

Lawson's problem to Chern's conjecture

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Abstract.

This is a survey of my main research. I avoid detailed descriptions and add some episodes for fun.

- §1 Lawson's problem and Calabi's rigidity theorem
- §2 Relation to the 2-dimensional Affine Toda equation
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- §4 Topology of stable minimal submanifolds
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- §6 Proper Dupin \Rightarrow taut \Rightarrow Dupin
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- §9 Lie contact structure and resolution of Sato's problem
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- §12 Chern's conjecture: Yau's 105th problem

The last one is new.

§1. Lawson's problem and Calabi's rigidity theorem

It is well-known that in the PhD thesis, B. Lawson constructed compact minimal surfaces of all topological types except for $\mathbb{R}P^2$ in S^3 . But here, we mention his another beautiful result on minimal surfaces, inspired by Calabi's "Rigidity of Kähler manifolds."

Theorem 1.1 (Calabi (1953)[2]). *The existence of an isometric embedding of a complex manifold with analytic Kähler metric into a complex space form is determined by the diastasis condition.*

The diastasis condition means a relation among coefficients of the analytic expansion of the embedding. By the way, in 2021, the collected

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works of Calabi was published [3] to celebrate his 100th birthday in 2023. However, he passed away soon after his birthday.

It is well known that the metric ds^2 of a minimal surface in \mathbb{R}^3 satisfies the Ricci condition: $\sqrt{-K}ds^2$ is flat (K is the Gauss curvature). Using Calabi's rigidity theorem, Lawson proves:

Theorem 1.2 (Lawson (1971) [27]). *If the metric of a minimal surface M in \mathbb{R}^n satisfies the Ricci condition, M lies fully in \mathbb{R}^3 , or in $\mathbb{R}^3 + \mathbb{R}^3$ as a 2-parameter family of associated minimal surfaces.*

How is this related to Calabi's rigidity theorem? A minimal surface $x : M \rightarrow \mathbb{R}^n$ is given by $x = \Re\psi$, where $\psi : M \rightarrow \mathbb{C}^n$ is a holomorphic curve (generalized Weierstrass representation). If the metric of M satisfies the Ricci condition, the Gauss map $\phi = \partial\psi : M \rightarrow Q_{n-2}(\mathbb{C}) \subset \mathbb{C}P^{n-1}$ is a holomorphic curve of curvature 1 (see §10 for the definition of $Q_{n-2}(\mathbb{C})$). Then, applying Calabi's rigidity theorem, Lawson shows ψ lies in \mathbb{C}^6 , and x lies in \mathbb{R}^3 or in $\mathbb{R}^3 + \mathbb{R}^3$.

Next, he poses a natural question:

The metric of a minimal surface in S^3 satisfies the spherical Ricci condition: $\sqrt{1-K}ds^2$ is flat. If the metric of a minimal surface M in S^n satisfies the spherical Ricci condition, does M decompose into minimal surfaces in S^3 ?

Note that minimal surfaces in spheres have no direct relation with holomorphic curves.

Using the analytic expansion, Calabi has begun studying minimal surfaces in S^n (1967). Later, it develops into the harmonic sequence, the twistor theory, etc. The origin is Calabi's work, which I recognized recently, looking back at his works.

I was interested in Lawson's problem as a graduate student and using the expansion without knowing Calabi's work, I obtained:

Theorem 1.3 (M. (1978) [29]). *If the metric of a minimal surface M in S^n satisfies the spherical Ricci condition, and if M is not a torus, then*

- (1) when $n < 6$, M fully lies in S^3 .
- (2) when M is compact and $n < 10$, then M is a direct sum of minimal surfaces in S^3 .

I gave a presentation on this result at the Japan-US seminar in Tokyo in 1977, which was organized by S.S. Chern and T. Otsuki, my supervisor, in the presence of Osserman, Lawson, Yau, Uhlenbeck, Atiyah, Berger, Almgren, Simons, Nitché and many other geometers. Then, on the spot, W. Klingenberg invited me to the University of Bonn, supported by Lawson's recommendation. The photo of the participants with each name is in the book [29] (my name was R. Naka at that time).

§2. Relation to the 2-dimensional Affine Toda equation

Almost 20 years later, while I was studying integrable systems, I noticed that Lawson's problem is essentially related to the periodic solutions of the 2-dimensional Affine Toda equation, and obtained the following result:

Theorem 2.1 (M.(1996) [38]). *The metric of the so-called superconformal minimal surface M in an odd dimensional sphere S^n corresponds to a periodic solution of the 2-dimensional Affine Toda equation. Then M is decomposed orthogonally into minimal surfaces in certain lower dimensional S^k (k relates to the period of the solution).*

Corollary 2.2. *Lawson's conjecture is the case $k = 3$, and true for non-flat superconformal minimal surfaces in an odd dimensional S^n .*

As is well-known, the relationship between the surface theory and the theory of integrable systems has become very close since the 1980s.

When A. Ros participated in the workshop in Takamatsu 2024, he asked me about superconformal minimal surfaces in spheres. His interest is in the normalized first eigenvalue estimate of compact surfaces by genus, and he says the bound must be taken by a special surface, such as a superconformal one. It's an honor he knew my result, and I expect the case to play an important role in the future.

§3. Exceptional values of the Gauss map of minimal surfaces

Around the 1980's, H. Fujimoto obtained the final result on the upper bound of the number of exceptional values of the Gauss map of complete minimal surfaces in \mathbb{R}^3 [21].

I remember that one day, Professor X came to my office, bringing Fujimoto's preprint, and asked me to read it. It was a hard analysis, and I felt there was no geometry. As time passed, it appeared in the journal of MSJ (1988) [21], and I noticed that Professor X was the referee. Later, I checked all the calculations and found that the estimate was just barely.

On the other hand, there remains a question in the case of finite total curvature, whether the number of possible exceptional values is limited to 2 or 3 (4 in general by Fujimoto). Since the only obvious known example whose Gauss map omits two values is a catenoid, I became interested in the construction of other examples.

I obtained examples in the case of punctured spheres and punctured tori, in the latter, I used the Weierstrass p function. Then, applying the covering method, I constructed minimal surfaces of almost all topological types whose Gauss maps omit two values.

As good luck would have it, a graduate student, K. Sato of Professor Y, is known to be very good at computer graphics. So I handed all the data to him and asked him to make graphics. He was so fast in obtaining interesting graphics, including punctured tori.

Then I sent the paper with those graphics to Professor Z to submit to *Archiv der Math.*, but no reply came. A year later, I asked Professor Z, and he replied that he was so sorry to have completely forgotten it. Soon after, the paper was accepted and published ([44] 1994).

Professor Y appreciated it as Sato had not yet published a paper needed to get a doctoral degree.

The reason why I describe this story is that, around 20 years later, the simplest example that I constructed played a very important role. Actually, a graduate student, Y. Kawakami of Professor R. Kobayashi, found out that the Gauss map of that surface not only omits two values but also has "totally ramified value number 2.5" [24]. This was amazing since, until then, we three (KKM) believed that the upper bound of the number of exceptional values and the totally ramified value number were the same. Because of this shocking matter, the problem turns out far beyond to solve and is still open [25].

§4. Topology of stable minimal submanifolds

Minimal submanifolds in a positively curved manifold are unlikely to be stable as great circles of S^2 [59]. As expected, stable minimal submanifolds have simple topology in non-negatively curved spaces. It was first proved independently:

Theorem 4.1 (Do Carmo-Peng [18], Fischer-Colbrie-Schoen [20]). *A smooth, orientable, complete, stable minimal surface in \mathbb{R}^3 is a plane.*

However, in \mathbb{R}^n , $n > 7$, Bombieri, De Giorgi, and Giusti show the existence of non-linear minimal graphs [1], i.e., stable ones. Then, it is natural to seek a topological restriction, which implies instability. B. Palmer shows:

Theorem 4.2 (Palmer (1991) [50]). *Let $M \subset \mathbb{R}^{n+1}$ be a complete, orientable minimal hypersurface. Suppose there exists a codimension one cycle C in M which does not separate M . Then M is unstable.*

This gives side evidence to Theorem 4.1, to the effect that a stable minimal surface in \mathbb{R}^3 has no genus, because if it had a handle, a cross section of the handle does not separate M . In the proof, Palmer uses L^2 harmonic 1-form as a dual of the cross section, or codimension one cycles in higher dimensional case, and shows a contradiction by finding an area decreasing direction. This method has been widely developed by many geometers after I show:

Theorem 4.3 (M. (1993) [37]). *Let M be a complete, non-compact, stable minimal hypersurface in a manifold with non-negative sectional curvature. Then there exist no non-trivial L^2 -harmonic 1-forms on M , and any codimension one cycle C disconnects M .*

§5. Study of wave fronts: Yau's 34th problem

Wave fronts developing at a constant speed are called isoparametric hypersurfaces. Typical examples are light waves, and the subject's origin is geometric optics in the early 20th in Italy [26], [60]. In the Euclidean and hyperbolic spaces, they are only umbilic hypersurfaces and cylinders. É. Cartan first treated the problem uniformly in the space forms and found more examples in the Euclidean sphere S^n [4].

Such hypersurfaces exist in a family of parallel constant mean curvature (CMC) hypersurfaces, and in fact, they have constant *principal* curvatures if the ambient space is a real space form. Each hypersurface

is a piece of a complete hypersurface, namely, extendable to a global wave front.

In the case S^n , the number g of distinct principal curvatures is restricted to 1,2,3,4,6, with multiplicities m_1, m_2 every other one for $g = 2, 4$, $m = m_i \in \{1, 2, 4, 8\}$ for $g = 3$, and $m = m_i \in \{1, 2\}$ for $g = 6$. They are completely classified (Yau's 34th problem [64]) and well investigated:

$g = 1$: a hypersphere

$g = 2$: a product of two spheres, called the Clifford hypersurface

$g = 3$: Isotropy orbits of the symmetric spaces, $SU(3)/SO(3)$, $SU(3) \times SU(3)/SU(3)$, $SU(6)/Sp(3)$ and E_6/F_4 , and are called Cartan hypersurfaces [4].

$g = 4$: Infinitely many homogeneous and non-homogeneous hypersurfaces of the so-called OT-FKM type [49], [19], other than two exceptional homogeneous ones [7], [14].

$g = 6$: Isotropy orbits of $G_2/SO(4)$ [17], and $G_2 \times G_2/G_2$ [40],[41]. My contribution is the last case:

Theorem 5.1 (M. (2013, 2016) [40],[41]). *An isoparametric hypersurfaces in S^{13} with $(g, m) = (6, 2)$ are G_2 orbits of the isotropy action of the symmetric space $G_2 \times G_2/G_2$ [40],[41].*

I was asked by Professor W three times to write a joint paper on the above theorem while trying to prove it. I always declined because we have never done joint work on it. I met him and talked with him several times, and I respect him as a geometer. However, when he said that more people would be interested in the paper if we would write it together, by which he meant I was no big deal, I felt bad, and, anyway, as we did not work together, I declined.

A complete reversal occurred after my paper appeared in Ann. Math. He and his student, Dr. A, found a mistake in it, which, to my regret, I should have noticed as the case was obtained by myself before in the case of multiplicity one [39]. Anyway, I recovered it soon and settled the problem [41].

§6. Proper Dupin \Rightarrow taut \Rightarrow Dupin

The notion of wave fronts is invariant under conformal transformations and dilations, i.e., belongs to the category of the Lie sphere geometry [5]. The Lie contact transformation group $O(n+1, 2)$ acts on the unit tangent bundle T_1S^n , which preserves the set of oriented

hypersurfaces of S^n , keeping oriented contact among them. It acts not on a hypersurface in S^n but on its Legendre image in T_1S^n .

My interest in wave fronts started from Dupin hypersurfaces, and then in isoparametric hypersurfaces, a more rigid object.

A hypersurface is called *Dupin*

(i) if each principal curvature is constant along its curvature distribution, and a *proper Dupin*

(ii) if furthermore, each principal curvature has constant multiplicity.

(i) implies that a leaf of each distribution is a sphere [48], [56].

Thus in the closed case, they are iterated sphere bundles over a sphere. However, in general, a Dupin hypersurface is not necessarily a piece of a complete one, in contrast to isoparametric hypersurfaces.

A closed submanifold M in S^n is called *taut*,

(*) if the square of spherical distance function from any generic point of S^n satisfies the equality in the Morse inequality.

The notions of Dupin and taut are invariant under Lie contact transformations.

Theorem 6.1 (Thorbergsson (1983) [61]). *A compact embedded proper Dupin hypersurface M in S^n has the same homology as that of an isoparametric hypersurface in S^n (obtained by Münzner [45],[46]). Thus, the number of distinct principal curvatures and the multiplicities coincide with those of isoparametric ones. Furthermore, M is taut.*

Conversely,

Theorem 6.2 (M. (1984) [31], Pinkall (1986) [54]). *A compact hypersurface M embedded in S^n is Dupin if M is taut.*

Thus, proper Dupin \Rightarrow taut \Rightarrow Dupin holds when M is closed, namely, compact embedded without boundary. In this respect, closed wave fronts have a beautiful feature from the viewpoint of Morse theory.

A closed proper Dupin hypersurface M in S^n has $g \in \{1, 2, 3, 4, 6\}$ distinct principal curvatures by Theorem 6.1. The following is easily seen by the statement after (*).

Fact. When $g = 1, 2$, M is a Lie image of an isoparametric hypersurface, i.e., an image of a sphere or an orthogonal product of two spheres.

However, the following is non-trivial:

Theorem 6.3 (M. (1984) [30]). *A closed proper Dupin hypersurface with $g = 3$ is a Lie image of an isoparametric hypersurface.*

Theorem 6.1 and 6.3 led Cecil and Ryan to:

Cecil-Ryan's conjecture (1985) [10]. *A closed proper Dupin hypersurface in S^n is a Lie image of an isoparametric hypersurface.*

By the way, Theorem 6.3 is extended to the local version:

Theorem 6.4 (T. Cecil-G. Jensen (1998) [8]). *A Dupin hypersurface with $g = 3$ is a Lie image of an isoparametric hypersurface.*

§7. Discovery of the Lie curvature

Theorem 7.1 (M. (1989) [32]). *When a hypersurface in S^n has more than three principal curvatures, the cross ratio of any four principal curvatures is a Lie invariant, i.e., invariant under the Lie contact transformations.*

I named it the *Lie curvature*. I came up with this invariant while I breastfed my baby. I learned projective geometry in an undergraduate course from Professor Otsuki, and I knew the cross ratio. Since the case $g \geq 4$ stuck in my head after I proved Theorem 6.3, it came back to me. This episode appeared in the newspaper on the new year's day in 2005.

Take $z_1, z_2, z_3, z_4 \in \mathbb{C}$ and define the cross ratio by

$$[z_1, z_2; z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)},$$

which takes value in \mathbb{R} if and only if z_1, z_2, z_3, z_4 are concircular.

At a point p of a hypersurface M in S^n , let γ be the normal geodesic. An oriented hypersphere C at p having oriented contact with $T_p M$ is called a *curvature sphere* if the contact order is bigger than one. The spherical radius θ of C corresponds to the principal curvature $\lambda = \cot \theta$ of M .

If M has four principal curvatures $\lambda_i = \cot \theta_i$, each λ_i corresponds to the curvature sphere C_i of radius θ_i , and C_i cuts $\gamma = S^1$ orthogonally at the point in the distance $2\theta_i$ from p . Expressing such points by $z_i \in \gamma = S^1 \subset \mathbb{C}$, we obtain the cross ratio among the four points, and this is nothing but the Lie curvature

$$\Phi(p) = \frac{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_3)} = [z_1, z_2; z_3, z_4]$$

in Theorem 7.1. This gives a judgment index to see if two hypersurfaces are Lie equivalent or not.

§8. Answer to Cecil-Ryan's conjecture

The Lie curvature plays an essential role in:

Theorem 8.1 (M. and T.Ozawa, (1989) [43]). *There exist counterexamples to the Cecil-Ryan conjecture for $g = 4, 6$.*

Pinkall and Thorbergsson also gave a counterexample in 1989 [55] but only for $g = 4$.

Ozawa considers a sophisticated bundle structure using Clifford systems. The simplest case is the Hopf fibration:

$$\begin{array}{c} M = \pi^{-1}(L) \subset S^7: \text{isoparametric hypersurface with } g = 4, \\ \pi \downarrow \\ L \subset S^4 \quad : \text{isoparametric hypersurface with } g = 2. \end{array}$$

Deform an isoparametric hypersurface L in S^4 by a conformal transformation φ . Since $\varphi(L)$ is taut, we can show $\pi^{-1}(\varphi(L))$ is also taut, and hence Dupin by Theorem 6.2, and in fact a proper Dupin. However, the Lie curvature of $\pi^{-1}(\varphi(L))$ does not coincide with that of an isoparametric hypersurface. Thus, $\pi^{-1}(\varphi(L))$ is a proper Dupin but not a Lie image of an isoparametric hypersurface. The fibration exists for any isoparametric hypersurfaces of OT-FKM type related to Clifford algebras. On the other hand, using the above Hopf fibration, I extended it to the case $g = 6$ ([33], cf. [36]), and obtained counterexamples for $g = 6$.

I received the Geometry Prize of MSJ in 2001 for discovering the Lie curvature and the resolution of the Cecil-Ryan conjecture. I appreciate U. Pinkall, who sent me his thesis (1981), by which I learned the Lie sphere geometry for the first time.

When S.S. Chern visited Tohoku University in 2001, I heard that he mentioned my work on Dupin hypersurfaces, because a decade ago, he worked with T. Cecil and wrote a joint paper [6], in which three of my papers were cited.

§9. Lie contact structure and resolution of Sato's problem

The n -dimensional sphere S^n has the isometry group $O(n+1)$, the conformal group $O(n+1, 1)$, and the Lie contact group $O(n+1, 2)$. The Levi-Civita connection is associated with the Riemannian structure, the

Weyl connection with the conformal structure, and the Tanaka connection with the Lie contact structure.

H. Sato first considers the Lie contact structure on the unit tangent bundle T_1M of an n -dimensional Riemannian manifold M , modelled by T_1S^n with structure group $O(n+1, 2)$ [57]. The Lie algebra of $O(n+1, 2)$ is a graded Lie algebra of the 2nd kind, in contrast to that of $O(n+1, 1)$ is of the 1st kind. To treat the equivalence problem between two structures, we need curvatures associated with suitable connections.

Remark 9.1 : It is not difficult to see that Riemannian flat induces conformally flat and Lie flat. But the converse is not necessarily true.

H. Sato's problem: *Does Lie flat induces conformally flat?*

I computed all curvatures of the Tanaka connection and obtained:

Theorem 9.2 (M. (1991) [34], [35]). *A Lie flat manifold is conformally flat.*

I was interested in Sato's problem, as I know the Lie sphere geometry. Reading N. Tanaka's long papers and computing all the curvatures with my best efforts, I wrote up the above result and sent it to Professor Sato. Then he replied that Tanaka's paper I referred to contains some mistake and I should not use it. So, I computed all the curvatures again using the right paper and had the same result.

Afterward, Professors Sato and K. Yamaguchi (N. Tanaka's former student) asked me to write a joint paper on it, as my method was too primitive and not attractive. However, I declined it, as I did not want to write a paper on the same results again. Their papers after mine are [58].

§10. Gauss map of isoparametric hypersurfaces

Choose a unit normal vector field η of an isoparametric hypersurface M^{n-1} in S^n . Then the Gauss map \mathcal{G} of M at $x \in M$ is given by

$$\mathcal{G}(x) = x \wedge \eta \in \text{Gr}_+(2, \mathbb{R}^{n+1})$$

where, the oriented 2-plane Grassmannian $\text{Gr}_+(2, \mathbb{R}^{n+1})$ is identified with the complex hyperquadric

$$Q_{n-1}(\mathbb{C}) = \{[z_0, \dots, z_n] \in \mathbb{C}P^n \mid z_0^2 + \dots + z_n^2 = 0\}$$

via

$$x \wedge \eta \mapsto [x + \sqrt{-1}\eta] \in Q_{n-1}(\mathbb{C}).$$

Remark 10.1 : $Q_{n-1}(\mathbb{C})$ is a Kähler (and so symplectic) Einstein manifold with positive Ricci curvature.

Theorem 10.2 (Palmer [51]). *When M^{n-1} is an isoparametric hypersurface in S^n , the Gauss image $L = \mathcal{G}(M)$ is a minimal Lagrangian submanifold of $Q_{n-1}(\mathbb{C})$.*

§11. Arnold-Givental conjecture

Arnold-Givental conjecture on the Lagrangian intersection of a compact Lagrangian submanifold L is

$$\sharp(L \cap \phi(L)) \geq \text{SB}(L, \mathbb{Z}_2),$$

where ϕ is a Hamiltonian diffeomorphism, the intersection is transversal, and SB is the sum of the Betti numbers (with coeff. \mathbb{Z}_2).

However, in general, this fails. In fact, for a small circle k in S^2 , there exists ϕ such that $\sharp(k \cap \phi(k)) = 0 < 2 = \text{SB}(k, \mathbb{Z}_2)$. On the other hand, a great circles S^1 always intersects with $\phi(S^1)$ for any Hamiltonian diffeomorphism ϕ , since ϕ preserves the area. Hence $\sharp(L \cap \phi(L)) \geq \text{SB}(L, \mathbb{Z}_2)$ holds. We need the Floer (co)homology $\text{HF}(L)$ to attack this delicate problem.

Let L be a compact Lagrangian submanifold of $Q_{n-1}(\mathbb{C})$. When $L \cap \phi(L) = \emptyset$ for some $\phi \in \text{Ham}(M, \omega)$, we say L is *Hamiltonian displaceable*. When $L \cap \phi(L) \neq \emptyset$ for any $\phi \in \text{Ham}(M, \omega)$, we say L is *Hamiltonian non-displaceable*.

Since $\text{HF}(L)$ is generated by the intersection $L \cap \phi(L)$, $\text{HF}(L) = 0$ follows from $L \cap \phi(L) = \emptyset$. (The converse is not necessarily true).

Now, we consider the Gauss image $L = \mathcal{G}(M)$ in $Q_{n-1}(\mathbb{C})$ of an isoparametric hypersurface M in S^n . We know $L = M/\mathbb{Z}_g$ [28], and so when $g = 1$, $L = S^{n-1} \subset Q_{n-1}(\mathbb{C})$ is the real form, and when $g = 2$, $L = M/\mathbb{Z}_2 = S^{k-1} \times S^{n-k}/\mathbb{Z}_2$ is the real form.

Theorem 11.1 (Y.G. Oh (1995) [47], H. Irie, T. Sakai and H. Tasaki (2013) [23]). *When $g = 1, 2$, they satisfy $\text{HF}(L) = H_*(L, \mathbb{Z}_2)$, and $\sharp(L \cap \phi(L)) \geq \text{SB}(L, \mathbb{Z}_2)$.*

For the cases $g \geq 3$, we obtain a partial answer:

Theorem 11.2 (H. Irie, H. Ma, R. M. and Y. Ohnita (2016) [22]).

- (i) *When $g = 3$, $L = \mathcal{G}(M)$ is the \mathbb{Z}_2 -homology sphere. If $m = m_i \geq 2$, $\text{HF}(L) = H_*(L, \mathbb{Z}_2)$ and $\sharp(L \cap \phi(L)) \geq \text{SB}(L, \mathbb{Z}_2)$ hold.*
- (ii) *When $g = 4, 6$ and $m_2 \geq m_1 \geq 2$, L is Hamiltonian non-displaceable.*

While doing our joint work, we stayed in the Yau center in Hainan Island and also in RIMS, each around a week, and spent wonderful research days from morning until night. We appreciate Professor Hui Ma, who arranged for a nice stay in Hainan, that was, needless to say, due to Professor Yau's favor. We should continue studying to finish the remaining cases, thanks to their support.



Fig. 1. Yau Cr. Hainan



Fig. 2. Pub near RIMS

§12. Chern's conjecture: Yau's 105th problem

Before finishing my report, I mention my latest result on:

Chern's conjecture [13]: *A closed minimal hypersurface in the Euclidean sphere is isoparametric if it has constant scalar curvature (CSC).*

This is Yau's 105th problem [63], and still open in general. The first non-trivial case is when $\dim M = 3$, which Almeida-Brito affirmatively solves [15] together with Peng-Terng [52], Chang [11] and Cheng-Wan [12]. When $\dim M \geq 4$ with $g \leq 3$, a local version is discussed by Almeida-Brito-Scherfner-Weiss in [16], and an affirmative answer is given by Chen-Li [9]. For $g \geq 4$, Tang-Yan obtained some result under further assumptions in [62]. Recently, I gave answers in the case of Dupin hypersurfaces:

Theorem 12.1 (M, 2025 [42]). *A closed CMC Dupin hypersurface M in S^n is isoparametric*

- (i) *if $g = 3$.*
- (ii) *if $g = 4$ and M has constant scalar curvature.*
- (iii) *if $g = 4$ and M has constant Lie curvature.*
- (iv) *if $g = 6$ and M has constant Lie curvatures.*

The originality of the proof is a use of the tautness and the Lie sphere geometry rather than an algebraic argument. In (i), we do not assume CSC. (ii) implies that Chern's conjecture is true for Dupin hypersurfaces with $g = 4$. In (iii) and (iv), instead of the scalar curvature, we use the Lie curvatures. When $g = 6$, there exist 15 Lie curvatures, but three of them determine all the others [33] if they are independent, i.e., if any one of the three cannot be expressed by the other two.

The Lie curvature I discovered has not been used so much, unfortunately. It is interesting to see if isoparametric hypersurfaces and closed Dupin hypersurfaces are Lie equivalent when all the Lie curvatures are constant when $g = 4, 6$ (cf. [32], [33]). Using the Lie curvatures in the proof of Theorem 12.1, I felt the answer may be affirmative.

Acknowledgement.



Fig. 3. BNU 2019

I would like to give special thanks to Professor Yoshihiro Ohnita on the occasion of his retirement from Osaka City University (now Osaka Metropolitan University) in March 2023.

Since around 1985, when I had a hard time raising three children, he has always been concerned not only with students but also with young researchers around him including me, although I am senior to him. I am always impressed by his power, organizing many valuable conferences, getting big funds, inviting important geometers from abroad, and above all, to achieve good research in geometry.

One of the events he organized was a long-continued Kofu study camp (1994-1999), where we gathered twice a year, reading new papers, reporting our new results, and discussing each other. Every time, he edited a report, including articles such as the explanation of Uhlenbeck's work, babble trees, DPW method, etc., etc., which remain as valuable materials.

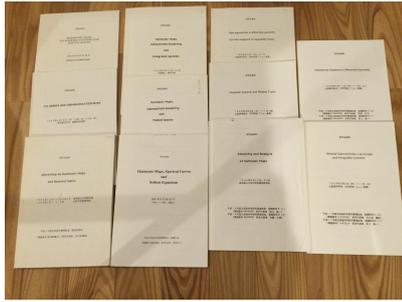


Fig. 4. Kofu Reports

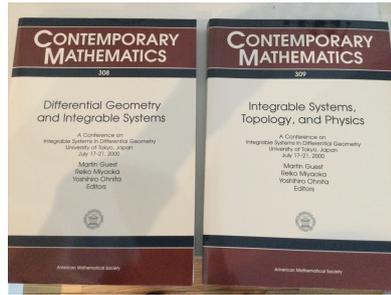


Fig. 5. Two of Proc. 2000

As one of the big events, Y. Ohnita, M. Guest, and I organized the 1st MSRI international Conference on Integrable Systems and Differential Geometry in Tokyo 2000 with more than 200 participants, including A. Bobenko, B. Dubrovin, K. Fukaya, D. Joyce, X. Liu, M. Mulase, K. Ono, R. Palais, S. Salamon, C.L. Terng, A. Veselov and many others. Three volumes of proceedings were published.

These are good memories for us all, and many young geometers left the nest and became now top researchers. Professor Ohnita is still working at Waseda University and OCAMI, and I wish him the best of luck and continued good health.

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