Stability of complete minimal Lagrangian submanifold and L^2 harmonic 1-forms

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Abstract We show that a non-compact complete stable minimal Lagrangian submanifold L in a Kähler manifold with positive Ricci curvature has no non-trivial L^2 harmonic 1-forms, which gives a topological and conformal constraint on L.

1 Introduction

In this paper, all manifolds are complete and oriented. It is well-known that there exist no compact stable minimal hypersurfaces in a Riemannian manifold with positive Ricci curvature [14]. In general, stable minimal submanifolds hardly exist in a positively curved manifold. On the stability of minimal Lagrangian submanifold in a Kähler manifold, we know:

Fact 1. [5] *A compact or compact with boundary minimal Lagrangian submanifold in a Kähler manifold with non-positive Ricci curvature is stable.*

Fact 2. [9] *A compact (Lagrangian-)stable minimal Lagrangian submanifold L in a Kähler manifold with positive Ricci curvature satisfies* $H^1(L,\mathbb{R}) = 0$.

Fact 2 suggests that there is a constraint on the topology of compact stable minimal Lagrangian submanifolds (Lagragian-stable is weaker than stable). However, both facts do not mention the complete non-compact case. The purpose of this paper is to investigate complete non-compact stable minimal Lagrangian submanifolds in a Kähler manifold with positive Ricci curvature. We obtain:

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Theorem 1. *There exist no non-trivial L^2 harmonic 1-forms on a non-compact complete stable minimal Lagrangian submanifold in a Kähler manifold M with positive Ricci curvature.*

By Dodziuk, it is shown:

Fact 3. [2] *When there are no non-trivial L^2 harmonic 1-forms on a complete non-compact Riemannian manifold N, any codimension one cycle of N disconnects N.*

Using this in the surface case, we obtain:

Theorem 2. *Let L be a complete stable minimal Lagrangian surface in a Kähler manifold (M,g) with positive Ricci curvature. Then L is conformally $S^2 \setminus \{s \text{ points}\} \setminus \{l \text{ disks}\}$, where one of the following occurs:*
(1) L is conformally S^2 ($s = l = 0$).
(2) L is conformally $\mathbb{C}$ ($s = 1$, $l = 0$), and there is no $b > 0$ such that $\overline{\mathrm{Ric}} \geq bg$.
(3) $s \geq 3$ and $l = 0$.
(4) $s \geq 1$ and $l = 1$.

Remark. We conjecture that the cases (3) and (4) do not occur.

Example: (1) Castro and Urbano [1] show that the diagonal S^2 in $Q_2(\mathbb{C}) = S^2 \times S^2$ is the unique stable minimal Lagrangian surface in $Q_2(\mathbb{C})$.

(2) Consider a complete positive metric on $\mathbb{C}$ (e.g., the induced metric on the paraboloid in $\mathbb{R}^3$) and take $M = \mathbb{C} \times \mathbb{C}$, so that M is a Kähler manifold with positive Ricci curvature. Then the diagonal set $\triangle_M$ is a complex submanifold, and hence volume minimizing by Wirtinger's inequality. Changing the complex and symplectic structure J and ω in the second term by $-J$ and $-\omega$, we see that $\triangle_M$ is a minimal Lagrangian submanifold which has the same volume as before, and hence is stable.

(3) In $M = \mathbb{C}P^1 \times \mathbb{C}$ with natural metric ($\overline{\mathrm{Ric}} \geq 0$), the standerd embedding of $S^1 \times \mathbb{R}$ is a totally geodesic Lagrangian surface with two parabolic ends. This is intuitively unstable, since S^1 could be shortened.

Here we remark that a non-compact complete Riemann surface N is classified in two ways; geometrically and function theoretically. In the former sense, if the universal covering of N is $\mathbb{C}$, we say N is parabolic, and if the universal covering is a complex disk $\mathbb{D}$, we say N is hyperbolic.

On the other hand, any dimensional non-compact complete Riemannian manifold N is called parabolic if any non-positive subharmonic function on N is constant, and nonparabolic (or hyperbolic) otherwise.

We call an unbounded component $\mathscr{E}$ of the complement of a sufficiently large compact subset of N an "end". Then $\mathscr{E}$ is called parabolic if there exists a parabolic manifold of whose only end is $\mathscr{E}$. Otherwise, $\mathscr{E}$ is called nonparabolic.

Theorem 1 is inspired by the following fact and is proved in [15]:

Fact 4. [10], [7] *Let M be a Riemannian manifold with non-negative sectional curvature and N be a complete non-compact stable minimal hypersurface in M. Then there exist no non-trivial L^2 harmonic 1-forms on N. When $\dim M = 3$, the curvature condition is weakened to non-negative scalar curvature.*

Using this, the first author [7] gives a partial proof to the results of Fischer-Colbrie and Scheon [3] to the effect that a complete orientable stable minimal surface in a Riemannian manifold M with non-negative scalar curvature are topologically S^2, T^2, $\mathbb{C}$ or $\mathbb{C}\setminus\{0\}$. Each case is realized in certain M.

The unique compact stable minimal submanifolds in $\mathbb{C}P^n$ are complex submanifolds [4]. Thus minimal Lagrangian submanifolds in $\mathbb{C}P^n$ are never stable, and the stability in this space should be argued in a weak sense, namely, the Hamiltonian stability (H-stability, for short), see [9]. The standard embeddings of T^n and $\mathbb{R}P^n$ in $\mathbb{C}P^n$ are H-stable [9]. For more results, see [8]. Concerning this, B. Palmer shows:

Fact 5. (1) [11] (§3) *The Gauss image in* $Q_2(\mathbb{C})$ *of a minimal surface in* S^3 *is minimal Lagrangian, and the only H-stable one is* S^2 *if* L *is compact.*

(2) [12] *If a non-compact complete minimal Lagrangian surface in a Kähler manifold* (M,g) *with* $\overline{\mathrm{Ric}}\geq bg$, $b>0$, *is H-stable, then the number of nonparabolic ends is less than two.*

Then Palmer conjectures:

Conjecture. [12] A non-compact complete minimal Lagrangian surface in a Kähler manifold (M,g) with $\overline{\mathrm{Ric}}\geq bg$ for some $b>0$ is *not* H-stable.

In this paper, we investigate the classical stability, and will discuss the H-stability in a separate paper.

2 Proof of Theorem 1

Let M be a real $2n$-dimensional Kähler manifold, and L be a minimal Lagrangian submanifold of M. We denote the Kähler form, the complex structure and the Kähler metric on M by ω, J and $\langle\,,\rangle$, respectively. We denote the connection, the curvature tensor and the Ricci tensor of M by $\bar{\nabla}$, $\bar{R}$ and $\overline{\mathrm{Ric}}$ respectively. We denote those of L without bar and the normal connection on L by $\nabla^{\perp}$. We adopt $\triangle=d\delta+\delta d$ as the definition of the Laplacian on L.

There is a natural correspondence between $\Lambda^1(L)$ and $\Gamma(T^{\perp}L)$ as follows: For $\alpha\in\Lambda^1(L)$, there exists $\xi\in\Gamma(T^{\perp}L)$ such that $\alpha(X)=\omega(\xi,X)$, $X\in TL$, and for $\xi\in\Gamma(T^{\perp}L)$, there exists $\alpha_\xi\in\Lambda^1(L)$ such that $\alpha_\xi(X)=\omega(\xi,X)$. Note that $\|\alpha_\xi\|=\|\xi\|$ holds.

We consider a deformation $\{\iota_t\}$ of L, namely a smooth family of immersions which satisfies $\iota_0=\iota$:

$$\iota_t : L\to M,\ t\in(-\varepsilon,\varepsilon).$$

In the following, we assume that the support of the deformation $\{\iota_t\}$ is compact and that the variation vector field

$$V_t := \frac{d}{dt}\iota_t$$

is normal to L. Let $\mathscr{A}_V(t):=\mathrm{Vol}(\iota_t(L))$. Then L is said to be *minimal* if the first variation $\mathscr{A}_V'(0)=(d/dt)|_{t=0}\mathscr{A}_V(t)$ vanishes for any compactly supported normal deformation $\{\iota_t\}$ of L. A minimal submanifold L is said to be *stable minimal* if the second variation $\mathscr{A}''_V(0) = (d^2/dt^2)|_{t=0}\mathscr{A}_V(t)$ is non-negative for any compactly supported normal deformation.

Fact 6. [9]. *Let M be a Kähler manifold and L be a minimal Lagrangian submanifold in M. Then the second variation formula of L w.r.t. a compactly supported normal variation V is given by*

$$\mathscr{A}''_V(0) = \int_L \{\langle \triangle \alpha_V, \alpha_V\rangle - \overline{\mathrm{Ric}}(V,V)\}.$$

When L is compact, Fact 1 and 2 immediately follows from this second variation formula and the Hodge theory.

When L is non-compact, we need the following fact on L^2 harmonic 1-forms.

Fact 7. [13] *Let α be an L^2 form on a Riemannian manifold L. Then α is harmonic, i.e., $\triangle\alpha = 0$, if and only if $d\alpha = 0$ and $\delta\alpha = 0$.*

Moreover, in order to obtain a variation vector field with compact support, we need a cut-off function. For any $r > 0$, we choose a function $f = f_r : L \to [0,1]$ with the following properties:

(1) f is continuous on L and smooth almost everywhere on L,

(2) $f = 1$ on $B_{r/2}$, $f = 0$ outside B_r,

(3) $\|df\|^2 \leq \dfrac{c}{r^2}$,

where B_r is a geodesic ball with radius r in L centered at a fixed point $p \in L$, and c is a constant independent of r. Such function is easily obtained by using the distance function.

Proof of Theorem 1: Let α be an L^2 harmonic 1-form on L, namely a smooth harmonic 1-form with

$$\int_L \|\alpha\|^2 < \infty.$$

We show that α must be trivial. We use ξ such that $\alpha = \omega(\xi,\ )$, the cut-off function f, and a variation vector field $V = f\xi$ with compact support. Putting $\alpha_V = \omega(V,\ ) = \omega(f\xi,\ ) = f\alpha$, we obtain from Fact 6

$$\begin{aligned}\mathscr{A}''_V(0) &= \int_L \{\langle \triangle \alpha_V, \alpha_V\rangle - \overline{\mathrm{Ric}}(V,V)\}\\ &= \int_L \{\langle (d\delta + \delta d)(f\alpha), f\alpha\rangle - \overline{\mathrm{Ric}}(V,V)\}\\ &= \int_L \{\|d(f\alpha)\|^2 + (\delta(f\alpha))^2 - \overline{\mathrm{Ric}}(V,V)\}\end{aligned}$$

since f has a compact support. For the first term, we have

$$\|d(f\alpha)\|^2 = \|df \wedge \alpha\|^2 = \|df\|^2\|\alpha\|^2 - \langle df, \alpha\rangle^2$$

since $d\alpha = 0$ holds by Fact 7. On the other hand, we have

$$\begin{aligned}(\delta(f\alpha)) &= -*d*(f\alpha)\\ &= -*(df\wedge *\alpha) - f*d*\alpha\\ &= -\langle df,\alpha\rangle\end{aligned}$$

since $\delta\alpha = 0$ holds by Fact 7. Combining these, we obtain

$$\begin{aligned}\mathscr{A}''_V(0) &= \int_L \{\|d(f\alpha)\|^2 + (\delta(f\alpha))^2 - \overline{\mathrm{Ric}}(V,V)\}\\ &= \int_L \{\|df\|^2\|\xi\|^2 - f^2\overline{\mathrm{Ric}}(\xi,\xi)\}.\end{aligned}$$

By the properties of the cut-off function f and the stability of L, we have

$$0 \le \mathscr{A}''_V(0) \le \frac{c}{r^2}\int_L \|\xi\|^2 - \int_{B_{r/2}} \overline{\mathrm{Ric}}(\xi,\xi).$$

Letting $r \to \infty$, we obtain

$$0 \le -\int_L \overline{\mathrm{Ric}}(\xi,\xi)$$

since ξ is L^2. Thus $\xi = 0$, namely, $\alpha = 0$ follows. □

3 Surface case

Proof of Theorem 2: When L is a surface, L cannot have positive genus by Theorem 1 and Fact 3. Hence L is conformally $S^2 \setminus \{s \text{ points}\} \setminus \{l \text{ disks}\}$. By Example (1) and (2), $L \cong S^2$ and $\mathbb{C}$ can be stable minimal Lagrangian in certain M.

(2) When $L \cong \mathbb{C}$, we have a harmonic 1-form $\alpha = dx$, where $z = x + iy$ is a complex coordinate of $\mathbb{C}$. Applying the cut-off function f as before to ξ such that $\alpha = \omega(\xi, \,)$, and using that the Dirichlet integral is a conformal invariant, we obtain

$$0 \le \mathscr{A}''_V(0) \le \frac{c}{r^2}\int_{B_r\setminus B_{r/2}} \|\xi\|^2 - \int_{B_{r/2}} \overline{\mathrm{Ric}}(\xi,\xi) = \frac{c}{r^2}\frac{3\pi r^2}{4} - \int_{B_{r/2}} \overline{\mathrm{Ric}}(\xi,\xi).$$

If $\overline{\mathrm{Ric}} \ge bg$, $b > 0$, the last term tends to $-b\int_L \|\xi\|^2$, which diverges since there are no L^2 harmonic 1-form on $L \cong \mathbb{C}$. Thus $\overline{\mathrm{Ric}}$ cannot be uniformly positive.

(3) When the universal covering of L is $\mathbb{C}$, L is either $\mathbb{C}$ or $\mathbb{C}\setminus\{0\}$. We show the latter is not stable. In fact, consider the holomorphic function

$$\varphi(z) = \frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

on L. Then

$$\alpha = \frac{x}{x^2+y^2}dx + \frac{y}{x^2+y^2}dx$$

is a harmonic 1-form on L. Since the L^2 norm depends only on the conformal structure, we obtain

$$\|\alpha\|_2^2 = \lim_{r\to\infty}\int_{1/r}^{r}\int_0^{2\pi}\|\alpha\|^2 r dr d\theta = 4\pi \lim_{r\to\infty}\log r.$$

The metric on L can be written as $ds^2 = \mu(p)|dz|^2$ where z is the complex parameter. Let f be the cut-off function as before w.r.t. the geodesic ball B_R around a point other than the origin, where we use the flat metric. Hence $R = r - a$ for some constant a. Since the Dirichlet integral depends only on the conformal structure, putting $V = f\xi$, $\alpha = \omega(\xi, \ )$ as before, we obtain

$$\mathscr{A}_V''(0) \le \frac{4\pi c}{R^2}\log r - \int_{B_{R/2}} \overline{\mathrm{Ric}}(\xi,\xi).$$

The first term on the right hand side tends to 0 as $R \to \infty$, and the second term tends to $\int_L \overline{\mathrm{Ric}}(\xi,\xi)$. Therefore, $L \cong \mathbb{C}\setminus\{0\}$ cannot be stable.

(4) When the universal covering of L is the disk, either $L \cong S^2 \setminus \{s \text{ points}\}$ where $s \ge 3$, or L has at least one nonparabolic end ($l \ge 1$). Since $\mathbb{D}$ has many L^2 harmonic 1-forms, $L \cong \mathbb{D}$ does not occur. Thus L has more than one ends.

When L has at least two nonparabolic ends ($l \ge 2$), Li-Tam prove that there exists a non-constant bounded harmonic function h with finite Dirichlet integral ([6], Theorem 2.1). Thus $\alpha = dh$ is an L^2 harmonic 1-form on L, and L cannot be stable by Theorem 1. □

References

1. I. Castro and F. Urbano, *Minimal Lagrangian surfaces in $S^2 \times S^2$*, Comm. Anal. and Geom. **15** (2007), 217-248.
2. J. Dodziuk, *L^2 harmonic forms on complete manifolds*, Ann. Math. Studies 102 (1982), 291-302.
3. D. Fischer-Colbrie and R. Schoen, *The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature*, Pure Appl. Math., **33** (1980), 199-211.
4. B. Lawson and J. Simons, *On stable currents and their applications to global problems in real and complex geometry*, Ann. of Math. **98** (1973), 427-450.
5. H. V. Le, *Minimal Φ-Lagrangian surfaces in almost Hermitian manifolds*, Mat. Sb. **180** (1989), 924–936, 991; translation in Math USSR-Sb. **67** (1990), 379-391.
6. P. Li and F. Tam, *Harmonic functions and the structure of complete manifolds*, J. Diff. Geom., **32** (1992), 359-383.
7. R. Miyaoka, *L^2 harmonic 1-forms in a complete stable minimal hypersurface*, Geometry and Global Analysis (Tohoku Univ.), (1993), 289-293.
8. H. Ma and Y. Ohnita, *Hamiltonian stability of the Gauss images of homogeneous isoparametric hypersurfaces*, J. Diff. Geom. (2013), to appear.
9. Y. G. Oh, *Second variation and stabilities of minimal lagrangian submanifolds in Kaähler manifolds*, Invent. Math. **101** (1990), 501-519.
10. B. Palmer, *Stability of minimal hypersurfaces*, Comment. Math. Helvetici **66** (1991), 185-188.

11. B. Palmer, *Buckling eigenvalues, Gauss maps and Lagrangian submanifolds*, Diff. Geom. Appl. **4** (1994), 391-403.
12. B. Palmer, *Biharmonic capacity and the stability of minimal Lagrangian submanifolds*, Tohoku Math. J. **55** (2003), 529-541.
13. G. de Rham, *Differentiable Manifolds, Forms, Currents, Harmonic forms*, Springer-Verlag (1984).
14. J. Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. **88** (1968), 62-105.
15. S. Ueki, *Stability of Lagrangian and isotropic submanifolds*, Dr. thesis (2013).