi_3-n(x_3))n(x_1)\\
&= \xi_2 n(x_1x_3,x_2)-\xi_3^2 n(x_2)-\xi_2\xi_3 n(x_1) \quad (\text{by } \mathbf{B.7}\ (5))\\
&= \xi_2(n(x_1x_3,x_2)-\xi_2 n(x_2)-\xi_3 n(x_1)),\\
x_3' &= (\xi_3\tilde{x}_1-x_3\tilde{x}_2)(-x_1x_3+\xi_2x_2)-(\xi_2\xi_3-n(x_3))\tilde{x}_1x_2\\
&= -\xi_3 n(x_1)x_3+(x_3\tilde{x}_2)(x_1x_3)-\xi_2 n(x_2)x_3+n(x_3)\tilde{x}_1x_2 \quad (\text{by } \mathbf{B.7}\ (4))\\
&= (-\xi_3 n(x_1)+n(x_3,\tilde{x}_1x_2)-\xi_2 n(x_2))x_3 \quad (\text{by } \mathbf{B.7}\ (10)),
\end{aligned}$$

and

$$\begin{aligned}
\xi_3' &= -n(\xi_3x_1-x_2\tilde{x}_3)-(\xi_2\xi_3-n(x_3))n(x_2)\\
&= -\xi_3^2 n(x_1)-\xi_3 n(x_1,x_2\tilde{x}_3)-\xi_2\xi_3 n(x_2) \quad (\text{by } \mathbf{B.7}\ (5))\\
&= \xi_2(-\xi_3 n(x_1)+n(x_1,x_2\tilde{x}_3)-\xi_2 n(x_2)).
\end{aligned}$$

However we have $n(\tilde{x}_1x_2,x_3)=n(x_2,x_1x_3)=n(x_2\tilde{x}_3,x_1)$ and $n(x_3,\tilde{x}_1x_2)=n(x_1x_3,x_2)=n(x_1,x_2\tilde{x}_3)$ by **B.7** (9). Thus we get (MJ3) and (MJ4). Finally, we compute

$$\begin{aligned}
q(a\times b,\bar{v}) &= q(\begin{pmatrix} -n(x_2,y_2) & \tilde{x}_1y_2+\tilde{y}_1x_2 \\ \tilde{x}_2y_1+\tilde{y}_2x_1 & -n(x_1,y_1) \end{pmatrix}, \begin{pmatrix} \xi_2 & x_3 \\ \tilde{x}_3 & \xi_3 \end{pmatrix}^{-}) \quad (\text{by } (10))\\
&= -\xi_2 n(x_2,y_2)-\xi_3 n(x_1,y_1)+n(\tilde{x}_1y_2,x_3)+n(\tilde{y}_1x_2,x_3) \quad (\text{by } (9))
\end{aligned}$$

and

$$\begin{aligned}
-T_1(a\circ\bar{v},b) &= -T_1((x_1,x_2)\circ \begin{pmatrix} \xi_3 & -x_3 \\ -\tilde{x}_3 & \xi_2 \end{pmatrix},(y_1,y_2)) \quad (\text{by } (5))\\
&= -T_1((\xi_3x_1-x_2\tilde{x}_3,-x_1x_3+\xi_2x_2),(y_1,y_2)) \quad (\text{by } (7))\\
&= -n(\xi_3x_1-x_2\tilde{x}_3,y_1)-n(-x_1x_3+\xi_2x_2,y_2) \quad (\text{by } (6))\\
&= -\xi_3 n(x_1,y_1)+n(x_2\tilde{x}_3,y_1)+n(x_1x_3,y_2)-\xi_2 n(x_2,y_2).
\end{aligned}$$

However we have $n(x_2\tilde{x}_3,y_1)=n(x_2,y_1x_3)=n(\tilde{y}_1x_2,x_3)$ and $n(x_1x_3,y_2)=n(x_3,\tilde{x}_1y_2)$ by **B.7** (9). Thus we get (MJ5).

B.9 In the situation of **B.8**, the k-module $J=k\oplus J_1\oplus J_0$ becomes precisely $H_3(\mathcal{C})$ after we make identification

$$\begin{pmatrix} \xi_1 & x_1 & x_2 \\ \tilde{x}_1 & \xi_2 & x_3 \\ \tilde{x}_2 & \tilde{x}_3 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & (x_1,x_2) \\ * & \begin{pmatrix} \xi_2 & x_3 \\ \tilde{x}_3 & \xi_3 \end{pmatrix} \end{pmatrix}$$

of elements. In particular, J carries a *bilinear* composition xy calculated by the *matrix product*. Direct calculation shows that $Q(x)e$, where

$$e := e_1 + e_0 = \begin{pmatrix} 1 & 0 \\ * & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(cf. **B.8** (11), **B.6** (6)), coincides with the square of x with respect to the matrix product. This result can be read as follows: Assume that we are in the *classical* situation where 2 is invertible over k. Then $J = H_3(\mathcal{C})$ becomes a *linear* Jordan algebra by means of the *bilinear* composition

$$x \cdot y := \frac{1}{2}(xy + yx). \tag{1}$$

Since we have $Q(x)e = x \cdot x$, the circle product $x \circ y$, which is the linearization $Q(x, y)e$ of $Q(x)e$, coincides with $2x \cdot y$. On the other hand, we have $2Q(x)y = x \circ (x \circ y) - x^2 \circ y$ by **B.6** and [Jac 1, I-§3, QJ 20]. Thus we get the formula

$$Q(x)y = 2x \cdot (x \cdot y) - x^2 \cdot y, \tag{2}$$

which recovers the quadratic composition by means of the classical Jordan product.

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Mathematical Institute
Tohoku University
Sendai 980-77
Japan

E-mail address: ikai@math.tohoku.ac.jp