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Isometric pluriharmonic immersions of Kähler manifolds into semi-Euclidean spaces

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ISOMETRIC PLURIHARMONIC IMMERSIONS OF KÄHLER MANIFOLDS INTO SEMI-EUCLIDEAN SPACES

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Abstract

In this thesis, we prove a classification of isometric pluriharmonic immersions of a Kähler manifold into a semi-Euclidean space, which establishes a generalization of Calabi-Lawson's theory concerning minimal surfaces in Euclidean spaces. Then we study these immersions for complete Kähler manifolds with low codimensions, and prove, in particular, a cylinder theorem and a Bernstein property. Moreover, we construct new examples of isometric pluriharmonic immersions.

Contents

1. Introduction

It has been a fundamental problem in the theory of minimal surfaces to determine the moduli spaces of those surfaces isometric to a given one. An answer to this problem was given in 1968 by E. Calabi [6] (See also H. B. Lawson [22]), who proved that the moduli space of isometric minimal immersions of a simply connected Riemann surface into a Euclidean space can be explicitly constructed as a set of certain complex matrices.

To be more precise, let M be a simply connected Riemann surface with a local complex coordinate z, and $f : M \to \mathbf{R}^3$ an isometric minimal immersion, that is, an isometric immersion of vanishing mean curvature of M into Euclidean 3-space $\mathbf{R}^3$. Since f gives rise to an isometric harmonic immersion, f is represented as $f = \sqrt{2}\,\mathrm{Re}\,\Phi$, where Φ is a holomorphic map from M into complex Euclidean 3-space $\mathbf{C}^3$ that satisfies the isotropic condition : $\langle \partial\Phi/\partial z, \partial\Phi/\partial z\rangle = 0$. We remark that $\Phi : M \to \mathbf{C}^3$ is an isometric immersion as well, and can be obtained by the well-known Weierstrass representation formula (See H. B. Lawson [22] or M. Spivak [26]). It is then proved by E. Calabi [5] that such isometric holomorphic immersions have rigidity, which means that for any two isometric holomorphic immersions Φ and Φ_0, there exists a unitary transformation U of $\mathbf{C}^3$ such that $\Phi = U\Phi_0$. Hence, if we fix an isometric holomorphic immersion Φ_0, then each isometric minimal immersion $f = \sqrt{2}\,\mathrm{Re}\,\Phi$ above is described in terms of the unitary transformation U. As a consequence, the conditions for two isometric minimal immersions $f_1 = \sqrt{2}\,\mathrm{Re}\,U_1\Phi_0$ and $f_2 = \sqrt{2}\,\mathrm{Re}\,U_2\Phi_0$ to be congruent, that is, they differ only by an isometry of $\mathbf{R}^3$, are also determined in terms of U_j. In summary, through this procedure we can obtain a parametrization of the congruence classes of minimal surfaces in $\mathbf{R}^3$ which are isometric to a given one, by a set of certain complex matrices.

In connection with the theory of relativity in physics, it has been an important subject to study spacelike surfaces of vanishing mean curvature in Minkowski 3-space $\mathbf{R}^3_1$. A surface in Minkowski 3-space $\mathbf{R}^3_1$ is said to be spacelike if the induced metric on it is positive definite. In this thesis, spacelike surfaces of vanishing mean curvature are referred to as spacelike minimal surfaces in $\mathbf{R}^3_1$, although they are usually called maximal surfaces in the literature. We note that each spacelike minimal surface in $\mathbf{R}^3_1$ is regarded as an isometric minimal immersion of a Riemann surface with positive definite Kähler metric into $\mathbf{R}^3_1$. Moreover, it should be remarked that such a Riemann surface has non-negative curvature, which contrasts with the fact that a Riemann surface isometrically and minimally immersed in $\mathbf{R}^3$ has non-positive curvature.

As in the case of minimal surfaces in $\mathbf{R}^3$, an isometric minimal immersion of a Riemann surface into $\mathbf{R}^3_1$ gives rise to a harmonic immersion. Moreover, a Weierstrass-type representation formula has been recently proved by O. Kobayashi [21] for such immersions. Based on these facts, it seems very plausible that fundamental methods for studying minimal surfaces in $\mathbf{R}^3$ can also work effectively for spacelike minimal surfaces in $\mathbf{R}^3_1$.

In this regard, we also recall that complex analysis has been a most essential ingredient in the research on minimal surfaces in Euclidean space

$\mathbf{R}^3$, as well as in Minkowski space $\mathbf{R}^3_1$. Therefore, when generalizing the theory of these surfaces to higher dimensions, it is natural to assume that a source domain has a Kähler structure. In the present thesis, we will in fact prove that the theory can be generalized to isometric pluriharmonic immersions of higher dimensional Kähler manifolds into real semi-Euclidean spaces. Here, following M. Dajczer and D. Gromoll [10], we say that an isometric immersion of a Kähler manifold is *pluriharmonic* if the $(1,1)$-component of the complexified second fundamental form of the immersion vanishes identically (Definition 2.3.1). It should be remarked that pluriharmonicity coincides with minimality of immersions provided the source Kähler manifold is complex one-dimensional. Moreover, it is immediate from the definition that any isometric pluriharmonic immersion is minimal. Conversely, we can prove that an isometric minimal immersion of a complex m-dimensional Kähler manifold into semi-Euclidean space $\mathbf{R}^{N+P}_N$ is pluriharmonic whenever $N = 0$ or $P = 2m$ (Proposition 2.3.4).

It has been shown by M. Dajczer and D. Gromoll, and the author that the geometry of isometric pluriharmonic immersions of Kähler manifolds into Euclidean spaces has many properties in common with that of minimal surfaces.

For instance, M. Dajczer and D. Gromoll [10] proved that for an isometric pluriharmonic immersion $f : M \to \mathbf{R}^P$ of a simply connected Kähler manifold M into Euclidean space $\mathbf{R}^P$, there exists an isometric holomorphic immersion $\Phi : M \to \mathbf{C}^P$ such that $f = \sqrt{2}\,\mathrm{Re}\,\Phi$. This result is further generalized by the author [19] to the case of semi-Euclidean ambient spaces (Proposition 2.3.7).

On the other hand, the author [18, 19] generalizes Calabi's classification theorem mentioned above. Namely, he has constructed a parametrization of the moduli space of full isometric pluriharmonic immersions of a simply connected Kähler manifold into a semi-Euclidean space, which is described in terms of certain complex matrices determined by a full isometric holomorphic immersion of the Kähler manifold into a complex semi-Euclidean space. More precisely, we have

Theorem 3.1.5. *Let M be a simply connected Kähler manifold and $\Phi : M \to \mathbf{C}_n^{n+p}$ a full isometric holomorphic immersion of M into $\mathbf{C}_n^{n+p}$, the complex semi-Euclidean space of dimension $n+p$ with index n. Then the set of congruence classes of full isometric pluriharmonic immersions of M into $\mathbf{R}_N^{N+P}$, the real semi-Euclidean space of dimension $N+P$ with index N, has a bijective correspondence with the set of $(n+p)\times(n+p)$-complex matrices satisfying the following conditions* (P1) – (P4)*:*

$$\text{(P1)} \qquad \frac{{}^t\partial\Phi}{\partial z^\alpha} P \frac{\partial\Phi}{\partial z^\beta} = 0 \quad (\alpha, \beta = 1, \dots, m),$$

$$\text{(P2)} \qquad {}^tP = P,$$

$$\text{(P3)} \qquad {}^*x_-(1_{np} - {}^tP1_{np}\overline{P})x_- \leq 0$$

$$\textit{for } x_- = {}^t(x_1, \dots, x_n, \underbrace{0, \dots, 0}_{p}),$$

$$\text{(P3)} \qquad {}^*x_+(1_{np} - {}^tP1_{np}\overline{P})x_+ \geq 0$$

$$\textit{for } x_+ = {}^t(\underbrace{0, \dots, 0}_{n}, x_{n+1}, \dots, x_{n+p}),$$

$$\text{(P4)} \qquad \textit{sign}(1_{np} - {}^tP1_{np}\overline{P}) = (N-n, P-p),$$

where $(z^1, \dots, z^m)$ is a local complex coordinate of M, and (P4) *means* *that the Hermitian matrix* $1_{np} - {}^tP1_{np}\overline{P}$ *has* $N-n$ *negative eigenvalues and* $P-p$ *positive eigenvalues.*

The global geometry of isometric pluriharmonic immersions has been studied by K. Abe, M. Dajczer and L. Rodriguez, and others in the case when real codimensions are one or two.

For instance, K. Abe [2] proved that an isometric pluriharmonic immersion of a complete Kähler manifold into a Euclidean space with real codimension one is a cylinder (Proposition 4.1.11). This asserts that the study of isometric pluriharmonic immersions of a complete Kähler manifold into a Euclidean space with real codimension one can be reduced to that of minimal surfaces in $\mathbf{R}^3$.

On the other hand, when the ambient space is an indefinite Euclidean space of real codimension one, S. -Y. Cheng and S. -T. Yau [8] proved that an isometric minimal immersion of a d-dimensional complete Riemannian manifold into $\mathbf{R}_1^{d+1}$ is totally geodesic. It then follows from these two results that, in the case of real codimension one, isometric pluriharmonic immersions of complete Kähler manifolds into semi-Euclidean spaces are simple.

In their paper [14], M. Dajczer and L. Rodriguez classified isometric pluriharmonic immersions with real codimension two of complete Kähler manifolds into Euclidean spaces in terms of the index of relative nullity, that is, the dimension of the kernel of the shape operator (Definition 4.1.1 and Proposition 4.1.14). However, it was left as an open problem to find nontrivial examples of these immersions, that is, whether there exists a non-holomorphic pluriharmonic immersion of a complete

Kähler manifold into a Euclidean space which is not a cylinder. The first affirmative answer to this problem is given by the author [18] by constructing explicitly such an immersion. As far as the author knows, examples of such immersions had not been previously obtained even locally. Subsequently, M. Dajczer and D. Gromoll [12] have also obtained many examples. In this thesis, we will also construct several examples of pluriharmonic immersions into indefinite Euclidean spaces (See 3.2).

In the case that the ambient spaces are indefinite Euclidean spaces with real codimensions greater than one, the following results are obtained.

Generalizing the result due to S. -Y. Cheng and S. -T. Yau mentioned above, T. Ishihara [20] proved that an isometric minimal immersion of a d-dimensional complete Riemannian manifold into $\mathbf{R}_N^{d+N}$ is totally geodesic (Proposition 4.2.1).

On the other hand, we prove that an isometric pluriharmonic immersion of a Kähler manifold into an indefinite Euclidean space of index one is totally geodesic if its tangent vectors are apart from the orthogonal complement of some timelike vector, under the assumption that the Kähler manifold is biholomorphic to $\mathbf{C}^m$ (Proposition 4.2.3).

We also prove the following cylinder theorem based on the result of M. Dajczer and L. Rodriguez [14].

Theorem 4.1.9. *Let M be a complete Kähler manifold of real dimension $2m$ and $f : M \to \mathbf{R}_N^{N+P}$ an isometric pluriharmonic immersion of M into $\mathbf{R}_N^{N+P}$. If the index of relative nullity ν is not less than $2m-2$, then f is $(2m-2)$-cylindrical.*

This thesis is organized as follows: In Chapter 2, we first review relevant basic properties of general isometric immersions, to fix our terminology and notation. Then we study pluriharmonic immersions and holomorphic immersions. We prove the existence of an isometric holomorphic immersion for a given pluriharmonic immersion, and the rigidity of isometric holomorphic immersions. We also obtain a criterion to count the substantial codimension of isometric holomorphic immersions. In Chapter 3, we study isometric pluriharmonic immersions in a local setting. In particular, we prove a classification theorem and illustrate some examples of such immersions. Chapter 4 is devoted to the study of these immersions in a global setting, assuming the completeness of source Kähler manifolds. Particular studies are made for the cases when these immersions reduce to being cylindrical or totally geodesic.

2. Preliminaries

2.1. Semi-Euclidean spaces

Let $\mathbf{R}_N^{N+P}$ denote a real vector space of dimension $N+P$ endowed with the standard metric

$$\langle\cdot,\cdot\rangle_{\mathbf{R}_N^{N+P}} \\ := -(dx^1)^2 - \cdots - (dx^N)^2 + (dx^{N+1})^2 + \cdots + (dx^{N+P})^2$$

of index N, and $\mathbf{C}_N^{N+P}$ a complex vector space of dimension $N+P$ endowed with the standard metric

$$\langle\cdot,\bar{\cdot}\rangle_{\mathbf{C}_N^{N+P}} \\ := -dz^1 dz^{\overline{1}} - \cdots - dz^N dz^{\overline{N}} + dz^{N+1} dz^{\overline{N+1}} + \cdots + dz^{N+P} dz^{\overline{N+P}}$$

of index N, respectively. Let l, t and s be integers such that

$$0 \le l \le \min(N,P), \quad 0 \le t \le N-l \text{ and } \quad 0 \le s \le P-l.$$

For each (l,t,s) we denote by $H(l,t,s)$ an $(l+t+s)$-dimensional subspace of $\mathbf{R}_N^{N+P}$ consisting of the elements

$$(\underbrace{X^1,\dots,X^l,X^{l+1},\dots,X^{l+t},0_{N-(l+t)}}_{N};\underbrace{X^1,\dots,X^l,X^{l+t+1},\dots,X^{l+t+s},0_{P-(l+s)}}_{P}),$$

where $X^j \in \mathbf{R}$ for $1 \le j \le l+t+s$. Also, by $H^{\mathbf{C}}(l,t,s)$ we denote an $(l+t+s)$-dimensional subspace of $\mathbf{C}_N^{N+P}$ consisting of the elements

$$(\underbrace{Z^1,\dots,Z^l,Z^{l+1},\dots,Z^{l+t},0_{N-(l+t)}}_{N};\underbrace{Z^1,\dots,Z^l,Z^{l+t+1},\dots,Z^{l+t+s},0_{P-(l+s)}}_{P})=:Z,$$

where $Z^j \in \mathbf{C}$ for $1 \leq j \leq l+t+s$. For each element $Z \in H^{\mathbf{C}}(l,t,s)$, we set

$$Z_0 := (Z^1, \dots, Z^l),$$

$$Z_- := (Z^{l+1}, \dots, Z^{l+t}), \quad Z_+ := (Z^{l+t+1}, \dots, Z^{l+t+s}),$$

which are called the *0-component*, the *$-$-component* and the *$+$-component* of Z, respectively. We often write $Z = (Z_0, Z_-, Z_+)$ for convenience.

Let $M_{(N+P)\times(N+P)}(\mathbf{F})$ denote the set of $(N+P)\times(N+P)$-matrices with entries in $\mathbf{F}(= \mathbf{R}$ or $\mathbf{C})$. Let $O(N,P)$ and $U(N,P)$ be the groups of isometries of $\mathbf{R}_N^{N+P}$ and $\mathbf{C}_N^{N+P}$, respectively, that is,

$$O(N,P) := \{O \in M_{(N+P)\times(N+P)}(\mathbf{R}); {}^tO 1_{NP} O = 1_{NP}\},$$

$$U(N,P) := \{U \in M_{(N+P)\times(N+P)}(\mathbf{C}); {}^*U 1_{NP} U = 1_{NP}\},$$

where

$$1_{NP} := \begin{bmatrix} -1_N & \\ & 1_P \end{bmatrix} \in M_{(N+P)\times(N+P)}(\mathbf{R}) \quad \text{and} \quad {}^*U = {}^t\overline{U}.$$

Note that each linear subspace of $\mathbf{R}_N^{N+P}$ can be written as $O(H(l,t,s))$ for some (l,t,s) and some $O \in O(N,P)$. As a result, when we discuss $O(N,P)$-congruence classes of maps into $\mathbf{R}_N^{N+P}$, we only have to consider $H(l,t,s)$ as subspaces of $\mathbf{R}_N^{N+P}$. We remark that the induced metric on $H(0,t,s)(\equiv \mathbf{R}_t^{t+s}) \subset \mathbf{R}_N^{N+P}$ is nondegenerate, while for $l > 0$ the induced metric on $H(l,t,s) \subset \mathbf{R}_N^{N+P}$ is degenerate.

We close this section by introducing some terminologies. A vector $v \in \mathbf{R}_N^{N+P}$ is called

spacelike	if	$\langle v,v\rangle_{\mathbf{R}_N^{N+P}} > 0$	or	$v = 0$,
lightlike or *null*	if	$\langle v,v\rangle_{\mathbf{R}_N^{N+P}} = 0$	and	$v \neq 0$,
timelike	if	$\langle v,v\rangle_{\mathbf{R}_N^{N+P}} < 0$.		

For a subspace W of $\mathbf{R}_N^{N+P}$, we define

$$W^{\perp} := \{v \in \mathbf{R}_N^{N+P};\ \langle v, w\rangle_{\mathbf{R}_N^{N+P}} = 0 \ \text{ for all } \ w \in W\}.$$

It is immediate that $\dim W + \dim W^{\perp} = N + P$, and $(W^{\perp})^{\perp} = W$. Also, a subspace W of $\mathbf{R}_N^{N+P}$ is nondegenerate, that is, $\langle \cdot, \cdot\rangle_{\mathbf{R}_N^{N+P}}|_W$ is nondegenerate, if and only if $\mathbf{R}_N^{N+P}$ is the direct sum of W and $W^{\perp}$.

2.2. Fundamental theory of isometric immersions

In this section, we fix our notations and review relevant basic properties of isometric immersions. Let M be a connected d-dimensional Riemannian manifold with metric g, and $f : M \to \mathbf{R}^{N+P}_N$ an isometric immersion. We remark that each vector $f_* v$ is spacelike for $v \in T_x M$, where f_* denotes the differential of f.

Let $\nabla : \Gamma(TM) \to \Gamma(TM \otimes T^*M)$ denote the Levi-Civita connection of M and $D : \Gamma(f^*T\mathbf{R}^{N+P}_N) \to \Gamma(f^*T\mathbf{R}^{N+P}_N \otimes T^*M)$ the connection induced by f from the Levi-Civita connection of $\mathbf{R}^{N+P}_N$.

Let $\alpha \in \Gamma(\mathit{Nor}\, f \otimes T^*M \otimes T^*M)$ be the second fundamental form of f defined by

$$\alpha(X,Y) := D_X f_* Y - f_* \nabla_X Y \qquad \text{for } X, Y \in \Gamma(TM),$$

where $\mathit{Nor}\, f$ is the normal bundle of f. We denote by $A \in \Gamma((\mathit{Nor}\, f)^* \otimes T^*M \otimes TM)$ the Weingarten operator of α, or the shape operator of f, which is defined by

$$g(A_\xi X, Y) = \langle \alpha(X,Y), \xi \rangle_{\mathbf{R}^{N+P}_N} \qquad \text{for } X, Y \in \Gamma(TM),\ \xi \in \Gamma(\mathit{Nor}\, f).$$

An isometric immersion f is called *minimal* if the mean curvature vector

$$H(x) := \frac{1}{d} \sum_{j=1}^{d} \alpha(e_j, e_j)$$

vanishes identically, where $\{e_j\}$ is an orthonormal basis for $T_x M$.

Let $\nabla^\perp : \Gamma(\mathit{Nor}\, f) \to \Gamma(\mathit{Nor}\, f \otimes T^*M)$ be the normal connection of f, which is defined by

$$\nabla^\perp_X \xi := D_X \xi + f_* A_\xi X \qquad \text{for } X \in \Gamma(TM),\ \xi \in \Gamma(\mathit{Nor}\, f).$$

We define the curvature tensor of ∇ by

$$R^{\nabla}(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

and the curvature tensor $R^{\nabla^{\perp}}$ of $\nabla^{\perp}$ in a similar way.

The following three propositions are often called the fundamental theorems of submanifolds.

Proposition 2.2.1. *Let $f : M \to \mathbf{R}^{N+P}_N$ be an isometric immersion of a Riemannian manifold into a semi-Euclidean space. Then we have the Gauss, Codazzi and Ricci equations:*

$$\begin{aligned} & g(R^{\nabla}(X,Y)Z,W) \\ =& \langle \alpha(X,W), \alpha(Y,Z) \rangle_{\mathbf{R}^{N+P}_N} - \langle \alpha(X,Z), \alpha(Y,W) \rangle_{\mathbf{R}^{N+P}_N}, \\ & (\nabla^{\perp}_X A)(Y,\xi) = (\nabla^{\perp}_Y A)(X,\xi), \\ & \langle R^{\nabla^{\perp}}(X,Y)\xi, \eta \rangle_{\mathbf{R}^{N+P}_N} = g([A_\xi, A_\eta]X, Y), \\ & \qquad\qquad\qquad \textit{for } X,Y,Z,W \in \Gamma(TM),\ \xi,\eta \in \Gamma(Nor\, f), \end{aligned}$$

where by definition $(\nabla^{\perp}_X A)(Y,\xi) = \nabla_X(A_\xi Y) - A_\xi(\nabla_X Y) - A_{\nabla^{\perp}_X \xi} Y$.

Concerning the converse to this proposition, we have the following existence theorem.

Proposition 2.2.2. *Let M be a simply connected d-dimensional Riemannian manifold and $\pi : E \to M$ a vector bundle over M of* rank *$N + P - d$ with a metric $\langle \cdot, \cdot \rangle$. Let $\overline{\nabla} : \Gamma(E) \to \Gamma(E \otimes T^*M)$ be a connection on E compatible with $\langle \cdot, \cdot \rangle$. Let s be a symmetric section of*

$E \otimes T^*M \otimes T^*M$. Suppose that $\overline{\nabla}$ and s satisfy

$$\begin{aligned} g(R^{\nabla}(X,Y)Z,W) =& \langle s(X,W), s(Y,Z)\rangle - \langle s(X,Z), s(Y,W)\rangle, \\ (\overline{\nabla}_X B)(Y,\xi) =& (\overline{\nabla}_Y B)(X,\xi), \\ \langle R^{\overline{\nabla}}(X,Y)\xi,\eta\rangle =& g([B_\xi, B_\eta]X,Y) \\ & \qquad\qquad \text{for } X,Y,Z,W \in \Gamma(TM),\ \xi,\eta \in \Gamma(E), \end{aligned}$$

where $B \in \Gamma(E^* \otimes T^*M \otimes TM)$ is defined by $g(B_\xi X, Y) = \langle s(X,Y), \xi\rangle$. Then there exists an isometric immersion $f : M \to \mathbf{R}^{N+P}_N$ and a bundle isomorphism $\phi : E \to Nor\, f$ covering f such that

$$\langle \phi(\xi), \phi(\eta)\rangle_{\mathbf{R}^{N+P}_N} = \langle \xi, \eta\rangle,$$
$$\phi(s(X,Y)) = \alpha(X,Y),$$
$$\phi\overline{\nabla}_X\xi = \nabla^\perp_X\phi(\xi),$$

where α and $\nabla^\perp$ are the second fundamental form and the normal connection of f, respectively.

The uniqueness of isometric immersions is treated by the following

Proposition 2.2.3. *Let $f, \widetilde{f} : M \to \mathbf{R}^{N+P}_N$ be isometric immersions with second fundamental forms α, $\widetilde{\alpha}$ and normal connections $\nabla^\perp$, $\widetilde{\nabla^\perp}$, respectively. Suppose that there is a bundle isomorphism $\phi : Nor\, f \to Nor\, \widetilde{f}$ such that*

$$\langle \phi(\xi), \phi(\eta)\rangle_{\mathbf{R}^{N+P}_N} = \langle \xi, \eta\rangle_{\mathbf{R}^{N+P}_N},$$
$$\phi(\alpha(X,Y)) = \widetilde{\alpha}(X,Y),$$
$$\phi\nabla^\perp_X\xi = \widetilde{\nabla^\perp}_X\phi(\xi),$$

for $X, Y \in \Gamma(TM)$, $\xi, \eta \in \Gamma(Nor\, f)$.

Then there exists a Euclidean motion τ *such that* $\widetilde{f} = \tau \circ f$ *and* $\tau_*|_{Nor\, f} = \phi$.

We now introduce two basic terminologies for the subsequent discussion.

Definition 2.2.4. An isometric immersion $f : M \to \mathbf{R}^{N+P}_N$ is said to be *full in* $H(l, t, s)$ if the image $f(M)$ of f is contained in $H(l, t, s)$ and if the coordinate functions $f^1, \ldots, f^l$, $f^{l+1}, \ldots, f^{l+t}, f^{l+t+1}, \ldots, f^{l+t+s}$ of f are linearly independent over $\mathbf{R}$.

Definition 2.2.5. An isometric immersion $f : M \to \mathbf{R}^{N+P}_N$ is said to be *m-cylindrical* if there exist a $(d-m)$-dimensional Riemannian manifold N and an isometric immersion $f' : N \to \mathbf{R}^{N+P-m}_N$ such that

$$\begin{array}{ccccc} M & = & N & \times & \mathbf{R}^m, \\ f & = & f' & \times & \mathrm{id}_{\mathbf{R}^m}. \end{array}$$

The following splitting theorem for isometric immersions of product Riemannian manifolds into Euclidean spaces is due to J. D. Moore [23].

Proposition 2.2.6. *Let* $f : M_1 \times M_2 \to \mathbf{R}^P$ *be an isometric immersion of a product Riemannian manifold. Suppose that*

$$\alpha(X, Y) = 0 \qquad \text{for } X \in \Gamma(TM_1),\ Y \in \Gamma(TM_2).$$

Then there exist vector subspaces E_j *and isometric immersions* $f_j : M_j \to E_j$ $(j = 1, 2)$ *such that*

$$f = f_1 \times f_2, \qquad E_1 \oplus E_2 \subset \mathbf{R}^P.$$

Proof. Take a point $(m_1, m_2) \in M_1 \times M_2$. We may assume, without loss of generality, $f(m_1, m_2) = 0 \in \mathbf{R}^P$. We claim that the two subspaces

$$E_1 := \mathrm{span}\{f_* X(x_2)\ ;\ X(x_2) \in T_{(m_1,x_2)}(M_1 \times \{x_2\}), \quad x_2 \in M_2\},$$
$$E_2 := \mathrm{span}\{f_* Y(x_1)\ ;\ Y(x_1) \in T_{(x_1,m_2)}(\{x_1\} \times M_2), \quad x_1 \in M_1\}$$

intersect orthogonally each other.

To see this, let σ_t be a curve on M_2 such that

$$\sigma_0 = m_2, \quad \sigma_1 = x_2, \quad \sigma_t \subset M_2 \quad (0 \le t \le 1),$$
$$Y(m_1) = \left.\frac{d}{dt}\right|_{t=0} (m_1 \times \sigma_t).$$

Then, by our assumption, we have

$$\begin{aligned} &D_{\frac{d}{dt}(m_1 \times \sigma_t)} f_* X(\sigma_t) \\ =&f_* \nabla_{\frac{d}{dt}(m_1 \times \sigma_t)} X(\sigma_t) + \alpha\left(\frac{d}{dt}(m_1 \times \sigma_t), X(\sigma_t)\right) \\ =&0 + 0 = 0, \end{aligned}$$

which, together with $\langle f_* X(m_2), f_* Y(m_1)\rangle_{\mathbf{R}^P} = 0$, implies that

$$\langle f_* X(x_2), f_* Y(m_1)\rangle_{\mathbf{R}^P} = 0 \qquad \text{for } x_2 \in M_2.$$

The same argument implies also that

$$\langle f_* X(x_2), f_* Y(x_1)\rangle_{\mathbf{R}^P} = 0 \qquad \text{for } x_1 \in M_1,\ x_2 \in M_2,$$

which means that E_1 and E_2 are orthogonal.

Now we construct isometric immersions $f_j : M_j \to E_j$ $(j = 1, 2)$. Let E_0 be a subspace complementary to $E_1 \oplus E_2$ and let $p_j : \mathbf{R}^P \to E_j$ denote the orthogonal projections. We then define

$$f_1(x_1) := p_1(f(x_1, m_2)),$$

and see that f_1 is independent of the choice of m_2, since

$$\frac{d}{dt}(p_1(f(x_1, \sigma_t))) = p_1(\frac{d}{dt}f(x_1, \sigma_t)) = p_1(f_*(0, \frac{d}{dt}\sigma_t)) = 0.$$

Similarly, $f_2(x_2) := p_2(f(m_1, x_2))$ is independent of the choice of m_1, and $f_0 := p_0 \circ f$ is constant. Therefore, we have $f_j : M_j \to E_j$ such that $f(x_1, x_2) = (\text{constant}, f_1(x_1), f_2(x_2)) \in E_0 \oplus E_1 \oplus E_2$. $\square$

In a similar fashion, we also obtain the following cylinder theorem for isometric immersions of the product of a Riemannian manifold and a Euclidean space into semi-Euclidean spaces.

Proposition 2.2.7. *Let $f : N \times \mathbf{R}^m \to \mathbf{R}^{N+P}_N$ be an isometric immersion of the product of a Riemannian manifold and $\mathbf{R}^m$ into a semi-Euclidean space. Suppose that*

$$\alpha(X, Y) = 0 \qquad \text{for } X \in \Gamma(T(N \times \mathbf{R}^m)),\ Y \in \Gamma(T\mathbf{R}^m).$$

Then f is m-cylindrical.

Proof. As in Proposition 2.2.6, we put

$$E_2 := \operatorname{span}\{f_*Y(x)\ ;\ Y(x) \in T_{(x,p)}(\{x\} \times \mathbf{R}^m), \quad x \in N\}.$$

We claim that $E_2 \equiv H(0, 0, m)(\equiv \mathbf{R}^m)$.

Let $Y(x) \in T_{(x,p)}(\{x\} \times \mathbf{R}^m)$ $(x \in N)$ and $X = X_1 + X_2 \in T_{(x,p)}(N \times \mathbf{R}^m)$ $(X_1 \in T_xN,\ X_2 \in T_p\mathbf{R}^m)$. It then follows from the flatness of $\mathbf{R}^m$ that

$$\nabla_X Y = \nabla_{X_1} Y + \nabla_{X_2} Y = 0 + 0 = 0,$$

which implies

$$D_X f_* Y = f_* \nabla_X Y + \alpha(X, Y) = 0 + 0 = 0.$$

Therefore, $f_*Y(x)$ is independent of (x, p), and E_2 is an m-dimensional vector subspace.

Since E_2 is a nondegenerate subspace, we can mimic the argument in Proposition 2.2.6 to complete the proof. $\square$

Now we deal with the problem of reducing the codimension of isometric immersions.

Definition 2.2.8. Let $f : M \to \mathbf{R}_N^{N+P}$ be an isometric immersion of a Riemannian manifold into a semi-Euclidean space. The subspace of $f^*T\mathbf{R}_N^{N+P}(x)$ spanned by

$$f_*X_1(x),\ D_{X_2}f_*X_1(x),\ D_{X_k}\cdots D_{X_2}f_*X_1(x), \qquad \text{for}\ \ X_1, \ldots, X_k \in \Gamma(TM)$$

is called the *k-th osculating space* of f at $x \in M$, and is denoted by $Osc^k f(x)$.

We remark that, by definition, $Osc^k f(x)$ is a subset of $Osc^{k+1} f(x)$.

Definition 2.2.9. Let $f : M \to \mathbf{R}_N^{N+P}$ be an isometric immersion with second fundamental form α. The subspace of $Nor\, f(x)$ defined by

$$Nor^1 f(x) := \operatorname{span}\{\alpha(X, Y)\ ;\ X, Y \in T_xM\}$$

is called the *first normal space* of f at $x \in M$.

Note that the first normal space $Nor^1 f(x)$ is the orthogonal complement of $Osc^1 f(x)$ in $Osc^2 f(x)$.

An isometric immersion f is called *nicely curved* if the dimension of the osculating space $Osc^k f(x)$ is constant for all $x \in M$ and for each

k. Under this assumption, we have a subbundle $Osc^k f$ of $f^*T\mathbf{R}^{N+P}_N$ for each k. Furthermore, it can be verified that $D_X\xi \in Osc^{k+1}f(x)$ for $\xi \in \Gamma(Osc^k f)$ and $X \in T_xM$, and that if $Osc^l f = Osc^{l+1} f$, then the subbundle $Osc^l f$ is parallel with respect to D and $Osc^l f = Osc^{l+1} f = Osc^{l+2} f = \cdots \subset f^*T\mathbf{R}^{N+P}_N$.

Proposition 2.2.10. *Let $f : M \to \mathbf{R}^{N+P}_N$ be an isometric immersion of a d-dimensional Riemannian manifold into a semi-Euclidean space, and L a nondegenerate subbundle of $Nor\, f$ of* rank q. *Suppose that L is parallel with respect to $\nabla^\perp$ and $Nor^1 f \subset L$. Then the substantial codimension of f is q, that is, there exists a $(d+q)$-dimensional subspace H of $\mathbf{R}^{N+P}_N$ such that $f(M) \subset H$.*

In particular, if f is nicely curved and if $Osc^l f$ is nondegenerate and satisfies $Osc^l f = Osc^{l+1} f$, then the substantial codimension of f coincides with rank $Osc^l f - d$.

Proof. It suffices to prove the first assertion, since the second one follows immediately from this. To prove the first assertion, we show that

$$f(M) \subset T_{x_0}M \oplus L(x_0)$$

for some fixed point $x_0 \in M$. Note that the subbundle $L^\perp$ consisting of the orthogonal complement $L^\perp(x)$ of $L(x)$ in $Nor\, f(x)$ is also parallel with respect to $\nabla^\perp$. Let $\eta \in L^\perp(x_0)$, and let γ be a curve on M through x_0. Since the parallel transport η_t of η along γ belongs to $L^\perp(\gamma(t))$ and L is nondegenerate, $A_{\eta_t} = 0$. Therefore,

$$\begin{aligned} D_{\dot\gamma}\eta_t &= - f_* A_{\eta_t}\dot\gamma + \nabla^\perp_{\dot\gamma}\eta_t \\ &= 0 + 0 = 0, \end{aligned}$$

which implies that $\eta_t = \eta$ is a constant vector in $\mathbf{R}_N^{N+P}$. Since

$$\frac{d}{dt}\langle f(\gamma(t)) - f(x_0), \eta\rangle_{\mathbf{R}_N^{N+P}} = \langle f_*\dot{\gamma}(t), \eta\rangle_{\mathbf{R}_N^{N+P}} = 0,$$

we see that $\langle f(\gamma(t)) - f(x_0), \eta\rangle_{\mathbf{R}_N^{N+P}} = 0$. Since γ and η are arbitrary, it follows that $f(M)$ is contained in the vector subspace $(L^\perp(x_0))^\perp$. $\square$

2.3. Pluriharmonic immersions

Let M be a connected Kähler manifold of real dimension $2m$ with Riemannian metric g and complex structure $J \in \Gamma(TM \otimes T^*M)$ satisfying

$$\begin{aligned} J_x^2 &= -\operatorname{id}_{T_xM}, \\ g(JX, JY) &= g(X, Y), \\ \nabla_X(JY) &= J\nabla_X Y \qquad \text{for } X, Y \in \Gamma(TM). \end{aligned}$$

Let T_xM^c be the complexification of the tangent space of M at x. Then we have a decomposition

$$T_xM^c = T_x^{(1,0)} \oplus T_x^{(0,1)},$$

where $T_x^{(1,0)}$ and $T_x^{(0,1)}$ denote the eigenspaces of J_x corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. This induces a decomposition of a symmetric tensor $\alpha_x \in \mathit{Nor}\, f(x) \otimes T_x^*M \otimes T_x^*M$ into the $(2,0)$, $(0,2)$ and $(1,1)$ components by restricting its complex bilinear extension to $T_x^{(1,0)} \otimes T_x^{(1,0)}$, $T_x^{(0,1)} \otimes T_x^{(0,1)}$ and $T_x^{(1,0)} \otimes T_x^{(0,1)} \oplus T_x^{(0,1)} \otimes T_x^{(1,0)}$, and these components are denoted by $\alpha^{(2,0)}$, $\alpha^{(0,2)}$ and $\alpha^{(1,1)}$, respectively.

It should be remarked that semi-Kähler manifolds can be defined similarly as in the positive definite case. The simplest example of semi-Kähler manifolds is provided by complex semi-Euclidean space $\mathbf{C}_N^{N+P}$, namely, a semi-Riemannian manifold $(\mathbf{R}_{2N}^{2N+2P}, \langle\cdot,\cdot\rangle_{\mathbf{R}_{2N}^{2N+2P}})$ with the standard complex structure J_0, defined by $J_0(\partial/\partial x^{2k-1}) := \partial/\partial x^{2k}$ and $J_0(\partial/\partial x^{2k}) := -\partial/\partial x^{2k-1}$.

Let $f : M \to \mathbf{C}_N^{N+P}$ be an isometric immersion of a Kähler manifold into a complex semi-Euclidean space. f is called *holomorphic* if $f_* \circ J = J_0 \circ f_*$. If f is holomorphic, then the second fundamental form satisfies

$$\alpha(X, JY) = \alpha(JX, Y) = J_0 \alpha(X, Y).$$

We are now in a position to define isometric pluriharmonic immersions as follows.

Definition 2.3.1. Let $f : M \to \mathbf{R}_N^{N+P}$ be an isometric immersion of a Kähler manifold into a semi-Euclidean space. f is said to be *pluriharmonic* if

$$\alpha(X, JY) = \alpha(JX, Y) \qquad \text{for } X, Y \in \Gamma(TM).$$

Remark 2.3.2. *Let $f : M \to \mathbf{R}_N^{N+P}$ be an isometric immersion of a Kähler manifold into a semi-Euclidean space. Then the following conditions are equivalent :*

(i) $\quad f$ *is pluriharmonic,*

(ii) $\quad \alpha^{(1,1)} = 0,$

(iii) $\quad A_\xi J = -J A_\xi \quad$ *for* $\xi \in \Gamma(\mathit{Nor}\, f)$,

(iv) $\quad \dfrac{\partial^2 f}{\partial z^\alpha \partial \overline{z^\beta}} = 0,$

where $(z^\alpha) := (z^1, \ldots, z^m)$ *is a local complex coordinate system on* M.

A pluriharmonic immersion is often called a $(1, 1)$-*geodesic immersion*, whose name comes from the condition (ii) as above. On the other hand, the term *pluriharmonic* refers the condition (iv). Our definition above is based on M. Dajczer and D. Gromoll [10], although they themselves called such immersions *circular*.

Example 2.3.3. An isometric holomorphic immersion $f : M \to \mathbf{C}^{N+P}_N$ of a Kähler manifold into a complex semi-Euclidean space is pluriharmonic, when regarded as an immersion into real semi-Euclidean space $\mathbf{R}^{2N+2P}_{2N}$.

It should be remarked that any pluriharmonic immersion is minimal. In fact, for an orthonormal basis $\{e_1, Je_1, \dots, e_m, Je_m\}$ for T_xM, we obtain

$$\begin{aligned}
H(x) =& \frac{1}{2m}\sum_{j=1}^{m}\{\alpha(e_j, e_j) + \alpha(Je_j, Je_j)\} \\
=& \frac{1}{2m}\sum_{j=1}^{m}\{\alpha(e_j, e_j) + \alpha(J^2e_j, e_j)\} \\
=& \frac{1}{2m}\sum_{j=1}^{m}\{\alpha(e_j, e_j) - \alpha(e_j, e_j)\} \\
=& 0.
\end{aligned}$$

Conversely, we can prove the following result concerning pluriharmonicity of minimal immersions.

Proposition 2.3.4. *Let $f : M \to \mathbf{R}^{N+P}_N$ be an isometric immersion of a Kähler manifold into a semi-Euclidean space. Suppose that $N = 0$ or $P = 2m$, the real dimension of M. If f is minimal, then f is pluriharmonic.*

Proof. We choose an orthonormal basis $\{e_1, \dots, e_m, Je_1, \dots, Je_m\}$ for T_xM and define $\sqrt{2}E_j := e_j + \sqrt{-1}Je_j \in T^{(0,1)}$, where J is the complex structure of M. It follows from the Gauss equation of f and the Kähler condition of M that for $k, r = 1, \dots, m$

$$\begin{aligned} 0 =& g(R(E_k, E_r)\overline{E_r}, \overline{E_k}) \\ =& \langle \alpha(E_k, \overline{E_k}), \alpha(E_r, \overline{E_r}) \rangle_{\mathbf{C}_N^{N+P}} - \langle \alpha(E_k, \overline{E_r}), \alpha(E_r, \overline{E_k}) \rangle_{\mathbf{C}_N^{N+P}}. \end{aligned}$$

Taking sums with respect to k and r then yields

$$0 = m^2 \langle H, \overline{H} \rangle_{\mathbf{C}_N^{N+P}} - \langle \alpha^{(1,1)}, \overline{\alpha^{(1,1)}} \rangle_{\mathbf{C}_N^{N+P}}.$$

Since H and $\alpha^{(1,1)}$ are not lightlike, this implies that $H = 0$ if and only if $\alpha^{(1,1)} = 0$. $\square$

In the case $N = 0$, Proposition 2.3.4 has been proved by M. Dajczer and L. Rodriguez [13], S. Udagawa [27] and M. J. Ferreira and R. Tribuzy [17]. In their paper, M. Dajczer and L. Rodriguez [13] claimed that the result is *quite surprising*.

Moreover, M. Dajczer [9] and S. Udagawa [27] proved the following result concerning holomorphicity of minimal immersions.

Proposition 2.3.5. *Let M be a Kähler manifold of real dimension $2m$. Let $f : M \to \mathbf{R}^{2m+2}$ be an isometric minimal immersion into a Euclidean space of real codimension two. If either of the following* (D) *or* (U) *holds, then f is holomorphic with respect to some orthogonal complex structure of $\mathbf{R}^{2m+2}$.*

(D) *The index of relative nullity ν is less than $2m - 4$ on M.*

(U) (i) *M is complete.* (ii) *M is parabolic, that is, M admits no positive non-constant superharmonic functions.* (iii) $|R^{\nabla}|^2 \geq \mathrm{scal}_g^2$ *on M.* (iv) *f is stable.*

It is well-known that a simply connected minimal surface in $\mathbf{R}^3$ has the so-called associated family, which is represented as the real part of a holomorphic immersion into $\mathbf{C}^3$. We show that a pluriharmonic immersion also has this property.

Proposition 2.3.6. *Let M be a simply connected Kähler manifold, and $f : M \to \mathbf{R}_N^{N+P}$ an isometric pluriharmonic immersion. Then there exists a 1-parameter family $f_\theta : M \to \mathbf{R}_N^{N+P}$, $\theta \in [0, \pi)$ of isometric pluriharmonic immersions such that $f_0 = f$.*

The family f_θ is called the *associated family* of f, and in particular, $f_{\pi/2}$ is called the *conjugate immersion* of f.

Proof. Define an endomorphism $J_{\theta x}$ of T_xM by

$$J_{\theta x} := \cos\theta \, \mathrm{id}_{T_xM} + \sin\theta J_x.$$

Then J_θ satisfies

$$J_\theta \circ J_{-\theta} = \mathrm{id}_{T_xM},$$
$$g(J_\theta X, J_\theta Y) = g(X, Y),$$
$$\nabla_X(J_\theta Y) = J_\theta \nabla_X Y,$$

which imply that

$$\begin{aligned} R^\nabla(X,Y) \circ J_\theta =& J_\theta \circ R^\nabla(X,Y), \\ R^\nabla(J_\theta X, J_\theta Y) =& R^\nabla(X,Y). \end{aligned}$$

Also, using the second fundamental form α of f, we define the symmetric section $\alpha_\theta \in \Gamma(\mathit{Nor}\, f \otimes T^*M \otimes T^*M)$ by

$$\alpha_\theta(X,Y) := \alpha(J_{\theta/2}X, J_{\theta/2}Y) \qquad \text{for } X, Y \in \Gamma(TM).$$

Let A^θ be the Weingarten operator of α_θ, that is, the section of $(Nor\, f)^* \otimes T^*M \otimes TM$ defined by

$$g(A^\theta_\xi X, Y) = \langle \alpha_\theta(X, Y), \xi \rangle_{\mathbf{R}^{N+P}_N} \qquad \text{for } X, Y \in \Gamma(TM),\ \xi \in \Gamma(Nor\, f).$$

Then the shape operator A of f and A^θ satisfy

$$A^\theta_\xi = A_\xi J_\theta = J_{-\theta} A_\xi.$$

In fact, by the definition of pluriharmonic immersions, we have

$$\begin{aligned} g(A^\theta_\xi X, Y) =& \langle \alpha(J_{\theta/2}X, J_{\theta/2}Y), \xi \rangle_{\mathbf{R}^{N+P}_N} \\ =& \langle \alpha(J_\theta X, Y), \xi \rangle_{\mathbf{R}^{N+P}_N} = g(A_\xi J_\theta X, Y) \\ =& \langle \alpha(X, J_\theta Y), \xi \rangle_{\mathbf{R}^{N+P}_N} \\ =& g(A_\xi X, J_\theta Y) = g(J_{-\theta} A_\xi X, Y). \end{aligned}$$

Now we construct isometric immersions f_θ by using Proposition 2.2.2. In Proposition 2.2.2, we take $Nor\, f$ as E, the normal connection $\nabla^\perp$ of f as $\overline{\nabla}$, and α_θ as s, respectively. Then we see that they satisfy the Gauss, Codazzi and Ricci equations.

In fact, since the second fundamental form α of f satisfies the Gauss equation,

$$\begin{aligned} & \langle \alpha_\theta(X, W), \alpha_\theta(Y, Z) \rangle_{\mathbf{R}^{N+P}_N} - \langle \alpha_\theta(X, Z), \alpha_\theta(Y, W) \rangle_{\mathbf{R}^{N+P}_N} \\ =& \langle \alpha(J_\theta X, W), \alpha(J_\theta Y, Z) \rangle_{\mathbf{R}^{N+P}_N} - \langle \alpha(J_\theta X, Z), \alpha(J_\theta Y, W) \rangle_{\mathbf{R}^{N+P}_N} \\ =& g(R^\nabla(J_\theta X, J_\theta Y)Z, W) \\ =& g(R^\nabla(X, Y)Z, W), \end{aligned}$$

which means that α_θ satisfies the Gauss equation.

Since the shape operator A of f satisfies the Codazzi equation,

$$\begin{aligned}(\nabla^{\perp}_X A^{\theta})(Y,\xi) =&(\nabla^{\perp}_X J_{-\theta} A)(Y,\xi)\\ =&J_{-\theta}(\nabla^{\perp}_X A)(Y,\xi)\\ =&J_{-\theta}(\nabla^{\perp}_Y A)(X,\xi)\\ =&(\nabla^{\perp}_Y A^{\theta})(X,\xi),\end{aligned}$$

which means that A^{θ} satisfies the Codazzi equation.

For the Ricci equation, we have

$$\begin{aligned}[A^{\theta}_{\xi}, A^{\theta}_{\eta}] =&A^{\theta}_{\xi}A^{\theta}_{\eta} - A^{\theta}_{\eta}A^{\theta}_{\xi}\\ =&(A^{\theta}_{\xi}J_{\theta})(J_{-\theta}A^{\theta}_{\eta}) - (A^{\theta}_{\eta}J_{\theta})(J_{-\theta}A^{\theta}_{\xi})\\ =&[A_{\xi}, A_{\eta}],\end{aligned}$$

from which the Ricci equation for A implies that for A^{θ}.

Consequently, we obtain a family of isometric immersions f_{θ} with second fundamental form α_{θ} and normal connection $\nabla^{\perp}$. Clearly $f_0 = f$. It remains to show that f_{θ} is pluriharmonic. Since J and J_{θ} commute,

$$\begin{aligned}\alpha_{\theta}(JX,Y) =&\alpha(J_{\theta/2}JX, J_{\theta/2}Y)\\ =&\alpha(JJ_{\theta/2}X, J_{\theta/2}Y)\\ =&\alpha(J_{\theta/2}X, JJ_{\theta/2}Y)\\ =&\alpha(J_{\theta/2}X, J_{\theta/2}JY)\\ =&\alpha_{\theta}(X,JY),\end{aligned}$$

which completes the proof. □

As a corollary of this result, we have the following proposition, which will play a key role in Chapter 3.

Proposition 2.3.7. *Let M be a simply connected Kähler manifold, and $f : M \to \mathbf{R}^{N+P}_N$ an isometric pluriharmonic immersion. Then there exists an isometric holomorphic immersion $\Phi : M \to \mathbf{C}^{N+P}_N$ such that $f = \sqrt{2}\,\mathrm{Re}\,\Phi$.*

Proof. Let $f_{\pi/2}$ be the conjugate immersion of f, whose existence is assured by Proposition 2.3.6. Then we can show without difficulty that a map from M to $\mathbf{C}^{N+P}_N$ defined by

$$\Phi := \frac{1}{\sqrt{2}} f - \sqrt{-1}\frac{1}{\sqrt{2}} f_{\pi/2}$$

is an isometric holomorphic immersion. $\square$

2.4. Holomorphic immersions

In this section we will prove a rigidity theorem concerning isometric holomorphic immersions of a Kähler manifold into complex semi-Euclidean spaces. A criterion for their substantial codimensions is also given.

K. Abe and M. A. Magid [4] and M. Umehara [28] generalize Calabi's rigidity theorem [5] in the following way.

Proposition 2.4.1. *Let $H^{\mathbf{C}}(l,t,s)$ and $H^{\mathbf{C}}(l',t',s')$ be linear subspaces of $\mathbf{C}_N^{N+P}$ as above. Let $\Phi = (\Phi_0, \Phi_-, \Phi_+) : M \to H^{\mathbf{C}}(l,t,s)$ and $\Psi = (\Psi_0, \Psi_-, \Psi_+) : M \to H^{\mathbf{C}}(l',t',s')$ be isometric holomorphic immersions, respectively. If Φ is full in $H^{\mathbf{C}}(l,t,s)$, then*

(1) *$s \leq s'$ and $t \leq t'$, and*

(2) *there exists a unitary transformation $U \in U(t',s')$ such that*

$$\begin{bmatrix} \Psi_- \\ \Psi_+ \end{bmatrix} = U \begin{bmatrix} \Phi_- \\ 0_{t'-t} \\ \Phi_+ \\ 0_{s'-s} \end{bmatrix}.$$

To prove this proposition in the indefinite case, we only have to apply Calabi's rigidity theorem, proved in the positive definite case, to the new isometric holomorphic immersions $(\Psi_-; \Phi_+, 0_{s'-s})$ and $(\Phi_-, 0_{t'-t}; \Psi_+) : M \to \mathbf{C}^{t'+s'}$ constructed from Φ and Ψ.

It should be remarked that we have no relation between Φ_0 and Ψ_0 in this case.

Definition 2.4.2. Let M be a Kähler manifold. A *full* isometric holomorphic immersion of M into $H^{\mathbf{C}}(l,t,s) \subset \mathbf{C}_N^{N+P}$ is called the *shape* of M if $l = 0$. The dimension $t+s$ of the ambient space is denoted by $\mathrm{sp}(M)$, and the index t by $\mathrm{sp}_-(M)$, respectively.

Note that, by Proposition 2.4.1, the *shape* of M is unique up to unitary transformations and that $\mathrm{sp} \times \mathrm{sp}_- : \{\text{Kähler manifolds}\} \to (\mathbf{N} \cup \{\infty\}) \times (\mathbf{N} \cup \{0\})$ is well-defined.

Let M be a Kähler manifold with $\mathrm{sp}(M) < \infty$ and $\mathrm{sp}_-(M) = 0$. Note that, in this case, M has non-positive Ricci curvature. We now give a method of computing $\mathrm{sp}(M)$, which is essentially due to E. Calabi [5].

Let $(U; z^1, \dots, z^m)$ be a local complex coordinate of M, and $\Phi : M \to \mathbf{C}^P$ an isometric holomorphic immersion which is nicely curved on U. On account of Proposition 2.2.10, we calculate the integer l such that $Osc^{l-1}\Phi \subsetneq Osc^l\Phi = Osc^{l+1}\Phi$.

Preparing the index sets

$$\Lambda_i := \{(\alpha_1, \dots, \alpha_i) \in \mathbf{N}^i \ ; \ 1 \leq \alpha_1 \leq \cdots \leq \alpha_i \leq m\},$$
$$\Lambda := \bigcup_{i \geq 1} \Lambda_i,$$

we describe the l-th osculating space of Φ as

$$Osc^l\Phi = \mathrm{span}\{\frac{\partial^{|a|}\Phi}{\partial z^a} \ ; \ a \in \bigcup_{1 \leq i \leq l} \Lambda_i\},$$

where we use multi-indices and $|a| := i$ if $a \in \Lambda_i$. For two elements $a := (\alpha_1, \dots, \alpha_k) \in \Lambda_k \subset \Lambda$, $b := (\beta_1, \dots, \beta_l) \in \Lambda_l \subset \Lambda$, we define a function on U by

$$g\{a, b\} := \frac{\partial^{k+l-2} g_{\alpha_1 \overline{\beta_1}}}{\partial z^{\alpha_2} \cdots \partial z^{\alpha_k} \partial z^{\overline{\beta_2}} \cdots \partial z^{\overline{\beta_l}}}.$$

Using this notation, we write an $m \times m$-Hermitian matrix $G := (g_{\alpha\overline{\beta}})$ as

$$G = \begin{bmatrix} g\{(1),(1)\} & \cdots & g\{(1),(m)\} \\ \vdots & & \vdots \\ g\{(m),(1)\} & \cdots & g\{(m),(m)\} \end{bmatrix} \quad \text{where } (i) \in \Lambda_1,$$

and for each index $a \in \Lambda$, we define an $(m+1)\times(m+1)$-Hermitian matrix $G[a]$ by

$$G[a] := \begin{bmatrix} & & & g\{(1),a\} \\ & G & & \vdots \\ & & & g\{(m),a\} \\ g\{a,(1)\} & \cdots & g\{a,(m)\} & g\{a,a\} \end{bmatrix}.$$

It then follows that $\det G[a] = 0$ if and only if

$$\frac{\partial\Phi}{\partial z^1}, \dots, \frac{\partial\Phi}{\partial z^m}, \frac{\partial^{|a|}\Phi}{\partial z^a}$$

are linearly dependent over $C^\infty(U)$.

In fact, for $a := (\alpha_1, \dots, \alpha_k),\ b := (\beta_1, \dots, \beta_l) \in \Lambda$, we have

$$\begin{aligned} &\langle \frac{\partial^{|a|}\Phi}{\partial z^a}, \ \overline{\frac{\partial^{|b|}\Phi}{\partial z^b}} \rangle_{\mathbf{C}^P} \\ =&\frac{\partial^{k+l-2}}{\partial z^{\alpha_2}\cdots\partial z^{\alpha_k}\partial z^{\overline{\beta_2}}\cdots\partial z^{\overline{\beta_l}}}\langle \frac{\partial\Phi}{\partial z^{\alpha_1}}, \ \overline{\frac{\partial\Phi}{\partial z^{\beta_1}}}\rangle_{\mathbf{C}^P} \\ =&g\{a,b\}, \end{aligned}$$

which implies that

$$\begin{aligned} &|\frac{\partial\Phi}{\partial z^1}\wedge\cdots\wedge\frac{\partial\Phi}{\partial z^m}\wedge\frac{\partial^{|a|}\Phi}{\partial z^a}|^2 \\ =&\begin{vmatrix} \langle \frac{\partial\Phi}{\partial z^1}, \overline{\frac{\partial\Phi}{\partial z^1}}\rangle_{\mathbf{C}^P} & \cdots & \langle \frac{\partial\Phi}{\partial z^1}, \overline{\frac{\partial\Phi}{\partial z^m}}\rangle_{\mathbf{C}^P} & \langle \frac{\partial\Phi}{\partial z^1}, \overline{\frac{\partial^{|a|}\Phi}{\partial z^a}}\rangle_{\mathbf{C}^P} \\ \vdots & & \vdots & \vdots \\ \langle \frac{\partial\Phi}{\partial z^m}, \overline{\frac{\partial\Phi}{\partial z^1}}\rangle_{\mathbf{C}^P} & \cdots & \langle \frac{\partial\Phi}{\partial z^m}, \overline{\frac{\partial\Phi}{\partial z^m}}\rangle_{\mathbf{C}^P} & \langle \frac{\partial\Phi}{\partial z^m}, \overline{\frac{\partial^{|a|}\Phi}{\partial z^a}}\rangle_{\mathbf{C}^P} \\ \langle \frac{\partial^{|a|}\Phi}{\partial z^a}, \overline{\frac{\partial\Phi}{\partial z^1}}\rangle_{\mathbf{C}^P} & \cdots & \langle \frac{\partial^{|a|}\Phi}{\partial z^a}, \overline{\frac{\partial\Phi}{\partial z^m}}\rangle_{\mathbf{C}^P} & \langle \frac{\partial^{|a|}\Phi}{\partial z^a}, \overline{\frac{\partial^{|a|}\Phi}{\partial z^a}}\rangle_{\mathbf{C}^P} \end{vmatrix} \\ =&\det G[a]. \end{aligned}$$

Similarly, $\det G[a_{m+1}, \dots, a_{m+k}, a] = 0$ if and only if

$$\frac{\partial \Phi}{\partial z^1}, \dots, \frac{\partial \Phi}{\partial z^m}, \frac{\partial^{|a_{m+1}|}\Phi}{\partial z^{a_{m+1}}}, \dots, \frac{\partial^{|a_{m+k}|}\Phi}{\partial z^{a_{m+k}}}, \frac{\partial^{|a|}\Phi}{\partial z^{a}}$$

are linearly dependent over $C^\infty(U)$, where $G[a_{m+1}, \dots, a_{m+k}, a]$ is defined as follows :

When an $(m+k) \times (m+k)$-Hermitian matrix $G[a_{m+1}, \dots, a_{m+k}]$ is already defined for given indices $a_{m+1}, \dots, a_{m+k} \in \Lambda$, an $(m+k+1) \times (m+k+1)$-Hermitian matrix $G[a_{m+1}, \dots, a_{m+k}, a]$ is defined for $a \in \Lambda$ by

$$\begin{aligned} &G[a_{m+1}, \dots, a_{m+k}, a] \\ :=&G[a_{m+1}, \dots, a_{m+k}][a] \\ =&\begin{bmatrix} & & & g\{(1), a\} \\ & G[a_{m+1}, \dots, a_{m+k}] & & \vdots \\ & & & g\{a_{m+k}, a\} \\ g\{a, (1)\} & \cdots & g\{a, a_{m+k}\} & g\{a, a\} \end{bmatrix}. \end{aligned}$$

We remark that Λ_i has a natural order, and so does Λ. Recall that each Λ_i is a finite set, and if $\det G[a_{m+1}, \dots, a_{m+k}, a] = 0$ for each $a \in \Lambda_i$, then it also holds for any $a \in \Lambda_j$ $(j \geq i)$. Consequently, the following proposition gives an algorithm to calculate $\mathrm{sp}(U)$.

Proposition 2.4.3. *Choose indices inductively by*

$$a_{m+l} := \min\{a \in \Lambda; \det G[a_{m+1}, \dots, a_{m+l-1}, a] > 0\} \quad \textit{for } l \geq 1.$$

If $\det G[a_{m+1}, \dots, a_{m+k}, a] = 0$ *on* U *for any* $a > a_{m+k}$, *then* $\mathrm{sp}(U) = m + k$.

Proof. This follows immediately from Proposition 2.2.10 and that

$$\begin{aligned}
&Osc^{|a_{m+k}|}\Phi \\
=&\operatorname{span}\{\frac{\partial\Phi}{\partial z^1},\dots,\frac{\partial\Phi}{\partial z^m},\frac{\partial^{|a_{m+1}|}\Phi}{\partial z^{a_{m+1}}},\dots,\frac{\partial^{|a_{m+k}|}\Phi}{\partial z^{a_{m+k}}},\} \\
=&Osc^{|a_{m+k}|+1}\Phi.
\end{aligned}$$

□

3. Moduli space of isometric pluriharmonic immersions (Local theory)

3.1. Classification theorem

Let M be a connected and simply connected Kähler manifold of complex dimension m. We denote by $\mathcal{M}^f(M;\mathbf{R}_N^{N+P})$ the moduli space of *full* isometric pluriharmonic immersions, that is, the set of $O(N,P)$-congruence classes of full isometric pluriharmonic immersions of M into $\mathbf{R}_N^{N+P}$.

Our aim of this section is to parametrize $\mathcal{M}^f(M;\mathbf{R}_N^{N+P})$ by the set $\mathcal{P}(\Phi;N,P)$ defined in the manner below.

We assume, throughout this section, that $\mathcal{M}^f(M;\mathbf{R}_N^{N+P})$ is not empty. Then it follows from Propositions 2.3.7 and 2.4.1 that there exists the *shape* $\Phi : M \to \mathbf{C}_n^{n+p}$ of M. For Φ and integers N and P, we define $\mathcal{P}(\Phi;N,P)$ to be the set of $(n+p)\times(n+p)$-complex matrices satisfying the following conditions (P1) – (P4):

(P1) $$\frac{{}^t\partial\Phi}{\partial z^\alpha} P \frac{\partial\Phi}{\partial z^\beta} = 0 \quad (\alpha,\beta = 1,\dots,m),$$

(P2) $${}^tP = P,$$

(P3) $${}^*x_-(1_{np} - {}^tP1_{np}\overline{P})x_- \leq 0 \quad \text{for } x_- \in H^{\mathbf{C}}(0,n,0),$$

(P3) $${}^*x_+(1_{np} - {}^tP1_{np}\overline{P})x_+ \geq 0 \quad \text{for } x_+ \in H^{\mathbf{C}}(0,0,p),$$

(P4) $$\mathrm{sign}(1_{np} - {}^tP1_{np}\overline{P}) = (N-n, P-p),$$

where $(z^1,\dots,z^m)$ is a local complex coordinate on M, and (P4) means that the Hermitian matrix $1_{np} - {}^tP1_{np}\overline{P}$ has $N-n$ negative eigenvalues and $P-p$ positive eigenvalues as well.

First, we give another description of $\mathcal{P}(\Phi; N, P)$ for later use.

Lemma 3.1.1. *An* $(n+p)\times(n+p)$*-complex matrix* P *belongs to* $\mathcal{P}(\Phi; N, P)$ *if and only if* P *satisfies* (P1) *and there exist a complex matrix*

$$U \in U(n)\times U(p) := \left\{ \begin{bmatrix} A & O \\ O & B \end{bmatrix} \in U(n,p) \quad ; \quad A \in U(n),\ B \in U(p) \right\},$$

and real numbers $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_p$ *satisfying*

(P2′) $\qquad P = {}^tU \mathrm{diag}(-\lambda_1, \dots, -\lambda_n; \mu_1, \dots, \mu_p) U,$

(P3′) $\qquad -1 \le -\lambda_1 \le \cdots \le -\lambda_n \le 0 \le \mu_p \le \cdots \le \mu_1 \le 1,$

(P4′) $\qquad -1 = -\lambda_1 = \cdots = -\lambda_{2n-N} < -\lambda_{2n-N+1},$

(P4′) $\qquad \mu_{2p-P+1} < \mu_{2p-P} = \cdots = \mu_1 = 1.$

Proof. In order to see that $P \in \mathcal{P}(\Phi; N, P)$ is diagonalized as in (P2′), we inductively define subsets $\mathcal{S}^{2n-(2j-1)}$ $(j = 1, \dots, n)$ of $H^{\mathbf{C}}(0, n, 0)$ and vectors $x_j \in \mathcal{S}^{2n-(2j-1)}$ as follows.

(Step 1) We set

$$\mathcal{S}^{2n-1} := \{x = (x_-; 0_p) \in \mathbf{C}_n^{n+p}; {}^*x 1_{np} x = -1\},$$

$$-\lambda_1 := \inf_{x \in \mathcal{S}^{2n-1}} \mathrm{Re}({}^t x P x).$$

Then there exists $x_1 \in \mathcal{S}^{2n-1}$ such that $-\lambda_1 = {}^t x_1 P x_1 \le 0$. In fact, since $\mathcal{S}^{2n-1}$ is compact, we have a vector $x_1 \in \mathcal{S}^{2n-1}$ such that $-\lambda_1 = \mathrm{Re}({}^t x_1 P x_1) \le 0$. Note that if $x \in \mathcal{S}^{2n-1}$ and $\theta := 1/2(\pi - \arg {}^t x P x)$, then the vector $e^{\sqrt{-1}\theta} x$ belongs to $\mathcal{S}^{2n-1}$ and $e^{2\sqrt{-1}\theta}({}^t x P x) \le \mathrm{Re}({}^t x P x)$. Hence, $\mathrm{Re}({}^t x_1 P x_1) = {}^t x_1 P x_1$.

(Step j) We set

$$\mathcal{S}^{2n-(2j-1)} := \{x = (x_-; 0_p) \in \mathcal{S}^{2n-(2j-3)}; {}^*x 1_{np} x_{j-1} = {}^*x\overline{Px_{j-1}} = 0\},$$
$$-\lambda_j := \inf_{x \in \mathcal{S}^{2n-(2j-1)}} \mathrm{Re}({}^t xPx).$$

Then the same argument as in Step 1 assures that there exists $x_j \in \mathcal{S}^{2n-(2j-1)}$ such that $-\lambda_j = {}^t x_j P x_j \leq 0$.

Consequently, we obtain vectors $x_1, \ldots, x_n \in H^{\mathbf{C}}(0, n, 0)$ such that

$$\begin{aligned}
{}^*x_j 1_{np} x_k &= -\delta_{jk}, \\
{}^t x_j P x_k &= -\lambda_j \delta_{jk}, \quad -\lambda_1 \leq \cdots \leq -\lambda_n \leq 0.
\end{aligned}$$

In a similar fashion we also obtain vectors $x_{n+1}, \ldots, x_{n+p} \in H^{\mathbf{C}}(0, 0, p)$ such that

$$\begin{aligned}
{}^*x_{n+j} 1_{np} x_{n+k} &= \delta_{jk}, \\
{}^t x_{n+j} P x_{n+k} &= \mu_j \delta_{jk}, \quad \mu_1 \geq \cdots \geq \mu_p \geq 0.
\end{aligned}$$

It is immediate from these that P is diagonalized as in (P2$'$):

$$\begin{aligned}
&U^{-1} := (x_1, \ldots, x_n; x_{n+1}, \ldots, x_{n+p}) \in U(n) \times U(p), \\
&{}^t U^{-1} P U^{-1} = \mathrm{diag}(-\lambda_1, \ldots, -\lambda_n; \mu_1, \ldots, \mu_p), \\
&-\lambda_1 \leq \cdots \leq -\lambda_n \leq 0 \leq \mu_p \leq \cdots \leq \mu_1.
\end{aligned}$$

Now we note that

$$1_{np} - {}^t P 1_{np} \overline{P} = {}^*\overline{U} \mathrm{diag}(-(1-\lambda_1^2), \ldots, -(1-\lambda_n^2); 1-\mu_1^2, \ldots, 1-\mu_p^2) \overline{U}.$$

Then (P3) means that $-(1-\lambda_j^2) \leq 0$ and $1-\mu_j^2 \geq 0$, which implies (P3$'$). (P4) means that $\mathrm{sign}(1_{np} - {}^t P 1_{np} \overline{P}) = (n-(2n-N), p-(2p-P))$, which is equivalent to (P4$'$).

Conversely, it is easy to see that matrices satisfying (P1) and (P2$'$) – (P4$'$) belong to $\mathcal{P}(\Phi; N, P)$. $\square$

In order to construct a bijection from $\mathcal{M}^f(M; \mathbf{R}^{N+P}_N)$ to $\mathcal{P}(\Phi; N, P)$, we prepare the following lemmas.

Lemma 3.1.2. *Let M be a simply connected Kähler manifold with shape $\Phi : M \to \mathbf{C}^{n+p}_n$. If $f : M \to \mathbf{R}^{N+P}_N$ is a full isometric pluriharmonic immersion, then there exists an $(N+P) \times (n+p)$-complex matrix S such that*

$$
\begin{array}{ll}
\text{(S0)} & f = \sqrt{2}\,\mathrm{Re}\, S\Phi, \\
\text{(S1)} & \dfrac{{}^t\partial\Phi}{\partial z^\alpha}\, {}^tS 1_{NP} S \dfrac{\partial\Phi}{\partial z^\beta} = 0 \qquad (\alpha, \beta = 1, \dots, m), \\
\text{(S2)} & {}^*S 1_{NP} S = 1_{np}, \\
\text{(S3)} & \mathrm{rank}(S, \overline{S}) = N + P,
\end{array}
$$

where $(S, \overline{S})$ denotes the $(N+P) \times 2(n+p)$-matrix consisting of S and its complex conjugate $\overline{S}$.

Proof. Recall that by Proposition 2.3.7, there exists an isometric holomorphic immersion $\Psi : M \to \mathbf{C}^{N+P}_N$ such that $f = \sqrt{2}\,\mathrm{Re}\,\Psi$. It also follows from Proposition 2.4.1 that for Φ and Ψ there exists $U = (u_{IJ}) \in U(N, P)$ $(I, J = 1, \dots, N+P)$ such that

$$
\begin{bmatrix} \Psi_- \\ \Psi_+ \end{bmatrix} = U \begin{bmatrix} \Phi_- \\ 0_{N-n} \\ \Phi_+ \\ 0_{P-p} \end{bmatrix}.
$$

Let S be the $(N+P) \times (n+p)$-matrix defined by

$$
S := \Big[\; \overbrace{S_1}^{n} \quad \overbrace{S_2}^{p} \; \Big] \; \}_{N+P},
$$

where

$$S_1 := \begin{bmatrix} u_{Ij} \end{bmatrix} \qquad (j = 1, \ldots, n),$$
$$S_2 := \begin{bmatrix} u_{I(N+a)} \end{bmatrix} \qquad (a = 1, \ldots, p).$$

Then we have

(i) $$f = \sqrt{2}\,\mathrm{Re}\,\Psi = \sqrt{2}\,\mathrm{Re}\,S\Phi = \frac{1}{\sqrt{2}}(S, \overline{S}) \begin{bmatrix} \Phi \\ \overline{\Phi} \end{bmatrix},$$

(ii) $$\partial f = \frac{1}{\sqrt{2}} \partial \Psi = \frac{1}{\sqrt{2}} S \partial \Phi.$$

Since f and Φ are isometric,

$$0 = 2 \frac{{}^t\partial f}{\partial z^\alpha} 1_{NP} \frac{\partial f}{\partial z^\beta}, \qquad 2 \frac{{}^*\partial f}{\partial z^\alpha} 1_{NP} \frac{\partial f}{\partial z^\beta} = \frac{{}^*\partial \Phi}{\partial z^\alpha} 1_{np} \frac{\partial \Phi}{\partial z^\beta},$$

which together with (ii) implies (S1) and (S2). By (i), the fullness of f in $\mathbf{R}^{N+P}_N$ is equivalent to (S3). $\square$

Conversely, by reversing the above process it is easy to see the following :

Lemma 3.1.3.

(1) *Let S be an $(N+P) \times (n+p)$-complex matrix satisfying* (S1), (S2) *and* (S3). *If we define f as in* (S0), *then the congruence class $[f]$ of f belongs to $\mathcal{M}^f(M; \mathbf{R}^{N+P}_N)$.*

(2) *Let $f_1 = \sqrt{2}\,\mathrm{Re}\,S_1\Phi$ and $f_2 = \sqrt{2}\,\mathrm{Re}\,S_2\Phi : M^{2n} \to \mathbf{R}^{N+P}_N$ be isometric pluriharmonic immersions. Then $[f_1] = [f_2]$ if and only if ${}^tS_1 1_{NP} S_1 = {}^tS_2 1_{NP} S_2$.*

We also have the following lemma.

Lemma 3.1.4. *If an $(N+P) \times (n+p)$-matrix S satisfies* (S1),(S2) *and* (S3), *then ${}^tS 1_{NP} S$ belongs to $\mathcal{P}(\Phi; N, P)$.*

Proof. (Step 1) By (S1), ${}^tS 1_{NP} S$ satisfies (P1).

(Step 2) By the same argument as in the proof of Lemma 3.1.1, we obtain $U \in U(n) \times U(p)$ such that

$$\begin{aligned} &{}^t S 1_{NP} S = {}^t U \mathrm{diag}(-\lambda_1, \dots , -\lambda_n; \mu_1, \dots , \mu_p) U, \\ &-\lambda_1 \leq \cdots \leq -\lambda_n \leq 0 \leq \mu_p \leq \cdots \leq \mu_1. \end{aligned}$$

It follows from (S2) that $-1 \leq -\lambda_1 \leq 0 \leq \mu_1 \leq 1$. In fact, let $V \in U(N, P)$ be a matrix such that

$$VS(\mathcal{S}^{2n-1}) \subset \{y \in H^{\mathbf{C}}(0, N, 0) \subset \mathbf{C}_N^{N+P}; {}^* y 1_{NP} y = -1\}.$$

Then we have

$$\begin{aligned} -\lambda_1 &= \inf_{x \in \mathcal{S}^{2n-1}} \mathrm{Re}({}^t x {}^t S 1_{NP} S x) = \inf_{y \in S(\mathcal{S}^{2n-1})} \mathrm{Re}({}^t y 1_{NP} y) \\ &= \inf_{y \in VS(\mathcal{S}^{2n-1})} \mathrm{Re}({}^t (V^{-1} y) 1_{NP} (V^{-1} y)) \\ &\geq \inf_{y \in VS(\mathcal{S}^{2n-1})} {}^* (V^{-1} y) 1_{NP} (V^{-1} y) = -1. \end{aligned}$$

Also, a similar argument applied to μ_1 implies $\mu_1 \leq 1$. Consequently, ${}^t S 1_{NP} S$ satisfies (P2′) and (P3′).

We proceed to prove that (S3) is equivalent to (P4′).

(Step 3A) Since $-1 \leq -\lambda_i \leq 0 \leq \mu_j \leq 1$, we can choose complex numbers a_i, b_i, c_j and d_j so that

$$\begin{aligned} \lambda_i &= a_i^2 + b_i^2, \qquad 1 = |a_i|^2 + |b_i|^2, \\ \mu_j &= c_j^2 + d_j^2, \qquad 1 = |c_j|^2 + |d_j|^2. \end{aligned} \tag{3.1.4 $*$}$$

In particular, if $\lambda_i = 1$ (resp. $\mu_j = 1$), we take $a_i = 1$, $b_i = 0$ (resp. $c_j = 1$, $d_j = 0$).

Note that a_i, b_i (resp. c_j, d_j) are linearly dependent over $\mathbf{R}$ if and only if $\lambda_i = 1$ (resp. $\mu_j = 1$).

(Step 3B) For these complex numbers a_i, b_i, c_j, d_j and the matrix

$$S = \overbrace{\begin{bmatrix} S_1 \\ S_2 \end{bmatrix}}^{n+p} \begin{matrix} \}_N \\ \}_P \end{matrix} \in M_{(N+P)\times(n+p)}(\mathbf{C}),$$

we consider $(2n+2p)\times(n+p)$-matrices $\widetilde{T}$ and $\widetilde{S}$ defined by

$$\widetilde{T} := \overbrace{\begin{bmatrix} \widetilde{T_1} \\ \widetilde{T_2} \end{bmatrix}}^{n+p} \begin{matrix} \}_{2n} \\ \}_{2p} \end{matrix} := \begin{bmatrix} a_1 & & & & & \\ b_1 & & & & & \\ & \ddots & & & & \\ & & a_n & & & \\ & & b_n & & & \\ & & & c_1 & & \\ & & & d_1 & & \\ & & & & \ddots & \\ & & & & & c_p \\ & & & & & d_p \end{bmatrix}$$

and

$$\widetilde{S} := \begin{bmatrix} S_1 \\ 0_{2n-N} \\ S_2 \\ 0_{2p-P} \end{bmatrix}.$$

By definition, we have

$$\begin{aligned} {}^t(\widetilde{T}U)1_{2n2p}(\widetilde{T}U) &= {}^t\widetilde{S}1_{2n2p}\widetilde{S}, \\ {}^*(\widetilde{T}U)1_{2n2p}(\widetilde{T}U) &= {}^*\widetilde{S}1_{2n2p}\widetilde{S} = 1_{np}, \end{aligned}$$

which implies that there exists $O \in O(2n,2p) = U(2n,2p) \cap O(2n,2p;\mathbf{C})$ such that $O\widetilde{S} = \widetilde{T}U$.

(Step 3C) (S3) holds if and only if $\mathrm{rank}(\widetilde{T_1}, \overline{\widetilde{T_1}}) = N$ and $\mathrm{rank}(\widetilde{T_2}, \overline{\widetilde{T_2}}) = P$.

In fact, $\mathrm{rank}(S, \overline{S}) = N + P$ if and only if we can choose N timelike vectors and P spacelike vectors from the image of $(S, \overline{S})$. By Step 3B, this is equivalent to being able to choose these vectors from the image of $(\widetilde{T}, \overline{\widetilde{T}})$, which means that $\mathrm{rank}(\widetilde{T_1}, \overline{\widetilde{T_1}}) = N$ and $\mathrm{rank}(\widetilde{T_2}, \overline{\widetilde{T_2}}) = P$.

(Step 3D) $\mathrm{rank}(\widetilde{T_1}, \overline{\widetilde{T_1}}) = N$ if and only if $1 = \lambda_1 = \cdots = \lambda_{2n-N} > \lambda_{2n-N+1}$, and $\mathrm{rank}(\widetilde{T_2}, \overline{\widetilde{T_2}}) = P$ if and only if $1 = \mu_1 = \cdots = \mu_{2p-P} > \mu_{2p-P+1}$.

In fact, by the definition of $\widetilde{T_1}$, $\mathrm{rank}(\widetilde{T_1}, \overline{\widetilde{T_1}}) = N$ if and only if there exist $2n - N$ pairs of $\mathbf{R}$-linearly dependent vectors $(a_i, \overline{a_i})$ and $(b_i, \overline{b_i})$. Step 3A then implies that this is equivalent to $1 = \lambda_1 = \cdots = \lambda_{2n-N} > \lambda_{2n-N+1}$. The proof for $\widetilde{T_2}$ is similar.

Step 3C combined with Step 3D now implies that (S3) and (P4′) are equivalent, which completes the proof of the lemma. □

We are now in a position to define a natural map $\mathcal{F}$ from $\mathcal{M}^f(M; \mathbf{R}_N^{N+P})$ to $\mathcal{P}(\Phi; N, P)$.

Let $[f]$ be an element of $\mathcal{M}^f(M; \mathbf{R}_N^{N+P})$. By Lemma 3.1.2, for each full isometric pluriharmonic immersion $f \in [f]$, we can choose an $(N + P) \times (n + p)$-matrix S satisfying (S0) – (S3). By Lemma 3.1.4, ${}^tS1_{NP}S$ belongs to $\mathcal{P}(\Phi; N, P)$. We then define the map $\mathcal{F}$ by

$$\mathcal{F}([f]) := {}^tS1_{NP}S,$$

which is well-defined by Lemma 3.1.3 (2).

With these preparations, we obtain a parametrization of the moduli space of full isometric pluriharmonic immersions [18, 19].

Theorem 3.1.5. *Let M be a connected and simply connected Kähler manifold with shape $\Phi : M \to \mathbf{C}_n^{n+p}$. Then the map $\mathcal{F} : \mathcal{M}^f(M; \mathbf{R}_N^{N+P}) \to \mathcal{P}(\Phi; N, P)$ is bijective.*

Proof. It follows from Lemma 3.1.3 (2) that $\mathcal{F}$ is injective. To show that $\mathcal{F}$ is surjective, we claim that for each $P \in \mathcal{P}(\Phi; N, P)$ there exists an $(N+P) \times (n+p)$-matrix S satisfying (S1), (S2) and (S3). First, by Lemma 3.1.1, there exist $U \in U(n,p)$ and $\lambda_i, \mu_j \in \mathbf{R}$ such that $P = {}^tU \mathrm{diag}(\underbrace{-1, \ldots, -1}_{2n-N}, -\lambda_{2n-N+1}, \ldots, -\lambda_n; \underbrace{1, \ldots, 1}_{2p-P}, \mu_{2p-P+1}, \ldots, \mu_p)U$. Choose complex numbers a_i, b_i, c_j and d_j such that (3.1.4 $*$) holds for these λ_i and μ_j. Then we define an $(N+P) \times (n+p)$-matrix S by

$$S := \begin{bmatrix} -1_{2n-N} & & & & & & & \\ & \begin{matrix} a_{2n-N+1} \\ b_{2n-N+1} \end{matrix} & & & & & & \\ & & \ddots & & & & & \\ & & & \begin{matrix} a_n \\ b_n \end{matrix} & & & & \\ & & & & 1_{2p-P} & & & \\ & & & & & \begin{matrix} c_{2p-P+1} \\ d_{2p-P+1} \end{matrix} & & \\ & & & & & & \ddots & \\ & & & & & & & \begin{matrix} c_p \\ d_p \end{matrix} \end{bmatrix} U.$$

It can be verified without difficulty that S satisfies (S1), (S2) and (S3), which together with Lemma 3.1.3 (1) implies that $\mathcal{F}$ is surjective. $\square$

Before closing this section, we now consider the moduli space without assuming the fullness of immersions. Let $\mathcal{M}(M; \mathbf{R}^{N+P}_N)$ denote the set of $O(N,P)$-congruence classes of isometric pluriharmonic immersions of a Kähler manifold M into $\mathbf{R}^{N+P}_N$. Then we have

$$\mathcal{M}(M; \mathbf{R}^{N+P}_N) = \coprod_{\substack{0 \le l \le \min(N,P), \\ 0 \le t \le N-l, \\ 0 \le s \le P-l}} \mathcal{M}^f(M; H(l,t,s)).$$

When N is zero, we have a natural bijection from $\mathcal{M}(M;\mathbf{R}^P)$ to the set of complex matrices satisfying conditions (P1) – (P3), by gathering $\mathcal{F}:\mathcal{M}^f(M;\mathbf{R}^{P'})\to\mathcal{P}(\Phi;0,P')$ for $P'\leq P$. In particular, the moduli space is finite dimensional in the positive definite case.

When N is not zero, since

$$\mathcal{M}(M;\mathbf{R}_N^{N+P})\supsetneq\coprod_{\substack{0\leq t\leq N,\\ 0\leq s\leq P}}\mathcal{M}^f(M;\mathbf{R}_t^{t+s}),$$

the moduli space $\mathcal{M}(M;\mathbf{R}_N^{N+P})$ is not finite dimensional in general. In fact, it is not true that $\mathcal{M}^f(M;H(l,t,s))$ is of finite dimension when $l\geq 1$.

3.2. Examples

In this section we give some explicit examples of isometric pluriharmonic immersions [18, 19].

First, we construct nontrivial isometric minimal immersions of Kähler manifolds of real dimension 4 into Euclidean space $\mathbf{R}^6$. It should be recalled that in this case minimal immersions are pluriharmonic (Proposition 2.3.4).

We choose a metric on $\mathbf{C}^2$ which admits a full isometric minimal immersion into $\mathbf{R}^6$ in the following way.

For this purpose we first choose a 6×6-matrix P as

$$P = \begin{bmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \end{bmatrix},$$

which satisfies (P2), (P3) and (P4).

We next choose a full holomorphic immersion Φ of $\mathbf{C}^2$ into $\mathbf{C}^6$ satisfying (P1). If we put $\partial\Phi/\partial z =: \zeta = {}^t(\zeta_1, \dots, \zeta_6)$, $\partial\Phi/\partial w =: \omega = {}^t(\omega_1, \dots, \omega_6)$, then ζ_i, ω_j must satisfy the following equations :

$$\begin{aligned}
&\frac{\partial\zeta_i}{\partial w} = \frac{\partial\omega_i}{\partial z}, \\
&\zeta_1\zeta_6 + \zeta_2\zeta_5 + \zeta_3\zeta_4 = 0, \\
&\zeta_1\omega_6 + \zeta_2\omega_5 + \zeta_3\omega_4 + \zeta_4\omega_3 + \zeta_5\omega_2 + \zeta_6\omega_1 = 0, \\
&\omega_1\omega_6 + \omega_2\omega_5 + \omega_3\omega_4 = 0.
\end{aligned}$$

It is easy to check that

$$\zeta := {}^t(z, zw, \frac{z^2}{2}, 0, -1, w), \quad \omega := {}^t(0, \frac{z^2}{2}, 0, -1, 0, z)$$

satisfy these equations. Hence,

$$\Phi(z,w) := {}^t\!\left(\frac{z^2}{2}, \frac{z^2 w}{2}, \frac{z^3}{6}, -w, -z, zw\right)$$

gives a full holomorphic immersion satisfying (P1). We now obtain the Kähler metric

$$g = \begin{bmatrix} \frac{1}{4}(|z|^2+2)^2 + (|z|^2+1)|w|^2 & \frac{1}{2}(|z|^2+2)w\overline{z} \\ \frac{1}{2}(|z|^2+2)\overline{w}z & \frac{1}{4}(|z|^2+2)^2 \end{bmatrix}$$

on $\mathbf{C}^2$ induced by Φ, for which $\mathcal{M}^f(\mathbf{C}^2;\mathbf{R}^6)$ is not empty.

Now, take a 6×6-matrix S such that ${}^tSS = P$ as in Theorem 3.1.5, and determine a minimal immersion f by Lemma 3.1.3 (1). With P chosen as above, we take S to be

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & & & & & 1 \\ & 1 & & & 1 & \\ & & 1 & 1 & & \\ & & \sqrt{-1} & -\sqrt{-1} & & \\ & \sqrt{-1} & & & -\sqrt{-1} & \\ \sqrt{-1} & & & & & -\sqrt{-1} \end{bmatrix},$$

from which f is determined to be

$$f(x+\sqrt{-1}y, u+\sqrt{-1}v) := \begin{bmatrix} \frac{1}{2}(x^2-y^2) + xu - yv \\ \frac{1}{2}(x^2-y^2)u - x(yv+1) \\ \frac{1}{6}(x^2-3y^2)x - u \\ -\frac{1}{6}(3x^2-y^2)y - v \\ -\frac{1}{2}(x^2-y^2)v - y(xu+1) \\ -xy + xv + yu \end{bmatrix} : \mathbf{C}^2 \to \mathbf{R}^6.$$

To sum up, we have a Kähler manifold biholomorphic to $\mathbf{C}^2$ and an isometric minimal immersion $f : \mathbf{C}^2 \to \mathbf{R}^6$, which have the following properties :

(1) The Kähler manifold $(\mathbf{C}^2, g)$ is complete.

(2) f is not holomorphic with respect to any orthogonal complex structure on $\mathbf{R}^6$.

(3) f is not cylindrical.

(4) f is completely complex ruled (cf. Definition 4.1.12).

Property (2) can be proved as follows: Assume that f is holomorphic. Then f is congruent to Φ in $\mathbf{R}^{12}$ by Calabi's rigidity theorem (Proposition 2.4.1), and hence the image of Φ lies in a real 6-dimensional affine subspace. This contradicts the fullness of Φ.

By a direct calculation, the second fundamental form α of f is given by

$$\alpha(\frac{\partial}{\partial x},\frac{\partial}{\partial x}) = \frac{1}{(|z|^2+2)^2}\begin{bmatrix} -(|z|^2-2)(|z|^2+2)-4(xu+yv) \\ 2x(|z|^2+2)+4u \\ 2x(|z|^2+2)-2\{(x^2-y^2)u+2xyv\} \\ -2y(|z|^2+2)-2\{(x^2-y^2)v-2xyu\} \\ -2y(|z|^2+2)-4v \\ -4(xv-yu) \end{bmatrix},$$

$$\alpha(\frac{\partial}{\partial x},\frac{\partial}{\partial y}) = \frac{1}{(|z|^2+2)^2}\begin{bmatrix} 4(xv-yu) \\ 2y(|z|^2+2)-4v \\ -2y(|z|^2+2)+2\{(x^2-y^2)v-2xyu\} \\ -2x(|z|^2+2)-2\{(x^2-y^2)u+2xyv\} \\ 2x(|z|^2+2)-4u \\ (|z|^2-2)(|z|^2+2)-4(xu+yv) \end{bmatrix},$$

$$\alpha(\frac{\partial}{\partial x}, \frac{\partial}{\partial u}) = \frac{(|z|^2+2)^2}{(|z|^2+2)^2+4|w|^2}\alpha(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}),$$
$$\alpha(\frac{\partial}{\partial x}, \frac{\partial}{\partial v}) = \frac{(|z|^2+2)^2}{(|z|^2+2)^2+4|w|^2}\alpha(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}),$$

$$\alpha(\frac{\partial}{\partial y}, \frac{\partial}{\partial u}) = \alpha(\frac{\partial}{\partial x}, \frac{\partial}{\partial v}), \quad \alpha(\frac{\partial}{\partial y}, \frac{\partial}{\partial v}) = -\alpha(\frac{\partial}{\partial x}, \frac{\partial}{\partial u}),$$
$$\alpha(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = -\alpha(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}),$$
$$\alpha(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}) = \alpha(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}) = \alpha(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}) = 0,$$

where $z = x + \sqrt{-1}y, w = u + \sqrt{-1}v$. Since the relative nullity space of f is vanishing, f is not cylindrical. Moreover, f is completely complex ruled, since the distribution spanned by $\{\partial/\partial u, \partial/\partial v\}$ is totally geodesic.

We can calculate the moduli space in this case. In fact, we have

$$\mathcal{P}(\Phi; 0, 6) = \left\{ e^{\sqrt{-1}\theta} \begin{bmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \end{bmatrix} \ ; \ \theta \in [0, 2\pi) \right\}.$$

Now, we illustrate a method of constructing pluriharmonic immersions into indefinite Euclidean spaces.

Let $f = \sqrt{2}\,\mathrm{Re}\,\Phi$ be an isometric minimal immersion of a simply connected Kähler manifold M into $\mathbf{R}^{N+P}(= \mathbf{R}_0^{N+P})$, where $\Phi : M \to \mathbf{C}^{N+P}$ is an isometric holomorphic immersion such that

$$(*) \qquad \frac{{}^t\partial\Phi}{\partial z^\alpha}\frac{\partial\Phi}{\partial z^\beta} = 0.$$

For $\Phi = {}^t(\Phi^1, \dots, \Phi^N, \Phi^{N+1}, \dots, \Phi^{N+P})$ we consider a new immersion

$$\widetilde{\Phi} := {}^t(\sqrt{-1}\Phi^1, \dots, \sqrt{-1}\Phi^N; \Phi^{N+1}, \dots, \Phi^{N+P}) : \widetilde{M} \to \mathbf{C}_N^{N+P},$$

where $\widetilde{M}$ is a Kähler manifold defined by

$$(\{z \in M : \widetilde{\Phi}^*\langle\cdot,\bar{\cdot}\rangle_{\mathbf{C}_N^{N+P}} > 0\}, \quad \widetilde{\Phi}^*\langle\cdot,\bar{\cdot}\rangle_{\mathbf{C}_N^{N+P}}).$$

Then the map $\widetilde{f}$ defined by $\widetilde{f} := \sqrt{2}\,\mathrm{Re}\,\widetilde{\Phi} : \widetilde{M} \to \mathbf{R}_N^{N+P}$ gives rise to a pluriharmonic immersion, since

$$\frac{{}^t\partial\widetilde{\Phi}}{\partial z^\alpha} 1_{NP} \frac{\partial\widetilde{\Phi}}{\partial z^\beta} = 0.$$

To sum up, in order to obtain (locally defined) pluriharmonic immersions into $\mathbf{R}_N^{N+P}$, we only have to construct holomorphic immersions into $\mathbf{C}^{N+P}$ satisfying the condition $(*)$.

As an example, we shall construct pluriharmonic immersions of subsets of $\mathbf{C}^2$ into $\mathbf{R}_1^5$, which are defined as cone immersions.

As remarked above, it suffices to define holomorphic immersions into $\mathbf{C}^{1+4}$ satisfying the condition $(*)$. Let C and D be simply connected domains of $\mathbf{C}$. Suppose that $\psi : C \to \mathbf{C}$ is a holomorphic function and $\phi : D \to \mathbf{C}^{1+4}$ is a holomorphic immersion such that

$$(**) \qquad {}^t\phi\phi = {}^t\phi\frac{\partial\phi}{\partial z} = \frac{{}^t\partial\phi}{\partial z}\frac{\partial\phi}{\partial z} = 0,$$

where z is a coordinate of D. Then the holomorphic immersion $\Phi(w,z) := \psi(w)\phi(z) : C \times D \to \mathbf{C}^{1+4}$ satisfies $(*)$, from which we obtain a pluriharmonic immersion of a subset of $C \times D$ into $\mathbf{R}_1^5$.

We can construct ϕ as follows. For any holomorphic function h on D we set

$$\begin{aligned} g(z) &:= {}^t(g^1(z), g^2(z), g^3(z)) \\ &:= \int^z {}^t(1 - h(\zeta)^2, \sqrt{-1}(1 + h(\zeta)^2), 2h(\zeta))d\zeta. \end{aligned}$$

Then

$$\phi(z) := {}^t(1 - {}^tg(z)g(z), \sqrt{-1}(1 + {}^tg(z)g(z)), 2g^1(z), 2g^2(z), 2g^3(z))$$

gives rise to a holomorphic immersion satisfying the condition $(**)$.

If we choose $\psi(w) := w$ and $h(z) := z$, the corresponding pluriharmonic immersion is

$$\begin{aligned} \widetilde{f}(w,z) =& \sqrt{2}\,\mathrm{Re}\,\widetilde{\Phi}(w,z) \\ =& \sqrt{2}\,\mathrm{Re}(w \begin{bmatrix} \sqrt{-1}(1+\frac{1}{3}z^4) \\ \sqrt{-1}(1-\frac{1}{3}z^4) \\ 2(z-\frac{1}{3}z^3) \\ \sqrt{-1}2(z+\frac{1}{3}z^3) \\ 2z^2 \end{bmatrix}) : \mathbf{C}^2 \supset \widetilde{C \times D} \to \mathbf{R}^5_1. \end{aligned}$$

It should be pointed out that we may use a class of complex ruled immersions obtained by M. Dajczer and D. Gromoll [12] as the above Φ, which provides us with a larger class containing cone immersions.

4. Isometric pluriharmonic immersions of complete Kähler manifolds (Global theory)

4.1. Cylinder theorem

In this section, we classify isometric pluriharmonic immersions of *complete* Kähler manifolds into semi-Euclidean spaces with low codimensions. In particular, we will prove a cylinder theorem concerning them.

In the beginning, we consider Riemannian manifolds in general.

Definition 4.1.1. Let $f : M \to \mathbf{R}^{N+P}_N$ be an isometric immersion with second fundamental form α. For each $x \in M$, the subspace of T_xM defined by

$$\triangle(x) := \{X \in T_xM;\ \alpha(X, Y) = 0, \quad Y \in T_xM\}$$

is called the *relative nullity space* of f at x, and its dimension $\nu(x)$ is called the *index of relative nullity* of f at x.

The following proposition is proved by K. Abe and M. Magid [3].

Proposition 4.1.2. *Let $f : M \to \mathbf{R}^{N+P}_N$ be an isometric immersion of a Riemannian manifold into a semi-Euclidean space with relative nullity space $\triangle(x)$. Then the following hold.*

(1) *The distribution $x \mapsto \triangle(x)$ is smooth on any open subset U where the index of relative nullity is constant.*

(2) *The relative nullity distribution $\triangle$ on U is integrable, and the leaves are totally geodesic in M and $\mathbf{R}^{N+P}_N$.*

(3) *The set $G := \{x \in M\ ;\ \nu(x) = \nu_0\}$ is open, where $\nu_0 := \min\{\nu(x);\ x \in M\}$.*

In what follows, we consider the relative nullity foliation $\triangle$ on G. Let $\triangle^{\perp}$ be the distribution on G given by the orthogonal complement $\triangle^{\perp}(x)$ of $\triangle(x)$ with respect to the Riemannian metric of M.

The following completeness result for the relative nullity foliations is proved by K. Abe [1], K. Abe and M. Magid [3], and is basic and well-known.

Proposition 4.1.3. *For an isometric immersion of a complete Riemannian manifold into a semi-Euclidean space, the relative nullity foliation is complete.*

In order to prove our cylinder theorem, we first define the splitting tensor field for an isometric immersion, more precisely, for its relative nullity distribution.

Definition 4.1.4. Let $f : M \to \mathbf{R}_N^{N+P}$ be an isometric immersion of a Riemannian manifold into a semi-Euclidean space. Let $\triangle$ denote its relative nullity distribution on $G \subset M$. For $T \in \Gamma(\triangle)$ and $X \in \triangle^{\perp}(x)$, we define

$$C_T X := -\operatorname{Pr}(\nabla_X T),$$

where $\operatorname{Pr} : T_x G \to \triangle^{\perp}(x)$ is the orthogonal projection. The tensor field $C \in \Gamma(\triangle^* \otimes \operatorname{End} \triangle^{\perp})$ is called the *splitting tensor* or the *conullity operator* of f.

To see $C \in \Gamma(\triangle^* \otimes \operatorname{End} \triangle^{\perp})$, it suffices to check that $C_{\phi T} X = \phi C_T X$ for any function ϕ. In fact, it can be verified that

$$C_{\phi T} X = -\operatorname{Pr}(\nabla_X(\phi T))$$

$$
\begin{aligned}
&= - \operatorname{Pr}\{(X\phi)T + \phi\nabla_X T\} \\
&= - \phi \operatorname{Pr}(\nabla_X T) \\
&=\phi C_T X.
\end{aligned}
$$

Proposition 4.1.5. *Let $f : M \to \mathbf{R}_N^{N+P}$ be an isometric immersion of a d-dimensional Riemannian manifold into a semi-Euclidean space. Let $\triangle$ be its relative nullity distribution on G, where the index of relative nullity is constant ν_0. Then the following hold.*

(1) *The distribution $\triangle^\perp$ is integrable if and only if*

$$g(C_T X, Y) = g(X, C_T Y) \qquad \text{for } X, Y \in \Gamma(\triangle^\perp),\ T \in \Gamma(\triangle).$$

(2) *The splitting tensor C of f vanishes identically on G if and only if each point of G has a neighborhood on which f is ν_0-cylindrical.*

Proof. (1) Since $T \in \Gamma(\triangle)$ and $Y \in \Gamma(\triangle^\perp)$ are orthogonal each other, we have

$$g(C_T X, Y) = -g(\nabla_X T, Y) = g(T, \nabla_X Y),$$

which implies

$$
\begin{aligned}
g(C_T X, Y) - g(X, C_T Y) =& g(T, \nabla_X Y) - g(T, \nabla_Y X) \\
=& g(T, [X, Y]).
\end{aligned}
$$

Therefore, $[X, Y] \in \Gamma(\triangle^\perp)$ for $X, Y \in \Gamma(\triangle^\perp)$ if and only if $g(C_T X, Y) = g(X, C_T Y)$.

(2) By Proposition 4.1.2 (2), $C \equiv 0$ if and only if $\nabla_X T$ and $\nabla_S T$ belong to $\Gamma(\triangle)$ for $S, T \in \Gamma(\triangle)$ and $X \in \Gamma(\triangle^\perp)$. It follows from this that $\triangle$ is parallel, and therefore so is $\triangle^\perp$. The rest of the proof follows from Propositions 2.2.7 and 4.1.3. $\square$

We now prepare some basic identities concerning splitting tensors and second fundamental forms.

Lemma 4.1.6. *Let $f : M \to \mathbf{R}^{N+P}_N$ be an isometric immersion of a Riemannian manifold into a semi-Euclidean space with splitting tensor C. If $S, T \in \Gamma(\triangle)$, $X, Y \in \Gamma(\triangle^\perp)$ and $\xi \in \Gamma(Nor\, f)$, then the following identities hold.*

(i) $\quad (\nabla_S C_T)X = C_T C_S X + C_{\nabla_S T} X.$

(ii) $\quad (\nabla_X C_T)Y - (\nabla_Y C_T)X = C_{\mathrm{Pr}(\nabla_X T)} Y - C_{\mathrm{Pr}(\nabla_Y T)} X.$

(iii) $\quad \nabla_T (A_\xi X) - A_\xi \nabla_T X = A_\xi C_T X + A_{\nabla^\perp_T \xi} X.$

(iv) $\quad \alpha(C_T X, Y) = \alpha(X, C_T Y),$

where $(\nabla_Z C_T)X := \mathrm{Pr}(\nabla_Z (C_T X)) - C_T \,\mathrm{Pr}(\nabla_Z X)$ *for* $Z \in \Gamma(TM)$.

Proof. Since the relative nullity distribution $\triangle$ is totally geodesic, $\nabla_S T \in \Gamma(\triangle)$ and $\nabla_S X \in \Gamma(\triangle^\perp)$. Hence, using the Gauss equation, we compute

$$
\begin{aligned}
&(\nabla_S C_T)X \\
=& -\mathrm{Pr}(\nabla_S \,\mathrm{Pr}(\nabla_X T)) - C_T \nabla_S X \\
=& -\mathrm{Pr}(\nabla_S \nabla_X T) - C_T \nabla_S X \\
=& -\mathrm{Pr}(R^\nabla(S, X)T + \nabla_X \nabla_S T + \nabla_{[S,X]} T) - C_T \nabla_S X \\
=& -\mathrm{Pr}(R^\nabla(S, X)T) + C_{\nabla_S T} X - \mathrm{Pr}(\nabla_{(\nabla_S X - \nabla_X S)} T) - C_T \nabla_S X \\
=& -\mathrm{Pr}(R^\nabla(S, X)T) + C_{\nabla_S T} X + C_T \nabla_S X - C_T \,\mathrm{Pr}(\nabla_X S) - C_T \nabla_S X \\
=& -\mathrm{Pr}(R^\nabla(S, X)T) + C_{\nabla_S T} X + C_T C_S X \\
=& \quad 0 + C_{\nabla_S T} X + C_T C_S X,
\end{aligned}
$$

which verifies (i).

Let $Q : T_xU \to \triangle(x)$ be the orthogonal projection. Then we have

$$
\begin{aligned}
&(\nabla_X C_T)Y \\
=&\operatorname{Pr}(\nabla_X(C_T Y)) - C_T \operatorname{Pr}(\nabla_X Y) \\
=&-\operatorname{Pr}(\nabla_X \operatorname{Pr}(\nabla_Y T)) - C_T \operatorname{Pr}(\nabla_X Y) \\
=&-\{\operatorname{Pr}(\nabla_X \nabla_Y T) - \operatorname{Pr}(\nabla_X Q(\nabla_Y T))\} + \operatorname{Pr}(\nabla_{\operatorname{Pr}(\nabla_X Y)} T) \\
=&-\operatorname{Pr}(\nabla_X \nabla_Y T) - C_{Q(\nabla_Y T)} X + \operatorname{Pr}(\nabla_{\operatorname{Pr}(\nabla_X Y)} T),
\end{aligned}
$$

which, together with the Gauss equation, implies that

$$
\begin{aligned}
&(\nabla_X C_T)Y - (\nabla_Y C_T)X \\
=&-\operatorname{Pr}(\nabla_X \nabla_Y T - \nabla_Y \nabla_X T) - (C_{Q(\nabla_Y T)} X - C_{Q(\nabla_X T)} Y) \\
&+ \operatorname{Pr}(\nabla_{(\operatorname{Pr}(\nabla_X Y) - \operatorname{Pr}(\nabla_Y X))} T) \\
=&-\operatorname{Pr}(R^\nabla(X,Y)T + \nabla_{[X,Y]} T) - (C_{Q(\nabla_Y T)} X - C_{Q(\nabla_X T)} Y) \\
&+ \operatorname{Pr}(\nabla_{\operatorname{Pr}([X,Y])} T) \\
=&-\operatorname{Pr}(R^\nabla(X,Y)T) - \operatorname{Pr}(\nabla_{Q([X,Y])} T) - (C_{Q(\nabla_Y T)} X - C_{Q(\nabla_X T)} Y) \\
=&\quad 0 + 0 + (C_{Q(\nabla_X T)} Y - C_{Q(\nabla_Y T)} X),
\end{aligned}
$$

verifying (ii).

To prove (iii) we compute, using the Codazzi equation, to get

$$
\begin{aligned}
&\nabla_T(A_\xi X) - A_\xi \nabla_T X \\
=&\nabla_X(A_\xi T) - A_\xi \nabla_X T - A_{\nabla^\perp_X \xi} T + A_{\nabla^\perp_T \xi} X \\
=&0 - A_\xi \nabla_X T - 0 + A_{\nabla^\perp_T \xi} X \\
=&A_\xi C_T X + A_{\nabla^\perp_T \xi} X.
\end{aligned}
$$

We proceed to prove (iv). It follows from (iii) that

$$\Pr(A_\xi C_T) = \Pr(\nabla_T A_\xi) - \Pr(A_{\nabla_T^\perp \xi}).$$

Hence $\Pr(A_\xi C_T)$ is symmetric, that is,

$$g(A_\xi C_T X, Y) = g(A_\xi C_T Y, X).$$

Therefore, we obtain

$$\alpha(C_T X, Y) = \alpha(X, C_T Y). \qquad \square$$

When the relative nullity foliation is complete, we obtain the following property for splitting tensors.

Lemma 4.1.7. *Let $f : M \to \mathbf{R}_N^{N+P}$ be an isometric immersion of a d-dimensional Riemannian manifold with splitting tensor C. Suppose that the relative nullity foliation $\triangle$ is complete. Then the only possible real eigenvalue of $C_{T_0} : \triangle^\perp(x_0) \to \triangle^\perp(x_0)$ $(T_0 \in \triangle(x_0))$ is zero.*

Proof. Let L be the leaf of $\triangle$ through x_0, and γ the geodesic in L such that $\gamma(0) = x_0$ and $\dot\gamma(0) = T_0$. We take a parallel frame field $\{e_1(t), \ldots, e_{d-\nu}(t)\}$ of $\triangle^\perp$ along γ. Then, by Lemma 4.1.6 (i) and the completeness of L, C satisfies the following ordinary differential equation for $t \in \mathbf{R}$:

$$\begin{cases} C'_{\dot\gamma(t)} = C^2_{\dot\gamma(t)}, \\ C_{\dot\gamma(0)} = C_{T_0}. \end{cases}$$

Now suppose that C_{T_0} has nonzero real eigenvalues $\lambda_1, \ldots, \lambda_k$, and set $\tau := (\max|\lambda_i|)^{-1} (> 0)$. Then we may define the operator C_t by

$$C_t := C_{T_0}(\mathrm{id}_{\triangle^\perp(x_0)} - tC_{T_0})^{-1} \quad \text{for } -\tau < t < \tau,$$

since $|t\lambda_i| < 1$ and the operator $\mathrm{id}_{\triangle^\perp(x_0)} - tC_{T_0}$ is invertible for $-\tau < t < \tau$. It is then verified that C_t satisfies the same differential equation for $-\tau < t < \tau$:

$$\begin{cases} C_t' = C_t^2, \\ C_0 = C_{T_0}, \end{cases}$$

and has an eigenvalue $(\tau - t)^{-1}$. In fact, it is easy to see

$$\begin{aligned} C_t' =& C_{T_0}\{-(\mathrm{id}_{\triangle^\perp(x_0)} - tC_{T_0})^{-1}(\mathrm{id}_{\triangle^\perp(x_0)} - tC_{T_0})'(\mathrm{id}_{\triangle^\perp(x_0)} - tC_{T_0})^{-1}\} \\ =& C_t^2, \end{aligned}$$

and

$$\begin{aligned} & |C_t - \frac{1}{\tau - t}\,\mathrm{id}_{\triangle^\perp(x_0)}| \\ =& |C_{T_0}(\mathrm{id}_{\triangle^\perp(x_0)} - tC_{T_0})^{-1} - \frac{1}{\tau - t}\,\mathrm{id}_{\triangle^\perp(x_0)}| \\ =& |C_{T_0} - \frac{1}{\tau - t}(\mathrm{id}_{\triangle^\perp(x_0)} - tC_{T_0})||\,\mathrm{id}_{\triangle^\perp(x_0)} - tC_{T_0}|^{-1} \\ =& \frac{\tau}{\tau - t}|C_{T_0} - \frac{1}{\tau}\,\mathrm{id}_{\triangle^\perp(x_0)}||\,\mathrm{id}_{\triangle^\perp(x_0)} - tC_{T_0}|^{-1} \\ =& 0. \end{aligned}$$

Then, by virtue of the uniqueness theorem of solutions for ordinary differential equations, we have $C_{\dot\gamma(t)} = C_t$ and hence C_t can be defined for all $t \in \mathbf{R}$. However, this is impossible, since the eigenvalue $(\tau - t)^{-1}$ of C_t blows up as $t \to \tau$. $\square$

We now prove that the splitting tensor of an isometric pluriharmonic immersion is complex linear.

Lemma 4.1.8. *Let $f : M \to \mathbf{R}_N^{N+P}$ be an isometric pluriharmonic immersion of a Kähler manifold with complex structure J. Then the* *splitting tensor C of f satisfies*

$$\text{(i)} \qquad C_{JT}X = JC_TX,$$

$$\text{(ii)} \qquad C_TJX = JC_TX \qquad \textit{for } X \in \Gamma(\triangle^{\perp}),\ T \in \Gamma(\triangle).$$

Proof. Since J is parallel and $\triangle$ is J-invariant by definition, we have

$$\begin{aligned}\text{(i)} \qquad C_{JT}X =& -\operatorname{Pr}(\nabla_X JT) = -\operatorname{Pr}(J\nabla_X T) = -J\operatorname{Pr}(\nabla_X T)\\ =& JC_TX.\end{aligned}$$

It then follows from Lemma 4.1.6 (iv) and (i) that for $Y \in \Gamma(\triangle^{\perp})$,

$$\begin{aligned}\alpha(C_TJX, Y) =& \alpha(JX, C_TY)\\ =& \alpha(X, JC_TY)\\ =& \alpha(X, C_{JT}Y)\\ =& \alpha(C_{JT}X, Y)\\ =& \alpha(JC_TX, Y),\end{aligned}$$

which implies $C_TJX - JC_TX \in \Gamma(\triangle)$, and hence (ii) follows. $\square$

We are now going to prove the following cylinder theorem for isometric pluriharmonic immersions, under appropriate assumptions on the index of relative nullity and the completeness of Kähler manifolds. In the positive definite case, this theorem has been obtained by M. Dajczer and L. Rodriguez [14].

Theorem 4.1.9. *Let M be a complete Kähler manifold of real dimension $2m$, and $f : M \to \mathbf{R}_N^{N+P}$ an isometric pluriharmonic immersion. If* *the index of relative nullity ν is not less than $2m-2$, then f is $(2m-2)$-cylindrical.*

Proof. Let G be an open set on which the index of relative nullity ν of f is equal to $2m-2$. We fix $x \in G$ and $T \in \triangle(x)$ arbitrarily.

Claim. The splitting tensor C_T is nilpotent.

To see the claim, we assume that $a + \sqrt{-1}b \in \mathbf{C}$ is an eigenvalue of C_T, that is,

$$C_T Y = (a + \sqrt{-1}b)Y = aY + bJY.$$

If we put $S := aT - bJT \in \triangle(x)$, then by Lemma 4.1.8 (i) we get

$$\begin{aligned} C_S Y =& aC_T Y - bJC_T Y \\ =& a(aY + bJY) - bJ(aY + bJY) \\ =& (a^2 + b^2)Y, \end{aligned}$$

and hence C_S has a real eigenvalue $a^2 + b^2$. Then it follows from Lemma 4.1.7 that $a^2 + b^2 = 0$, which implies that the eigenvalue of C_T is zero.

Since $\dim \triangle^{\perp}(x) = 2$, we have $C_T^2 = 0$. Consequently, $C_T = 0$. To see this, using a basis such that

$$J|_{\triangle^{\perp}(x)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

we write C_T as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then it is immediate from Lemma 4.1.8 (ii) that

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = 0,$$

which implies $a = b = c = d = 0$.

To sum up, we conclude that $C = 0$ on G. Hence, by Proposition 4.1.5 (2) and the analyticity of f, the isometric pluriharmonic immersion f is $(2m-2)$-cylindrical. $\Box$

For isometric minimal immersions of complete Kähler manifolds of codimension one, we can prove a stronger cylinder theorem. To prove this, we first show the following proposition, which has been proved by K. Abe [2] in the positive definite case.

Proposition 4.1.10. *Let M be a Kähler manifold of real dimension $2m$. If $f : M \to \mathbf{R}_N^{2m+1}$ is an isometric immersion of real codimension one, then the index of relative nullity ν of f is not less than $2m - 2$.*

Proof. We take a normal vector field ξ of f, and denote the shape operator A_ξ by A for simplicity. Let $\lambda_1, \dots, \lambda_{2m}$ be the principal curvatures of f, that is, the eigenvalues of A, and let $\{e_1, \dots, e_{2m}\}$ be the corresponding principal frame, that is, the frame consisting of eigenvectors of A. Then, by the Gauss equation, we get

$$\begin{aligned}
&g(R^\nabla(e_i, e_j)Je_i, e_k) \\
=&\langle \alpha(e_k, e_i), \alpha(Je_i, e_j)\rangle_{\mathbf{R}_N^{2m+1}} - \langle \alpha(e_k, e_j), \alpha(Je_i, e_i)\rangle_{\mathbf{R}_N^{2m+1}} \\
=&g(Ae_i, e_k)g(Ae_j, Je_i) - g(Ae_j, e_k)g(Ae_i, Je_i) \\
=&\lambda_i \delta_{ik} \lambda_j g(e_j, Je_i),
\end{aligned}$$

which implies

$$R^\nabla(e_i, e_j)Je_i = \lambda_i \lambda_j g(e_j, Je_i)e_i.$$

In a similar fashion, for $i \neq j$, we also get

$$
\begin{aligned}
& g(R^\nabla(e_i, e_j)e_i, Je_k) \\
=& g(Ae_i, Je_k)g(Ae_j, e_i) - g(Ae_j, Je_k)g(Ae_i, e_i) \\
=& -\lambda_i \lambda_j g(e_j, Je_k),
\end{aligned}
$$

which implies

$$JR^\nabla(e_i, e_j)e_i = -\lambda_i \lambda_j Je_j.$$

Since $R^\nabla(X, Y)J = JR^\nabla(X, Y)$, we then obtain

$$\lambda_i \lambda_j (g(e_j, Je_i)e_i + Je_j) = 0 \quad \text{for } i \neq j.$$

If λ_1 is not zero, then for $j \neq 1$, we have either $\lambda_j = 0$ or $g(e_j, Je_1)e_1 + Je_j = 0$. The latter can be true for at most one j, say $j = 2$, and then $\lambda_j = 0$ for $j \geq 3$. Therefore, we conclude that $\operatorname{rank} A \leq 2$. $\square$

Combining Propositions 2.3.4, 4.1.9 and 4.1.10, we obtain

Proposition 4.1.11. *Let M be a complete Kähler manifold of real dimension $2m$. If $f : M \to \mathbf{R}_N^{2m+1}$ is an isometric minimal immersion of real codimension one, then f is $(2m-2)$-cylindrical.*

Before proceeding to the case of codimension two, the following definition is in order.

Definition 4.1.12. Let $f : M \to \mathbf{R}_N^{N+P}$ be an isometric immersion of a Kähler manifold M of real dimension $2m$ into $\mathbf{R}_N^{N+P}$. f is called *completely complex ruled* if M has a real codimension two foliation such that each leaf is a Kähler submanifold of M and its image under f is an affine subspace of real dimension $2m-2$.

The following proposition has been proved by M. Dajczer and L. Rodriguez [14] in the positive definite case.

Proposition 4.1.13. *Let M be a complete Kähler manifold of real dimension $2m \geq 4$, and $f : M \to \mathbf{R}_N^{N+P}$ an isometric pluriharmonic immersion. Suppose that the index of relative nullity ν is not less than $2m-4$. Then f is either completely complex ruled or $(2m-4)$-cylindrical.*

Sketch of proof. It follows from Theorem 4.1.9 and its proof that if $\nu \geq 2m-2$ everywhere, then f is $(2m-2)$-cylindrical, and that if M has a non-empty open subset on which $\nu = 2m-4$ and the splitting tensor $C = 0$, then f is $(2m-4)$-cylindrical.

Let U be a connected component of the open set on which $\nu = 2m-4$ and where there exists a vector $T \in \triangle$ such that $C_T \neq 0$. Given any point $x \in U$ and any vector $T \in \triangle(x)$ such that $C_T \neq 0$, it can be verified that $\dim \ker C_T = 2$, by Lemma 4.1.8 (2) together with the claim in the proof of Theorem 4.1.9. It also holds that $\alpha(X, Y) = 0$ for $X, Y \in \ker C_T$ by Lemma 4.1.6 (iv). For any other vector $S \in \triangle(x)$ such that $C_S \neq 0$, we can prove $\ker C_T = \ker C_S$. Since the $(2m-2)$-dimensional distribution $\triangle \oplus \ker C_T$ is integrable and totally geodesic, it then follows that f is completely complex ruled. $\square$

We remark that, combining Proposition 2.3.5 with Proposition 4.1.13, M. Dajczer and L. Rodriguez [14] prove the following theorem, which classifies isometric minimal immersions of complete Kähler manifolds into Euclidean spaces of real codimension two.

Proposition 4.1.14. *Let M be a complete Kähler manifold of real dimension $2m \geq 4$, and $f : M \to \mathbf{R}^{2m+2}$ an isometric minimal immersion of real codimension two with the index of relative nullity ν.*

(1) *If there exists a point $x \in M$ such that $\nu(x) < 2m - 4$, then f is holomorphic with respect to some orthogonal complex structure of $\mathbf{R}^{2m+2}$.*

(2) *If $\nu \geq 2m - 4$ everywhere, then f is either completely complex ruled or $(2m - 4)$-cylindrical.*

4.2. Bernstein property

It has been proved by E. Calabi [7] that the only complete spacelike minimal surface in $\mathbf{R}^3_1$ is a plane (See also O. Kobayashi [21]). This proves that the classical Bernstein theorem for minimal surfaces in $\mathbf{R}^3$ is also true when the ambient space is replaced by Minkowski 3-space $\mathbf{R}^3_1$. In this section, we are concerned with some generalizations of this Bernstein property.

In his paper [20], T. Ishihara proves the following proposition by a standard technique which has been used in S. -Y. Cheng and S. -T. Yau [8], and S. Nishikawa [24], for instance.

Proposition 4.2.1. *Let M be a d-dimensional Riemannian manifold, and $f : M \to \mathbf{R}^{d+N}_N$ an isometric minimal immersion. If M is complete, then f is totally geodesic.*

This result can not be extended further when the index of the ambient space is less than the codimension N. In fact, F. J. M. Estudillo and A. Romero [16] give the following example.

Example 4.2.2. We put

$$\phi(z) := \frac{1}{2}{}^t(e^z - 2e^{-z}, e^z + 2e^{-z}, -3\sqrt{-1}, -1)$$

$$\text{for } z = x + \sqrt{-1}y \in \mathbf{C}.$$

Then, we have

$$f(x, y) := \mathrm{Re} \int \phi dz$$

$$=\frac{1}{2}{}^t((e^x+e^{-x})\cos y,(e^x-2e^{-y})\cos y,3y,-x),$$

which gives rise to a nontrivial minimal immersion of a complete Kähler manifold biholomorphic to $\mathbf{C}$ into $\mathbf{R}^4_1$.

In fact, it is verified that $\langle\phi(z),\phi(z)\rangle_{\mathbf{C}^5_1}=0$ and $\langle\phi(z),\overline{\phi(z)}\rangle_{\mathbf{C}^5_1}=10/4+2\cos 2y$.

Roughly speaking, the following proposition means that an isometric pluriharmonic immersion is totally geodesic if its tangent vectors are apart from the orthogonal complement of some timelike vector.

Proposition 4.2.3. *Let M be a Kähler manifold biholomorphic to $\mathbf{C}^m$ with a global complex coordinate system $(z^1,\dots,z^m)$ on M. Let $f: M\to\mathbf{R}^{2m+p}_N$ be an isometric pluriharmonic immersion, where $N=0$ or 1. Suppose that there exist a constant unit vector $e\in\mathbf{R}^{2m+p}_N$ and a positive constant ϵ such that*

$$\begin{aligned}&e \text{ is timelike}\quad \text{when } N=1,\\ &|\langle e,\overline{\frac{\partial f}{\partial z^j}}\rangle_{\mathbf{C}^{2m+p}_1}|^2>0,\\ &|\langle e,\overline{\frac{\partial f}{\partial z^j}}\rangle_{\mathbf{C}^{2m+p}_1}|^2\geq\epsilon|\langle\frac{\partial f}{\partial z^j},\overline{\frac{\partial f}{\partial z^j}}\rangle_{\mathbf{C}^{2m+p}_1}|\quad \text{for } j=1,\dots,m.\end{aligned}$$

Then f is totally geodesic.

Proof. We may consider, without loss of generality, that the constant vector e is ${}^t(1,0,\dots,0)$. Then, from $f={}^t(f^1,\dots,f^{2m+p})$, we can define functions ψ^k_j on M by

$$\psi^k_j(z):=\frac{\dfrac{\partial f^k}{\partial z^j}(z)}{\dfrac{\partial f^1}{\partial z^j}(z)}\quad \text{for } j=1,\dots,m \text{ and } k=1,\dots,2m+p,$$

because M has global coordinates and $|\frac{\partial f^1}{\partial z^j}| = |\langle e, \overline{\frac{\partial f}{\partial z^j}} \rangle_{\mathbf{C}_1^{2m+p}}| > 0$. Since f is pluriharmonic, ψ_j^k are holomorphic. Moreover, it is observed that ψ_j^k are bounded by the assumption. In fact, we see that when $N = 0$,

$$\left| \frac{\frac{\partial f^k}{\partial z^j}}{\frac{\partial f^1}{\partial z^j}} \right|^2 \leq \frac{\langle \frac{\partial f}{\partial z^j}, \overline{\frac{\partial f}{\partial z^j}} \rangle_{\mathbf{C}^{2m+p}}}{|\frac{\partial f^1}{\partial z^j}|^2} \leq \frac{1}{\epsilon},$$

and when $N = 1$,

$$\left| \frac{\frac{\partial f^k}{\partial z^j}}{\frac{\partial f^1}{\partial z^j}} \right|^2 \leq \frac{\langle \frac{\partial f}{\partial z^j}, \overline{\frac{\partial f}{\partial z^j}} \rangle_{\mathbf{C}_1^{2m+p}} + |\frac{\partial f^1}{\partial z^j}|^2}{|\frac{\partial f^1}{\partial z^j}|^2} \leq 1 + \frac{1}{\epsilon}.$$

Therefore, ψ_j^k are constant functions, and hence there exist constants $c_j^k \in \mathbf{C}$ such that

$$\frac{\partial f^k}{\partial z^j}(z) = c_j^k \frac{\partial f^1}{\partial z^j}(z) \quad \text{for } j = 1, \dots, m \text{ and } k = 1, \dots, 2m + p.$$

Hence we have

$$\begin{aligned} f(z) =& 2\,\mathrm{Re} \int_0^z \sum_{j=1}^m \frac{\partial f}{\partial z^j} dz^j = 2\,\mathrm{Re} \int_0^z \sum_{j=1}^m \begin{bmatrix} 1 \\ c_j^2 \\ \vdots \\ c_j^{2m+p} \end{bmatrix} \frac{\partial f^1}{\partial z^j} dz^j \\ =& 2\,\mathrm{Re} \sum_{j=1}^m \begin{bmatrix} 1 \\ c_j^2 \\ \vdots \\ c_j^{2m+p} \end{bmatrix} F_j(z), \end{aligned}$$

where F_j is a holomorphic function. This implies that f is totally geodesic. $\square$

References

[1] Abe, K., *A characterization of totally geodesic submanifolds in S^N and $\mathbf{C}P^N$ by an inequality*, Tôhoku Math. J. **23** (1971), 219–244.

[2] Abe, K., *On a class of hypersurfaces of* $\mathbf{R}^{2n+1}$, Duke Math. J. **41** (1974), 865–874.

[3] Abe, K. and Magid, M. A., *Relative nullity foliations and indefinite isometric immersions*, Pacific J. Math. **124** (1986), 1–20.

[4] Abe, K. and Magid, M. A., *Indefinite rigidity of complex submanifolds and maximal surfaces*, Math. Proc. Camb. Phil. Soc. **106** (1989), 481–494.

[5] Calabi, E., *Isometric imbeddings of complex manifolds*, Ann. of Math. **58** (1953), 1–23.

[6] Calabi, E., *Quelques applications de l'analyse complexe aux surfaces d'aire minima*, Topics in complex manifolds (by Rossi, H.), Univ. Montreal, 1968, pp. 59–81.

[7] Calabi, E., *Examples of Bernstein problems for some nonlinear equations*, Proc. Sympos. Pure Appl. Math. **15** (1970), 223–230.

[8] Cheng, S. -Y. and Yau, S. -T., *Maximal space-like hypersurfaces in the Lorentz-Minkowski spaces*, Ann. of Math. **104** (1976), 407–419.

[9] Dajczer, M., *A characterization of complex hypersurfaces in* $\mathbf{C}^n$, Proc. Amer. Math. Soc. **105** (1989), 425–428.

[10] Dajczer, M. and Gromoll, D., *Real Kaehler submanifolds and uniqueness of the Gauss maps*, J. Differential Geom. **22** (1985), 13–28.

[11] Dajczer, M. and Gromoll, D., *Rigidity of complete Euclidean hypersurfaces*, J. Differential Geom. **31** (1990), 401–416.

[12] Dajczer, M. and Gromoll, D., *The Weierstrass representation for complete minimal real Kaehler submanifolds of codimension two,*

Invent. Math. **119** (1995), 235–242.

[13] Dajczer, M. and Rodriguez, L., *Rigidity of real Kaehler submanifolds*, Duke Math. J. **53** (1986), 211–220.

[14] Dajczer, M. and Rodriguez, L., *Complete real Kaehler minimal submanifolds*, J. Reine Angew. Math. **419** (1991), 1–8.

[15] Dajczer, M., Antonucci, M., Oliveira, G., Lima-Filho, P. and Tojeiro, R., *Submanifolds and Isometric Immersions*, Publish or Perish, Inc., Houston, 1990.

[16] Estudillo, F. J. M. and Romero, A., *On maximal surfaces in the n-dimensional Lorentz-Minkowski space*, Geom. Dedicata **38** (1991), 167–174.

[17] Ferreira, M. J. and Tribuzy, R., *On the type decomposition of the second fundamental form of a Kähler submanifold*, Preprint.

[18] Furuhata, H., *Construction and classification of isometric minimal immersions of Kähler manifolds into Euclidean spaces*, Bull. London Math. Soc. **26** (1994), 487–496.

[19] Furuhata, H., *Moduli space of isometric pluriharmonic immersions of Kähler manifolds into indefinite Euclidean spaces*, Pacific J. Math. (to appear).

[20] Ishihara, T., *Maximal spacelike submanifolds of a pseudoriemannian space of constant curvature*, Michigan Math. J. **35** (1988), 345–352.

[21] Kobayashi, O., *Maximal surfaces in the 3-dimensional Minkowski space* L^3, Tokyo J. Math. **6** (1983), 297–309.

[22] Lawson, H. B. Jr., *Lectures on Minimal Submanifolds Volume I*, Publish or Perish, Inc., Berkeley, 1980.

[23] Moore, J. D., *Isometric immersions of Riemannian products*, J. Differential Geom. **5** (1971), 159–168.

[24] Nishikawa, S., *On maximal spacelike hypersurfaces in a Lorentzian manifold*, Nagoya Math. J. **95** (1984), 117–124.

[25] O'Neill, B., *Semi-Riemannian Geometry with applications to relativity*, Academic Press, Inc., San Diego, 1983.

[26] Spivak, M., *A Comprehensive Introduction to Differential Geometry*, Volume IV, Second edition, Publish or Perish, Inc., Wilmington, 1979.

[27] Udagawa, S., *Minimal immersions of Kaehler manifolds into complex space forms*, Tokyo J. Math. **10** (1987), 227–239.

[28] Umehara, M., *Diastases and real analytic functions on complex manifolds*, J. Math. Soc. Japan **40** (1988), 519–539.